Mathematics III (RMA3A001)

Module I

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Lecture - 6

Numerical Solutions of System of Linear Equations

- Simultaneous linear equations have great importance in the field of engineering and science.
- In the field of science the analysis of electronic circuits having number of invariant elements, analysis of a network under sinusoidal steady stady state conditions and determination of output of a chemical plant are some of the problems which depends on the solution of system of linear algebraic equations.
- We shall discuss the solution of system of m linear equations with n unknowns where $m = n$ by different iterative and direct methods.

- Some of the iterative methods are
  - Gauss seidel method
  - Successive over relaxation (SOR) method
- Some of the direct methods are
  - Doolittle's method
  - Crout's method
  - Cholesky's method

- Let us explain the Gauss seidel method in the case of three linear equations with three unknowns. Similarly we can extend the method into n linear equations with n unknowns.
- Consider the system of equations

$$a_1x + b_1y + c_1z = d_1 \tag{1}$$

$$a_2x + b_2y + c_2z = d_2 \tag{2}$$

$$a_3x + b_3y + c_3z = d_3 \tag{3}$$

- Verify that,

$$|a_1| > |b_1| + |c_1|$$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

- Then the Gauss Seidel iterative method can be used for the given system. Solving equation (1), (2) and (3) for x, y and z respectively we get

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

- Now we can start the solution process with initial values $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ for x, y and z respectively.

- We calculate

$$x^{(1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(0)} - c_1 z^{(0)}\right)$$

$$y^{(1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(1)} - c_2 z^{(0)}\right)$$

$$z^{(1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(1)} - b_3 y^{(1)}\right)$$

- Thus as soon as a new values for a variable is found, it is used immediately in the following equations.
- If $x^{(k)}$, $y^{(k)}$, $z^{(k)}$ are the k^{th} iterates then the $(k+1)^{th}$ iterates will be

- $$x^{(k+1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(k)} - c_1 z^{(k)}\right)$$
$$y^{(k+1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(k+1)} - c_2 z^{(k)}\right)$$
$$z^{(k+1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(k+1)} - b_3 y^{(k+1)}\right)$$
$$k = 0,\ 1,\ 2,\ \ldots.$$
- The process is continued until the convergence is assured.
- **NOTE :** In the absence of initial approximation $x^{(0)},\ y^{(0)},\ z^{(0)}$, they are taken as $(0, 0, 0)$.

Example 1

Solve the following system of linear equations by using Gauss seidel iterative method correct up to three decimal places.

$$10x - 5y - 2z = 3$$

$$x + 6y + 10z = -3$$

$$4x - 10y + 3z = -3$$

Solution : The above system of equations can be rewrite as

$$10x - 5y - 2z = 3 \tag{4}$$

$$4x - 10y + 3z = -3 \tag{5}$$

$$x + 6y + 10z = -3 \tag{6}$$

It can be easily verified that,

$$|10| > |-5| + |-2|$$

$$|-10| > |4| + |3|$$

$$|10| > |1| + |6|$$

We are from the equation (4), (5) and (6)

$$x = \frac{1}{10}(3 + 5y + 2z)$$

$$y = \frac{1}{10}(3 + 4x + 3z)$$

$$z = -\frac{1}{10}(3 + x + 6y)$$

Thus the iteration scheme for Gauss seidel method is

$$x^{(k+1)} = \frac{1}{10}\left(3 + 5y^{(k)} + 2z^{(k)}\right)$$

$$y^{(k+1)} = \frac{1}{10}\left(3 + 4x^{(k+1)} + 3z^{(k)}\right)$$

$$z^{(k+1)} = -\frac{1}{10}\left(3 + x^{(k+1)} + 6y^{(k+1)}\right)$$

$$k = 0,\ 1,\ 2,\ \ldots .$$

Let $x^0 = y^0 = z^0 = 0$ be the initial approximation to the solution.

First approximation $(k=0)$:

$$x^{(1)}=\frac{1}{10}\left(3+5y^{(0)}+2z^{(0)}\right)=\frac{1}{10}(3+5\times 0+2\times 0)=0.3$$

$$y^{(1)}=\frac{1}{10}\left(3+4x^{(1)}+3z^{(0)}\right)=\frac{1}{10}(3+4\times 0.3+3\times 0)=0.42$$

$$z^{(1)}=-\frac{1}{10}\left(3+x^{(1)}+6y^{(1)}\right)=-\frac{1}{10}(3+0.3+6\times 0.42)=-0.582$$

Second approximation $(k=1)$:

$$x^{(2)}=\frac{1}{10}\left(3+5y^{(1)}+2z^{(1)}\right)=\frac{1}{10}\left(3+5\times 0.42+2\times(-0.582)\right)=0.3936$$

$$y^{(2)}=\frac{1}{10}\left(3+4x^{(2)}+3z^{(1)}\right)=\frac{1}{10}\left(3+4\times 0.3936+3\times(-0.582)\right)=0.28284$$

$$z^{(2)}=-\frac{1}{10}\left(3+x^{(2)}+6y^{(2)}\right)=-\frac{1}{10}(3+0.3936+6\times 0.28284)=-0.509064$$

