Mathematics III (RMA3A001)

Module I

Ramesh Chandra Samal

Department of Mathematics
Ajay Binay Institute of Technology
Cuttack, Odisha

Lecture - 14

Lagrange's interpolation formula

Let $y = f(x)$ be a polynomial in x which takes the values $y_0 = f(x_0)$, $y_1 = f(x_1)$, ... $y_n = f(x_n)$ corresponding to $x_0, x_1, x_2, \ldots\ldots\ldots\ldots x_n$. There are $(n+1)$ values of $f(x)$. So $(n+1)^{th}$ difference is zero. Thus $f(x)$ is a polynomial in x of degree n. Let this polynomial be

$$\begin{aligned} y = f(x) \;=\; & a_0(x-x_1)(x-x_2)\ldots\ldots(x-x_n) \\ & + a_1(x-x_0)(x-x_2)\ldots\ldots(x-x_n) \\ & + a_2(x-x_0)(x-x_1)(x-x_3)\ldots\ldots(x-x_n) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + a_n(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1}) \end{aligned} \tag{1}$$

where $a_0, a_1, a_2, \ldots\ldots a_n$ are the constants can be determined by putting $y = y_0$ at $x = x_0$, $y = y_1$ at $x = x_1, \ldots\ldots\ldots\ldots\ldots$ etc.

Now putting $y = y_0$ and $x = x_0$ in equation (1), we get

$$y_0 = a_0(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)$$

$$\implies a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}$$

Similarly putting $y = y_1$ and $x = x_1$ in equation (1), we get

$$\begin{aligned} y_1 =& a_1(x_1 - x_0)(x_1 - x_2)\ \ldots\ldots\ (x_1 - x_n) \\ \implies a_0 =& \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ \ldots\ldots\ (x_1 - x_n)} \end{aligned}$$

Proceeding in this way, we get $a_2, a_3, a_4, \ \ldots\ldots\ a_n$. Now putting all the values of $a_0, a_1, a_2, \ \ldots\ldots\ a_n$ into equation (1) we get,

$$\begin{aligned} f(x) =& \frac{(x - x_1)(x - x_2)\ \ldots\ldots\ (x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ \ldots\ldots\ (x_0 - x_n)} y_0 \\ &+ \frac{(x - x_0)(x - x_2)\ \ldots\ldots\ (x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ \ldots\ldots\ (x_1 - x_n)} y_1 \\ &+ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \frac{(x - x_0)(x - x_1)\ \ldots\ldots\ (x - x_{n-1})}{(x_{n-1} - x_0)(x_{n-1} - x_2)\ \ldots\ldots\ (x_{n-1} - x_n)} y_n \end{aligned}$$

is known as Lagranges interpolating polynomial of degree n.

Example 1

Find a unique polynomial of degree two or less by using Lagranges interpolation given that, $f(0) = 1,\ f(1) = 3,\ f(3) = 55$.

Solution :

Here $x_0 = 0,\ x_1 = 1,\ x_2 = 3$ and $y_0 = f(x_0) = 1,\ y_1 = f(x_1) = 3,\ y_2 = f(x_2) = 55$

By Lagranges interpolation

$$
\begin{aligned}
f(x) =& \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2 \\
=& \frac{(x-1)(x-3)}{(0-1)(0-3)}(1) + \frac{(x-0)(x-3)}{(1-0)(1-3)}(3) + \frac{(x-0)(x-1)}{(3-0)(3-1)}(55) \\
=& \frac{1}{3}(x-1)(x-3) - \frac{3}{2}x(x-3) + \frac{55}{6}x(x-1) \\
=& 8x^2 - 6x + 1
\end{aligned}
$$

Example 2

Find the $f(4)$ from the following table by using Lagranges interpolation

x :	0	1	2	5
$f(x)$:	2	5	7	8

Solution :

Here $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 5$

and $y_0 = f(x_0) = 2,\ y_1 = f(x_1) = 5,\ y_2 = f(x_2) = 7,\ y_3 = f(x_3) = 8$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-3)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(5) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(7) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(8)$$

$$
\begin{aligned}
\implies f(4) =& \frac{(4-1)(4-2)(4-5)}{(0-1)(0-3)(0-5)}(2) + \frac{(4-0)(4-2)(4-5)}{(1-0)(1-2)(1-5)}(5) + \\
& \frac{(4-0)(4-1)(4-5)}{(2-0)(2-1)(2-5)}(7) + \frac{(4-0)(4-1)(4-2)}{(5-0)(5-1)(5-2)}(8) \\
=& 1.2 - 10 + 14 + 3.2 \\
=& 8.4
\end{aligned}
$$

Any Questions?

Thank You