1

$V = -x^2$

$\ddot{x} = 2x$

$\dot{x} = p \qquad (m=1)$

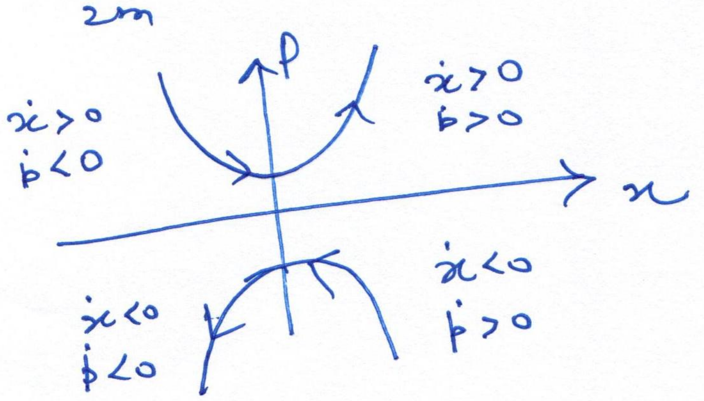
$\dot{p} = 2x$

F.p. $= (0,0)$

$E = \frac{p^2}{2} - x^2$

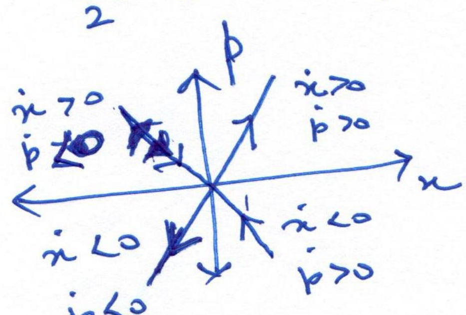
$E = 5$

$\Rightarrow \frac{p^2}{2m} - x^2 = 5$



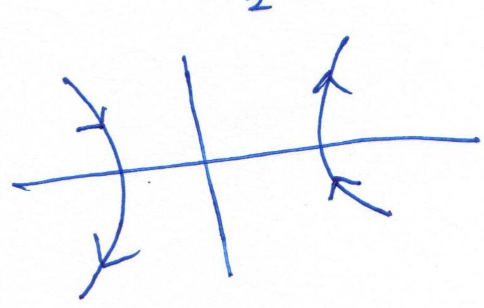

$E = 0$

$\frac{p^2}{2} = x^2 \Rightarrow p = \pm\sqrt{2}\,x$




$E = -5$

$x^2 - \frac{p^2}{2} = 5$



(2) (i)

$$\ddot{x} = ax - x^2$$

$$\dot{x} = y$$

$$\dot{y} = ax - x^2 \qquad \text{F. P.} = (a,0)\ ;\ (0,0)$$

$$J = \begin{pmatrix} 0 & 1 \\ a-2x_1 & 0 \end{pmatrix}$$

$$J\big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm\sqrt{a}\ ;\ v_1 = \begin{pmatrix} 1 \\ \sqrt{a} \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ -\sqrt{a} \end{pmatrix}$$

$$J\big|_{(a,0)} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}$$

$a > 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{a}$ center

$a = 0 \Rightarrow \lambda = 0$ degenerate eigenvalues

$a < 0 \Rightarrow \lambda = \pm\sqrt{-a}$ saddle points

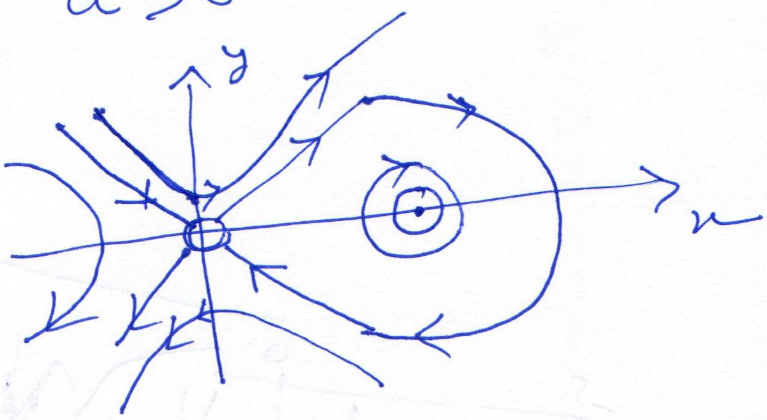
$$v_1 = \begin{pmatrix} 1 \\ \sqrt{-a} \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ \sqrt{-a} \end{pmatrix}$$

