Mathematics III (RMA3A001)

Module II

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Lecture - 15

Numerical Differentiation

- It is the process for determining the value of the derivative of a function at some values of the arguments from the given set of values of function. To determine the values of the derivative of $y = f(x)$ we have to approximate the function $y = f(x)$ by interpolating polynomial and then differentiate this function an may times.
- If the value of x are equally spaces and the derivative is near the beginning of the arguments we use Newton's forword formula.
- If the derivative is near the and of the given arguments we apply Newton's backward formula.
- To find the derivative at the middle of the given arguments we apply Stirling's or Bessel's formula.
- If the arguments are not equally spaced then we use Newton's divided difference formula or Lagranges formula for finding the derivatives.

Derivative of the function at the given arguments equally spaced

Derivative using Newton's Forward difference formula

Consider Newtons forward difference formula

$$y = y_0 + U\Delta y_0 + \frac{U(U-1)}{2!}\Delta^2 y_0 + \frac{U(U-1)(U-2)}{3!}\Delta^3 y_0 + \ldots\ldots \quad (1)$$

Where

$$U = \frac{x - x_0}{h} \quad \text{and} \quad \frac{dU}{dx} = \frac{1}{h}$$

Differentiation both sided of (1) w.r.t x we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[y_0 + \frac{U(U-1)}{2!}\Delta^2 y_0 + \frac{U(U-1)(U-2)}{3!}\Delta^3 y_0 + \ldots\ldots\right] \\ &= \frac{d}{dU}\left[y_0 + \frac{U(U-1)}{2!}\Delta^2 y_0 + \frac{U(U-1)(U-2)}{3!}\Delta^3 y_0 + \ldots\ldots\right]\frac{dU}{dx}\end{aligned}$$

$$\implies \frac{dy}{dx} = \left[\Delta y_0 + \frac{(2U-1)}{2!}\Delta^2 y_0 + \frac{(3U^2 - 6U + 2)}{3!}\Delta^3 y_0 + \ldots\ldots\right] \quad (2)$$

At $x = x_0, U = 0$ thus

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ \ldots\ldots\ldots\right]$$

Now differentiating (2) w.r t x we get

$$\frac{d^2y}{dx^2} = \frac{1}{h}\left[\Delta^2 y_0 - \frac{(6U-6)}{3!}\Delta^3 y_0 + \frac{(12U^2-36U+22)}{4!}\Delta^2 y_0 + \ \ldots\ldots\ldots\right]\frac{dU}{dx}$$

$$\implies \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \frac{(6U-6)}{6}\Delta^3 y_0 + \frac{(12U^2-36U+22)}{24}\Delta^2 y_0 + \ \ldots\ldots\right]$$

At $x = x_0, \ \ U = 0$ we get

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ \ldots\ldots\ldots\right]$$

Proceeding in this way, we get successive derivative at the required points.

Derivative using Newtons Backward difference formula

Consider Newtons backward difference formula

$$y = y_0 + U\nabla y_0 + \frac{U(U+1)}{2!}\nabla^2 y_0 + \frac{U(U+1)(U+2)}{3!}\nabla^3 y_0 + \ldots\ldots \quad (3)$$

Where

$$U = \frac{x - y_n}{h}$$

Differentiating both side of equation (3) w.r.t x we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2U+1)}{2!}\nabla^2 y_n + \frac{(3U^2+6U+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\right] \quad (4)$$

At $x = x_n, U = 0$ we get

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \ldots\ldots\ldots\right]$$

Differentiating both side of equation (4) w.r.t x we get

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{(6U+6)}{6}\nabla^3 y_n + \frac{(12U^2+36U+22)}{24}\nabla^4 y_n + \ \ldots\ldots\ldots\right]$$

At $x = x_n, U = 0$ we get

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ \ldots\ldots\ldots\right]$$

Derivatives of the functions at the given arguments unequally spaced

Derivative using Newtons divided difference formula:
By Newtons divided difference formula we have

$$
\begin{aligned}
y = f(x) =& f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \ .\ .\ .\ .\ . \\
&+ (x - x_0)(x - x_1)\ .\ .\ .\ .\ .\ (x - x_{n-1}) f[x_0, x_1,\ .\ .\ .\ .\ .\ x_n]
\end{aligned} \tag{5}
$$

Differentiating the equation w.r.t x as many times as we required and put $x = x_0$ we get the required derivatives.

Derivative using Lagrange's formula:

By Lagranges formula we have

$$
\begin{aligned}
y = f(x) = &\frac{(x-x_1)(x-x_2)\,.\,.\,.\,.\,.\,(x-x_n)}{(x_0-x_1)(x_0-x_2)\,.\,.\,.\,.\,.\,(x_0-x_n)} f(x_0) \\
&+\frac{(x-x_0)(x-x_2)\,.\,.\,.\,.\,.\,(x-x_n)}{(x_1-x_0)(x_1-x_2)\,.\,.\,.\,.\,.\,(x_1-x_n)} f(x_1)\,.\,.\,.\,.\,.\,. \\
&+\frac{(x-x_0)(x-x_1)\,.\,.\,.\,.\,.\,(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\,.\,.\,.\,.\,.\,(x_n-x_{n-1})} f(x_n)
\end{aligned} \tag{6}
$$

Differentiating equation (6) w.r.t x as many times as we required and put $x = x_0$, we get the required derivatives.

Example 1

Find $f'(2.5)$ from the following table

x	1.5	1.9	2.5	3.2	4.3	5.9
$f(x)$	3.375	6.059	13.625	29.368	73.907	196.579

Solution :
Here the arguments are not equally spaced. Therefore applying Newtons divided difference formula the difference table is as below.

x	$f(x)$					
1.5	3.375					
1.9	6.059	6.71 $f[x_0, x_1]$				
2.5	13.625	12.61	5.90 $f[x_0, x_1, x_2]$			
3.2	29.368	22.49	7.6	1 $f[x_0, ..x_3]$		
4.3	73.903	40.49	10.00	1	0 $f[x_0, ..x_4]$	
5.9	196.579	76.67	13.40	1	0	0 $f[x_0, ..x_5]$

Newtons divided difference formula is

$$
\begin{aligned}
f(x) =& f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\
&+ (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] \\
&+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f[x_0, x_1, x_2, x_3, x_4] \\
&+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4) f[x_0, x_1, x_2, x_3, x_4, x_5] \\
\implies f(x) =& 3.375 + (x - 1.5)6.71 + (x - 1.5)(x - 1.9)5.90 \\
&+ (x - 1.5)(x - 1.9)(x - 2.5)1 + 0 + 0 \\
f'(x) =& 6.71 + [(x - 1.5) + (x - 1.9)]5.90 \\
&+ [(x - 1.9)(x - 2.5) + (x - 1.5)(x - 2.5) + (x - 1.5)(x - 1.9)]1 \\
f'(2.5) =& 6.71 + [(2.5 - 1.5) + (2.5 - 1.9)]5.90 \\
&+ [(2.5 - 1.9)(2.5 - 2.5) + (2.5 - 1.5)(2.5 - 2.5) + (2.5 - 1.5)(2.5 - \\
=& 6.71 + 9.44 + 0.6 \\
=& 16.75
\end{aligned}
$$

Example 2

Find $f'(3)$ and $f''(3)$ from the following table

x	0	1	2	5
$f(x)$	2	3	12	147

Solution : Here the arguments are not equally spaced Now we apply Lagranges interpolation formula.

Here $x_0 = 0 \quad x_1 = 1 \quad x_2 = 2 \quad x_3 = 5$ and

$f(x_0) = 2 \quad f(x_1) = 3 \quad f(x_2) = 12 \quad f(x_3) = 147$

By Lagrange interpolation formula

$$\begin{aligned}
f(x) =& \frac{(x-x_0)(x-x_1(x-x_3)}{(x_2-x_0)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\
&+ \frac{(x-x_1)(x-x_2(x-x_3)}{(x_0-x_1)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_1)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \\
=& \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) \\
&+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-1)}(147) \\
=& x^3 + x^2 - x + 2
\end{aligned}$$

$$f'(x) = 3x^2 + 2x - 1, \qquad f'(3) = 32$$

$$f''(x) = 6x + 2, \qquad f''(3) = 20$$

Any Questions?

Thank You

Lecture - 16

The general form of numerical integration is to find an approximate value of the integral

$$I = \int_a^b \omega(x) f(x) dx \tag{1}$$

Where $\omega(x) > 0$ in $[a, b]$ is the weight function. The limits of integration may be finite, semi-finite or infinite.

The integral (1) is approximated by a finite linear combination of values of $f(x)$ in the form

$$I = \int_a^b \omega(x) f(x) dx \approx \sum_{k=0}^{n} \lambda_k f(x_k)$$

Where $x_0,\ x_1,\ x_2,\ .\ .\ .\ .\ .\ x_n$ are called the abscissas or nodes within the limits of integration $[a, b]$ and $\lambda_k,\ \ k = 0, 1, 2,\ .\ .\ .\ .\ n$ are called the weights of the integration rule. The error of approximation is given as

$$R_n = \int_a^b \omega(x) f(x) dx - \sum_{k=0}^{n} \lambda_k f(x_k)$$

When $\omega(x) = 1$ and the nodes x_k's are equispaced with $x_0 = a,\ x_n = b$ with spacing $h = \frac{b-a}{n}$ the methods are called Newton-cotes interpolation methods. The weights λ_k's are called cotes numbers.
Where,

$$\lambda_k = \frac{(-1)^{n-k}}{k!(n-k)!} \int_0^n s(s-1)\ldots\ldots(s-k+1)(s-k-1)\ldots\ldots(s-n)ds$$

$$R_n = \frac{h^{n+2}}{(n+1)!} \int_0^n s(s-1)\ldots\ldots\ldots(s-n)f^{n+1}(\eta)ds$$

In Newton cotes method if $n = 1$ then the preceding rule is known as Trapezoidal rule. Thus we have, $x_0 = a,\ x_1 = b,\ h = b - a$. Now the rule becomes

$$\begin{aligned} \int_a^b f(x)dx &= \sum_{k=0}^{1} \lambda_k f(x_k) \\ \implies \int_a^b f(x)dx &= \lambda_0 f(x_0) + \lambda_1 f(x_1) \\ \implies \int_a^b f(x)dx &= \lambda_0 f(a) + \lambda_1 f(b) \end{aligned} \tag{2}$$

$$\lambda_0 = \frac{(-1)^{1-0}}{(1-0)!0!} h \int_0^1 \frac{s(s-1)}{s} ds = -h \int_0^1 (s-1)ds = \frac{h}{2}$$

$$\lambda_1 = \frac{(-1)^{1-1}}{(1-1)!1!} h \int_0^1 \frac{s(s-1)}{s} ds = h \int_0^1 (s-1)ds = \frac{h}{2}$$

