Solutions to selected exercise of "Randomized Algorithms"

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1 Chapter 4

1.1 Problem 4.7

1. We first prove that
$$e^{\lambda x} - 1 \leq \lambda(e^x - 1)$$
When $x = 0$, $LHS = RHS = 0$. And when $x > 0$,
$$(e^{\lambda x} - 1)' = \lambda e^{\lambda x} \leq \lambda e^x = [\lambda(e^x - 1)]'$$
The left work is easy
$$\begin{array}{ll} & e^{\lambda x} - 1 \leq \lambda(e^x - 1) \\ \Rightarrow & e^{\lambda(x_1 - x_2)} e^{x_2} \leq \lambda(e^{x_1} - e^{x_2}) + e^{x_2} \ (x = x_1 - x_2) \\ \Rightarrow & e^{\lambda x_1 + (1-\lambda)x_2} \leq \lambda e^{x_1} + (1-\lambda)e^{x_2} \end{array}$$

2. This part is a bit tricky.
$$\begin{aligned} E[f(z)] &= \sum_z f(z) Pr[Z = z] \\ &= \sum_z f[0 \cdot (1-z) + 1 \cdot z] Pr[Z = z] \\ &\leq \sum_z [z \cdot f(1) + 1 - z f(0)] Pr(Z = z) \\ &= \sum_z z Pr(Z = z) f(1) + \sum_z (1-z) Pr(Z = z) f(0) \\ &= E(Z) f(1) + (1 - E(Z)) f(0) \\ &= p f(1) + (1-p) f(0) \\ &= E[f(x)] \end{aligned}$$

3. Here comes that final step.

$$\begin{aligned}
& Pr(Y > \lambda + \mu) \\
< \ & \frac{E(e^{tY})}{e^{t(\lambda+\mu)}} = \frac{\prod_{i=1}^{n} E(e^{tY_i})}{e^{t(\lambda+\mu)}} \\
< \ & \frac{\prod_{i=1}^{n} E(e^{tZ_i})}{e^{t(\lambda+\mu)}} \text{ (by (1) and (2), } Z_i \text{ is the Bernoulli random variable)} \\
= \ & \ldots = \frac{e^{(e^t-1)\mu}}{e^{t(\lambda+\mu)}} \\
\leq \ & \left[\frac{e^{\frac{\lambda}{\mu}}}{(1+\frac{\lambda}{\mu})^{(1+\frac{\lambda}{\mu})}} \right]^{\mu} \\
= \ & \frac{e^{\lambda}}{\left[e^{\frac{\lambda}{p}} (1 - \frac{(\frac{\lambda}{p})^2}{n}) \right]^{(p+\frac{\lambda}{n})}} \ (\mu = pn) \\
\approx \ & e^{-\frac{\lambda^2}{pn}} = e^{-\frac{2\lambda^2}{n}} \text{ (here } p = 1/2)
\end{aligned}$$

One can also use Azuma's inequality, but the result will be $e^{-\frac{\lambda^2}{2n}}$, which is a little worse than Hoeffding's bound.

2 Chapter 5

2.1 Problem 5.5

For every vertex $v \in L$, we choose $n^{\frac{3}{4}}$ vertices uniformly and randomly from R as v's neighbors. Let A be the event of "every vertex in R has degree $> 3n^{\frac{3}{4}}$", and let B be the event of "every subset of $n^{\frac{3}{4}}$ vertices in L that has fewer than $n - n^{\frac{3}{4}}$ neighbors in R. $Pr[A] < \frac{1}{2}$ could be verified by Chernoff bound easily. Now we are going to prove that $Pr[B] < \frac{1}{2}$.

$$\begin{aligned}
Pr[B] \ & \leq \ \binom{n}{n^{\frac{3}{4}}} \binom{n}{n - n^{\frac{3}{4}}} \left(\frac{n - n^{\frac{3}{4}}}{n} \right)^{(n^{\frac{3}{4}} \cdot n)/2} \\
& \leq \ \left(\frac{en}{n^{\frac{3}{4}}} \right)^{n^{\frac{3}{4}}} \left(\frac{en}{n - n^{\frac{3}{4}}} \right)^{n - n^{\frac{3}{4}}} \left(\frac{n - n^{\frac{3}{4}}}{n} \right)^{\frac{n^{7/4}}{2}} \\
& \leq \ \cdots \\
& \leq \ e^{n + \frac{1}{4} n^{\frac{3}{4}} \ln n - \frac{n^{\frac{3}{2}}}{2}} < \frac{1}{2}
\end{aligned}$$

Therefore, $Pr(A \cup B) < 1$.

