ASSIGNMENT-2

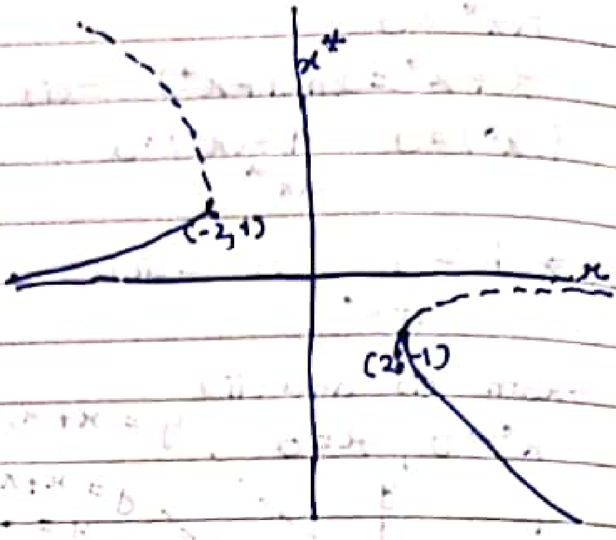
KARTIKEYA SAXENA
180101034

Ans-1- $\dot{x} = 1 + rx + x^2 = f(x)$

$f(x^*) = 0$

$x^{*2} + rx^* + 1 = 0$ —(i)

$x^* = \dfrac{-r \pm \sqrt{r^2-4}}{2}$



$\dfrac{d}{dx^*}(f(x^*)) = 0$, $f(x^*) = 0$
for bifurcation pt.

$2x^* + r = 0$

$r = -2x^*$ —(ii)

from (i) & (ii)

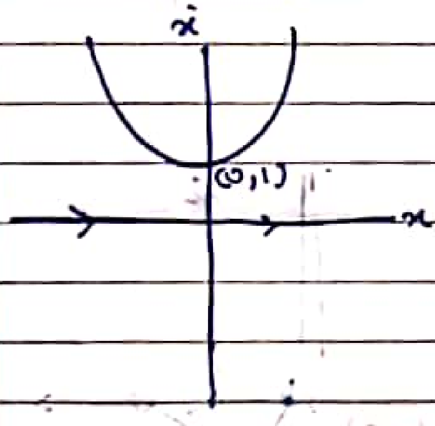
$x^* = -1, r = 2$ or

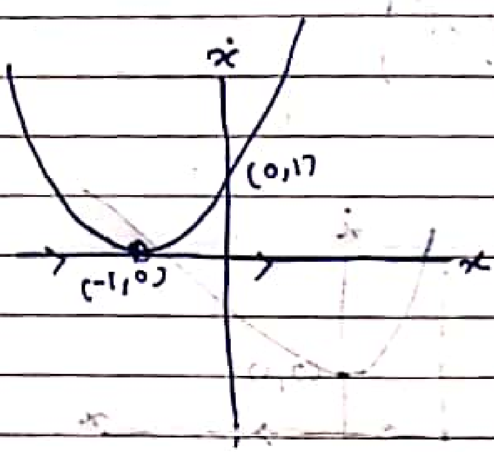
$x^* = 1, r = -2$

$|r| < 2$

$r = 0$

$\dot{x} = x^2 + 1$





$|r| = 2$

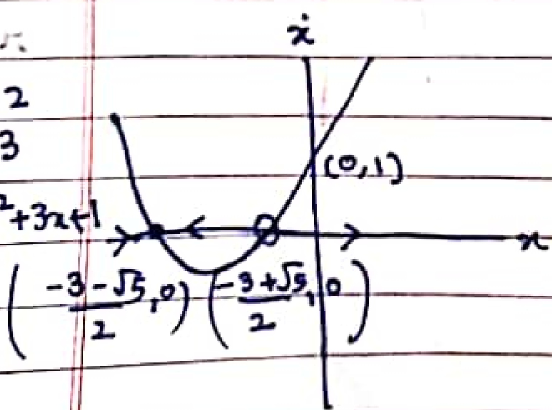
$r = 2$

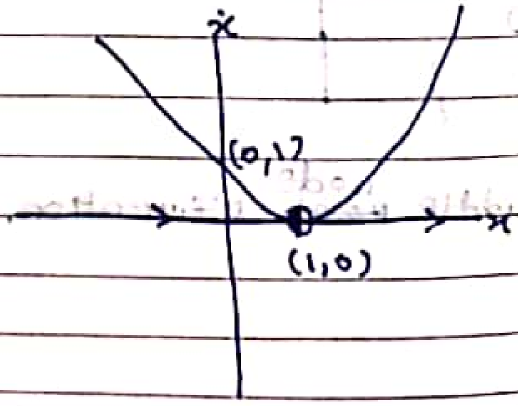
$\dot{x} = x^2 + 2x + 1$

$|r| > 2$

$r = 3$

$\dot{x} = x^2 + 3x + 1$





$|r| = 2$

$r = -2$

$\dot{x} = x^2 - 2x + 1$

Saddle node bifurcation at both points.

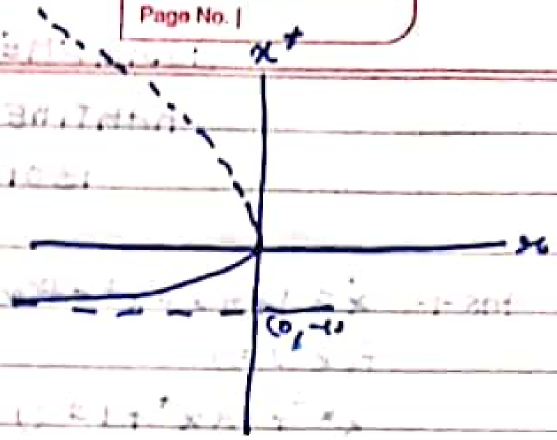
Ans-2. $\dot{x} = r + x - \ln(1+x) = f(x)$

$f(x^*) = 0$

$r + x^* = \ln(1+x^*)$ —(i)

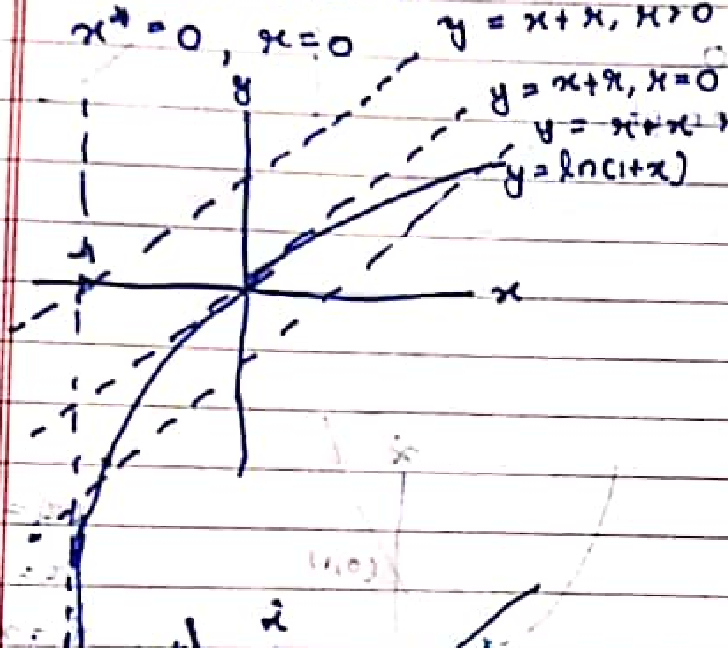
$f(x^*) = 0$, $\frac{d}{dx^*} f(x^*) = 0$

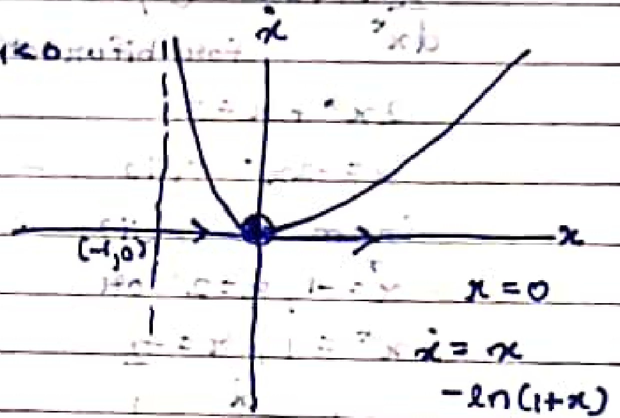
$\Rightarrow 1 - \frac{1}{1+x^*} = 0$ —(ii)



from (i) and (ii)

$x^* = 0$, $r = 0$





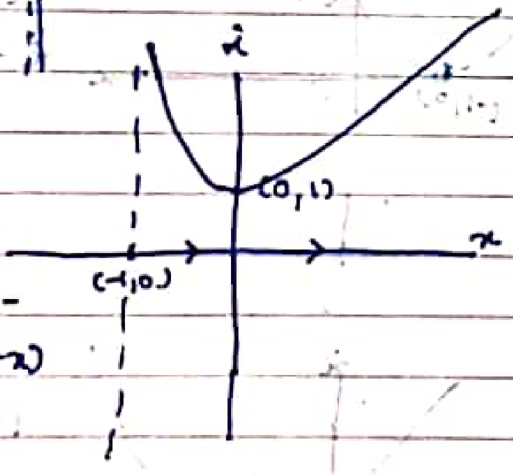
$r = 0$

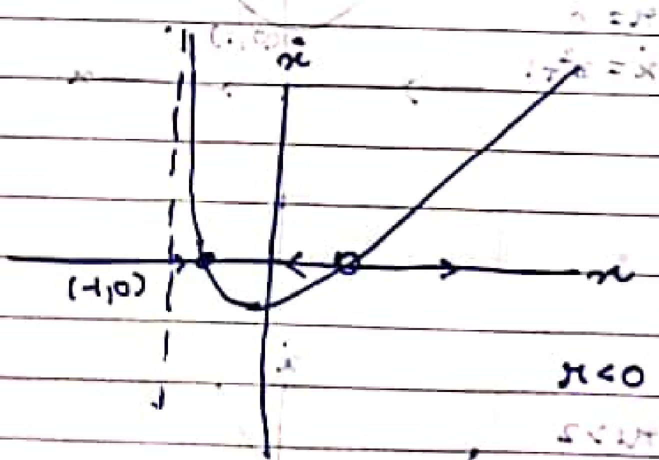
$\dot{x} = x - \ln(1+x)$

$r > 0$

$r = 1$

$\dot{x} = 1 + x - \ln(1+x)$





$r < 0$

Saddle ~~point~~ node bifurcation.

