Linear Motion   Module-3 (Rectilinear Motion)

Speed: The Speed of a body may be defined as its rate of change of displacement w.r.t its surrounding. It is a Scalar quantity. (s)

Velocity:
It is defined as rate of change of displacement w.r.t its surroundings in a perticular direction. It is a vector quantity. (v)

Acceleration: It is defined as the rate of change of its velocity. It is said to be positive, when the velocity of a body increases with time, and negative when the velocity decreases with time. (a)

Motion under uniform acceleration

let u = Initial Velocity
v = final velocity
t = Time taken by the particle to change its velocity from u to v. (In second)
a = Uniform positive acceleration
S = distance travelled in t seconds.

$$\boxed{v = u + at} \quad \text{------} \quad (1)$$

$$\text{Avg. Velocity} = \frac{u+v}{2}$$

$$S = \text{Avg. velocity} \times \text{Time}$$

$$= \left(\frac{u+v}{2}\right) \times t \quad \text{------} \quad (11)$$

Put the value of v in eqn (11)

$$S = \left(\frac{u+u+at}{2}\right) \times t$$

$$= \left(\frac{2u+at}{2}\right) t$$

$$= \frac{2ut}{2} + \frac{1}{2}at^2$$

$$\boxed{S = ut + \frac{1}{2}at^2} \quad \text{------} \quad (111)$$

From eq^n ① we find out,

$$t = \frac{v-u}{a}$$

Now substituting this value of t in eq^n ⑪

$$S = \left(\frac{u+v}{2}\right) \times \left(\frac{v-u}{a}\right)$$

$$= \frac{(v+u)(v-u)}{2a}$$

$$= \frac{v^2-u^2}{2a}$$

$$v^2-u^2 = 2as \Rightarrow \boxed{v^2 = u^2+2as}$$

Q// A motor car takes 10 seconds to cover 30 ~~minutes~~ meters and 12 sec to cover 42m. Find the uniform acceleration of the car and its velocity at the end of 15 sec.

When $t = 10$ sec

$S = 30$ m.

When $t = 12$ sec.

$S = 42$m

We know the distance travelled by the car in 10 sec

$$30 = ut + \frac{1}{2}at^2$$

$$= 10u + \frac{1}{2} \times a(10)^2$$

$$= 10u + 50a \Rightarrow 10u+50a = 30 \quad ①$$

Similarly distance travelled by the car in 12 sec

$$(S)\ 42 = ut + \frac{1}{2}at^2$$

$$42 = 12u + \frac{1}{2} \times a(12)^2$$

$$12u + 72a = 42 \quad ⑪$$

Solve eq^n ① & ⑪

$$a = 0.5\ m/s^2$$

Velocity at the end of 15 sec.

Substituting the value of a in eq^n ①

$$180 = 60u + (300 \times 0.5)$$

$$= 60u + 150$$

$\boxed{u = 0.5 m/s}$

Velocity of the car after 15 sec

We know that the velocity of the car

$$V = u + at$$

$$= 0.5 + (0.5 \times 15)$$

$$\boxed{V = 8 m/s.}$$

Motion under force of gravity

It is a perticular case of motion, under a constant acceleration of (g) where its value is taken as $9.8 m/s^2$. If there is a fall under gravity, the expression for velocity and distance travelled in terms of initial velocity, time and gravity acceleration will be

$$V = u + gt$$

$$S = ut + \frac{1}{2}gt^2$$

$$v^2 = u^2 + 2gs$$

But if the motion takes place against the force of gravity i.e the particle is projected upwards. the corresponding equations will be

$$V = -u + gt$$

$$S = -ut + \frac{1}{2}gt^2$$

$$V^2 = -u^2 + 2gs.$$

→ In this case, the value of u is taken as negative due to upward motion.

→ the distances in upward direction are taken as negative, while those in the downward direction are taken as positive.

