Chaos in Evolutionary Game Theory Dynamics

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1 Introduction

In Classical Game Theory, players make decisions in a game and based on the decisions made by them and all other players, their payoff is decided. Each player's main concern is about optimising their payoff. These so called players are considered optimal adversaries with a completely logical mind and extreme reasoning. Classical Game Theory mainly deals with the expected behaviour of these players and the decisions made by them in this all adverse environment. In nature however, such optimal adversaries don't exist. But interestingly enough, game theory still find applications in evolutionary biology where there is no self conscious, logical player making decisions explicitly. Section

Evolutionary biology is based on the idea that an organism's traits are largely determined by genes and that genes are the fundamental units of natural selection. A gene is a pattern of DNA whose copies exists inside organisms and determine its one trait. So while the organism only lives once, gene is potentially immortal (as it is passed down from the parent to offspring with a very little probability of mutation) and in the long run the competition is between genes. So the genes which determine traits that are more suitable for survival and reproduction in a given environment will tend to win (have more "fitness" in darwinian notion) by increasing the organisms' representation in the overall population in the environment. So traits and characteristics determined by genes are analogous to strategy in a game and fitness is analogous to payoff and payoff is depended on the interactions between these different strategies.

In this report, the dynamics of these evolutionary games are considered at various complexities. The report is organized as follows: In Section 2 dynamics of a simple two player evolutionary game is discussed and useful terminologies are introduced. Then In Section 3 and 4 various ways of looking at fixed points in evolutionary games are discussed and then the relation between them is compared in Section 5. In Section 6 and 7 occurrence and classification of bifurcations and strange attractors is explored in three and four player evolutionary games. Finally Section 8 concludes the report along with all the findings.

2 A Simple Beetle Model

2.1 Model Description

Consider an environment where a beetle's fitness is determined by its ability to find food and extract nutrients from it. Suppose initially there's a single species of beetle but a mutation occurs introducing beetles with larger body sizes. The beetles compete with each other for food when they stumble upon it. Assume that at a time such competition for food takes place between two beetles only for simple considerations. So this battle for food between these beetles can be considered as a game where their genes have inherently fixed the strategies for these beetles and payoffs after these battles would decide their survival and thus their proportion in the population as a whole.

2.2 Game Dynamics

Let x be the proportion of small beetles and $1-x$ be the proportion of large beetles in the population. Assume that beetles are paired at random and each beetle engages in only one competition (per unit time). Let the

		Beetle Y	
		Small	*Large*
Beetle X	*Small*	$(5,5)$	$(1,8)$
	Large	$(8,1)$	$(3,3)$

Table 1: Payoff matrix

strategies used by small and large beetles as a consequence of their traits and characteristics be S_1 and S_2 and the payoff of playing strategy S_i against S_j be U_{ij} as shown in the Table 1. This payoff could be imagined as the energy a beetle would get on acquiring the food after battle. So the Expected payoffs of S_1 and S_2 are:

$U(S_i) = \Sigma_j pr(S_j)U_{ij}$

$U(S_1) = xU_{11} + (1-x)U_{12} = 5x + (1-x) = 4x + 1$

$U(S_2) = xU_{21} + (1-x)U_{22} = 8x + 3(1-x) = 5x + 3$

where $pr(S_j)$ is the proportion of population using strategy S_j.
Average Expected Payoff of the entire population is:

$U(avg) = \Sigma_i pr(S_i)U(S_i)$

$U(avg) = xU(S_1) + (1-x)U(S_2) = x(4x+1) + (1-x)(5x+3) = -x^2 + 3x + 3$

The proportion of beetles changes depending upon how much expected payoff they have relative to the average expected payoff and is governed by the following equation proposed by Taylor and Jonker(1978)[3]:

