ME16B166.

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ASSIGNMENT - 1

1)

a) Given $X_1, X_2, \dots X_N$ & $Y_1, Y_2, \dots Y_N$ be a set of N. measurements of two variables x and y.

Errors, with standard deviation in X $\rightarrow \sigma_\delta$.

Errors, with standard deviation in Y $\rightarrow \sigma_\epsilon$.

a) Ratio of variances in errors,

$$\lambda = \frac{\sigma_\epsilon^2}{\sigma_\delta^2}$$

Objective function is,

$$J = \min_{\alpha, \beta, \hat{x}_i} \sum_{i=1}^{N} \frac{(y_i - \alpha \hat{x}_i - \beta)^2}{\lambda} + \sum_{i=1}^{N} (x_i - \hat{x}_i)^2.$$

sub. to

$$\frac{\partial J}{\partial \alpha} = 0, \quad \frac{\partial J}{\partial \hat{x}_i} = 0, \quad i = 1, 2, \dots N.$$

$$\frac{\partial J}{\partial \beta} = 0$$

Note:

N+2

$$\frac{\partial J}{\partial \alpha} = 0 \Rightarrow \sum_{i=1}^{N} \frac{(y_i - \alpha \hat{x}_i - \beta) \times (2)(-\hat{x}_i)}{\lambda} = 0 \; ; \; —(1)$$

$$\frac{\partial J}{\partial \beta} = 0 \Rightarrow \sum_{i=1}^{N} \frac{(2)(y_i - \alpha \hat{x}_i - \beta) \times (-1)}{\lambda} = 0 \; ; \; —(2)$$

$$\frac{\partial J}{\partial \hat{x}_i} = 0 \Rightarrow \cancel{\sum_{i=1}^{N}} \frac{-2\alpha \cdot (y_i - \alpha \hat{x}_i - \beta)}{\lambda} + 2(x_i - \hat{x}_i)(-1) = 0$$

$i = 1, 2, \dots N$.

$$\Rightarrow \frac{\cancel{\alpha}\,\alpha(y_i - \alpha\hat{x}_i - \beta)}{\lambda} + \cancel{2}(x_i - \hat{x}_i) = 0$$

$$\Rightarrow \hat{x}_i \cancel{\left(\frac{\alpha^2}{\lambda} + 1\right)} = \frac{x_i + \frac{\alpha}{\lambda} y_i - \frac{\alpha\beta}{\lambda}}{\frac{\alpha^2}{\lambda} + 1}$$

$$\boxed{\therefore \hat{x}_i = \frac{x_i + \frac{\alpha}{\lambda} y_i - \frac{\alpha\beta}{\lambda}}{\frac{\alpha^2}{\lambda} + 1}} \quad \text{— ③}$$

From eq ②

$$\Rightarrow \sum_{i=1}^{N} y_i - \alpha \sum_{i=1}^{N} \hat{x}_i - \beta \sum_{i=1}^{N} 1 = 0 \Rightarrow \boxed{\bar{y} - \alpha\bar{x} - \beta = 0}$$

$$\Rightarrow N\bar{y} - \alpha \sum_{i=1}^{N} \frac{x_i + \frac{\alpha}{\lambda} y_i - \frac{\alpha\beta}{\lambda}}{\frac{\alpha^2}{\lambda} + 1} - \beta N = 0$$

$$\Rightarrow N\bar{y}\left(\frac{\alpha^2}{\lambda} + 1\right) - \alpha \sum_{i=1}^{N} x_i - \cancel{\frac{\alpha^2}{\lambda} \sum y_i} + \cancel{\frac{\alpha^2\beta N}{\lambda}} - \cancel{\frac{\alpha^2\beta N}{\lambda}} - \beta N = 0$$

