Mathematics III (RMA3A001)

Module I

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Lecture - 5

Fixed point iteration method

- Let the equation be $f(x) = 0$ and (a, b) be the interval in which the root of the equation $f(x)$ lies.
- Write down the equation $f(x)$ in the form of $x = \varphi(x)$ such that $|\varphi'(x)| < 1$ for all $x \in (a, b)$ i.e the interval in which the root of the equation $f(x)$ lies.
- Now the iteration method is

$$x_{k+1} = \varphi(x_k), \qquad k = 0,\ 1,\ 2,\ \ldots\ldots$$

- *First approximation* $(k = 0)$

$$x_1 = \varphi(x_0)$$

- *Second approximation* $(k = 1)$

$$x_2 = \varphi(x_1)$$

- *Third approximation* $(k = 2)$

$$x_3 = \varphi(x_2)$$

and so on.

- Where x_0 is the initial approximation to the root of the equation. This is the one point formula. Since one initial approximation x_0 is required.
- **NOTE :** Fixed point iteration method has a linear rate of convergence.

Example 1

Find the real root of the equation $f(x) = x^3 - 5x + 1 = 0$ by using fixed point iteration method.

Solution : We have $f(x) = x^3 - 5x + 1 = 0$

$$f(0) = 1 > 0, \qquad f(1) = -3 < 0$$

So the root of the equation lies in the interval $(0, 1)$.
Now $f(x) = x^3 - 5x + 1 = 0$ can be written as

$$x = -\frac{1}{(x^2 - 5)}$$

where

$$\varphi(x) = -\frac{1}{(x^2 - 5)}$$

Now

$$\varphi'(x) = \frac{2x}{(x^2 - 5)^2}$$

$$\left|\varphi'(x)\right| = \left|\frac{2x}{\left(x^2-5\right)^2}\right| < 0, \qquad \text{for all} \quad x \in (0,1)$$

i.e the interval in which the real root of $f(x)$ lies.
Thus the iteration scheme is

$$x_{k+1} = -\frac{1}{\left(x_k^2-5\right)} \qquad k = 0,\ 1,\ 2,\ \ldots\ldots$$

Let $x_0 = 0.2$ be the initial approximation to the root of the equation.
First approximation $(k = 0)$

$$\begin{aligned} x_1 &= -\frac{1}{\left(x_0^2-5\right)} \\ &= -\frac{1}{\left[(0.2)^2-5\right]} \\ &= 0.2016129 \end{aligned}$$

Second approximation $(k = 1)$

$$\begin{aligned} x_2 &= -\frac{1}{\left(x_1^2 - 5\right)} \\ &= -\frac{1}{\left[(0.2016129)^2 - 5\right]} \\ &= 0.2016392 \end{aligned}$$

Third approximation $(k = 2)$

$$\begin{aligned} x_3 &= -\frac{1}{\left(x_2^2 - 5\right)} \\ &= -\frac{1}{\left[(0.2016392)^2 - 5\right]} \\ &= 0.20016396 \end{aligned}$$

Thus the root of the equation by fixed point iteration after three steps is 0.2016396

Example 2

Find the real root of the equation $f(x) = 2x - \cos x - 3 = 0$ correct up to three decimal places by using fixed point iteration method.

Solution : We have $f(x) = 2x - \cos x - 3 = 0$

$$f(0) = -4 < 0, \qquad f(1) = -1.54 < 0, \qquad f(2) = 1.416 > 0$$

So the root of the equation lies in the interval $(1, 2)$.
Now $f(x) = 2x - \cos x - 3 = 0$ can be written as

$$x = \frac{\cos x + 3}{2}$$

where

$$\varphi(x) = \frac{\cos x + 3}{2}$$

Now

$$\varphi'(x) = -\frac{1}{2}\sin x$$

$$\left|\varphi'(x)\right| = \left|-\frac{1}{2}\sin x\right| = \left|\frac{1}{2}\sin x\right| < 0, \qquad \text{for all} \quad x \in (1, 2)$$

i.e the interval in which the real root of $f(x)$ lies. Thus the iteration scheme is applicable

So the iteration scheme is

$$\begin{aligned} x_{k+1} &= \varphi(x_k) \\ \implies x_{k+1} &= \frac{\cos x_k + 3}{2} \qquad k = 0,\ 1,\ 2,\ \ldots\ldots \end{aligned}$$

Let $x_0 = 1.5$ be the initial approximation to the root of the equation.

First approximation $(k = 0)$

$$\begin{aligned} x_1 &= \frac{\cos x_0 + 3}{2} \\ &= \frac{\cos(1.5) + 3}{2} = 1.5354 \end{aligned}$$

Second approximation $(k = 1)$

$$\begin{aligned} x_2 &= \frac{\cos x_1 + 3}{2} \\ &= \frac{\cos(1.5354) + 3}{2} = 1.5177 \end{aligned}$$

Third approximation $(k = 2)$

$$\begin{aligned} x_3 &= \frac{\cos x_2 + 3}{2} \\ &= \frac{\cos(1.5177) + 3}{2} = 1.5265 \end{aligned}$$

Fourth approximation $(k = 3)$

$$\begin{aligned} x_4 &= \frac{\cos x_3 + 3}{2} \\ &= \frac{\cos(1.5265) + 3}{2} = 1.5221 \end{aligned}$$

Fifth approximation $(k = 4)$

$$\begin{aligned} x_5 &= \frac{\cos x_4 + 3}{2} \\ &= \frac{\cos(1.5221) + 3}{2} = 1.5243 \end{aligned}$$

Sixth approximation $(k = 5)$

$$\begin{aligned} x_6 &= \frac{\cos x_5 + 3}{2} \\ &= \frac{\cos(1.5243) + 3}{2} = 1.5232 \end{aligned}$$

Seventh approximation $(k = 6)$

$$\begin{aligned} x_7 &= \frac{\cos x_6 + 3}{2} \\ &= \frac{\cos(1.5232) + 3}{2} = 1.5237 \end{aligned}$$

Thus the root of the equation $f(x) = 2x - \cos x - 3 = 0$ correct up to three decimal places by fixed point iteration method is 1.523

Any Questions?

Thank You