PERIOD OF THE CONTINUED FRACTION OF $\sqrt{n}$

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Abstract. This paper seeks to recapitulate the known facts about the length of the period of the continued fraction expansion of $\sqrt{n}$ as a function of n and to make a few (possibly) original contributions. I have established a result concerning the average period length for $k < \sqrt{n} < k + 1$, where k is an integer, and, following numerical experiments, tried to formulate the best possible bounds for this average length and for the maximum length of the period of the continued fraction expansion of $\sqrt{n}$, with $\lfloor \sqrt{n} \rfloor = k$.

Many results used in the course of this paper are borrowed from [1] and [2].

1. Preliminaries

Here are some basic definitions and results that will prove useful throughout the paper. They can also be probably found in any number theory introductory course, but I decided to include them for the sake of completeness.

Definition 1.1. *The integer part of x, or $\lfloor x \rfloor$, is the unique number $k \in \mathbb{Z}$ with the property that $k \leq x < k + 1$.*

Definition 1.2. *The continued fraction expansion of a real number x is the sequence of integers $(a_n)_{n \in \mathbb{N}}$ obtained by the recurrence relation*

$$x_0 = x, \ a_n = \lfloor x_n \rfloor, \ x_{n+1} = \frac{1}{x_n - a_n}, \ \textit{for } n \in \mathbb{N}.$$

Let us also construct the sequences

$$\begin{aligned} P_0 &= a_0, & Q_0 &= 1, \\ P_1 &= a_0 a_1 + 1, & Q_1 &= a_1, \end{aligned}$$

and in general

$$P_n = P_{n-1} a_n + P_{n-2}, \qquad Q_n = Q_{n-1} a_n + Q_{n-2},$$

for $n \geq 2$. It is obvious that, since a_n are positive, P_n and Q_n are strictly increasing for $n \geq 1$ and both are greater than or equal to F_n

(the n-th Fibonacci number). Let us define the n-th convergent

$$R_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n.}}}}$$

Theorem 1.3. *The following relations hold for $n \geq 2$:*

$$R_n = \frac{P_n}{Q_n} = \frac{P_{n-1}a_n + P_{n-2}}{Q_{n-1}a_n + Q_{n-2}};\ x = \frac{P_{n-1}x_n + P_{n-2}}{Q_{n-1}x_n + Q_{n-2}}.$$

The proof can be easily made by induction, and is also to be found in [2, Sect. 5.2.4]. Other well-known facts are that

$$P_{n-1}Q_n - Q_{n-1}P_n = (-1)^n.$$

(also proven by induction) and that

$$\left|x - \frac{P_n}{Q_n}\right| = \left|\frac{P_n x_{n+1} + P_{n-1}}{Q_n x_{n+1} + Q_{n-1}} - \frac{P_n}{Q_n}\right| = \left|\frac{(-1)^n}{Q_n(Q_n x_{n+1} + Q_{n-1})}\right| \leq \frac{1}{Q_n Q_{n+1}}.$$

It follows that $\lim_{n\to\infty} R_n = x$. In particular, the last result implies that two numbers whose continued fraction expansions coincide must be equal.

2. Periodicity of continued fractions

Theorem 2.1. *The continued fraction expansion of a real number x is periodic from a point onward iff x is the root of some quadratic equation $ax^2 + bx + c = 0$ with integer coefficients.*

Proof. The sufficiency is easier to prove. Indeed, if we know that $a_n = a_{n+p}$ for all $n \geq N$, then let us note that x_N and x_{N+p} have the same continued fraction and thus are equal. On the other hand, we know that

$$x_N = a_N + \cfrac{1}{a_{N+1} + \cfrac{1}{a_{N+2} + \cfrac{1}{\ddots + \cfrac{1}{x_{N+p}}}}} = \frac{\tilde{P}_{p-1}x_{N+p} + \tilde{P}_{p-2}}{\tilde{Q}_{p-1}x_{N+p} + \tilde{Q}_{p-2}}$$

and therefore it satisfies the second-degree equation

$$\tilde{Q}_{p-1}x_N^2 + (\tilde{Q}_{p-2} - \tilde{P}_{p-1})x_N - \tilde{P}_p = 0.$$

At the same, let us remember that $x = \frac{P_{N-1}x_N+P_{N-2}}{Q_{N-1}x_N+Q_{N-2}}$. By a trivial computation we obtain that $x_N = \frac{-Q_{N-2}x+P_{N-2}}{Q_{N-1}x-P_{N-1}}$. Therefore x satisfies a second-degree equation with integer coefficients too, namely

$$\tilde{Q}_{p-1}(-Q_{N-2}x+P_{N-2})^2+(\tilde{Q}_{p-2}-\tilde{P}_{p-1})(-Q_{N-2}x+P_{N-2})(Q_{N-1}x-P_{N-1}) \\ -\tilde{P}_p(Q_{N-1}x-P_{N-1})^2 = 0.$$

