Bifurcation in 1D flow

Saddle node bifurcation

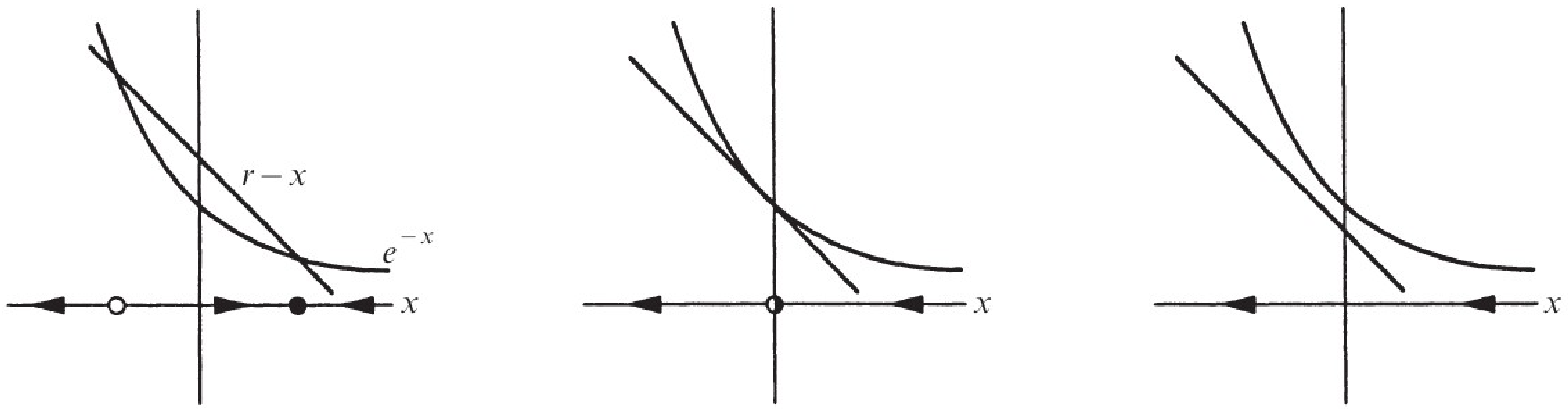
Example:

$$\dot{x} = r - x - e^{-x}$$

The fixed points satisfy $f(x) = r - x - e^{-x} = 0$.

Fixed point can't by find quite easily due to its dependence on the complex function.

We will find the solution using the graphical method.



(a) (b) (c)

Bifurcation in 1D flow

Example:

$$\dot{x} = r - x - e^{-x}$$

To find the bifurcation point r_c, we impose the condition that the graphs of $r - x$ and e^{-x} intersect *tangentially*. Thus we demand equality of the functions *and* their derivatives:

$$e^{-x} = r - x$$

and

$$\frac{d}{dx}e^{-x} = \frac{d}{dx}(r - x).$$

The second equation implies $-e^{-x} = -1$, so $x = 0$. Then the first equation yields $r = 1$. Hence the bifurcation point is $r_c = 1$, and the bifurcation occurs at $x = 0$. ■

Bifurcation in 1D flow

Example:

$$\dot{x} = r - x - e^{-x}$$

In a certain sense, the examples $\dot{x} = r - x^2$ or $\dot{x} = r + x^2$ are representative of *all* saddle-node bifurcations; that's why we called them "prototypical." The idea is that, close to a saddle-node bifurcation, the dynamics typically look like $\dot{x} = r - x^2$ or $\dot{x} = r + x^2$.