$$\Rightarrow \boxed{\bar{y} - \hat{\alpha}\bar{x} = \hat{\beta}}$$

$$\Rightarrow \hat{\beta} = \bar{y} - \hat{\alpha}\bar{x}$$

From eq ①

$$\Rightarrow \sum_{i=1}^{N} \hat{x}_i (y_i - \alpha\hat{x}_i - \beta) = 0 \Rightarrow \sum_{i=1}^{N} (y_i\hat{x}_i - \alpha\hat{x}_i^2 - \beta\hat{x}_i) = 0$$

$$\Rightarrow \sum_{i=1}^{N} \left[\frac{x_i + \frac{\alpha}{\lambda} y_i - \frac{\alpha\beta}{\lambda}}{\left(\frac{\alpha^2}{\lambda} + 1\right)} \times y_i - \alpha \left(\frac{x_i + \frac{\alpha}{\lambda} y_i - \frac{\alpha\beta}{\lambda}}{\frac{\alpha^2}{\lambda} + 1} \right)^2 - \beta \left(\frac{x_i + \frac{\alpha}{\lambda} y_i - \frac{\alpha\beta}{\lambda}}{\frac{\alpha^2}{\lambda} + 1} \right) \right] = 0$$

$$\Rightarrow \sum_{i=1}^{N} (\lambda x_i + y_i\alpha + \alpha^2\bar{x} - \bar{y}\alpha)(\alpha x_i + \bar{y} - \alpha\bar{x} - y_i) = 0$$

$$\Rightarrow \alpha\lambda \sum x_i(x_i - \bar{x}) + \alpha^3\bar{x} \cancelto{0}{\sum (x_i - \bar{x})} - \alpha^2\bar{x} \cancelto{0}{\sum (y_i - \bar{y})}$$

$$- \lambda \sum x_i(y_i - \bar{y}) + \alpha^2 N \cdot S_{xy} - \alpha N S_{yy} = 0$$

$\Rightarrow N S_{xy}\alpha^2 + (\lambda N S_{xx} - N S_{yy})\alpha - \lambda N S_{xy} = 0$

$\Rightarrow S_{xy}\alpha^2 + (\lambda S_{xx} - S_{yy})\alpha - \lambda S_{xy} = 0$

This is a Quadratic equation,

$$\boxed{\hat{\alpha}_{WTLS} = \frac{(S_{yy} - \lambda S_{xx}) \pm \sqrt{(S_{yy} - \lambda S_{xx})^2 + 4\lambda S_{xy}^2}}{2 S_{xy}}}$$

$$\boxed{\hat{\beta}_{WTLS} = \bar{y} - \hat{\alpha}_{WTLS}\,\bar{x}}$$

$$\boxed{\hat{x}_i = \frac{\lambda x_i + (y_i - \beta)\alpha}{(\alpha^2 + \lambda)}}$$

b) when $\beta = 0$

The model becomes $\hat{y}_i = \alpha \hat{x}_i$

~~sub~~ formulation is

$$J = \min \sum_{i=1}^{N} \frac{(y_i - \alpha\hat{x}_i)^2}{\sigma_\delta^2} + \sum_{i=1}^{N} \frac{(x_i - \hat{x}_i)^2}{\sigma_\epsilon^2}$$

sub. to

$$\frac{\partial J}{\partial \alpha} = 0 \quad \& \quad \frac{\partial J}{\partial \hat{x}_i} = 0$$

$$\frac{\partial J}{\partial \alpha} = 0 \Rightarrow \sum \hat{x}_i(\alpha\hat{x}_i - y_i) = 0 \quad \text{--- (4)}$$

$$\frac{\partial J}{\partial \hat{x}_i} = 0 \Rightarrow \hat{x}_i = \frac{\lambda x_i + y_i\alpha}{(\alpha^2 + \lambda)} \quad \text{--- (5)} \quad \left[\begin{array}{c}\text{put } \beta = 0 \\ \text{in (3)}\end{array}\right]$$

put $\hat{x}_i$ value in (4)