$\frac{dpr(S_i)}{dt} = pr(S_i)(U(S_i) - U(avg))$

$\frac{dx}{dt} = x(U(S_1) - U(avg)) = x(4x + 1 - (-x^2 + 3x + 3)) = x(x-1)(x+2)$

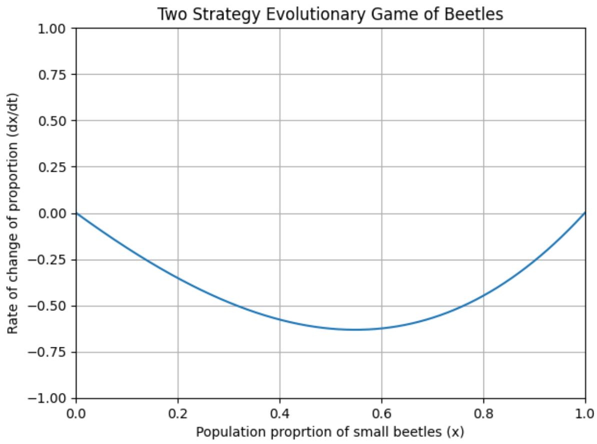


Figure 1: Phase portrait

The dynamics show that eventually all the beetles will be large beetles in this environment. Another significant observation is that average payoff of 5 per small beetle is more in an all small beetles environment still the evolution stabilise at smaller average payoff of 3 per large beetle. This means that environment would become more and more hostile over time as large beetles battle over food. This situation is analogous to the Prisoner's dilemma situation in classical game theory.

3 Evolutionarily Stable Strategies

A strategy is said to be evolutionarily stable if, when the entire population is using this strategy, a small group of invaders using a different strategy will die out eventually. More formally, Let U_{ij} be the payoff of playing strategy S_i against S_j. Then Strategy S_i is evolutionarily stable iff $U_{ii} > U_{ji}$ or $U_{ii} = U_{ji}$ and $U_{ij} > U_{jj}$ $\forall j \neq i$.

4 Mixed Strategies

It is possible that both of the strategies considered in the 2 player game are not evolutionarily stable. In such a case a mixed strategy is evolutionarily stable. There are basically two viewpoints in defining what a mixed strategy is. First is that each organism is hardwired to follow a pure strategy (S_1 or S_2), but some portion of population follows one while the rest follows the other and have the same payoff. It would be evolutionarily stable if any other proportion of strategy in population quickly dies off. Second is that each organism follows either of the strategy with a certain probability. If invaders follow a different probability and they quickly die off then it is evolutionarily stable.

5 Relation Between Dynamical equilibrium, Evolutionarily Stable Strategies and Nash Equilibrium

In game theoretic jargon, Nash equilibrium is a situation when no participant can gain by a unilateral change of strategy if the strategies of the others remain unchanged. If Strategy S is Evolutionarily Stable Strategy then [S, S] is a Nash Equilibrium but the converse is not true. If [S, S] is a Nash Equilibrium then S is a fixed point in Taylor and Jonkers flow dynamics equations but the converse is not true. Although biologist study these systems for Evolutionarily Stable Strategy, Taylor Jonkers flow fixed point is a good representation for most cases.

6 Three Strategy Evolutionary Games

E.C. Zeeman looked into various Three Strategy Evolutionary Games in his paper on Population dynamics [4]. Although chaos cannot occur on such systems, interesting hopf bifurcations and limit cycles do emerge. Consider the following game:

	S1	S2	S3
S1	2	1	5
S2	5	a	0
S3	1	4	3

Table 2: Payoff matrix

where a is real valued parameter.
Let x be the portion of population using the strategy S_1 and y be the portion of the population using strategy S_2 implying $1 - x - y$ to be the portion of population using S_3.
Again using the Taylor and Jonkers flow analysis :

$U(S_1) = 2x + y + 5(1 - x - y) = 5 - 3x - 4y$

$U(S_2) = 5x + ay + 0(1 - x - y) = 5x + ay$

$U(S_3) = x + 4y + 3(1 - x - y) = 3 - 2x + y$

$U(avg) = xU(S_1) + yU(S_2) + (1 - x - y)U(S_3)$

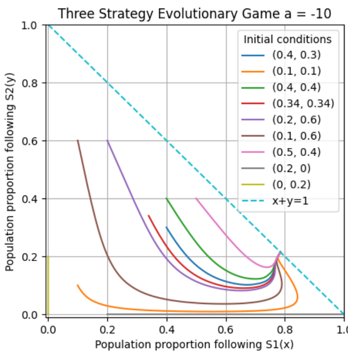
$= x(5 - 3x - 4y) + y(5x + ay) + (1 - x - y)(3 - 2x + y)$

