Quantum Error Correction

Sender — Alice

$\boxed{\text{Noisy channel}}$

Receiver — Bob

$\boxed{1\ 0\ 0\ 1\ 1\ 0\ 1\ 0}$ — Bits is sent ✓

1 0 0 1 0 0 1 0 — single bit error ✓

1 0 1 1 0 0 1 0

1 0 0 1 0 1 1 0 — multiple bit error ✓

Consecutive bits are corrupted — Burst error

Parity bit

$\boxed{1\,0\,1\,1\,0\,0\,1}\ 0$ — no. of 1 = 4

1 0 1 0 0 0 1 0 → not acceptable

Logical bits

$$\begin{cases} 0_L = 000 \\ 1_L = 111 \end{cases}$$

Suppose you get : 101 → then correct it to 111

Errors in quantum communication

Decoherence

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \longrightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$\boxed{\text{Bit flips}}$

continuous evolution of quantum states also introduce phase errors

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad ; \quad |\psi'\rangle = e^{i\alpha}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$$

Say, a phase perturbation converts it to

$$\boxed{|\psi'\rangle = \frac{|0\rangle + e^{i\alpha}|1\rangle}{\sqrt{2}}}$$

A msmt of σ_z will still give you $|0\rangle$ or $|1\rangle$ with probability $\frac{1}{2}$

But if you make a msmt of $\underline{\sigma_x}$ it will project $+1$ with probability

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|\psi'\rangle = \frac{|0\rangle + e^{i\alpha}|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\alpha}\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\, e^{i\alpha/2}\begin{pmatrix}e^{-i\alpha/2}\\e^{i\alpha/2}\end{pmatrix}$$

$$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad x = \pm 1$$

$$x = +1, \quad |x=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$P_{+1} = |+1\rangle\langle +1| = \frac{1}{2}\begin{pmatrix}1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

$$P_{+1}|\psi'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\alpha/2\\\cos\alpha/2\end{pmatrix} e^{i\alpha/2} = \underline{\underline{\cos\alpha/2\, e^{i\alpha/2}}}\ \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}$$

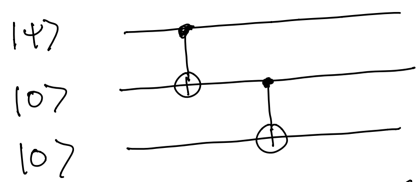
$$\boxed{\cos^2\frac{\alpha}{2}}$$

$0_L = 000$

Sender

Alice

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



Logical

$|\psi_L\rangle$

1st bit   2nd   3rd

$$|\psi\rangle_L = (\alpha|0\rangle + \beta|1\rangle)\,|0\rangle\,|0\rangle$$

$$= (\alpha|00\rangle + \beta|11\rangle)\,|0\rangle$$

$$\boxed{|\psi_L\rangle = \alpha|000\rangle + \beta|111\rangle}$$

Alice is sending ψ_L

Say, noise in the channel flips a bit with probability p and leaving it unchanged is $1-p$

The state received by Bob via the noisy channel is $|\psi_2\rangle_L$

We will have the following possibilities

Bob receives	Probability of occurrence
$\alpha|000\rangle + \beta|111\rangle$	$(1-p)^3$
$\alpha|100\rangle + \beta|011\rangle$	$p(1-p)^2$
$\alpha|010\rangle + \beta|101\rangle$	$p(1-p)^2$
$\beta|110\rangle$	$p(1-p)^2$

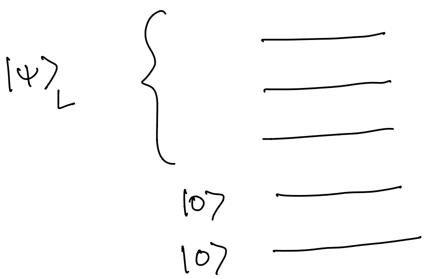
$\alpha\, |001\rangle + \beta|110\rangle$ Total $= 3p(1-p)^2$

