ASSIGNMENT-3
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Ans-1- $\dot{x} = y$
$\dot{y} = -2x - 3y$

$\dot{\underline{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A\underline{x}$

$\tau = -3$, $\Delta = 2$

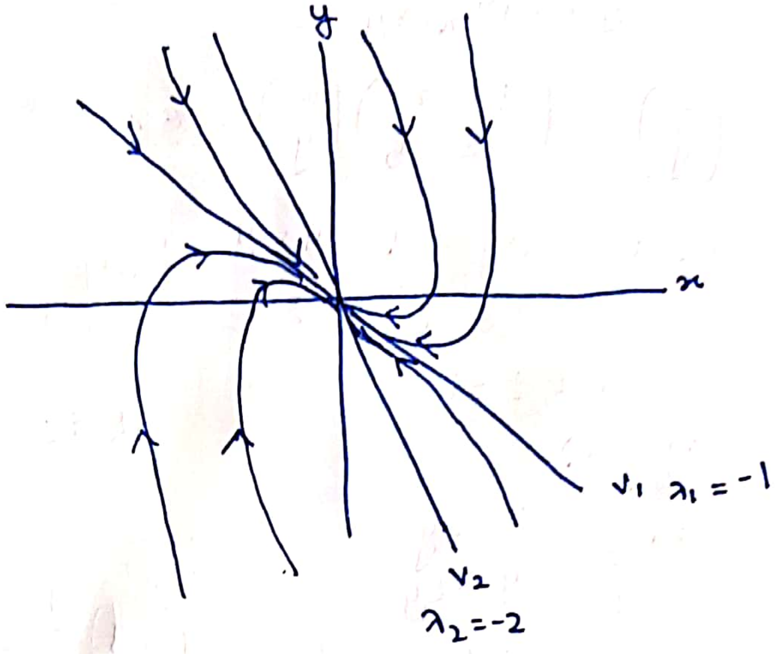
$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+1)(\lambda+2) = 0$

$\lambda_1 = -1$, $\lambda_2 = -2$

$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(Stable node)
(0,0)



Ans-2- $\dot{x} = 3x - 4y$
$\dot{y} = x - y$

$\dot{\underline{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A\underline{x}$

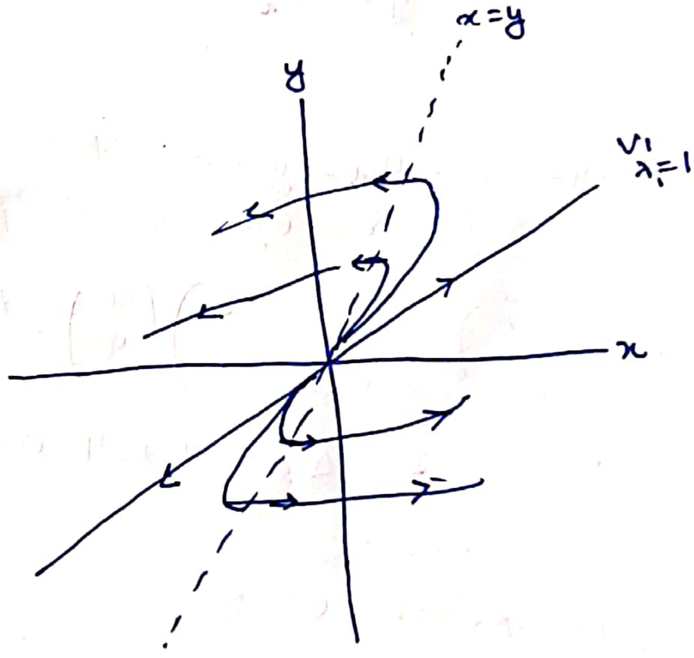
$\tau = 2$, $\Delta = -3 + 4 = 1$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda_1 = \lambda_2 = 1$

$v_1 = v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



① $x = y$
$\dot{x} = -x$
$\dot{y} = 0$

(degenerate unstable node)
(0,0)

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Ans-3- $\dot{x} = 5x + 2y$

$\dot{y} = -17x - 5y$

$$\dot{\underline{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A\underline{x}$$

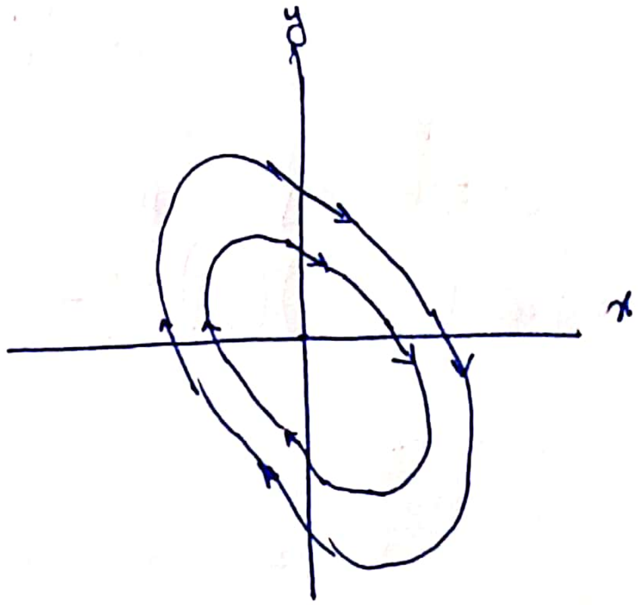
$\tau = 0, \quad \Delta = -25 + 34 = 9$

$\lambda^2 + 9 = 0$

$\lambda = \pm 3i$

(center)

(0,0)



① $x = 0, \quad \dot{x} = 2y$

$\dot{y} = -5y$

② $y = 0, \quad \dot{x} = 5x$

$\dot{y} = -17x$

Ans-4- $\dot{x} = 4x - 3y$

$\dot{y} = 8x - 6y$

$$\dot{\underline{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A\underline{x}$$

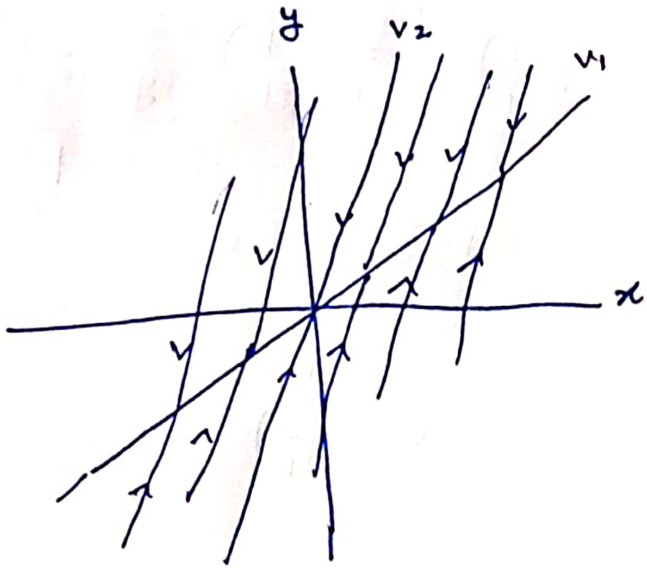
$\tau = -2, \quad \Delta = -24 + 24 = 0$

$\lambda^2 + 2\lambda = 0$

$\lambda(\lambda + 2) = 0$

$\lambda_1 = 0, \quad \lambda_2 = -2$

$v_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



Infinite fixed points of the form $(3\mu, 4\mu)$, $\mu \in \mathbb{R}$

Each is a stable fixed pt.