The converse is slightly more difficult to prove. Assume that x satisfies the equation $f(x) = ax^2+bx+c = 0$. Then, since $x = \frac{P_{n-1}x_n+P_{n-2}}{P_{n-1}x_n+P_{n-2}}$, it follows that each of the remainders x_n, $n \geq 2$, also satisfies the second-degree equation

$$a(P_{n-1}x_n+P_{n-2})^2+b(P_{n-1}x_n+P_{n-2})(Q_{n-1}x_n+Q_{n-2}) \\ +c(Q_{n-1}x_n+Q_{n-2})^2 = 0$$

or $f_n(x_n) = A_nx_n^2+B_nx_n+C_n = 0$, where I have denoted

$A_n = aP_{n-1}^2+bP_{n-1}Q_{n-1}+cQ_{n-1}^2$,
$B_n = 2aP_{n-1}P_{n-2}+b(P_{n-1}Q_{n-2}+P_{n-2}Q_{n-1})+2cQ_{n-1}Q_{n-2}$, and
$C_n = aP_{n-2}^2+bP_{n-2}Q_{n-2}+cQ_{n-2}^2$.

To help with the subsequent computations, let us evaluate

$$\begin{aligned}|f(R_n)| = |f(R_n)-f(x)| &= \left|(R_n-x)f'(x)+\frac{(R_n-x)^2}{2}f''(x)\right| \\ &< \frac{|f'(x)|}{Q_{n+1}Q_n}+\frac{|f''(x)|}{Q_{n+1}^2Q_n^2} \\ &\leq \frac{2|a||x|+|b|}{Q_{n+1}Q_n}+\frac{|a|}{Q_{n+1}^2Q_n^2}.\end{aligned}$$

Let us note that $A_n = Q_{n-1}^2f(R_{n-1})$ and $C_n = A_{n-1}$; then,

$$\begin{aligned}|A_n| = Q_{n-1}^2|f(R_{n-1})| &\leq \frac{Q_{n-1}}{Q_n}|f'(x)|+\frac{|a|}{Q_n^2} \\ &\leq 2|a||x|+|b|+|a|\end{aligned}$$

and the same goes for C_n. With respect to B_n, we can say that

$$B_n = Q_{n-1}Q_{n-2}(f(R_{n-1})+f(R_{n-2})-a(R_{n-1}-R_{n-2})^2), \text{ so}$$

$$\begin{aligned}|B_n| &\leq Q_{n-1}Q_{n-2}(|f(R_{n-1}|+|f(R_{n-2})|+\frac{|a|}{Q_{n-1}^2Q_{n-2}^2}) \\ &\leq \left(\frac{Q_{n-2}}{Q_n}+1\right)(2|a||x|+|b|)+\left(\frac{1}{Q_{n-1}^2Q_n^2}+\frac{2}{Q_{n-1}^2Q_{n-2}^2}\right)|a| \\ &\leq \frac{3}{2}(2|a||x|+|b|)+\frac{5}{2}|a|.\end{aligned}$$

Then, we have proven that all of A_n, B_n, and C_n can take a limited number of values. Eventually, such a triple is bound to reoccur twice, making some x_k, x_l, and x_m roots of the same second-degree equation for distinct k, l, and m. Since a second-degree equation only has two roots, two of those numbers will have to be equal, say $x_k = x_l$. Then, $a_{k+i} = a_{l+i}$ for $i \geq 0$. $\square$

Actually, there is no need to wait for a triple to reoccur twice, because x_n cannot be both roots of the given equation. The original equation, $ax^2 + bx + c = 0$, has two roots, and the rational $-\frac{b}{2a}$ lies between them. Then, for $n > c \log |2a|$ (such that $Q_n > |2a|$), the law of the best approximations says that R_n is closer to x than $-\frac{b}{2a}$ (and than the other root). Therefore, the sign of $f(R_n)$ only depends on the sign of $R_n - x$, from a point onward, and therefore it alternates. Then A_n and C_n have distinct signs, and, since x_n is positive, the other root of f_n must be negative. If the equation has two roots of different signs, x_n must be the positive root. Then, if we repeat the above reasoning only counting the triples for $n > c \log 2a$, we have proved the following

Theorem 2.2. *If x is the solution of the equation $f(x) = ax^2+bx+c = 0$ with integer coefficients, the length of the period of the continued fraction expansion of x cannot exceed $(|f'(x)| + |a|)^2(\frac{3}{2}|f'(x)| + \frac{5}{2}|a|) + \mathcal{O}(\log |a|)$.*

If not much can be said in general about the period of a quadratic irrational x (after all, every periodic sequence of integers determines one such irrational), a lot is known about the continuous fraction expansion of irrationals of the form $\sqrt{D}$, for rational $D = \frac{p}{q}$.

