Mathematics III (RMA3A001)

Module I

Ramesh Chandra Samal

Department of Mathematics
Ajay Binay Institute of Technology
Cuttack, Odisha

Lecture - 7

Successive over relaxation (SOR) method

- Let us explain the SOR method in the case of three linear equations with three unknowns with relaxation parameter ω.
- Similarly we can extend the method into n linear equations with n unknowns.
- Consider the system of equations

$$a_1x + b_1y + c_1z = d_1 \tag{1}$$

$$a_2x + b_2y + c_2z = d_2 \tag{2}$$

$$a_3x + b_3y + c_3z = d_3 \tag{3}$$

- In matrix form the above system of equations can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

STEP I : Verify the sufficient condition for the SOR method i.e the coefficient matrix $\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is symmetric positive definite.

- For symmetric verify $\mathbf{A}^\mathbf{T} = \mathbf{A}$.
- For positive definite show that

$$\begin{vmatrix} a_1 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} > 0$$

STEP II : Verify that the relaxation parameter ω satisfies $0 < \omega < 2$.

STEP III : Write down the Gauss seidal iteration scheme for the above system of equations.

$$x^{(k+1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(k)} - c_1 z^{(k)}\right) \tag{4}$$

$$y^{(k+1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(k+1)} - c_2 z^{(k)}\right) \tag{5}$$

$$z^{(k+1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(k+1)} - b_3 y^{(k+1)}\right) \tag{6}$$

STEP IV : Multiply the RHS of the equation (4), (5) and (6) by ω and adding to the vectors $x^{(k)}$, $y^{(k)}$ and $z^{(k)}$ by multiplying $(1-\omega)$ respectively equation (4), (5) and (6) becomes

$$\begin{aligned}
x^{(k+1)} &= (1-\omega)x^{(k)} + \omega\frac{1}{a_1}\left(d_1 - b_1 y^{(k)} - c_1 z^{(k)}\right)\\
y^{(k+1)} &= (1-\omega)y^{(k)} + \omega\frac{1}{b_2}\left(d_2 - a_2 x^{(k)} - c_2 z^{(k)}\right)\\
z^{(k+1)} &= (1-\omega)z^{(k)} + \omega\frac{1}{c_3}\left(d_3 - a_3 x^{(k+1)} - b_3 y^{(k+1)}\right)
\end{aligned}$$

$$k = 0, 1, 2, \ldots$$

The process is continued until the convergence is assured.
NOTE : In the absence of initial approximation $x^{(0)}$, $y^{(0)}$, $z^{(0)}$, they are taken as $(0, 0, 0)$.

Example 1

Solve the following system of linear equations

$$3x - y + z = -1$$

$$-x + 3y - z = 7$$

$$x - y + 3z = -7$$

Check that the SOR method with the value $\omega = 1.25$ as the relaxation parameter can be used to solve the system of equations and then solve it.

Solution :

STEP I : Here the coefficient matrix $\mathbf{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Now $\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Thus $\mathbf{A}^{\mathbf{T}} = \mathbf{A}, \implies \mathbf{A}$ is a symmetric matrix.

Now $\begin{vmatrix} 3 \end{vmatrix} = 3 > 0, \quad \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8 > 0, \quad \begin{vmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 20 > 0$

So, the coefficient matrix $\mathbf{A}$ is symmetric positive definite.

STEP II :

The relaxation parameter $\omega = 1.25$ lies in the interval $0 < \omega < 2$. Thus SOR method is applicable for the given system of linear equations.

STEP III :

The iteration scheme by Gauss seidal iteration scheme for the above system of equations.

$$x^{(k+1)} = \frac{1}{3}\left(-1 + y^{(k)} - z^{(k)}\right) \tag{7}$$

$$y^{(k+1)} = \frac{1}{3}\left(7 + x^{(k+1)} + z^{(k)}\right) \tag{8}$$

$$z^{(k+1)} = \frac{1}{3}\left(-7 - x^{(k+1)} + y^{(k+1)}\right) \tag{9}$$

STEP IV :

Multiply the RHS of the equation (7), (8) and (9) by ω and adding to the vectors $x^{(k)}$, $y^{(k)}$ and $z^{(k)}$ by multiplying $(1 - \omega)$ respectively equation (7), (8) and (9) becomes

