Quantum Fourier Transform

Last class

get $|0\rangle$ or $|4\rangle$ $\longrightarrow$ periodicity 2
$\downarrow$
determines the non-vanishing states in the 1st register

$$|\psi\rangle = \sum_x a_x |x\rangle$$

Take QFT : $$|\psi'\rangle = U|\psi\rangle = \sum_x a_x U|x\rangle = \underline{\underline{\sum_y \tilde{a}_y |y\rangle}}$$

$$\tilde{a}_y = \frac{1}{\sqrt{N}} \sum \left(e^{2\pi i/N}\right)^{xy} a_x$$

$$= \frac{1}{\sqrt{N}} \sum \omega^{xy} a_x \qquad \underline{\omega = e^{2\pi i/N}}$$

$$U \sum_x f(x)|x\rangle$$
$$= \sum_x f(x)\, U|x\rangle$$
$$= \sum_x f(x) \sum_y K(x,y)|y\rangle$$
$$= \sum_y \left(\sum_x K(x,y) f(x)\right)|y\rangle$$
$$\Rightarrow \sum_y \tilde{f}(y)|y\rangle$$
$$K(x,y) = \frac{1}{\sqrt{N}} e^{2\pi i(xy)/N}$$

$$\boxed{U = \sum_{yz} \frac{e^{2\pi i yz/N}}{\sqrt{N}} |y\rangle\langle z|}$$

$$U|x\rangle = \sum_{yz} \frac{e^{2\pi i yz/N}}{\sqrt{N}} |y\rangle \underbrace{\langle z|x\rangle}_{\delta_{zx}}$$

$$= \sum_y \frac{e^{2\pi i yx/N}}{\sqrt{N}} |y\rangle$$

$$|\tilde{x}\rangle = U|x\rangle$$

$$\boxed{|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i xy/N} |y\rangle}$$

Implementation of QFT $\qquad \underline{\underline{N = 2^n}}$

(i) Single qubit $\quad n = 1, \; N = 2$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{2\pi i xy/2} |y\rangle$$

$$= \frac{1}{\sqrt{2}} \left[|0\rangle + e^{2\pi i x/2} |1\rangle \right]$$

let us denote $\frac{x}{2} = 0{\cdot}x$; $\frac{x}{2^2} = 0{\cdot}0x$

In analogy $0{\cdot}3 = \frac{3}{10}$; $0{\cdot}03 = \frac{3}{10^2}$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i(0{\cdot}x)}|1\rangle\right]$$

$x=0$: $|0\rangle \longrightarrow \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

$x=1$: $|1\rangle \longrightarrow \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

$$e^{2\pi i(0{\cdot}1)} = e^{2\pi i \frac{1}{2}} = e^{\pi i} = (-1)$$

(ii) $n=2 \Rightarrow N=4$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{2^2}}\sum_{y=0}^{3} e^{2\pi i x y/4}|y\rangle$$

Write $y = 2y_1 + y_0$ $\quad 2^1 \times y_1 + 2^0 \times y_0$

$y_1, y_0 \in 0, 1$

$|y\rangle = |y_1 y_0\rangle$

$$|\tilde{x}\rangle = \frac{1}{2}\sum_{y_1, y_0} e^{2\pi i x(2y_1+y_0)/4}|y_1 y_0\rangle$$

$$= \frac{1}{\sqrt{2}}\sum_{y_1\in 0,1} e^{2\pi i x(2y_1/4)}|y_1\rangle \otimes \frac{1}{\sqrt{2}}\sum_{y_0\in 0,1} e^{2\pi i x y_0/4}|y_0\rangle$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i x/2}|1\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i x/4}|1\rangle\right]$$

Now, take $x = 2x_1 + x_0$

$$e^{2\pi i x/2} = e^{2\pi i(2x_1+x_0)/2} = e^{2\pi i x_1} \cdot e^{2\pi i x_0/2}$$

for both $x_1 = 0$?

$$|\tilde{x}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i x_0/2}|1\rangle\right]$$

$$\otimes \frac{1}{\sqrt{2}}\left[|0\rangle + \underbrace{(-1)^{x_1}}_{e^{2\pi i x_1/2}} e^{2\pi i x_0/4}|1\rangle\right]$$

$x_1 = 1$

$$\begin{aligned} & e^{2\pi i x/4} \\ &= e^{2\pi i(2x_1+x_0)/4} \\ &= e^{\pi i x_1} \cdot e^{2\pi i x_0/4} \\ &= (-1)^{x_1} \cdot e^{2\pi i x_0/4} \end{aligned}$$

$$\boxed{|\tilde{x}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i(0.x_0)}|1\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i(0.x_1x_0)}|1\rangle\right]}$$

