Assignment 1

CS203B

Abhishek Pardhi | 200026 | apardhi20@iitk.ac.in

Question 1

A company asks you to set up a mechanism to send single bits (0 or 1) between two points A and B. The company has n identical communication channels available. However, these channels are not perfect. Each of these channels flips the bits send through it (0 becomes 1 or 1 becomes 0) with probability p. The company gives you two options for using these channels to connect A and B: either the channels can be connected in series between A and B or they can be connected in parallel between A and B. In the series mode, the output of the channel at the receiving end is taken as the received bit, while in the parallel mode, the bit which appears in the majority of the channel outputs is taken as the received bit. Answer the following questions:

1A

If $p = 1/3$ and $n = 5$, which mode of connecting the channels maximizes the probability that the correct bit is received at the receiving end?

Answer: Due symmetry in this problem, let us assume that A send bit 1, then:

1. In case of parallel arrangement of channels, B will receive bit 1 only when no more than 2 flips occur in channels. Then Probability that B receives bit 1 is:

$$\mathbb{P} = \binom{5}{2} \cdot p^2 \cdot (1-p)^3 + \binom{5}{1} \cdot p \cdot (1-p)^4 + \binom{5}{0} \cdot (1-p)^5$$

$$\Rightarrow \mathbb{P} = 10 \cdot 1/9 \cdot 8/27 + 5 \cdot 1/3 \cdot 16/81 + 32/273$$

$$\Rightarrow \mathbb{P} = \frac{80 + 80 + 32}{273} = \frac{192}{273}$$

2. In case of series arrangement of channels, B will receive bit 1 only when even number of flips occur in channels. Then Probability that B receives bit 1 is:

$$\mathbb{P} = \binom{5}{0} \cdot (1-p)^5 + \binom{5}{2} \cdot p^2 \cdot (1-p)^3 + \binom{5}{4} \cdot p^4 \cdot (1-p)$$

$$\Rightarrow \mathbb{P} = 32/273 + 10 \cdot 1/9 \cdot 8/27 + 5 \cdot 1/81 \cdot 2/3$$

$$\Rightarrow \mathbb{P} = \frac{32 + 80 + 10}{273} = \frac{122}{273}$$

Hence, **Parallel** mode of connecting the channels maximizes the probability that the correct bit is received at the receiving end.

1B

Assume that $p = 1/3$, $n = 3$ and the parallel mode is used for connecting A and B. A chooses the bit 0 with probability $2/3$ and the bit 1 with probability $1/3$ and sends this random bit to B. The majority of the bits received at B turns out to be equal to 1. Given this fact, what is the probability that the original bit sent by A was also equal to 1?

Answer: lets X and Y are events where A send bit 1 and B receives bit 1 respectively. Few points to be noted:

1. If A send bit 1 then B can receive bit 1 only when no more than 1 flip occur in channels.
2. If A send bit 0 then B can receive bit 1 only when at least 2 flips occur in channels.

Then Probability that A send bit 1 given that B receives bit 1 is:

$$\mathbb{P}(X|Y) = \frac{\mathbb{P}(X \cap Y)}{\mathbb{P}(Y)} = \frac{\mathbb{P}(Y|X) \cdot \mathbb{P}(X)}{\mathbb{P}(Y)}$$

$$\Rightarrow \mathbb{P}(X|Y) = \frac{\left(\binom{3}{0} \cdot (1-p)^3 + \binom{3}{1} \cdot p \cdot (1-p)^2\right) \cdot 1/3}{\left(\binom{3}{0} \cdot (1-p)^3 + \binom{3}{1} \cdot p \cdot (1-p)^2\right) \cdot 1/3 + \left(\binom{3}{2} \cdot p^2 \cdot (1-p) + \binom{3}{3} \cdot p^3\right) \cdot 2/3}$$

$$\Rightarrow \mathbb{P}(X|Y) = \frac{(8/27 + 3 \cdot 1/3 \cdot 4/9) \cdot 1/3}{(8/27 + 3 \cdot 1/3 \cdot 4/9) \cdot 1/3 + (3 \cdot 1/9 \cdot 2/3 + 1/9) \cdot 2/3}$$

$$\Rightarrow \mathbb{P}(X|Y) = \frac{(8/27 + 4/9) \cdot 1/3}{(8/27 + 4/9) \cdot 1/3 + (2/9 + 1/27) \cdot 2/3}$$

$$\Rightarrow \mathbb{P}(X|Y) = 10/17$$

Hence, answer is $10/17$.

Question 2

Suppose U is a continuous random variable with the probability density function ($c \in R$)

$$g(u) = \begin{cases} c - |u|, & \text{if } |u| < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

2A

Find the constant c.