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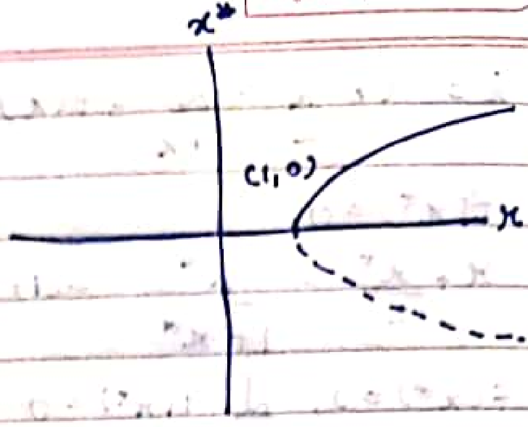


Ans-8 - $\dot{x} = r - \cosh x = f(x)$

$f(x^*) = 0$

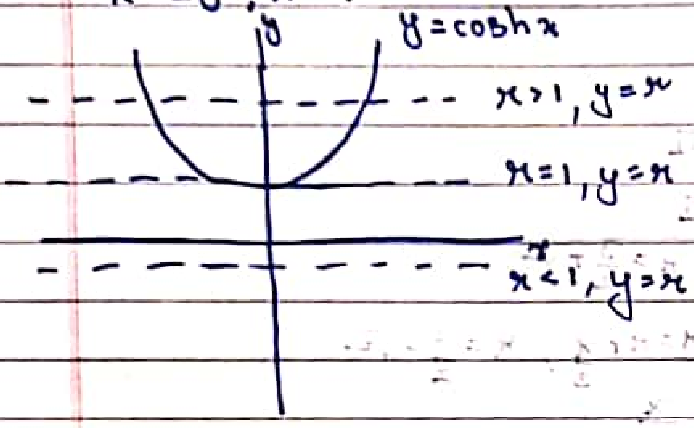
$r = \cosh x^*$ — (i)

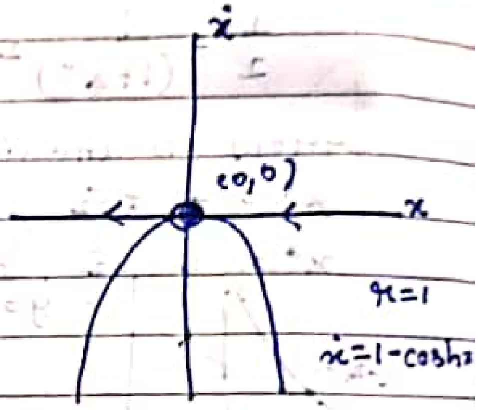
$f(x^*) = 0, \ \frac{d}{dx^*} f(x^*) = 0$

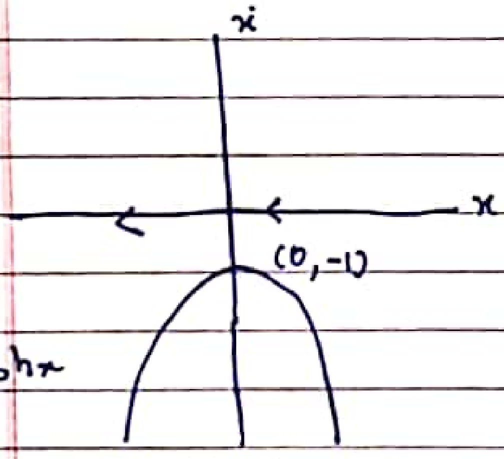
$\sinh x^* = 0$ — (ii)

from (i) and (ii)

$x^* = 0, \ r = 1$



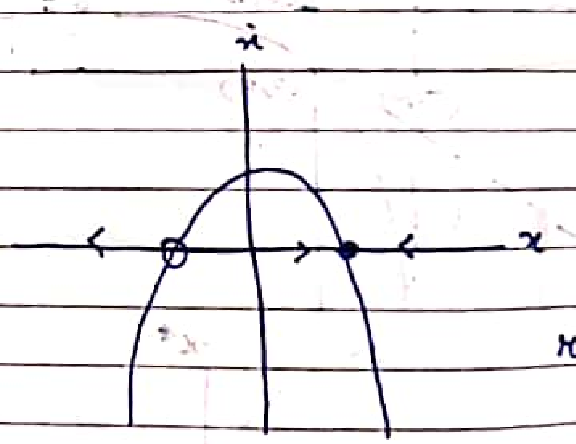




$r < 1$

$r = 0$

$\dot{x} = -\cosh x$



saddle node bifurcation

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Ans-4- $\dot{x} = r + \frac{x}{2} - \frac{x}{1+x} = f(x)$

$f(x^*) = 0,$

$r + \frac{x^*}{2} = \frac{x^*}{1+x^*}$ —(1)

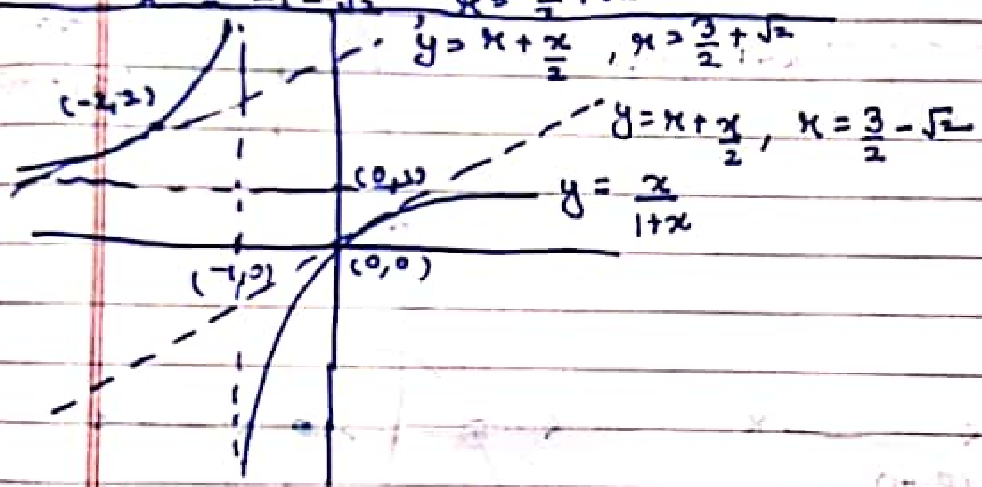
$f(x^*) = 0, \quad \frac{d}{dx^*} f(x^*) = 0$

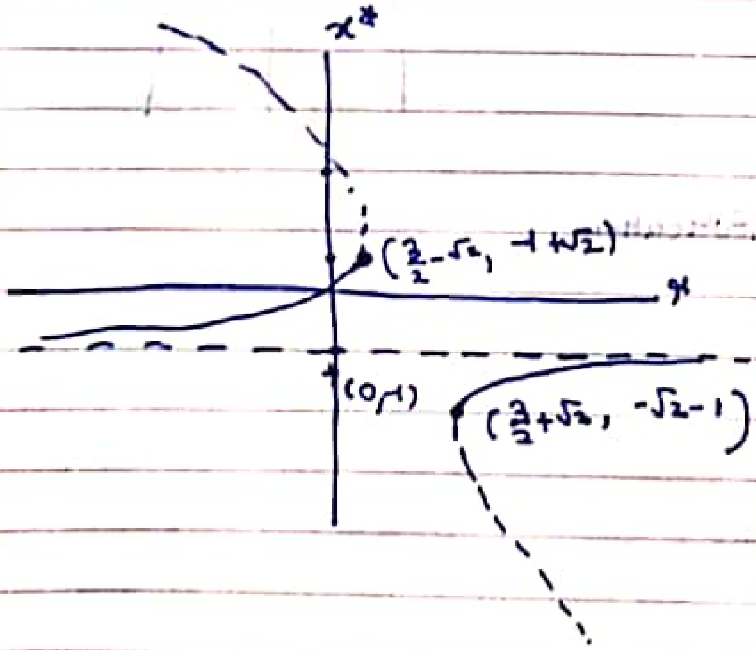
$\frac{1}{2} = \frac{1}{(1+x^*)^2}$ —(ii)

from (i) and (ii)

$x^* = -1 + \sqrt{2}, \quad r = \frac{3}{2} - \sqrt{2}$

$x^* = -1 - \sqrt{2}, \quad r = \frac{3}{2} + \sqrt{2}$





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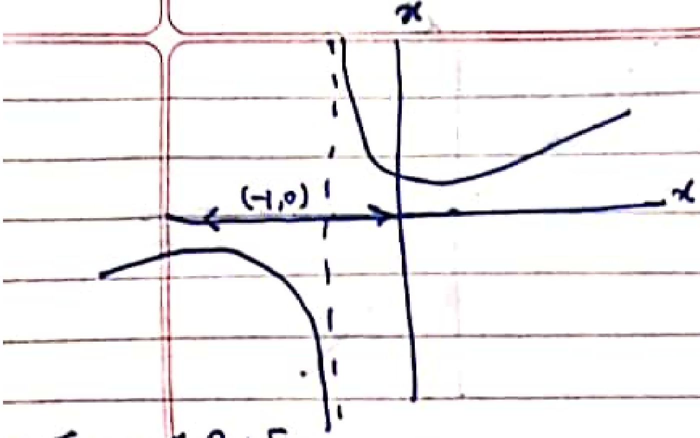
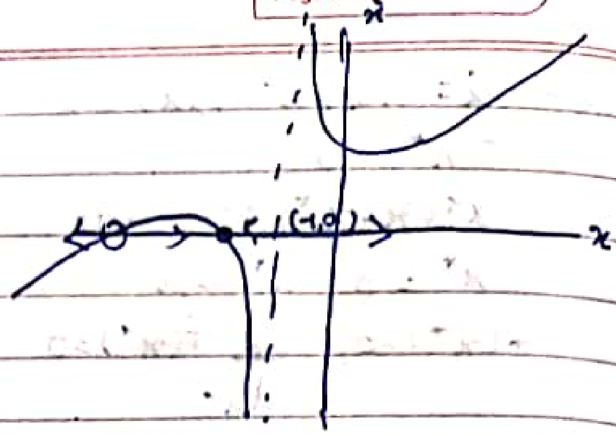
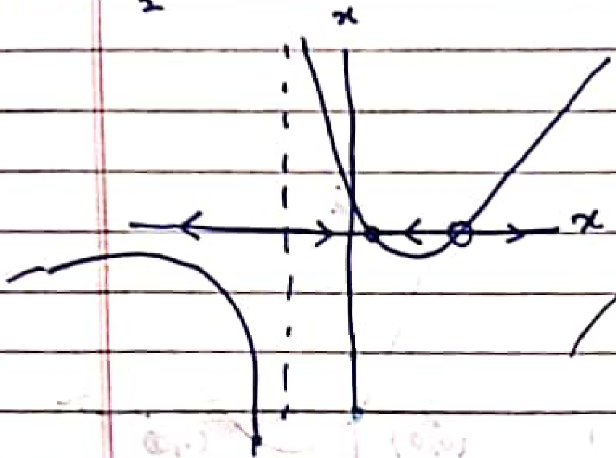