Putting these values in equation (2)

$$
\begin{aligned}
\int_a^b f(x)dx &= \frac{h}{2} f(a) + \frac{h}{2} f(b) \\
\implies \int_a^b f(x)dx &= \frac{h}{2} \left[f(a) + f(b) \right] \\
\implies \int_a^b f(x)dx &= \left(\frac{b-a}{2} \right) \left[f(a) + f(b) \right]
\end{aligned}
$$

Which is known as Trapezoidal rule for numerical integration

NOTE : The error in Trapezoidal rule is given by

$$
-\frac{(b-a)^3}{12} f''(\eta), \qquad a < \eta < b
$$

Simpsion's $\frac{1}{3}$ rule

In Newton's cotes method if $n = 2$ then the preceding rule is known as Simpsion's $\frac{1}{3}$ rule. Thus we have

$$x_0 = 1,\ x_1 = \frac{a+b}{2},\ x_2 = b,\ h = \frac{b-a}{2}$$

Now the rule becomes

$$\int_a^b f(x)dx = \sum_{k=0}^{2} \lambda_k f(x_k)$$

$$\implies \int_a^b f(x)dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

$$\implies \int_a^b f(x)dx = \lambda_0 f(a) + \lambda_1 f\left(\frac{a+b}{2}\right) + \lambda_2 f(b) \tag{3}$$

Now

$$\lambda_0 = \frac{(-1)^{2-0}}{(2-0)!0!} h \int_0^2 \frac{s(s-1)(s-2)}{(s-0)} ds = \frac{h}{3}$$

$$\lambda_1 = \frac{(-1)^{2-1}}{(2-1)!1!} h \int_0^2 \frac{s(s-1)(s-2)}{(s-1)} ds = \frac{4h}{3}$$

$$\lambda_2 = \frac{(-1)^{2-2}}{(2-2)!2!} h \int_0^2 \frac{s(s-1)(s-2)}{(s-2)} ds = \frac{h}{3}$$

Putting these values in equation (3)

$$\begin{aligned}
\int_a^b f(x)dx &= \frac{h}{3} f(a) + \frac{4h}{3} f\left(\frac{a+b}{2}\right) + \frac{h}{3} f(b) \\
&= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \\
&= \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]
\end{aligned}$$

is known a simpsions $\frac{1}{3}$ rule for numerical integration
NOTE : The error in Simpsions $\frac{1}{3}$ rule is given by

$$-\frac{(b-a)^5}{2880} f''(\eta), \qquad a < \eta < b$$

For $n = 3$ in Newton cotes rule the preceding rule is known as Simpson's $\frac{3}{8}$ rule for numerical integration.
Here

$$x = a + h, \quad x_2 = a + 2h, \quad x_3 = b$$

For $n = 3$ the rule becomes

$$\int_a^b f(x)dx = \sum_{k=0}^{3} \lambda_k f(x_k)$$

$$\implies \int_a^b f(x)dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3)$$

$$\implies \int_a^b f(x)dx = \lambda_0 f(a) + \lambda_1 f\,(a+h) + \lambda_2 f\,(a+2h) + \lambda_3 f(b) \tag{4}$$

Now

$$\lambda_0 = \frac{(-1)^{3-0}}{(3-0)!0!} h \int_0^3 \frac{s(s-1)(s-2)(s-3)}{(s-0)} ds = \frac{3h}{8}$$

$$\lambda_1 = \frac{(-1)^{3-1}}{(3-1)!1!} h \int_0^3 \frac{s(s-1)(s-2)(s-3)}{(s-1)} ds = \frac{9h}{8}$$

$$\lambda_2 = \frac{(-1)^{3-2}}{(3-2)!2!} h \int_0^3 \frac{s(s-1)(s-2)(s-3)}{(s-2)} ds = \frac{9h}{8}$$

$$\lambda_3 = \frac{(-1)^{3-3}}{(3-3)!3!} h \int_0^3 \frac{s(s-1)(s-2)(s-3)}{(s-3)} ds = \frac{3h}{8}$$

Cont ...

Putting these values in equation (4)

$$
\begin{aligned}
\int_a^b f(x)dx =& \frac{3h}{8} f(a) + \frac{9h}{8} f(a+h) + \frac{9h}{8} f(a+2h) + \frac{3h}{8} f(b) \\
=& \frac{3h}{8} \left[f(a) + 3f(a+h) + +3f(a+2h) + f(b) \right]
\end{aligned}
$$

is known a simpsions $\frac{3}{8}$ rule for numerical integration

Example 1

Evaluate

$$\int_0^1 \frac{dx}{1+x}$$

by using

(i) Trapezoidal rule

(ii) Simpsions $\frac{1}{3}$ rule

(iii) Simpsions $\frac{3}{8}$ rule

Solution:

Here

$$f(x) = \frac{1}{1+x}, \quad a = 0, \quad b = 1$$

(i) Trapezoidal rule

$$a = 0, \quad b = 1, \quad h = 1 - 0 = 0$$

By Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x} = \frac{1-0}{2}[f(0) + f(1)] = \frac{1}{2}\left[1 + \frac{1}{2}\right] = \frac{3}{4} = 0.75$$

(ii) Simpsions $\frac{1}{3}$ rule

$$a = 0, \quad \frac{a+b}{2} = \frac{1}{2}, \quad b = 1, \quad h = \frac{1-0}{2} = \frac{1}{2}$$

Simpsions $\frac{1}{3}$ rule

$$\begin{aligned}\int_0^1 \frac{dx}{1+x} =& \frac{1-0}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] \\ =& \frac{1}{6}\left[1 + 4 \times \frac{2}{3} + \frac{1}{3}\right] \\ =& \frac{1}{6} \times \frac{25}{6} = \frac{25}{36} = 0.6947\end{aligned}$$

(iii) Simpsions $\frac{3}{8}$ rule

$$a = 0, \quad b = 1, \quad h = \frac{1-0}{3} = \frac{1}{3}, \quad a + h = \frac{1}{3}, \quad a + 2h = \frac{2}{3}$$

By Simpsion's rule

$$\begin{aligned}
\int_0^1 \frac{dx}{1+x} &= \frac{3 \times \frac{1}{3}}{8} \left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1) \right] \\
&= \frac{1}{8} \left[1 + 3 \times \frac{3}{4} + 3 \times \frac{3}{5} + \frac{1}{2} \right] \\
&= \frac{1}{8} \left[1 + \frac{9}{4} + \frac{9}{5} + \frac{1}{2} \right] \\
&= 0.69375
\end{aligned}$$

Any Questions?

Thank You

Lecture - 17

Let the curve by $y = f(x)$ the limit of integration is form a to b. We divide the interval $[a, b]$ into n equal subinterval by taking the nodes $x_0,\ x_2,\ .\ .\ .\ .\ .\ .\ .\ x_n$ where $x_0 = a$ and $x_n = b$.

Composite Trapezoidal Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \ .\ .\ .\ .\ .\ y_{n-1}) + y_n\right]$$

Composite Simpsion's $\frac{1}{3}$ Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx = \frac{h}{3}[y_0 + 2(y_2 + y_4 + \;.\;.\;.\;.\;.\; y_{n-2}) + 4(y_1 + y_3 + \;.\;.\;.\;.\;.\; y_{n-1}) + y_n]$$

Composite Simpsion's $\frac{3}{8}$ Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx = \frac{3h}{8}[y_0 + 2(y_1 + y_2 + y_4 + y_5 + \;.\;.\;.\;.\;.\; y_{n-1}) + 2(y_3 + y_6 + \;.\;.\;.\;.\;.\; y_{n-3}) + y_n]$$

Example 1

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by taking $h = 1$ or $n = 6$ by Trapezoidal rule.

Solution :

Divide the interval (0,6) into six equal parts with step length $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ at these points are given below.

x	0	1	2	3	4	5	6
$y = f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027

Here $y_0 = 1,\ y_1 = 0.5,\ y_2 = 0.2,\ y_3 = 0.1,\ y_4 = 0.0588,\ y_5 = 0.0385,\ y_6 = 0.027$.

By composite Trapezoidal rule

$$\begin{aligned}\int_0^6 \frac{dx}{1+x^2} &= \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right) + y_6\right] \\ &= \frac{1}{2}\left[1 + 2\left(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385\right) + 0.027\right] \\ &= 1.4108\end{aligned}$$

Example 2

Evaluate $\int_0^1 \frac{1}{1+x}dx$ by taking $h = 0.125$ or $n = 8$ by Trapezoidal rule.

Solution :
Divide the interval (0,1) into eight equal parts with step length $h = 0.125$. The values of $f(x) = \frac{1}{1+x}$ at these points are given below.

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1.0
$y = f(x)$	1.0	0.8889	0.8000	0.7273	0.6667	0.6154	0.5714	0.533	0.5

Here $y_0 = 1.0,\ y_1 = 0.8889,\ y_2 = 0.8000,\ y_3 = 0.7273,\ y_4 = 0.6667,\ y_5 = 0.6154,\ y_6 = 0.5714,\ y_7 = 0.533,\ y_8 = 0.5.$

By composite Trapezoidal rule

$$\int_0^1 \frac{1}{1+x}dx = \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right) + y_8\right]$$

$$= \frac{0.125}{2}[1.0 + 2(0.8889 + 0.8000 + 0.7273 + 0.6667$$

$$+ 0.6154 + 0.5714 + 0.5333) + 0.5] = 0.69413$$

Example 3

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by taking $h = \frac{1}{6}$ or $n = 6$ by Simpson's rule, obtain the approximate value of π.

Solution :

We have $\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

Divide the interval (0,1) into six equal parts with step length $h = \frac{1}{6}$. The values of $f(x) = \frac{1}{1+x^2}$ at these points are given below.

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = f(x)$	1.0000	0.9729	0.9000	0.8000	0.6923	0.5901	0.5000

Here $y_0 = 1.0000,\ y_1 = 0.9729,\ y_2 = 0.9000,\ y_3 = 0.8000,\ y_4 = 0.6923,\ y_5 = 0.5901,\ y_6 = 0.5000.$

By composite Simpsons rule

$$\begin{aligned}\int_0^1 \frac{dx}{1+x^2} =& \frac{h}{3}\left[y_0 + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right) + y_6\right] \\ =& \frac{1}{8}\left[1 + 4\left(0.9729 + 0.8 + 0.5902\right) + 2\left(0.9 + 0.6923\right) + 0.5\right] \\ =& 0.785397\end{aligned}$$

By Simpsons rule

$$\begin{aligned}\frac{\pi}{4} =& 0.785397 \\ \implies \pi =& 3.141588\end{aligned}$$

Example 4

Evaluate $\int_0^4 e^x dx$ by taking $h = 1$ or $n = 4$ by Simpson's rule.