Answer: The total probability should be equal to 1. Then

$$\Rightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} (c - |u|)\, du = 1$$

$$\Rightarrow 2 \cdot \int_{0}^{\frac{1}{2}} (c - |u|)\, du = 1$$

$$\Rightarrow \left(cu - \frac{u^2}{2}\right)_0^{\frac{1}{2}} = \frac{1}{2}$$

$$\Rightarrow \boxed{c = \frac{5}{4}} \tag{1}$$

2B

The cumulative distribution function of a random variable X is the function $F_X(x) = P(X \leq x)$ for every $x \in R$. Find the cumulative distribution function F_U of U.

Answer: Clearly, if $x \leq -\frac{1}{2}$ then $F_X(x) = 0$. Also, if $x \geq \frac{1}{2}$ then $F_X(x) = 1$ since the former case covers no probability and the latter case covers total probability which is 1. Let's handle the remaining two cases:

1. If $x \in (-\frac{1}{2}, 0]$ then

$$\Rightarrow F_X(x) = \int_{-\frac{1}{2}}^{x} (c - |u|)\, du$$

$$\Rightarrow F_X(x) = \left(cu + \frac{u^2}{2}\right)_{-\frac{1}{2}}^{x}$$

$$\Rightarrow F_X(x) = c\left(x + \frac{1}{2}\right) + \frac{1}{2}\left(x^2 - \frac{1}{4}\right)$$

putting value of c from equation 1:

$$\Rightarrow F_X(x) = \frac{2x^2 + 5x + 2}{4}$$

2. If $x \in [0, \frac{1}{2})$ then

$$\Rightarrow F_X(x) = \int_{-\frac{1}{2}}^{x} (c - |u|)\, du$$

since $x \geq 0$ we can write:

$$\Rightarrow F_X(x) = \int_{-\frac{1}{2}}^{0} (c - |u|)\, du + \int_{0}^{x} (c - |u|)\, du$$

$$\Rightarrow F_X(x) = \frac{1}{2} + \int_{0}^{x} (c - |u|)\, du$$

since g is symmetric about y axis and total area of $g(u)$ is 1, therefore area of $g(u)$ in region where x is negative is $\frac{1}{2}$.

$$\Rightarrow F_X(x) = \frac{1}{2} + \left(cu - \frac{u^2}{2}\right)_{0}^{x}$$

$$\Rightarrow F_X(x) = \frac{1}{2} + cx - \frac{x^2}{2}$$

putting value of c from equation 1:

$$\Rightarrow F_X(x) = \frac{-2x^2 + 5x + 2}{4}$$

Therefore, definition of function F_U is:

$$\boxed{\mathrm{F}_X(x) = \begin{cases} 1, & \text{if } x \geq \frac{1}{2} \\ \frac{-2x^2+5x+2}{4}, & \text{if } 0 < x < \frac{1}{2} \\ \frac{2x^2+5x+2}{4}, & \text{if } -\frac{1}{2} < x < 0 \\ 0, & \text{if } x \leq \frac{1}{2} \end{cases}}$$

2C

Evaluate the conditional probability $Pr\left(\frac{1}{8} < U < \frac{2}{5} | \frac{1}{10} < U < \frac{1}{5}\right)$.

Answer: $Pr\left(\frac{1}{8} < U < \frac{2}{5} | \frac{1}{10} < U < \frac{1}{5}\right) =$

$$\Rightarrow \frac{Pr\left(\frac{1}{8} < U < \frac{2}{5} \bigcap \frac{1}{10} < U < \frac{1}{5}\right)}{Pr\left(\frac{1}{10} < U < \frac{1}{5}\right)}$$

$$\Rightarrow \frac{Pr\left(\frac{1}{8} < U < \frac{1}{5}\right)}{Pr\left(\frac{1}{10} < U < \frac{1}{5}\right)}$$

$$\Rightarrow \frac{\int_{1/8}^{1/5} F_U(u)du}{\int_{1/10}^{1/5} F_U(u)du}$$

$$\Rightarrow \frac{-\frac{2}{3} \cdot \left(\frac{1}{125} - \frac{1}{512}\right) + \frac{5}{2} \cdot \left(\frac{1}{25} - \frac{1}{64}\right) + 2 \cdot \left(\frac{1}{5} - \frac{1}{8}\right)}{-\frac{2}{3} \cdot \left(\frac{1}{125} - \frac{1}{1000}\right) + \frac{5}{2} \cdot \left(\frac{1}{25} - \frac{1}{100}\right) + 2 \cdot \left(\frac{1}{5} - \frac{1}{10}\right)}$$

Therefore the conditional probability $\boxed{\Pr\left(\frac{1}{8} < U < \frac{2}{5} | \frac{1}{10} < U < \frac{1}{5}\right) = \frac{261}{352}}$

Question 3

Alice has an unbiased 5-sided die and 5 different coins with her. The probabilities of obtaining a head on tosses of these coins are $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$, respectively. She likes to observe patterns in subsequent tosses of these coins. Alice performs the following experiment. She rolls the 5-sided die and if the i^{th} side turns up, she chooses the i^{th} coin and starts tossing this coin repeatedly.