Theorem 2.3. *In the continued fraction expansion of $\sqrt{D}$, the remainders always take the form $x_n = \frac{\sqrt{D}+b_n}{c_n}$, where the numbers b_n, c_n, as well as the continued fraction digits a_n can be obtained by means of the following algorithm: set $a_0 = \lfloor D \rfloor$, $b_1 = a_0$, $c_1 = D - a_o^2$, and then compute*

$$a_{n-1} = \left\lfloor \frac{a_0 + b_{n-1}}{c_{n-1}} \right\rfloor, b_n = a_{n-1}c_{n-1} - b_{n-1}, c_n = \frac{D - b_n^2}{c_{n-1}}.$$

Proof. We already know that $\sqrt{D} = \frac{P_{n-1}x_n + P_{n-2}}{Q_{n-1}x_n + Q_{n-2}}$, or equivalently

$$\begin{aligned} x_n &= \frac{-Q_{n-2}\sqrt{D} + P_{n-2}}{Q_{n-1}\sqrt{D} - P_{n-1}} = \frac{(Q_{n-2}\sqrt{D} - P_{n-2})(Q_{n-1}\sqrt{D} + P_{n-1})}{P_{n-1}^2 - DQ_{n-1}^2} \\ &= \frac{(-1)^{n-1}\sqrt{D} + DQ_{n-1}Q_{n-2} - P_{n-1}P_{n-2}}{P_{n-1}^2 - DQ_{n-1}^2}. \end{aligned}$$

Then, we know precisely the values of b_n and c_n (which, in case they exist, must be unique, being rational), namely

$$b_n = (-1)^n(P_{n-1}P_{n-2} - DQ_{n-1}Q_{n-2}) \text{ and} \\ c_n = (-1)^n(DQ_{n-1}^2 - P_{n-1}^2). \tag{2.1}$$

The claims concerning the recurrence relation can be verified direcly. First, though, we need the following:

Lemma 2.1. *If k is a natural number and x a real number, then*

$$\left\lfloor \frac{x}{k} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{k} \right\rfloor.$$

The lemma's proof lies in [1, pp. 295-296]. By using this lemma, one can easily find that

$$a_n = \lfloor x_n \rfloor = \left\lfloor \frac{\sqrt{D}+b_n}{c_n} \right\rfloor = \left\lfloor \frac{\lfloor \sqrt{D}+b_n \rfloor}{c_n} \right\rfloor = \left\lfloor \frac{a_0+b_n}{c_n} \right\rfloor,$$

while a simple computation shows that

$$\frac{\sqrt{D}+b_{n-1}}{c_{n-1}} = x_{n-1} = a_{n-1}+\frac{1}{x_n} = a_{n-1}+\frac{c_n}{\sqrt{D}+b_n} = \frac{a_{n-1}\sqrt{D}+a_{n-1}b_n+c_n}{\sqrt{D}+b_n}$$

and equivalently

$$(\sqrt{D}+b_{n-1})(\sqrt{D}+b_n) = c_{n-1}(a_{n-1}\sqrt{D}+a_{n-1}b_n+c_n)$$

whence we get (since $\sqrt{D}$ is irrational) that

$$b_{n-1}+b_n = c_{n-1}a_{n-1}$$

and

$$D + b_{n-1}b_n = c_{n-1}a_{n-1}b_n + c_{n-1}c_n \Leftrightarrow \\ \Leftrightarrow D + b_n(b_{n-1} - c_{n-1}a_{n-1}) = b_n + c_{n-1}c_n \Leftrightarrow D - b_n^2 = c_{n-1}c_n.$$

Then, the first terms of the sequences are easy to find: $x_1 = \frac{1}{\sqrt{D}-a_0} = \frac{\sqrt{D}+a_0}{D-a_0^2}$. So $b_1 = a_0$ and $c_1 = D - a_0^2$. □

Theorem 2.4. *The numbers b_n and c_n are positive and satisfy $\sqrt{D} - b_n < c_n < \sqrt{D} + b_n$. Furthermore, we have that $b_n < \sqrt{D}$ and $c_n < 2\sqrt{D}$.*

Proof. First, we are going to prove by induction that

$$0 < \frac{\sqrt{D}-b_n}{c_n} < 1. \tag{2.2}$$

Indeed, for $n = 1$

$$0 < \frac{\sqrt{D} - a_0}{D - a_0^2} < \sqrt{D} - a_0 < 1.$$

Suppose this statement is true for a natural number n. It is always true that

$$\frac{\sqrt{D} - b_{n+1}}{c_{n+1}} = \frac{D - b_{n+1}^2}{c_{n+1}(\sqrt{D} + b_{n+1})} = \frac{c_n}{\sqrt{D} + b_{n+1}} = \frac{c_n}{\sqrt{D} + a_n c_n - b_n}$$
$$= \frac{1}{\dfrac{\sqrt{D} - b_n}{c_n} + a_n},$$

and by the induction hypothesis $\frac{\sqrt{D}-b_n}{c_n} + a_n > a_n \geq 1$, whence $\frac{\sqrt{D}-b_{n+1}}{c_{n+1}} < 1$ as well. Thus, the proof is complete.