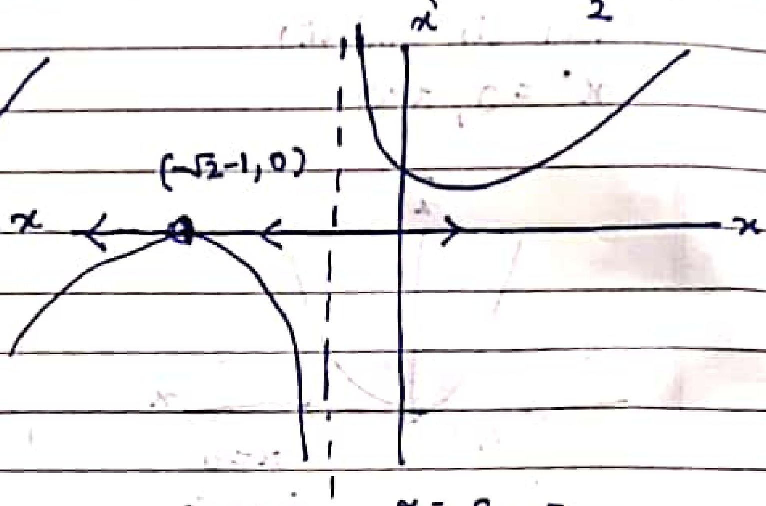
$\frac{3}{2}-\sqrt{2} < r < \frac{3}{2}+\sqrt{2}$



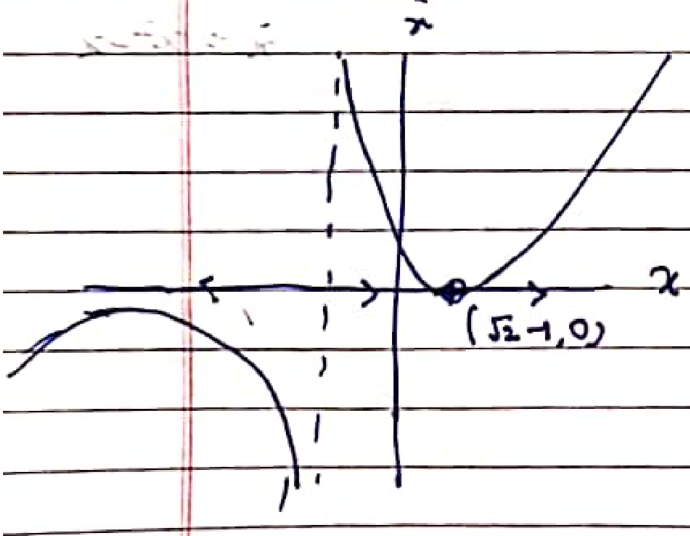
$r > \frac{3}{2}+\sqrt{2}$



$r < \frac{3}{2}-\sqrt{2}$



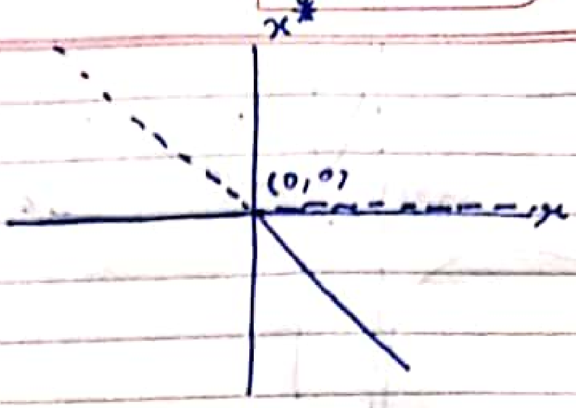
$r = \frac{3}{2}+\sqrt{2}$



$r = \frac{3}{2}-\sqrt{2}$

Saddle node bifurcation at both pts.

Ans-5 – $\dot{x} = rx + x^2 = f(x)$



$f(x^*) = 0$

$x^*(r + x^*) = 0 \quad -(i)$

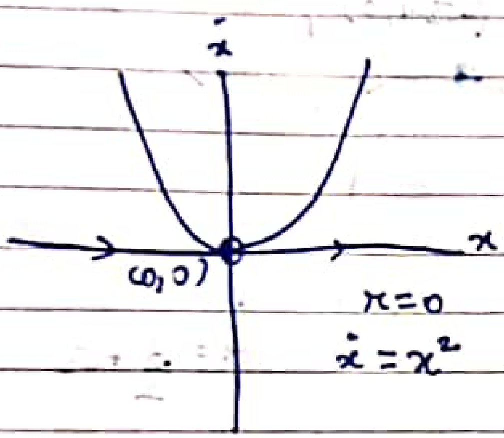
$x^* = 0, \; x^* = -r$

$f(x^*) = 0, \quad \frac{d}{dx^*} f(x^*) = 0$

$r + 2x^* = 0 \quad -(ii)$

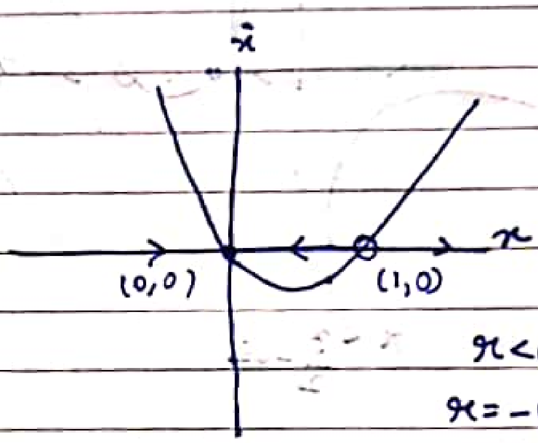
from (i) and (ii)

$x^* = 0, \; r = 0$



$r = 0$

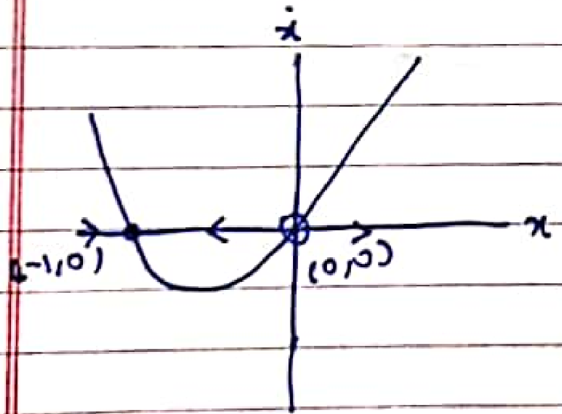
$\dot{x} = x^2$



$r < 0$

$r = -1$

$\dot{x} = x^2 - x$



$r > 0$

$r = 1$

$\dot{x} = x^2 + x$

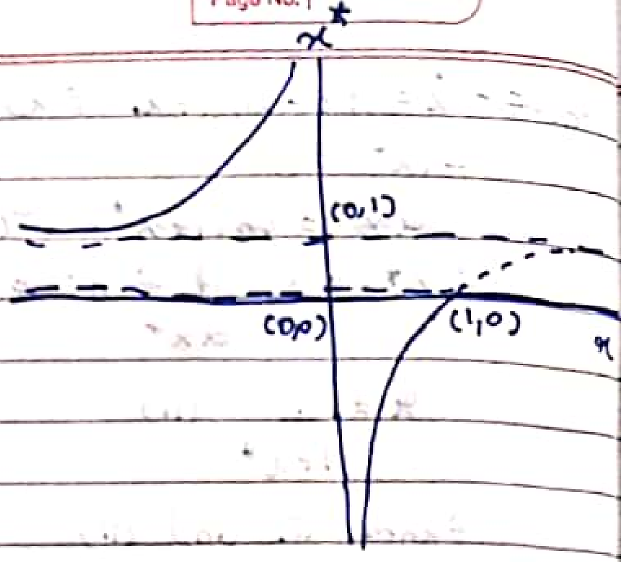
Transcritical bifurcation

Ans-6- $\dot{x} = x - r x(1-x) = f(x)$

$f(x^*) = 0$

$x^*(1 - r(1-x^*)) = 0 \quad -(i)$

$x^* = 0, \quad x^* = 1 - \frac{1}{r}$



$f(x^*) = 0, \ \frac{d}{dx^*} f(x^*) = 0$

$1 - r + 2x^* r = 0 \quad -(ii)$

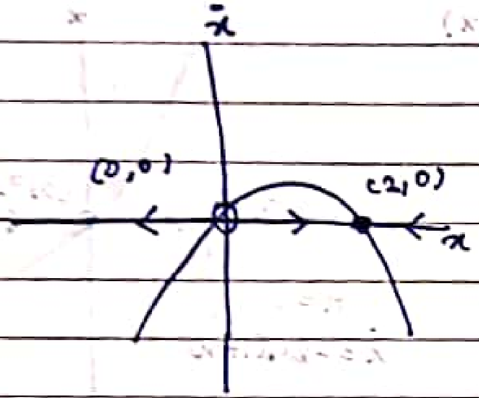
from (i) and (ii)