Solution :

Divide the interval (0,4) into four equal parts with step length $h = 1$. The values of $f(x) = e^x$ at these points are given below.

x	0	1	2	3	4
$y = f(x)$	1	2.72	7.39	20.09	54.60

Here $y_0 = 1, \; y_1 = 2.72, \; y_2 = 7.39, \; y_3 = 20.09, \; y_4 = 54.60.$

By composite Simpsons rule

$$\int_0^4 e^x dx = \frac{h}{3}\left[y_0 + 4\left(y_1 + y_3\right) + 2y_2 + y_4\right]$$

$$= \frac{1}{3}\left[1 + 4\left(2.72 + 20.09\right) + 2\left(7.39\right) + 54.60\right]$$

$$= 53.87$$

Any Questions?

Thank You

Lecture - 18

Gauss Legendre two point formula for numerical integration

In Newton cotes formula if $n = 1$ and the range of interpolation is from $[-1, 1]$ then the rule is known as Gauss Ligendre 2-point formula for numerical integration.

Any finite interval $[a, b]$ can be transformed to $[-1, 1]$ by using the formula

$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

When

$$\begin{aligned} x =& a, \qquad t = \frac{a - \left(\frac{b+a}{2}\right)}{\left(\frac{b-a}{2}\right)} = -1 \\ x =& b, \qquad t = \frac{b - \left(\frac{b+a}{2}\right)}{\left(\frac{b-a}{2}\right)} = 1 \end{aligned}$$

Now the rule becomes

$$\int_{-1}^{1} f(x)dx = \sum_{k=0}^{1} \lambda_k f(x_k)$$

$$\implies \int_{-1}^{1} f(x)dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) \tag{1}$$

Here we have four unknowns i.e $\lambda_0,\ \lambda_1,\ x_0$ and x_1. So the rule is exact for all polynomial of degree ≤ 3 i.e $1,\ x,\ x^2$, and x^3. Let $f(x) = 1$, from equation (1)

$$2 = \lambda_0 + \lambda_1 \tag{2}$$

Let $f(x) = x$, from equation (1)

$$0 = \lambda_0 x_0 + \lambda_1 x_1 \tag{3}$$

Let $f(x) = x^2$, from equation (1)

$$\frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2 \tag{4}$$

Let $f(x) = x^3$, from equation (1)

$$0 = \lambda_0 x_0^3 + \lambda_1 x_1^3 \tag{5}$$

Solving equation (2) and (5) we get

$$\lambda_0 = 1, \quad \lambda_1 = 1, \quad x_0 = -\sqrt{\frac{1}{3}}, \quad x_1 = \sqrt{\frac{1}{3}}$$

Putting these values in equation (1) we get

$$\int_{-1}^{1} f(x)dx = f\left(-\sqrt{\frac{1}{3}}\right) + f\left(\sqrt{\frac{1}{3}}\right)$$

Which is known as Gauss Legendre two point formula for numerical integration.

Gauss Legendre three point formula for numerical integration

In newton cotes formula if $n = 2$ and the range of integration is from $[-1, 1]$ then the rule is known as Gauss Legendre three point foormula for numerical integration.

Any finite interval $[a, b]$ can be transformed to $[-1, 1]$ by using the formula.

$$x = \left(\frac{b-a}{2}\right) t + \left(\frac{b+a}{2}\right)$$

When

$$x = a, \qquad t = \frac{a - \left(\frac{b+a}{2}\right)}{\left(\frac{b-a}{2}\right)} = -1$$

$$x = b, \qquad t = \frac{b - \left(\frac{b+a}{2}\right)}{\left(\frac{b-a}{2}\right)} = 1$$

Now the rule becomes,

$$\int_{-1}^{1} f(x)dx = \sum_{k=0}^{2} \lambda_k f(x_k)$$

$$\implies \int_{-1}^{1} f(x)dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2) \tag{6}$$

Here we have six unknown $\lambda_0,\ \lambda_1,\ \lambda_2,\ x_0,\ x_1,\ x_2$
So the rule is exact for all polynomial of degree ≤ 5. i.e $1,\ x,\ x^2,\ x^3,\ x^4,\ x^5$.
Let $f(x) = 1$ from equation (6)

$$2 = \lambda_0 + \lambda_1 + \lambda_2 \tag{7}$$

Let $f(x) = x$ from equation (6)

$$0 = \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 \tag{8}$$

Let $f(x) = x^2$ from equation (6)

$$\frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 \tag{9}$$

Let $f(x) = x^3$ from equation (6)

$$0 = \lambda_0 x_0^3 + \lambda_1 x_1^3 + \lambda_2 x_2^3 \tag{10}$$

Let $f(x) = x^4$ from equation (6)

$$\frac{2}{5} = \lambda_0 x_0^4 + \lambda_1 x_1^4 + \lambda_2 x_2^4 \tag{11}$$

Let $f(x) = x^5$ from equation (6)

$$0 = \lambda_0 x_0^5 + \lambda_1 x_1^5 + \lambda_2 x_2^5 \tag{12}$$

Solving equation (7) to (12), we get

$$\lambda_0 = \frac{5}{9}, \quad \lambda_1 = \frac{8}{9}, \quad \lambda_2 = \frac{5}{9}$$

$$x_0 = -\sqrt{\frac{3}{5}}, \quad x_1 = 0, \quad x_2 = \sqrt{\frac{3}{5}}$$

Putting these values in equation (6) we get

$$\int_{-1}^{1} f(x)dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

is called Gauss Legendre three point formula for numerical integration.

Example 1

Evaluate

$$\int_0^1 \frac{1}{1+x}$$

by using

(i) Gauss Legendre two point formula $(n = 1)$

(ii) Gauss Legendre two point formula $(n = 2)$

Solution :

Any finite interval $[a, b]$ can be transformed to $[-1, 1]$ by using the formula

$$\begin{aligned} x =& \left(\frac{b-a}{2}\right) t + \left(\frac{b+a}{2}\right) \\ \implies \quad x =& \frac{1-0}{2} t + \frac{1+0}{2} \\ \implies \quad x =& \frac{t+1}{2} \\ \implies \quad t =& 2x - 1, \qquad dt = 2dx \end{aligned}$$

Now

$$\int_0^1 \frac{1}{1+x} = \int_{-1}^1 \frac{\frac{1}{2}dt}{1+\left(\frac{t+1}{2}\right)} = \int_{-1}^1 \frac{dt}{t+3}$$

(i) Gauss Legendre two point formula $(n = 1)$

$$\begin{aligned}
\int_0^1 \frac{1}{1+x} = \int_{-1}^1 \frac{dt}{t+3}, \qquad f(t) = \frac{1}{t+3} \\
=& f\left(-\sqrt{\frac{1}{3}}\right) + f\left(\sqrt{\frac{1}{3}}\right) \\
=& \frac{1}{\left(-\sqrt{\frac{1}{3}}+3\right)} + \frac{1}{\left(\sqrt{\frac{1}{3}}+3\right)} \\
=& 0.692307
\end{aligned}$$

(ii) Gauss Legendre two point formula $(n = 2)$

$$\int_0^1 \frac{1}{1+x} = \int_{-1}^{1} \frac{dt}{t+3}, \qquad f(t) = \frac{1}{t+3}$$

$$= \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{1}{\left(-\sqrt{\frac{3}{5}}+3\right)} + \frac{8}{9}\left(\frac{1}{0+3}\right) + \frac{1}{\left(\sqrt{\frac{3}{5}}+3\right)}$$

$$= 0.693121$$

Example 2

Evaluate

$$\int_1^2 \frac{1}{1+x^2}$$

by using

(i) Gauss Legendre two point formula $(n = 1)$

(ii) Gauss Legendre two point formula $(n = 2)$

Solution :

Any finite interval $[a, b] = [1, 2]$ can be transformed to $[-1, 1]$ by using the formula.

$$x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right) = \frac{2-1}{2}t + \frac{2+1}{2}$$

$$\implies 2x = t + 3$$

$$\implies x = \frac{t+3}{2}, \quad dx = \frac{1}{2}dt$$

When $x = 1, \quad t = -1$ and $x = 2, \quad t = 1$

Thus,

$$\int_1^2 \frac{dx}{1+x^2} = \int_{-1}^1 \frac{1}{1+\left(\frac{t+3}{2}\right)^2}\frac{1}{2}dt = \int_{-1}^1 \frac{2}{4+(t+3)^2}dt$$

(i) Gauss Legendre two point formula $(n = 1)$

$$\int_1^2 \frac{dx}{1+x^2} = \int_{-1}^1 \frac{2}{4+(t+3)^2}dt, \qquad f(t) = \frac{2}{4+(t+3)^2}$$

$$= f\left(-\sqrt{\frac{1}{3}}\right) + f\left(\sqrt{\frac{1}{3}}\right)$$

$$= \frac{2}{4+\left(-\sqrt{\frac{1}{3}}+3^2\right)} + \frac{2}{4+\left(\sqrt{\frac{1}{3}}+3^2\right)}$$

$$= 0.3217158$$

(ii) Gauss Legendre two point formula $(n = 2)$

$$\int_1^2 \frac{dx}{1+x^2} = \int_{-1}^{1} \frac{2}{4+(t+3)^2}\,dt$$

$$f(t) = \frac{2}{4+(t+3)^2} = \frac{5}{9} f\left(-\sqrt{\frac{1}{3}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{1}{3}}\right)$$

$$= \frac{5}{9}\left[\frac{2}{4+\left(-\sqrt{\frac{3}{5}}+3^2\right)}\right] + \frac{8}{9}\left[\frac{2}{4+(0+3)^2}\right] + \frac{5}{9}\left[\frac{2}{4+\left(\sqrt{\frac{3}{5}}+3^2\right)}\right]$$

$$= 0.3217559$$

Any Questions?