We know that $\frac{\sqrt{D}+b_n}{c_n} = x_n > 1$. By adding this to inequality (2.2) we obtain that $\frac{2\sqrt{D}}{c_n} > 1$, or $0 < c_n < 2\sqrt{D}$. Then, multiplying by the denominator, $\frac{\sqrt{D}+b_n}{c_n} > 1$ implies $\sqrt{D} + b_n > c_n$, and $0 < \frac{\sqrt{D}-b_n}{c_n} < 1$ implies$\sqrt{D} - b_n < c_n$. Finally, combining these two inequalities we get $\sqrt{D} - b_n < \sqrt{D} + b_n$, so $b_n > 0$. Thus, we have proved, albeit in a different order, all the promised inequalities. □

An immediate consequence is

Corollary 2.5. *The continued fraction expansion of $\sqrt{D}$ is periodic, with a period of at most pq, if $D = \frac{p}{q}$.*

Indeed, in virtue of their representation (2.1), we can write $b_n = \frac{\tilde{b}_n}{q}$ and $c_n = \frac{\tilde{c}_n}{q}$, where $\tilde{b}_n$ and $\tilde{c}_n$ are integers. For them, the following relations hold: $0 < \tilde{b}_n < \sqrt{pq}$ and $\sqrt{pq} - \tilde{b}_n < \tilde{c}_n < \sqrt{pq} + \tilde{b}_n$. Thus, $\tilde{c}_n$ can take at most $2\tilde{b}_n - 1$ value for each $\tilde{b}_n$, and the number of possible distinct pairs $(\tilde{b}_n, \tilde{c}_n)$ is no greater than

$$\sum_{\tilde{b}_n=1}^{\lfloor\sqrt{pq}\rfloor} 2\tilde{b}_n - 1 = \lfloor\sqrt{pq}\rfloor^2 < pq.$$

Whenever that number of consecutive pairs is considered, two must coincide. Thus $x_k = x_l$ for some $0 < k,\ l < pq$, resulting in a period of length at most pq.

Theorem 2.6. *The period of the continued fraction expansion of $\sqrt{D}$ starts with the second term, that is, $\exists p\ a_k = a_{k+p}\ \forall k > 0$. Furthermore,* *if the period consists of the* p *terms* $a_1, a_2, \dots, a_p$*, then* $a_p = 2\lfloor\sqrt{D}\rfloor$ *and the sequence* $a_1, a_2, \dots, a_{p-1}$ *is symmetric.*

Proof. Let us also consider the numbers $x'_n = \frac{\sqrt{D}-b_n}{c_n}$ for $n > 0$. They obey the recurrence relation

$$x'_n + a_n = \frac{\sqrt{D} - b_n + a_n c_n}{c_n} = \frac{\sqrt{D} + b_{n+1}}{c_n} = \frac{D - b_{n+1}^2}{c_n(\sqrt{D} - b_{n+1})} = \frac{c_{n+1}}{\sqrt{D} - b_{n+1}}$$
$$= \frac{1}{x'_{n+1}}.$$

Since $x'_n < 1$, we obtain that $a_n = \left\lfloor \frac{1}{x'_{n+1}} \right\rfloor$, for $n > 0$. Then, assume that the periodicity starts not at 0, that the smallest n_0 for which $a_n = a_{n+p}$ $\forall n \geq n_0$ is not 1. Since $a_n = a_{n+p}$ $\forall n \geq n_0$, it follows that $x'_{n_0} = x'_{n_0+p}$, and, by the relation above (if $n_0 > 0$) that $a_{n_0-1} = a_{n_0+p-1}$, which contradicts our assumption about n_0. By this contradiction, we have proved that the period indeed starts with the second term a_1.

By conjugation from $\sqrt{D} = a_0 + \frac{1}{x_1}$ we obtain that $-\sqrt{D} = a_0 - \frac{1}{x'_1}$, or $\frac{1}{x'_1} = a_0 + \sqrt{D}$. This gives us a continued fraction expansion of $\frac{1}{x'_1}$ as

$$\frac{1}{x'_1} = 2a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, a_{p-1} + \frac{1}{x_n}}}}$$

On the other hand, we know that

$$\frac{1}{x'_1} = \frac{1}{x'_{p+1}} = a_p + x'_p = a_p + \frac{1}{\cfrac{1}{x'_p}} = a_p + \frac{1}{a_{p-1} + x'_{p-1}}$$

and by recurrence we obtain

$$\frac{1}{x'_1} = a_p + \cfrac{1}{a_{p-1} + \cfrac{1}{a_{p-2} + \cfrac{1}{\ddots\, a_1 + x'_1}}}.$$

Since $x'_1 < 1$, both are portions of the continued fraction expansion of $\frac{1}{x'_1}$, and therefore they must coincide. By identifying the coefficients we obtain that $2a_0 = a_p$, $a_1 = a_{p-1}$, $\cdots$, $a_k = a_{p-k}$. □

3. Numerical experiments

I tried to find out the relation between n and the size of the period in the continued fraction expansion of $\sqrt{n}$, in brief $l(n)$. I used for this purpose the algorithm described in Theorem 2.3, implemented in C, in the program attached to this paper. I tried to evaluate both the average size of $l(n)$ and its maximal values.