$x^* = 0, \ r = 1$

$r < 0$

$r = -1$

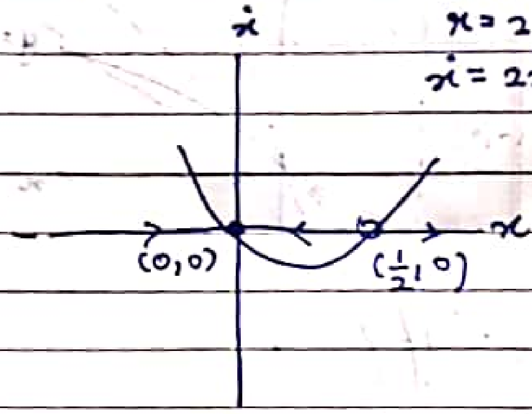
$\dot{x} = 2x - x^2$



$r > 1$

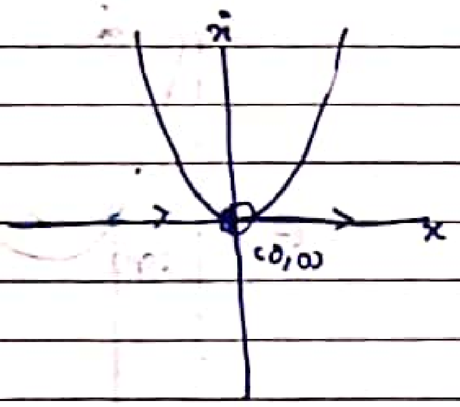
$r = 2$

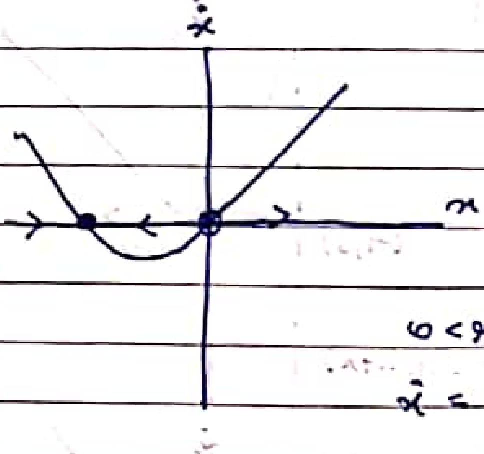
$\dot{x} = 2x^2 - x$



$r = 1$

$\dot{x} = x^2$





$0 < r < 1$

$\dot{x} = \frac{x^2 + x}{2}$

Transcritical bifurcation

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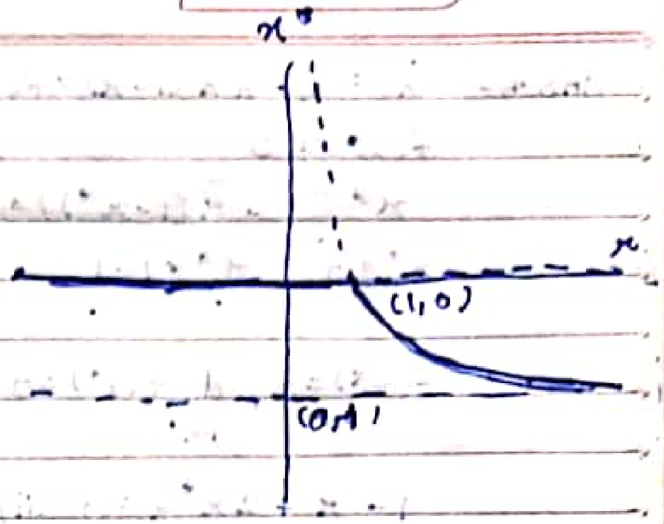
Ans-7 $\dot{x} = rx - \ln(1+x) = f(x)$

$f(x^*) = 0$

$rx^* = \ln(1+x^*)$ — (i)

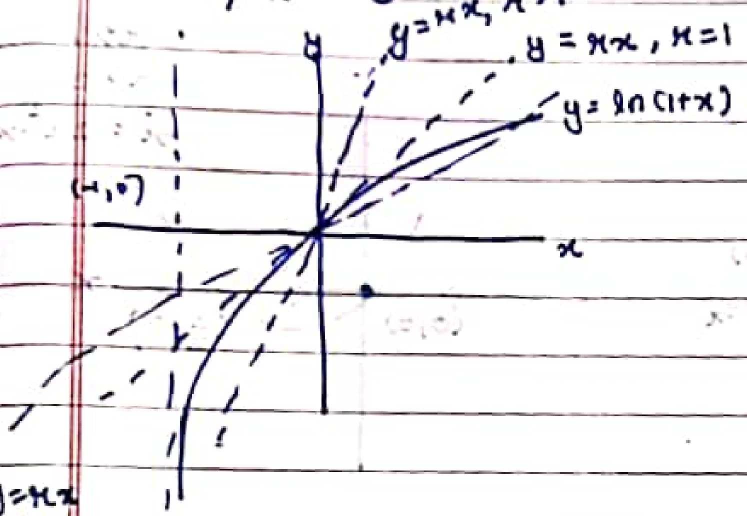
$f(x^*) = 0, \quad \frac{d}{dx^*} f(x^*) = 0$

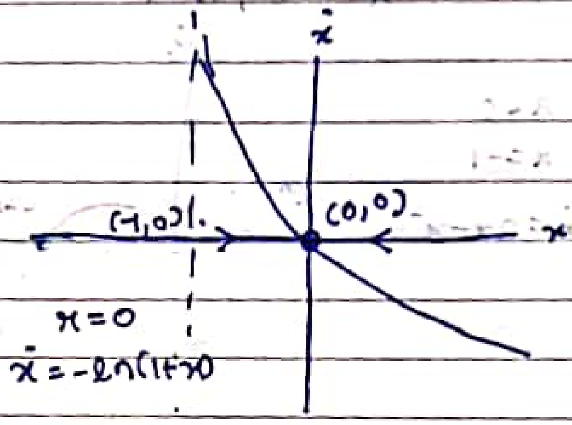
$r = \frac{1}{1+x^*}$ (ii)



from (i) and (ii)

$x = 1, \ x^* = 0$



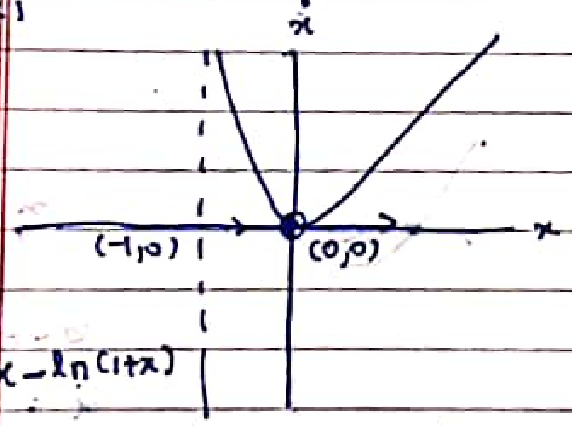


$r = 0$

$\dot{x} = -\ln(1+x)$

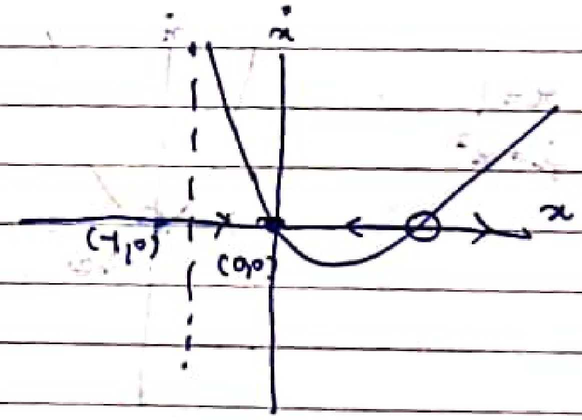
$y = rx$

$0 < r < 1$



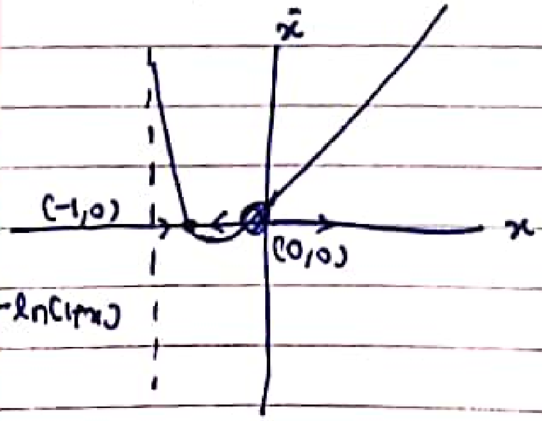
$r = 1$

$\dot{x} = x - \ln(1+x)$



$r > 1$

$r = 2$



$\dot{x} = 2x - \ln(1+x)$

$0 < r < 1$

$r = 1/2$

$\dot{x} = \frac{x}{2} - \ln(1+x)$

Transcritical bifurcation.

Ans-8- $\dot{x} = x(r - e^{x}) = f(x)$

$f(x^*) = 0$

$x^*(r - e^{x^*}) = 0$ —(i)

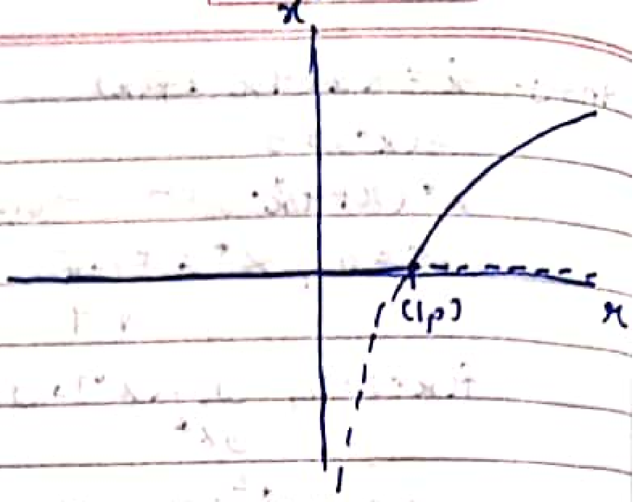
$x^* = 0 \qquad x^* = \ln r$

$f(x^*) = 0 \quad , \frac{d}{dx^*} f(x^*) = 0$

$r = (x^*+1)e^{x^*}$ — (ii)

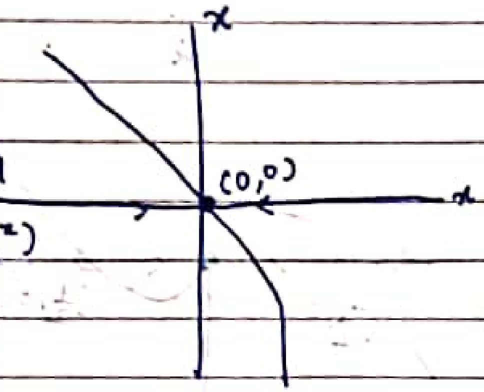
from (i) and (ii)