For example

A stone is thrown vertically upward with a velocity of 29.4 m/s from the top of a tower 34.3 m high. Find the total time taken by the stone to reach the foot of the tower.

Initial velocity $u = -29.4 m/s$ (− due to upward motion)

and height of tower

$$h = 34.3 m$$

t = time taken by the stone to reach the foot of the tower.

$$h = ut + \frac{1}{2}gt^2$$

$$34.3 = ut + \frac{1}{2}gt^2$$

$$34.3 = (-29.4 \times t) + \frac{1}{2}9.8t^2$$

$$t^2 - 6t - 7 = 0$$

$$\boxed{t = 7s}$$

Distance travelled in the nth second

$$S_n = un + \frac{1}{2}an^2$$

$$S_{n-1} = u(n-1) + \frac{1}{2}a(n-1)^2$$

$$S = S_n - S_{n-1}$$

$$= un + \frac{1}{2}an^2 - \left[u(n-1) + \frac{1}{2}a(n-1)^2\right]$$

$$\boxed{S = u + \frac{a}{2}(2n-1)}$$

Q11 A train starts from rest with an acceleration 'a' and describes distances S_1 & S_2 & S_3 in the first, second & third second of its journey. Find the ratio $S_1 : S_2 : S_3$.

Initial velocity of train $(u) = 0$

acceleration $= a$

distance described in 1st second $= S_1$

distance described in 2nd second $= S_2$

distance described in 3rd second $= S_3$

We know that distance described by the train in 1st second

$$S_1 = u + \frac{a}{2}(2n_1 - 1) = 0 + \frac{a}{2}[(2 \times 1) - 1] = \frac{a}{2}$$

Similarly distance described in second second

$$S_2 = u + \frac{a}{2}(2n_2 - 1) = 0 + \frac{a}{2}[(2 \times 2) - 1] = \frac{3a}{2}$$

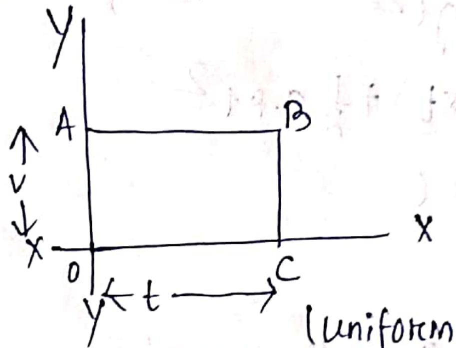
distance described in third second

$$S_3 = u + \frac{a}{2}(2n_3 - 1) = 0 + \frac{a}{2}[(2 \times 3) - 1] = \frac{5a}{2}$$

Ratio of distances $S_1 : S_2 : S_3 = \frac{a}{2} : \frac{3a}{2} : \frac{5a}{2} = 1 : 3 : 5$.

Graphical representation of velocity, time & distance travelled by a body.

When the body is moving with a uniform velocity

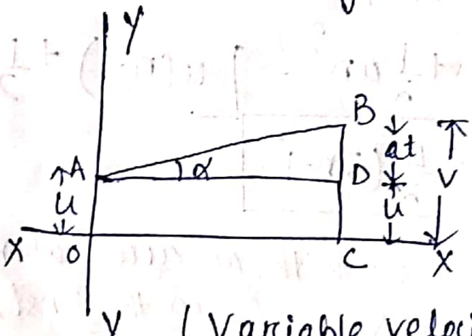


(uniform velocity)

Consider the motion of a body, which is represented by the graph OABC. we know the distance travelled by the body

$$\boxed{S = \text{velocity} \times \text{Time}}$$

When the body is moving with a variable velocity



(Variable velocity)

we know that the distance travelled by a body

$$S = ut + \frac{1}{2}at^2$$

From the above figure, we know that area of the figure OABC, $= \text{Area}(OADC + ABD)$