This is ordinary Hadamard transformation

$$|x\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i(0.x)}|1\rangle\right]$$

$$\frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i(0.x_1)} e^{2\pi i x_0/4}|1\rangle\right]$$

phase rotation in the 2nd term depends on x_0

We need x_0 to provide a control

$$B_{12} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/4} \end{pmatrix}$$

Controlled B_{jk} gate

$$B_{jk} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^{k-j+1}} \end{pmatrix} \qquad e^{2\pi i/2^{k-j+1}} = e^{i\theta_{jk}}$$

$$\theta_{jk} = \frac{2\pi}{2^{k-j+1}}$$

$$U_{jk} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes B_{jk}$$

$$U_{jk}|x,y\rangle = |0\rangle\langle 0|x\rangle \otimes |y\rangle + |1\rangle\langle 1|x\rangle \otimes B_{jk}|y\rangle$$

$$U_{jk}|x,y\rangle = \begin{cases} |x\rangle \otimes |y\rangle & \text{if } x=0 \\ |x\rangle \otimes B_{jk}|y\rangle & \text{if } x=1 \end{cases}$$

$$B_{jk}|y\rangle = \begin{cases} |y\rangle, & y=0 \\ e^{i\theta_{jk}}|y\rangle, & y=1 \end{cases}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_{jk}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_{jk}} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\theta_{jk}} \end{pmatrix} = e^{i\theta_{jk}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$x=y=1$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i(0.x_0)}|1\rangle\right] \quad \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i(0.x_1)} e^{2\pi i x_0/4}|1\rangle\right]$$

$$= \underbrace{\frac{1}{\sqrt{2}}\left[|0\rangle + (-1)^{x_0}|1\rangle\right]}_{\text{Simply Hadamard gate}} \otimes \underbrace{\frac{1}{\sqrt{2}}\left[|0\rangle + (-1)^{x_1} e^{\frac{2\pi i x_0}{4}}|1\rangle\right]}_{\text{Controlled phase rotation gate}}$$

$$\boxed{|\tilde{x}\rangle = \underbrace{\frac{1}{\sqrt{2}}\left[|0\rangle + (-1)^{x_0}|1\rangle\right]}\ \underbrace{B_{12}^{0}\left[|0\rangle + (-1)^{x_1}|1\rangle\right]}}$$

$$B_{12}^{0}\left[|0\rangle + (-1)^{x_1}|1\rangle\right]$$

$$B_{12}^{x_0}|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/4} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$B_{12}^{x_0}(-1)^{x_1}|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/4} \end{pmatrix}(-1)^{x_1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (-1)^{x_1}\begin{pmatrix} 0 \\ e^{2\pi i/4} \end{pmatrix}$$

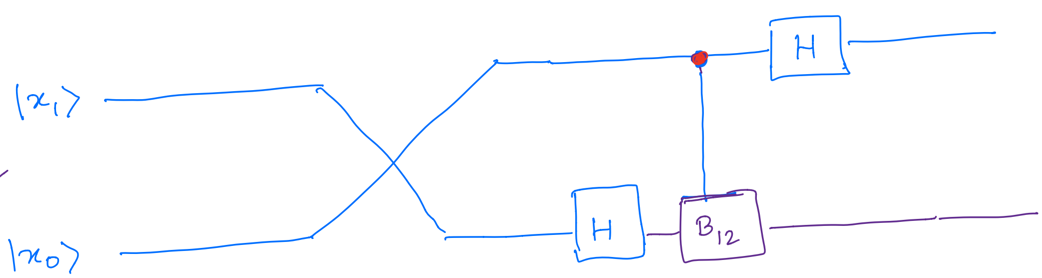
Before you write down the quantum circuit note that 1st qubit has a power $\underline{(-1)^{x_0}}$ while the 2nd one has $\underline{(-1)^{x_1}}$ when the $\underline{\text{i/p}}$ input state is $\underline{|x_1 x_0\rangle}$

$$U_{QFT}|x_1 x_0\rangle = |\tilde{x}\rangle$$

$$= \frac{1}{2}\left(|0\rangle + (-1)^{x_0}|1\rangle\right) \otimes B_{12}^{x_0}\left(|0\rangle + (-1)^{x_1}|1\rangle\right)$$

$$= (U_H \otimes I)\, U_{12}\, (I \otimes U_H)\, |x_0, x_1\rangle$$

$$U_{QFT}|x_1 x_0\rangle = (U_H \otimes I)\, U_{12}\, (I \otimes U_H)\, U_{SWAP}|x_1 x_0\rangle$$



$\underline{e^{2\pi i x_0/4}}$

$$(I \otimes U_H)\, |x_1 x_0\rangle = |x_1\rangle \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + (-1)^{x_0}|1\rangle\right]$$

$n = 2, \quad |x\rangle = |x_1 x_0\rangle$

$\underline{n = 3}$, $N = 2^3 = 8$

$$|x\rangle = |x_2 x_1 x_0\rangle$$

$$\boxed{|\tilde{x}\rangle = \frac{1}{\sqrt{8}} \sum_{y_2, y_1, y_0 \in \{0,1\}} e^{2\pi i x (4y_2 + 2y_1 + y_0)/8}\, |y_2 y_1 y_0\rangle}$$