H_r : r-dimensional hypercube

π of $\{1, \ldots, r\}$

the $\pi(i)^{th}$ dimension to be mapped to the i^{th} dimension

$u \longrightarrow u'$

automorphism σ of $G = (V, E)$

is $\sigma : V \to V$ so that

σ is 1–1

$\{\sigma(u), \sigma(v)\} \in E$ for every $\{u, v\} \in E$

$$u = u_1 u_2 \ldots u_r$$
$$u' = u'_1 u'_2 \ldots u'_r$$
} realise π

Two vertices should be adjacent iff they differ in exactly 1 bit as per the new name

$$\sigma(x_1 \ldots x_r) = x_{\pi(1)} \oplus u_{\pi(1)} \oplus u'_1 \;\Big|\; \ldots \;\Big|\; x_{\pi(r)} \oplus u_{\pi(r)} \oplus u'_r$$

$\underline{\underline{u, u'}}$ $\quad u = u_1 \ldots u_r$

$$\sigma(u) = \underbrace{\underbrace{u_{\pi(1)} \oplus u_{\pi(1)}}_{0} \oplus u'_1}_{u'_1} \;\Big|\; u'_2 \;\Big|\; \ldots \;\Big|\; u'_r$$

$$\underline{\sigma(u) = u'} \quad \text{—————} \quad (1)$$

$$\sigma(x_1 \ldots x_r) = \underbrace{x_{\pi(1)} \oplus u_{\pi(1)} \oplus u'_1} \Big| \ldots$$

$$x_1 \ldots x_{\pi(1)} \ldots x_r$$

$$x_1 \ldots x_{\pi(2)} \ldots x_{\pi(1)} \ldots x_r$$

a $\pi(i)^{th}$ dimensional edge of H_r

$$x = x_1 \ldots x_{\pi(i)} \ldots x_r$$

$$x' = x_1 \ldots \overline{x_{\pi(i)}} \ldots x_r$$

$$\sigma(x) = \underbrace{x_{\pi(1)} \oplus u_{\pi(1)} \oplus u'_1}_{=} \Big| \ldots \Big| x_{\pi(i)} \oplus u_{\pi(i)} \oplus u'_i \Big| \ldots$$

$$\sigma(x') = \qquad \Big| \quad \Big| \overline{x_{\pi(i)}} \oplus u_{\pi(i)} \oplus u'_i \Big| \ldots$$

automorphism σ

satisfies the req. s.

$u \longrightarrow u'$

it realises π of $\{1 .. r\}$

(u, v) and (u', v')

we want to map (u, v) to (u', v')

using an automorphism

$u - v$ is along dim k

$u' - v'$ is along dim k'

Construct π so that $\pi(k') = k$

$\left\{\begin{array}{l}\text{realise } \pi \\ \text{using } u \to u' \text{ mapping}\end{array}\right.$

$$(u, v) \to (u', v')$$

Example

H_3 : $\pi(1) = 1 \quad \pi(2) = 3 \quad \pi(3) = 2$

$$\begin{array}{ccc} 000 & \longrightarrow & 110 \\ u & & u' \end{array}$$

$$u = 000$$
$$u' = 110$$

$$\sigma(x) = x_1 \oplus \underbrace{0 \oplus u_1'} \;\Big|\; x_3 \oplus \underbrace{0 \oplus u_2'} \;\Big|\; x_2 \oplus \underbrace{0 \oplus u_3'}$$

$$= x_1 \oplus 1 \;\Big|\; x_3 \oplus 1 \;\Big|\; x_2 \oplus 0$$

$$= \underline{\underline{\overline{x_1}\ \overline{x_3}\ x_2}}$$

✓ $000 \to 110$

$001 \to 100$

$010 \to 111$

$011 \to 101$

$100 \to 010$

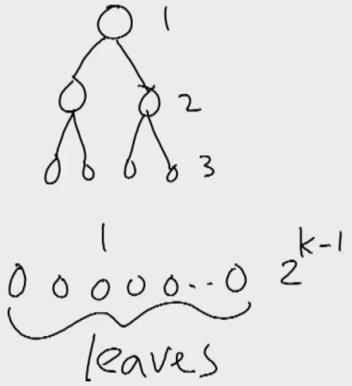
$101 \to 000$

$110 \to 011$

$111 \to 001$

Complete binary tree of

$n-1$ nodes. $\quad n = 2^k$



H_k has 2^k nodes

CBT_k can be embedded in H_k ?

No!