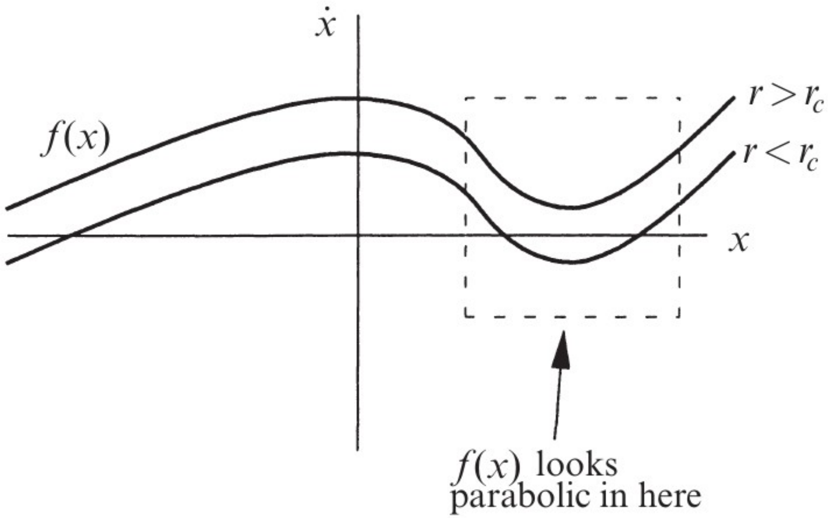
Using the Taylor expansion for e^{-x} about $x = 0$, we find

$$\begin{aligned}\dot{x} &= r - x - e^{-x} \\ &= r - x - \left[1 - x + \frac{x^2}{2!} + \cdots\right] \\ &= (r-1) - \frac{x^2}{2} + \cdots\end{aligned}$$

For appropriate scalins of r and x it can be shown that it has the typical form as

$$\dot{x} = r - x^2$$

Bifurcation in 1D flow



$$\dot{x} = f(x, r)$$

$$= f(x^*, r_c) + (x - x^*)\left.\frac{\partial f}{\partial x}\right|_{(x^*, r_c)} + (r - r_c)\left.\frac{\partial f}{\partial r}\right|_{(x^*, r_c)} + \tfrac{1}{2}(x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r_c)} + \cdots$$

Bifurcation in 1D flow

where we have neglected quadratic terms in $(r - r_c)$ and cubic terms in $(x - x^*)$. Two of the terms in this equation vanish: $f(x^*, r_c) = 0$ since x^* is a fixed point, and $\partial f/\partial x|_{(x^*, r_c)} = 0$ by the tangency condition of a saddle-node bifurcation. Thus

$$\dot{x} = a(r - r_c) + b(x - x^*)^2 + \cdots$$

where $a = \partial f/\partial x|_{(x^*, r_c)}$ and $b = \frac{1}{2}\partial^2 f/\partial x^2|_{(x^*, r_c)}$

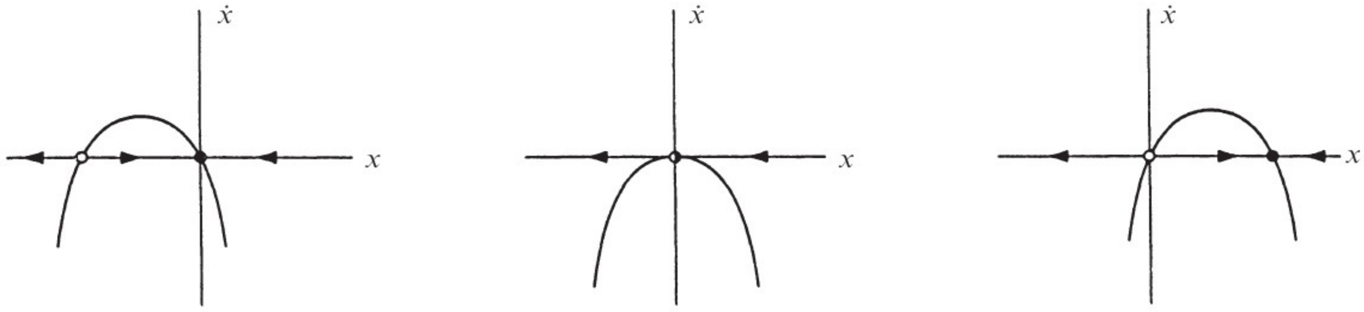
Bifurcation in 1D flow

Transcritical Bifurcation

There are certain cases in which the fixed point will remain for all the parameter range and may change its stability as the parameter is varied. The transcritical bifurcation describes mechanism of such change in the stability.

