Mathematics III (RMA3A001)

Module I

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Lecture - 13

Let $f(x_0),\ f(x_{1)},\ f(x_2),\ .\ .\ .\ .\ .\ .\ .\ f(x_n)$ be the values of the function $f(x)$ corresponding the points $x_0,\ x_1,\ .\ .\ .\ .\ .\ .\ .\ .\ x_n$ not equally spaced.
Then we define the first divided difference of $f(x)$ between x_0 and x_1 as follows.

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The second divided difference of $f(x)$ between $x_0,\ x_1$ and x_2 as follows.

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Where

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

In similar way, the n^{th} divided difference is given by

$$f[x_0, x_1,\ .\ .\ .\ .\ .\ .\ x_{n-1},\ x_n] = \frac{f[x_1, x_2,\ .\ .\ .\ .\ x_n] - f[x_0, x_1,\ .\ .\ .\ .\ x_{n-1}]}{x_n - x_0}$$

Newton's divided difference formula

Let $f(x_0),\ f(x_{1)},\ f(x_2),\ ....\ f(x_n)$ be the values of the function $f(x)$ at the points $x_0,\ x_1,\ ....\ x_n$ respectively which are not equally spaced.
By the definition of divided difference, the first divided difference of $f(x)$ is given by

$$f[x, x_0] = \frac{f(x_0) - f(x)}{x_0 - x}$$

$$\implies f(x) = f(x_0) + (x - x_0) f[x, x_0] \tag{1}$$

Again the second divided difference is given by

$$f[x, x_0, x_1] = \frac{f[x_0, x_1] - f[x, x_0]}{x_1 - x}$$

$$\implies f[x, x_0] = f[x_0, x_1] + (x - x_1) f[x, x_0, x_1]$$

$$\implies \frac{f(x) - f(x_0)}{x - x_0} = f[x_0, x_1] + (x - x_1) f[x, x_0, x_1]$$

$$\implies f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x, x_0, x_1] \tag{2}$$

Similarly,

$$
\begin{aligned}
f(x) =& f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\
&+ (x - x_0)(x - x_1)(x - x_2) f[x, x_0, x_1, x_2]
\end{aligned}
\tag{3}
$$

Proceeding in the same way

$$
\begin{aligned}
f(x) =& f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \ \ldots\ldots \\
&+ (x - x_0)(x - x_1)(x - x_2) \ldots\ldots (x - x_{n-1}) f[x_0, x_1, \ \ldots\ldots x_n] \\
&+ (x - x_0)(x - x_1)(x - x_2) \ldots\ldots (x - x_n) f[x, x_0, x_1, \ \ldots\ldots x_n]
\end{aligned}
\tag{4}
$$

Since, $f(x)$ is a polynomial of degree n so

$$
f[x, x_0, x_1, \ \ldots\ldots x_n] = 0
$$

Thus the equation (4) becomes

$$
\begin{aligned}
f(x) =& f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \ \ldots\ldots \\
&+ (x - x_0)(x - x_1)(x - x_2) \ldots\ldots (x - x_{n-1}) f[x_0, x_1, \ \ldots\ldots x_n]
\end{aligned}
$$

This formula is known as Newtons divided difference interpolation formula.

Example 1

Find the unique polynomial of fegree two or less such that $f(0) = 1,\ f(1) = 3,\ f(3) = 55$ by using Newtons divided difference interpolation.

Solution :

Here, $x_0 = 0,\ x_1 = 1,\ x_2 = 3$

and $f(x_0) = 1,\ f(x_1) = 3,\ f(x_2) = 55$

By Newtons divided difference interpolation

$$f(x) = P(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{3-1}{1-0} = 2$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{55-3}{3-1} = 26$$

$$f[x_0, x_1, x_1] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{26-2}{3-0} = 8$$

x	$f(x)$				
0	1				
1	3	2	$f[x_0, x_1]$		
3	55	26		8	$f[x_0, x_1, x_2]$

Putting these values in equation (1) we get,

$$f(x) = P(x) = 1 + (x - 0)2 + (x - 0)(x - 1)8$$
$$= 8x^2 - 6x + 1$$

Example 2

Use Newton divided difference formula to calculate $f(3)$ from the following table

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Solution :

Here $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 5,\ x_5 = 6$

and $f(x_0) = 1,\ f(x_1) = 14,\ f(x_2) = 15,\ f(x_3) = 5,\ f(x_4) = 6,\ f(x_5) = 19$

We have from the Newtons divided difference interpolation formula

$$\begin{aligned} f(x) = P(x) = & f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ & + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] \\ & + (x - x_0)(x - x_1)(x - x_2)(x - x_3) f[x_0, x_1, x_2, x_3, x_4] \\ & + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4) f[x_0, x_1, x_2, x_3, x_4, x_5] \end{aligned} \tag{5}$$

Divided difference table

x	$f(x)$					
0	1					
1	14	13	$f[x_0, x_1]$			
2	15	1	-6	$f[x_0, x_1, x_2]$		
4	5	-5	-2	1	$f[x_0, x_1, x_2, x_3]$	
5	6	1	2	1	0	$f[x_0, x_1, x_2, x_3, x_4]$
6	19	13	6	1	0	0
						$f[x_0, x_1, x_2, x_3, x_4, x_5]$

Putting the values in equation (5)

$$\begin{aligned} f(x) = P(x) =& 1 + (x-0)13 + (x-0)(x-1)(6) + (x-0)(x-1)(x-2)(1) \\ &+ (x-0)(x-1)(x-2)(x-4)(0) + (x-0)(x-1)(x-2)(x-4)(x \\ =& 1 + 13x - 6x(x-1) + x(x-1(x-2)) \end{aligned}$$

Now

$$\begin{aligned} f(3) = P(3) =& 1 + 39 - 18 \times 2 + 3(3-1)(3-2) \\ =& 1 + 39 - 36 + 6 \\ =& 10 \end{aligned}$$

Any Questions?

Thank You