Table 1 contains the successive peaks of $\frac{l(n)}{\sqrt{n}}$, for $1 < n < 10^8$, that is to say the values n for which $\frac{l(n)}{\sqrt{n}} > \frac{l(m)}{\sqrt{m}}$ for all nonsquare $m < n$. Apparently, the ratio gets ever larger, by increasingly smaller increments. However, the table does not make it clear whether the ratio is bounded from above. One conclusion that can be drawn from it, though, is that neither $\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} d(n-k^2)$, nor $\sqrt{n}\log n$ are good upper bounds for $l(n)$; both are much too large.

The graph below tries to bring an intuitive answer to the question whether the quantity $\frac{l(n)}{\sqrt{n}}$ is bounded, by considering the peaks of this function as given in Table 1 (and disregarding the precise points where they are attained).

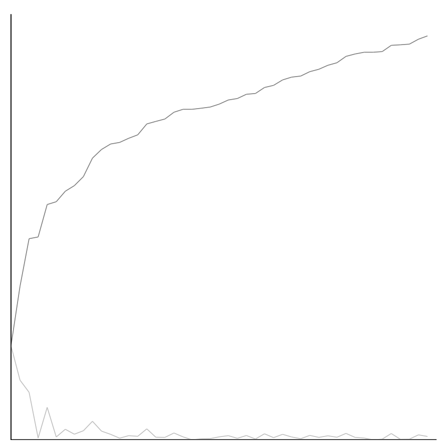

Figure 1. The peaks of $\frac{l(n)}{\sqrt{n}}$, plotted in successive order, for $1 < n < 10^8$.

Here, the upper line represents the peaks, and the lower one the differences between consecutive peaks; the x-coordinates of the points are equally spaced. This picture seems to indicate that the ratio is not bounded.

The subsequent graph (Figure 2) illustrates the evolution of $l(n)$ versus $\sqrt{n}$, for $1 < \sqrt{n} < 1001$. The upper line represents local maxima, while the lower line represents averages of $l(n)$, both computed on intervals of length 1 ($k < \sqrt{n} < k+1$, $k \in \mathbb{N}$). The red line represents $k = \lfloor\sqrt{n}\rfloor$ for comparison. It is clear that $l(n)$ and $\sqrt{n}$ have the same

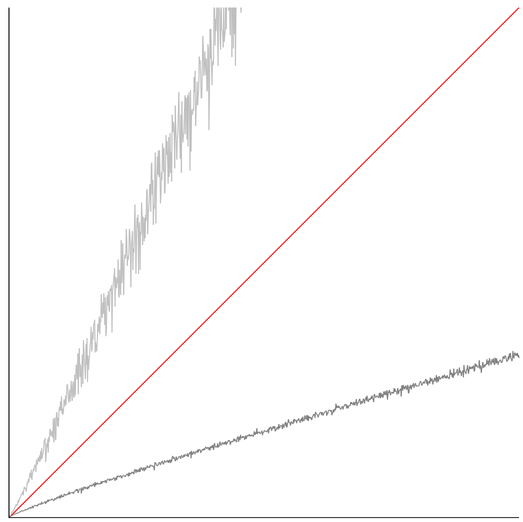

FIGURE 2. $l(n)$ plotted against $\lfloor\sqrt{n}\rfloor$, $1 < \sqrt{n} < 1001$.

order of magnitude, on average.

The next diagram (Figure 3) represents the ratio $\frac{l(n)}{\lfloor\sqrt{n}\rfloor}$, both in average and the maximum value for $\lfloor\sqrt{n}\rfloor = k$, as a function of k. The upper line represents local maxima and the lower line averages of $\frac{l(n)}{\lfloor\sqrt{n}\rfloor}$, both taken on the intervals $k < \sqrt{n} < k+1$. The red line represents $y = 1$ for comparison. It can be clearly seen that the average of $\frac{l(n)}{\lfloor\sqrt{n}\rfloor}$ for $\lfloor\sqrt{n}\rfloor = k$ is less than k, while the maximum of this function on the same interval is bigger than k.

Finally, the last graph (4) represents the maximum value of the ratio, namely $\max_{\lfloor\sqrt{n}\rfloor=k} \frac{l(n)}{k}$, and the inverse of the ratio's average value, $\frac{k}{\frac{l(k^2+1)+\ldots+l(k^2+2k)}{2k}}$, versus $\log\log k$.