$$\Rightarrow \sum (\lambda x_i + y_i\alpha)\left(\alpha(\lambda x_i + y_i\alpha) - (\alpha^2 + \lambda) y_i\right) = 0$$

$$\Rightarrow \sum (\lambda x_i + y_i\alpha)\left(\alpha\lambda x_i + \cancel{y_i\alpha^2} - (\cancel{\alpha^2 y_i} + \lambda y_i)\right) = 0$$

$$\Rightarrow \sum_{i=1}^{N} (\lambda x_i + y_i\alpha).(\alpha x_i - y_i) = 0.$$

$$\Rightarrow \alpha^2 \sum_{i=1}^{N} x_i y_i + \alpha \sum_{i=1}^{N} (\lambda x_i^2 - y_i^2) - \lambda \sum_{i=1}^{N} x_i y_i = 0.$$

This is a Q.E

$$\therefore \boxed{\hat{\alpha} = \frac{(\sum y_i^2 - \lambda \sum x_i^2) + \sqrt{(\sum y_i^2 - \lambda \sum x_i^2)^2 + 4\lambda(\sum x_i y_i)^2}}{2\sum x_i y_i}}$$

b). OLS :

Assumption: 'x_i' is perfect. Model: $y_i = \alpha x_i + \beta$

Optimization formulation:

$$J = \min_{\alpha,\beta} \sum_{i=1}^{N} (y_i - \alpha x_i - \beta)^2$$

sub. to

$$\frac{\partial J}{\partial \alpha} = 0 \Rightarrow \sum (\alpha x_i + \beta - y_i) x_i = 0 \quad -(1)$$

$$\frac{\partial J}{\partial \beta} = 0 \Rightarrow \sum (\alpha x_i + \beta - y_i) = 0$$

$$\Rightarrow \boxed{\alpha \bar{x} + \beta = \bar{y}} \quad -(2)$$

sub. $\beta = \bar{y} - \alpha\bar{x}$ in (1)

$$\alpha \sum x_i^2 + (\bar{y} - \alpha\bar{x}) \sum x_i - \sum y_i x_i = 0.$$

$$\Rightarrow \alpha\left(\sum x_i^2 - N\left(\frac{\sum x_i}{N}\right)^2\right) + N\bar{x}\bar{y} - \sum x_i y_i = 0$$

$$\Rightarrow \alpha(\sum x_i^2 - N\bar{x}^2) + N\bar{x}\bar{y} - \sum x_i y_i = 0$$

$$\Rightarrow N\alpha S_{xx} - N S_{xy} = 0 \Rightarrow \boxed{\hat{\alpha}_{OLS} = \frac{S_{xy}}{S_{xx}}}$$

$$\boxed{\hat{\alpha}_{OLS} = \frac{S_{xy}}{S_{xx}}}$$

$$\boxed{\hat{\beta}_{OLS} = \bar{y} - \hat{\alpha}_{OLS}\,\bar{x}} \qquad \boxed{\hat{y}_i = \hat{\alpha}_{OLS}\, x_i + \hat{\beta}_{OLS}}$$

IOLS: (y_i's are perfect.)

$x = y\alpha' + \beta'$ ⇝ model.

Swaping x & y and following similar procedure ~~for~~, as followed for OLS, we get.

$$\boxed{\hat{\alpha}' = \frac{S_{xy}}{S_{yy}}, \quad \hat{\beta}' = \bar{x} - \hat{\alpha}'\bar{y}} \qquad \boxed{\hat{x}_i = \hat{\alpha}' y_i + \hat{\beta}'}$$

$$y = \frac{x}{\alpha'} - \frac{\beta'}{\alpha'} \implies \boxed{\hat{\alpha}_{IOLS} = \frac{1}{\hat{\alpha}'}}; \quad \boxed{\hat{\beta}_{IOLS} = \frac{-\beta'}{\alpha'}}$$

TLS:

from part ⓐ (WTLS calculation, $\lambda = 1$ for TLS.

