PH551: Nonlinear Dynamics and Chaos

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Class Time Table:

Monday: 12:00-1:00PM

Tuesday: 12:00-1:00PM

Friday: 11:00-12:00PM

Two Quizes: 20

Assignments: 20

Project: 30

End Sem:30

Syllabus

Historical Development of Chaos: Newton [1642-1727], Laplace [1749-1827]-Determinism, Poincare [1854-1912]-Chaos in Three-Body Problem, Fluid Motion-Weather Prediction [1950] , Lorenz - Reincarnation of Chaos (1961), Robert May - Chaos in Population Dynamics, Universality of chaos and later developments, Deterministic Chaos- Main ingradients, Current Problems of Interest.

Dynamical systems: Importance of concepts of chaos, Fractals, and nonlinear dynamics in different natural and engineering processes. Introduction to dynamical systems, state space: continuous state with discrete time or continuous time variable, discrete state with discrete or continous time variable.

One-dimensional system: Fixed points and their local and global stability analysis, converting the dynamical problem into equivalent problem of potentials. Two-dimensional system: Fixed points and linear stability analysis. Nonlinear analysis with examples of pendulum. Dissipation and the divergence theorem, Poincare-Bendixon’s Theorem, weakely nonlinear oscillators.

Syllabus

Three-dimensional system: Linear and nonlinear stability analysis with examples of Lorentz system, forced nonlinear oscillator, Poincare section and maps. Bifurcation theory: Bifurcations in 1D and 2D flows with examples of sadle-node, transcritical, pitchfork bifurcations in different physical systems. Hopf-bifurcations. Homoclinic and hetroclinic bifurcations.

One dimensional Maps and Chaos: Stability of fixed point and periodic orbits, quadratic maps, bifurcation in maps, characterization of chaos using Lyapunov exponents and Fourier spectrum.

Different Routes to Chaos: Quasiperiodic, intermittency, period doubling, etc.

Fractals and attractors: Introduction to countable and non-countable sets, Cantor set, Dimension of self-similar Fractals. Henon map, Rossler systems, Chemical chaos, forced-double well oscillators.

A brief phenomenology of turbulent flow: Phenomenology of Turbulent flow in classical (Kolgmogorov phenomenology for energy cascade) and quantum system (especially generation and phenomenology of turbulence in Bose-Einstein condensation and superfluid Helium).

Books

1. **Strogatz, S. Nonlinear Dynamics and Chaos. Reading, MA: Addison-Wesley, 2007.**

2. Lakshmanan, M and R. Rajasekar, Nonlinear Dynamics: Integrability, Chaos and Patterns, Springer, 2003.

3. Hilborn, Robert C. Chaos and Nonlinear Dynamics. Oxford University Press, Second edition, 2000.

4. Guckenheimer, J., and P. Holmes. Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. New York, NY: Springer-Verlag, 2002.

5. Drazin, P. G. Nonlinear systems. Cambridge, UK: Cambridge University Press, 1992.

6. Berge, P., Y. Pomeau, and C. Vidal. Order Within Chaos. New York, NY: Wiley 1987.

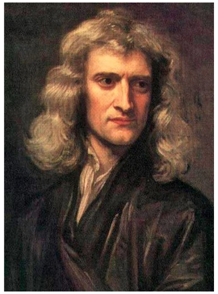
Possible project topics

- Nonlinear Dynamics in Physiological and Biological modelling.
- Studying spread of disease through mathematical models using linear stability analysis.
- Nonlinear Chemical Dynamics Oscillations and Patterns (belousov zhabotinsky reaction).
- Nonlinear dynamics of weather prediction model.
- Time series data for the Bitcoin to USD market for the presence of Chaotic attractors and its implications.
- Evolutionary Game Theory.
- Stochastic resonance in Lorenz model.
- Quantum Chaos.
- Kolmogorov-Arnold-Moser Theory.
- Poincare work on Chaos.