Ans-5- $\dot{x} = 5x + 10y$

$\dot{y} = -x - y$

$$\underline{\dot{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

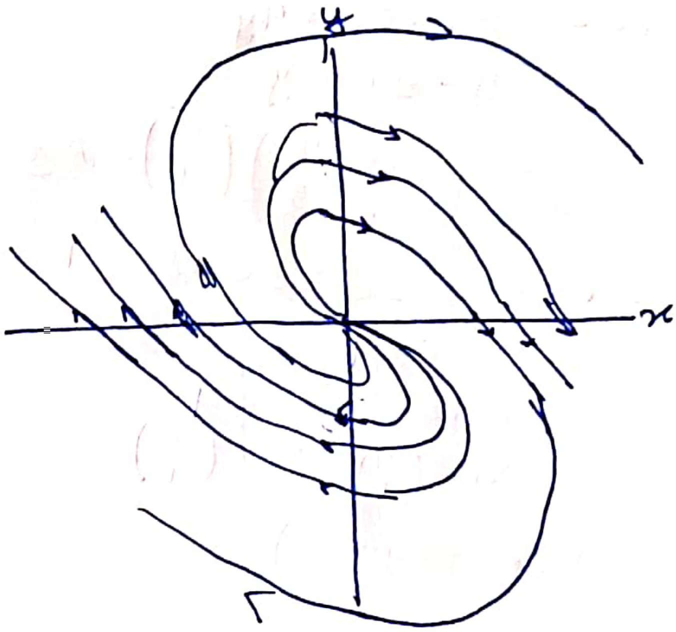
$\tau = 4$, $\Delta = -5 + 10 = 5$

$$\lambda^2 - 4\lambda + 5 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$\lambda_1 = 2 + i$, $\lambda = 2 - i$

(Unstable spiral node (0,0))



① $x = 0$, $\dot{x} = 10y$

$\dot{y} = -y$

② $y = 0$, $\dot{x} = 5x$

$\dot{y} = -x$

Ans-6- $\dot{x} = -3x + 2y$

$\dot{y} = x - 2y$

$$\underline{\dot{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\tau = -5$, $\Delta = 6 - 2 = 4$

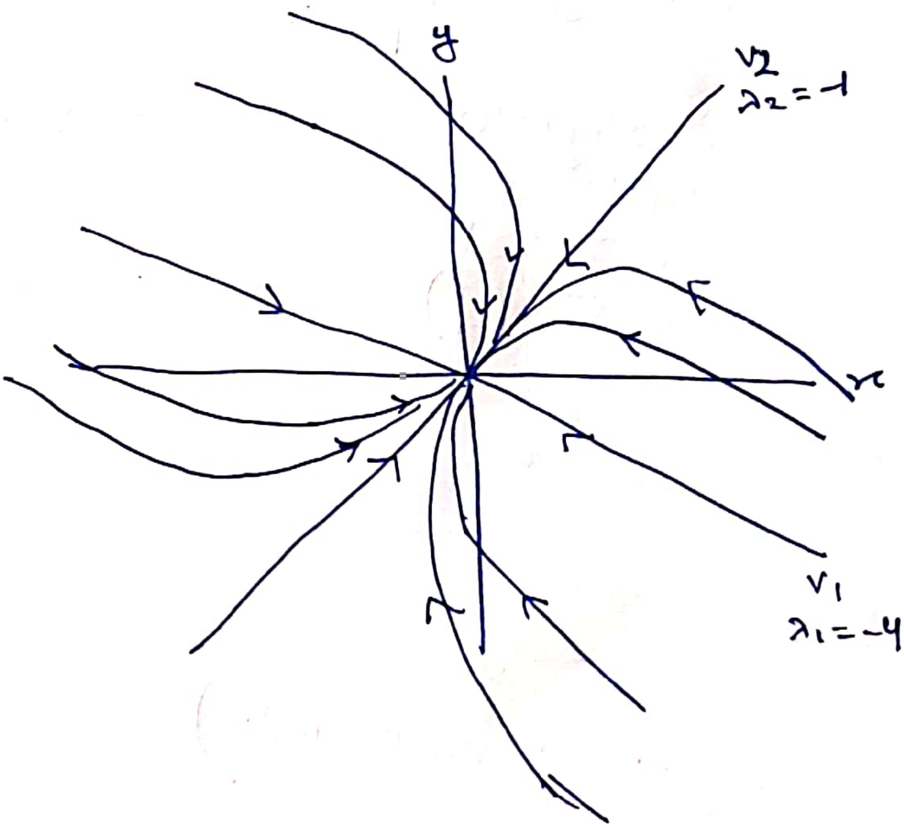
$$\lambda^2 + 5\lambda + 4 = 0$$

$$(\lambda + 1)(\lambda + 4) = 0$$

$\lambda_1 = -4$, $\lambda_2 = -1$

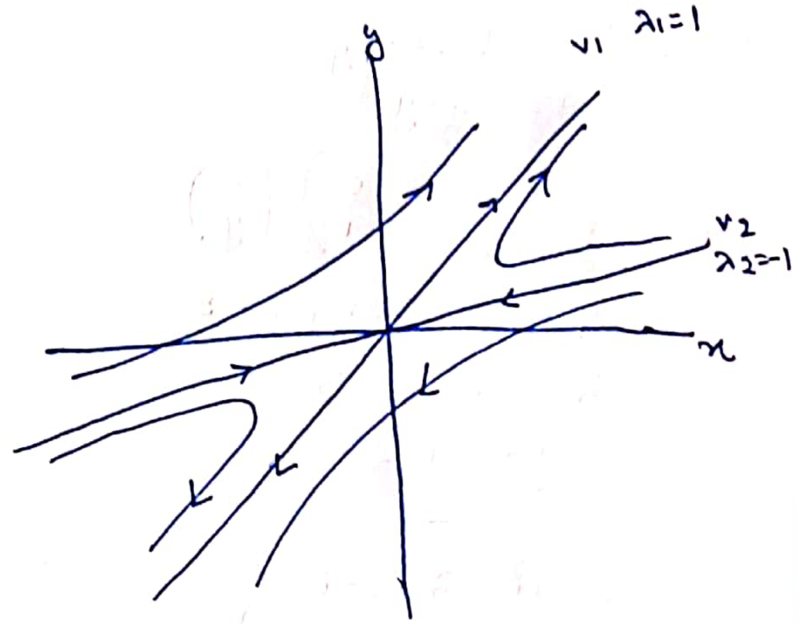
$v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(Stable node at (0,0))



Ans-7- $\dot{x} = -3x + 4y$

$\dot{y} = 2x + 3y$



$$\dot{\underline{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A\underline{x}$$

$\tau = 0 \qquad \Delta = -9 + 8 = -1$

$\lambda^2 = 1$

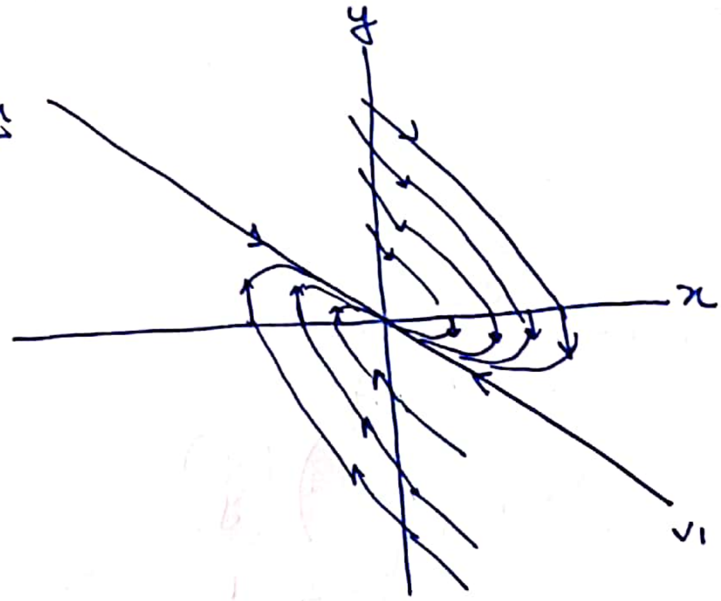
$\lambda_1 = 1, \quad \lambda_2 = -1$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(Saddle point (0,0))

Ans-8- $\dot{x} = y$

$\dot{y} = -x - 2y$



$$\dot{\underline{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A\underline{x}$$

$\tau = -2, \quad \Delta = 1$

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda + 1)^2 = 0$

$\lambda_1 = \lambda_2 = -1$

$v_1 = v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

① $y = 0, \quad \dot{x} = 0$
$\qquad \dot{y} = -x$

② $x = 0, \quad \dot{x} = y$
$\qquad \dot{y} = -2y$

(Degenerate stable node (0,0))

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Ans-1- $\dot{x} = x - y = f(x,y)$
$\dot{y} = 1 - e^{x} = g(x,y)$

Fixed pts.:

$\dot{x} = 0 \qquad \dot{y} = 0$
$x = y \qquad e^{x} = 1$
$\qquad\qquad x = 0$

Fixed pts: $(0,0)$

$$J_{(0,0)} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & -1 \\ -e^{x} & 0 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