i.e. $$J = \min_{\alpha, \beta, \hat{x}_i} \sum_{i=1}^{N} (y_i - \alpha\hat{x}_i - \beta)^2 + \sum_{i=1}^{N} (x_i - \hat{x}_i)^2$$

$$\boxed{\begin{aligned} \hat{\alpha}_{TLS} &= \frac{(S_{yy} - S_{xx}) + \sqrt{(S_{yy} - S_{xx})^2 + 4S_{xy}^2}}{2S_{xy}} \\ \therefore \hat{\beta}_{TLS} &= \bar{y} - \hat{\alpha}_{TLS}\bar{x} \end{aligned}}$$

$$\therefore \hat{y}_i = \hat{\alpha}_{TLS}\hat{x}_i + \hat{\beta}_{TLS} \qquad \boxed{\hat{x}_i = \frac{\lambda x_i + (y_i - \hat{\beta}_{TLS})\hat{\alpha}_{TLS}}{(\hat{\alpha}_{TLS}^2 + 1)}}$$

2 a) Assuming standard ~~Et~~

Extraction photometric — X

Catalytic Fluorimeter — Y.

1 Using OLS :

Assumptions: 'X' is 'perfect' (No. error in measurement of x_i's).

Model : $\hat{y}_i = \alpha x_i + \beta$

where $\hat{\alpha}_{OLS} = \frac{S_{xy}}{S_{xx}}$

where. $\left[S_{xx} = \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \bar{x}^2 \quad S_{xy} = \frac{1}{N}\sum_{i=1}^{N} x_i y_i - \bar{x}\bar{y} \right]$

$\hat{\beta}_{OLS} = \bar{y} - \hat{\alpha}\bar{x}$.

All calculations are done in excel, attached at end of document

$\bar{x} = 2.0155$; $\bar{y} = 1.9505$. , $S_{xx} = 0.931135$.

$S_{yy} = 0.923245$.

$S_{xy} = 0.924072$

$\hat{\alpha}_{OLS} = \frac{S_{xy}}{S_{xx}} = 0.99241$.

$\boxed{\hat{\alpha}_{OLS} = 0.99241}$

$\hat{\beta}_{OLS} = \bar{y} - \hat{\alpha}_{OLS}\bar{x} = 1.9505 - 0.99241 \times 2.0155$.

$\boxed{\hat{\beta}_{OLS} = -0.0495}$

$\boxed{y = 0.99241 x - 0.0495}$ ⇝ OLS model.

$\hat{\alpha}_{OLS} \sim 1$. , $\hat{\beta}_{OLS} \sim 0$

(2) a) (ii) IOLS method :

Assumption : 'y' is perfect.

model : $x = \alpha' y + \beta'$.

Estimates :

$$\hat{\alpha}' = \frac{S_{xy}}{S_{yy}} = \frac{0.924072}{0.923244} \doteq 1.0009$$

$$\hat{\beta}' = \bar{x} - \hat{\alpha}' \bar{y} = 0.06325$$

$\hat{\alpha}' \sim 1$, $\hat{\beta}' \simeq 0$

(iii) TLS method :

Assumption : Both x & y have same error mean & variance in their measurement.

$$\hat{\alpha}_{TLS} = \frac{(S_{yy} - S_{xx}) + \sqrt{(S_{yy} - S_{xx})^2 + 4S_{xy}^2}}{2 S_{xy}}$$

$$= \frac{(0.923244 - 0.931135) + \sqrt{(0.923244 - 0.931135)^2 + 4 \times 0.924072^2}}{2 \times 0.924072}$$

$\hat{\alpha}_{TLS} = 0.995737$, $\hat{\beta}_{TLS} = \bar{y} - \bar{\alpha}_{TLS} \bar{x}$

$\hat{\beta}_{TLS} = -0.05639$.