Historical Developments of Nonlinear Dynamics

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytical studies of planetary motion
1890s	Poincaré	Geometric approach, nightmares of chaos
1920–1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920–1960	Birkhoff	Complex behavior in Hamiltonian mechanics
	Kolmogorov	
	Arnol'd	
	Moser	
1963	Lorenz	Strange attractor in simple model of convection
1970s	Ruelle & Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators, and their applications

Historical Developments of Nonlinear Dynamics



Newton [1642-1727]

Newton's Laws: The equation of motion for a particle of mass m under a force field $\mathbf{F}(\mathbf{x}, t)$ is given by

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}, t)$$

Given initial condition $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, we can determine $\mathbf{x}(t)$ *in principle.*

Given initial condition $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, we can determine $\mathbf{x}(t)$ *in principle.*

Using Newton's laws we can understand dynamics of many complex dynamical systems, and predict their future quantitively. For example, the equation of a simple oscillator is

$$m\ddot{x} = -kx$$

Historical Developments of Nonlinea Dynamics

whose solution is

$$x(t) = A\cos\left(\sqrt{k/m}t\right) + B\sin\left(\sqrt{k/m}t\right),$$

with A and B to be determined using initial condition. The solution is simple oscillation.

Planetary motion (2 body problem)

$$\mu\ddot{\mathbf{r}} = -(\alpha/r^2)\hat{\mathbf{r}},$$

the solution is elliptical orbit for the planets. In fact the astronomical data matched quite well with the predictions. Newton's laws could explain dynamics of large number of systems, e.g., motion of moon, tides, motion of planets, etc.

He was unable to solve the three body problem!

Historical Developments of Nonlinear Dynamics

Laplace [1749-1827]- Determinism

Newton's law was so successful that the scientists thought that the world is deterministic. In words of Laplace

"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at any given moment knew all of the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit the data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain and the future just like the past would be present before its eyes."

Historical Developments of Nonlinear Dynamics

Poincare [1854-1912]-Chaos in Three-Body Problem

One of the first glitch to the dynamics came from three-body problem. The question posed was whether the planetary motion is stable or not. It was first tackled by Poincare towards the end of ninteenth century. He showed that we cannot write the trajectory of a particle using simple function. In fact, the motion of a planet could become random or disorderly (unlike ellipse). This motion was called chaotic motion later. In Poincare's words itself.

Historical Developments of Nonlinear Dynamics

"If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. but even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon. - in a 1903 essay "Science and Method"."

Clearly determinism does not hold in nature in the classical sense...

Historical Developments of Nonlinear Dynamics

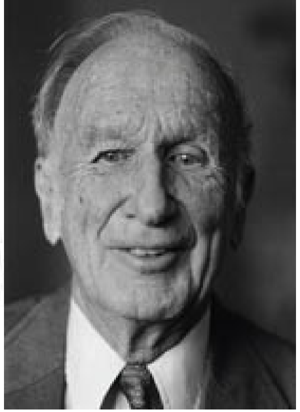
Fluid Motion- Weather Prediction [1950]

Motion of fluid parcel is given by

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p + \nu\nabla^2\mathbf{u}.$$

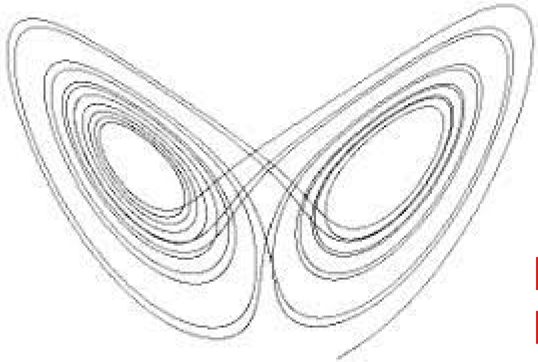
where ρ, $\mathbf{u}$, and p are the density, velocity, and pressure of the fluid, and ν is the kinetic viscosity of the fluid. The above equation is Newton's equation for fluids. There are some more equations for the pressure and density. These complex set of equations are typically solved using computers. The first computer solution was attempted by a group consisting of great mathematician named Von Neumann. Von Neumann thought that using computer program we could predict weather of next year, and possibly plan out vacation accordingly. However his hope was quickly dashed by Lorenz in 1963.