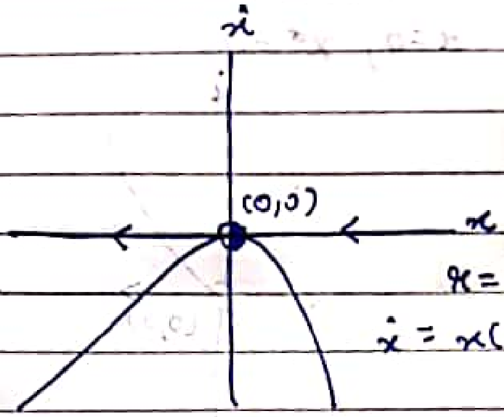
$r = 1 , x^* = 0$



$r<0, r=-1$

$\dot{x} = x(-1-e^{x})$





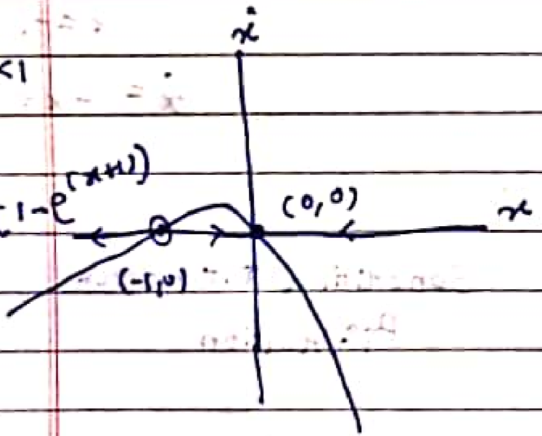
$r = 1$

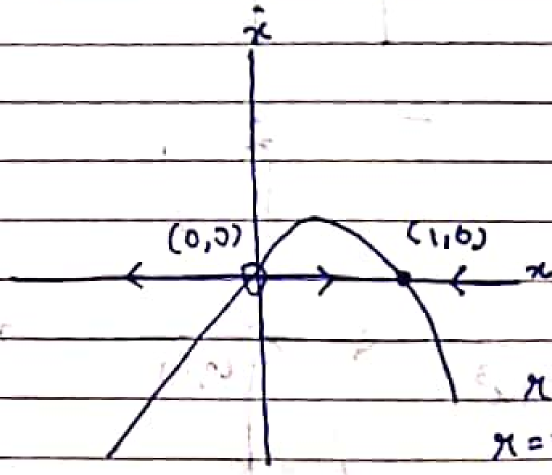
$\dot{x} = x(1-e^{x})$

$0<r<1$

$r = \frac{1}{e}$

$\dot{x} = \frac{x}{e}(1-e^{(x+1)})$





$r>1$

$r = e$

$\dot{x} = x(1-e^{x-1})e$

Transcritical bifurcation

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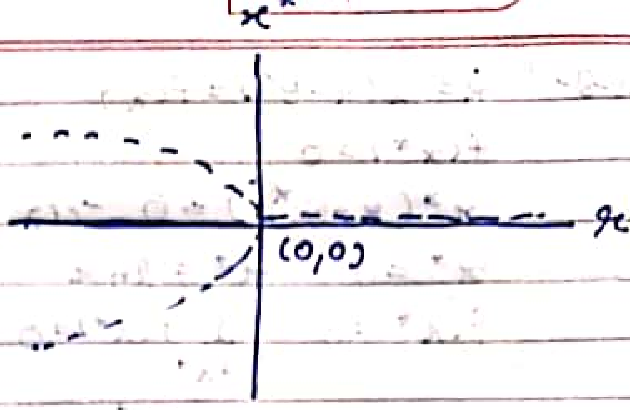
Ans-9- $\dot{x} = rx + 4x^3 = f(x)$

$f(x^*) = 0$

$x^*(r + 4x^{*2}) = 0 \quad -(i)$

$x^* = 0, \quad x^* = \pm\sqrt{\frac{-r}{4}}$



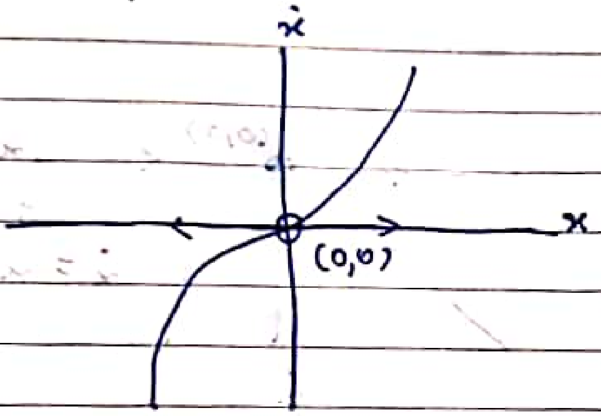
$f(x^*) = 0, \quad \frac{d}{dx^*} f(x^*) = 0$

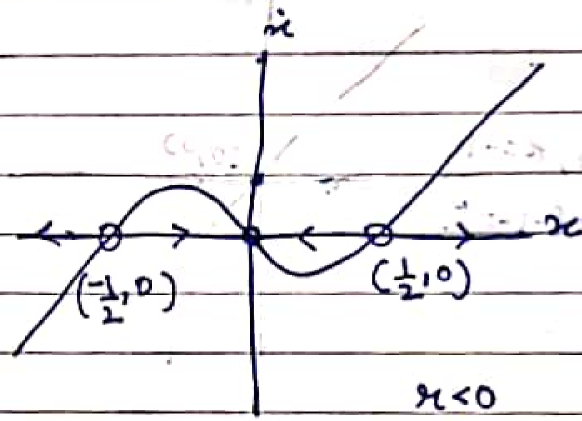
$r + 12x^{*2} = 0 \quad -(ii)$

from (i) and (ii)

$r = 0, \ x^* = 0$

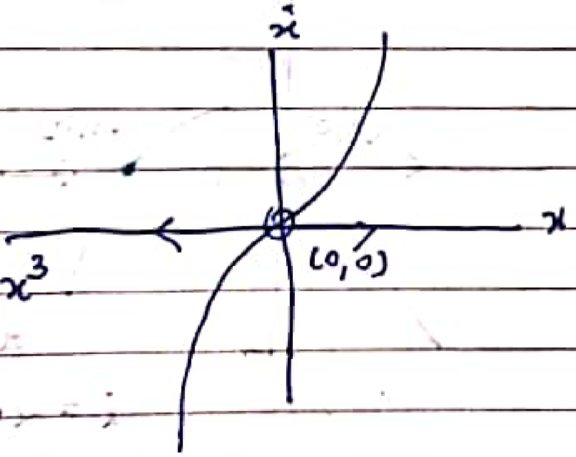
$r = 0$
$\dot{x} = 4x^3$





$r < 0$
$r = -1$
$\dot{x} = -x + 4x^3$

$r > 0$
$r = 1$
$\dot{x} = x + 4x^3$



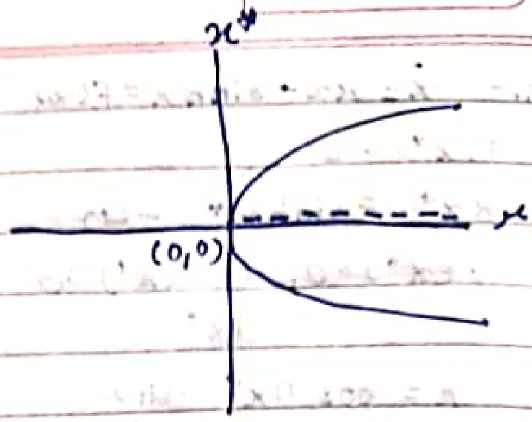
Subcritical Pitchfork Bifurcation

Ans-10- $\dot{x} = rx - 4x^3 = f(x)$

$f(x^*) = 0$

$x^*(r - 4x^{*2}) = 0 \quad -(i)$

$x^* = 0, \quad x^* = \sqrt{\frac{r}{4}}$

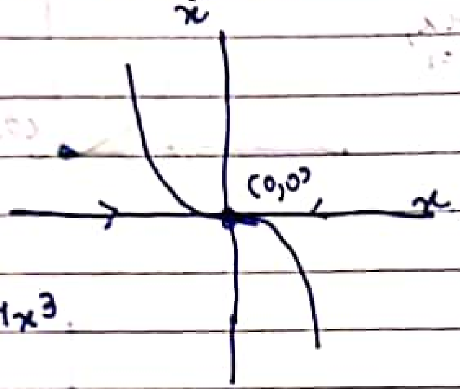


$f(x^*) = 0, \quad \frac{d}{dx^*} f(x^*) = 0$

$r - 12x^{*2} = 0 \quad -(ii)$

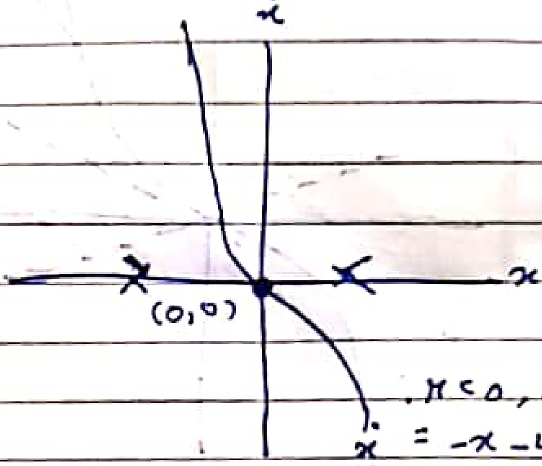
from (i) and (ii)

$r = 0, \quad x^* = 0$



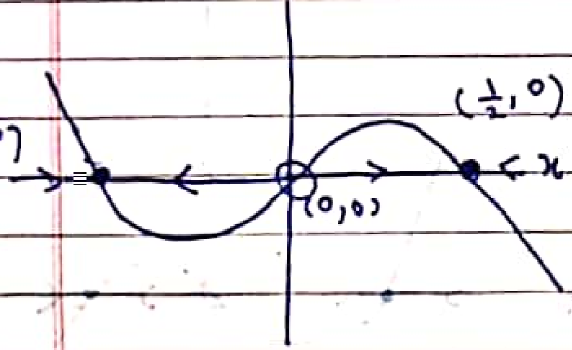
$r = 0$

$\dot{x} = -4x^3$



$r < 0, \ r = -1$

$\dot{x} = -x - 4x^3$



supercritical pitchfork bifurcation

$r > 0, \ r = 1$

$\dot{x} = x - 4x^3$

Ans-11- $\dot{x} = rx - \sinh x = f(x)$

$f(x^*) = 0$

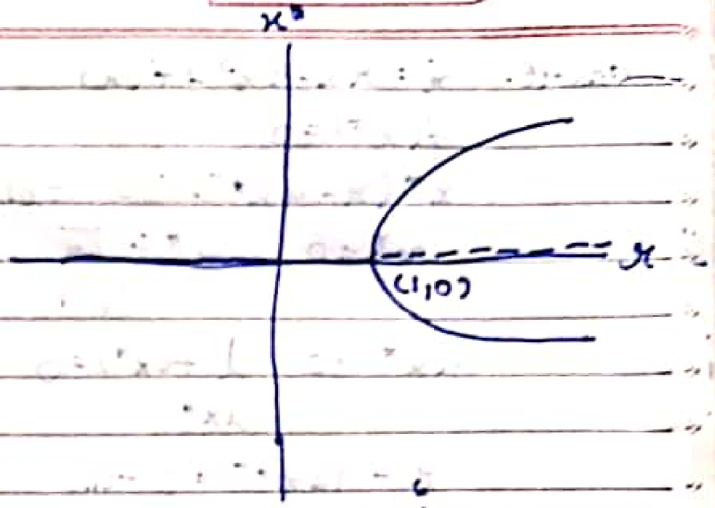
$rx^* = \sinh x^*$ —(i)