$\Uparrow$

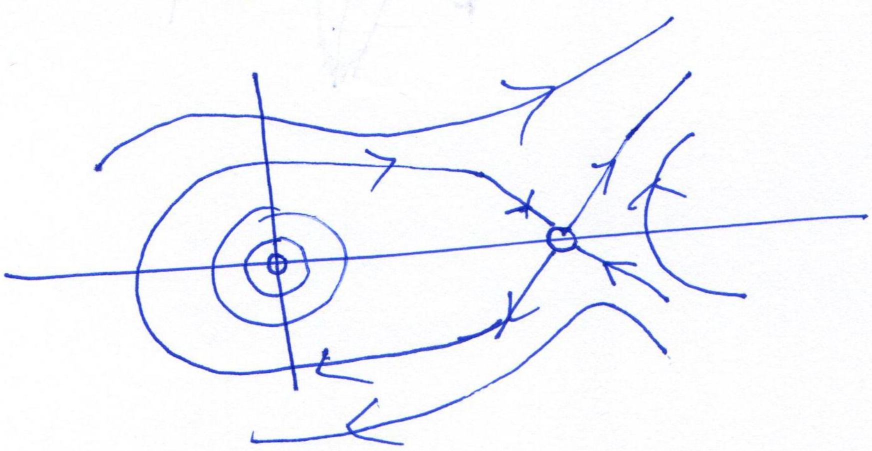
converging

diverging.

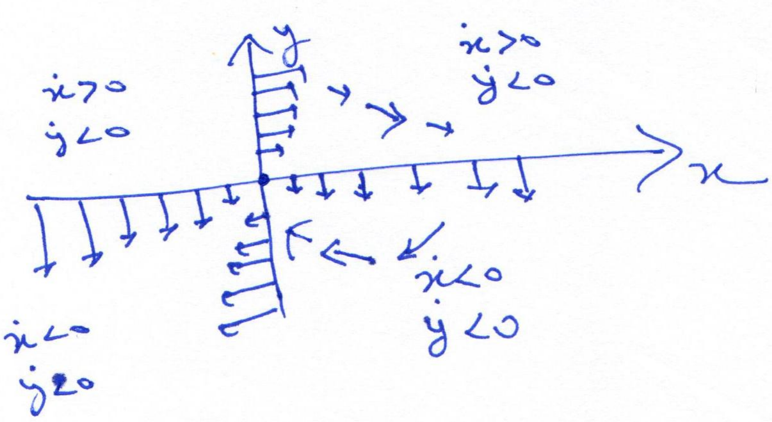
$a > 0$




$a < 0$



$a = 0$



(2) (ii)

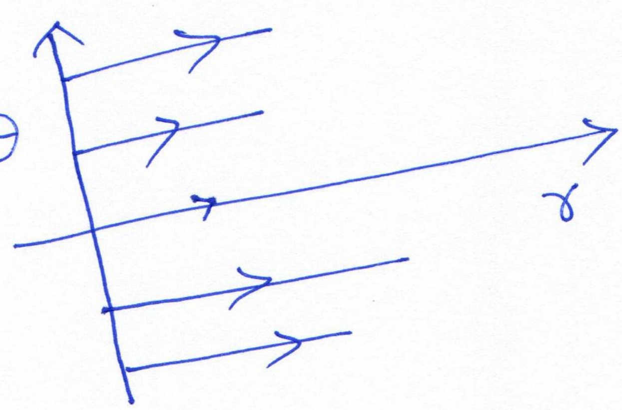
$\dot{r} = r(1-r^2) \qquad \dot{\theta} = 1 - \cos\theta$

