PH 441

Lecture 4

~~part 1~~

There are certain matrices that are extremely important in Quantum mechanics.
We already know about <u>Hermitian matrices</u>.

E.g.

$$A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$A^+ = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Eigenvalues:

$$\begin{vmatrix} -\lambda & i \\ -i & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 = 1$$

$$\Rightarrow \lambda_1 = 1, -1 \qquad \text{(real)}$$

Unitary ~~matrices~~ matrix

If the matrix U satisfies $U^+ = U^{-1}$ then U is called a unitary matrix.

Example

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \qquad |U| = \frac{1}{2} + \frac{1}{2} = 1$$

$$U^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$U^{-1} = \frac{\text{adj } U}{|U|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$\text{adj } U = (U^c)^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

Orthogonal matrix

If $A^T = A^{-1}$ or $AA^T = I$, A is called an orthogonal matrix.

Example $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$AA^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Normal matrix

A matrix A is normal if it satisfies

$$AA^{+} = A^{+}A$$

Theorem: Let A be a normal matrix. Then its eigenvectors corresponding to different eigenvalues are orthogonal.

~~Proof~~

$$A\,\cancel{|\lambda_i\rangle = \lambda_i |\lambda_i\rangle}$$

$$\Rightarrow \cancel{(A - \lambda_i)\,|\lambda_i\rangle} = 0$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_y^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_y \sigma_y^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_y^{+} \sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

PH 441

Lecture 4

~~part I~~

Spectral decomposition of a normal matrix is quite a powerful technique in several applications.

Let A be a normal matrix with eigenvalues $\{\lambda_i\}$ and eigenvectors $\{|\lambda_i\rangle\}$ which are assumed to be orthonormal. Then A is decomposed as :

$$\boxed{A = \sum_i \lambda_i \, |\lambda_i\rangle\langle\lambda_i|}$$

which is called the spectral decomposition of A.

Proof

We know, $\quad I = \sum_{i=1}^{n} |\lambda_i\rangle\langle\lambda_i|$

$$A = \sum_{i=1}^{n} A\,|\lambda_i\rangle\langle\lambda_i|$$

$$= \sum_{i=1}^{n} \lambda_i \,|\lambda_i\rangle\langle\lambda_i|$$

Proved

$P_i = |\lambda_i\rangle\langle\lambda_i|$ is called the projection operator in the direction of λ_i.

Thus,

$$\boxed{A = \sum_{i=1}^{n} \lambda_i P_i}$$

Show that

$AP_i = \lambda_i P_i$

$$\cancel{AP_j = \sum_i (\lambda_i P_i) P_j^2 = \sum_i \lambda_i (|\lambda_i\rangle\langle\lambda_i|)(|\lambda_j\rangle\langle\lambda_j|)}$$

$$\cancel{= \sum \lambda_i |\lambda_i\rangle\langle\lambda_i|\lambda_j\rangle\langle\lambda_j|}$$

$$\cancel{= \sum \lambda_i |\lambda_i\rangle \cancel{\langle\lambda_i|}\, \delta_{ij} \langle\lambda_j|}$$

$= \lambda_i P_i$

Show that if A is a normal matrix then for an arbitrary $n \in N$

$$A^n = \sum_i \lambda_i^n P_i$$

Proof

$$\cancel{A^n P_i = A^{n-1} A P_i}$$

$$\cancel{= \lambda_i A^{n-1} \lambda_i}$$

$$A P_i = \lambda_i P_i$$

$$A^n P_i = A^{n-1} A P_i = \lambda_i A^{n-1} P_i$$

$$= \lambda_i^2 A^{n-2} P_i$$

$$= \lambda_i^n P_i$$

Thus,

$$A^n = A^n \sum_i P_i = \sum_i A^n P_i$$

$$\Rightarrow \boxed{A^n = \sum_i \lambda_i^n P_i}$$

Examples

Find the spectral decomposition of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Solution

Eigenvalues of $\sigma_y = 1, -1$

Eigenvector corresponding to $\lambda_1 = 1$

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Sly, for $\lambda_2 = -1$

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$\cancel{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x \end{pmatrix}}$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$\Rightarrow -i y_2 = y_1$

$\Rightarrow y_2 = i y_1$

$y = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Thus,

$$P_1 = |\lambda_1\rangle\langle\lambda_1| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix}$$

$$P_2 = |\lambda_2\rangle\langle\lambda_2| = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \end{pmatrix}$$

$$\Rightarrow P_1 = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$P_2 = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Thus,

$\cancel{\sigma_y =} \cancel{\#} \cancel{P_1} \lambda$

$$\sigma_y = \lambda_1 P_1 + \lambda_2 P_2$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + (-1)\frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Show that

$$e^{i\alpha \sigma_y} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

Proof:

$$e^{i\alpha\sigma_y} = \sum_{K=0}^{\alpha} \frac{(i\alpha\sigma_y)^K}{K!}$$

$$= \sum_{K=0}^{\alpha} \frac{(i\alpha)^K}{K!} (\sigma_y)^K$$

$$= \sum_{K=0}^{\alpha} \frac{(i\alpha)^K}{K!} \left[\lambda_1^K P_1 + \lambda_2^K P_2 \right]$$

$$= \sum \frac{(i\alpha)^K}{K!} \left[P_1 + (-1)^K P_2 \right]$$

$$= \sum_K \frac{(i\alpha)^K}{K!} P_1 + \sum \frac{(-i\alpha)^K}{K!} P_2$$

$$= e^{i\alpha} P_1 + e^{-i\alpha} P_2$$

$$= \frac{1}{2} e^{i\alpha} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{1}{2} e^{-i\alpha} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{i\alpha}+e^{-i\alpha}}{2} & -i\,\frac{e^{i\alpha}-e^{-i\alpha}}{2} \\ i\,\frac{e^{i\alpha}-e^{-i\alpha}}{2} & \frac{e^{i\alpha}+e^{-i\alpha}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

Pauli matrices

These are very important class of matrices for us! Any 2×2 matrix could be written or expressed in terms of ~~Pat~~ Pauli matrices and the unit matrix.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Prove that

$$e^{i\alpha \hat{n}\cdot\vec{\sigma}} = \cos\alpha \, I + i(\hat{n}\cdot\vec{\sigma})\sin\alpha$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, $\hat{n}$ is a unit vector and $\alpha \in R$

Proof

~~Let~~ $\vec{n} = n_x \hat{x} + n_y \hat{y} + n_z \hat{z}$

Say, $A = \vec{n}\cdot\vec{\sigma} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$

$$\Rightarrow \quad A = \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

Eigenvalues of A are $\pm\sqrt{n_x^2 + n_y^2 + n_z^2} = \pm 1$

$\Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = -1$

$$P_1 = |\lambda_1\rangle\langle\lambda_1| = \frac{1}{2}\begin{pmatrix} n_z + 1 & n_x - i n_y \\ n_x + i n_y & 1 - n_z \end{pmatrix}$$

$$P_2 = |\lambda_2\rangle\langle\lambda_2| = \frac{1}{2}\begin{pmatrix} 1 - n_z & -n_x + i n_y \\ -n_x - i n_y & 1 + n_z \end{pmatrix}$$

Thus, $$e^{i\alpha A} = \sum \frac{(i\alpha A)^k}{k!} = \sum \frac{(i\alpha)^k}{k!}\left(\lambda_1^k P_1 + \lambda_2^k P_2\right)$$

$$= \cos\alpha \, I + i(\vec{n}\cdot\vec{\sigma})\sin\alpha$$