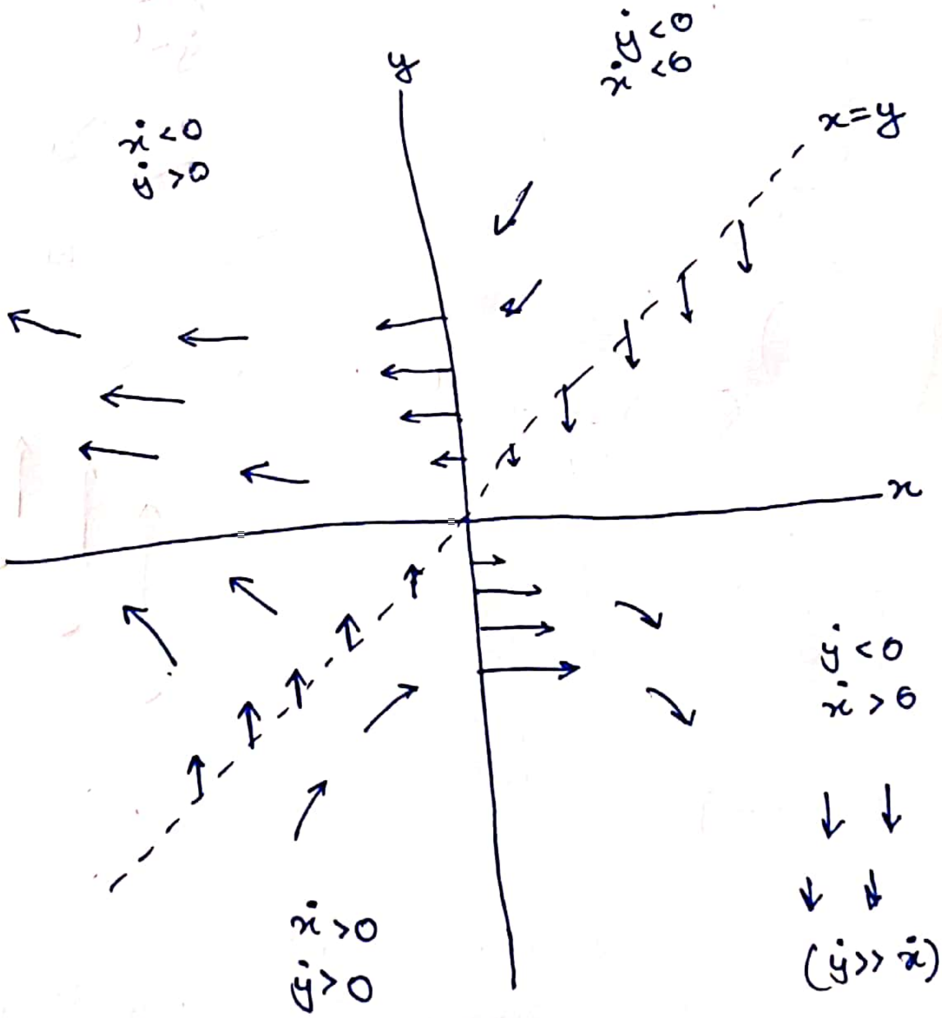
$\tau = 1, \quad \Delta = 0 - 1 = -1$

Saddle pt. at $(0,0)$

Nullclines

$\dot{x} = 0$
$x = y$
$\dot{y} = 1 - e^{x}$

$\dot{y} = 0$
$1 - e^{x} = 0$
$x = 0$
$\dot{x} = -y$



Ans-2- $\dot{x} = x(x-y) = f(x,y)$
$\dot{y} = y(2x-y) = g(x,y)$

Fixed pts:

$\dot{x} = 0$ — $x=0$ or $x=y$

$\dot{y} = 0$ — $y = 0$ or $2x = y$

Fixed pt. $(0,0)$

$$J_{(0,0)} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 2x-y & -x \\ 2y & 2x-2y \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

No comments. (Maybe infinite fixed pts.)

<u>Null clines</u>

$\dot{x} = 0$

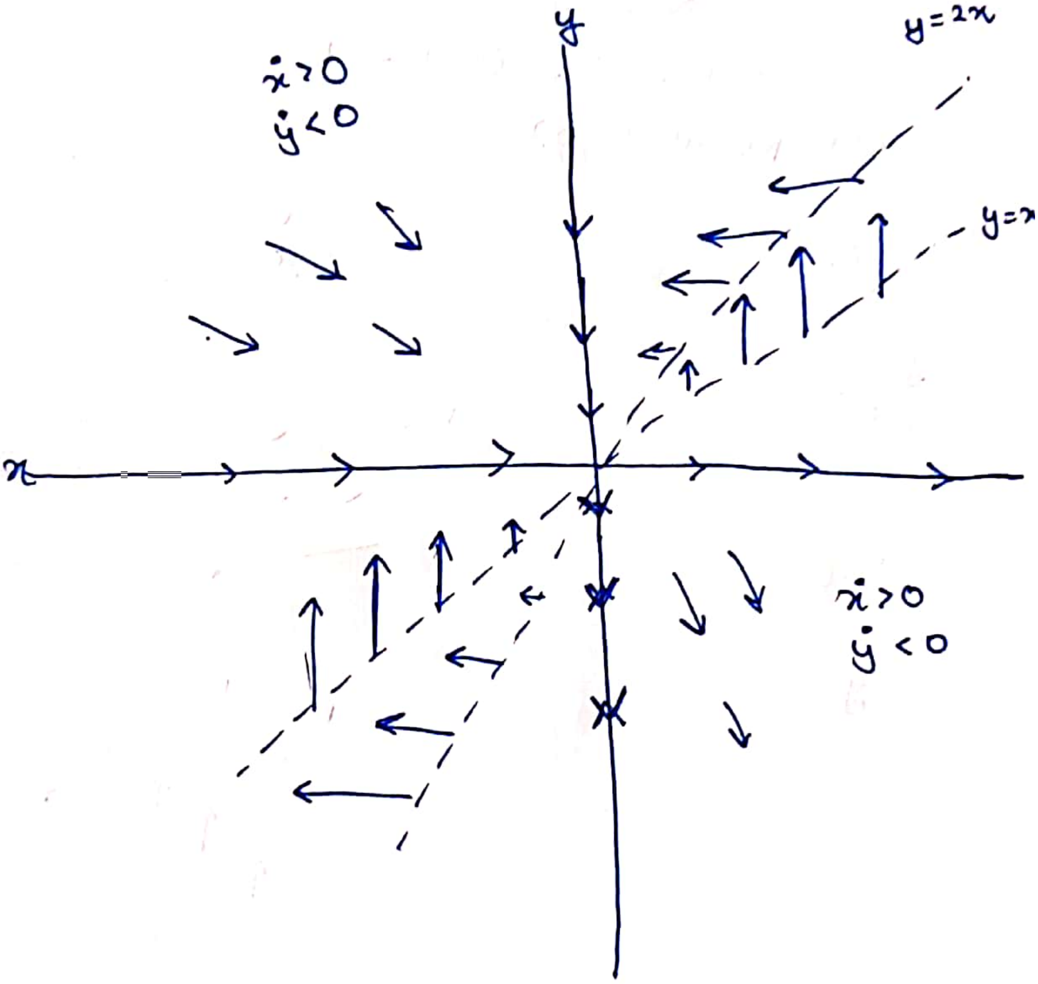
$x=0$
$\left(\begin{array}{l} \dot{x} = 0 \\ \dot{y} = -y^2 \end{array}\right)$

$x=y$
$\left(\begin{array}{l} \dot{x} = 0 \\ \dot{y} = y^2 \end{array}\right)$

$\dot{y} = 0$

$y=0$
$\left(\begin{array}{l} \dot{x} = x^2 \\ \dot{y} = 0 \end{array}\right)$

$2x=y$
$\left(\begin{array}{l} \dot{x} = -x^2 \\ \dot{y} = 0 \end{array}\right)$



For pts. in IInd quadrant approach $(0,0)$
Pts. in IVth quadrant move away from $(0,0)$
Ist and IIIrd form homodinic orbits.

Ans-3- $\dot{x} = x(2-x-y) = f(x,y)$

$\dot{y} = x-y = g(x,y)$

$\dot{x}=0$ $\qquad$ $\dot{y}=0$

$x=0$ or $x+y=2$ $\qquad$ $x=y$.