F. Pts. $\equiv \quad r^* = 0, 1$

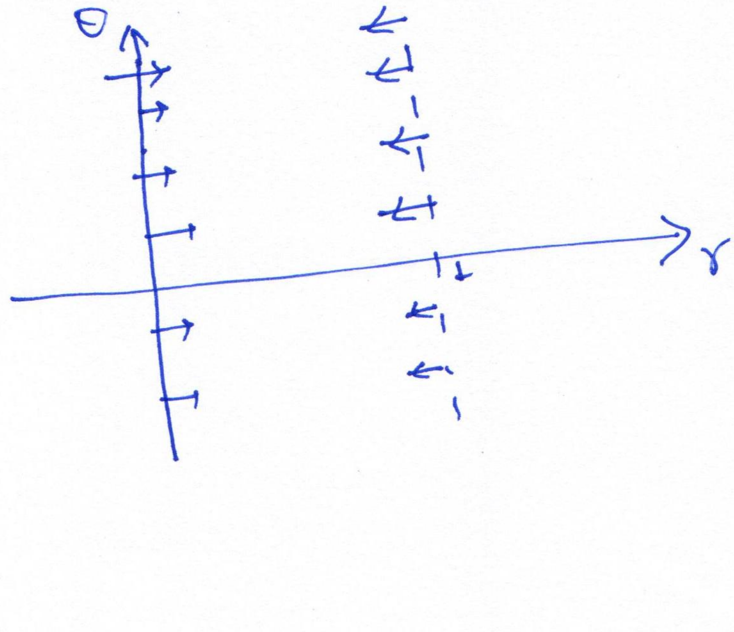
$\theta^* = 2n\pi$

$$J = \begin{pmatrix} 1-3r^2 & 0 \\ 0 & \sin\theta \end{pmatrix}$$

$$J\Big|_{\substack{(0,0) \\ \text{or} \\ (0,2n\pi)}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \lambda_{1,2} = 0, 1$$




$$J\Big|_{(1,2n\pi)} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \lambda = 0, -2$$



2 (iii)

$$\ddot{x} + x\dot{x} + x = 0$$

$$\dot{x} = y$$

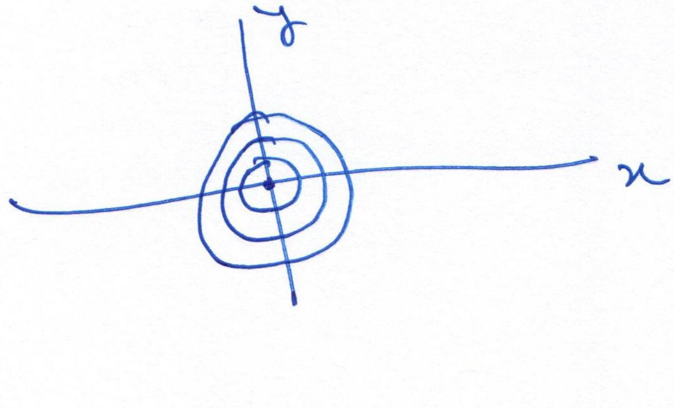
$$\dot{y} = -x(y+1)$$

f. pts. $\equiv$ $(0,0)$.

