Mathematics III (RMA3A001)

Module I

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Lecture - 9

Let us consider a system of three linear equations with three unknowns and is given by,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

In matrix form the above set of equations can be represented as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

i.e.

$$A\mathbf{x} = B \tag{1}$$

Where, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Let

$$A = LU \tag{2}$$

Where L is a lower triangular matrix and U is a unit upper triangular matrix

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Using equation (2) in equation (1)

$$LU\mathbf{x} = B \tag{3}$$

Put

$$U\mathbf{x} = \mathbf{y} \quad \text{where} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{4}$$

Using equation (4) in equation (3)

$$L\mathbf{y} = B \tag{5}$$

Solving equation (5) by forward substitution method find out the value of $\mathbf{y}$. Putting the value of $\mathbf{y}$ in equation (4) find out the value of $\mathbf{x}$ in backward substitution method.

Example 1

Solve the following system of equations by using crouts method.

$$3x + y + z = 4$$

$$x + 2y + 2z = 3$$

$$2x + y + 3z = 4$$

Solution :

In matrix form the above set of equations can be written as,

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$\implies A\mathbf{x} = B \tag{6}$$

Let

$$A = LU \tag{7}$$

Where L is a lower triangular matrix and U is a unit upper triangular matrix

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Putting in equation (7)

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Equating both sides, we have

$$l_{11} = 3, \qquad l_{21} = 1, \qquad l_{31} = 2$$

$$l_{11}u_{12} = 1, \qquad \implies 3u_{12} = 1, \qquad \implies u_{12} = \frac{1}{3}$$

$$l_{11}u_{13} = 1, \qquad \implies 3u_{13} = 1, \qquad \implies u_{13} = \frac{1}{3}$$

$$l_{21}u_{12} + l_{22} = 2, \qquad \implies 1 \times \frac{1}{3} + l_{22} = 2, \qquad \implies l_{22} = \frac{5}{3}$$

$$l_{31}u_{12} + l_{32} = 1, \qquad \implies 2 \times \frac{1}{3} + l_{32} = 1, \qquad \implies l_{32} = \frac{1}{3}$$

$$l_{21}u_{13} + l_{22}u_{23} = 2, \qquad \implies 1 \times \frac{1}{3} + 1 \times \frac{1}{3}u_{23} = 2, \qquad \implies u_{23} = 1$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 3, \qquad \implies 2 \times \frac{1}{3} + \frac{1}{3} \times 1 + l_{33} = 2, \qquad \implies l_{33} = 2$$

Thus,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & \frac{5}{3} & 0 \\ 2 & \frac{1}{3} & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Using equation (7) in equation (6)

$$LU\mathbf{x} = B \tag{8}$$

Put

$$U\mathbf{x} = \mathbf{y} \quad \text{where} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{9}$$

Using equation (9) in equation (8)

$$L\mathbf{y} = B \tag{10}$$

From equation (10) we have,

$$Ly = B$$

$$\implies \begin{bmatrix} 3 & 0 & 0 \\ 1 & \frac{5}{3} & 0 \\ 2 & \frac{1}{3} & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3y_1 \\ y_1 + \frac{5}{3}y_2 \\ 2y_1 + \frac{1}{3}y_2 + 2y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

Equating both sides and solving it by forward substitution method

$$3y_1 = 4, \qquad \implies y_1 = \frac{4}{3}$$

$$y_1 + \frac{5}{3}y_2 = 3, \qquad \implies \frac{4}{3} + \frac{5}{3}y_2 = 3 \qquad \implies y_2 = 1$$

$$2y_1 + \frac{1}{3}y_2 + 2y_3 = 4, \qquad \implies 2 \times \frac{4}{3} + \frac{1}{3} \times 1 + 2y_3 = 4 \qquad \implies y_3 = \frac{1}{2}$$

Thus

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Putting the value of $\mathbf{y}$ in equation (9)

$$U\mathbf{x} = \mathbf{y}$$

$$\implies \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$\implies \begin{bmatrix} x + \frac{1}{3}y + \frac{1}{3}z \\ y + z \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Equating both sides and solving it by backward substitution method, we have

$$z = \frac{1}{2}$$

$$y + z = 1 \quad \implies y + \frac{1}{2} = 1 \quad \implies y = \frac{1}{2}$$

$$x + \frac{1}{3}y + \frac{1}{3}z = \frac{4}{3} \quad \implies x + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{4}{3} \implies x = 1$$

Thus $x = 1$, $y = \frac{1}{2}$ and $z = \frac{1}{2}$ is the solution of the above system of equations by crouts method.

Any Questions?

Thank You