Area of the figure OADC $= u \times t$

area of figure ABD $= \frac{1}{2} \times t \times at = \frac{1}{2}at^2$

Total Area OABC $= ut + \frac{1}{2}at^2$

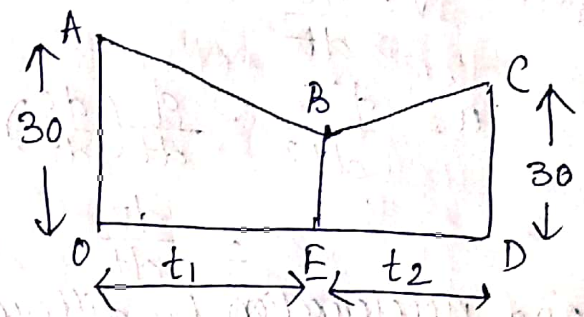
We see that the area of the OABC represents the distance traversed by the body to some scale, from the figure

$$\tan\alpha = \frac{at}{t} = a$$

$\tan\alpha$ represents the acceleration of the body.

for example

A train moving with a velocity of 30Kmph has to slow down to 15Kmph due to repairs along the road. If the distance covered during retardation be one Km and that covered during acceleration be half a Km, find the time lost in the journey.



let OABCD be the velocity-time graph in which AB ~~represent~~ represent the period of retardation & BC period of acceleration.

First of all, consider motion of the train from A to B. In this case distance travelled $(S_1) = 1$ Km.

Initial velocity $(u_1) = 30$ Kmph

final velocity $(v_1) = 15$ Kmph

t_1 = Time taken by the train to move from A to B.

Area of the trapezium OABE(S_1)

$$1 = \frac{30+15}{2} \times t_1 = 22.5 t_1$$

$$t_1 = \frac{1}{22.5} \text{ hr} = 2.67 \text{ min}$$

Now consider motion of the train from B to C. In this case, distance travelled $(S_2) = 0.5$ Km, Initial velocity $(u_2) = 15$ Kmph and final velocity $(v_2) = 30$ Kmph.

let t_2 = time taken by the train to move from B to C.

We also know that area of trapezium BCDE(S_2)

$$\frac{1}{2} = \frac{15+30}{2} \times t = 22.5 t_2$$

$$t_2 = \frac{1}{45} \text{ hr} = 1.33 \text{ min}$$

Total time $(t) = t_1 + t_2 = 2.67 + 1.33 = 4$ min.

If the train had moved uniformly with a velocity of 30Km/hr, then the time required to cover 1.5Km

$$= \frac{60}{30} \times 3/2 = 3 \text{ min}$$

Total Lost = 4 - 3 = 1 min.

Motion under variable Acceleration

Velocity and acceleration at any instant

$$V = \frac{ds}{dt}$$

$$a = \frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}(V)$$

$$= \frac{dV}{dt}$$

Velocity and acceleration by differentation

$$S = 3t^3 + 2t^2 + 6t + 4 \quad \text{—— (I)}$$

$$S = 6 + 5t^2 + 6t^3 \quad \text{—— (II)}$$

$$S = 2t^3 + 4t - 15 \quad \text{—— (III)}$$

Now differenciating both sides of the eq^n's w.r.t t.

$$\frac{ds}{dt} = \frac{d}{dt}(3t^3 + 2t^2 + 6t + 4)$$

$$= \frac{d}{dt}(3t^3) + \frac{d}{dt}(2t^2) + \frac{d}{dt}(6t) + \frac{d}{dt}(4)$$

$$\boxed{\frac{ds}{dt} = 9t^2 + 4t + 6} \quad \text{—— (I)}$$

For eq^n - (II) $\quad \frac{ds}{dt} = \frac{d}{dt}(6 + 5t^2 + 6t^3)$

$$\boxed{\frac{ds}{dt} = 10t + 18t^2}$$

For eq^n - (III) $\quad \frac{ds}{dt} = \frac{d}{dt}(2t^3 + 4t - 15)$

$$\boxed{\frac{ds}{dt} = 6t^2 + 4}$$

This eq^n again differenciate

For eq^n (I) $\quad \boxed{\frac{d^2s}{dt^2} = 18t + 4}$

For eq^n (II) $\quad \boxed{\frac{d^2s}{dt^2} = 10 + 36t}$

For eqⁿ (III)

