Mathematics III (RMA3A001)

Module I

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Lecture - 12

Let $y = f(x)$ be a function which takes the values $f(a),\ f(a+h),\ f(a+2h), \dots\dots\dots f(a+nh)$ for $x = a,\ a+h,\ a+2h, \dots\dots\dots\dots (a+nh)$. Where h is the step size of the arguments.
Let us consider the function $f(x)$

$$\begin{aligned} f(x) =& a_0 + a_1(x-a-nh) + a_2(x-a-nh)(x-a-(n-1)h) \\ & + a_3(x-a-nh)(x-a-(n-1)h)(x-a-(n-1)h) \\ & + \dots\dots\dots\dots + a_n(x-a-nh)(x-a-(n-1)h) \dots\dots (x-a-h) \end{aligned} \tag{1}$$

Where $a_0,\ a_1,\ a_2, \dots\dots\dots\dots a_n$ are constants.
Putting $x = a,\ a+h,\ a+2h, \dots\dots\dots\dots a+nh$ in to equation (1) respectively, we get

$$
\begin{aligned}
a_0 &= f(a+nh) \\
a_0 - a_1 h &= f(a+(n-1)h) \\
\implies \ a_1 &= \frac{f(a+nh) - f(a+(n-1)h)}{h} \\
\implies \ a_1 &= \frac{\nabla f(a+nh)}{h}
\end{aligned}
$$

and $$a_0 - 2ha_1 + 2h^2 a_2 = f(a-(n-2)h)$$

$$\implies \ a_2 = \frac{\nabla^2 f(a+nh)}{2!\ h^2}$$

Continuing in this way, we get

$$a_3 = \frac{\nabla^3 f(a+nh)}{3!\ h^3} \ldots\ldots\ldots\ldots a_n = \frac{\nabla^n f(a+nh)}{n!\ h^n}$$

Now substituting these values of $a_0,\ a_1,\ a_2,\ \ldots\ldots\ldots\ a_n$ into equation (1), we get

$$
\begin{aligned}
f(x) = & f(a+nh) + \frac{\nabla f(a+nh)}{h}(x-a-nh) \\
& + \frac{\nabla^2 f(a+nh)}{2!\, h^2}(x-a-nh)(x-a-(n-1)h) \\
& + \frac{\nabla^3 f(a+nh)}{3!\, h^3}(x-a-nh)(x-a-(n-1)h)(x-a-(n-2)h) + \ \ldots \\
& + \frac{\nabla^n f(a)}{n!\, h^n}(x-a-nh)(x-a-(n-1)h)\ \ldots\ (x-a-h)
\end{aligned}
\tag{2}
$$

Further put $x = a + nh + Uh,\ x - a - (n-1)h = (U+1)h,$
$x - a - (n-2)h = (U+2)h,\ \ldots\ldots\ldots x - a - h = (U+n-1)h$
Using these values in equation (2)

$$
\begin{aligned}
f(a+nh+Uh) = & f(a+nh) + U\nabla f(a+nh) + \frac{U(U+1)}{2!}\nabla^2 f(a+nh) + \ \ldots\ldots \\
& + \frac{U(U+1)(U+2)\ \ldots\ldots\ (U+n-1)}{n!}\nabla^n f(a+nh)
\end{aligned}
$$

This formula is known as Newton's backward interpolation with equal intervals.

Example 1

For the following data

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

Solution : The difference table for the given data is as follows

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$	$\nabla^6 f(x)$	$\nabla^7 f(x)$
1	1							
		7						
2	8		12					
		19		6				
3	27		18		0			
		37		6		0		
4	64		24		0		0	
		61		6		0		0
5	125		30		0		0	
		91		6		0		
6	216		36		0			
		127		6				
7	343		42					
		169						
8	512							

Here $a + nh = 8$, $h = 1$, and $x = 7.5$ then

$$U = \frac{x - (a + nh)}{h} = \frac{7.5 - 8}{1} = -0.5$$

By Newtons backward interpolation formula

$$\begin{aligned}
f(a + nh + Uh) =& f(a + nh) + U\nabla f(a + nh) + \frac{U(U+1)}{2!}\nabla^2 f(a + nh) \\
&+ \frac{U(U+1)(U+2)}{3!}\nabla^3 f(a + nh) \\
\implies f(7.5) =& f(8) + (-0.5)\nabla f(8) + \frac{(-0.5)(-0.5+1)}{2!}\nabla^2 f(8) \\
&+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}\nabla^3 f(8) \\
=& 512 - 84.5(169) - \frac{(0.5)(0.5)}{2}(42) - \frac{(0.5)(0.5)(1.5)}{6}(6) \\
=& 512 - 84.5 - 5.25 - 0.375 \\
=& 421.875
\end{aligned}$$

Example 2

Given the following table

x	0.1	0.2	0.3	0.4	0.5
e^x	1.10517	1.2140	1.34986	1.49182	1.64872

Find $e^{0.411}$ by using Newtons backward interpolation.

Solution : The difference table is given by

x	$y = e^x$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0.1	1.10517				
		0.10883			
0.2	1.2140		0.02703		
		0.13586		-0.02093	
0.3	1.34986		0.0061		0.02997
		0.14196		0.00884	
0.4	1.49182		0.01494		
		0.1569			
0.5	1.64872				

Here $a + nh = 0.5, \quad h = 1, \quad x = 0.411$

$$U = \frac{x - (a + nh)}{h} = \frac{0.411 - 0.5}{0.1} = -0.89$$

By Newtons backward interpolation formula

$$
\begin{aligned}
y(x) =& y_n + U\nabla y_n + \frac{U(U+1)}{2!}\nabla^2 y_n + \frac{U(U+1)(U+2)}{3!}\nabla^3 y_n \\
&+ \frac{U(U+1)(U+2)(U+3)}{4!}\nabla^4 y_n
\end{aligned}
$$

$$
\begin{aligned}
\implies \quad y(0.411) =& 1.64872 + (-0.89)(0.1569) + \frac{(-0.89)(-0.89+1)}{2!}0.01494 \\
&+ \frac{(-0.89)(-0.89+1)(-0.89+2)}{3!}(0.00884) \\
&+ \frac{(-0.89)(-0.89+1)(-0.89+2)(-0.89+3)}{4!}(0.02977) \\
=& 1.507903164
\end{aligned}
$$

Hence, $e^{0.411} = 1.507903164$

Any Questions?

Thank You