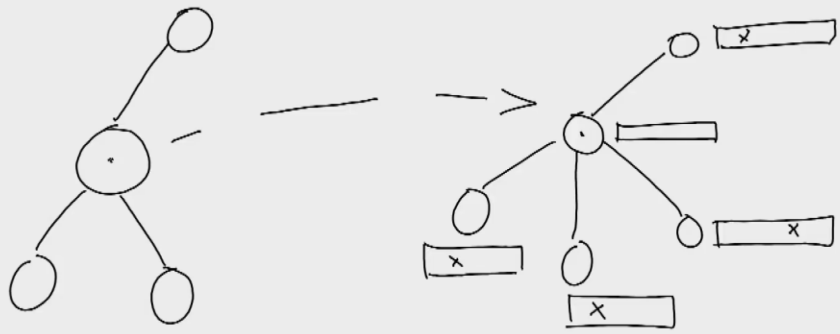
A CBT of k levels $\quad \left.\begin{array}{c}0\\ \vdots\\ k-1\end{array}\right\} \begin{array}{c}1\\ 2\\ \vdots\\ 2^{k-1}\end{array}$

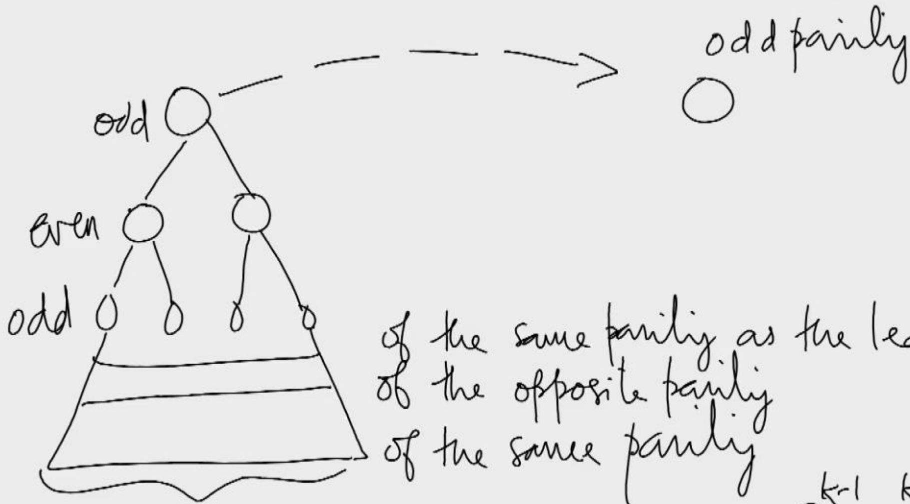
$2^k = n$

suppose CBT_k embeds in H_k

Two nodes of a Hypercube
differ by exactly one bit
iff
they are adjacent

if the label of node has
even parity then
it's children & parents
have odd parity &
Vice Versa





of the same family as the leaves

of the opposite family

of the same family

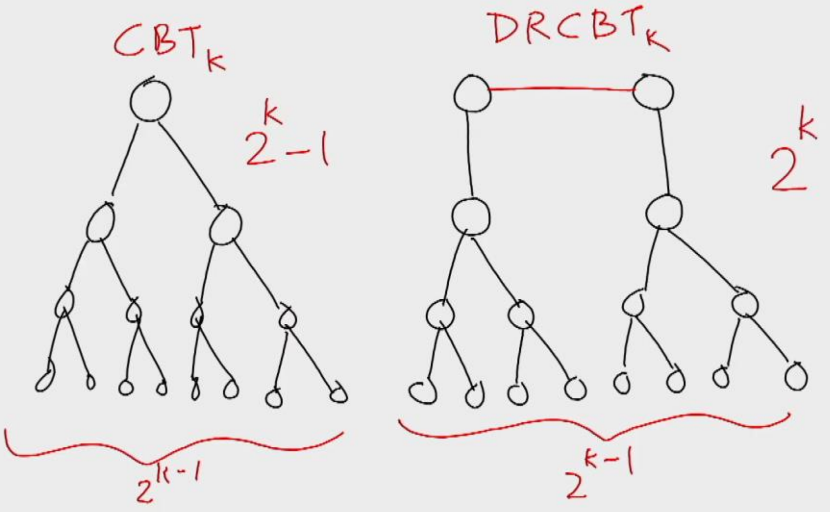
$$2^{k-1} + 2^{k-3}$$

$$\frac{n}{2} + \frac{n}{8} = \frac{5n}{8} > \frac{n}{2}$$

$> \frac{n}{2}$ nodes of H_k are of the same parity

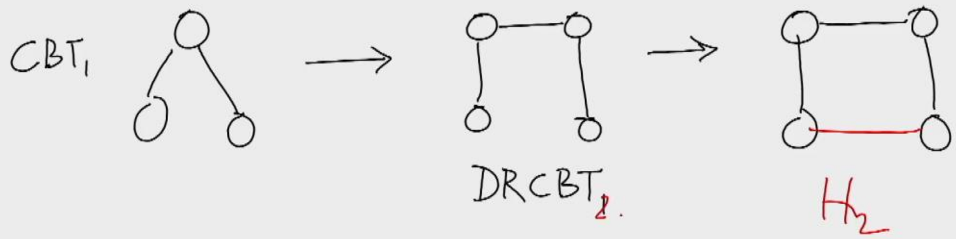
<u>Contradiction</u>

Double Rooted CBT (DRCBT)



A DRCBT can be embedded in a Hypercube of the same no. of nodes.

