VIT Vellore Institute of Technology

Final Assessment Test – April 2019

Course: MAT3004 - Applied Linear Algebra

Class NBR(s): 0732 /0735/ 0736 / 0776 / 0777 / 0787 / 0788 / 0789 / 0790 / 0791 /0908 / 1759 / 1817 / 5189 / 5958 / 5964

Slot: C1+TC1+TCC1+V2

Time: Three Hours

Max. Marks: 100

(5 X 20 = 100 Marks)

Answer any FIVE Questions

1. a) Find the LU factorization of the matrix $A = \begin{bmatrix} 2 & 8 & 0 \\ 2 & 2 & -3 \\ 1 & 2 & 7 \end{bmatrix}$ where L is a lower triangular matrix with diagonal entries 1 and U is an upper triangular matrix. Using this, solve $A\bar{x} = \bar{b}$, where $\bar{x}^T = [x\ y\ z]$ and $\bar{b}^T = [18\ 3\ 12]$. **[10]**

   b) Find all values of a for which the following linear system has solutions. **[10]**

$$x + 2y + z = a^2$$
$$x + y + 3z = a$$
$$3x + 4y + 7z = 8$$

   Find the solutions for the system of equations for these values of a.

2. a) Let W be the subspace of $\mathbb{R}^4$ spanned by $w_1 = (2,0,3,-4), w_2 = (4,2,-5,1), w_3 = (2,-2,14,-13), w_4 = (6,2,-2,-3)$. Is $W = \mathbb{R}^4$? If not, find a basis of W and extend it to a basis of $\mathbb{R}^4$. **[10]**

   b) Prove that a vector space V is the direct sum of subspaces U and W, that is, $V = U \oplus W$ if and only if for any $v \in V$ there exist unique $u \in U$ and $w \in W$ such that $v = u + w$. **[10]**

3. a) Find a basis for the Row space, Column space and the null space of the matrix **[10]**

$$A = \begin{bmatrix} -2 & 2 & 3 & 7 & 1 \\ -2 & 2 & 4 & 8 & 0 \\ -3 & 3 & 2 & 8 & 4 \\ 4 & -2 & 1 & -5 & -7 \end{bmatrix}.$$

   b) Is the polynomial $p(t) = 3t^2 - 3t + 1$ a linear combination of $p_1(t) = t^2 - t, p_2(t) = t^2 - 2t + 1, p_3(t) = -t^2 + 1$? Can you conclude that $\{p_1(t), p_2(t), p_3(t)\}$ is a basis of $P_2(\mathbb{R})$? **[10]**

4. a) Let V and W be vector spaces. Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V and let $w_1, w_2, \dots, w_n$ be any vectors (possibly repeated) in W. Prove that there exists a unique linear transformation $T: V \longrightarrow W$ such that $T(v_i) = w_i$, for $i = 1, 2, \dots, n$. **[10]**

   b) Let T: on $\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, -x_2, x_1 + 4x_3)$. Let $\alpha = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and $\beta = \{v_1, v_2, v_3\}$ be another ordered basis consisting of $v_1 = (1,0,0), v_2 = (1,1,0)$, and $v_3 = (1,1,1)$ for $\mathbb{R}^3$. Find the associated matrix of T with respect to α and the associated matrix of T with respect to β. Are they similar? **[10]**

$3t^2$

$5t^2 - 5t$

$-t^2 + 2t - 1$

$-t^2 + 1$

5. a) Prove that for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, the function defined by $< x, y > = x_1 y_1 + 3x_2 y_2 + 5x_3 y_3$ is an inner product on $\mathbb{R}^3$. Find the angle between the two vectors $(2,1,1)$ and $(1,0,-1)$ with respect to this inner product. [10]

b) Let W be the subspace of the Euclidean space $\mathbb{R}^3$ spanned by the vectors $v_1 = (1,1,2)$ and $v_2 = (1,1,-1)$. [10]

(i) Find the orthogonal projection $\mathrm{Proj}_W(\mathbf{b})$ of the vector $\mathbf{b} = (1,3,-2)$ onto the subspace W.

(ii) Also find the shortest distance between $\mathbf{b}$ and the subspace W.

6. a) Find all the least squares solutions of $A\,\mathbf{x} = \mathbf{b}$ where $= \begin{bmatrix} 1 & 3 & -3 \\ 2 & 4 & -2 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. [10]

b) Find the QR factorization of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & -2 & 2 \end{bmatrix}$. [10]

7. a) The alphabets A to Z are encoded using $A \leftrightarrow 0, B \leftrightarrow 1, \ldots Z \leftrightarrow 25$. The encrypted cipher text is the sequence of numbers 50, 33, 26, 34, 22, 22. The matrix used to encrypt is $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. [10]

Find the original message.

b) What is the geometric effect of each one of the matrices $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ on a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$? [10]

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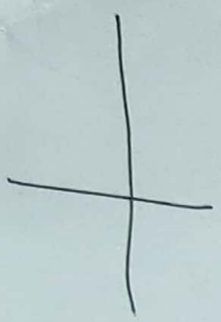

$y = x$

$y = -x$

$y = m_1 x + c_1$

$x = m_2 y + c_2$

$y = m_1 m_2 y + m_1 c_2 + c_1$

$m_1 = 1$

$m_2 = -1$

$m_1 m_2 = -1$

$m_1 \perp m_2$

A B C D E F G H I J

0 1 2 3 4 5 6 7 8 9

$0^2 + 2^2 + EW^2$

$= 4 + 4 = 8$

$\sqrt{8}$

$0(\sqrt{8} + 2\sqrt{5}$

$m = 2$

$2 = mx + 0$

$2 = mx$

$2 = 2 \cdot 1 \quad x = 1$