$= -x^2 - (1 - a)y^2 + 2xy - 2y + 3$

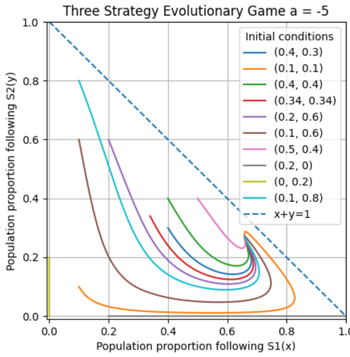
$\frac{dx}{dt} = x(U(S_1) - U(avg)) = x(x^2 + (1 - a)y^2 - 2xy - 3x - 2y + 2)$

$\frac{dy}{dt} = y(U(S_2) - U(avg)) = y(x^2 + (1 - a)y^2 - 2xy + 5x + (a + 2)y - 3)$

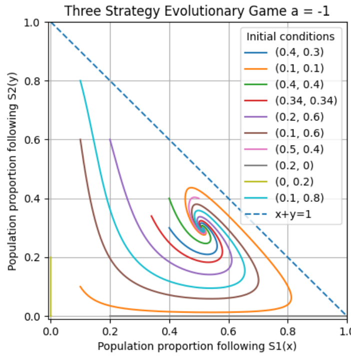
The above dynamics when solved numerically showed denerate hopf bifurcation near a = 3. It is structurally unstable and any small pertubation of the value of a destroys the closed orbits. This degenerate case typically arises when a nonconservative system suddenly becomes conservative at the bifurcation point. The fixed point becomes a nonlinear center, rather than the weak spiral required by a Hopf bifurcation. It is very difficult to demonstrate such system numerically since any approximation tend to blow up the closed orbit structure.