$\hat{\alpha}_{TLS} \sim 1$, $\hat{\beta}_{TLS} \sim 0$

Conclusion:

Ideally, $\alpha = 1$, $\beta = 0$ in $y = \alpha x + \beta$;

In OLS, IOLS & TLS, the values of α — close to 1.

and value of β ~ close to 0.

TLS is a line in between IOLS & OLS.

even for it α — close to 1 & β — close to 0.

* Among the methods the '$\hat{\alpha}$' estimate of IOLS is very close to '1'. So it can be the most accurate method, followed by TLS., followed by OLS.

* The new method (CF) can be a good substitute for EP

EP_X	CF_y	Xi-X_	Yi-Y_	(Xi-X_)^2	(Yi-Y_)^2	(Xi-X_)(Yi-Y_)
1.98	1.87	-0.0355	-0.0805	0.00126	0.00648	0.00285775
2.31	2.2	0.2945	0.2495	0.08673	0.06225	0.07347775
3.29	3.15	1.2745	1.1995	1.62435	1.4388	1.52876275
3.56	3.42	1.5445	1.4695	2.38548	2.15943	2.26964275
1.23	1.1	-0.7855	-0.8505	0.61701	0.72335	0.66806775
1.57	1.41	-0.4455	-0.5405	0.19847	0.29214	0.24079275
2.05	1.84	0.0345	-0.1105	0.00119	0.01221	-0.00381225
0.66	0.68	-1.3555	-1.2705	1.83738	1.61417	1.72216275
0.31	0.27	-1.7055	-1.6805	2.90873	2.82408	2.86609275
2.82	2.8	0.8045	0.8495	0.64722	0.72165	0.68342275
0.13	0.14	-1.8855	-1.8105	3.55511	3.27791	3.41369775
3.15	3.2	1.1345	1.2495	1.28709	1.56125	1.41755775
2.72	2.7	0.7045	0.7495	0.49632	0.56175	0.52802275
2.31	2.43	0.2945	0.4795	0.08673	0.22992	0.14121275
1.92	1.78	-0.0955	-0.1705	0.00912	0.02907	0.01628275
1.56	1.53	-0.4555	-0.4205	0.20748	0.17682	0.19153775
0.94	0.84	-1.0755	-1.1105	1.1567	1.23321	1.19434275
2.27	2.21	0.2545	0.2595	0.06477	0.06734	0.06604275
3.17	3.1	1.1545	1.1495	1.33287	1.32135	1.32709775
2.36	2.34	0.3445	0.3895	0.11868	0.15171	0.13418275

E(X)	E(Y)	SXX	SYY	SXY
2.0155	1.9505	0.931135	0.923245	0.924072

(2) b) (i) OLS :

$x_i = 2.31$ mg/l.

$y_i = 2.20$ mg/l.

$\hat{y}_i = 0.9924 \times 2.31 - 0.0497$

$\boxed{\hat{y}_i = 2.243 \text{ mg/l}}$

95% Confidence interval for estimates:

$\sum (y_i - \hat{y}_i)^2 = 0.123629$

$\hat{\sigma}^2_{\varepsilon y} = \frac{\sum (y_i - \hat{y}_i)}{18} = 0.006868$ [Error mean = 0] [Error - Variance]

$Y_i \sim N(\hat{y}_i, \sigma^2_{\varepsilon y})$

$\hat{\sigma}^2_{\hat{\alpha}_{OLS}} = \frac{\hat{\sigma}_{\varepsilon y}^2}{N S_{xx}} \approx 0.0003688$

$\hat{\sigma}_{\hat{\alpha}_{OLS}} = 0.0192$

$\hat{\alpha}_{OLS} = 0.9924$

True mean of -

$\alpha_{ols} \in \left[\hat{\alpha}_{OLS} - t_{18, 95\%} \times \frac{\hat{\sigma}_{\alpha_{OLS}}}{\sqrt{n}}, \hat{\alpha}_{OLS} + t_{18} \times \frac{\hat{\sigma}_{\hat{\alpha}_{OLS}}}{\sqrt{n}} \right]$