The normal form for a transcritical bifurcation is

$$\dot{x} = rx - x^2.$$



(a) $r<0$ (b) $r=0$ (c) $r>0$

Bifurcation in 1D flow

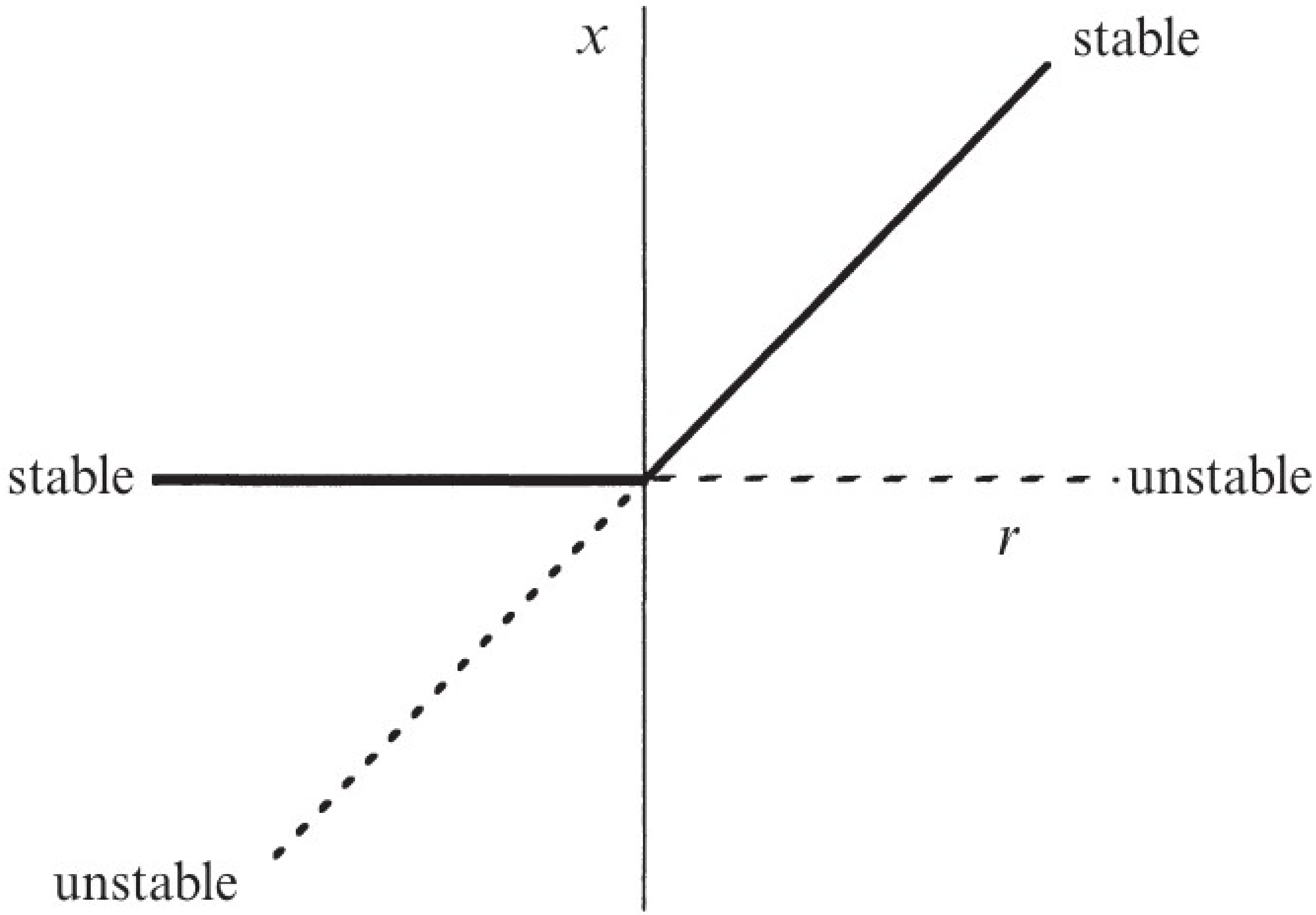
Transcritical Bifurcation

For $r < 0$, there is an unstable fixed point at $x^* = r$ and a stable fixed point at $x^* = 0$. As r increases, the unstable fixed point approaches the origin, and coalesces with it when $r = 0$. Finally, when $r > 0$, the origin has become unstable, and $x^* = r$ is now stable. Some people say that an ***exchange of stabilities*** has taken place between the two fixed points.