Fixed pts. $(0,0)$ $\quad$ $(1,1)$

$$J_{(0,0)} = \left.\begin{bmatrix} 2-2x-y & -x \\ 1 & -1 \end{bmatrix}\right|_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

$\tau = 1, \quad \Delta = -2$

(Saddle point $(0,0)$)

$$J_{(1,1)} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$\tau = -2, \quad \Delta = 1-(-1) = 2$

$\lambda^2 + 2\lambda + 2 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$ $\qquad$ ($\cancel{un}$stable spiral $(1,1)$)

<u>Null clines</u>

$\dot{x}=0$

$x=0$

$\begin{pmatrix} \dot{x}=0 \\ \dot{y}=-y \end{pmatrix}$

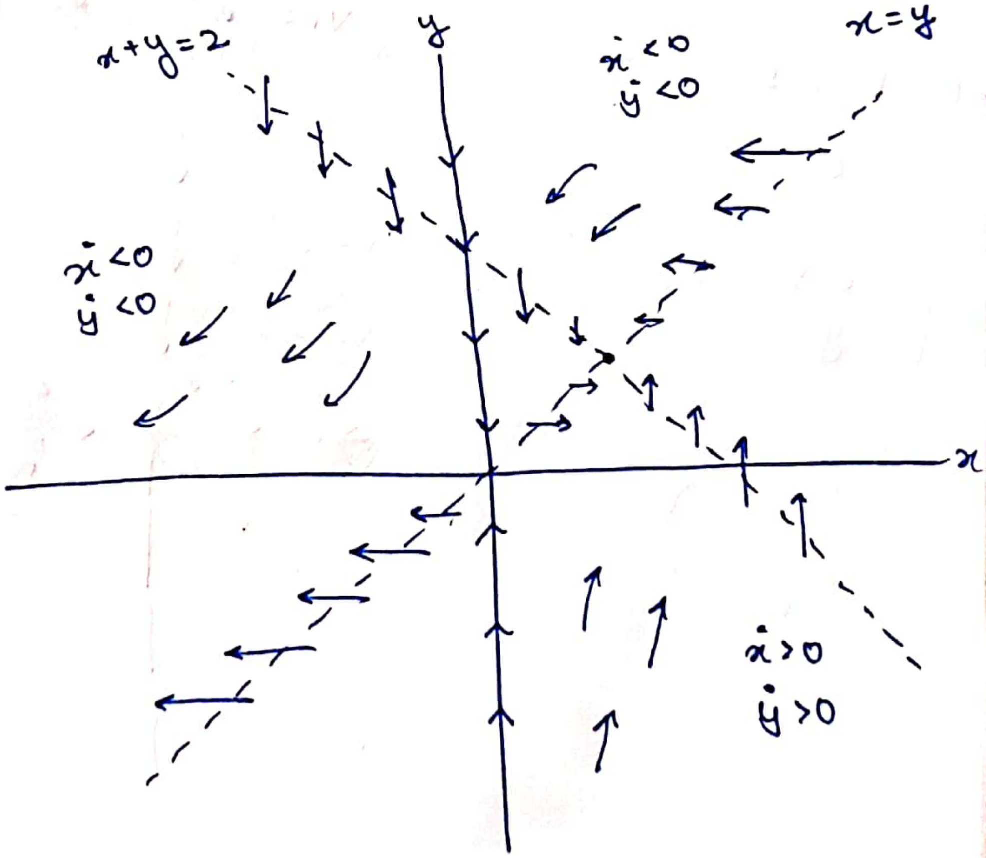
$x+y=2$

$\begin{pmatrix} \dot{x}=0 \\ \dot{y}=2-2y \end{pmatrix}$

$\dot{y}=0$

$x=y$

$\begin{pmatrix} \dot{x}=x(2-2x) \\ \dot{y}=0 \end{pmatrix}$



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Ans-4- $\dot{x} = x - x^3 = f(x,y)$

$\dot{y} = -y = g(x,y)$

$\dot{x} = 0 \qquad \dot{y} = 0$

$x = 0, \pm 1 \qquad y = 0$

Fixed pts. $(0,0)$, $(1,0)$, $(-1,0)$

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-3x^2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$J_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\tau = 0$, $\Delta = -1$

(saddle point $(0,0)$)

$$J_{(\pm 1,0)} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$\tau = -3$, $\Delta = 2$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+1)(\lambda+2) = 0$

$\lambda_1 = -1$, $\lambda_2 = -2$

$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(stable node $(\pm 1, 0)$)

Null clines

$\dot{x} = 0$

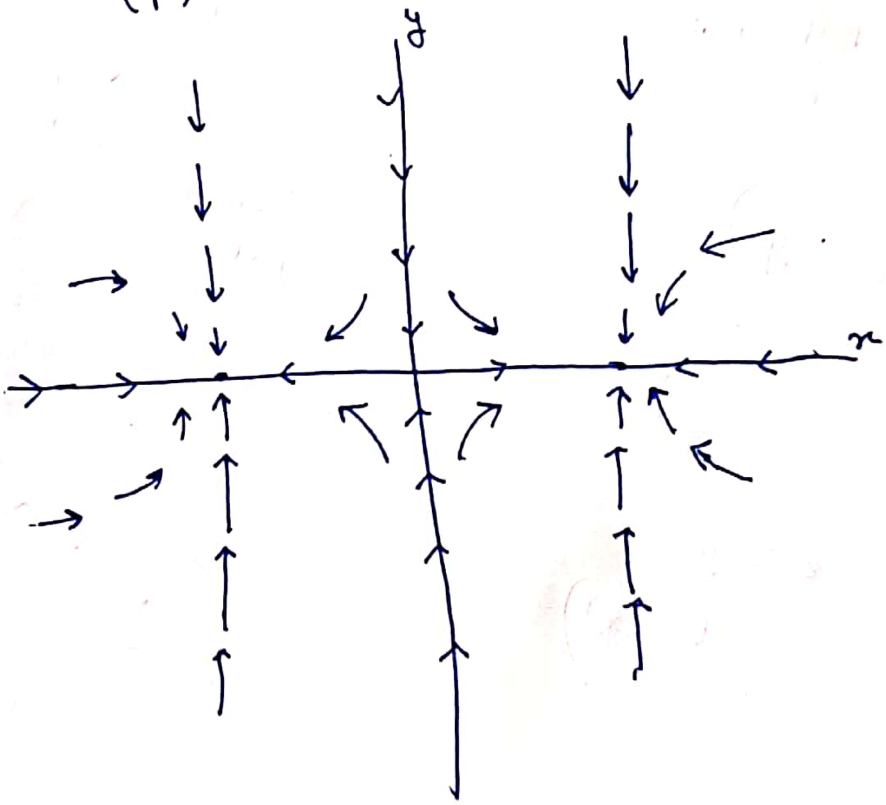
$x = 0, \pm 1$

$\begin{pmatrix} \dot{x} = 0 \\ \dot{y} = -y \end{pmatrix}$

$\dot{y} = 0$

$y = 0$

$\begin{pmatrix} \dot{x} = x - x^3 \\ \dot{y} = 0 \end{pmatrix}$



Ans-5- $\dot{x} = y = f(x,y)$

$\dot{y} = x(1+y) - 1 = g(x,y)$

$\dot{x} = 0$ $\qquad \dot{y} = 0$

$y = 0$ $\qquad x(1+y) = 1$

Fixed pt. $(1,0)$

$$J_{(1,0)} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\Bigg|_{(1,0)} = \begin{bmatrix} 0 & 1 \\ 1+y & x \end{bmatrix}\Bigg|_{(1,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$\tau = 1, \quad \Delta = -1$

(Saddle point $(1,0)$)