Thank You

Lecture - 19

Euler methods

Let is consider the first order differential equation

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

We have the Euler's method is

$$y_{n+1} = y(x_{n+1}) = y_n + hf(x_n, y_n) \qquad n = 0, 1, 2, \ \ldots\ldots$$

Where h is the step length

First approximation $(n = 0)$

$$y_1 = y(x_1) = y_0 + hf(x_0, y_0)$$

Second approximation $(n = 1)$

$$y_2 = y(x_2) = y_1 + hf(x_1, y_1)$$

Third approximation $(n = 2)$

$$y_3 = y(x_3) = y_2 + hf(x_2, y_2)$$

Example 1

Solve $y' = -2xy^2,\ y(0) = 1$ by Euler's method by taking $h = 0.2$

Solution :

Here $f(x, y) = -2xy^2,\ x_0 = 0,\ y_0 = 1,\ h = 0.2$

So, $x_1 - 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8,\ x_5 = 1$

We have the Euler's method is

$$y_{n+1} = y(x_{n+1}) = y_n + hf(x_n, y_n) \qquad n = 0, 1, 2, \ldots\ldots$$

First approximation $(n = 0)$

$$\begin{aligned} y_1 &= y(x_1) = y_0 + hf(x_0, y_0) \\ \implies y_1 &= y(0.4) = 1 + 0.2f(0, 1) \\ &= 1 + 0.2(-2 \times 0 \times 1^2) = 1 \end{aligned}$$

Second approximation $(n = 1)$

$$\begin{aligned} y_2 &= y(x_2) = y_1 + hf(x_1, y_1) \\ \implies y_2 &= y(0.4) = 1 + (0.2)f(0.2, 1) \\ &= 1 + 0.2(-2 \times 0.2 \times 1^2) = 0.92 \end{aligned}$$

Third approximation $(n = 2)$

$$\begin{aligned} y_3 =& y(x_3) = y_2 + hf(x_2, y_2) \\ \implies y_3 =& y(0.6) = 0.92 + (0.2)f(0.4, 0.92) \\ =& 0.92 + 0.2(-2 \times 0.4 \times (0.92)^2) = 0.784576 \end{aligned}$$

Fourth approximation $(n = 3)$

$$\begin{aligned} y_4 =& y(x_4) = y_3 + hf(x_3, y_3) \\ \implies y_4 =& y(0.8) = 0.784576 + (0.2)f(0.6, 0.784576) \\ =& 0.784576 + 0.2(-2 \times 0.6 \times (0.784576)^2) = 0.636841 \end{aligned}$$

Fifth approximation $(n = 4)$

$$\begin{aligned} y_5 =& y(x_5) = y_4 + hf(x_4, y_4) \\ \implies y_5 =& y(1) = 0.636841 + (0.2)f(0.8, 0.636841) \\ =& 0.636841 + 0.2(-2 \times 0.8 \times (0.636841)^2) = 0.507059 \end{aligned}$$

Example 2

Solve by Euler's method $y' = x + y, \quad y(1) = 3.$

Solution :

Here $f(x, y) = x + y, \quad x_0 = 1, \quad y_0 = 3$

Let $h = 0.25$ then $x_1 = 1.25, \quad x_2 = 1.5, \quad x_3 = 1.75, \quad x_4 = 2$

We have the Euler's method is

$$y_{n+1} = y(x_{n+1}) = y_n + hf(x_n, y_n) \qquad n = 0, 1, 2, \ldots\ldots$$

First approximation $(n = 0)$

$$\begin{aligned} y_1 &= y(x_1) = y_0 + hf(x_0, y_0) \\ \implies y_1 &= y(1.25) = 3 + 0.25f(1, 3) \\ &= 3 + 0.25(1 + 3) = 4 \end{aligned}$$

Second approximation $(n = 1)$

$$\begin{aligned} y_2 =& y(x_2) = y_1 + hf(x_1, y_1) \\ \implies y_2 =& y(1.5) = 4 + 0.25f(1.25, 4) \\ =& 4 + 0.25(1.25 + 4) = 5.3125 \end{aligned}$$

Third approximation $(n = 2)$

$$\begin{aligned} y_3 =& y(x_3) = y_2 + hf(x_2, y_2) \\ \implies y_3 =& y(1.75) = 5.3125 + 0.25f(1.5, 5.3125) \\ =& 5.3125 + 0.25(1.5 + 5.3125) = 7.015625 \end{aligned}$$

Fourth approximation $(n = 3)$

$$\begin{aligned} y_4 =& y(x_4) = y_3 + hf(x_3, y_3) \\ \implies y_4 =& y(2) = 7.015625 + 0.25f(1.75, 7.015625) \\ =& 7.015625 + 0.25(1.75 + 3) = 9.207031 \end{aligned}$$

Example 3

Solve $y' = x^2 + y^2, \quad y(0) = 0$ by Euler's method to evaluate $y(0.5)$

Solution :

Here $f(x, y) = x^2 + y^2, \quad x_0 = 0, \quad y_0 = 0$

Let $h = 0.1$ then $x_1 = 0.1, \quad x_2 = 0.2, \quad x_3 = 0.3, \quad x_4 = 0.4, \quad x_5 = 0.5$

We have the Euler's method is

$$y_{n+1} = y(x_{n+1}) = y_n + hf(x_n, y_n) \qquad\qquad n = 0, 1, 2, \ .\ .\ .\ .\ .$$

First approximation $(n = 0)$

$$\begin{aligned} y_1 &= y(x_1) = y_0 + hf(x_0, y_0) \\ \implies y_1 &= y(0.1) = 0 + (0.1)f(0, 0) \\ &= 0 + (0.1)(0^2 + 0^2) = 0 \end{aligned}$$

Second approximation $(n = 1)$

$$\begin{aligned} y_2 &= y(x_2) = y_1 + hf(x_1, y_1) \\ \implies y_1 &= y(0.2) = 0 + (0.1)f(0.1, 0) \\ &= 0 + (0.1)((0.1)^2 + 0^2) = 0.001 \end{aligned}$$

Third approximation $(n = 2)$

$$\begin{aligned}
& y_3 = y(x_3) = y_2 + hf(x_2, y_2) \\
\implies & y_2 = y(0.3) = 0.001 + (0.1)f(0.2, 0.001) \\
& = 0.001 + (0.1)((0.2)^2 + (0.001)^2) = 0.005
\end{aligned}$$

Fourth approximation $(n = 3)$

$$\begin{aligned}
& y_4 = y(x_4) = y_3 + hf(x_3, y_3) \\
\implies & y_3 = y(0.4) = 0.005 + (0.1)f(0.3, 0.005) \\
& = 0.005 + (0.1)((0.3)^2 + (0.005)^2) = 0.014
\end{aligned}$$

Fifth approximation $(n = 4)$

$$\begin{aligned}
& y_5 = y(x_5) = y_4 + hf(x_4, y_4) \\
\implies & y_4 = y(0.5) = 0.014 + (0.1)f(0.4, 0.014) \\
& = 0.014 + (0.1)((0.4)^2 + (0.014)^2) = 0.03
\end{aligned}$$

Let is consider the first order differential equation

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

We have the improved Euler's method is

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y^*_{n+1})\right] \qquad n = 0, 1, 2, \; \ldots\ldots$$

Where

$$y^*_{n+1} = hf(x_n, y_n)$$

First approximation $(n = 0)$

$$y^*_1 = hf(x_0, y_0)$$

$$y_1 = y(x_1) = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y^*_1)\right]$$

Second approximation $(n = 1)$

$$y_2^* = hf(x_1, y_1)$$

$$y_2 = y(x_2) = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_1 + h, y_2^*)\right]$$

Third approximation $(n = 2)$

$$y_3^* = hf(x_2, y_2)$$

$$y_3 = y(x_3) = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_2 + h, y_3^*)\right]$$

and so on.

Example 4

Solve $y' = x + y, \quad y(0) = 1$ by improved Euler's method.

Solution :

Here $f(x, y) = x + y, \quad x_0 = 0, \quad y_0 = 1$

Let $h = 0.25$

$x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1$

We have the improved Euler's method is

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y^*_{n+1})\right] \qquad n = 0, 1, 2, \ldots\ldots$$

Where

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

First approximation $(n = 0)$

$$
\begin{aligned}
y_1^* &= y_0 + hf(x_0, y_0) \\
&= 1 + 0.25 f(0, 1) \\
&= 1 + 0.25(0 + 1) = 1.25 \\
y_1 = y(x_1) &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1^*)\right] \\
\implies \; y_1 = y(0.25) &= 1 + \frac{0.25}{2}\left[f(0, 1) + f(0 + 0.25, 1.25)\right] \\
&= 1 + \frac{0.25}{2}\left[(0 + 1) + (0.25 + 1.25)\right] \\
&= 1.3125
\end{aligned}
$$

Second approximation $(n = 1)$

$$\begin{aligned}
y_2^* =& y_1 + hf(x_1, y_1) \\
=& 1.3125 + 0.25f(0.25, 1.3125) \\
=& 1.3125 + 0.25(0.25 + 1.3125) = 1.703125 \\
y_2 = y(x_2) =& y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_1 + h, y_2^*)\right] \\
\implies y_2 = y(0.5) =& 1.3125 + \frac{0.25}{2}\left[f(0.25, 1.3125) + f(0.25 + 0.25, 1.703125)\right] \\
=& 1.3125 + \frac{0.25}{2}\left[(0.25 + 1.3125) + (0.5 + 1.703125)\right] \\
=& 1.783203
\end{aligned}$$

Third approximation $(n = 2)$

$$
\begin{aligned}
y_3^* =& y_2 + hf(x_2, y_2) \\
=& 1.783203 + 0.25 f(0.5, 1.783203) \\
=& 1.783203 + 0.25(0.5 + 1.783203) = 2.354003 \\
y_3 = y(x_3) =& y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_2 + h, y_3^*)\right] \\
\implies \quad y_3 = y(0.75) =& 1.783203 + \frac{0.25}{2}[f(0.5, 1.783203) \\
& + f(0.5 + 0.25, 2.354003)] \\
=& 1.783203 + \frac{0.25}{2}\left[(0.5 + 1.783203) + (0.75 + 2.354003)\right] \\
=& 2.456603
\end{aligned}
$$

Fourth approximation $(n = 3)$

$$\begin{aligned}
y_4^* =& y_3 + hf(x_3, y_3)\\
=&2.456603 + 0.25f(0.75, 2.456603)\\
=&2.456603 + 0.25(0.75 + 2.456603) = 3.258253\\
y_4 = y(x_4) =& y_3 + \frac{h}{2}\left[f(x_3, y_3) + f(x_3 + h, y_4^*)\right]\\
\implies \quad y_4 = y(1) =&2.456603 + \frac{0.25}{2}[f(0.75, 2.456603)\\
&+f(0.75 + 0.25, 3.258253)]\\
=&2.456603 + \frac{0.25}{2}\left[(0.75 + 2.456603) + (1 + 3.258253)\right]\\
=&3.38971
\end{aligned}$$

Example 5

Solve $y' = -2xy^2, \quad y(0) = 1$ by improved Euler's method to find $y(0.6)$

Solution :

Here $f(x, y) = -2xy^2, \quad x_0 = 0, \quad y_0 = 1$

Let $h = 0.2$

$x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6$

We have the improved Euler's method is

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y^*_{n+1})\right] \qquad n = 0, 1, 2, \ldots\ldots$$

Where

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

First approximation $(n = 0)$

$$
\begin{aligned}
y_1^* =& y_0 + hf(x_0, y_0) \\
=& 1 + (0.2)f(0, 1) \\
=& 1 + (0.2)(-2 \times 0 \times 1^2) = 1
\end{aligned}
$$

$$
\begin{aligned}
y_1 = y(x_1) =& y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1^*)\right] \\
\implies y_1 = y(0.2) =& 1 + \frac{0.2}{2}\left[f(0, 1) + f(0 + 0.2, 1)\right] \\
=& 1 + \frac{0.2}{2}\left[-2 \times 0 \times 1^2 - 2 \times 0.2 \times 1^2\right] \\
=& 0.992
\end{aligned}
$$