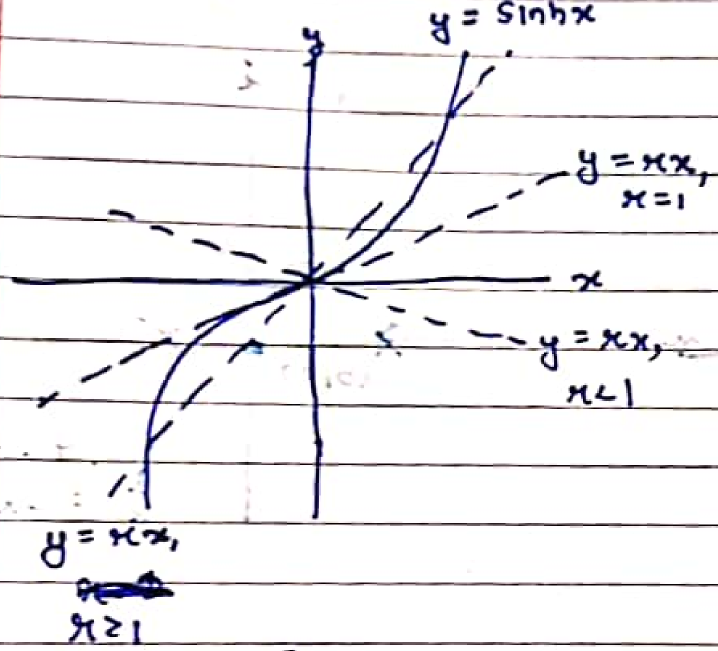
$f(x^*) = 0, \quad \frac{d}{dx^*} f(x^*) = 0$

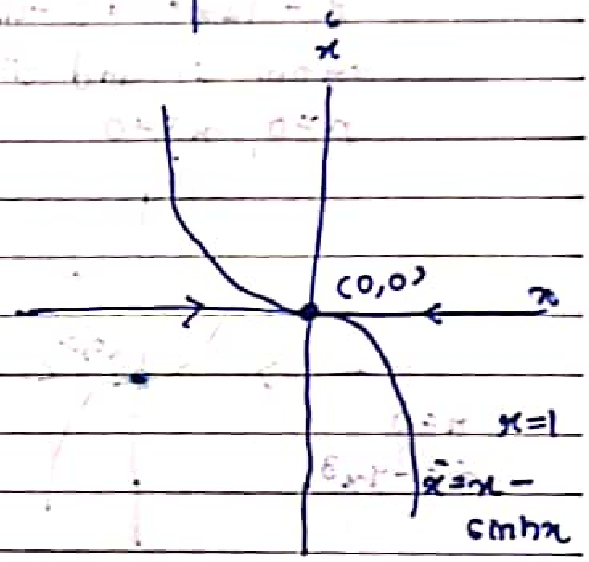
$r = \cosh x^*$ —(ii)

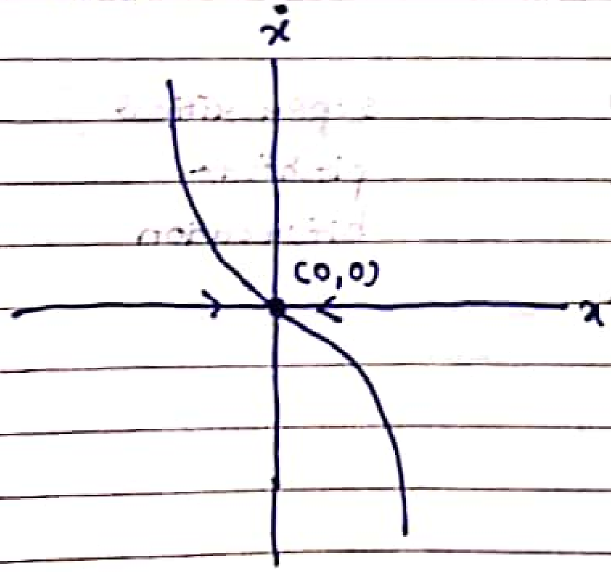
from (i) and (ii)

$x^* = 0, \; r = 1$



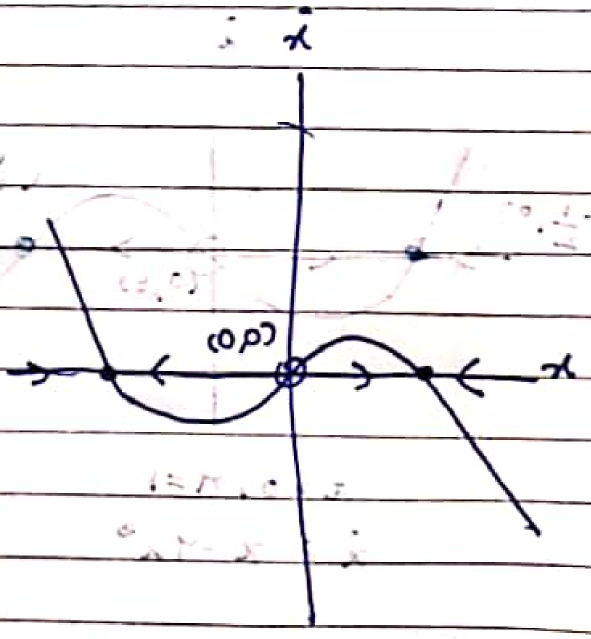






$r<1, \; r=0$

$\dot{x} = -\sinh x$



$r>1$

supercritical pitchfork bifurcation

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Ans-12 $\dot{x} = x + \frac{rx}{1+x^2} = f(x)$

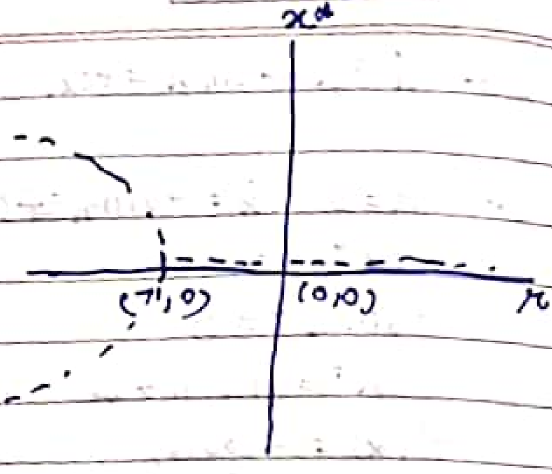
$f(x^*) = 0$

$x^*\left(1 + \frac{r}{1+x^{*2}}\right) = 0$ -(i)

$x^* = 0 \quad x^* = \sqrt{-1-r}$

$f(x^*) = 0, \frac{d}{dx^*} f(x^*) = 0$

$1 + r\left(\frac{1-x^{*2}}{(1+x^{*2})^2}\right) = 0$ -(ii)

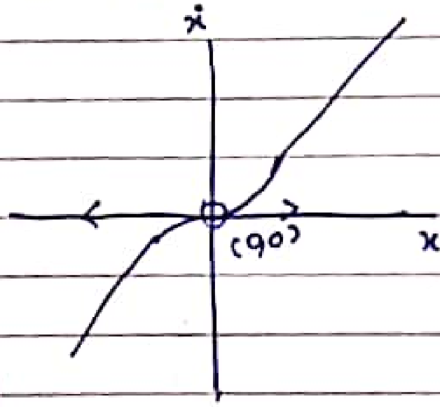


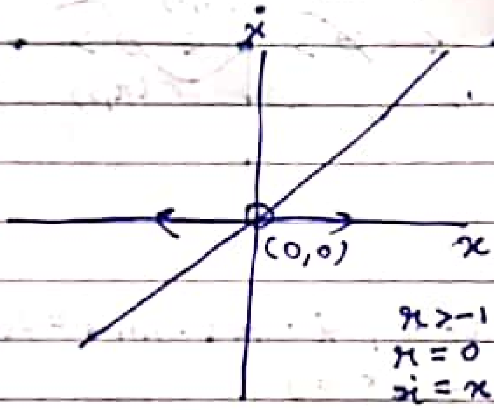
from (i) and (ii)

$r = -1, \; x^* = 0$

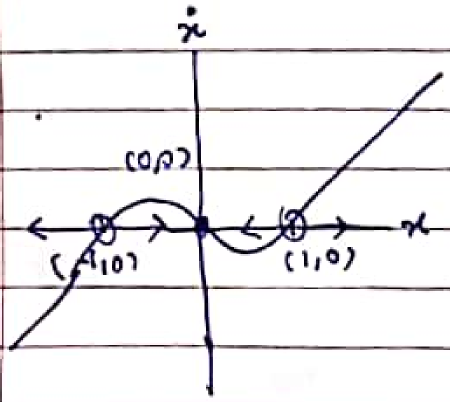
$r = -1$

$\dot{x} = \frac{x^3}{1+x^2}$





$r > -1$
$r = 0$
$\dot{x} = x$



Subcritical pitchfork bifurcation.

