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CS203 Quiz

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1. (2+3 points) You are going to play 2 games of chess with an opponent who is equally likely to be a beginner, intermediate, or a master. Depending on which, your chances of winning an individual game are 90%, 50%, or 30%, respectively.

   17/30 1. What is your probability of winning the first game?
   2. Assume that the outcomes of the games are independent and only depend on the skill level of your opponent. What is the probability that you will win the second game, given that you have won the first game?

2. (5 points) Show that

$$Var[X|Y = y] = E[X^2|Y = y] - (E[X|Y = y])^2.$$

3. (6 points) Consider the set $S = \{0, 1, \ldots, 6\}$, let X, Y be two independent random variables which are uniformly distributed over S. Define seven new random variables: $Z_j = X + j \cdot Y \mod 7$, for each $j \in S$. Prove that the random variables $\{Z_j : j \in S\}$ are pairwise independent.

4. (4+4 points) Consider a decision problem F (it has YES/NO answer). You are given a randomized algorithm A, which on any input x and a random string r, outputs the correct answer with probability $\geq 5/6$. In other words,

$$\forall\, x : \quad P_r[A(x, r) = F(x)] \geq \frac{5}{6}.$$

   To decrease the failure probability (1/6), we devise an algorithm B which runs the algorithm A n times independently (the outcomes of A in different runs are mutually independent). Finally, B outputs the majority answer among the n runs of A. If we want the failure probability of algorithm B to be < 1/100,

   17 1. what is the minimum number of trials n needed if we use the law of large numbers.
   $\log_6 100$ 2. what is the minimum number of trials n needed if we use the Chernoff bound. Recall the Chernoff bound for n independent copies of random variable X is:

$$P[\Sigma_{i=1}^{n} X_i < (1 - \delta)nE[X]] \leq e^{-nE[X]\delta^2/2}$$