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega\frac{1}{3}\left(-1 + y^{(k)} - z^{(k)}\right) \tag{10}$$

$$y^{(k+1)} = (1-\omega)y^{(k)} + \omega\frac{1}{3}\left(7 + x^{(k+1)} + z^{(k)}\right) \tag{11}$$

$$z^{(k+1)} = (1-\omega)z^{(k)} + \omega\frac{1}{3}\left(-7 - x^{(k+1)} + y^{(k+1)}\right) \tag{12}$$

$$k = 0,\ 1,\ 2,\ \ldots .$$

Taking the initial approximation as $x^{(0)} = y^{(0)} = z^{(0)} = 0$

First approximation $(k = 0)$

$$
\begin{aligned}
x^{(1)} &= (1-\omega)x^{(0)} + \omega\frac{1}{3}\left(-1 + y^{(0)} - z^{(0)}\right) \\
&= (1-1.25)\times 0 + \frac{1.25}{3}(-1+0-0) \\
&= -0.416666 \\
y^{(1)} &= (1-\omega)y^{(0)} + \omega\frac{1}{3}\left(7 + x^{(1)} + z^{(0)}\right) \\
&= (1-1.25)\times 0 + \frac{1.25}{3}(7-0.41666+0) \\
&= 2.743055 \\
z^{(1)} &= (1-\omega)z^{(0)} + \omega\frac{1}{3}\left(-7 - x^{(1)} + y^{(1)}\right) \\
&= (1-1.25)\times 0 + \frac{1.25}{3}(-7+0.41666+2.743055) \\
&= -1.600116
\end{aligned}
$$

Second approximation $(k = 1)$

$$x^{(2)} = (1-\omega)x^{(1)} + \omega\frac{1}{3}\left(-1 + y^{(1)} - z^{(1)}\right)$$

$$= (1-1.25)(-0.416666) + \frac{1.25}{3}(-1 + 2.743055 + 1.600116)$$

$$= 1.497154$$

$$y^{(2)} = (1-\omega)y^{(1)} + \omega\frac{1}{3}\left(7 + x^{(2)} + z^{(1)}\right)$$

$$= (1-1.25)(2.743055) + \frac{1.25}{3}(7 + 1.497154 - 1.600116)$$

$$= 2.188002$$

$$z^{(2)} = (1-\omega)z^{(1)} + \omega\frac{1}{3}\left(-7 - x^{(2)} + y^{(2)}\right)$$

$$= (1-1.25)(-1.600116) + \frac{1.25}{3}(-7 - 1.497154 + 2.188002)$$

$$= -2.228784$$

Third approximation $(k = 2)$

$$x^{(3)} = (1-\omega)x^{(2)} + \omega\frac{1}{3}\left(-1 + y^{(2)} - z^{(2)}\right)$$

$$= (1-1.25)(1.497154) + \frac{1.25}{3}(-1 + 2.1880022 + 2.228784)$$

$$= 1.049372$$

$$y^{(3)} = (1-\omega)y^{(2)} + \omega\frac{1}{3}\left(7 + x^{(3)} + z^{(2)}\right)$$

$$= (1-1.25)(2.188002) + \frac{1.25}{3}(7 + 1.049372 - 2.228784)$$

$$= 1.878244$$

$$z^{(3)} = (1-\omega)z^{(2)} + \omega\frac{1}{3}\left(-7 - x^{(3)} + y^{(3)}\right)$$

$$= (1-1.25)(-2.228784) + \frac{1.25}{3}(-7 - 1.049372 + 1.878244)$$

$$= -2.014107$$

Thus the solution of above system of equations by SOR method with relaxation parameter $\omega = 1.25$ after three steps is given by $x = 1.049372, y = 1.878244, z = -2.014107$

Example 2

Solve the following system of linear equations

$$10x + y - z = 2$$

$$x + 10y - 2z = 5$$

$$-x - 2y + 10z = 3$$

Check that the SOR method with the value $\omega = 1.25$ as the relaxation parameter can be used to solve the system of equations and then solve it.

Solution :

STEP I : Here the coefficient matrix $\mathbf{A} = \begin{bmatrix} 10 & 1 & -1 \\ 1 & 10 & -2 \\ -1 & -2 & 10 \end{bmatrix}$

Now $\mathbf{A^T} = \begin{bmatrix} 10 & 1 & -1 \\ 1 & 10 & -2 \\ -1 & -2 & 10 \end{bmatrix}$

Thus $\mathbf{A^T} = \mathbf{A}, \implies \mathbf{A}$ is a symmetric matrix.

Now $\begin{vmatrix} 10 \end{vmatrix} = 10 > 0, \quad \begin{vmatrix} 10 & 1 \\ 1 & 10 \end{vmatrix} = 99 > 0, \quad \begin{vmatrix} 10 & 1 & -1 \\ 1 & 10 & -2 \\ -1 & -2 & 10 \end{vmatrix} = 944 > 0$

So, the coefficient matrix $\mathbf{A}$ is symmetric positive definite.

STEP II :

The relaxation parameter $\omega = 1.25$ lies in the interval $0 < \omega < 2$. Thus SOR method is applicable for the given system of linear equations.