$r < -1, \; r = -2$

$\dot{x} = x\left(\frac{x^2-1}{x^2+1}\right)$

Ans-18) $\dot{x} = rx - \sin x = f(x)$

a) $r = 0$, $\dot{x} = -\sin x = f(x)$

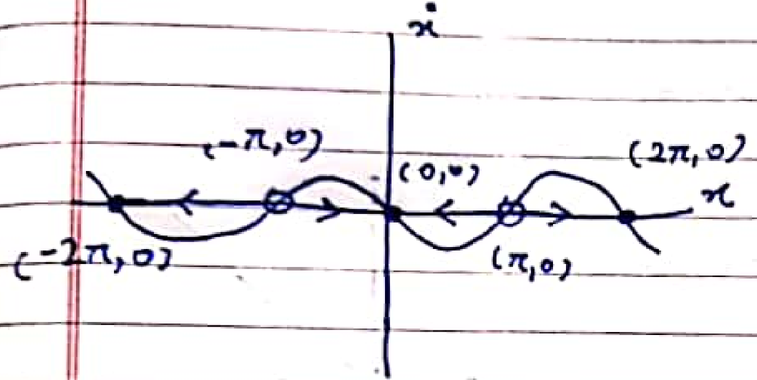
$f(x^*) = 0$

$-\sin x^* = 0$

$x^* = n\pi, n \in \mathbb{Z}$

$f'(x) = -\cos x$

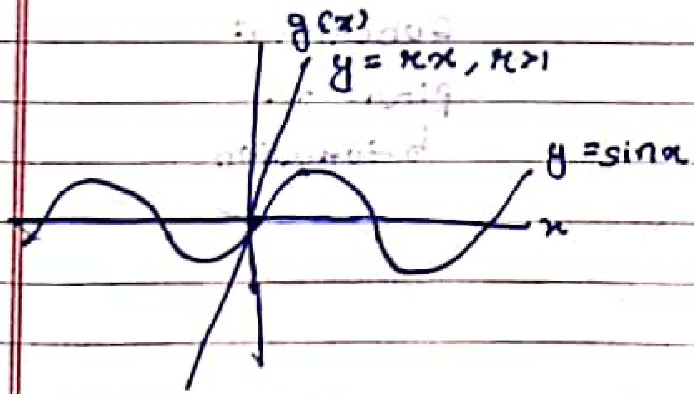
$f'(x^*) = \begin{cases} -1, & x^* = 2n\pi \quad n \in \mathbb{Z} \quad \text{stable} \\ 1, & x^* = (2n+1)\pi \quad n \in \mathbb{Z} \quad \text{unstable} \end{cases}$



b) $r > 1$, $\dot{x} = rx - \sin x = f(x)$

$f(x^*) = 0$

$rx^* = \sin x^*$



1 solution $x^* = 0$

$f'(x) = r - \cos x$

$f'(x^*) = f'(0) = r - 1 > 0$

$x^* = 0$ is unstable fixed pt.

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c) $\dot{x} = rx - \sin x = f(x)$

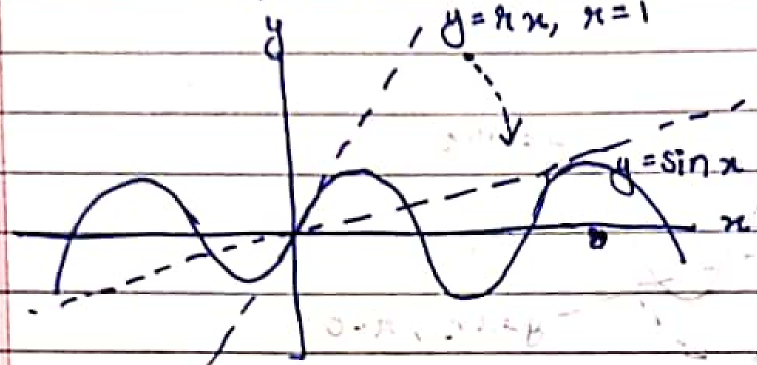
Bifurcation occurs when

~~at~~ $f(x^*) = 0$, $\frac{d}{dx^*} f(x^*) = 0$

$rx^* = \sin x^*$ & $r = \cos x^*$

$\Rightarrow$ $|r| \leq 1$

$\Rightarrow$ $x^* = \tan x^*$ & $r = \cos x^*$



as $r \to \infty$ to 0 the first pitchfork bifurcation occurs at $r=1$. This is subcritical pitchfork bifurcation as the single fixed pt. $x^* = 0$ which was unstable $r>1$ becomes stable at $r<1$ and two symmetric unstable fixed pts. are created. Then as r decreases new fixed pts are created ~~as~~ in pairs of two one stable and other unstable i.e saddle ~~pt~~ node bifurcations. Fixed pts. keep on increasing with saddle node bifurcations ~~at~~ and become infinite at $r=0$

d) $0 < r << 1$

Bifurcation occurs when

$f(x^*) = 0$, $\frac{d}{dx^*} f(x^*) = 0$

~~$\Rightarrow$~~ $r = \frac{\sin x^*}{x^*} << 1$, $r = \cos x^* << 1$

$\Rightarrow x^* = 2n\pi \pm \frac{\pi}{2}$, $n \in \mathbb{Z}$

~~$\Rightarrow$~~ $r = \cos x^* > 0$

$\Rightarrow x^* = 2n\pi + \frac{\pi}{2}$, $n \in \mathbb{Z}$ Substituting in other eq^n.

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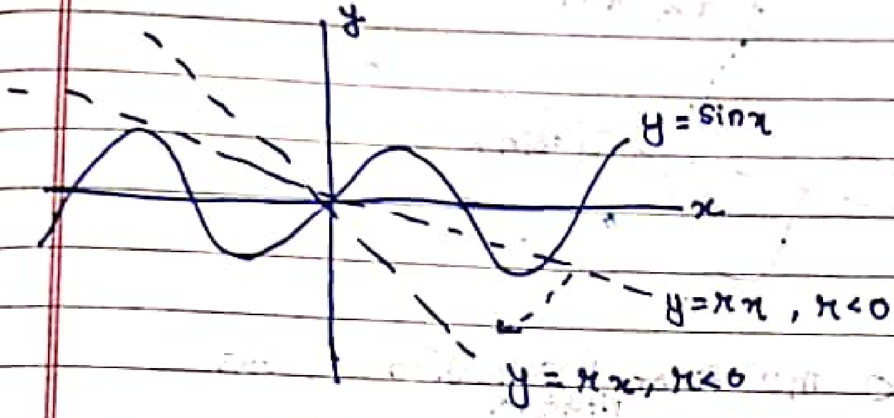
$$r = \frac{\sin\left(2n\pi + \frac{\pi}{2}\right)}{2n\pi + \frac{\pi}{2}} = \frac{1}{2n\pi + \frac{\pi}{2}}, \quad n \in \mathbb{Z}, \; n \gg 1$$

since $r \ll 1$

Bifurcation occur approx. at these r.