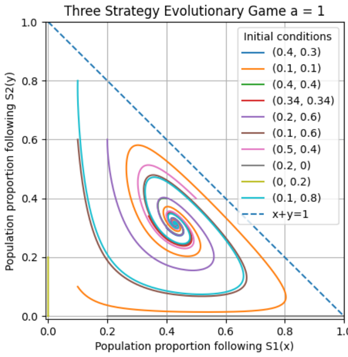
(a) $a = -10$



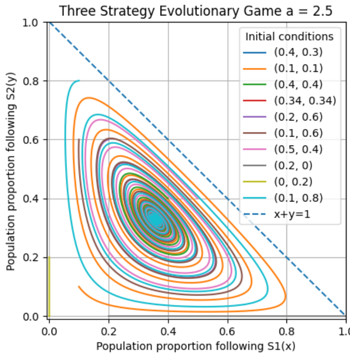
(b) $a = -5$



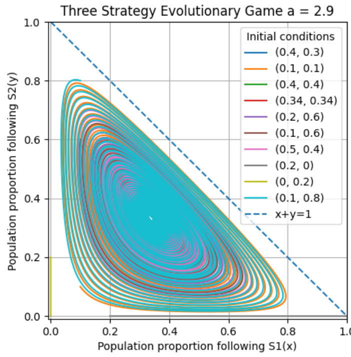
(c) $a = -1$



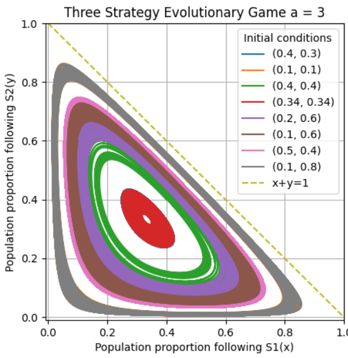
(d) $a = 1$



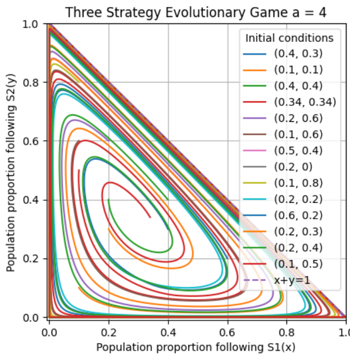
(e) $a = 2.5$



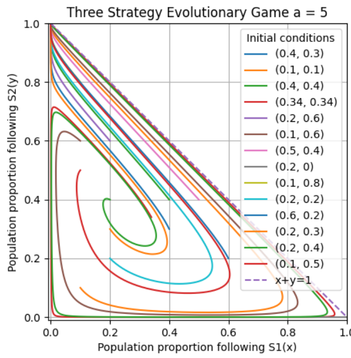
(f) $a = 2.9$



(g) $a = 3$



(h) $a = 4$



(i) $a = 5$

Figure 2: Phase portraits at different values of a

7 Four Strategy Evolutionary Games

Strange attractors and chaotic behaviour have been seen in four or more player evolutionary games. I analysed a particular four strategy game given below which takes a route to chaos through hopf bifurcation and period doubling of the limit cycle.

	S1	S2	S3	S4
S1	-1	-1	-10	1000
S2	-1.5	-1	-1	1000
S3	a	0.5	0	-1000
S4	0	0	0	0

Table 3: Payoff matrix

So the above evolutionary game has the given payoff matrix with a single parameter a. a is varied from 2.4 to 6. The following phase portraits snapshots at different values of a shows the route to chaos.

Let x, y and z be the proportion of the population using strategy $S1$, $S2$ and $S3$ respectively implying $1 - x - y - z$ to be the portion of population using $S4$. Following Taylor and Jonkers flow analysis, following differential equations are derived:

$U(S_1) = -x - y - 10z + 1000(1 - x - y - z) = 1000 - 1001x - 1001y - 1010z$

$U(S_2) = -1.5x - y - z + 1000(1 - x - y - z) = 1000 - 1001.5x - 1001y - 1001z$

$U(S_3) = ax + 0.5y + 0z - 1000(1 - x - y - z) = -1000 + (1000 + a)x + 1000.5y + 1000z$

$U(S_4) = 0x + 0y + 0z + 0(1 - x - y - z) = 0$

$U(avg) = xU(S_1) + yU(S_2) + zU(S_3) + (1 - x - y - z)U(S_4)$

$= x(1000 - 1001x - 1001y - 1010z) + y(1000 - 1001.5x - 1001y - 1001z) + z(-1000 + (1000 + a)x + 1000.5y + 1000z) + (1 - x - y - z)(0)$

$= -1001x^2 - 1001y^2 + 1000z^2 - 2002.5xy + (a - 10)xz - 0.5yz + 1000x + 1000y - 1000z$

$\frac{dx}{dt} = x(U(S_1) - U(avg)) = -x(-1001x^2 - 1001y^2 + 1000z^2 - 2002.5xy + (a - 10)xz - 0.5yz + 2001x + 2001y + 10z - 1000)$

$\frac{dy}{dt} = y(U(S_2) - U(avg)) = -y(-1001x^2 - 1001y^2 + 1000z^2 - 2002.5xy + (a - 10)xz - 0.5yz + 2001.5x + 2001y + z - 1000)$

$\frac{dz}{dt} = z(U(S_3) - U(avg)) = -z(-1001x^2 - 1001y^2 + 1000z^2 - 2002.5xy + (a - 10)xz - 0.5yz - ax - 0.5y - 2000z + 1000)$

The phase portrait snapshots given below show the proportion x, y and z of strategy S_1, S_2 and S_3 at different values of a. The last player will simply have the proportion $1 - x - y - z$.

At a = 2.4 the system, for most trajectories, spirals to a fixed point at $(x, y, z, 1-x-y-z) = (0.36403639486087613, 0.6145453382835914, 0.02023699978781856, 0.00118126706771391)$. This can be verified by the eigenvalues of the jacobian at this fixed point which turn out to be:

$e_1 = -0.975777446278604$

$e_2 = -0.00211522973261581 - 0.150224998796878\sqrt{3}i$

$e_3 = -0.00211522973261581 + 0.150224998796878\sqrt{3}i$

where $i = \sqrt{-1}$ and e_1, e_2 and e_3 are the eigenvalues of the jacobian at the fixed point. Real portion of all three eigenvalues is negative indicating a stable fixed point and complex eigenvalues indicate the spiraling inwards of the trajectories nearby.