Please note the important difference between the saddle-node and transcritical bifurcations: in the transcritical case, the two fixed points don't disappear after the bifurcation—instead they just switch their stability.

Bifurcation in 1D flow

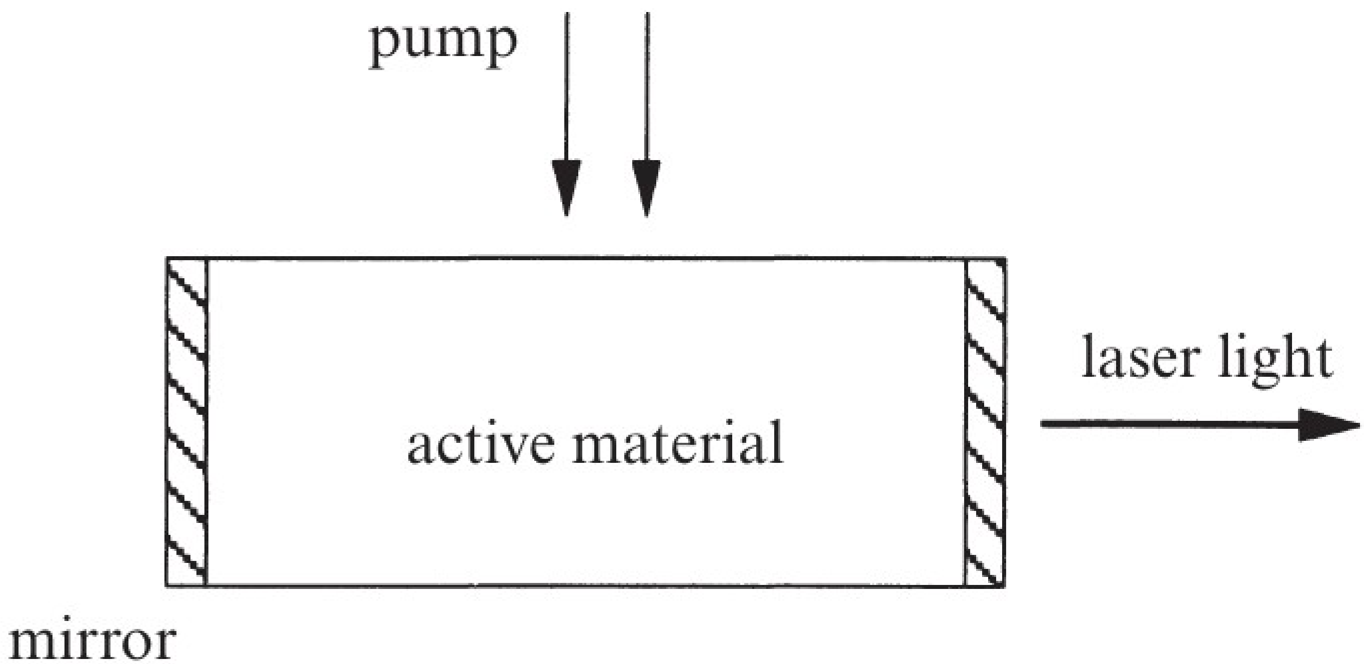
Transcritical Bifurcation



Bifurcation in 1D flow

Transcritical Bifurcation

Physical Example



Typical setup for the laser.

<u>Bifurcation in 1D flow</u>

Transcritical Bifurcation

$$\begin{aligned} \dot{n} &= \text{gain} - \text{loss} \\ &= GnN - kn. \end{aligned}$$

n(t) number of photon emmited

The gain term comes from the process of *stimulated emission*, in which photons stimulate excited atoms to emit additional photons. Because this process occurs via random encounters between photons and excited atoms, it occurs at a rate proportional to n and to the number of excited atoms, denoted by $N(t)$. The parameter $G > 0$ is known as the gain coefficient. The loss term models the escape of photons through the endfaces of the laser. The parameter $k > 0$ is a rate constant; its reciprocal $\tau = 1/k$ represents the typical lifetime of a photon in the laser.

