Mathematics III (RMA3A001)

Module I

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Lecture - 10

Let us consider a system of three linear equations with three unknowns and is given by,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

In matrix form the above set of equations can be represented as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

i.e.

$$A\mathbf{x} = B \tag{1}$$

Where,

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let

$$A = LL^T \tag{2}$$

Where L is a lower triangular matrix and L^T is transpose of the matrix L.

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Using equation (2) in equation (1)

$$LL^T\mathbf{x} = B \tag{3}$$

Let

$$L^T\mathbf{x} = \mathbf{y} \quad \text{where} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{4}$$

Using equation (4) in equation (3)

$$L\mathbf{y} = B \tag{5}$$

Solving equation (5) by forward substitution method find out the value of $\mathbf{y}$. Putting the value of $\mathbf{y}$ in equation (4) find out the value of $\mathbf{x}$ by backward substitution method.

Example 1

Solve the following system of equations by using cholesky method.

$$x + 2y + 3z = 5$$

$$2x + 8y + 22z = 6$$

$$3x + 22y + 82z = -10$$

Solution :

In matrix form the above set of equations can be written as,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$$\implies A\mathbf{x} = B \tag{6}$$

Let

$$A = LL^T \tag{7}$$

Where L is a lower triangular matrix and L^T is a the transpose of the matrix L.

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

From equation (7)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\implies \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{ll}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{ll}l_{31} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}$$

Equating both sides, we have

$$l_{11}^2 = 1 \quad \Longrightarrow \; l_{11} = 1, \quad l_{11}l_{21} = 2 \quad \Longrightarrow \; l_{21} = 2, \quad l_{11}l_{31} = 3 \quad \Longrightarrow \; l_{31} = 3$$

$$l_{21}^2 + l_{11}^2 = 8, \qquad \Longrightarrow \; 4 + l_{22}^2 = 8, \qquad \Longrightarrow \; l_{22} = 2$$

$$l_{31}l_{21} + l_{32}l_{22} = 22, \qquad \Longrightarrow \; 2 \times 3 + 2l_{32} = 22, \qquad \Longrightarrow \; l_{32} = 8$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 82, \qquad \Longrightarrow \; 9 + 64 + l_{33}^2 = 82, \qquad \Longrightarrow \; l_{33}^2 = 9, \qquad \Longrightarrow \; l_{33} = 3$$

Thus,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}$$

$$L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

Using equation (7) in equation (6)

$$LL^T\mathbf{x} = B \tag{8}$$

Let

$$L^T\mathbf{x} = \mathbf{y} \quad \text{where} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{9}$$

Using equation (9) in equation (8)

$$L\mathbf{y} = B \tag{10}$$

From equation (10) we have,

$$L\mathbf{y} = B$$

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$$\implies \begin{bmatrix} y_1 \\ 2y_1 + 2y_2 \\ 3y_1 + 8y_2 + 3y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

Equating both sides and solving it by forward substitution method

$$y_1 = 5$$

$$2y_1 + 2y_2 = 6, \qquad \implies 2 \times 5 + 2y_2 = 6 \qquad \implies y_2 = -2$$

$$3y_1 + 8y_2 + 3y_3 = -10, \qquad \implies 3 \times 5 + 8 \times (-2) + 3y_3 = -10 \qquad \implies y_3 = -3$$

Thus

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

Putting the value of $\mathbf{y}$ in equation (9)

$$L^T \mathbf{x} = \mathbf{y}$$

$$\implies \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

$$\implies \begin{bmatrix} x + 2y + 3z \\ 2y + 8z \\ 3z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

Equating both sides and solving it by backward substitution method, we have

$$3z = -3, \qquad \implies z = -1$$

$$2y + 8z = -2 \qquad \implies 2y + 8 \times (-1) = -2 \qquad \implies y = 3$$

$$x + 2y + 3z = -3 \qquad \implies x + 2 \times 3 + 3 \times (-1) = -3 \implies x = 2$$

Thus $x = 2$, $y = 3$ and $z = -1$ is the solution of the above system of equations by choleskys method.

Any Questions?

Thank You