Second approximation $(n = 1)$

$$
\begin{aligned}
y_2^* =& y_1 + hf(x_1, y_1) \\
=& 0.992 + (0.2)f(0.2, 0.992) \\
=& 0.992 + (0.2)(-2 \times 0.2 \times 0.992^2) = 0.976128
\end{aligned}
$$

$$
\begin{aligned}
y_2 = y(x_2) =& y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_1 + h, y_2^*)\right] \\
\implies y_2 = y(0.4) =& 0.992 + \frac{0.2}{2}\left[f(0.2, 0.992) + f(0.2 + 0.2, 0.976128)\right] \\
=& 0.992 + \frac{0.2}{2}\left[-2 \times 0.2 \times 0.992^2 - 2 \times 0.4 \times 0.976128^2\right] \\
=& 0.8764113
\end{aligned}
$$

Third approximation $(n = 2)$

$$
\begin{aligned}
y_3^* =& y_2 + hf(x_2, y_2) \\
=& 0.8764113 + (0.2)f(0.4, 0.8764113) \\
=& 0.8764113 + (0.2)(-2 \times 0.2 \times 0.8764113^2) = 0.753515
\end{aligned}
$$

$$
\begin{aligned}
y_3 = y(x_3) =& y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_2 + h, y_3^*)\right] \\
\implies y_3 = y(0.6) =& 0.8764113 + \frac{0.2}{2}[f(0.4, 0.8764113) \\
& + f(0.4 + 0.2, 0.753515)] \\
=& 0.8764113 + \frac{0.2}{2}[-2 \times 0.2 \times 0.8764113^2 \\
& -2 \times 0.6 \times 0.753515^2] \\
=& 0.746829
\end{aligned}
$$

Example 6

Solve $y' = y + e^x, \quad y(0) = 0$. Find y(0.4) by improved Euler's method by taking $h = 0.2$

Solution :

Here $f(x, y) = y + e^x, \quad x_0 = 0, \quad y_0 = 1, \quad h = 0.2$

$x_1 = 0.2, \quad x_2 = 0.4 = 0.6$

We have the improved Euler's method is

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y^*_{n+1})\right] \qquad n = 0, 1, 2, \ldots\ldots$$

Where

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

First approximation $(n = 0)$

$$\begin{aligned}
y_1^* =& y_0 + hf(x_0, y_0) \\
=& 0 + (0.2)f(0, 0) \\
=& 0 + (0.2)(0 + e^0) = 0.2
\end{aligned}$$

$$\begin{aligned}
y_1 = y(x_1) =& y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1^*)\right] \\
\implies \quad y_1 = y(0.2) =& 0 + \frac{0.2}{2}\left[f(0, 0) + f(0 + 0.2, 0.2)\right] \\
=& 0 + \frac{0.2}{2}\left[(0 + e^0)(0.2 + e^{0.2})\right] \\
=& 0.24214
\end{aligned}$$

Second approximation $(n = 1)$

$$
\begin{aligned}
y_2^* =& y_1 + hf(x_1, y_1) \\
=& 0.24214 + (0.2)f(0.2 + 0.2, 0.24214) \\
=& 0.24214 + (0.2)(0.4 + e^{0.24214}) = 1.62667
\end{aligned}
$$

$$
\begin{aligned}
y_2 = y(x_2) =& y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_1 + h, y_2^*)\right] \\
\implies y_2 = y(0.4) =& 0.24214 + \frac{0.2}{2}\left[f(0.2, 0.24214) + f(0.2 + 0.2, 1.62667)\right] \\
=& 0.24214 + \frac{0.2}{2}\left[(0.2 + e^{0.24214})(0.4 + e^{1.62667})\right] \\
=& 0.59116
\end{aligned}
$$

Any Questions?

Thank You

Lecture - 20

Let us consider the first order differential equations

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

Runga Kutta method of first order

It is equivalent to Eulers method.

Runge Kutta method of second order

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{1}{2}(k_1 + k_2)$$

Where,

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + h, y_n + k_1) \end{aligned}$$

$$n = 0, 1, 2, \ \ldots.$$

Runge Kutta method of third order

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Where,

$$\begin{aligned} k_1 =& hf(x_n, y_n) \\ k_2 =& hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ k_3 =& hf(x_n + h, y_n + k_2) \end{aligned}$$

$$n = 0, 1, 2, \ \ldots .$$

Runge Kutta method of fourth order

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Where,

$$\begin{aligned}
k_1 =& hf(x_n, y_n) \\
k_2 =& hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
k_3 =& hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\
k_4 =& hf(x_n + h, y_n + k_3)
\end{aligned}$$

$$n = 0, 1, 2, \ \ldots .$$

Example 1

Solve $y' = -2xy^2, \;\; y(0) = 1$ by Runge Kutta method to evaluate $y(0.1)$ and $y(0.2)$.

Solution :

Here $f(xy) = -2xy^2, \;\; x_0 = 0, \;\; y_0 = 1$, Let $h = 0.1$ then $x_1 = 0.1, \;\; x_2 = 0.2$.
We have the Runge Kutta method of fourth order is

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Where,

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\ k_4 &= hf(x_n + h, y_n + k_3) \end{aligned}$$

$$n = 0, 1, 2, \ldots$$

First approximation $(n = 0)$

$$
\begin{aligned}
k_1 =& hf(x_0, y_0) = (0.1)f(0, 1) = (0.1)(-2 \times 0 \times 1^2) = 0 \\
k_2 =& hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f\left(0 + \frac{0.1}{2}, 1 + \frac{1}{2}.0\right) \\
=& (0.1)f(0.05, 1) = (0.1)(-2 \times 0.05 \times 1^2) = -0.01 \\
k_3 =& hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
=& (0.1)f\left(0 + \frac{0.1}{2}, 1 - \frac{0.01}{2}\right) = (0.1)f(0.05, 0.995) \\
=& (0.1)(-2 \times 0.05 \times 0.995^2) = -0.00990025
\end{aligned}
$$

$$
\begin{aligned}
k_4 =& hf(x_0 + h, y_0 + k_3) = (0.1)f(0 + 0.1, 1 - 0.00990025) \\
=& (0.1)f(0.1, 0.99009975) = -0.019605
\end{aligned}
$$

Now

$$
\begin{aligned}
y_1 = y(x_1) =& y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
\implies y_1 = y(0.1) =& 1 + \frac{1}{6}\Big(0 + 2(-0.01) + 2(-0.00990025) - 0.019605\Big) \\
=& 0.990099
\end{aligned}
$$

Second approximation $(n = 1)$

$$
\begin{aligned}
k_1 =& hf(x_1, y_1) = (0.1)f(0.1, 0.990099) \\
=& (0.1)(-2 \times 0.1 \times 0.990099^2) = -0.019605 \\
k_2 =& hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)f\left(0.1 + \frac{0.1}{2}, 0.990099 - \frac{0.019605}{2}\right) \\
=& (0.1)f(0.15, 0.9802965) = (0.1)(-2 \times 0.15 \times 0.9802965^2) = -0.028829 \\
k_3 =& hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) \\
=& (0.1)f\left(0 + \frac{0.1}{2}, 0.990099 - \frac{0.028829}{2}\right) = (0.1)f(0.15, 0.9756845) \\
=& (0.1)(-2 \times 0.15 \times 0.9756845^2) = -0.028558 \\
k_4 =& hf(x_1 + h, y_1 + k_3) = (0.1)f(0.1 + 0.1, 0.990099 - 0.028558) \\
=& (0.1)f(0.2, 0.961541) = (0.1)\left(-2 \times 0.2 \times 0.961541^2\right) \\
=& -0.036982
\end{aligned}
$$

Now

$$
\begin{aligned}
y_2 = y(x_2) &= y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
\implies y_2 = y(0.2) &= 0.990099 + \frac{1}{6}\big(-0.019605 - 2 \times 0.028829 - 2(-0.028558) - \\
&= 0.9044185
\end{aligned}
$$

Any Questions?

Thank You

Lecture - 21

Method for system of ordinary differential equations

Let us consider a system of ordinary differential equations

$$\frac{dy}{dx} = f_1(x, y, z)$$
$$\frac{dz}{dx} = f_2(x, y, z)$$
$$y(x_0) = y_0, \qquad x(z_0) = z_0$$

By R-K method of fourth order

$$k_1 = hf_1(x_n, y_n, z_n)$$
$$l_1 = hf_2(x_n, y_n, z_n)$$
$$k_2 = hf_1\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_1, z_n + \frac{1}{2}l_1\right)$$
$$l_2 = hf_2\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_1, z_n + \frac{1}{2}l_1\right)$$

$$
\begin{aligned}
k_3 =& hf_1\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_2, z_n + \frac{1}{2}l_2\right)\\
l_3 =& hf_2\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_2, z_n + \frac{1}{2}l_2\right)\\
k_4 =& hf_1(x_n + h, y_n + k_3, z_n + l_3)\\
l_4 =& hf_2(x_n + h, y_n + k_3, z_n + l_3)\\
y_{n+1} =& y(x_{n+1}) = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
z_{n+1} =& y(x_{n+1}) = z_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)
\end{aligned}
$$

$$n = 0, 1, 2, \; \ldots\ldots.$$

Example 1

Solve the system of differential equations

$$\frac{dy}{dx} = -0.5y, \qquad \frac{dz}{dx} = 4 - 0.3z - 0.1y$$
$$y(0) = 4, \qquad z(0) = 6$$

to find y and z at $x = 0.5$ by taking $h = 0.5$ using Runge Kutta method of 4^{th} order.

Solution :

Here

$$f_1(x, y, z) = -0.5y$$
$$f_2(x, y, z) = 4 - 0.3z - 0.1y$$

Let $x_0 = 0, \quad y_0 = 4, \quad z_0 = 6, \quad h = 0.5, \quad x_1 = 0.5$

By R-K method of fourth order.