3A

What is the expected number of tosses required for obtaining 6 consecutive heads given that the side 1 turned up during the roll of the die?

Answer: Let E_n be the expected number of tosses required for obtaining n consecutive heads given that the side 1 turned up during the roll of the die Notice that after finding $n-1$ consecutive heads:

1. If we get a head, then we have found n consecutive heads with number of tosses $E_{n-1}+1$.
2. If we get a tails, then we have to do the experiment again from scratch and then the number of tosses required will be $E_{n-1}+1+E_n$.

Therefore, we can write:

$$\Rightarrow E_n = \frac{1}{6}\cdot(E_{n-1}+1) + \frac{5}{6}\cdot(E_{n-1}+1+E_n)$$

$$\Rightarrow E_n = 6\cdot E_{n-1} + 6$$

Let $E'_n = E_n + \frac{6}{5}$, then:

$$\Rightarrow E'_n = 6\cdot E'_{n-1}$$

$$\Rightarrow E'_n = 6^{n+k}$$

$$\Rightarrow E_n = 6^{n+k} - \frac{6}{5}$$

We know that $E_0 = 0$

$$\Rightarrow 6^k = \frac{6}{5}$$

$$\Rightarrow E_n = 6^n\cdot\frac{6}{5} - \frac{6}{5}$$

$$\Rightarrow E_n = \frac{6}{5}\cdot(6^n - 1)$$

For $n = 6$:

$$\Rightarrow E_n = \frac{6}{5}(6^6 - 1) = 55986$$

Therefore, expected number of tosses is $\boxed{55986}$

3B

What is the expected number of tosses required for obtaining 6 consecutive heads while performing this random experiment?

Answer: Let i^{th} side turned up during the roll of the die. Then similarly

$$\Rightarrow E_n = \frac{i}{6} \cdot (E_{n-1} + 1) + \frac{6-i}{6} \cdot (E_{n-1} + 1 + E_n)$$
$$\Rightarrow i \cdot E_n = 6 \cdot E_{n-1} + 6$$

Let $i \cdot E'_n = E_n + \frac{6}{6-i}$, then:

$$\Rightarrow E'_n = \frac{6}{i} \cdot E'_{n-1}$$
$$\Rightarrow E'_n = \left(\frac{6}{i}\right)^{n+k}$$
$$\Rightarrow E_n = \left(\frac{6}{i}\right)^{n+k} - \frac{6}{6-i}$$
$$\Rightarrow E_n = \frac{6}{6-i}\left[\left(\frac{6}{5}\right)^{n} - 1\right]$$

This E_n was for a given i. Since i can be 1,2,3,4,5 with equal probability, we can write the expected number of tosses required for obtaining 6 consecutive heads while performing this random experiment as:

$$\Rightarrow E_n = \frac{\sum_{i=1}^{5} \frac{6}{6-i}\left[\left(\frac{6}{5}\right)^{n} - 1\right]}{5}$$

putting $n = 6$ we get:

$$\rightarrow E_n = 11449.42$$

Therefore, expected number of tosses is $\boxed{\approx 11450}$

3C

What is the probability that in the first n tosses, she obtains n consecutive heads?

Answer: Let X be the event where in the first n tosses, Alice obtains n consecutive heads and E_i be the event where i^{th} side turned up during the roll of the die. Then

$$\Rightarrow \mathbb{P}(X) = \sum_{i=1}^{5} \mathbb{P}(X|E_i) \cdot \mathbb{P}(E_i)$$
$$\Rightarrow \mathbb{P}(X) = \frac{1}{5} \cdot \left(\frac{1}{6}\right)^n + \frac{1}{5} \cdot \left(\frac{2}{6}\right)^n + \frac{1}{5} \cdot \left(\frac{3}{6}\right)^n + \frac{1}{5} \cdot \left(\frac{4}{6}\right)^n + \frac{1}{5} \cdot \left(\frac{5}{6}\right)^n$$
$$\Rightarrow \mathbb{P}(X) = \frac{1^n + 2^n + 3^n + 4^n + 5^n}{5 \cdot 6^n} \tag{2}$$

Therefore the required probability is $\boxed{\text{P(X)} = \left(\frac{1^n+2^n+3^n+4^n+5^n}{5 \cdot 6^n}\right)}$

3D

In the first n-tosses, she obtains n consecutive heads. Given this outcome, calculate the probability that the i^{th} side turned up during the roll of the die (the closed form expression for arbitrary i). How does these probabilities behave as $n \to \infty$?