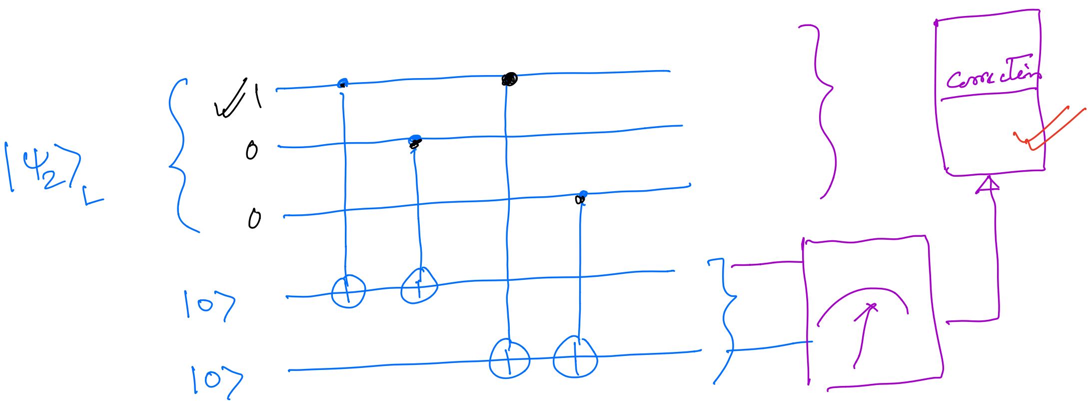
$\alpha\, |110\rangle + \beta\, |001\rangle$

$\alpha\, |101\rangle + \beta\, |010\rangle$ $\quad p^2(1-p)$

$\alpha\, |011\rangle + \beta\, |100\rangle$ Total $= 3p^2(1-p)$

$\alpha\, |111\rangle + \beta\, |000\rangle$ $\quad p^3$





$$\left(\alpha|000\rangle + \beta|111\rangle\right)|00\rangle = \left(\alpha|000\rangle + \beta|111\rangle\right)|00\rangle$$

$\Rightarrow$ ancilla bits are not affected at all.

$$|\psi_2\rangle_L = \left(\alpha\,|100\rangle + \beta\,|011\rangle\right)|00\rangle$$

$=$

Bob's action results in $|\psi_3\rangle_L$

State received $|\psi_2\rangle_L$	After Bob's gate operation $|\psi_3\rangle_L$	Probability
$\alpha|000\rangle+\beta|111\rangle$	$(\alpha|000\rangle+\beta|111\rangle)\,|00\rangle$	$(1-p)^3$
$\alpha|100\rangle+\beta|011\rangle$	$(\alpha|100\rangle+\beta|011\rangle)\,|11\rangle$	$p(1-p)^2$
$\alpha|010\rangle+\beta|101\rangle$	$(\alpha|010\rangle+\beta|101\rangle)\,|10\rangle$	$p(1-p)^2$
$\alpha|001\rangle+\beta|110\rangle$	$(\alpha|001\rangle+\beta|110\rangle)\,|01\rangle$	$p(1-p)^2$
$\alpha(|110\rangle+\beta|001\rangle)$	$(\alpha|110\rangle+\beta|011\rangle)\,|01\rangle$	$p^2(1-p)$
$\alpha|101\rangle+\beta|010\rangle$	——— $|10\rangle$	$p^2(1-p)$
$\alpha|011\rangle+\beta|100\rangle$	——— $|11\rangle$	$p^2(1-p)$
$\alpha|111\rangle+\beta|000\rangle$	——— $|00\rangle$	p^3

msmt of ancilla may yield

$|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$

Say Bob get $|00\rangle$

$\alpha|000\rangle+\beta|111\rangle$ with prob. $(1-p)^3$ (without errors)

$\alpha|111\rangle+\beta|000\rangle$ with prob. p^3 (errors)

$\boxed{\text{Bob take's no action}}$

Now, Say Bob gets $|01\rangle$

$\alpha|001\rangle+\beta|110\rangle$ with prob. $p(1-p)^2$ (with one error)

$\alpha|110\rangle+\beta|001\rangle$ with prob. $p^2(1-p)$ (with two errors)

H.O.

Then

Bob applies a σ_x on the third qubit

$$\sigma_x \xrightarrow{3rd} \left(\alpha|001\rangle + \beta|110\rangle\right) \longrightarrow \underline{\underline{\alpha|000\rangle + \beta|111\rangle}}$$

$$\sigma_x \xrightarrow{3rd} \left(\alpha|110\rangle + \beta|001\rangle\right) \longrightarrow \underline{\underline{\alpha|111\rangle + \beta|000\rangle}}$$

One is getting corrected fully while the other one is fully getting wrong!