MA 225 | Quiz II
Probability Theory and Random Processes | 17:00–18:00 IST
IIT GUWAHATI | October 23, 2019

Model Answers of Quiz II

[5pnts.] 1. Show that

$$\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} = \frac{1}{2}.$$

Solution: Consider a sequence of *i.i.d.* random variables $\{X_n\}$, where $X_n \sim Poisson(1)$ for all $n = 1, 2\ldots$. Then $E(X_n) = Var(X_n) = 1$ and $\sum_{i=1}^{n} X_i \sim Poisson(n)$ for all n.
Now, using central limit theorem, $\sqrt{n}(\overline{X}_n - 1) \to Z \sim N(0, 1)$ in distribution.

$$\begin{aligned}\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} &= \lim_{n\to\infty} P\left(\sum_{i=1}^{n} X_i \le n\right)\\ &= \lim_{n\to\infty} P\left(\overline{X}_n \le 1\right)\\ &= \lim_{n\to\infty} P\left(\sqrt{n}(\overline{X}_n - 1) \le 0\right)\\ &= \frac{1}{2}.\end{aligned}$$

[5pnts.] 2. A probability class takes two exams, Exam 1 and Exam 2. Let X and Y denote the scores in Exam 1 and Exam 2, respectively. Suppose (X, Y) follow a bivariate normal distribution with parameters $\mu_X = 70, \mu_Y = 60, \sigma_X = 10, \sigma_Y = 15$ and $\rho = 0.6$. If we select a student at random, what is the probability (in term of $\Phi(\cdot)$, the cumulative distribution function of a $N(0, 1)$ random variable) that

(a) the student scores more than 75 in Exam 2, given that the student scored 80 in Exam 1?

Solution: We need to find $P(Y > 75|X = 80)$. It is known that the conditional distribution of Y given $X = x$ is normal with mean $\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$ and variance $\sigma_X^2(1 - \rho^2)$. Here $x = 80$, and $Y|X = 80 \sim N(69, 144)$.
Hence the required probability is

$$\begin{aligned}P(Y > 75|X = 80) &= P\left(\frac{Y - 69}{12} > \frac{75 - 69}{12}\middle| X = 80\right)\\ &= 1 - \Phi(0.5) \text{ or } \Phi(-0.5).\end{aligned}$$

(b) the sum of his/her Exam 1 and Exam 2 scores is over 150?

Solution: Here $X + Y \sim N(\mu_X + \mu_Y,\, \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y) \equiv N(130, 505)$.
Hence the required probability is

$$\begin{aligned}P(X + Y > 150) &= P\left(\frac{X + Y - 130}{\sqrt{505}} > \frac{20}{\sqrt{505}}\right)\\ &= 1 - \Phi(0.89) \text{ or } \Phi(-0.89).\end{aligned}$$

[5pnts.] 3. Let $\{X_n\}$ and X be random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Suppose X_n converges to X in the 1st mean. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \log(1 + e^x)$. Show that $f(X_n)$ converges to $f(X)$ in the 1st mean.

Solution: Here $f(\cdot)$ is differentiable and $f'(x) = \frac{e^x}{1+e^x} \implies |f'(x)| \leq 1$. Fix $x < y$. Then, using mean value theorem, there exists a $c \in (x, y)$ such that

$$\begin{aligned}
&\frac{f(x) - f(y)}{x - y} = f'(c) \\
\implies &\frac{|f(x) - f(y)|}{|x - y|} = |f'(c)| \leq 1 \\
\implies &|f(x) - f(y)| \leq |x - y|.
\end{aligned}$$

Note that this statement is true for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$.
Hence

$$\begin{aligned}
&|f(X_n) - f(X)| \leq |X_n - X| \\
\implies &0 \leq E|f(X_n) - f(X)| \leq E|X_n - X| \to 0 \text{ as } n \to \infty \text{ since } X_n \to X \text{ in 1st mean} \\
\implies &E|f(X_n) - f(X)| \to 0 \text{ as } n \to \infty \\
\implies &f(X_n) \to f(X) \text{ in 1st mean.}
\end{aligned}$$