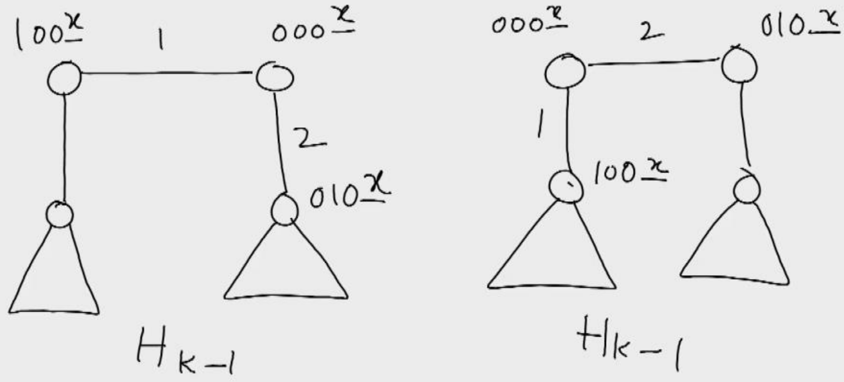
<u>Proof</u> : Basis



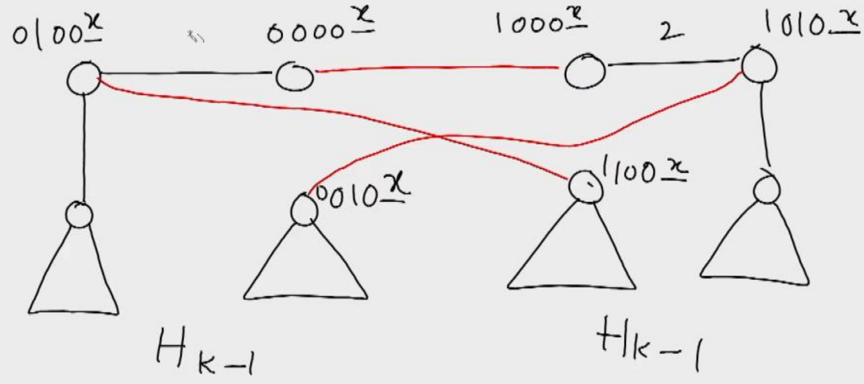
H_K.

Construct H_{k-1}'s by removing the 1st dimension edges

DRCBTs of 2^{k-1} nodes embedded in the H_{k-1}'s



DRCBTs of 2^{k-1} nodes embedded in the H_{k-1}'s



CBT algorithms

Can be executed on DRCBT

at $O(1)$ extra cost

Linear Array algorithms

K-D meshes

CBT algorithms. MoT algorithms

The degree of every node in an N-node Hypercube is ~~θ~~ log N.

impractical !

Butterfly network

r-D butterfly has

$(r+1) \cdot 2^r$ nodes

2^r rows

$r+1$ columns

$r \cdot 2^{r+1}$ edges

nodes

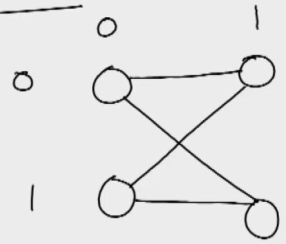
$\langle w, i \rangle$ —— $\langle w', i' \rangle$

are adjacent iff

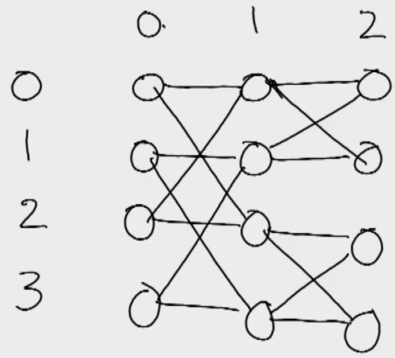
$i' = i+1$ and

($w = w'$ or w & w' differ exactly in the i^{th} bit)

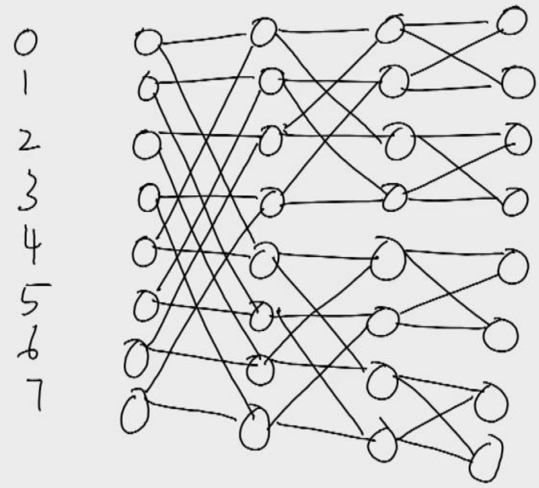
1 — D



2 — D



3-D Butterfly



every node has a degree ≤ 4
constant degree graph

Butterfly$_r$ & H_r

collapse every row of a bfly

every collapsed vertex is adj to exactly those c.v.s that differ exactly in 1 bit is a hypercube of

r- dimension

$B.fly_r \rightarrow H_r$

A T- time algorithm on H_r

runs in $O(Tr)$ time on $B.fly_r$