PH441

Lecture 03

Qubit

A qubit – a quantum bit – is the fundamental unit of quantum information. At any given instant of time, it is in a superposition of state represented by a linear combination of vectors $|0\rangle$ and $|1\rangle$ in two-dimensional complex vector space.

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle$$

where $|a|^2 + |b|^2 = 1$, a and b are called probability amplitudes.

Through measurement, a qubit is forced to collapse irreversibly through projection to either $|0\rangle$ or $|1\rangle$. The probability of its doing either is $|a|^2$ and $|b|^2$ respectively.

A single qubit is <u>not</u> interesting alone. We need many more qubits to do useful things!

Before we proceed further, let me introduce some mathematical tools which we will need.

Consider the three-dimensional vector

$$\vec{w} = \hat{i}\, w_1 + \hat{j}\, w_2 + \hat{k}\, w_3$$

It can be represented in several forms:

$$\vec{w} = (w_1, w_2, w_3)$$

$$= [w_1 \;\; w_2 \;\; w_3] \quad \text{as a row vector}$$

$$= \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad \text{as a column vector}$$

Let us introduce an elegant notation introduced by Paul Dirac, an English theoretical physicist. The vector $\vec{w}$ can be denoted by ~~a~~ the ~~vector~~ notation, $|w\rangle$ called ket-vector:

$$|w\rangle = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \quad ; \text{ here } |w\rangle \text{ is a three-dimensional vector}$$

If $\vec{w}$ is n-dimensional, then:

$$|w\rangle = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

~~The complex~~ On the other hand, given a vector $\vec{v} = (v_1, v_2, \cdots v_m)$, we denote by $\langle v|$, pronounced "bra-v", the <u>row vector</u> $[\bar{v}_1 \; \bar{v}_2 \; \cdots \; \bar{v}_m]$, where we take the complex conjugate of each entry.

$|v\rangle\langle w|$ is known as the outer product.

When $n = m$, the bra-ket

$$\langle v|w\rangle = \langle v|\,|w\rangle = (\langle v|)(|w\rangle) \text{ of}$$

$\vec{v}$ and $\vec{w}$ is the usual inner product.

$$\langle v|w\rangle = \overline{v_1}w_1 + \overline{v_2}w_2 + \cdots + \overline{v_n}w_n$$

The length of the vector $\vec{v}$ is $\|\vec{v}\| = \sqrt{\langle v|v\rangle}$

Some examples

1. Say $\vec{v} = (3, i)$ and $\vec{w} = (2+i, 4)$

then

$$|v\rangle = \begin{bmatrix} 3 \\ i \end{bmatrix}$$

$$\langle v| = \begin{bmatrix} 3 & -i \end{bmatrix}$$

$$|w\rangle = \begin{bmatrix} 2+i \\ 4 \end{bmatrix}$$

$$\langle w| = \begin{bmatrix} 2-i & 4 \end{bmatrix}$$

$$\langle v|w\rangle = \begin{bmatrix} 3 & -i \end{bmatrix} \cdot \begin{bmatrix} 2+i \\ 4 \end{bmatrix}$$

$$= 6 + 3i - 4i$$

$$= 6 - i$$

$$|v\rangle\langle w| = \begin{bmatrix} 3 \\ i \end{bmatrix} \begin{bmatrix} 2-i & 4 \end{bmatrix} = \begin{bmatrix} 6-3i & 12 \\ 2i+1 & 4i \end{bmatrix}$$

2. $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\langle 0|1\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 + 0 = 0$$

sly, $\langle 1|0\rangle = 0$

$$\langle 0|0\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

sly, $\langle 1|1\rangle = 1$

3. $|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\langle +|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 1 - 1 \end{bmatrix} = 0$$

$$\langle +|+\rangle = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$$

sly, $\cancel{\langle -|+\rangle = 0\ 1}$

sly, $\langle -|-\rangle = 1$

<u>Note</u>: $|+\rangle$ and $|-\rangle$ are known as standard basis

More about bra and ket

There is a one-to-one correspondence between a ket space and a bra space.