$$\boxed{\frac{d^2s}{dt^2} = 12t}$$

Q11 A car moves along a straight line whose eqⁿ of motion is given by $s = 12t + 3t^2 - 2t^3$ where (s) is in metre and (t) is in seconds. Calculate
(i) velocity and acceleration at start
(ii) acceleration, when the velocity is zero.

Eqⁿ of displacement, $s = 12t + 3t^2 - 2t^3$ —— (I)

Velocity at start

Differenciating the above eqⁿ w.r.t t,

$$\frac{ds}{dt} = 12 + 6t - 6t^2$$

$$v = 12 + 6t - 6t^2 \quad \text{—— (II)}$$

Substituting $t = 0$ in the above eqⁿ

$$v = 12 + (6 \times 0) - (6 \times 0^2) = 12 \text{ m/s}.$$

Acceleration at start

Again differenciating eqⁿ (II) w.r.t t

$$\frac{dv}{dt} = \frac{d}{dt}(12 + 6t - 6t^2)$$

$$\frac{dv}{dt} = 6 - 12t$$

$$a = 6 - 12t \quad \text{—— (III)}$$

Put the value $t = 0$ in eqⁿ (III)

$$a = 6 - (12 \times 0) = 6 \text{ m/s}^2.$$

Acceleration, when the velocity is zero

Substituting eqⁿ (II) equal to zero

$$12 + 6t - 6t^2 = 0$$

$$t^2 - t - 2 = 0$$

$$t = 2 \text{ sec.}$$

It means that velocity of the car after two second will be zero. Now put the value of $t = 2$ in eqⁿ (III)

$$a = 6 - 12t = 6 * (12 \times 2) = -18 \text{ m/s}^2.$$

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Velocity & displacement by Integration

$$a = 4t^3 - 3t^2 + 5t + 6$$

$$a = \frac{dv}{dt}$$

$$\int dv = \int a\,dt$$

$$v = \int 4t^3 - 3t^2 + 5t + 6\,dt$$

$$= 4 \times \frac{t^4}{4} - 3 \times \frac{t^3}{3} + 5 \times \frac{t^2}{2} + 6t + C_1$$

$$= t^4 - t^3 + \frac{5}{2}t^2 + 6t + C_1$$

Again integrating

$$\frac{ds}{dt} = v$$

$$\int ds = \int v\,dt$$

$$\Rightarrow s = \int \left(t^4 - t^3 + \frac{5}{2}t^2 + 6t + C_1\right) dt$$

$$\Rightarrow s = \frac{t^5}{5} - \frac{t^4}{4} + \frac{5}{2} \times \frac{t^3}{3} + 6\frac{t^2}{2} + C_1 t + C_2$$

$$\Rightarrow s = \frac{t^5}{5} - \frac{t^4}{4} + \frac{5}{6}t^3 + 3t^2 + C_1 t + C_2.$$

for ex;

A train, starting from rest, is uniformly accelerated. The acceleration at any instant is $\frac{10}{V+1}$ m/s^2 where (V) is the velocity of the train in m/s at the instant. Find the distance in which the train will attain a velocity of 35 kmph.