$\alpha_{OLS} \in \left[0.9924 - \frac{2.1009 \times 0.0192}{\sqrt{20}}, 0.9924 + \frac{2.1009 \times 0.9924}{\sqrt{20}} \right]$

$\boxed{\tilde{\alpha}_{OLS_{true}} \in [0.985, 1.0002]}$

For a particular x_i, y_i

$\alpha_{OLS} \in \left[\hat{\alpha}_{OLS} - t_{18} \times \hat{\sigma}_{\hat{\alpha}_{OLS}}, \hat{\alpha}_{OLS} + t_{18} \times \hat{\sigma}_{\hat{\alpha}_{OLS}} \right]$

$\alpha_{OLS} \in [0.9924 - 2.1009 \times 0.0192, 0.9924 + 2.1009 \times 0.0192]$

$\boxed{\alpha_{OLS} \in [0.9521, 1.0328]}$

$$\cancel{\hat{\sigma}_{\hat{\beta}_{OLS}}} = \cancel{\hat{\sigma}_{\varepsilon_y} \cdot x \cdot \sqrt{\frac{\sum x_i^2}{n \sum (x_i - \bar{x})^2}}}$$

<u>Estimate for y_i:</u>

$$Y_i \sim N(\hat{y}_i, \sigma^2_{\varepsilon_y}).$$

$$\boxed{\frac{Y_i - \hat{y}_i}{\hat{\sigma}_{\varepsilon_y}} \sim t_{N-2}.}$$

$$Y_i^* \in [\hat{y}_i - \hat{\sigma}_{\varepsilon_y} t_{n-2}, \hat{y}_i + \hat{\sigma}_{\varepsilon_y} t_{n-2}]$$

$$\in [2.243 - 0.0828 \times 2.1009, 2.243 + 0.0828 \times 2.1009]$$

$$Y_i^* \in [2.0688, 2.416] \quad \boxed{Y_i^* \sim \text{true value}}$$

$\rightarrow y_i = 2.2$, lies in the interval.

(ii) - <u>IOLS:</u> $\boxed{\hat{x}_i = 1.0009\, y_i + 0.06325}$ $\boxed{\hat{x}_i = 2.265\ \text{mg/l.}}$

$$n = 20 \; ; \; \hat{\sigma}^2_{\varepsilon_x} = \frac{\sum (x_i - \hat{x})^2}{18} = 0.0069. \rightsquigarrow [\text{Error variance in } x].$$

$$X_i \sim N(\hat{x}_i, \sigma^2_{\varepsilon_x}).$$

<u>Estimates:</u>

$$\hat{\sigma}_{\hat{\alpha}_{IOLS}} = \frac{\hat{\sigma}_{\varepsilon_x}}{\sqrt{N S_{yy}}} = 0.01933$$

True mean of

$$\alpha'_{IOLS} \in \left[\hat{\alpha}'_{IOLS} - \frac{\hat{\sigma}_{\alpha'_{IOLS}} \times t_{n-2}}{\sqrt{n}}, \hat{\alpha}'_{IOLS} + \frac{\hat{\sigma}_{\alpha'_{IOLS}} \times t_{n-2}}{\sqrt{n}} \right]$$

$$\boxed{\underset{\text{true mean}}{\alpha'^*_{IOLS}} \in [0.993, 1.009]}$$

for a particular $x_i; y_i$

$$\alpha'^*_{IOLS} \in [\hat{\alpha}'_{IOLS} - \hat{\sigma}_{\alpha'_{IOLS}} \times t_{n-2}, \hat{\alpha}'_{IOLS} + \hat{\sigma}_{\alpha'_{IOLS}} \times t_{n-2}]$$

$$\boxed{\alpha'^*_{IOLS} \in [0.9683, 1.0496]}$$

Estimate for . x_i:

$$X_i \sim N(\hat{X}_i, \sigma^2_{\varepsilon_x})$$

$$\hat{\sigma}^2_{\varepsilon_x} = \frac{\sum (x_i - \hat{x}_i)^2}{18} = 0.0069.$$

$$X_i \sim [2.265 - 2.1009 \times 0.08306, \; 2.265 + 2.1009 \times 0.08306]$$

true value.

