PH 441
End-Sem. Part II
27/11/2020
Marks: 20

1. (a) In using Grover's algorithm for N=8 to identify one marked state, after some iterations, the amplitude of the marked state is found to be $k_i = \sqrt{6/7}$ and that of each unmarked state to have the value $1/7$. Find the amplitude of the marked state, k_{i+2} after two more iterations. $\frac{1}{8}\left(3 + \sqrt{6/7}\right)$
(b) Work out the probability of failure of Grover's algorithm for identifying a single marked state out of 8 items after two applications of Grover's rotation. $7/128$

2+2=4

2. (a) Find the quantum Fourier transform (QFT) of

$$\sqrt{\frac{2}{N}} \sum_{x=0}^{N-1} sin\left(\frac{2\pi x}{N}\right)|x\rangle$$

$-\frac{i}{\sqrt{2}}\left(|1\rangle - |N-1\rangle\right)$

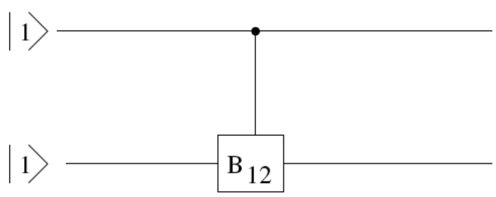
(b)Find the QFT of the Bell state: $\frac{|01\rangle+|10\rangle}{\sqrt{2}}$. $\frac{1}{2\sqrt{2}}\left[2|00\rangle - (1-i)|01\rangle - (1+i)|11\rangle\right]$
(c) The order in which the elementary gates are applied in implementing QFT on a four qubit register, (after the swap operation which relabels the qubits) are as follows:

$$H_3(B_{23}H_2)(B_{13}B_{12}H_1)(B_{03}B_{02}B_{01}H_0)$$

Based on the above information, deduce how many B_{jk} gates are required for implementing QFT on a m qubit register? (Note that: B_{jk} represents the gate applied on the j-th qubit with k-th qubit as control and H_i represents a Hadamard gate applied on the i-th qubit.) $m(m-1)/2$

2+2+2=6

3. (a) Find the output of the following circuit:



$-|11\rangle$

(b) Find the continued fraction representation of $\frac{61}{45}$. $[1, 2, 1, 4, 3]$

1+1=2

4. (a) In a three-qubit code, assume that the probability of a single bit flip error is 0.05. If the ancilla bits are measured to be $|01\rangle$, what is the probability that the measured state is error free? 0.95
(b) Consider a three-qubit code for correcting a bit flip where $|\psi\rangle = a|0\rangle + b|1\rangle$ is encoded as $|\psi'\rangle = a|000\rangle + b|111\rangle$. The third qubit of the encoded state is distorted by a rotation of 60^0 about the x-axis. Find the resultant encoded state.

$\frac{\sqrt{3}}{2}\left(a|000\rangle + ib|111\rangle\right) + \frac{1}{2}\left(ia|001\rangle + b|110\rangle\right)$

2+3=5

5. Briefly discuss how you will encode, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, using Shor's 9-qubit error code. Draw the relevant quantum circuit.

__Solution__

1. (a) Here $k_i = \sqrt{\frac{6}{7}} = a_w$

$$a_x = \frac{1}{7}$$

__1st iteration__ $U_s \, U_w$

After applying U_w : $\text{mean} = \frac{1}{8}\left(7 \cdot \frac{1}{7} - \sqrt{\frac{6}{7}}\right) = \frac{1}{8}\left(1 - \sqrt{\frac{6}{7}}\right)$

$$a_w = 2\bar{a} - a_w$$

After applying U_s

$$\Rightarrow a_w = \frac{1}{4}\left(1 - \sqrt{\frac{6}{7}}\right) - \left(-\sqrt{\frac{6}{7}}\right) = \frac{1}{4} + \frac{3}{4}\sqrt{\frac{6}{7}}$$

$$a_x = \frac{1}{4}\left(1 - \sqrt{\frac{6}{7}}\right) - \frac{1}{7} = \frac{3}{28} - \frac{1}{4}\sqrt{\frac{6}{7}}$$

__2nd iteration__

$$\text{mean} = \frac{1}{8}\left(7\left\{\frac{3}{28} - \frac{1}{4}\sqrt{\frac{6}{7}}\right\} - \frac{1}{4} - \frac{3}{4}\sqrt{\frac{6}{7}}\right)$$

$$= \frac{1}{8}\left(\frac{1}{2} - \frac{5}{2}\sqrt{\frac{6}{7}}\right)$$

After applying U_s

$$a_w = \frac{1}{8}\left(1 - 5\sqrt{\frac{6}{7}}\right) + \frac{1}{4} + \frac{3}{4}\sqrt{\frac{6}{7}}$$

$$= \frac{1}{8}\left(3 + \sqrt{\frac{6}{7}}\right)$$

Thus, The amplitude of the marked state k_{i+2} after two more iterations will be

$$\boxed{\frac{1}{8}\left(3 + \sqrt{\frac{6}{7}}\right)}$$

(b) If you follow the steps, you will get

$$a_w = \frac{11}{8\sqrt{2}}$$

$$\Rightarrow \text{Probability of success} = \frac{121}{128}$$

Thus, $\boxed{\text{Probability of failure} = 1 - \frac{121}{128} = \frac{7}{128}}$