On increasing a further the system goes through a super critical hopf bifurcation and a new limit cycle is born.

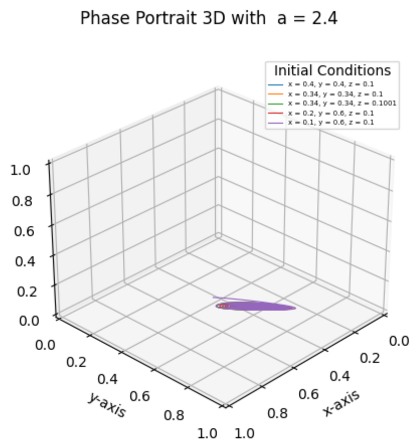
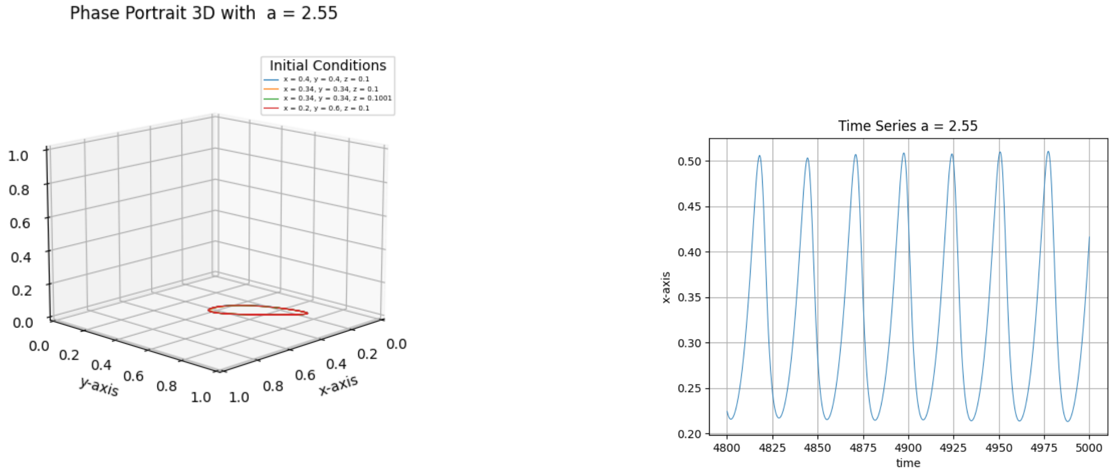


Figure 3: $a = 2.4$

At a = 2.55 we have an elliptical stable limit cycle surrounding the earlier fixed point which becomes unstable now.