2.2 Problem 5.6

In the **Lazy Select** algorithm, it is guaranteed that given n selections, at least $n \cdot (1 - O(n^{-\frac{1}{4}})) > n^{\frac{3}{4}}$ selections will be successful. Thus $\log n$ random bits are enough since the probability of failure is at most $n^{\frac{3}{4}}/n = n^{-\frac{1}{4}}$.

2.3 Problem 5.7

1. First I would like to show a wrong attempt. With the idea of "add vertex one by one".

$$\begin{aligned}
& \#\text{ (connected subgraph of size } r) \\
= & \binom{n}{r} Pr[\text{the } r \text{ vertices are connected}] \\
\leq & \binom{n}{r} \prod_{i=1}^{r-1} d \left(\frac{i}{n}\right) \\
\approx & \left(\frac{ne}{r}\right)^r \left(\frac{d}{n}\right)^r \sqrt{2\pi r} \left(\frac{r}{e}\right)^r \frac{n}{rd} \\
= & nd^r \sqrt{2\pi r}/rd \\
\leq & nd^r
\end{aligned}$$

   Unfortunately, this approach is wrong. The crux lies in that when considering "add vertex one by one" we have assumed that there is an order list $(v_1, v_2, \ldots, v_r)$, in which v_j is adjacent to some v_i with $i < j$. It is also corresponding to a spanning tree. But we didn't count the number of such trees.

   The correct solution.
   First, we know that every connected subgraph with r vertices has a spanning tree. And we know that the number of spanning trees is less than 4^r up to isomorphism [1]. Now we fix one of such tree structures, label the vertices with $(A_1, A_2, \ldots, A_r)$ and look for how many subgraphs of the n vertices graph have such a structure. Notice that A_j is adjacent with A_i for some $i < j$. One can also think that A_i is the parent of A_j and A_1 is the root.

   We choose A_1 first, there are n possibilities, and next we choose A_2, there are at most d possibilities, and $A_3, \ldots, A_r$. Therefore the total number of such subgraphs is less than $4^r nd^{r-1}$, which is less than nd^{2r} if one likes.

2.

$$\begin{aligned}
& Pr[\text{surviving connected subgraph of size} \geq \alpha \log n \text{ for some } \alpha > 1] \\
= & \sum_{l \geq \alpha \log n} \left[\#(\text{connected subgraph of size } l) \left(\frac{1}{2d^2}\right)^l\right] \\
= & \frac{1}{d} \sum_{l \geq \alpha \log n} \left[n(4d)^l \left(\frac{1}{2d^2}\right)^l\right]
\end{aligned}$$

[1] No close formula for the number t(n) of trees with n vertices up to isomorphism is known, however, we have approximation

$$t(n) \sim \beta \alpha^n n^{-5/2}$$

where $\beta \approx 0.5$ and $\alpha \approx 3$

$$
\begin{aligned}
&= n\frac{\left(\frac{2}{d}\right)^{\alpha\log n}}{d-2} \\
&= \beta n^{\alpha(1-\log d)+1} \ (\beta = 1/(d-2))
\end{aligned}
$$

If using nd^{2r} instead of $4^r nd^{r-1}$, one will get

$$
\ldots = n\sum_{l\geq\alpha\log n}\left[(d^2)^l\left(\frac{1}{2d^2}\right)^l\right] = n\sum_{l\geq\alpha\log n}\left(\frac{1}{2}\right)^l = 2n^{-\alpha+1} = o(1)
$$

2.4 Problem 5.8

1. For any edge $e \in E$,

$$
Pr[e\ survives] = (\frac{1}{3}n^{\frac{3}{4}})^2 = \frac{1}{9\sqrt{n}}
$$

   Thus $e\frac{1}{9\sqrt{n}}(\sqrt{n}+1) \leq 1$, the Lovász Local Lemma tells us that such an independent set exists.

2. Since $E(|V|) = \frac{1}{3}n^{\frac{3}{4}}$, by the Chernoff bound

$$
Pr[|V| > (1+\epsilon)\frac{1}{3}n^{\frac{3}{4}}] < \left[\frac{e^\epsilon}{(1+\epsilon)^{(1+\epsilon)}}\right]^{\frac{1}{3}n^{\frac{3}{4}}}
$$

   Again, we have $E(|E|) = \frac{n\sqrt{n}}{2}\frac{1}{9\sqrt{n}} = \frac{n}{18}$ and

$$
Pr[|E| < (1-\epsilon)\frac{n}{18}] < e^{-\frac{n}{18}\epsilon^2/2}
$$

   From above we know that the $Pr[|V| < (1+\epsilon)\frac{1}{3}n^{\frac{3}{4}} \cup |E| > (1-\epsilon)\frac{n}{18}]$ is very high. Given $(1+\epsilon)\frac{1}{3}n^{\frac{3}{4}}$ vertices, if $|IS| \geq \alpha\sqrt{n}$ for some $\alpha > 1$, the maximum number of edges we could expect in the graph is (all form cliques with equal size)

$$
\alpha\sqrt{n}\cdot\frac{1}{2}\left(\frac{(1+\epsilon)\frac{1}{3}n^{\frac{3}{4}}}{\alpha\sqrt{n}}\right)^2 = (1+\epsilon)\frac{n}{\alpha 18} < (1-\epsilon)\frac{n}{18}
$$

   Which is very unlikely.