$$\boxed{X_i \in [2.0904, 2.4395]}$$

$$\boxed{\text{i.e. } X_i \sim [\hat{x}_i - t_{18} \times \hat{\sigma}_{\varepsilon x}, \; \hat{x}_i + t_{18} \times \hat{\sigma}_{\varepsilon x}]}$$

(iii) TLS :

$$\hat{x}_i = \frac{x_i + (y_i - \beta)\alpha}{(1 + \alpha^2)}$$

$$\hat{x}_i = \frac{2.31 + (2.2 + 0.05641)(0.99574)}{(1 + 0.99574^2)}$$

$$\boxed{\hat{x}_i = 2.288 \text{ mg/l}}$$

$$\hat{y}_i = \hat{\alpha}\hat{x}_i + \hat{\beta}$$

$$\hat{y}_i = 0.9957 \times 2.288 - 0.05639$$

$$\boxed{\hat{y}_i = 2.222 \text{ mg/l}}$$

(3) $(\Delta T)_{max} = 1.5^\circ C$

$\Downarrow$

$(\Delta T)_{max} = 2.7^\circ F$

$\therefore {}^\circ F = \frac{9}{5} \times ({}^\circ C) + 32$

$\boxed{\Delta F = \frac{9}{5} \Delta C}$

We will use the OLS & IOLS models first.

Let $CO_2 \longrightarrow x$.

$T \longrightarrow y$.

All calculations are done in Excel.

$\bar{x} = 369.4197$ $\quad S_{xx} = 255.7407$

$\bar{y} = 0.851226$ $\quad S_{yy} = 0.0875$

$S_{xy} = 4.19215$

(i) By OLS:

Assumption: The measurement in CO_2 value is made perfectly (i.e. Instrument is very accurate)

Model: $y = \alpha x + \beta$

Estimates:

$$\hat{\alpha}_{OLS} = \frac{S_{xy}}{S_{xx}} = 0.0164$$

$$\hat{\beta}_{OLS} = \bar{y} - \hat{\alpha}_{OLS}\bar{x} = -5.2072$$

$$\therefore \boxed{y = 0.0164x - 5.2072}$$

Now,

max Global increase, $y_i = 2.7°F$.

$$(x_i)_{correl.} = \frac{2.7 + 5.2072}{0.0164} = \underline{482.25 \text{ ppm } CO_2}$$

IOLS :

Assumption: Measurement in Temperature is made perfectly.

Model : $x = \alpha' y + \beta'$.

Estimates :-

$$\hat{\alpha}' = \frac{S_{xy}}{S_{yy}} = 47.91$$

$$\hat{\beta}' = \bar{x} - \hat{\alpha}' \bar{y} = 328.64$$

$$\boxed{x = 47.91 y + 328.64}$$

max Global increase, $y_i = 2.7°F$

$$(x_i)_{correl.} = 2.7 \times 47.91 + 328.64 = \underline{457.9 \text{ ppm } CO_2}$$

TLS :-

$$\hat{\alpha} = 0.01639 \qquad \hat{\beta} = -5.20782$$

$$\hat{x}_i \Rightarrow \frac{\hat{\alpha} y_i + x_i - \hat{\beta}\hat{\alpha}}{1 + \hat{\alpha}^2} = \underline{482.19 \text{ ppm } CO_2}$$

Conclusion:

IOLS is the best method suitable for this because by IOLS, the max. limit of CO_2 value is 457.9 ppm which is less than corresponding to that derived by OLS & TLS.

So, we use IOLS.