First approximation $(n = 0)$

$$
\begin{aligned}
k_1 =& hf_1(x_0, y_0, z_0) = 0.5f_1(0, 4, 6) = 0.5(-0.5 \times 4) = -1 \\
l_1 =& hf_2(x_0, y_0, z_0) = 0.5f_2(0, 4, 6) = 0.5[4 - 0.3 \times 6 - 0.1 \times 4] = 0.9 \\
k_2 =& hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \\
=& 0.5f_1(0.25, 3.5, 6.45) = 0.5[-0.5 \times 3.5] = -0.875 \\
l_2 =& hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \\
=& 0.5f_2(0.25, 3.5, 6.45) = 0.5[4 - 0.3 \times 6.45 - 0.1 \times 3.5] = 0.86 \\
k_3 =& hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) \\
=& 0.5f_1(0.25, 3.56, 6.43) = 0.5[-0.5 \times 3.56] = -0.89 \\
l_3 =& hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) \\
=& 0.5f_2(0.25, 3.56, 6.43) = 0.5[4 - 0.3.43 - 0.1 \times 3.56] = 0.86
\end{aligned}
$$

$$
\begin{aligned}
k_4 =& h f_1(x_0 + h, y_0 + k_3, z_0 + l_3) \\
=& 0.5 f_1(0.5, 3.11, 6.86) = 0.5[-0.5 \times 3.11] = -0.78 \\
l_4 =& h f_2(x_0 + h, y_0 + k_3, z_0 + l_3) \\
=& 0.5 f_2(0.5, 3.11, 6.86) = 0.5[4 - 0.3 \times 6.86 - 0.1 \times 3.11] = 0.82 \\
y_1 =& y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
\Longrightarrow \quad y_1 =& y(0.5) = 4 + \frac{1}{6}[-1 - 2 \times 0.875 - 2 \times 0.89 - 0.78] \\
=& 4 - 0.885 = 3.115 \\
z_1 =& z(x_1) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\
\Longrightarrow \quad z_1 =& z(0.5) = 6 + \frac{1}{6}[0.9 + 2 \times 0.86 + 2 \times 0.86 + 0.82] \\
=& 6 + 0.86 = 6.86
\end{aligned}
$$

Example 2

Solve the system of differential equations

$$\frac{dy}{dx} = xz + 1, \qquad \frac{dz}{dx} = -xy$$
$$y(0) = 0, \qquad z(0) = 1$$

for $x = 0.3, 0.6$ using Runge Kutta method of 4^{th} order.

Solution :

Here

$$f_1(x, y, z) = xz + 1$$
$$f_2(x, y, z) = -xy$$

$x_0 = 0, \quad y_0 = 0, \quad z_0 = 1,$ Let $h = 0.3, \quad x_1 = 0.3, \quad x_2 = 0.6$

By R-K method of fourth order.

First approximation $(n = 0)$

$$
\begin{aligned}
k_1 &= hf_1(x_0, y_0, z_0) = 0.3f_1(0, 0, 1) = 0.3(0 + 1) = 0.3 \\
l_1 &= hf_2(x_0, y_0, z_0) = 0.3f_2(0, 0, 1) = 0.3 \times 0 = 0 \\
k_2 &= hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \\
&= 0.3f_1(0.15, 0.15, 1) = 0.3[0.15 \times 1 + 1] = -0.345 \\
l_2 &= hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \\
&= 0.2f_2(0.15, 0.15, 1) = 0.3[-0.15 \times 1.5] = -0.00675 \\
k_3 &= hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) \\
&= 0.3f_1(0.15, 0.1725, 0.9966) = 0.3[0.15 \times 0.9966 + 1] = 0.345 \\
l_3 &= hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) \\
&= 0.3f_2(0.15, 0.1725, 0.9966) = 0.3[-0.15 \times 0.1725] = -0.00776
\end{aligned}
$$

$$
\begin{aligned}
k_4 =& h f_1(x_0 + h, y_0 + k_3, z_0 + l_3)\\
=& 0.3 f_1(0.3, 0.345, 0.992) = 0.3[0.3 \times 0.992 + 1] = 0.3893\\
l_4 =& h f_2(x_0 + h, y_0 + k_3, z_0 + l_3)\\
=& 0.3 f_2(0.3, 0.345, 0.992) = 0.3[-0.3 \times 0.345] = -0.031\\
y_1 =& y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
\implies y_1 =& y(0.3) = 0 + \frac{1}{6}[0.3 + 2 \times 0.345 + 2 \times 0.345 + 0.3893]\\
=& 0 + 0.3448 = 0.3448\\
z_1 =& z(x_1) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)\\
\implies z_1 =& z(0.3) = 1 + \frac{1}{6}[0 - 2 \times 0.00675 - 2 \times 0.00776 - 0.031]\\
=& 1 - 0.01 = 0.99
\end{aligned}
$$

Second approximation $(n = 1)$

$$
\begin{aligned}
k_1 =& hf_1(x_1, y_1, z_1)\\
=& 0.3 f_1(0.3, 0.3448, 0.99)\\
=& 0.3(0.3 \times 0.99 + 1)\\
=& 0.3891\\
l_1 =& hf_2(x_1, y_1, z_1)\\
=& 0.3 f_2(0.3, 0.3448, 0.99)\\
=& 0.3(-0.3 \times 0.3448)\\
=& -0.031032
\end{aligned}
$$

$$
\begin{aligned}
k_2 =& h f_1\left(x_1+\frac{h}{2}, y_1+\frac{1}{2} k_1, z_0+\frac{1}{2} l_1\right) \\
=& 0.3 f_1\left(0.3+\frac{0.3}{2}, 0.3448+\frac{0.3891}{2}, 0.99-\frac{0.031032}{2}\right) \\
=& 0.3\,[(0.3+0.15)(0.99-0.015516)+1]=0.43155 \\
l_2 =& h f_2\left(x_1+\frac{h}{2}, y_1+\frac{1}{2} k_1, z_0+\frac{1}{2} l_1\right) \\
=& 0.3 f_2\left(0.3+\frac{0.3}{2}, 0.3448+\frac{0.3891}{2}, 0.99-\frac{0.031032}{2}\right) \\
=& -0.3\,[(0.3+0.15)(0.3448 \times 0.19455)]=-0.07281
\end{aligned}
$$

$$
\begin{aligned}
k_3 =& hf_1\left(x_1+\frac{h}{2}, y_1+\frac{1}{2}k_2, z_1+\frac{1}{2}l_2\right)\\
=& 0.3f_1\left(0.3+\frac{0.3}{2}, 0.3448+\frac{0.43155}{2}, 0.99-\frac{0.07281}{2}\right)\\
=& 0.3\left[(0.3+0.15)(0.99-0.03641)+1\right] = 0.42873\\
l_3 =& hf_2\left(x_1+\frac{h}{2}, y_1+\frac{1}{2}k_2, z_1+\frac{1}{2}l_2\right)\\
=& 0.3f_2\left(0.3+\frac{0.3}{2}, 0.3448+\frac{0.43155}{2}, 0.99-\frac{0.07281}{2}\right)\\
=& -0.3\left[(0.3+0.15)(0.3448\times 0.2158)\right] = -0.07568
\end{aligned}
$$

$$
\begin{aligned}
k_4 =& h f_1(x_1 + h, y_1 + k_3, z_1 + l_3) \\
=& 0.3 f_1(0.3 + 0.3, 0.3448 + 0.42873, 0.99 - 0.07568) \\
=& 0.3[0.6(0.99 - 0.07568) + 1] = 0.4646 \\
l_4 =& h f_2(x_1 + h, y_1 + k_3, z_1 + l_3) \\
=& 0.3 f_1(0.3 + 0.3, 0.3448 + 0.42873, 0.99 - 0.07568) \\
=& -0.3[(0.3 + 0.3)(0.3448 + 0.42873)] = -0.1393 \\
y_2 =& y(x_2) = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
\implies y_2 =& y(0.6) = 0.3448 + \frac{1}{6}[0.3891 + 2 \times 0.43155 + 2 \times 0.342873 + 0.4646] \\
=& 0.3448 + 0.4290 = 0.7738 \\
z_2 =& z(x_2) = z_1 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\
\implies z_2 =& z(0.6) = 0.99 + \frac{1}{6}[-0.031032 - 2 \times 0.07281 - 2 \times 0.07568 - 0.1393] \\
=& 0.99 - 0.077885 = 0.9121
\end{aligned}
$$

Any Questions?

Thank You

Lecture - 22

Methods for higher order differential equations

Consider the differential equation

$$\frac{d^2y}{dx^2} = f(x, y, y') \tag{1}$$

Given $y(x_0) = y_0, \;\; y'(x_0) = y'_0$

Putting

$$y' = z \tag{2}$$

$$y'' = z' \tag{3}$$

Using equation (2) and (3) in equation (1) it reduces to

$$z' = f(x, y, z) \tag{4}$$

and $y(x_0) = y_0, \;\; y'(x_0) = z(x_0) = z_0$

Solving equation (2) and (4) with condition $y(x_0) = y_0, \;\; z(x_0) = z_0$ simultaneously using the method for system of ordinary differential equation we can get the solution.

Example 1

Using Range Kutta metod of fourth order solve $y'' - 2y' + 2y = e^{2x} \sin x$ with conditions $y(0) = -0.4, \;\; y'(0) = -0.6$ to find $y(0.2)$ by taking $h = 0.2$.

Solution :

Given

$$y'' - 2y' + 2y = e^{2x} \sin x \tag{5}$$

$$y(0) = -0.4, \;\; y'(0) = -0.6$$

Put

$$y' = z \tag{6}$$

Using equation (6) in equation (5) it becomes

$$z' - 2z + 2y = e^{2x} \sin x \tag{7}$$

with conditions $y(0) = -0.4, \;\; z(0) = -0.6$

Now we have to solve the system of differential equations

$$y' = z$$

$$z' = 2z - 2y + e^{2x} \sin x$$

with conditions $y(0) = -0.4, \ \ z(0) = -0.6$

Now $f_1(x, y, z) = z, \ f_2(x, y, z) = 2z - 2y + e^{2x} \sin x$

Here $x_0 = 0, \ y_0 = -0.4, \ z_0 = -0.6, \ h = 0.2, \ x_1 = 0.4$

First approximation $(n = 0)$

$$\begin{aligned}
k_1 =& h f_1(x_0, y_0, z_0) \\
=& 0.2 f_1(0, -0.4, -0.6) = (0.2 \times 0.6) = -0.12 \\
l_1 =& h f_2(x_0, y_0, z_0) = 0.2 f_2(0, -0.4, -0.6) \\
=& (0.2)[2(-0.6) - 2(-0.4) + e^0 \sin 0] = -0.08 \\
k_2 =& h f_1 \left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2} k_1, z_0 + \frac{1}{2} l_1 \right) \\
=& 0.2 f_1 \left(0 + \frac{0.2}{2}, -0.4 - \frac{0.12}{2}, -0.6 - \frac{0.08}{2} \right) \\
=& 0.2 f_1(0.1, -0.46, -0.64) \\
=& 0.2(-0.64) = -0.128
\end{aligned}$$

$$
\begin{aligned}
l_2 =& hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)\\
=& 0.2f_2\left(0 + \frac{0.2}{2}, -0.4 - \frac{0.12}{2}, -0.6 - \frac{0.08}{2}\right)\\
=& 0.2f_2(0.1, -0.46, -0.64)\\
=& (0.2)[2(-0.64) - 2(-0.46) + e^{2(0.1)}\sin(0.1)] = -0.072\\
k_3 =& hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)\\
=& 0.2f_1\left(0 + \frac{0.2}{2}, -0.4 - \frac{0.128}{2}, -0.6 - \frac{0.072}{2}\right)\\
=& 0.2f_1(0.1, -0.464, -0.636)\\
=& 0.2(-0.636) = -0.1272
\end{aligned}
$$

$$
\begin{aligned}
l_3 =& h f_2\left(x_0+\frac{h}{2}, y_0+\frac{1}{2} k_2, z_0+\frac{1}{2} l_2\right) \\
=& 0.2 f_2\left(0+\frac{0.2}{2},-0.4-\frac{0.128}{2},-0.6-\frac{0.072}{2}\right) \\
=& 0.2 f_2(0.1,-0.464,-0.636) \\
=& (0.2)\left[2(-0.636)-2(-0.464)+e^{2(0.1)} \sin (0.1)\right]=-0.0684 \\
k_4 =& h f_1\left(x_0+h, y_0+k_3, z_0+l_3\right) \\
=& 0.2 f_1(0+0.2,-0.4-0.1272,-0.6-0.0684) \\
=& 0.2 f_1(0.2,-0.5272,-0.6684) \\
=& 0.2(-0.6684)=-0.1337
\end{aligned}
$$

$$
\begin{aligned}
l_4 =& h f_2(x_0 + h, y_0 + k_3, z_0 + l_3)\\
=& 0.2 f_2(0 + 0.2, -0.4 - 0.1272, -0.6 - 0.0684)\\
=& 0.2 f_2(0.2, -0.5272, -0.6684)\\
=& (0.2)[2(-0.6684) - 2(-0.5272) + e^{2(0.2)} \sin(0.2)] = -0.0684
\end{aligned}
$$

$$
\begin{aligned}
y_1 =& y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
\implies y_1 =& y(0.2) = -0.4 + \frac{1}{6}[-0.08 + 2 \times (-0.128) + 2 \times (-0.1272) - 0.1337]\\
=& -0.4 - 0.1207 = -0.5207
\end{aligned}
$$