Historical Developments of Nonlinear Dynamics



Edward N. Lorenz

Lorenz - Reincarnation of Chaos

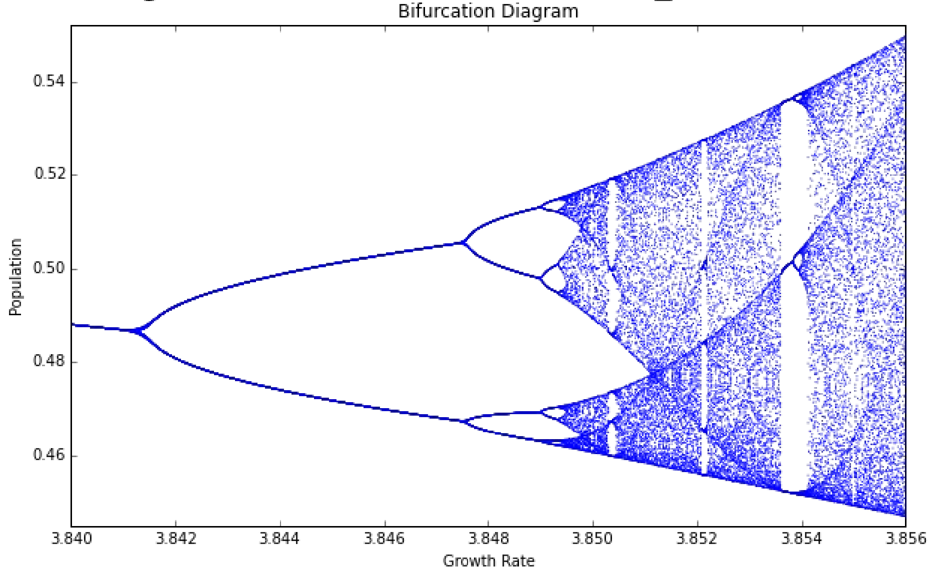


In 1961, Edward Lorentz discovered the butterfly effect while trying to forecast the weather. He was essentially solving the convection equation. After one run, he started another run whose initial condition was a truncated one. When he looked over the printout, he found an entirely new set of results. The results was expected to be same as before.

Lorenz believed his result, and argued that the system is sensitive to the initial condition. This accidental discovery generated a new wave in science after a while. Note that the equations used by Lorenz do not conserve energy unlike three-body problem. These two kinds of systems are called dissipative and conservative systems, and both of them show chaos.

Historical Developments of Nonlinear Dynamics

Robert May - Chaos in Population Dynamics



Robert May

In 1976, May was studying population dynamics using simple equation

$$P_{n+1} = aP_n(1 - P_n)$$

where P_n is the population on the nth year. May observed that the time series of P_n shows constant, periodic, and chaotic solution.

Historical Developments of Nonlinear Dynamics

Universality of chaos and later developments

In 1979, Feigenbaum showed that the behaviour of May's model for population dynamics is shared by a class of systems. Later scientists discovered that these features are also seen in many experiments. After this discovery, scientists started taking chaos very seriously. Some of the pioneering experiments were done by Gollub, Libchaber, Swinney, and Moon.

Deterministic Chaos- Main ingradients

- Nonlinearity: Response not proportional to input forcing (somewhat more rigourous definition a bit later
- Sensitivity to initial conditions.
- Deterministic systems too show randomness (deterministic chaos). Even though noisy systems too show many interesting stochastic or chaotic behaviour, we will focus on deterministic chaos in these notes.

Dynamical System

A dynamical system is specified by a set of variables called state variables and evolution rules. The state variables and the time in the evolution rules could be discrete or continuous. Also the evolution rules could be either deterministic or stochastic. Given initial condition, the system evolves as

$$\mathbf{x}(0) \to \mathbf{x}(t).$$

The evolution rules for dyanamical systems are quite precise. Contrast this with psychological laws where the rules are not precise. In the present course we will focus on dynamical systems whose evolution is deterministic.

Dynamical System

The most generic way to characterize such systems is through differential equations. Some of the examples are

1. **One dimensional Simple Oscillator**: The evolution is given by

$$m\ddot{x} = -kx,$$

We can reduce the above equation to two first-order ODE. The ODEs are

$$\begin{aligned} \dot{x} &= p/m, \\ \dot{p} &= -kx. \end{aligned}$$

The state variables are x and p.