Answer: Let X be the event where in the first n tosses, Alice obtains n consecutive heads and E_i be the event where i^{th} side turned up during the roll of the die. Then using Bayesian Formula:

$$\Rightarrow \mathbb{P}(E_i|X) = \frac{\mathbb{P}(X|E_i \cdot \mathbb{P}(E_i)}{\sum_{i=1}^{5} \mathbb{P}(X|E_i) \cdot \mathbb{P}(E_i)}$$

using 2:

$$\Rightarrow \mathbb{P}(E_i|X) = \frac{\left(\frac{i}{6}\right)^n \cdot \frac{1}{5}}{\frac{1^n+2^n+3^n+4^n+5^n}{5 \cdot 6^n}}$$

$$\Rightarrow \mathbb{P}(E_i|X) = \frac{i^n}{1^n + 2^n + 3^n + 4^n + 5^n}$$

Therefore the required conditional probability is $\boxed{\mathrm{P}(\mathrm{E}_i|X) = \frac{i^n}{1^n+2^n+3^n+4^n+5^n}}$

1. Probability in 3C converges to:

$$\Rightarrow \lim_{n\to\infty} \frac{1^n + 2^n + 3^n + 4^n + 5^n}{5 \cdot 6^n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{5} \cdot \left[\left(\frac{1}{6}\right)^n + \left(\frac{2}{6}\right)^n + \left(\frac{3}{6}\right)^n + \left(\frac{4}{6}\right)^n + \left(\frac{5}{6}\right)^n\right]$$

   We know that, if $0 < a < 1$ then:

$$\Rightarrow \lim_{n\to\infty} a^n = 0 \tag{3}$$

   Using this, the Probability in 3D will tend to $\boxed{0}$.

2. Probability in 3C converges to:

$$\Rightarrow \lim_{n\to\infty} \frac{i^n}{1^n + 2^n + 3^n + 4^n + 5^n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\left(\frac{i}{5}\right)^n}{\left(\frac{1}{5}\right)^n + \left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n + \left(\frac{4}{5}\right)^n + 1}$$

   Using 3: Probability in 3D converges to: $\boxed{\mathrm{P}= \begin{cases} 1, & \text{if } i = 5 \\ 0, & \text{otherwise} \end{cases}}$

Question 4

In this programming exercise, let us explore the behavior of the averages,

$$\frac{X_1 + X_2 + X_3 + \cdots + X_n}{n}$$

of independent and identically distributed random variables X_i as $n \to \infty$

4A

Implement the following program using any language and graph plotting library of your choice (You are encouraged to use Python with Matplotlib for this assignment). Consider the random variable X which takes the values $0, 1, 2, 3, \ldots m-1$ with the respective probabilities $p_0, p_1, p_2, p_3 \ldots p_{m-1}$ such that $p_0 + p_1 + p_2 + p_3 \cdots + p_{m-1} = 1$. Your program should take as inputs, the value m, the probabilities $p_0, p_1, p_2, p_3 \ldots p_{m-1}$ and n which is the number of samples to be generated. Generate n samples according to the distribution of X and calculate the average value of the samples generated. Repeat this sampling and averaging process for a fairly large number of iterations and store the average values obtained. Round each of the average values to the nearest integer and generate a plot of the frequency of the rounded averages thus obtained against the range of possible values.

Answer: Provided in the zipped folder as '200026.py'.

4B

The following questions should be answered in your main answer script. Give a brief account of how you implemented random sampling according to the required distributions in your program. Also, use your program to answer the following questions:

i

How does the frequency plot of the averages behave as $n \to \infty$?

Answer: I used a np.random.choice() function from the numpy library which generates a random sample from a given 1-D array.
The Average random variable behaves as a Normal random variable and the graph looks like a bell curve as $n \to \infty$. This is also proven by the central limit theorem which states that 'The distribution of sample means approximates a normal distribution as the sample size gets larger, regardless of the population's distribution'. And the peak value of this bell curve happens at the expected value of this experiment.

ii

Does the shape of the frequency plot change on varying m or the values of the probabilities? Can you interpret the shape of the plots for these distributions in terms of any of the concepts that were discussed in class?

Answer: No, the shape of the frequency plot does not change on varying m or the values of the probabilities. On changing the value of probabilities the graph may get shifted and on changing the value of m the graph may get squeezed but all these changes won't change the bell shaped curve.
The shape of the graph is like a bell curve. This is interpreted in terms of the concept of central limit theorem that is stated in the above written answer.