Example 2

Using Range Kutta metod of fourth order solve $y'' + xy' + y = 0$ with conditions $y(0) = 1, \quad y'(0) = 0$ to find $y(0.1)$ by taking $h = 0.1$.

Solution :

Given

$$y'' + xy' + y = 0 \tag{8}$$

$$y(0) = 1, \quad y'(0) = 0$$

Put

$$y' = z \tag{9}$$

Using equation (8) in equation (9) it becomes

$$z' + xz + y = 0 \tag{10}$$

with conditions $y(0) = 1, \quad z(0) = 0$

Now we have to solve the system of differential equations

$$y' = z$$

$$z' = -xz - y$$

with conditions $y(0) = 1, \;\; z(0) = 0$
Now $f_1(x, y, z) = z, \;\; f_2(x, y, z) = -xz - y$
Here $x_0 = 0, \;\; y_0 = 1, \;\; z_0 = 0, \;\; h = 0.1, \;\; x_1 = 0.1$
First approximation $(n = 0)$

$$\begin{aligned} k_1 =& h f_1(x_0, y_0, z_0) = 0.1 f_1(0, 1, 0) = (0.1 \times 0) = -0 \\ l_1 =& h f_2(x_0, y_0, z_0) = 0.1 f_2(0, 1, 0) \\ =& (0.1)(-0 \times 0 - 1) = -0.1 \end{aligned}$$

$$
\begin{aligned}
k_2 =& h f_1\left(x_0+\frac{h}{2}, y_0+\frac{1}{2}k_1, z_0+\frac{1}{2}l_1\right)\\
=& 0.1 f_1\left(0+\frac{0.1}{2}, 1+\frac{0}{2}, 0-\frac{0.1}{2}\right)\\
=& 0.1 f_1(0.05, 1, -0.05)\\
=& 0.1(-0.05) = -0.005\\
l_2 =& h f_2\left(x_0+\frac{h}{2}, y_0+\frac{1}{2}k_1, z_0+\frac{1}{2}l_1\right)\\
=& 0.1 f_2\left(0+\frac{0.1}{2}, 1+\frac{0}{2}, 0-\frac{0.1}{2}\right)\\
=& 0.1 f_2(0.05, 1, -0.05)\\
=& (0.1)[-0.05(-0.05) - 1] = -0.09975
\end{aligned}
$$

$$
\begin{aligned}
k_3 =& h f_1\left(x_0+\frac{h}{2}, y_0+\frac{1}{2}k_2, z_0+\frac{1}{2}l_2\right)\\
=& 0.1 f_1\left(0+\frac{0.1}{2}, 1-\frac{0.005}{2}, 0-\frac{0.09975}{2}\right)\\
=& 0.1 f_1(0.05, 0.9975, -0.0499)\\
=& 0.1(-0.0499) = -0.00499\\
l_3 =& h f_2\left(x_0+\frac{h}{2}, y_0+\frac{1}{2}k_2, z_0+\frac{1}{2}l_2\right)\\
=& 0.1 f_2\left(0+\frac{0.1}{2}, 1-\frac{0.005}{2}, 0-\frac{0.09975}{2}\right)\\
=& 0.1 f_2(0.05, 0.9975, -0.0499)\\
=& (0.1)[-0.05(-0.0499) - 0.9975] = -0.0995
\end{aligned}
$$

$$
\begin{aligned}
k_4 =& h f_1(x_0 + h, y_0 + k_3, z_0 + l_3) \\
=& 0.1 f_1(0 + 0.1, 1 - 0.00499, 0 - 0.0995) \\
=& 0.1 f_1(0.1, 0.99511, -0.0995) \\
=& 0.1(-0.0995) = -0.00995
\end{aligned}
$$

$$
\begin{aligned}
l_4 =& h f_2(x_0 + h, y_0 + k_3, z_0 + l_3) \\
=& 0.1 f_2(0 + 0.1, 1 - 0.00499, 0 - 0.0995) \\
=& 0.1 f_2(0.1, 0.99511, -0.0995) \\
=& (0.1)[-0.1(-0.0995) - 0.99511] = -0.0985
\end{aligned}
$$

$$
\begin{aligned}
y_1 =& y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
\implies y_1 =& y(0.1) = 1 + \frac{1}{6}[-0.1 + 2 \times (-0.005) + 2 \times (-0.00499) - 0.00995] \\
=& 1 - 0.00498 = 0.99502
\end{aligned}
$$

Any Questions?

Thank You

Lecture - 23

Multistep Method

We will discuss two type of multi step methods for solving first order differential equations.

1. Milnes method
2. Adams Basforth method

Let us consider the ordinary differential equation with initial condition as

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

We have to find an approximate value of y for $x_n = x_0 + nh$ starting with (x_0, y_0)
Let $x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ x_3 = x_0 + 3h,\ \ldots\ldots\ x_n = x_0 + nh.$

STEP-I

Find the value of $y_1 = y(x_1),\ y_2 = y(x_2),\ y_3 = y(x_3)$ by using any one of the method previously discussed if it is not given.

STEP-II

Find

$$\begin{aligned} y_1' &= f_1 = f(x_1, y_1) \\ y_2' &= f_2 = f(x_2, y_2) \\ y_3' &= f_3 = f(x_3, y_3) \end{aligned}$$

STEP-III

By Milne's predictor formula

$$y_4 = y(x_4) = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

or

$$y_4 = y(x_4) = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

Find

$$y_4' = f_4 = f(x_4, y_4)$$

STEP-IV

By Milne's corrector formula

$$y_4 = y(x_4) = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

or

$$y_4 = y(x_4) = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

STEP-V

Find

$$y'_4 = f_4 = f(x_4, y_4)$$

By Milne's predictor formula

$$y_5 = y(x_5) = y_1 + \frac{4h}{3}[2y'_2 - y'_3 + 2y'_4]$$

or

$$y_5 = y(x_5) = y_1 + \frac{4h}{3}[2f_2 - f_3 + 2f_4]$$

Find

$$y'_5 = f_5 = f(x_5, y_5)$$

By Milne's corrector formula

$$y_5 = y(x_5) = y_3 + \frac{h}{3}[y'_3 + 4y'_4 + y'_5]$$

or

$$y_5 = y(x_5) = y_3 + \frac{h}{3}[f_3 + 4f_4 + f_5]$$

Continuing this process we can get the result.
In general, Milne's predictor formula is

$$y_{n+1} = y(x_{n+1}) = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

or

$$y_{n+1} = y(x_{n+1}) = y_{n-3} + \frac{4h}{3}[2f_{n-2} - f_{n-1} + 2f_n]$$

Milne's corrector formula

$$y_{n+1} = y(x_{n+1}) = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

or

$$y_{n+1} = y(x_{n+1}) = y_{n-1} + \frac{h}{3}[f_{n-1} + 4f_n + f_{n+1}]$$

Example 1

Find $y(2)$ if $y(x)$ is the solution of

$$\frac{dy}{dx} = \frac{x+y}{2}, \qquad y(0) = 2$$

Given that $y(0.5) = 2.636, \;\; y(1.0) = 3.595$ and $y(1.5) = 4.968$ using Milne's method

Solution :

Given $f(x,y) = \frac{x+y}{2}, \;\; x_0 = 0, \;\; y_0 = 2$

Let $h = 0.5, \;\; x_1 = 0.5, \;\; x_2 = 1.0, \;\; x_3 = 1.5, \;\; x_4 = 2.0$

$y_1 = 2.636, \;\; y_2 = 3.595, \;\; y(3) = 4.968$

$$\begin{aligned} y_1' &= f_1 = f(x_1, y_1) = f(0.5, 2.636) = 1.568 \\ y_2' &= f_2 = f(x_2, y_2) = f(1.0, 3.595) = 2.2975 \\ y_3' &= f_3 = f(x_3, y_3) = f(1.5, 4.968) = 3.234 \end{aligned}$$

By Milnes Predictor formula we have

$$\begin{aligned} y_4 =& y(x_4) = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right] \\ \implies y_4 =& y(2) = 2 + \frac{4}{3}(0.5)\left[2(1.568) - 2.975 + 2(3.234)\right] \\ =& 6.871 \end{aligned}$$

$$y_4' = f_4 = f(x_4, y_4) = f(2, 6.871) = 4.4355$$

By Milnes corrector formula

$$\begin{aligned} y_4 =& y(x_4) = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right] \\ \implies y_4 =& y(2) = 3.595 + \frac{0.5}{3}\left[2.2975 + 4(3.234) + 4.4355\right] \\ =& 6.8732 \end{aligned}$$

Example 2

Use Milnes method to find $y(0.4)$ and $y(0.5)$ given that $y' = x - y^2, \quad y(0) = 1$ taking $h = 0.1$

Solution :

Here $f(x, y) = x - y^2, \quad x_0 = 0, \quad y_0 = 1$ and $h = 0.1$

$x_1 = 0.1, \quad x_2 = 0.2, \quad x_3 = 0.3, \quad x_4 = 0.4 = 1$

Now we have to find $y_1 = y(x_1) = y(0.1), \quad y_2 = y(x_2) = y(0.2), \quad y_3 = y(x_3) = y(0.3)$ by using R-K method of fourth order.