2. (a) (A similar problem was done in the class. Follow the same procedure.)

$$|\psi\rangle = \sqrt{\frac{2}{N}} \sum_{x=0}^{N-1} \sin\left(\frac{2\pi x}{N}\right)|x\rangle$$

$$QFT\{|\psi\rangle\} = \sqrt{\frac{2}{N}}\sqrt{\frac{1}{N}} \sum_{y=0}^{N-1}\sum_{x=0}^{N-1} \sin\left(\frac{2\pi x}{N}\right) e^{2\pi i x y/N} |y\rangle$$

$$= \sum_{y=0}^{N-1} c_y |y\rangle$$

Here,
$$c_y = \frac{\sqrt{2}}{N} \sum_{x=0}^{N-1} \sin\left(\frac{2\pi x}{N}\right) e^{2\pi i x y/N}$$

$$= \frac{\sqrt{2}}{N} \sum_{x=0}^{N-1} \left(\frac{e^{2\pi i x/N} - e^{-2\pi i x/N}}{2i}\right) e^{2\pi i x y/N}$$

$$= \frac{1}{\sqrt{2} N i} \sum_{x=0}^{N-1} \left(e^{2\pi i x(y+1)/N} - e^{-2\pi i x(y-1)/N}\right)$$

$$\Rightarrow c_y = \begin{cases} \frac{1}{\sqrt{2} i}, & \text{when } y = 1 \\ \frac{-1}{\sqrt{2} i}, & \text{when } y = N-1 \\ 0 & \text{else} \end{cases}$$

Thus, $\boxed{QFT\{|\psi\rangle\} = -\frac{i}{\sqrt{2}}\left(|1\rangle - |N-1\rangle\right)}$

(b)
$$|\psi\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$\boxed{QFT(|\psi\rangle) = \frac{1}{2\sqrt{2}}\left[2|00\rangle - (1-i)|01\rangle - (1+i)|11\rangle\right]}$$

(c) $\boxed{m(m-1)/2}$

3. (a) $-|11\rangle$

(b) $[1, 2, 1, 4, 3]$

4. (a)

		Prob.
$\|01\rangle \longrightarrow$	$a\|001\rangle + b\|110\rangle$	$p(1-p)^2$
	$a\|110\rangle + b\|001\rangle$	$p^2(1-p)$

$$p = 0.05$$

$$\text{Prob. of error free} = \frac{p(1-p)^2}{p(1-p)^2 + p^2(1-p)}$$

$$= \frac{1-p}{(1-p)+p}$$

$$= 0.95 \quad //$$

(b)

$$|\psi'\rangle = a|00\rangle|0\rangle + b|11\rangle|1\rangle$$

Due to rotation in the third qubit:

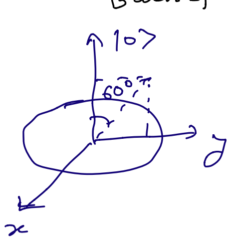
$$|0\rangle \longrightarrow \frac{\sqrt{3}}{2}|0\rangle + i\frac{1}{2}|1\rangle$$

$$|1\rangle \longrightarrow \frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle$$

$\left\{ \begin{array}{l} \theta = 60^\circ \\ \phi = \pi/2 \end{array} \right.$

$\left\{ |\psi\rangle \rightarrow \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle \right.$

(Recall Bloch sphere!)



Thus the transformed state is

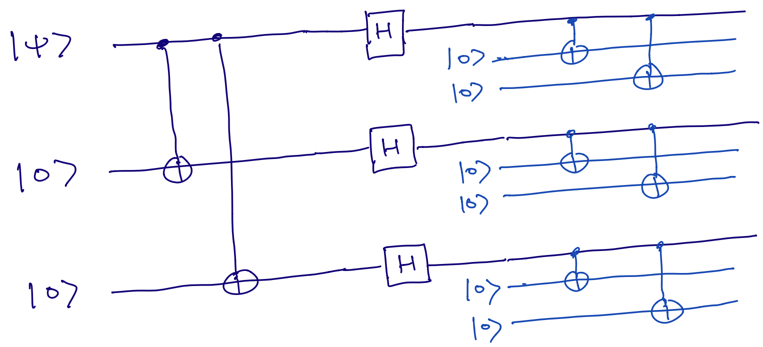
$$a|00\rangle\left(\frac{\sqrt{3}}{2}|0\rangle + i\frac{1}{2}|1\rangle\right) + b|11\rangle\left(\frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle\right)$$

$$= \frac{\sqrt{3}}{2}\left[a|000\rangle + ib|111\rangle\right]$$

$$+ \frac{1}{2}\left[\, ia\,|001\rangle + b\,|110\rangle \,\right]$$

5. It was discussed in details in the class.

The relevant circuit is



The circuit does the following

$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{Quantum ckt}} \alpha\,|{+}{+}{+}\rangle + \beta\,|{-}{-}{-}\rangle$$