   After the random deletion, there are still about $\frac{1}{3}n^{\frac{3}{4}}$ vertices left, which is considerably larger than $\sqrt{n}$, so the positive probability proved by Lovász Local Lemma is indeed very small.

2.5 Problem 5.10

We choose a complete graph $G = (V, E)$, and delete every edge with probability $\frac{1}{2}$. Given any subset $T \subseteq V$ with size $n/2$, we can obtain the expect value of $c(T) = n^2/8$ [*that is* , $\#(vertices\ v_i \in T) \times \#(its\ neighbors\ u_i \notin T) \times (Pr[e = (v_i, u_i)]\ not\ deleted)$].

Next we define event A_T to be

$$\left|c(T) - \frac{n^2}{8}\right| > f(n)$$

where $f(n)$ will be specified later. It is obviously that A_T is a Binomial random variable with expectation $\frac{n^2}{8}$. By the standard estimates for Binomial distribution give in A.1.14 in Book [1], we obtain

$$Pr\left[\left|c(T) - \frac{n^2}{8}\right| > f(n)\right] < 2 \cdot exp\left(-\frac{f^2(n)}{2\left(\frac{n^2}{8}\right)^2} \cdot \frac{n^2}{8}\right) = 2 \cdot e^{-4\left(\frac{f(n)}{n}\right)^2}$$

There are $\binom{n}{\frac{n}{2}} \approx 2^n$ subsets with size of $\frac{n}{2}$, so A_T is dependent on at most 2^n other events (Yeah, quite large!). Now we plug everything to the symmetric case of the local lemma, we want

$$e \cdot (2^n + 1) \cdot 2e^{-4\left(\frac{f(n)}{n}\right)^2} < 1$$

It is true for $f(n) = \alpha \cdot n^{\frac{3}{2}}$ for some $\alpha > \frac{\sqrt{\ln 2}}{2}$.

Therefore we can make $f(n) = O(n^{\frac{3}{2}})$.

Another perspective

On the other hand, we can also use Martingales to show that on a random graph $G(n, 1/2)$, with high probability, a fixed subset T with size of $\frac{n}{2}$ renders a $c(T)$ with $|c(T) - \frac{n^2}{8}| \leq O(n^{\frac{3}{2}})$. We use a general setting introduced by Alon, see [1] p.101. The theorem asserts that for all $\epsilon > 0$ there exists $\delta > 0$ so that the following holds. Suppose Paul has a strategy for finding Y such that every line of questioning has total variance at most σ^2. Then

$$Pr[|Y - E[Y]| > \alpha\sigma] \leq 2 \cdot e^{-\frac{\alpha^2}{2(1+\epsilon)}}$$

for all positive α with $\alpha C < \sigma(1+\epsilon)\delta$. Here we choose an Edge Exposure Martingale, such that an edge is deleted or not will change the value of $c(T)$ at most 1, so the variance of the total "line of the question" would be $\sigma = \sum P_i(1-P_i)c_i^2 < \frac{1}{4} \cdot 1 \cdot \frac{n^2}{2} = \frac{n^2}{8}$. We choose $\epsilon = 1$, $\delta = 1$, $\alpha = \sqrt{2n}$, and then

$$Pr\left[\left|c(T) - \frac{n^2}{8}\right| > \frac{n^{\frac{3}{2}}}{4}\right] \leq 2 \cdot e^{-n}$$

Since there are at most $\binom{n}{n/2} \approx 2^n$ many subsets with $n/2$ vertices in G, and $2^n \cdot 2\,e^{-n} < 1$, we know that a random graph has a positive probability to satisfy all the subsets.

Summary

The first solution shows that we could construct a graph by deleting every edge from a complete graph with probability 1/2. The second approach shows that a random graph $G = (n, 1/2)$ has a positive probability to meet the requirement. The second result is stronger than the first in some sense.

2.6 Problem 5.11

See book [1] Chapter 15, p.251-253.

References

[1] Noga Alon and Joel H. Spencer. *The probabilistic method (2ed).* New York: Wiley-Interscience, 2000.