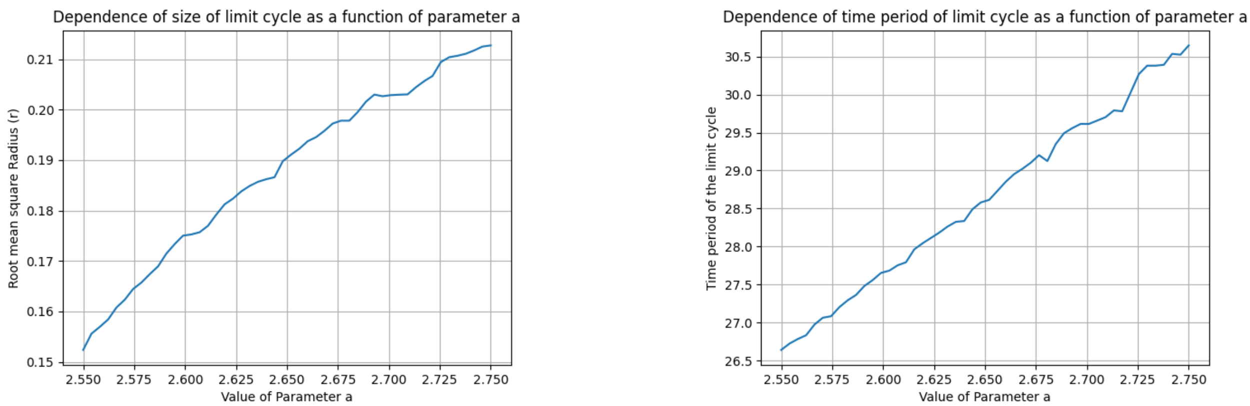


(a) $a = 2.55$

(b) Time series for $a = 2.55$ along x-axis

Figure 4: Snapshot at $a = 2.55$

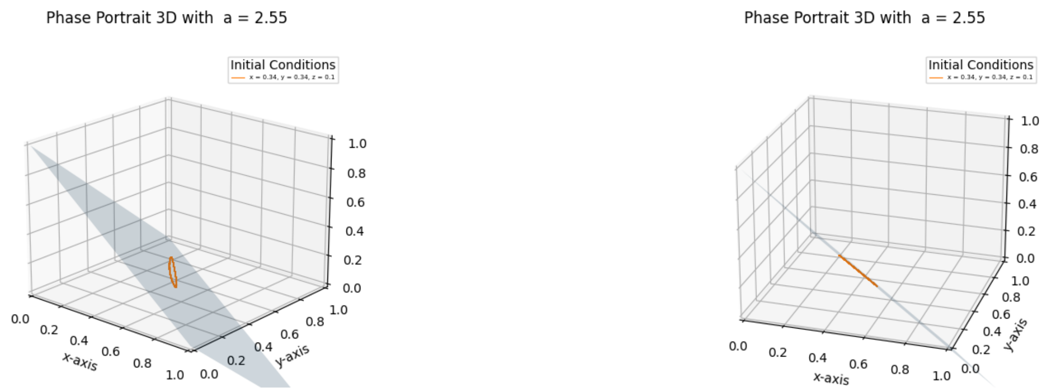
The limit cycle is almost planer lying on the plane $x + y + z = 1$ with a root mean square radius of about 0.154. On further varying a from 2.55 to 2.75, the size of the limit cycle (represented as the root mean square distance from the centroid) grows proportional to $\sqrt{a}$ and time period grows linearly with a. In Hopf bifurcations radius of the limit cycle generally have $O(\sqrt{\mu})$ dependence on the parameter μ, this system also follows this trend.



(a) size of limit cycle vs a

(b) time period of limit cycle vs a

Figure 5: Dependence of size and time period of limit cycle on the parameter a

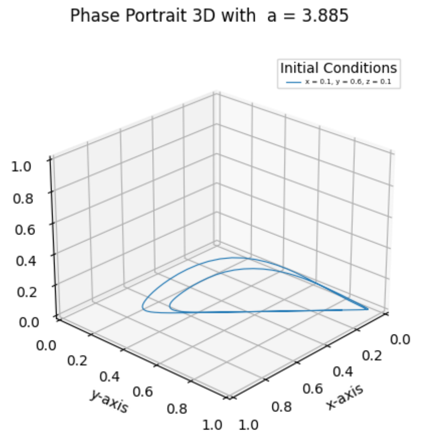


(a) planar limit cycle

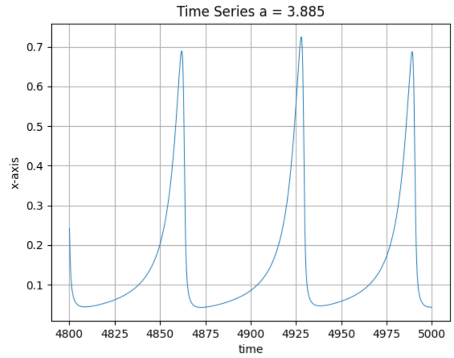
(b) Planar ellipse

Figure 6: Planar Limit Cycle

On increasing a further, it goes period doubling bifurcation and the snapshots at $a = 3.885$ and $a = 4$ suggests the splitting of limit cycle into multiple orbit structure rather than a simple elliptical structure.

