PH 441

Lecture 5

Qubit states

A general qubit state can be written as:

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$$

where $|c_0|^2 + |c_1|^2 = 1$, $c_i \in \mathbb{C}$

Using polar representation: $c_0 = r_0 e^{i\phi_0}$, $c_1 = r_1 e^{i\phi_1}$

$$|\psi\rangle = r_0 e^{i\phi_0} |0\rangle + r_1 e^{i\phi_1} |1\rangle$$

Now given two components c_i, one can conclude that we have 4 unknowns (2 phases and 2 angles) amplitudes) that uniquely determine the components.

However, in case of quantum bits, we know that a quantum state does not change if we multiply it with any number of unit norm.

i.e. $|\psi\rangle \equiv e^{i\phi} |\psi\rangle$

Thus, if we take $\phi = -\phi_0$ then our equivalent state is:

$$e^{-i\phi_0} |\psi\rangle = e^{-i\phi_0} \left(r_0 e^{i\phi_0} |0\rangle + r_1 e^{i\phi_1} |1\rangle \right)$$

$$= r_0 |0\rangle + r_1 e^{i(\phi_1 - \phi_0)} |1\rangle$$

So, from 4 parameters, now we end up with 3 parameters! r_0, r_1 and $\phi = \phi_1 - \phi_0$. ~~It~~ Again, $|r_0|^2 + |r_1|^2 = 1 \Rightarrow r_0^2 + r_1^2 = 1$, thus reducing the unknowns to 2! Taking $r_0 = \cos\theta$ and $r_1 = \sin\theta$

we obtain the equivalent representation of $|\psi\rangle$:

$$\boxed{|\psi\rangle = \cos\theta\,|0\rangle + e^{i\phi}\sin\theta\,|1\rangle}$$

The Bloch sphere representation:

The above results into the Bloch sphere representation, named after Felix Bloch.

But before we go further, let us discuss about Pauli matrices yet again! It is going to be very relevant. In fact

" what about Pauli matrix along any arbitrary direction? "

$\longrightarrow$

Quantum Computation and Quantum Cryptography

PH 441

Lecture 5

The Qubit : Bloch Sphere representation

The smallest unit of ~~s~~ quantum ~~state~~ information is called a qubit, which corresponds to classical bit or cbit 0 and 1

: $\alpha|0\rangle + \beta|1\rangle$

We learnt that any 2×2 matrix could be written/expressed in terms of Pauli matrices and the unit matrix.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

~~Clearly, a qubit, being a two state system, to can be represented by Pauli matrices.~~

What about Pauli matrix along any arbitrary direction, say $\hat{n}$?

$$\hat{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

~~σ_n~~

$$\sigma_n = \vec{\sigma}\cdot\hat{n} = \sigma_x \sin\theta\cos\phi + \sigma_y \sin\theta\sin\phi + \sigma_z\cos\theta$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$\boxed{\sigma_n = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}}$$

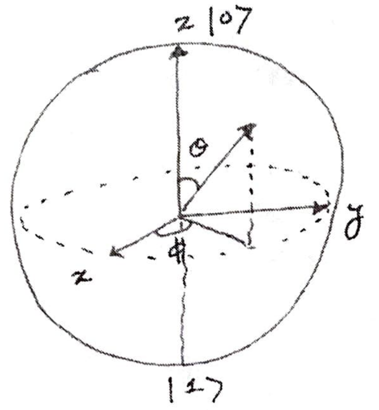
Eigenvalues $\lambda = \pm\sqrt{\cos^2\theta + \sin^2\theta} = \pm 1$

Eigenstates associated with ~~$\lambda = +1$~~ $\lambda = +1$

$$|\theta, \phi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \cos\frac{\theta}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\phi}\sin\frac{\theta}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

Bloch sphere representation: (Representation of a Pauli matrix in an arbitrary direction)



Associated with every point on the unit sphere, there is a unique state, having a value (θ, ϕ).

Take $\theta = 0$, ~~$\phi = 0$~~ $\phi = 0$, it corresponds to the state $|0\rangle$

$\Rightarrow$ North pole corresponds to the state $|0\rangle$

Take, ~~$\theta = 0$~~, $\theta = \pi$, ~~ϕ~~ $\phi = 0$, it corresponds to the state $|1\rangle$

Consider the point where +ve x-axis meets the equator:

In this case, $\theta = \frac{\pi}{2}$, $\phi = 0$

Thus this point corresponds to $\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$

On the other hand, the point where −ve x-axis meets the equator refers to $\theta = \frac{\pi}{2}$, ~~$\phi = 0$~~ $\phi = -\pi$

Thus, it represents the state:

$$\frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

Thus, every point on the Bloch sphere stands for a unique quantum state. All these states that lie on the surface of a unit sphere are called pure states.

Q. How much information is there in a qubit?

A. In principle, a qubit contains infinite amount of information. However, much of this information is not available to us, as if we make measurement, either we get 0 or 1.

Let us dig a bit further!

Consider a qubit state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\phi}}{\sqrt{2}}|1\rangle$$

If we make a measurement on $|\psi\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$, we get either $|0\rangle$ or $|1\rangle$ each with probability $\frac{1}{2}$. However we loss the information on ϕ. But if we make a measurement on a diagonal basis, then we will be in a position to find out the information about the relative phase ϕ.