<u>Null clines</u>

$\dot{x} = 0$

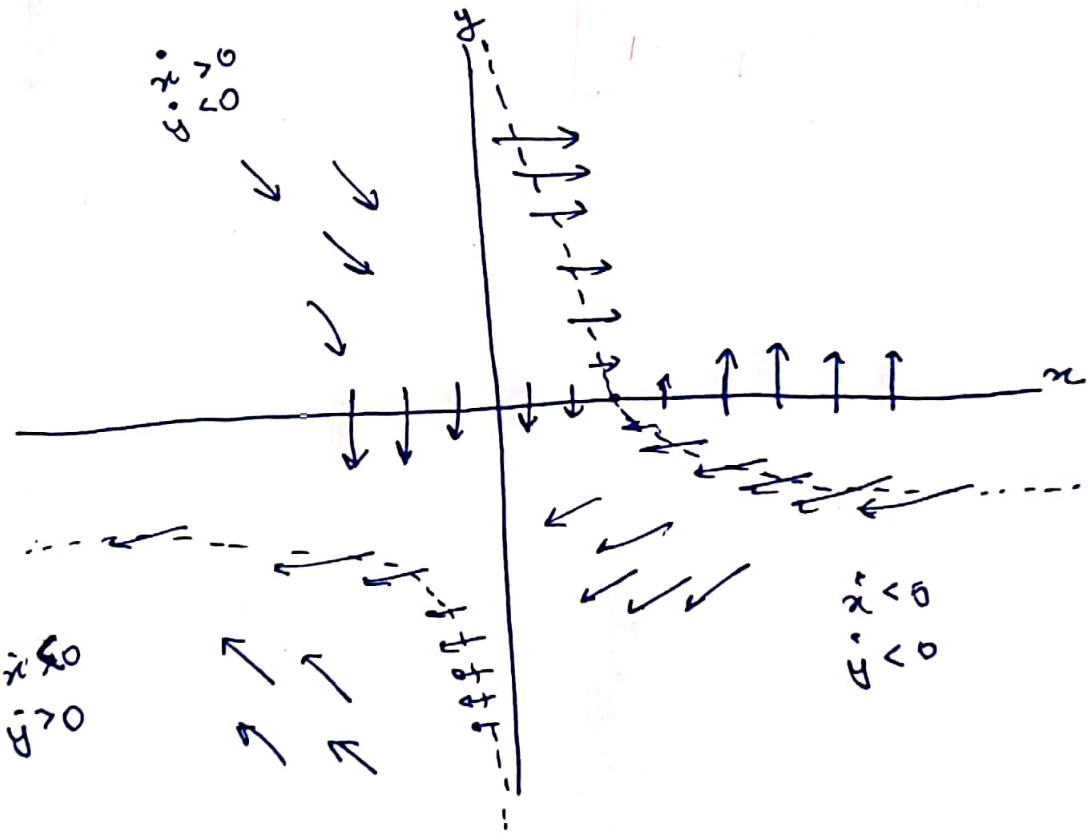
$y = 0$

$\left(\begin{array}{l} \dot{x} = 0 \\ \dot{y} = x - 1 \end{array}\right)$

$\dot{y} = 0$

$x(1+y) = 1$

$\left(\begin{array}{l} \dot{x} = y \\ \dot{y} = 0 \end{array}\right)$



Ans-6- $\dot{x} = x^2 - y = f(x,y)$

$\dot{y} = x - y = g(x,y)$

$\dot{x} = 0 \qquad \dot{y} = 0$

$x^2 = y \qquad x = y$

Fixed pts. $(0,0)$ $(1,1)$

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & -1 \\ 1 & -1 \end{bmatrix}$$

$$J_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$\tau = -1$, $\Delta = 1$

$\lambda^2 + \lambda + 1 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

(Stable spiral $(0,0)$)

$$J_{(1,1)} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$$

$\tau = 1$, $\Delta = -2 + 1 = -1$

(saddle point $(1,1)$)

Nullclines:

$\dot{x} = 0$

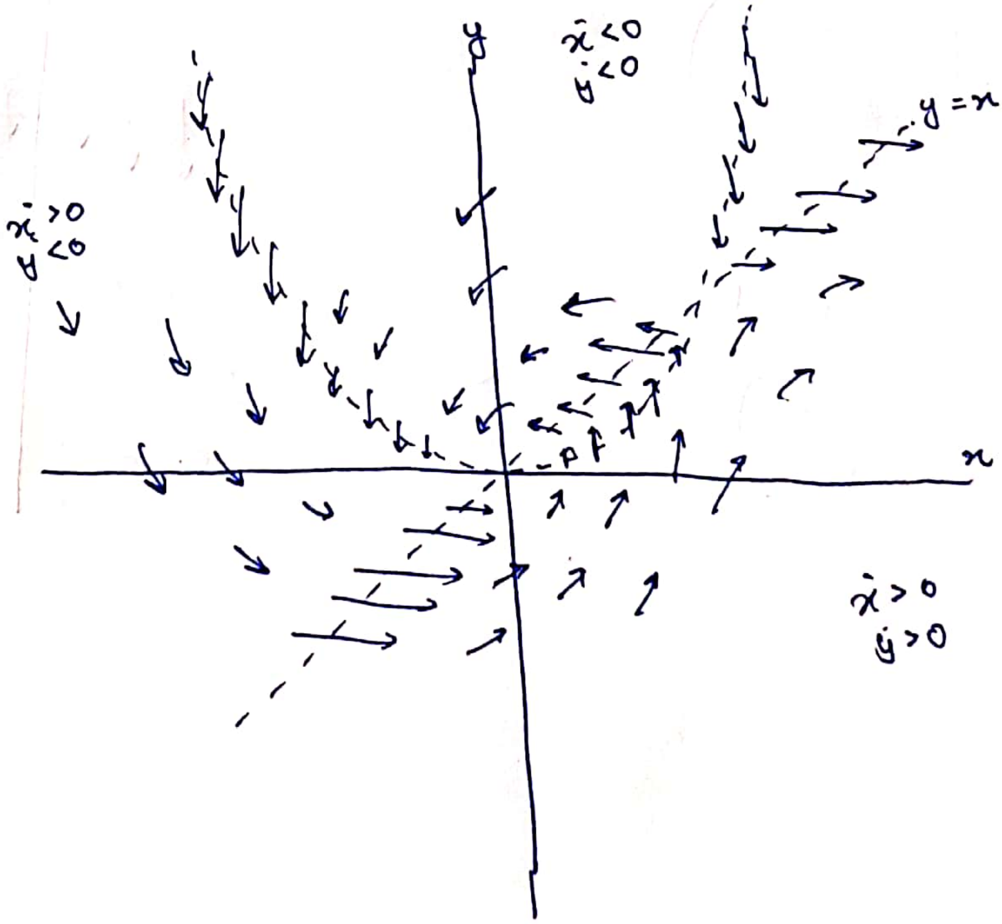
$x^2 = y$

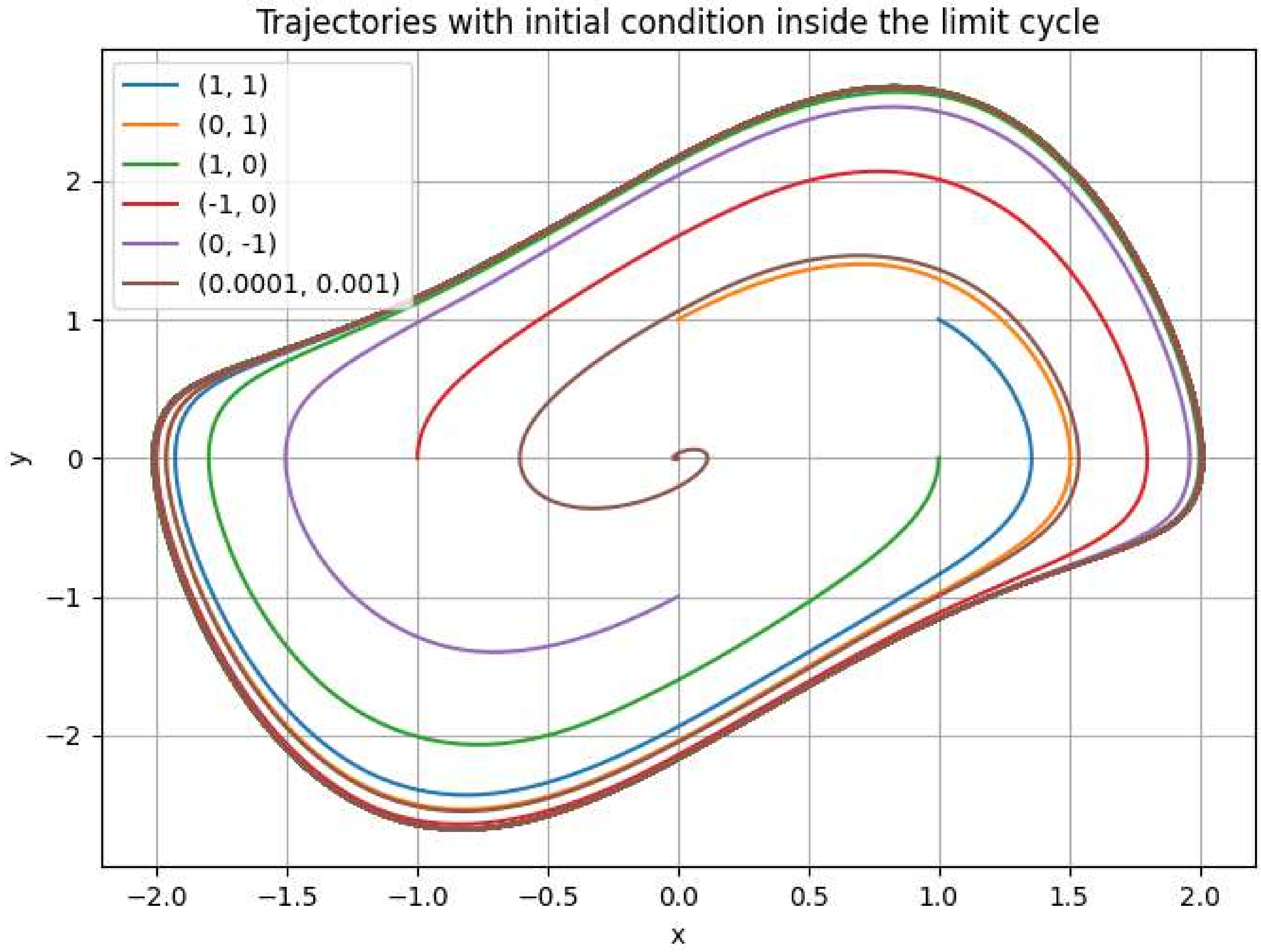
$\left(\begin{array}{l} \dot{x} = 0 \\ \dot{y} = x - x^2 \end{array} \right)$