$$J = \begin{pmatrix} 0 & 1 \\ -(x_2+1) & -x_1 \end{pmatrix}$$

$$J|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\lambda = \pm i$$



3

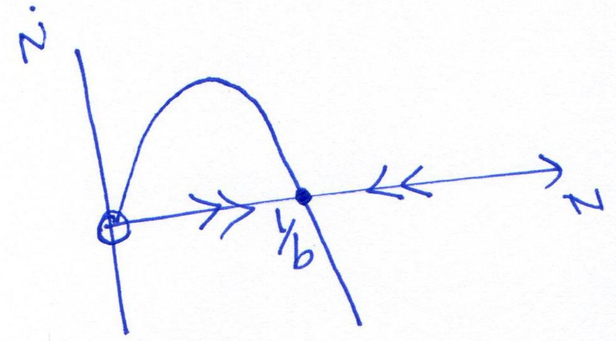
$$\dot{N} = -aN\ln(bN)$$

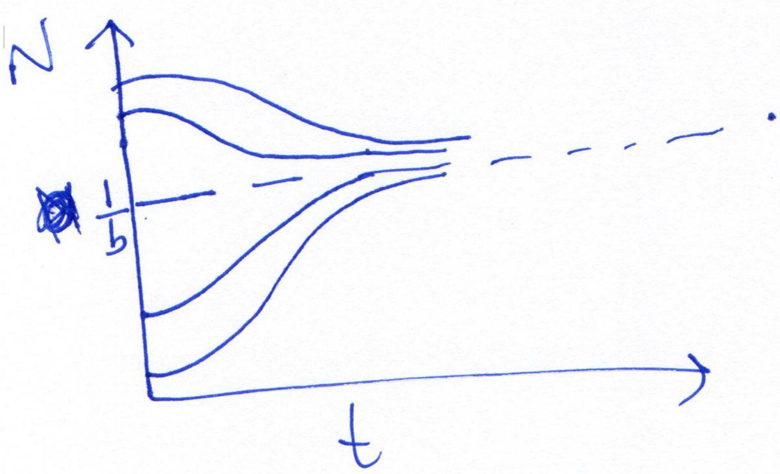
Fixed pt. $\Rightarrow$ $-Na\ln(bN) = 0$

$N = 0$ or, $\ln(bN) = 0$

$$\Downarrow$$

$$\boxed{N = 1/b}$$



4 a

$$\dot{x} = \mu x - x^2$$

$$\dot{y} = -y \qquad y^* = 0 \ ; \quad \mu x^* - x^{*2} = 0 \Rightarrow x^* = 0, \mu .$$

Fixed points are $(0,0)$ & $(\mu, 0)$.

$$J = \begin{pmatrix} \mu - 2x & 0 \\ 0 & -1 \end{pmatrix}$$

$$J\Big|_{(0,0)} = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix}$$

$\lambda_1 = -1, \quad \lambda_2 = \mu$

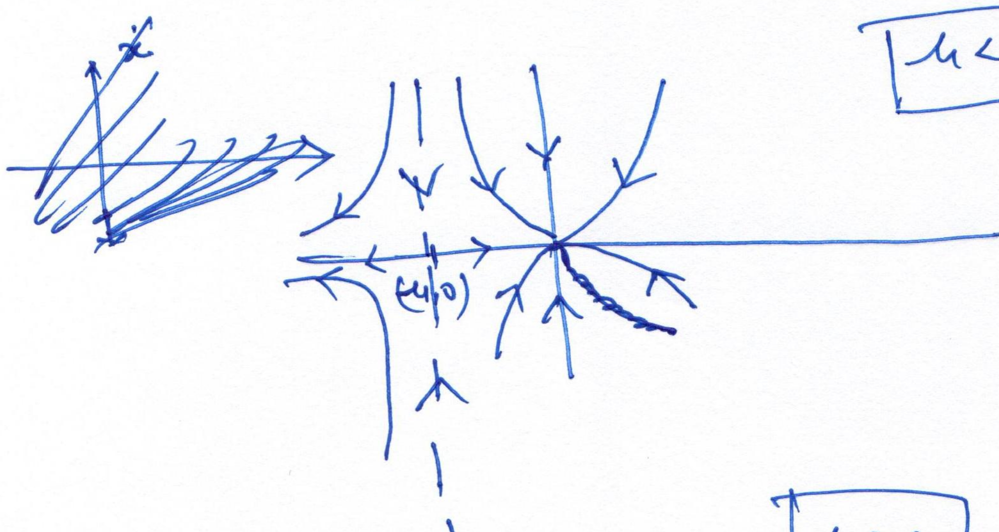
$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$J\Big|_{(\mu,0)} = \begin{pmatrix} -\mu & 0 \\ 0 & -1 \end{pmatrix}$$