Third approximation $(k = 2)$:

$$x^{(3)} = \frac{1}{10}\left(3 + 5y^{(2)} + 2z^{(2)}\right) = \frac{1}{10}\left(3 + 5 \times 0.28284 + 2 \times (-0.509064)\right) = 0.3396072$$

$$y^{(3)} = \frac{1}{10}\left(3 + 4x^{(3)} + 3z^{(2)}\right) = \frac{1}{10}\left(3 + 4 \times 0.3396072 + 3 \times (-0.509064)\right) = 0.283123$$

$$z^{(3)} = -\frac{1}{10}\left(3 + x^{(3)} + 6y^{(3)}\right) = -\frac{1}{10}(3 + 0.3396072 + 6 \times 0.283123) = -0.5038345$$

Fourth approximation $(k = 3)$:

$$x^{(4)} = \frac{1}{10}\left(3 + 5y^{(3)} + 2z^{(3)}\right) = \frac{1}{10}\left(3 + 5 \times 0.283123 + 2 \times (-0.5038345)\right) = 0.340794$$

$$y^{(4)} = \frac{1}{10}\left(3 + 4x^{(4)} + 3z^{(3)}\right) = \frac{1}{10}\left(3 + 4 \times 0.340794 + 3 \times (-0.5038345)\right) = 0.285167$$

$$z^{(4)} = -\frac{1}{10}\left(3 + x^{(4)} + 6y^{(4)}\right) = -\frac{1}{10}(3 + 0.340794 + 6 \times 0.285167) = -0.505179$$

Fifth approximation $(k = 4)$:

$$x^{(5)} = \frac{1}{10}\left(3 + 5y^{(4)} + 2z^{(4)}\right) = \frac{1}{10}\left(3 + 5 \times 0.285167 + 2 \times (-0.505179)\right) = 0.341548$$

$$y^{(5)} = \frac{1}{10}\left(3 + 4x^{(5)} + 3z^{(4)}\right) = \frac{1}{10}\left(3 + 4 \times 0.341548 + 3 \times (-0.505179)\right) = 0.285065$$

$$z^{(5)} = -\frac{1}{10}\left(3 + x^{(5)} + 6y^{(5)}\right) = -\frac{1}{10}\left(3 + 0.341548 + 6 \times 0.285065\right) = -0.505193$$

Sixth approximation $(k = 5)$:

$$x^{(6)} = \frac{1}{10}\left(3 + 5y^{(5)} + 2z^{(5)}\right) = \frac{1}{10}\left(3 + 5 \times 0.285065 + 2 \times (-0.505193)\right) = 0.341493$$

$$y^{(6)} = \frac{1}{10}\left(3 + 4x^{(6)} + 3z^{(5)}\right) = \frac{1}{10}\left(3 + 4 \times 0.341493 + 3 \times (-0.505193)\right) = 0.285039$$

$$z^{(6)} = -\frac{1}{10}\left(3 + x^{(6)} + 6y^{(6)}\right) = -\frac{1}{10}\left(3 + 0.341493 + 6 \times 0.285039\right) = -0.505172$$

Thus the solution of the above system of linear equations by Gauss Seidel iteration method correct up to three decimal point each $x = 0.341$, $y = 0.285$, $z = -0.505$

Example 2

Solve the following system of linear equations by using Gauss seidel iterative method by taking the initial approximations as $x = 2$, $y = 1$ and $z = 1$.

$$
\begin{aligned}
4x + 11y - z &= 33 \\
6x + 3y + 12z &= 35 \\
8x - 3y + 2z &= 10
\end{aligned}
$$

Solution : The above system of equations can be rewrite as

$$8x - 3y + 2z = 10 \tag{7}$$

$$4x + 11y - z = 33 \tag{8}$$

$$6x + 3y + 12z = 35 \tag{9}$$

It can be easily verified that,

$$
\begin{aligned}
|8| &> |-3| + |2| \\
|11| &> |4| + |-1| \\
|12| &> |6| + |3|
\end{aligned}
$$

We are from the equation (7), (8) and (9)

$$x = \frac{1}{8}(10 + 3y - 2z)$$
$$y = \frac{1}{11}(33 - 4x + z)$$
$$z = \frac{1}{12}(35 - 6x - 3y)$$

Thus the iteration scheme for Gauss seidel method is

$$x^{(k+1)} = \frac{1}{8}\left(10 + 3y^{(k)} - 2z^{(k)}\right)$$

$$y^{(k+1)} = \frac{1}{11}\left(33 - 4x^{(k+1)} + z^{(k)}\right)$$

$$z^{(k+1)} = \frac{1}{12}\left(35 - 6x^{(k+1)} - 3y^{(k+1)}\right)$$

$$k = 0,\ 1,\ 2,\ \ldots .$$

Given that $x^{(0)} = 2,\ y^{(0)} = 1$ and $z^{(0)} = 0$ is the initial approximation to the solution.