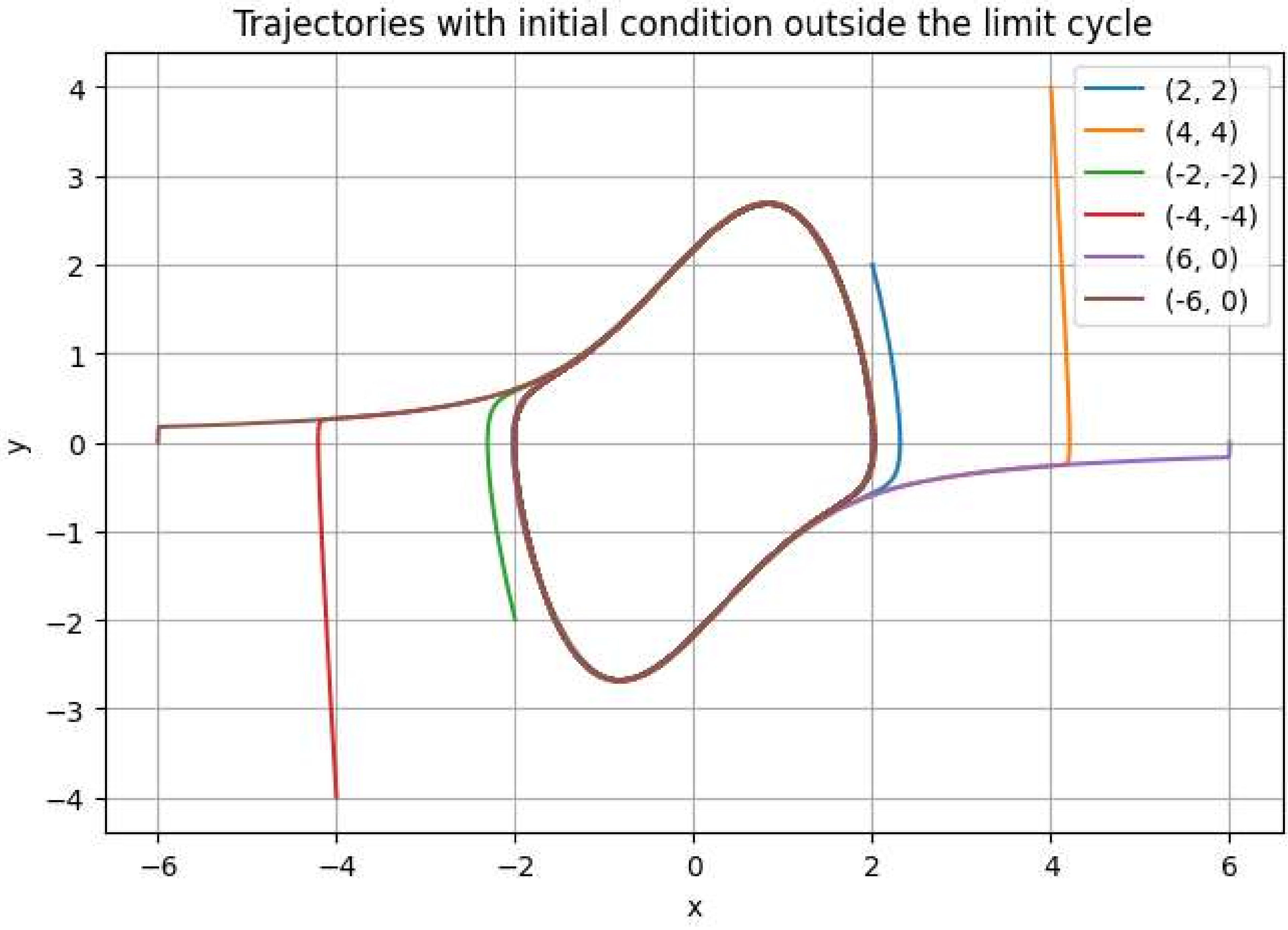
$\dot{y} = 0$

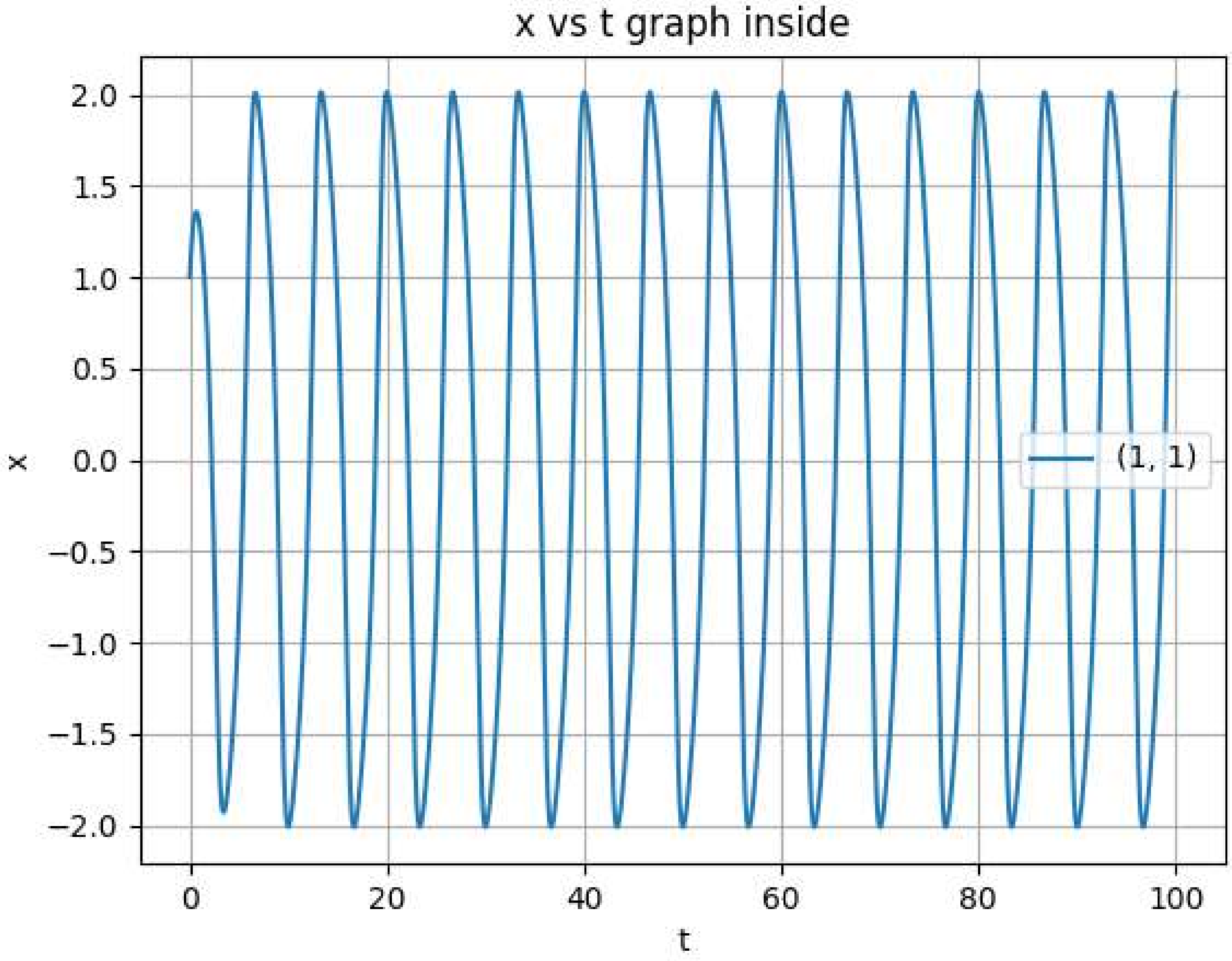
$x = y$

$\left(\begin{array}{l} \dot{x} = x^2 - x \\ \dot{y} = 0 \end{array} \right)$

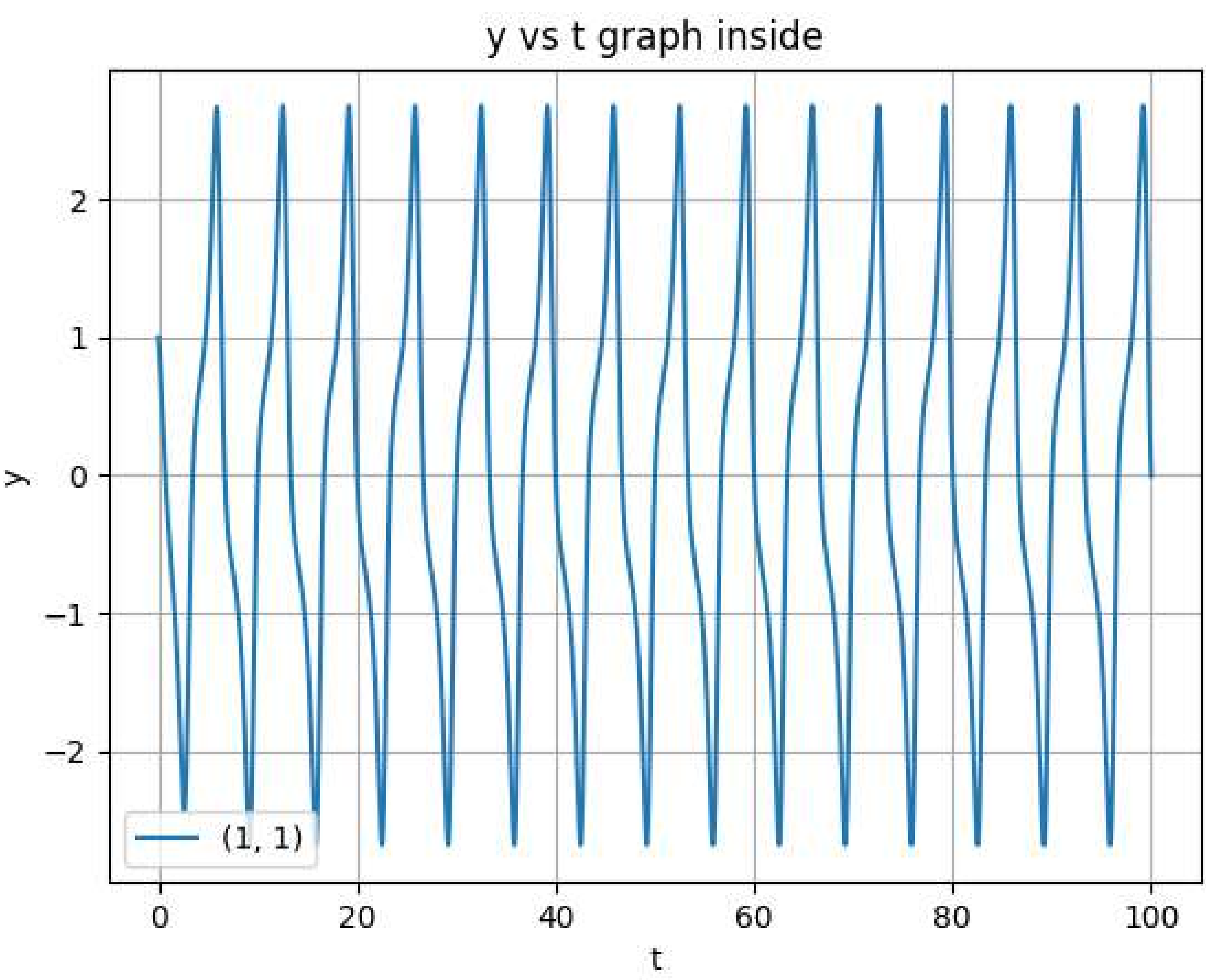