This graph starts at $k = 3$ in order for $\log\log k$ to be a positive number. The red line is $y = 1$ and is provided for comparison. The vertical coordinates have been divided by 4, while the horizontal have been divided by $\log\log 10^4 - \log\log 3 \cong 2.13$.

Only now it is clear that both curves have linear growth, with the slope of $\frac{2k^2}{l(k^2+1)+\ldots+l(k^2+2k)}$ being approximately 2.2 and the slope of $\max_{\lfloor\sqrt{n}\rfloor=k} \frac{l(n)}{k}$ being somewhat lower, probably around 1.3. The following conjecture arises naturally:

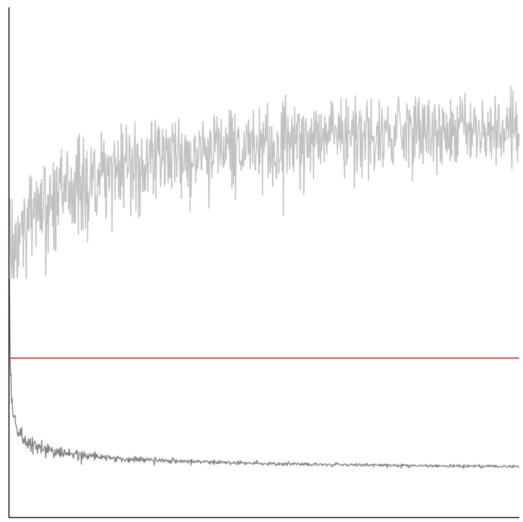

FIGURE 3. $\frac{l(n)}{\lfloor\sqrt{n}\rfloor}$ plotted versus $\lfloor\sqrt{n}\rfloor$, $1 < \sqrt{n} < 1001$.

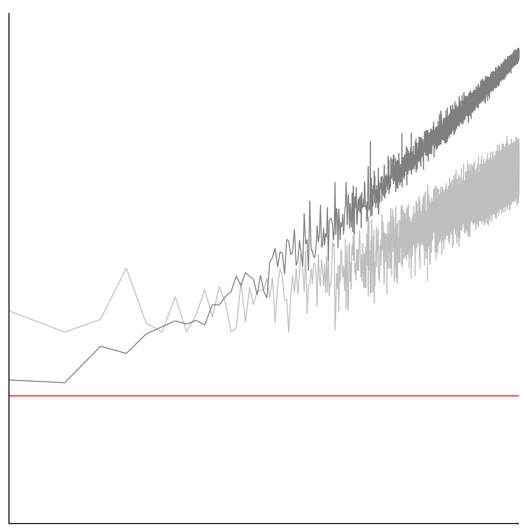

FIGURE 4. $\max_{\lfloor\sqrt{n}\rfloor=k} \frac{l(n)}{k}$ and $\frac{2k^2}{l(k^2+1)+\ldots+l(k^2+2k)}$ plotted versus $\log\log k$, $3 \leq k \leq 10000$.

Conjecture 3.1. *There exist constants $a_1 \cong 2.2$, b_1, c_1, $a_2 \cong 1.3$, b_2, and c_2, such that for every sufficiently large k*

$$k(a_1 \log\log k + b_1) \leq \max_{\lfloor\sqrt{n}\rfloor=k} l(n) \leq k(a_1 \log\log k + c_1)$$

and

$$\frac{k}{a_2 \log\log k + b_2} \leq \frac{\sum_{n=k^2+1}^{k^2+2k} l(n)}{2k} \leq \frac{k}{a_2 \log\log k + c_2}.$$

4. Theoretical results

Though by far worse than the bounds which seem attainable in practice, these are my results concerning $l(n)$.

Let $d(n)$ denote, in the sequel, the number of positive (integer) divisors of n.

Theorem 4.1. *For every $\epsilon > 0$, there exists $C(\epsilon)$ such that*

$$l(n) < C(\epsilon) + \sqrt{D} 2^{(1+\epsilon)\frac{\log D}{\log\log D}}.$$

Here a bound of the form $l(n) = \mathcal{O}(\sqrt{n}(A + B \log\log n))$ is probably attainable.

Proof. For each pair b_n, c_n, we have $c_n c_{n-1} = D - b_n^2$, or $c_n | D - b_n^2$. Thus, the number of possible pairs cannot exceed

$$\sum_{b=1}^{\lfloor\sqrt{D}\rfloor} d(D - b^2).$$

By [3, Theorem 317], for any $\epsilon > 0$, $d(n) < 2^{(1+\epsilon)\frac{\log n}{\log\log n}}$ for all sufficiently large n ($n > n_0(\epsilon)$). Then, since the function that bounds $d(n)$ in the inequality above is increasing, it follows that