Bifurcation in 1D flow

Transcritical Bifurcation

Now comes the key physical idea: after an excited atom emits a photon, it drops down to a lower energy level and is no longer excited. Thus N decreases by the emission of photons. To capture this effect, we need to write an equation relating N to n. Suppose that in the absence of laser action, the pump keeps the number of excited atoms fixed at N_0. Then the *actual* number of excited atoms will be reduced by the laser process. Specifically, we assume

$$N(t) = N_0 - \alpha n,$$

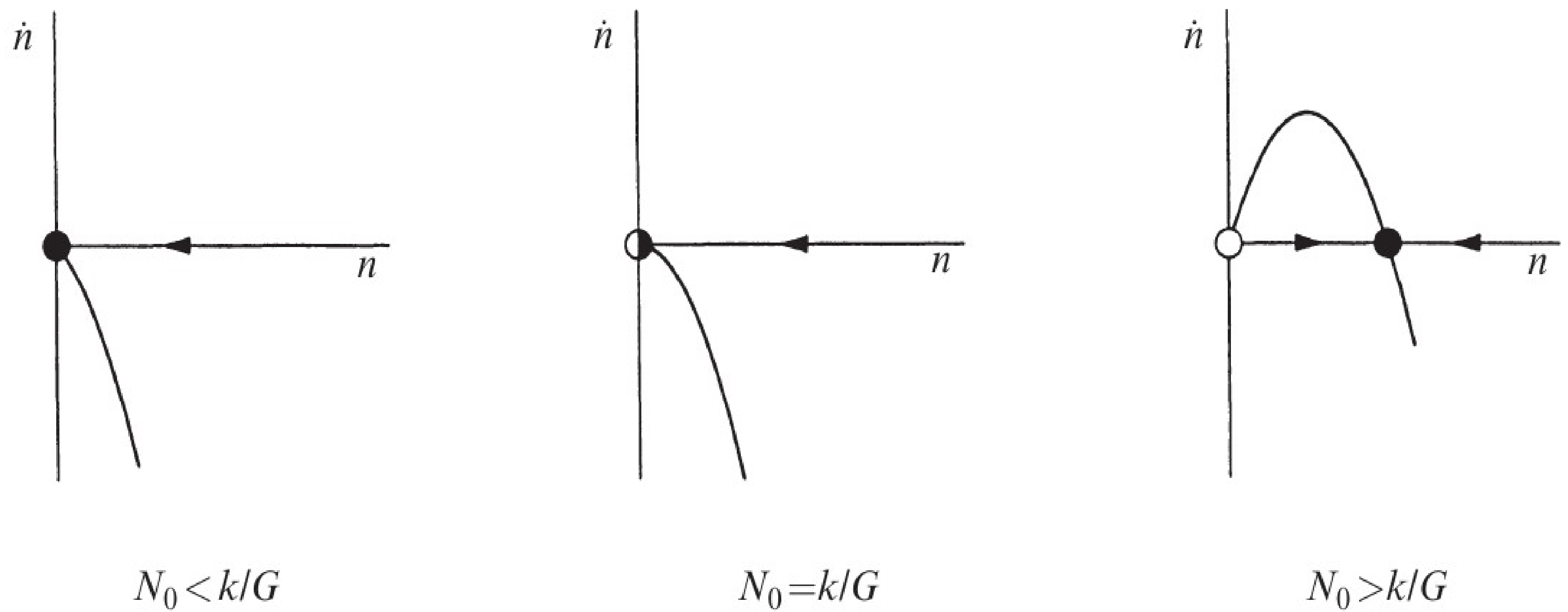
where $\alpha > 0$ is the rate at which atoms drop back to their ground states. Then

$$\begin{aligned} \dot{n} &= Gn(N_0 - \alpha n) - kn \\ &= (GN_0 - k)n - (\alpha G)n^2. \end{aligned}$$

Bifurcation in 1D flow

Transcritical Bifurcation

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$
$$= (GN_0 - k)n - (\alpha G)n^2.$$



Bifurcation in 1D flow

Transcritical Bifurcation

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$
$$= (GN_0 - k)n - (\alpha G)n^2 .$$