First approximation $(k = 0)$:

$$x^{(1)} = \frac{1}{8}\left(10 + 3y^{(0)} - 2z^{(0)}\right) = \frac{1}{8}\,(10 + 3 \times 1 - 2 \times 1) = 1.375$$

$$y^{(1)} = \frac{1}{11}\left(33 - 4x^{(1)} + z^{(0)}\right) = \frac{1}{11}\,(33 - 4 \times 1.375 + 1 \times 1) = 2.590909$$

$$z^{(1)} = \frac{1}{12}\left(35 - 6x^{(1)} - 3y^{(1)}\right) = \frac{1}{12}\,(35 - 6 \times 1.375 - 3 \times 2.590909) = 1.581439$$

Second approximation $(k = 1)$:

$$x^{(2)} = \frac{1}{8}\left(10 + 3y^{(1)} - 2z^{(1)}\right) = \frac{1}{8}\left(10 + 3 \times 2.590909 - 2 \times (1.581439)\right) = 1.826231$$

$$y^{(2)} = \frac{1}{11}\left(33 - 4x^{(2)} + z^{(1)}\right) = \frac{1}{11}\left(33 - 4 \times 1.826231 + 1 \times (1.581439)\right) = 2.479683$$

$$z^{(2)} = -\frac{1}{12}\left(35 - 6x^{(2)} - 3y^{(2)}\right) = \frac{1}{12}\,(33 - 6 \times 1.826231 - 3 \times 2.47968) = 1.383630$$

Third approximation $(k = 2)$:

$$x^{(3)} = \frac{1}{8}\left(10 + 3y^{(2)} - 2z^{(2)}\right) = \frac{1}{8}\left(10 + 3 \times 2.479683 - 2 \times 1.383630\right) = 1.833973$$

$$y^{(3)} = \frac{1}{11}\left(33 - 4x^{(3)} + z^{(2)}\right) = \frac{1}{11}\left(33 - 4 \times 1.833973 + 1 \times 1.383630\right) = 2.458885$$

$$z^{(3)} = \frac{1}{12}\left(35 - 6x^{(3)} - 3y^{(3)}\right) = -\frac{1}{12}\left(35 - 6 \times 1.833973 - 3 \times 2.458885\right) = 1.384958$$

Fourth approximation $(k = 3)$:

$$x^{(4)} = \frac{1}{8}\left(10 + 3y^{(3)} - 2z^{(3)}\right) = \frac{1}{8}\left(10 + 3 \times 2.45885 - 2 \times 1.384958\right) = 1.825842$$

$$y^{(4)} = \frac{1}{11}\left(33 - 4x^{(4)} + z^{(3)}\right) = \frac{1}{11}\left(33 - 4 \times 1.825842 + 1 \times 1.384958\right) = 2.459919$$

$$z^{(4)} = \frac{1}{12}\left(35 - 6x^{(4)} - 3y^{(4)}\right) = -\frac{1}{12}\left(35 - 6 \times 1.825842 - 3 \times 2.459919\right) = 1.389747$$

Fifth approximation $(k = 4)$:

$$x^{(5)} = \frac{1}{8}\left(10 + 3y^{(4)} - 2z^{(4)}\right) = \frac{1}{8}(10 + 3 \times 2.459919 - 2 \times 1.389747) = 1.825032$$

$$y^{(5)} = \frac{1}{11}\left(33 - 4x^{(5)} + z^{(4)}\right) = \frac{1}{11}(33 - 4 \times 1.8255032 + 1 \times 1.389747) = 2.462692$$

$$z^{(5)} = \frac{1}{12}\left(35 - 6x^{(5)} - 3y^{(5)}\right) = -\frac{1}{12}(35 - 6 \times 1.825032 - 3 \times 2.462692) = 1.388477$$

Sixth approximation $(k = 5)$:

$$x^{(6)} = \frac{1}{8}\left(10 + 3y^{(5)} - 2z^{(5)}\right) = \frac{1}{8}(10 + 3 \times 2.462692 - 2 \times 1.388477) = 1.826390$$

$$y^{(6)} = \frac{1}{11}\left(33 - 4x^{(6)} + z^{(5)}\right) = \frac{1}{11}(33 - 4 \times 1.826390 + 1 \times 1.388477) = 2.462083$$

$$z^{(5)} = \frac{1}{12}\left(35 - 6x^{(5)} - 3y^{(5)}\right) = -\frac{1}{12}(35 - 6 \times 1.826390 - 3 \times 2.462083) = 1.387950$$

Thus the solution of the above system of linear equations by Gauss Seidel iteration method after sixth step is $x = 1.826390$, $y = 2.462083$, $z = 1.387950$

Any Questions?

Thank You