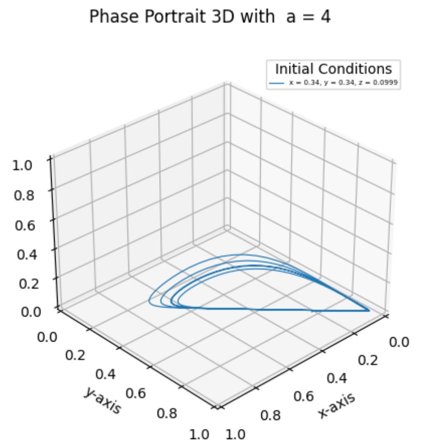


(a) $a = 3.885$

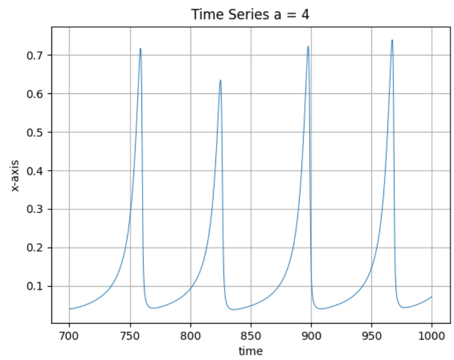


(b) Time series for $a = 3.885$ along x-axis

Figure 7: Snapshot at $a = 3.885$



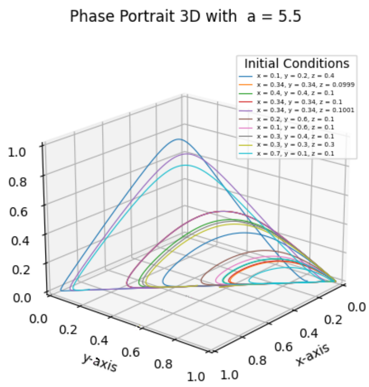
(a) $a = 4$



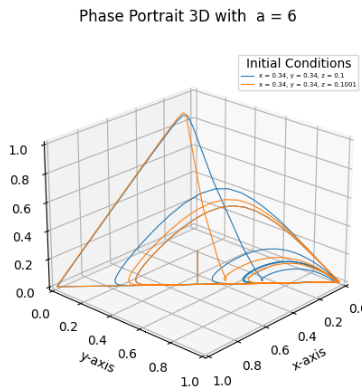
(b) Time series for $a = 4$ along x-axis

Figure 8: Snapshot at $a = 4$

On further increasing a the limit cycle goes through multiple period doubling and chaotic behaviour is observed at $a = 5.5$. At $a = 5.5$ a strange attractor similar to a single flap of lorentz attractor is observed as can be seen in Figure 9a. The phase portrait becomes even more complicated as a increases till $a = 6$. Figure 9b shows the sensitive dependence on initial condition with two trajectories starting with very close initial conditions $(x_0, y_0, z_0) = (0.34, 0.34, 0.1)$ and $(x_1, y_1, z_1) = (0.34, 0.34, 0.1001)$ and diverging exponentially fast.



(a) $a = 5.5$



(b) $a = 5.5$

Figure 9: Snapshot at $a = 5.5$ and $a = 6$

8 Analysis and Conclusion

All the phase portrait snapshots are drawn using python's Matplotlib library. Numerical Integrations are performed using scipy's integrate module with default RK45 (Explicit Runge-Kutta method of order 5). Eigenvalues are found using the sympy module which computes them by solving the equation $det(A - \lambda I) = 0$ using quadratic and cubic formula for 2D and 3D matrix and using newton's method for higher dimension matrices. Python floats have a 16 digit precision and all the calculations are performed under this precision. The four strategy evolutionary game discussed in Section 7 becomes quite sensitive to precision after the parameter a becomes greater than 4. Therefore in very long time trajectories errors stack up and give totally different results making it difficult to analyse after a certain length trajectory. I wanted to perform fractal analysis on this strange attractor but couldn't due to this limitation. All the code of the analysis can be found at my github repository for chaos [7]. In conclusion this report gives analysis of one dynamical system from two, three and four strategy evolutionary games each. Two strategy games turned out to be pretty simple with stable fixed equilibrium point at either a pure strategy or a mixed strategy. Three strategy games had some interesting hopf bifurcation (especially degenerate hopf bifurcation) but no chaotic behaviour. Chaos emerged in the four strategy game through hopf bifurcation and period doubling of the limit cycle. It resulted in a strange attractor which looked similar to a single flap of the lorentz attractor.

References

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