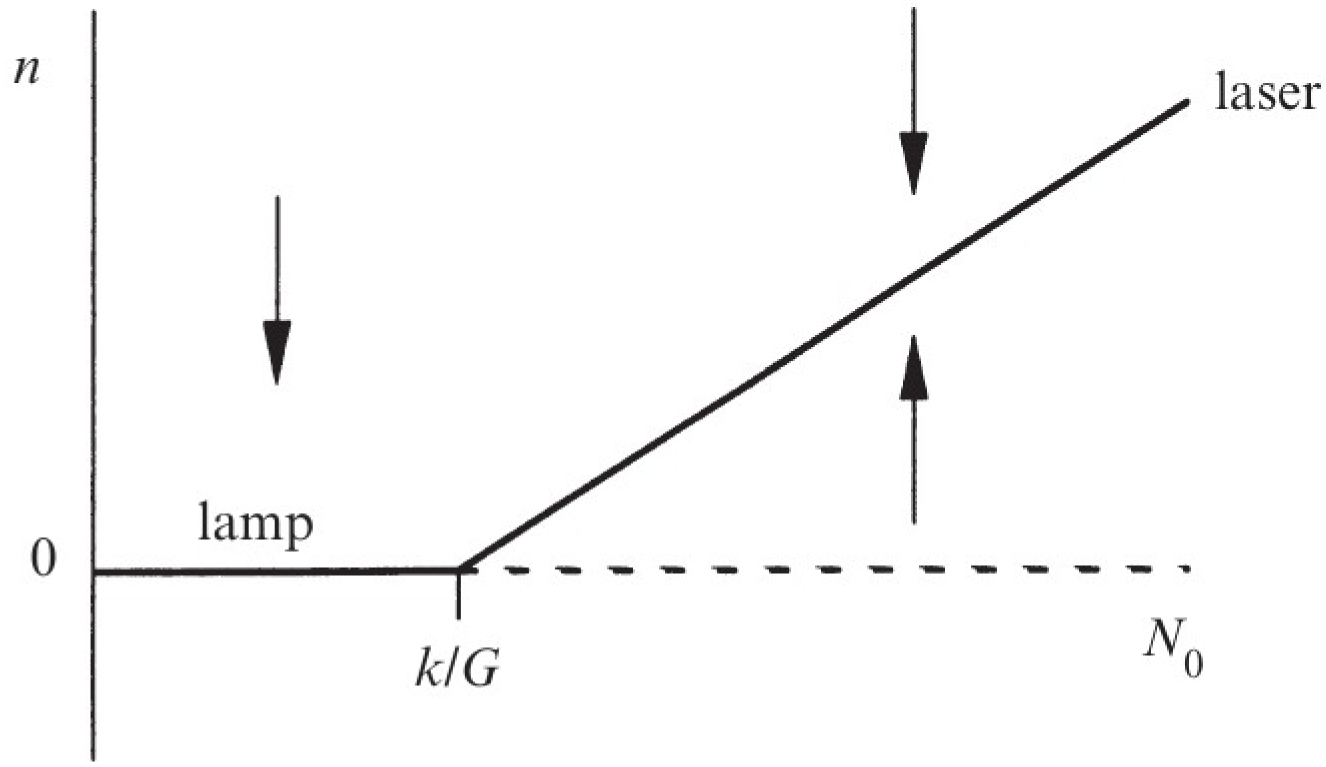
When $N_0 < k/G$, the fixed point at $n^* = 0$ is stable. This means that there is no stimulated emission and the laser acts like a lamp. As the pump strength N_0 is increased, the system undergoes a transcritical bifurcation when $N_0 = k/G$. For $N_0 > k/G$, the origin loses stability and a stable fixed point appears at $n^* = (GN_0 - k)/\alpha G > 0$, corresponding to spontaneous laser action. Thus

$N_0 = k/G$ can be interpreted as the ***laser threshold*** in this model.

Bifurcation in 1D flow

Transcritical Bifurcation

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$
$$= (GN_0 - k)n - (\alpha G)n^2.$$



Although this model perdicts the threshold value of pump beyond which the laser action begins it has lots of limitations (not consider the spontaneous emission, quantum mechanical effect, etc..)

Bifurcation in 1D flow

Pitchfork Bifurcation

Supercritical Pitchfork Bifurcation

$$\dot{x} = rx - x^3 .$$