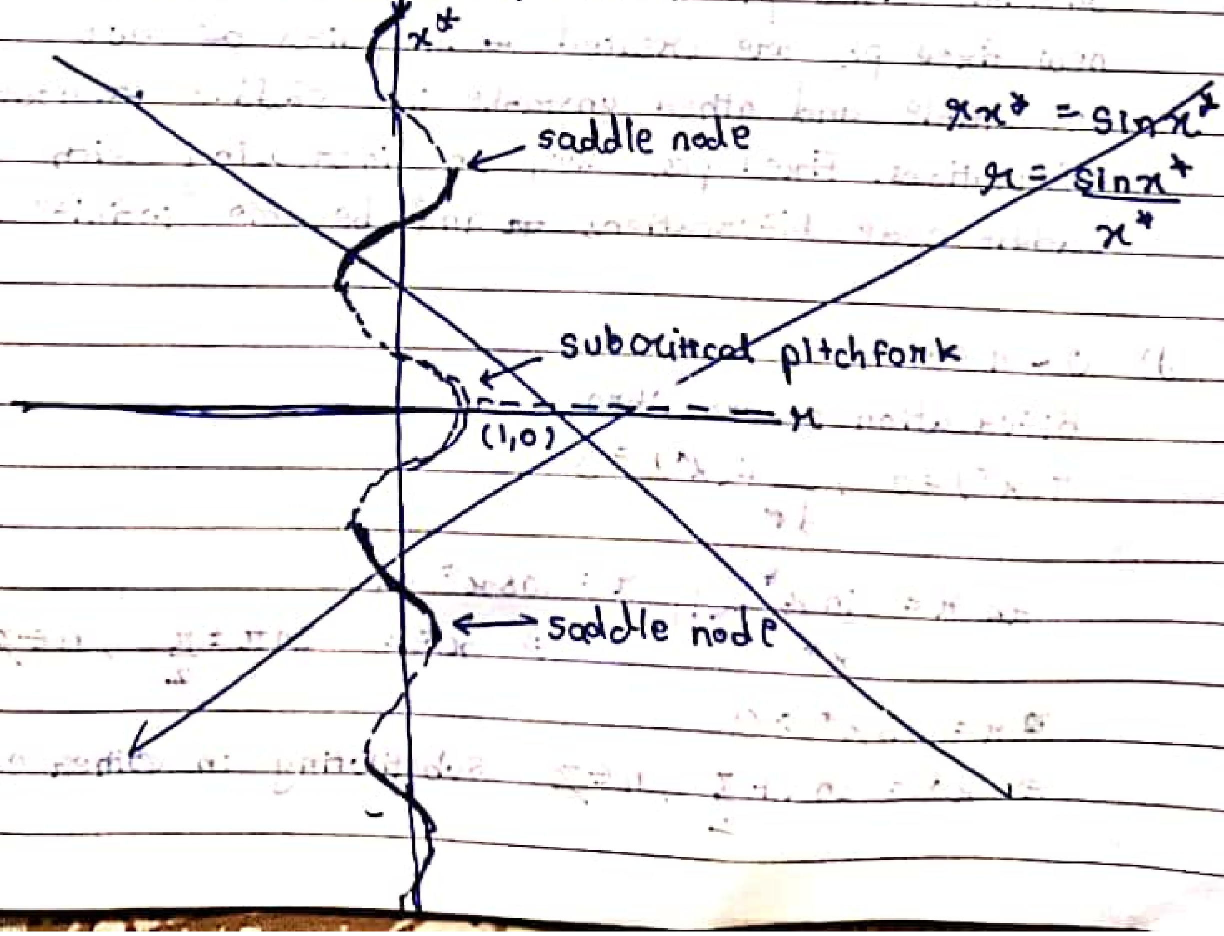
Q) as $r \to 0$ to $-\infty$

$\dot{x} = rx - \sin x$

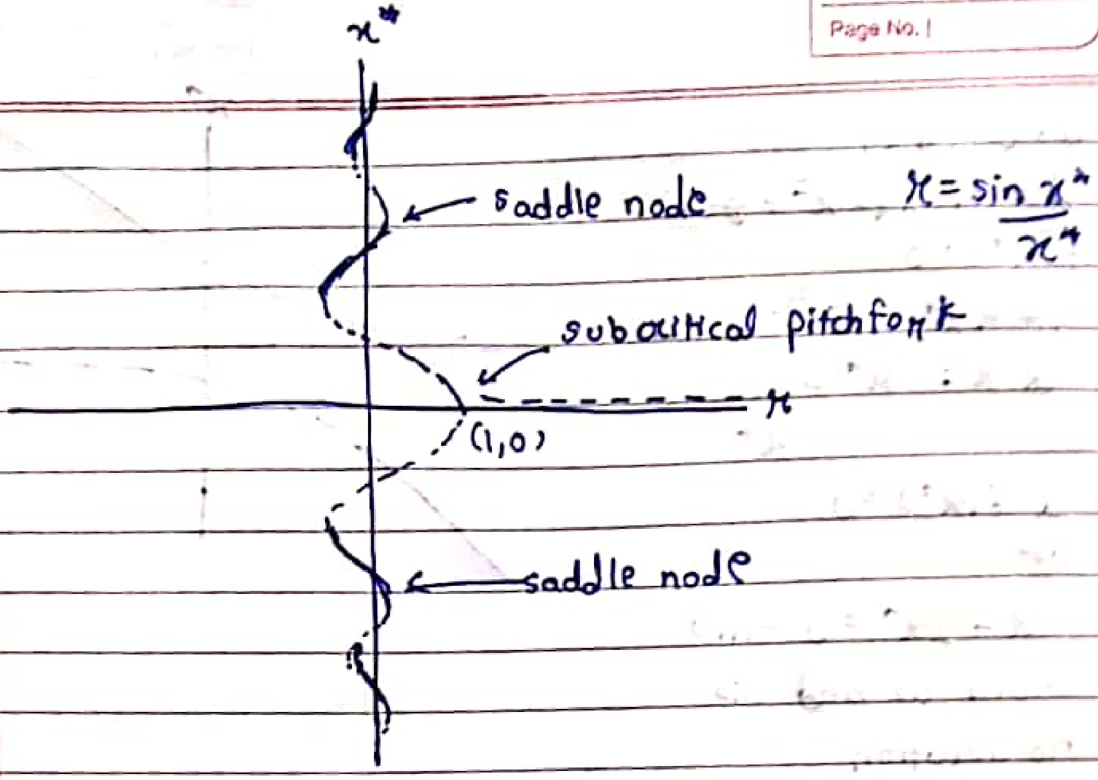


as $r \to 0$ to $-\infty$ the no. of fixed pts. decreases through saddle node bifurcations until only one stable fixed $x^* = 0$ remains.

f)



f)



Ans-14- $\dot{x} = h + rx - x^2$

a) $h < 0, h = -1$

$\dot{x} = rx - x^2 - 1 = f(x)$

$f(x^*) = 0$

$x^{*2} + 1 = r x^*$ -(i)

$r = x^* + \frac{1}{x^*}$

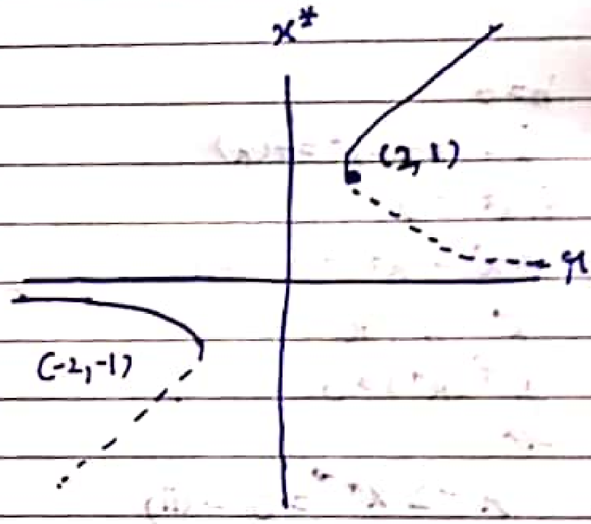
$\frac{d}{dx^*} f(x^*) = 0$

$r - 2x^* = 0$ -(ii)

from (i) and (ii)

$x^* = 1, r = 2$ or

$x^* = -1, r = -2$



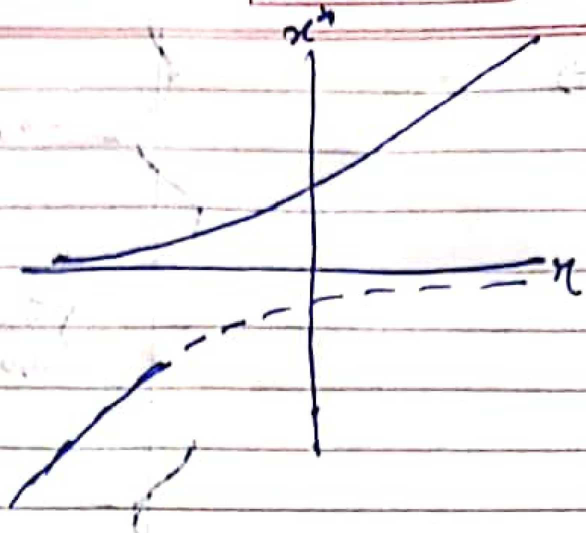
$h>0$, $h=1$

$\dot{x} = 1 + rx - x^2 = f(x)$

$f(x^*) = 0$

$1 + rx^* - x^{*2} = 0$ —(i)

$r = x^* - \frac{1}{x^*}$



$\frac{d}{dx} f(x^*) = 0$

$r - 2x^* = 0$ —(ii)

from (i) and (ii)

no solution

$h=0$

$\dot{x} = rx - x^2 = f(x)$

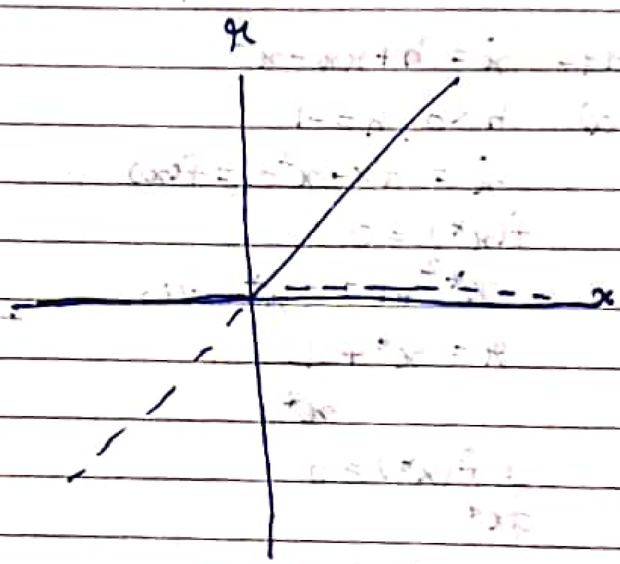
$f(x^*) = 0$

$rx^* - x^{*2} = 0$ —(i)

$x^* = 0, r$

$\frac{d}{dx} f(x^*) = 0$

$r - 2x^{*} = 0$ —(ii)



from (i) and (ii)

$r = 0, x^* = 0$

b) $\dot{x} = h + rx - x^2 = f(x)$

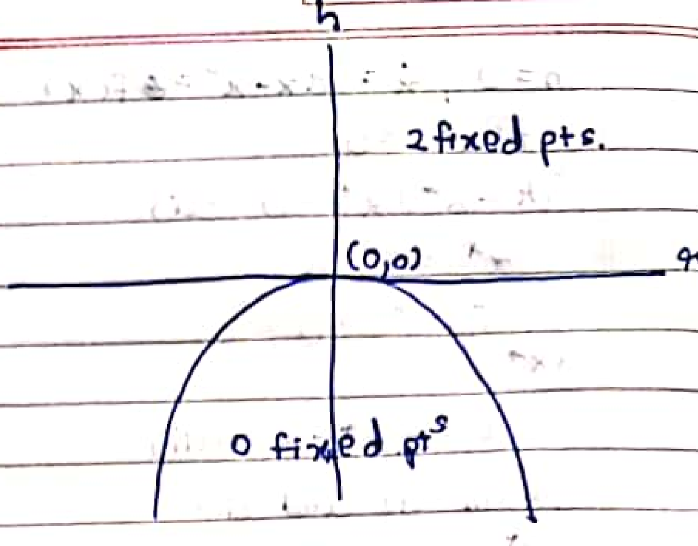
$f(x^*) = 0 \;,\; \frac{d}{dx} f(x^*) = 0$

$h + rx^* - x^{*2} = 0$

$r = 2x^*$

$\Rightarrow \; h = -x^{*2}$

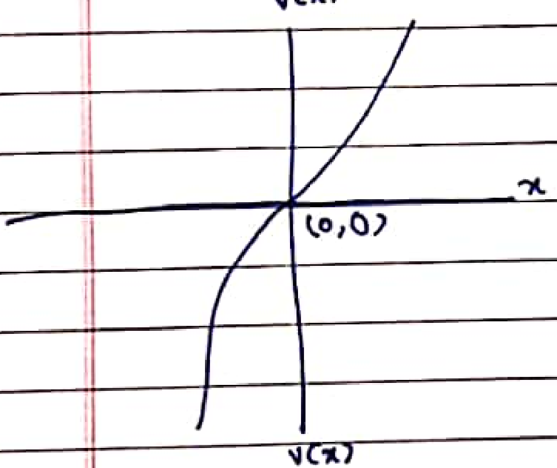
$\Rightarrow \; h = -\left(\frac{r}{2}\right)^2$



c) $-\frac{dV}{dx} = h + rx - x^2$

$V = \frac{x^3}{3} - \frac{rx^2}{2} - hx$

Two ~~reg~~ regions.

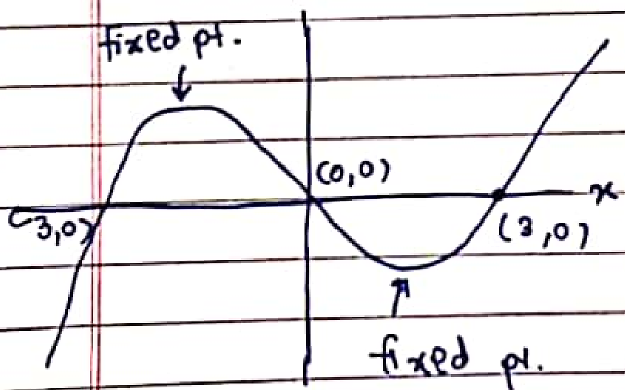


$h = -3$

$r = 0$

$V(x) = \frac{x^3}{3} + 3x$

No fixed pt.



$h = 3$

$r = 0$

$V(x) = \frac{x^3}{3} - 3x$

Two fixed pts