Dynamical System

2. **LRC Circuit:** The equation for a LRC circuit in series is given by

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = V_{applied}.$$

The above equation can be reduced to

$$\begin{aligned} \dot{Q} &= I, \\ L\dot{I} &= V_{applied} - RI - \frac{Q}{C}. \end{aligned}$$

The state variables are Q and I.

3. **Population Dynamics:** One of the simplest model for the volution of population P over time is given by

$$\dot{P} = \alpha P - P^2,$$

where α is a costant.

Dynamical System

A general dynamical system is given by $|x(t)\rangle = (x_1, x_2, ...x_n)^T$. Its evolution is given by

$$\frac{d}{dt}|x(t)\rangle = |f(|x(t)\rangle, t)\rangle$$

where **f** is a continuous and differentiable function. In terms of components the equations are

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2, ..., x_n, t), \\ \dot{x_2} &= f_2(x_1, x_2, ..., x_n, t) \\ . & \quad . \\ \dot{x_n} &= f_n(x_1, x_2, ..., x_n, t), \end{aligned}$$

where f_i are continuous and differentiable functions. When the functions f_i are independent of time, the system is called *autonomous* system. However, when f_i are explicit function of time, the system is called *nonautonomous*. The three examples given above are autonomous sysetms.

Dynamical System

A nonautonomous system can be converted to an autonomous oney by renaming $t = x_{n+1}$ and

$$\dot{x}_{n+1} = 1.$$

An example of nonautonomous system is

$$\begin{aligned} \dot{x} &= p \\ \dot{p} &= -x + F(t). \end{aligned}$$

The above system can be converted to an autonomous system using the following procedure.

$$\begin{aligned} \dot{x} &= p \\ \dot{p} &= -x + F(t) \\ \dot{t} &= 1. \end{aligned}$$

In the above examples, the system variables evolve with time, and the evolution is described using ordinary differential equation. There are however many situations when the system variables are fields in which case the evolution is described using partial differential equation. We illustrate these kinds of systems using examples.

Dynamical System

1. **Diffusion Equation**

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$

   Here the state variable is field $T(x)$. We can also describe $T(x)$ in Fourier space using Fourier coefficients. Since there are infinite number of Fourier modes, the above system is an infinite-dimensional. In many situations, finite number of modes are sufficient to describe the system, and we can apply the tools of nonlinear dynamics to such set of equations. Such systems are called low-dimensional models.

2. **Navier-Stokes Equation**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}.$$

   Here the state variables are $\mathbf{u}(\mathbf{x})$ and $p(\mathbf{x})$.

Dynamical System

Number of variables →

Nonlinearity ↓	$n = 1$	$n = 2$	$n \geq 3$	$n >> 1$	Continuum
Linear	*Growth, decay, or equilibrium* Exponential growth RC circuit Radioactive decay	*Oscillations* Linear oscillator Mass and spring RLC circuit 2-body problem (Kepler, Newton)	Civil engineering, structures Electrical engineering	*Collective phenomena* Coupled harmonic oscillators Solid-state physics Molecular dynamics Equilibrium statistical mechanics	*Waves and patterns* Elasticity Wave equations Electromagnetism (Maxwell) Quantum mechanics (Schrödinger, Heisenberg, Dirac) Heat and diffusion Acoustics Viscous fluids
				The frontier	
Nonlinear	Fixed points Bifurcations Overdamped systems, relaxational dynamics Logistic equation for single species	Pendulum Anharmonic oscillators Limit cycles Biological oscillators (neurons, heart cells) Predator-prey cycles Nonlinear electronics (van der Pol, Josephson)	*Chaos* Strange attractors (Lorenz) 3-body problem (Poincaré) Chemical kinetics Iterated maps (Feigenbaum) Fractals (Mandelbrot) Forced nonlinear oscillators (Levinson, Smale) Practical uses of chaos Quantum chaos ?	Coupled nonlinear oscillators Lasers, nonlinear optics Nonequilibrium statistical mechanics Nonlinear solid-state physics (semiconductors) Josephson arrays Heart cell synchronization Neural networks Immune system Ecosystems Economics	*Spatio-temporal complexity* Nonlinear waves (shocks, solitons) Plasmas Earthquakes General relativity (Einstein) Quantum field theory Reaction-diffusion, biological and chemical waves Fibrillation Epilepsy Turbulent fluids (Navier-Stokes) Life