Eq^n of acceleration $a = \frac{10}{V+1}$

$$\boxed{a = v.\frac{dv}{ds}}$$

$$v.\frac{dv}{ds} = \frac{10}{v+1}$$

$$v(v+1)dv = 10ds \quad \text{———— (I)}$$

$$(v^2+v)dv = 10ds$$

Integrating both sides of eq^n (II)

$$\frac{v^3}{3} + \frac{v^2}{2} = 10s + C_1 \quad \text{———— (II)}$$

where C_1 is the first constant of integration. substituting the values of $s=0$ and $v=0$ in eq^n - (II)

$$C_1 = 0$$

Put the value of $C_1 = 0$ in eq^n - (II)

$$\frac{v^3}{3} + \frac{v^2}{2} = 10s$$

$$2v^3 + 3v^2 = 60s \text{ ——— (III)}$$

Now for distance travelled by the train, put $v = 36$ kmph or 10 m/s in eq^n - (III)

$$2 \times (10)^3 + 3(10)^2 = 60s$$

$$2000 + 300 = 60s$$

$$s = \frac{2300}{60} = 38.3 \text{ m.}$$

Module-4 (Curvilinear Motion)

$$1 \text{ revolution/min} = 2\pi \text{ rad/min}$$

$$N \text{ revolution/min} = 2\pi N \text{ rad/min}$$

and Angular velocity $\omega = 2\pi N$ rad/min

$$= \frac{2\pi N}{60} \text{ rad/sec}$$

Angular velocity

It is the rate of change of angular displacement of a body and is expressed in r.p.m. (revolution per min.) or in radian per second. It is denoted by (ω.)

Angular acceleration

It is the rate of change of angular velocity and is expressed in radian per second per sec (rad/s^2) and is usually denoted by (α). It may be constant or variable.

Angular displacement

It is the total angle, through which a body has rotated, and is usually denoted by (θ). If a body is rotating with a uniform angular velocity (ω) then in 't' second, $\boxed{\theta = \omega t}$

Motion of rotation under constant angular acceleration

ω_0 = Initial angular velocity
ω = Final angular velocity
t = time taken by the particle to change its velocity from ω_0 to ω → (omega)

α = Constant angular acceleration in rad/s^2
θ = Total angular displacement in radian

$$\boxed{\omega = \omega_0 + \alpha t} \text{ --- (I)}$$

Avg. angular velocity $= \frac{\omega_0 + \omega}{2}$

total angular displacement,

$$\theta = \text{Avg. velocity} \times \text{Time} = \left(\frac{\omega_0+\omega}{2}\right) \times t \text{ --- (II)}$$

Put the value of ω from eqⁿ - (I)

$$\theta = \frac{\omega_0 + (\omega_0 + \alpha t)}{2} \times t = \frac{2\omega_0 + \alpha t}{2} \times t$$

$$\boxed{\theta = \omega_0 t + \frac{1}{2}\alpha t^2} \text{ --- (III)}$$

From eqⁿ (I), we find that

$$t = \frac{\omega - \omega_0}{\alpha}$$

Put the value of t in eqⁿ (II)

$$\theta = \left(\frac{\omega_0+\omega}{2}\right) \times \left(\frac{\omega - \omega_0}{\alpha}\right) = \frac{\omega^2 - \omega_0^2}{2\alpha}$$

$$\boxed{\omega^2 = \omega_0^2 + 2\alpha\theta} \text{ --- (IV)}$$

Q1) A wheel increases its speed from 45 r.p.m in 30 seconds. Find (a) Angular acceleration of the wheel.
(b) No. of revolutions made by the wheel in these 30 seconds.

Initial angular velocity (ω_0) = 45 r.p.m = 1.5π rad/sec
Final angular velocity (ω) = 90 r.p.m = 3π rad/sec
t = 30 sec.

(a) Angular acceleration of the wheel

let α = Angular acceleration of the wheel

we know that the final Angular velocity of the wheel (ω)

$$3\pi = \omega_0 + \alpha t = 1.5\pi + (\alpha \times 30)$$

$$= 1.5\pi + 30\alpha$$

$$\alpha = \frac{3\pi - 1.5\pi}{30} = \frac{1.5\pi}{30} = 0.05\pi \text{ rad/sec}^2.$$

(b) No. of revolutions made by the wheel in 30 sec

We also know that total angle turned by the wheel in 30 sec.