To find $y_1 = y(x_1) = y(0.1)$

$$
\begin{aligned}
k_1 = hf(x_0, y_0) &= 0.1f(0,1) = 0.1(0 - 1^2) = -0.1 \\
k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) &= 0.1f(0.05, 0.95) \\
&= 0.0(0.05 - 0.95^2) = -0.08525 \\
k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) &= 0.1f(0.05, 0.9574) \\
&= 0.1(0.05 - 0.9574^2) = -0.0867 \\
k_4 = hf(x_0 + h, y_0 + k_3) &= 0.1f(0.1, 0.9137) \\
&= 0.1(0.1 - 0.9137^2) = -0.07348
\end{aligned}
$$

$$
\begin{aligned}
y_1 = y(x_1) = y(0.1) &= y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \\
&= 1 + \frac{1}{6}[-0.1 + 2(-0.08525) + 2(-0.0867) - 0.07348] \\
&= 0.9138
\end{aligned}
$$

Similarly we can find

$$y_2 = y(x_2) = y(0.2) = 0.85112$$
$$y_3 = y(x_3) = y(0.3) = 0.8076$$

$$y_1' = f_1 = f(x_1, y_1) = f(0.1, 0.9138) = -0.7350$$
$$y_2' = f_2 = f(x_2, y_2) = f(0.2, 0.85112) = -0.5248$$
$$y_3' = f_3 = f(x_3, y_3) = f(0.3, 0.8076) = -0.3522$$

By Milnes Predictor formula

$$
\begin{aligned}
y_4 =& y(x_4) = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right] \\
\implies y_4 =& y(0.4) = 1 + \frac{4(0.1)}{3}\left[2(-0.7350) + (0.5248) + 2(-0.3522)\right] \\
=& 0.78006
\end{aligned}
$$

$$
y_4' = f_4 = f(x_4, y_4) = f(0.4, 0.78006) = -0.20849
$$

By Milnes corrector formula

$$
\begin{aligned}
y_4 =& y(x_4) = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right] \\
\implies y_4 =& y(0.4) = 0.85112 + \frac{0.1}{3}\left[-0.5248 + 4(-0.3522) - 0.20849\right] \\
=& 0.77972
\end{aligned}
$$

$$
y_4' = f_4 = f(x_4, y_4) = f(0.4, 0.77972) = -0.20796
$$

By Milnes Predictor formula

$$
\begin{aligned}
y_5 =& y(x_5) = y_1 + \frac{4h}{3}\left[2f_2 - f_3 + 2f_4\right] \\
\implies y_5 =& y(0.5) = 0.9138 + \frac{4(0.1)}{3}\left[2(-0.5248) + 0.3522 + 2(-0.20796)\right] \\
=& 0.76534
\end{aligned}
$$

$$
y_5' = f_5 = f(x_5, y_5) = f(0.5, 0.76534) = -0.08575
$$

By Milnes corrector formula

$$
\begin{aligned}
y_5 =& y(x_5) = y_3 + \frac{h}{3}\left[f_3 + 4f_4 + f_5\right] \\
\implies y_5 =& y(0.5) = 0.8076 + \frac{0.1}{3}\left[(-0.3522) + 4(-0.20796) - 0.08575\right] \\
=& 0.76528
\end{aligned}
$$

Any Questions?

Thank You

Lecture - 24

Adams-Bashforth method

To solve

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

We have to find an approximate value of y for $x_n = x_0 = nh$ starting with (x_0, y_0)

Adams-Bashforth prediction formula

$$\begin{aligned} y_{n+1} =& y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right] \\ i.e \qquad y_{n+1} =& y_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right] \end{aligned}$$

Adams-Moulten correction formula

$$\begin{aligned} y_{n+1} =& y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right] \\ i.e \qquad y_{n+1} =& y_n + \frac{h}{24}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right] \end{aligned}$$

Continuing the procedure of Milnes's method and applying the above formula we can solve an ordinary differential equation by Adams Bashforth method.

Example 1

Solve $y' = 2e^x - y$ given
$y(0) = 2,\ y(0.1) = 2.010,\ y(0.2) = 2.040,\ y(0.3) = 2.090$ to get $y(0.4)$ and $y(0.5)$ by using Adams Bashforth method.

Solution :

Given $f(x, y) = 2e^x - y, \quad x_0 = 0, \quad y_0 = 2$
Let $h = 0.1, \quad x_1 = 0.1, \quad x_2 = 0.2, \quad x_3 = 0.3, \quad x_4 = 0.4, \quad x_5 = 0.5$
$y_1 = 2.010, \quad y_2 = 2.040, \quad y(3) = 2.090$

$$
\begin{aligned}
y_0' &= f_0 = f(x_0, y_0) = f(0, 2) = 0 \\
y_1' &= f_1 = f(x_1, y_1) = f(0.1, 2.010) = 0.2003 \\
y_2' &= f_2 = f(x_2, y_2) = f(0.2, 2.040) = 0.4028 \\
y_3' &= f_3 = f(x_3, y_3) = f(0.3, 2.090) = 0.6097
\end{aligned}
$$

By Adams Basforth Predictor formula

$$y_4 = y(x_4) = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$\implies y_4 = y(0.4) = 2.090 + \frac{0.1}{24}\left[55(0.6097) - 59(0.4028) + 37(0.2003) - 9(0)\right]$$

$$= 2.16158$$

$$y_4' = f_4 = f(x_4, y_4) = f(0.4, 2.16158) = 0.8221$$

By Adams Moulten corrector formula

$$y_4 = y(x_4) = y_3 + \frac{h}{24}\left[9f_4 + 19f_3 - 5f_2 + f_1\right]$$

$$\implies y_4 = y(0.4) = 2.090 + \frac{0.1}{24}\left[9(0.8221) + 19(0.6097) - 5(0.4028) + 0.2003\right]$$

$$= 2.16153$$

$$y_4' = f_4 = f(x_4, y_4) = f(0.4, 2.16153) = 0.82147$$

By Adams Basforth Predictor formula

$$
\begin{aligned}
y_5 =& y(x_5) = y_4 + \frac{h}{24}\left[55 f_4 - 59 f_3 + 37 f_2 - 9 f_1\right] \\
\implies y_5 =& y(0.5) = 2.16153 \\
&+ \frac{0.1}{24}\left[55(0.82147) - 59(0.6097) + 37(0.4028) - 9(0.2003)\right] \\
=& 2.25459
\end{aligned}
$$

$$
y_5' = f_5 = f(x_5, y_5) = f(0.5, 2.25459) = 1.0428
$$

By Adams Moulten corrector formula

$$
\begin{aligned}
y_5 =& y(x_5) = y_4 + \frac{h}{24}\left[9 f_5 + 19 f_4 - 5 f_3 + f_2\right] \\
\implies y_5 =& y(0.5) = 2.16153 \\
&+ \frac{0.1}{24}\left[9(1.0428) + 19(0.82147) - 5(0.6097) + 0.4028\right] \\
=& 2.25472
\end{aligned}
$$

Example 2

Use Adams Basforth method to find $y(0.8)$ and $y(1.0)$ given that $y' = 1 + y^2, \quad y(0) = 0.$

***Solution*:**

Here $f(x, y) = 1 + y^2, \quad x_0 = 0, \quad y_0 = 0$. Let $h = 0.2$

$x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6, \quad x_4 = 0.8, \quad x_5 = 1$

Now we have to find $y_1 = y(x_1) = y(0.2), \quad y_2 = y(x_2) = y(0.4), \quad y_3 = y(x_3) = y(0.6)$ by using R-K method of fourth order.

To find $y_1 = y(x_1) = y(0.2)$

$$k_1 = hf(x_0, y_0) = 0.2f(2, 0) = 0.2(1 + 0^2) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 0.1)$$

$$= 0.2(1 + 0.1^2) = 0.202$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1, 0.101)$$

$$= 0.2(1 + 0.101^2) = 0.20204$$

$$
\begin{aligned}
k_4 = hf(x_0+h, y_0+k_3) =& 0.2f(0.2, 0.20204) \\
=& 0.2(1+0.20204^2) = 0.20816 \\
y_1 = y(x_1) = y(0.2) =& y_0 + \frac{1}{6}[k_1+2k_2+2k_3+k_4] \\
=& 0 + \frac{1}{6}[0.2+2(0.202)+2(0.20204)+0.20816] \\
=& 0.2027
\end{aligned}
$$

Similarly we can find

$$
\begin{aligned}
y_2 =& y(x_2) = y(0.4) = 0.4228 \\
y_3 =& y(x_3) = y(0.6) = 0.6841
\end{aligned}
$$

$$
\begin{aligned}
y_0' =& f_0 = f(x_0, y_0) = f(0,0) = 1 \\
y_1' =& f_1 = f(x_1, y_1) = f(0.2, 0.2027) = 1.04109 \\
y_2' =& f_2 = f(x_2, y_2) = f(0.4, 0.4228) = 1.1788 \\
y_3' =& f_3 = f(x_3, y_3) = f(0.6, 0.6841) = 1.4679
\end{aligned}
$$

By Adams Basforth Predictor formula

$$
\begin{aligned}
y_4 =& y(x_4) = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right] \\
\implies y_4 =& y(0.8) = 0.6841 \\
&+ \frac{0.2}{24}\left[55(1.4679) - 59(1.1788) + 37(1.04109) - 9(1)\right] \\
=& 1.02331
\end{aligned}
$$

$$y'_4 = f_4 = f(x_4, y_4) = f(0.8, 1.02331) = 2.0473$$

By Adams Moulten corrector formula

$$
\begin{aligned}
y_4 =& y(x_4) = y_3 + \frac{h}{24}\left[9f_4 + 19f_3 - 5f_2 + f_1\right] \\
\implies y_4 =& y(0.8) = 0.6841 \\
&+ \frac{0.2}{24}\left[9(2.0473) + 19(1.4679) - 5(1.1788) + 1.04109\right] \\
=& 1.0296
\end{aligned}
$$

$$y'_4 = f_4 = f(x_4, y_4) = f(0.8, 1.0296) = 2.06007$$

By Adams Basforth Predictor formula

$$
\begin{aligned}
y_5 =& y(x_5) = y_4 + \frac{h}{24}\left[55f_4 - 59f_3 + 37f_2 - 9f_1\right] \\
\implies y_5 =& y(1.0) = 1.0296 \\
&+\frac{0.1}{24}\left[55(2.06007) - 59(1.4679) + 37(1.1788) - 9(1.04109)\right] \\
=&1.5375
\end{aligned}
$$

$$y_5' = f_5 = f(x_5, y_5) = f(1, 1.5375) = 3.3659$$

By Adams Moulten corrector formula

$$
\begin{aligned}
y_5 =& y(x_5) = y_4 + \frac{h}{24}\left[9f_5 + 19f_4 - 5f_3 + f_2\right] \\
\implies y_5 =& y(1.0) = 1.0296 \\
&+\frac{0.2}{24}\left[9(3.3639) + 19(2.06007) - 5(1.4679) + 1.1788\right] \\
=&1.567
\end{aligned}
$$

Any Questions?

Thank You