Let us now measure $|\psi\rangle$ in a diagonal basis:

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$\Rightarrow \quad |0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \quad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Now, ~~we can express~~ ~~$|\psi\rangle =$~~

Thus,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right] + \frac{e^{i\phi}}{\sqrt{2}}\left[\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\left[|+\rangle + |-\rangle\right] + \frac{e^{i\phi}}{2}\left[|+\rangle - |-\rangle\right]$$

$$= \frac{1}{2}\left(1 + e^{i\phi}\right)|+\rangle + \frac{1}{2}\left(1 - e^{i\phi}\right)|-\rangle$$

$$= \frac{e^{i\phi/2}}{2}\left[2\cos\frac{\phi}{2}\,|+\rangle - 2i\,\sin\frac{\phi}{2}\,|-\rangle\right]$$

$$|\psi\rangle = \underbrace{e^{i\phi/2}}\left[\cos\frac{\phi}{2}\,|+\rangle - i\,\sin\frac{\phi}{2}\,|-\rangle\right]$$

overall phase factor, which does not matter in a quantum state.

Now if we make a measurement in the diagonal basis $\{|+\rangle, |-\rangle\}$ we will get

$|+\rangle$ with probability $\cos^2\frac{\phi}{2}$

$|-\rangle$ with probability $\sin^2\frac{\phi}{2}$

Thus, this ~~information~~ measurement gives us some information about the relative phase.

Digression

Diagonal basis and computational basis

A qubit: $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$

with $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$\Rightarrow \quad |\psi\rangle = \begin{pmatrix}a\\b\end{pmatrix}$

$\{|0\rangle, |1\rangle\}$ is called the computational basis (CB)

If a measurement is made in CB, we are not going to get a linear combination, rather we will get either $|0\rangle$ or $|1\rangle$.

physical examples of quantum bits :

(i) Spin - $\frac{1}{2}$ atoms / system

$$S_z = +\frac{1}{2} \quad \text{or} \quad -\frac{1}{2}$$

$$\uparrow$$

$$|0\rangle \quad \text{or} \quad |1\rangle$$

$$\text{or } |\uparrow\rangle \quad \text{or} \quad |\downarrow\rangle$$

(ii) Polarization state of a photon

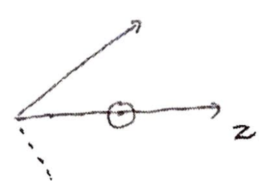
A photon propagating along the z-direction has its polarization either along x-direction or y-direction, basically in the X-Y plane.

Polarization direction :

$$|x\rangle \equiv |\leftrightarrow\rangle \equiv |H\rangle$$

$$|y\rangle \equiv |\updownarrow\rangle \equiv |V\rangle$$

We can take ~~d~~ different polarization directions as well. Say ~~45~~ one polarization direction making 45° with the z-direction and the other 135° to it.



This type of basis are called diagonal basis.

In diagonal basis, one of the ~~axes~~ axis makes 45° with computational basis, while the other one makes 135° with CB.

$\cancel{|\pm\rangle \equiv}$ Let's represent the state making 45° with the Horizontal direction, i.e. CB by

$$|+\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] \quad \left\{= \frac{1}{\sqrt{2}}\left[|\leftrightarrow\rangle + |\updownarrow\rangle\right]\right\}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right] \quad \Rightarrow |\pm\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle \pm |1\rangle\right]$$

Properties of Bloch sphere

(1) Orthogonality of opposite points:

consider a general qubit state

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

Say $|\chi\rangle$ corresponds to an opposite point on the Bloch sphere

$$|\chi\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\phi+\pi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$$

$$= \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle - e^{i\phi}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$$

So,

$$\langle\chi|\psi\rangle = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\pi-\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi-\theta}{2}\right)$$

$$= \cos\left(\frac{\theta}{2} + \frac{\pi-\theta}{2}\right)$$

$$= \cos\frac{\pi}{2}$$

$$= 0$$

$\Rightarrow$ Opposite points corresponds to orthogonal qubit states.

(2) Rotations on the Bloch sphere :

The Pauli matrices σ_x, σ_y and σ_z give rise to rotation operators, which rotate the Bloch vector $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ about $\hat{x}$, $\hat{y}$ and $\hat{z}$:

$$R_x(\theta) = e^{-i\frac{\theta}{2}\sigma_x}$$

$$R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y}$$

$$R_z(\theta) = e^{-i\frac{\theta}{2}\sigma_z}$$

We know from last lecture:

$$e^{i\alpha\sigma} = \cos\alpha\, I + i\,(\vec{n}\cdot\vec{\sigma})\sin\alpha$$

$$R_x(\theta) = e^{-i\frac{\theta}{2}\sigma_x} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\sigma_x$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_z(\theta) = e^{-i\frac{\theta}{2}\sigma_z} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\sigma_z$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

Consider:

$$R_x(\pi) = \begin{pmatrix} \cos\frac{\pi}{2} & -i\sin\frac{\pi}{2} \\ -i\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$= -i\,\sigma_x$$

$\Rightarrow$ σ_x operator is equivalent to a rotation of 180° about the x-axis.

The rotation operators do not in general keep the co-efficient of the $|0\rangle$ component of the qubit state real.