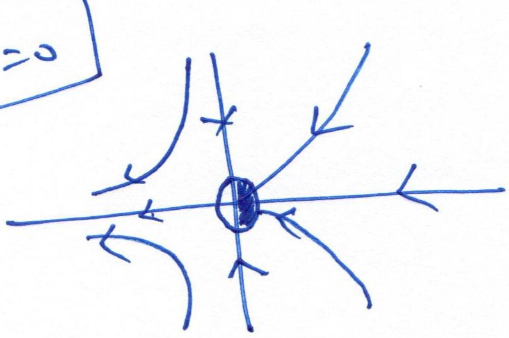
$\lambda_1 = -\mu, \quad \lambda_2 = -1$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

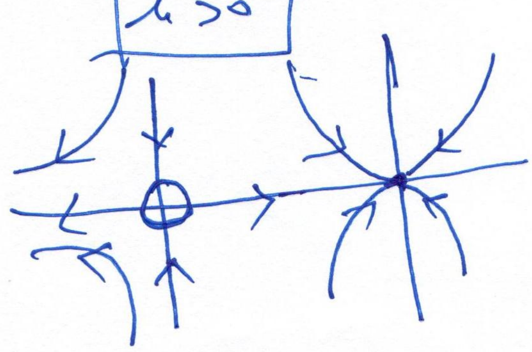
$\boxed{\mu < 0}$




$\boxed{\mu = 0}$



$\boxed{\mu > 0}$



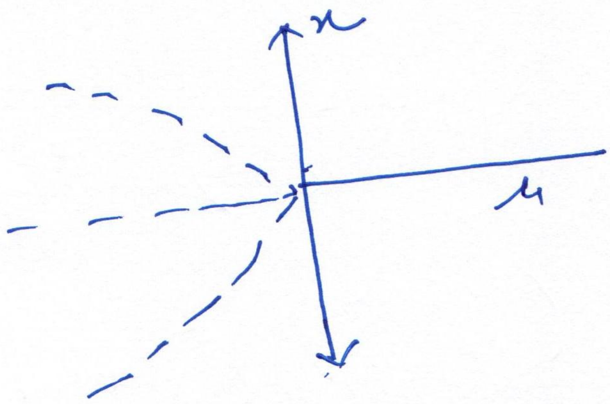
Transcritical bifurcation

(b)

$\dot{x} = \mu x + x^3$

$\dot{y} = -y$

Fixed points. $\equiv$ $(0,0)$, $(\sqrt{-\mu}, 0)$ & $(-\sqrt{-\mu}, 0)$



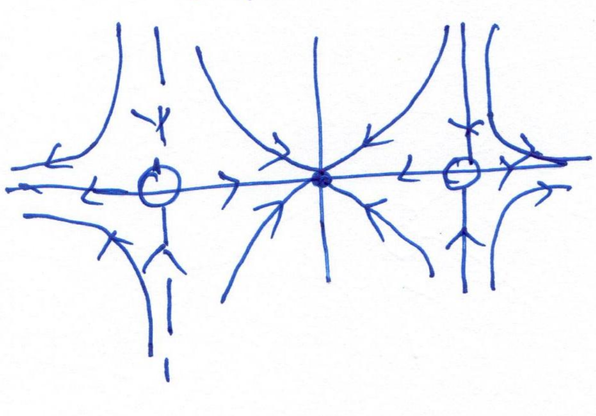

Subcritical pitchfork bifurcation.

$$J = \begin{pmatrix} \mu + 3x^2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J\big|_{(0,0)} = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \lambda_1 = \mu, \ \lambda_2 = -1$$

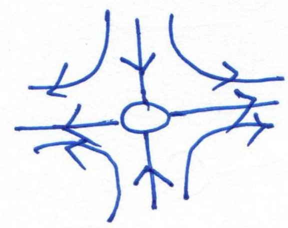
$$J\big|_{(\pm\sqrt{\mu}, 0)} = \begin{pmatrix} -2\mu & 0 \\ 0 & -1 \end{pmatrix} \quad \lambda_1 = -2\mu, -1$$

$\mu < 0$



$\mu = 0$

$\mu = 0$



$\mu > 0$