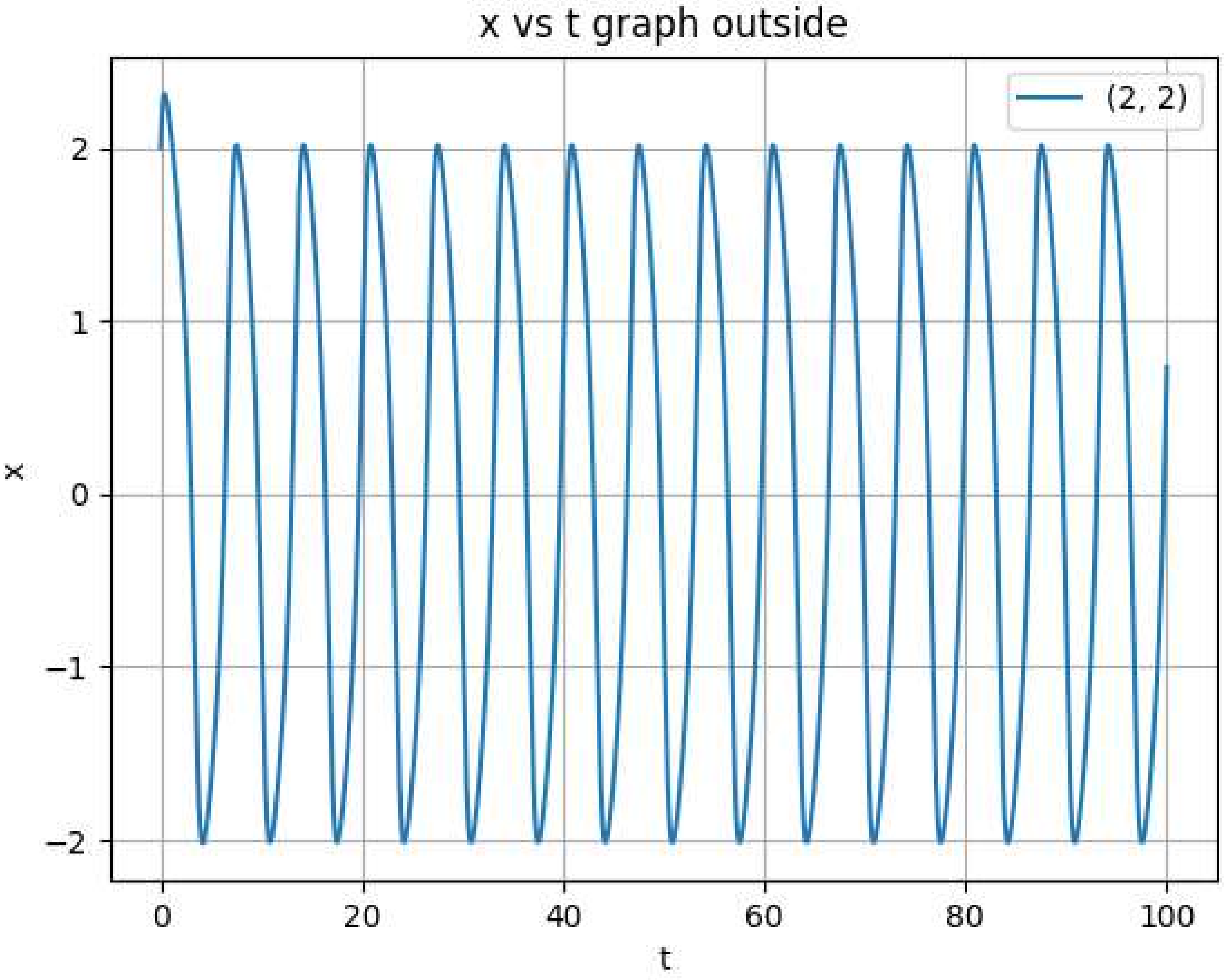

Trajectories with initial condition inside the limit cycle
(1, 1)
(0, 1)
(1, 0)
(-1, 0)
(0, -1)
(0.0001, 0.001)
y
x
2
1
0
−1
−2
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