$$|\alpha\rangle \xrightarrow{\text{Dual correspondence}} \langle\alpha|$$

$$|\alpha\rangle + |\beta\rangle \xrightarrow{DC} \langle\alpha| + \langle\beta|$$

$$c\,|\alpha\rangle \xrightarrow{DC} c^* \langle\alpha|$$

$$a|\alpha\rangle + b|\beta\rangle \xrightarrow{DC} a^* \langle\alpha| + b^* \langle\beta|$$

ket space   bra-space

$$\left\{ \begin{array}{l} a = a^* \\ \quad = \bar{a} \\ *,- \uparrow \text{ complex conjugate} \end{array} \right.$$

Inner product of ket $|\alpha\rangle$ and bra $\langle\beta|$ is defined as

$$\langle\beta|\alpha\rangle$$

This product is in general a complex number.

Please note that : (1) $\langle\beta|\alpha\rangle = (\langle\alpha|\beta\rangle)^*$

(2) $\langle\alpha|\alpha\rangle \geqslant 0$

Two kets $|\alpha\rangle$ and $|\beta\rangle$ are said to be orthogonal if

$$\langle\alpha|\beta\rangle = 0$$

or $\langle\beta|\alpha\rangle = 0$

Normalized ket $|\tilde{\alpha}\rangle$ from $|\alpha\rangle$:

$$|\tilde{\alpha}\rangle = \frac{1}{\sqrt{\langle\alpha|\alpha\rangle}} |\alpha\rangle$$

with the property $\langle\tilde{\alpha}|\tilde{\alpha}\rangle = 1$

I said earlier that if a qubit is in the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

then the Qubit can be in both $|0\rangle$ and $|1\rangle$ state simultaneously, if we do not make any measurement! However if we make a measurement to know, then the measurement will force the state $|\psi\rangle$ to ~~go~~ take either $|0\rangle$ or $|1\rangle$. This measurement is generally represented by operators, mathematically speaking. In fact for every physically observable quantities in classical physics there is a corresponding operator in Quantum physics.

If an operator, A, acts on a state say $|\psi\rangle$ it may go over into an another state say $|\phi\rangle$:

$$A|\psi\rangle = |\phi\rangle$$

Example say $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, now due to application of operator A it turns into $|\phi\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then what how you will represent A?

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is easy to see that A will be represented by the 2x2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus, if you are in 2-dimensional linear vector space, then the operators will be represented by 2×2 matrix.

Say, if you are in n-dimensional LVS, then the operators will be represented by $n \times n$ matrix.

An operator acts on a ket from left side:

$$A \cdot (|\alpha\rangle) = A|\alpha\rangle$$ and the resulting product is another ket.

On the other hand, an operator always acts on a bra from ~~left~~ the right side:

$$(\langle\alpha|) \cdot A = \langle\alpha| A$$, resulting in another bra.

The ket $A|\alpha\rangle$ and the bra $\langle\alpha|A$ are, in general not dual to each other. We define symbol A^{+} as

$$A|\alpha\rangle \xleftrightarrow{DC} \langle\alpha| A^{+}$$

The operator A^{+} is said to be Hermitian adjoint or simply adjoint of A. An operator A is said to be Hermitian if

$$A = A^{+}.$$

In Quantum physics the operators represents physically observables quantities. So these operators, represented by matrix, must be Hermitian!

Because: (i) The Hermian operators has real eigenvalues.

(ii) The eigenkets of the operator A ~~rep~~ corresponding to different eigenvalues are orthogonal.

It is explained in details in the class!

Recall that:

If $A|a\rangle = a|a\rangle$

then $|a\rangle$ is called the eigenket of the operator A and 'a' is the eigenvalue.

Now, given an arbitrary ket $|\alpha\rangle$ in the ket space spanned by the eigenkets of A, then it is represented as:

$$|\alpha\rangle = \sum_a c_a |a\rangle$$

$\Rightarrow$ the arbitrary ket is a linear superposition of eigenkets $|a\rangle$ with ~~cor~~ corresponding amplitudes c_a s.

Remember: $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$?

Here $|0\rangle$ and $|1\rangle$ are eigenkets with corresponding $c_a = \frac{\sqrt{3}}{2}$ and $c_a = \frac{1}{2}$ respectively.