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

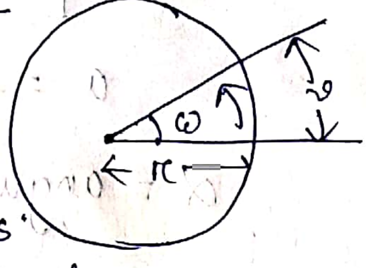
$$= (1.5\pi \times 30) + \left(\frac{1}{2} \times 0.05\pi (30)^2\right)$$

$$= 67.5\pi \text{ rad} = \frac{67.5\pi}{2\pi} = 33.75 \text{ rev}$$

$$\boxed{1 \text{ rev} = 2\pi \text{ rad}}$$

Linear or tangential velocity of a rotating body

Consider a body rotating about its Axis



ω = Angular velocity of the body in rad/s.

r = Radius of the Circular path in metres.

v = Linear velocity of the particle on the periphery in m/s.

length of arc = Radius of Arc × Angle subtended in rad.

$$\boxed{v = r\omega}$$

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A wheel of 1.2m diameter starts from rest and is accelerated at the rate of $0.8\ rad/s^2$. Find the linear velocity of a point on its periphery after 5sec.

dia of wheel = 1.2m

radius(r) = 0.6m

Initial angular velocity (ω_0) = 0 (starts from rest)

Angular acceleration (α) = $0.8\ rad/s^2$

$t = 5s.$

We know that angular velocity of the wheel after 5 sec

$$\omega = \omega_0 + \alpha t = 0 + (0.8 \times 5) = 4\ rad/s.$$

Linear velocity of a point on its periphery after 5 sec,

$$v = r\omega = 0.6 \times 4 = 2.4\ m/s.$$

<u>linear or tangential acceleration of a rotating body</u>

$$a = \frac{dv}{dt} = \frac{d}{dt}(v)$$

$$\boxed{V = r\omega}$$

$$a = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\alpha.$$

α = angular acc^n in rad/sec^2 & is equal to $\frac{d\omega}{dt}$.

NOTE

Angular acc^n may also be

$$\boxed{\alpha = \frac{a}{r}}$$

Q// The equation for angular displacement of a body moving on a circular path is given by

$$\theta = 2t^3 + 0.5$$

where θ is in rad and t in sec. Find angular velocity, displacement and acc^n after 2sec.

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Eq^n for angular displacement $\theta = 2t^3 + 0.5$ —— (I)

Angular displacement after 2 sec

Put $t = 2$ in eq^n (I)

$$\theta = 2\times(2)^3 + 0.5 = 16.5 \text{ rad.}$$

Angular velocity after 2 sec.

$$\frac{d\theta}{dt} = \frac{d}{dt}(2t^3 + 0.5)$$

$$\omega = 6t^2 \quad \text{—— (II)}$$

Put $t=2$ in eq^n (II)

$$\omega = 6\times(2)^2 = 6\times 4 = 24 \text{ rad/sec.}$$

Angular acc^n after 2 sec.

$$\frac{d\omega}{dt} = \frac{d}{dt}(6t^2)$$

$$\alpha = 12t \quad \text{—— (III)}$$

Put $t=2$ in eq^n (III)

$$\alpha = 12\times 2 = 24 \text{ rad/s}^2.$$

Relation between Linear Motion & Angular Motion.

Sl No.	Particulars	Linear Motion or Rectilinear Motion	Angular Motion or Curvilinear Motion
1.	Initial velocity	u	ω_0
2.	Final velocity	v	ω
3.	Constant acceleration	a	α
4.	Total distance covered	S	θ
5.	Final velocity	$v = u + at$	$\omega = \omega_0 + \alpha t$
6.	distance traversed	$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
7.	Final velocity	$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
8.	Differential formula for velocity	$v = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$
9.	Differential formula for acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$