$$\begin{aligned} l(D) &\leq \sum_{b=1}^{\lfloor\sqrt{D}\rfloor} d(D - b^2) \\ &\leq \sum_{\substack{1 \leq b \leq \lfloor\sqrt{D}\rfloor \\ D - b^2 \leq n_0(\epsilon)}} d(D - b^2) + \sum_{\substack{1 \leq b \leq \lfloor\sqrt{D}\rfloor \\ D - b^2 > n_0(\epsilon)}} d(D - b^2) \\ &\leq \sum_{k=1}^{n_0(\epsilon)} d(k) + \sum_{\substack{1 \leq b \leq \lfloor\sqrt{D}\rfloor \\ D - b^2 > n_0(\epsilon)}} 2^{(1+\epsilon)\frac{\log D - b^2}{\log\log D - b^2}} \\ &\leq C(\epsilon) + \sum_{\substack{1 \leq b \leq \lfloor\sqrt{D}\rfloor \\ D - b^2 > n_0(\epsilon)}} 2^{(1+\epsilon)\frac{\log D}{\log\log D}} \\ &< C(\epsilon) + \sqrt{D} \cdot 2^{(1+\epsilon)\frac{\log D}{\log\log D}}. \end{aligned}$$

$\square$

n	$l(n)$	$\sqrt{n}$	$\frac{l(n)}{\sqrt{n}}$	$\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} d(n-k^2)$	$\sqrt{n}\log n$
2	1	1.414	0.707	0	1.0
3	2	1.732	1.155	0	1.9
7	4	2.646	1.512	4	5.1
43	10	6.557	1.525	26	24.7
46	12	6.782	1.769	28	26.0
211	26	14.526	1.790	98	77.7
331	34	18.193	1.869	140	105.6
631	48	25.120	1.911	218	162.0
919	60	30.315	1.979	278	206.8
1726	88	41.545	2.118	418	309.7
4846	152	69.613	2.183	820	590.7
7606	194	87.212	2.224	1130	779.4
10399	228	101.975	2.236	1368	943.2
10651	234	103.204	2.267	1438	957.0
10774	238	103.798	2.293	1420	963.8
18379	322	135.569	2.375	2034	1331.1
19231	332	138.676	2.394	2052	1367.9
32971	438	181.579	2.412	2916	1889.0
48799	544	220.905	2.463	3720	2384.8
61051	614	247.085	2.485	4278	2722.7
78439	696	280.070	2.485	5048	3156.4
82471	716	287.178	2.493	5124	3250.9
111094	834	333.308	2.502	6188	3872.4
162094	1016	402.609	2.524	7720	4829.7
187366	1106	432.858	2.555	8460	5255.3
241894	1262	491.827	2.566	9916	6096.8
257371	1318	507.317	2.598	10340	6320.3
289111	1400	537.690	2.604	10964	6761.2
294694	1438	542.857	2.649	11308	6836.6
799621	2383	894.215	2.665	25834	12154.1
969406	2664	984.584	2.706	22740	13571.9
1234531	3030	1111.095	2.727	26544	15584.4
1365079	3196	1168.366	2.735	28240	16505.2
1427911	3308	1194.952	2.768	29026	16934.5
1957099	3898	1398.964	2.786	34808	20266.7
2237134	4212	1495.705	2.816	38004	21868.3
2847079	4784	1687.329	2.835	43964	25076.8
5715319	6892	2390.673	2.883	66394	37195.7
10393111	9352	3223.835	2.901	93342	52086.4
12843814	10442	3583.827	2.914	105148	58661.4
14841766	11226	3852.501	2.914	114260	63616.2
18461899	12542	4296.731	2.919	129960	71889.6
20289091	13358	4504.341	2.966	138224	75788.2
23345326	14348	4831.700	2.970	149824	81974.2
28473454	15876	5336.052	2.975	167880	91590.6
39803611	19002	6309.010	3.012	204556	110404.3
40781911	19396	6386.072	3.037	208528	111907.9

TABLE 1. Maxima of $l(n), 1 < n < 10^8$

By using the result in [4], namely that $\sum_{1\le k<\sqrt{n}} d(n-k^2) = \mathcal{O}(\sqrt{n}\log^3 n)$, it is possible to improve this result to

$$l(n) = \mathcal{O}(\sqrt{n}\log^3 n).$$

Finally, it has been proven in [5], by a different methd involving an estimate of the number of primitive classes of solutions of $x^2-Dy^2 = N$, that

$$l(D) \le \frac{7}{2\pi^2}\sqrt{D}\log D + \mathcal{O}(\sqrt{D}).$$

Another method, involving a bound on the size of ϵ, the fundamental unit in $\mathbb{Z}[\sqrt{D}]$, is employed in [6] to show that $l(D) < 3.76\sqrt{D}\log D$.

Observation 4.2. *The period length of the continued fraction expansion of $\sqrt{k^2+1}$, $k \in \mathbb{N}$, is always* 1*, and is* 1 *only for numbers of that form (see* [1, p. 298]*). Indeed,*

$$\sqrt{k^2+1} = k + \cfrac{1}{2k + \cfrac{1}{2k + \ddots}}.$$

It follows trivially that the minimum of $l(n)$ is 1 *on each interval $\lfloor\sqrt{n}\rfloor = k$ and is attained at $n = k^2+1$.*

Concerning the next theorem, the best possible bound is probably better than the one presented below, probably in the order of $\frac{k}{a_2+c_2\log\log k}$ for the average of $l(D)$ on the interval $[k^2+1, k^2+2k]$. Nevertheless, I have not managed to find the proof for a better bound than the one below.

