Bitonic Sequence

$x_0 \quad x_1 \quad \ldots \quad x_{n-1}$

for $0 \le i \le \frac{n}{2} - 1$, $\quad a_i = \min\{x_i, x_{i+\frac{n}{2}}\}$

$\qquad\qquad\qquad\qquad\quad b_i = \max\{x_i, x_{i+\frac{n}{2}}\}$

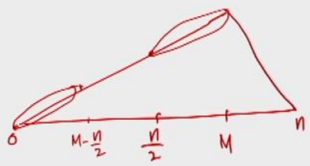
$\langle a_0, \ldots, a_{\frac{n}{2}-1} \rangle \quad \langle b_0, \ldots b_{\frac{n}{2}-1} \rangle$ are both bitonic.

Moreover, $\forall\, i, j \quad a_i \le b_j$

<u>Proof</u>: The trough is at 0 (Assumption)

$x_0 < x_1 < x_2 < \ldots x_M > x_{M+1} \ldots > x_{n-1} > x_0$

Assume that $M \geq \frac{n}{2}$ (wlg.)



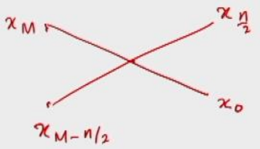
$a_0 \ldots a_{M-n/2} \qquad b_0 \ldots b_{M-n/2}$

$x_0 \qquad x_{M-n/2} \qquad x_{\frac{n}{2}} \qquad x_M$

$a_0 < \ldots < a_{M-n/2} \qquad b_0 < \ldots < b_{M-n/2}$

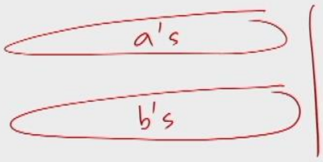
$$x_{M-n/2} \quad \cdots \quad x_{n/2-1} \quad x_{\frac{n}{2}} \qquad\qquad x_M \quad \cdots \quad x_{n-1} \quad x_0$$

$$x_{M-n/2} < x_M \qquad\qquad x_{\frac{n}{2}} > x_0$$



$\exists k$ so that

$$\begin{array}{lll} x_{M-\frac{n}{2}} < \cdots < x_k & & < x_{k+1} < \cdots < x_{n/2} \\ \qquad\quad \wedge & & \qquad\qquad\quad \vee \\ x_M > \cdots > x_{k+\frac{n}{2}} & & > x_{k+\frac{n}{2}+1} > \cdots > x_0 \end{array}$$



$$a_0 < \cdots < a_{M-\frac{n}{2}} < \cdots < \underline{a_k} \; \square \; \underline{a_{k+1}} \; \cancel{>} \cdots \; a_{\frac{n}{2}} > a_0$$

$$b_0 > \cdots > b_{M-\frac{n}{2}} > \cdots > b_k \bigcirc b_{k+1} < \cdots \; b_{\frac{n}{2}} < b_0$$

$\square$ $\bigcirc$ are dependent on the inputs

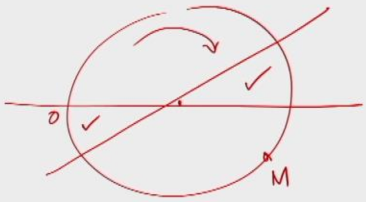
$a_k \mid a_{k+1}$: peak of the a sequence

$b_k \mid b_{k+1}$: trough of the b sequence

$\langle a_i \rangle$ and $\langle b_i \rangle$ are bitonic sequences

peak of $\langle a_i \rangle$ $<$ trough of $\langle b_i \rangle$

Division line passes through the trough



bitonicity
is maintained

Analysis

$$T_M(n,n) = T_M\left(\frac{n}{2},\frac{n}{2}\right) + 1 \qquad \text{(OEM)}$$

$$= \log n + 1$$

$$T_S(n) = T_S\left(\frac{n}{2}\right) + \underbrace{T_M\left(\frac{n}{2},\frac{n}{2}\right)}_{\log\frac{n}{2}+1}$$

identical to OEMS

$O(\log^2 n)$ time

Cost

$$C_M(n,n) = 2C_M\left(\frac{n}{2},\frac{n}{2}\right) + n$$
$$= n\log n + n = O(n\log n)$$

$$C_S(n) = 2C_S\left(\frac{n}{2}\right) + C_M\left(\frac{n}{2},\frac{n}{2}\right)$$
$$= \frac{n\log^2 n + n\log n}{4} = O(n\log^2 n)$$

$O(\log^2 n)$ time at $O(n\log^2 n)$ cost

Can we sort faster on comparator n/w?