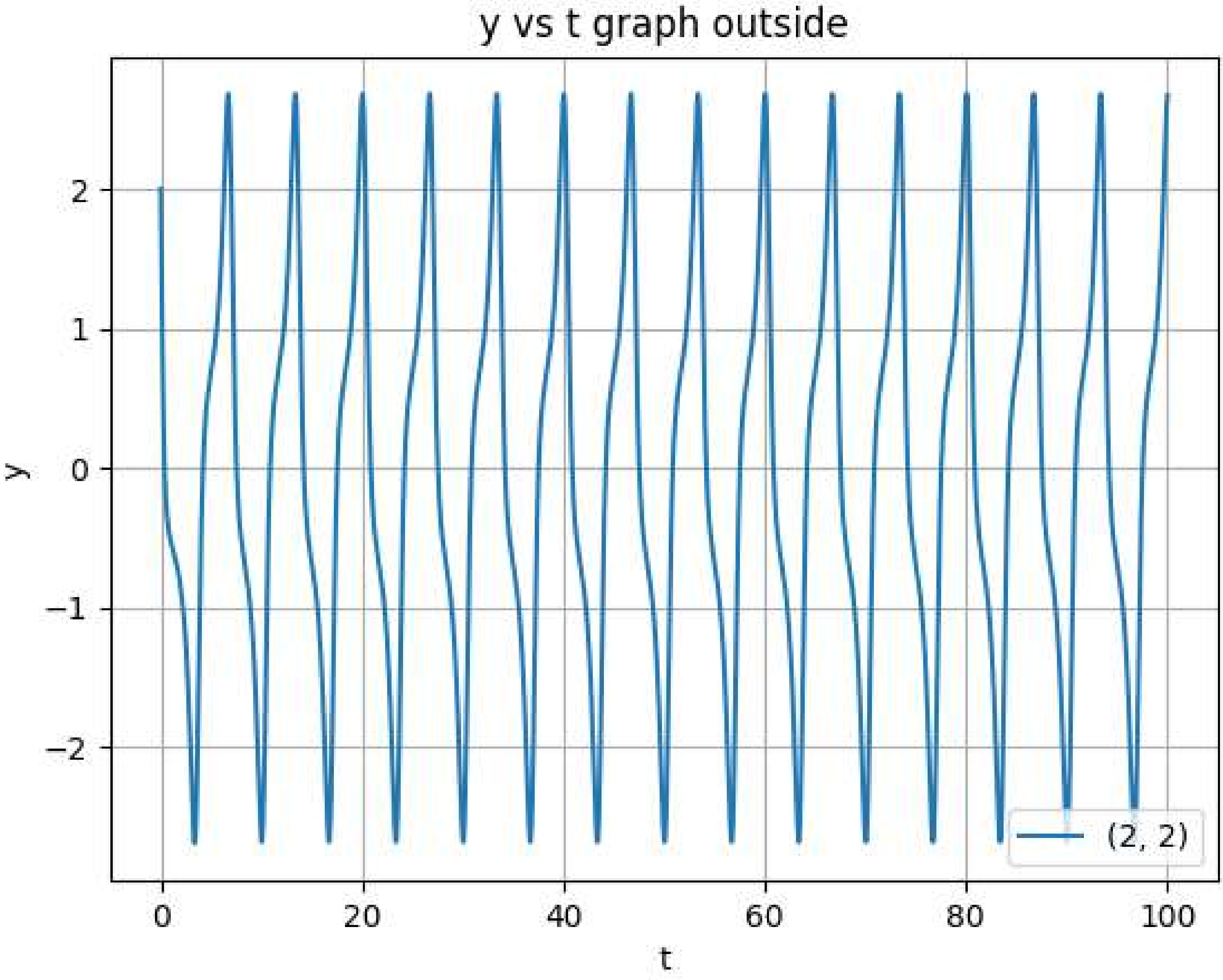

Trajectories with initial condition outside the limit cycle
(2, 2)
(4, 4)
(-2, -2)
(-4, -4)
(6, 0)
(-6, 0)
x
y
−6
−4
−2
0
2
4
6
−4
−3
−2
−1
0
1
2
3
4



x vs t graph inside
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
x
0
20
40
60
80
100
t
(1, 1)


y vs t graph inside
y
t
2
1
0
−1
−2
0
20
40
60
80
100
(1, 1)


x vs t graph outside
(2, 2)
x
t
2
1
0
−1
−2
0
20
40
60
80
100



y vs t graph outside
y
t
2
1
0
−1
−2
0
20
40
60
80
100
(2, 2)

```python
"""Vander pol oscillator"""
import numpy as np
import matplotlib.pyplot as plt
from scipy import integrate


"""Function for calculating the vector field at different x and y"""
def f(t, r):
    x,y = r
    fx = y
    gx = -x + y * (1 - x**2)
    return fx, gx

"""Time span for which the tragectory is calculated"""
tspan = np.linspace(0, 100, 5000)

"""Solving with integration method RK45 (default)"""
def solve(r0):
    return integrate.solve_ivp(f, [tspan[0], tspan[-1]], r0, t_eval=tspan)

"""Trajectories with initial condition inside the limit cycle"""
def inside_limit_cycle():
    initial_conditions = [(1, 1), (0, 1), (1, 0), (-1, 0), (0, -1), (0.0001, 0.001)]

    for initial_condition in initial_conditions:
        graph = solve(initial_condition)
        x,y = graph.y
        t = graph.t
        plt.plot(x, y, label='{}'.format(initial_condition))

    plt.xlabel('x')
    plt.ylabel('y')
    plt.title('Trajectories with initial condition inside the limit cycle')

    plt.legend()
    plt.grid(True)
    plt.show()

def outside_limit_cycle():
    initial_conditions = [(2, 2), (4, 4), (-2, -2), (-4, -4), (6, 0), (-6, 0) ]

    for initial_condition in initial_conditions:
        graph = solve(initial_condition)
        x,y = graph.y
        t = graph.t
        plt.plot(x, y, label='{}'.format(initial_condition))

    plt.xlabel('x')
    plt.ylabel('y')
    plt.title('Trajectories with initial condition outside the limit cycle')

    plt.legend()
    plt.grid(True)
    plt.show()

def x_vs_t_inside():
    initial_conditions = [(1, 1)]

    for initial_condition in initial_conditions:
        graph = solve(initial_condition)
        x,y = graph.y
        t = graph.t
        plt.plot(t, x, label='{}'.format(initial_condition))

    plt.xlabel('t')
    plt.ylabel('x')
```

```python
    plt.title('x vs t graph')

    plt.legend()
    plt.grid(True)
    plt.show()

def y_vs_t_inside():
    initial_conditions = [(1, 1)]

    for initial_condition in initial_conditions:
        graph = solve(initial_condition)
        x,y = graph.y
        t = graph.t
        plt.plot(t, y, label='{}'.format(initial_condition))

    plt.xlabel('t')
    plt.ylabel('y')
    plt.title('y vs t graph')

    plt.legend()
    plt.grid(True)
    plt.show()

def x_vs_t_outside():
    initial_conditions = [(2, 2)]

    for initial_condition in initial_conditions:
        graph = solve(initial_condition)
        x,y = graph.y
        t = graph.t
        plt.plot(t, x, label='{}'.format(initial_condition))

    plt.xlabel('t')
    plt.ylabel('x')
    plt.title('x vs t graph')

    plt.legend()
    plt.grid(True)
    plt.show()

def y_vs_t_outside():
    initial_conditions = [(2, 2)]

    for initial_condition in initial_conditions:
        graph = solve(initial_condition)
        x,y = graph.y
        t = graph.t
        plt.plot(t, y, label='{}'.format(initial_condition))

    plt.xlabel('t')
    plt.ylabel('y')
    plt.title('y vs t graph')

    plt.legend()
    plt.grid(True)
    plt.show()



inside_limit_cycle()
outside_limit_cycle()
x_vs_t_inside()
y_vs_t_inside()
x_vs_t_outside()
y_vs_t_outside()
```