Theorem 4.3. *The average size of $l(D)$ for $k^2 < D < (k+1)^2$ is no greater than $\frac{7}{4}k + \frac{3}{4}$.*

Proof In order that $x_n = \frac{\sqrt{D}+b_n}{c_n}$, where x_n is a remainder in the continuous fraction decomposition of $\sqrt{D}$, we need to have $D - b_n^2 \vdots c_n$ and $\sqrt{D} > b_n > |\sqrt{D} - c_n|$ (by Theorem 2.4).

For fixed b_n and c_n, the number of D such that $c_n \mid D - b_n^2$ cannot exceed $\frac{2k}{c_n}+1$, because $D - b_n^2$ takes values in an interval of $2k$ integers. Then, let us distinguish two cases: $c_n \le k$ and $c_n > k$.
In the first case, we have $k - c_n < b_n \le k$, so the number of possible values for b_n is c_n.
In the second case, we have $c_n - k < b_n \le k$, so the number of possible b_n cannot exceed $2k - c_n$. Furthermore, since $c_n > k$, the number of multiples of c_n in each interval $k^2+1-b_n^2, \dots, k^2+2k-b_n^2$ (otherwise said, the number of D with $k^2 < D < (k+1)^2$ and $c_n \mid D - b_n^2$) cannot

exceed 2.
Then, the total number of possible triples is no greater than

$$\begin{aligned}\sum_{D=k^2+1}^{k^2+2k} l(D) &\leq \sum_{c_n=1}^{k}(\frac{2k}{c_n}+1)c_n + \sum_{c_n=k+1}^{2k} 2(2k-c_n)\\ &\leq 2k^2 + 3\sum_{t=1}^{k} t\\ &= \frac{7}{2}k^2 + \frac{3}{2}k.\end{aligned}$$

Dividing by $2k$ in order to obtain the average, we get the desired result, q. e. d.

Appendix

This is the program that I wrote in order to evaluate the size of $l(D)$.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>

int period_length(mint n) {
   int a_0, a, b, c, b_0, c_0, result=0;
   a_0=sqrt(n*1.0);
   b=b_0=a_0;
   c=c_0=n-a_0*a_0;
   do {
      a=(a_0+b)/c;
      b=a*c-b;
      c=(n-b*b)/c;
      result++;
   } while ((b!=b_0)||(c!=c_0));
   return result;
}

int nodiv(int n) {
   int i, j, result=1;
   for (i=2; i*i<=n; i++)
      if ((n%i)==0) {
         j=1;
         while ((n%i)==0) {
            j++;
            n=n/i;
```

```
        }
        result=result*j;
     }
   if (n!=1) return result<<1;
   return result;
}

int estim(int n) {
   int i, result=0, j=sqrt(n*1.0);
   for (i=1; i<j; i++)
      result+=nodiv(n-i*i);
   return result;
}

int main() {
   int i, j, i2, imax;
   int l, lmax;
   double r, r0=0.0, s, lavg;
   int beginning_number=9332, end_number=10001;
   printf("%d %d\n", beginning_number, end_number);
   for (i=beginning_number*beginning_number,
        i2=beginning_number, imax=i+(i2<<1)+1;
        i2<end_number;
        i=imax, i2++, imax+=(i2<<1)+1) {
      lavg=0.0;
      lmax=0;
      for (j=i+1; j<imax; j++) {
         l=period_length(j);
         if (l>lmax) lmax=l;
         s=sqrt(j*1.0);
         r=l/s;
         lavg+=l*1.0;
         if (r>r0) {
            r0=r;
            fprintf(stderr,
               "n=%d l(n)=%d sqrt(n)=%.3f r=%.3f "
               "est=%d 2nd est=%.1f\n",
               j, l, s, r, estim(j), sqrt(j*1.0)*log(j*1.0));
         }
      }
      printf("Square root= %d Maximum= %d Average= %.3f\n",
             i2, lmax, lavg/((imax-i-1)*1.0));
```

```
    }
    return 0;
}
```

In order to compute $l(n)$ for $\sqrt{n}$ between 1 and k, the program takes

$$\mathcal{O}\left(\sum_{n=2}^{k^2-1} l(n)\right)$$

time. According to my best estimates, this does not exceed $\mathcal{O}(k^3)$, but is probably even lower (probably $\mathcal{O}\big(\frac{k^3}{\log\log k}\big)$). The program took almost a day (23 hours) to run for $1 < \sqrt{n} < 10^4$.

Different C programs were also employed to produce the diagrams included in the paper.

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