AKS Sorting network. $O(\log n)$ time
$O(n\log n)$ cost

Optimal Algorithm for list colouring

3-colouring in $O(\log^* n)$ time using
n processors on EREW PRAM

$O(\log^{(k)} n)$ time using $\frac{n}{\log^{(k)} n}$ processors

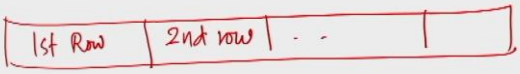
1. Colour the list using $\log^{(k)} n$ colours.

   $k+1$ steps of symmetry breaking

   $\log^{(k)} n$ - colouring

n processors	$k+1$ steps	:	$O(n)$ steps
$n/\log^{(k)} n$ pr.	$O(k \cdot \log^{(k)} n)$	:	$O(\log^{(k)} n)$ time

2. List is in array



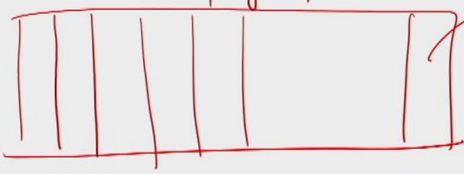
n

Visualize this as a 2D array.

$n/\log^{(k)} n$ columns $\log^{(k)} n$ rows

$n/\log^{(k)} n$

$\log^{(k)} n$



$\to$ one processors sort using radix sort. on colours

logical structure of the list is intact.

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1. Colour the list using $\log^{(k)} n$ colours.

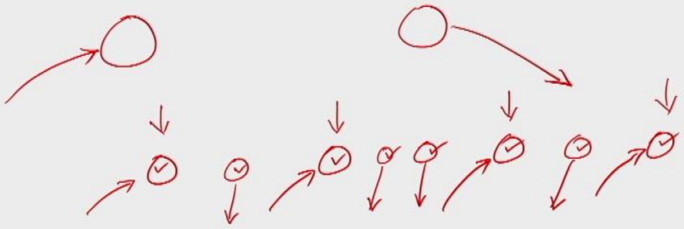
$k+1$ steps of symmetry breaking

$\log^{(k)} n$ - colouring

n processors	$k+1$ steps	:	$O(n)$ steps
$n/\log^{(k)} n$ pr.	$O(k \cdot \log^{(k)} n)$	:	$O(\log^{(k)} n)$ time

$4 \cdot \cdot \log^{(k)} n + 3$

3. Handle 'inter-row' vertices /* intra-row vertices */



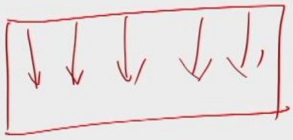
Row 0 :

Row 1 :

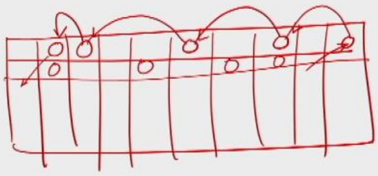
$\vdots$

$O(1)$

$O(\log^{(k)} n)$



4



Row 0.

Intra Row vertices

<u>Step 1</u> : Bring alive all vertices in Row 0 Colour 4, Row 0 colour 5

<u>Step 2</u> : Bring alive ______ Row 1 Colour 4

<u>Step 3</u> ______ Row 0 Colour 6
1 5
2 4

$2\log^{(k)} n - 1$ steps (why?)

$O(\log^{(k)} n)$ time $\frac{n}{\log^{(k)} n}$ processors

on an EREW PRAM

Cost : $O(n)$

Optimal Algorithm for list ranking