STEP III :

The iteration scheme by Gauss seidal iteration scheme for the above system of equations.

$$x^{(k+1)} = \frac{1}{10}\left(2 - y^{(k)} + z^{(k)}\right) \tag{13}$$

$$y^{(k+1)} = \frac{1}{10}\left(5 - x^{(k+1)} + 2z^{(k)}\right) \tag{14}$$

$$z^{(k+1)} = \frac{1}{10}\left(3 + x^{(k+1)} + 2y^{(k+1)}\right) \tag{15}$$

STEP IV :

Multiply the RHS of the equation (13), (14) and (15) by ω and adding to the vectors $x^{(k)}$, $y^{(k)}$ and $z^{(k)}$ by multiplying $(1-\omega)$ respectively equation (13), (14) and (15) becomes

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega\frac{1}{10}\left(2 - y^{(k)} + z^{(k)}\right) \tag{16}$$

$$y^{(k+1)} = (1-\omega)y^{(k)} + \omega\frac{1}{10}\left(5 - x^{(k+1)} + 2z^{(k)}\right) \tag{17}$$

$$z^{(k+1)} = (1-\omega)z^{(k)} + \omega\frac{1}{10}\left(3 + x^{(k+1)} + 2y^{(k+1)}\right) \tag{18}$$

$$k = 0,\ 1,\ 2,\ \ldots.$$

Taking the initial approximation as $x^{(0)} = y^{(0)} = z^{(0)} = 0$

First approximation $(k = 0)$

$$x^{(1)} = (1-\omega)x^{(0)} + \omega\frac{1}{10}\left(2 - y^{(0)} + z^{(0)}\right)$$
$$= (1-1.25)\times 0 + \frac{1.25}{10}(2-0+0)$$
$$= 0.25$$
$$y^{(1)} = (1-\omega)y^{(0)} + \omega\frac{1}{10}\left(5 - x^{(1)} + 2z^{(0)}\right)$$
$$= (1-1.25)\times 0 + \frac{1.25}{10}(5-0.25+0)$$
$$= 0.59375$$
$$z^{(1)} = (1-\omega)z^{(0)} + \omega\frac{1}{10}\left(3 + x^{(1)} + 2y^{(1)}\right)$$
$$= (1-1.25)\times 0 + \frac{1.25}{10}(3+0.25+2\times 0.59375)$$
$$= 0.55468$$

Second approximation $(k = 1)$

$$x^{(2)} = (1-\omega)x^{(1)} + \omega\frac{1}{10}\left(2 - y^{(1)} + z^{(1)}\right)$$

$$= (1-1.25)\times 0.25 + \frac{1.25}{10}(2 - 0.59375 + 0.55468)$$

$$= 0.182616$$

$$y^{(2)} = (1-\omega)y^{(1)} + \omega\frac{1}{10}\left(5 - x^{(2)} + 2z^{(1)}\right)$$

$$= (1-1.25)\times 0.59375 + \frac{1.25}{10}(5 - 0.182616 + 2\times 0.55468)$$

$$= 0.592404$$

$$z^{(2)} = (1-\omega)z^{(1)} + \omega\frac{1}{10}\left(3 + x^{(2)} + 2y^{(2)}\right)$$

$$= (1-1.25)\times 0.555468 + \frac{1.25}{10}(3 + 0.182616 + 2\times 0.592405)$$

$$= 0.407258$$

Third approximation $(k = 2)$

$$x^{(3)} = (1-\omega)x^{(2)} + \omega\frac{1}{10}\left(2 - y^{(2)} + z^{(2)}\right)$$

$$= (1-1.25)\times 0.182616 + \frac{1.25}{10}(2 - 0.592405 + 0.407258)$$

$$= 0.181202$$

$$y^{(3)} = (1-\omega)y^{(2)} + \omega\frac{1}{10}\left(5 - x^{(3)} + 2z^{(2)}\right)$$

$$= (1-1.25)\times 0.592405 + \frac{1.25}{10}(5 - 0.181202 + 2\times 0.407258)$$

$$= 0.556063$$

$$z^{(3)} = (1-\omega)z^{(2)} + \omega\frac{1}{10}\left(3 + x^{(2)} + 2y^{(3)}\right)$$

$$= (1-1.25)\times 0.407258 + \frac{1.25}{10}(3 + 0.181202 + 2\times 0.556063)$$

$$= 0.434851$$

Thus the solution of above system of equations by SOR method with relax-

Thus the solution of above system of equations by SOR method with relaxation parameter $\omega = 1.25$ after three steps is given by $x = 0.181202, y = 0.556063, z = 0.434851$

Any Questions?

Thank You