MA 225 — Quiz I

Probability Theory and Random Processes — 11:00–12:00 IST

IIT GUWAHATI — August 17, 2019

Model Answers of Quiz I

[5pnts.] 1. Consider four coding machines M_1, M_2, M_3, and M_4 producing binary codes 0 and 1. The machine M_1 produces codes 0 and 1 with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$. The code produced by machine M_k is fed into machine M_{k+1}, ($k = 1, 2, 3$), which may either leave the received code unchanged or may change it. Suppose that each of the machines M_2, M_3, and M_4 change the code with probability $\frac{3}{4}$. Given that the machine M_4 has produced code 1, find the conditional probability that the machine M_1 produced code 0.

Solution: Let A_{ij} denotes the event that jth machine produces the code i, $i = 0, 1$ and $j = 1, 2, 3, 4$. Using Bayes theorem, the required probability

$$P(A_{01}|A_{14}) = \frac{P(A_{14}|A_{01})P(A_{01})}{P(A_{14}|A_{01})P(A_{01}) + P(A_{14}|A_{11})P(A_{11})}.$$

Now, each code can change maximum 3 times. If M_1 produce code 0 and M_4 produce code 1, then the code has changed odd number times, *i.e.* either once or thrice. Similarly, if both M_1 and M_4 produce code 1, then the code has changed even number times, *i.e.* either twice or none. Hence

$$P(A_{14}|A_{01}) = \binom{3}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \binom{3}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^0 = \frac{9}{16},$$
$$P(A_{14}|A_{11}) = \binom{3}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^3 + \binom{3}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) = \frac{7}{16}.$$

Hence the required probability is $P(A_{01}|A_{14}) = \frac{3}{10}$.

[5pnts.] 2. A biased coin is tossed repeatedly. Each time there is a probability p of a head turning up independent of other tosses. Let p_n be the probability that an even number of heads has occurred after n tosses (zero is an even number). Thus $p_0 = 1$.

(a) Find the recurrence relation between p_n and p_{n-1} for $n \geq 1$.

Solution: Let A_n denote the event that even number of heads turn up after n tosses. Then

$$\begin{aligned} p_n &= P(A_n) \\ &= P(A_n|A_{n-1})P(A_{n-1}) + P\left(A_n|A_{n-1}^c\right)P(A_{n-1}^c) \\ &= (1-p)p_{n-1} + p(1-p_{n-1}) \\ &= p + (1-2p)p_{n-1}. \end{aligned}$$

(b) Find the value of p_{21} in terms of p.

Solution:

$$\begin{aligned} p_{21} &= p + (1-2p)p_{20} \\ &= p + (1-2p)p + (1-2p)^2 p_{19} \\ &\vdots \\ &= p + (1-2p)p + \ldots + (1-2p)^{20}p + (1-2p)^{21} \\ &= \frac{1}{2} + \frac{1}{2}(1-2p)^{21}. \end{aligned}$$

[5pnts.] 3. Let $\mathcal{S}$ be a sample space and $\mathcal{F}$ be a σ-algebra defined on $\mathcal{S}$. Let $P : \mathcal{F} \to \mathbb{R}$ be a set function satisfying the following properties:

1. $P(E) \geq 0$ for all $E \in \mathcal{F}$.
2. $P(\mathcal{S}) = 1$.
3. For $n \geq 2$ and for disjoint events $E_1, E_2, \ldots, E_n$, $P\left(\cup_{i=1}^{n} E_i\right) = \displaystyle\sum_{i=1}^{n} P(E_i)$.
4. If $\{E_n\}_{n\geq 1}$ is a sequence of decreasing events such that $\cap_{i=1}^{\infty} E_i = \phi$, then $\displaystyle\lim_{n\to\infty} P(E_n) = 0$.

Show that P is a probability function.

Solution: Here we need to show the following: For disjoint $A_1, A_2 \ldots \in \mathcal{F}$, $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

$$\begin{aligned} P(\cup_{i=1}^{\infty} A_i) &= P\left[(\cup_{i=1}^{n} A_i) \cup (\cup_{i=n+1}^{\infty} A_i)\right] \\ &= \sum_{i=1}^{n} P(A_i) + P(\cup_{i=n+1}^{\infty} A_i). \end{aligned} \tag{1}$$

Now define $B_n = \cup_{i=n}^{\infty} A_i$ for $n \geq 1$. Clearly $\{B_n\}_{n\geq 1}$ is a decreasing sequence of sets. We claim that $\cap_{n=1}^{\infty} B_n = \phi$. If not, then there exist a $\omega \in \cap_{n=1}^{\infty} B_n$ and hence $\omega \in B_n$ for all $n \geq 1$. In particular, $\omega \in B_1 = \cup_{i=1}^{\infty} A_i$. As A_i's are disjoint, $\omega \in A_i$ for exactly one i, say i_0. Then $\omega \notin A_i$ for $i > i_0$ and hence $\omega \notin B_n$ for $n > i_0$. This is a contradiction to the fact that $\omega \in B_n$ for all $n \geq 1$.

This implies that $\displaystyle\lim_{n\to\infty} P\left(\cup_{i=n+1}^{\infty} A_i\right) = \lim_{n\to\infty} P(B_{n+1}) = 0$. Taking limit $n \to \infty$ on both sides of (1), we get

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$