Mathematics III (RMA3A001)

Module I

Ramesh Chandra Samal

Department of Mathematics
Ajay Binay Institute of Technology
Cuttack, Odisha

Lecture - 11

Interpolation and approximation

Let the explicity nature of a function $f(x)$ is not known. But the value of the function at $(n+1)$ distinct points $x_0,\ x_1,\ x_2,\ \ldots\ldots\ldots x_n$ where $x_0 < x_1 < x_2,\ \ldots\ldots\ldots < x_n$ is known. The process of interpolation is to find out another function $P(x)$ of degree n such that

$$P(x_i) = f(x_i) \qquad\qquad i = 0,\ 1,\ 2,\ \ldots\ldots\ldots\ldots n$$

The process by which $P(x)$ is determined is called interpolation. $P(x)$ is called the interpolating polynomial of function $f(x)$.
If $P(x)$ is a polynomial then it is called polynomial interpolation.
NOTE : The interpolating polynomial of a function $f(x)$ is unique.

Types of interpolation

There are several types of interpolation. Some of them are listed below.

(i) Newton's forward interpolation

(ii) Newton's backward interpolation

(iii) Newton's divided difference interpolation

(iv) Lagrange's interpolation

NOTE :

(a) Newton's forward interpolation and Newton's backward interpolation are interpolation with equal intervals.

(b) Newton's divided difference interpolation and Lagrange's interpolation are interpolation with unequal intervals.

Finite differences

Suppose the function $y = f(x)$ has the values $y_0,\ y_1,\ y_2,\ \ldots\ldots\ldots\ y_n$ for the values of $x = x_0 + h,\ x_0 + 2h,\ x_0 + 3h,\ \ldots\ldots\ldots\ x_0 + nh$. To determine the values of $f(x)$ is based on the principle of finite differences.

Forward differences (Δ)

The differences $y_1 - y_0,\ y_2 - y_1,\ \ldots\ldots\ldots\ y_n - y_{n-1}$ are called first forward differences are denoted by $\Delta y_0,\ \Delta y_1,\ \Delta y_2,\ \ldots\ldots\ldots\ \Delta y_{n-1}$, where Δ is known as first forward difference operator.

Thus the first forward difference is given by

$$\Delta y_k = y_{k+1} - y_k$$

The second forward difference is given by

$$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$$

In general the r^{th} forward difference is given by

$$\Delta^r y_k = \Delta^{r-1} y_{k+1} - \Delta^{r-1} y_k$$

Forward difference table

$$
\begin{array}{ccccccc}
x & y & \Delta & \Delta^2 & \Delta^3 & \Delta^4 & \Delta^5 \\
x_0 & y_0 & & & & & \\
 & & \Delta y_0 & & & & \\
x_0+h & y_1 & & \Delta^2 y_0 & & & \\
 & & \Delta y_1 & & \Delta^3 y_0 & & \\
x_0+2h & y_2 & & \Delta^2 y_1 & & \Delta^4 y_0 & \\
 & & \Delta y_2 & & \Delta^3 y_1 & & \Delta^5 y_0 \\
x_0+3h & y_3 & & \Delta^2 y_2 & & \Delta^4 y_1 & \\
 & & \Delta y_3 & & \Delta^3 y_2 & & \\
x_0+4h & y_4 & & \Delta^2 y_3 & & & \\
 & & \Delta y_4 & & & & \\
x_0+5h & y_5 & & & & &
\end{array}
$$

The difference $\Delta^k y_0$ lies on a straight line sloping downward to the right.

Backward differences (∇)

The differences $y_1 - y_0,\ y_2 - y_1,\ \ldots\ldots\ldots\ y_n - y_{n-1}$ are called first backward differences are denoted by $\nabla y_1,\ \nabla y_2,\ \nabla y_3,\ \ldots\ldots\ldots\ \nabla y_n$, where ∇ is known as first backward difference operator.
Thus the first backward difference is given by

$$\nabla y_k = y_k - y_{k-1}$$

The second backward difference is given by

$$\nabla^2 y_k = \nabla y_k - \Delta y_{k-1}$$

In general the r^{th} backward difference is given by

$$\nabla^r y_k = \nabla^{r-1} y_k - \nabla^{r-1} y_{k-1}$$

Backward difference table

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		∇y_1				
x_0+h	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_0+2h	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_4$
x_0+3h	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_0+4h	y_4		$\nabla^2 y_5$			
		∇y_5				
x_0+5h	y_5					

The difference $\Delta^k y_5$ lies on a straight line sloping upward to the right.

Newton forward interpolation formula

This formula is applicable when the arguments are given with equal spaced.
Let $y = f(x)$ be a function which takes the values $f(a)$, $f(a+h)$, $f(a+2h), \ldots\ldots\ldots f(a+nh)$ for $x = a,\ a+h,\ a+2h,\ \ldots\ldots\ldots\ldots (a+nh)$. Where h is the step size of the arguments.
Here $(n+1)$ arguments are given therefore the $(n+1)^t h$ difference is zero. Thus $f(x)$ is a polynomial of degree n. So $f(x)$ can be written as

$$
\begin{aligned}
f(x) =& a_0 + a_1(x-a) + a_2(x-a)(x-a-h) \\
& + a_3(x-a)(x-a-h)(x-a-2h) \\
& + \ \ldots\ldots\ldots\ldots + a_n(x-a)(x-a-h)\ldots\ldots(x-a-(n-1)h)
\end{aligned}
\tag{1}
$$

Where $a_0,\ a_1,\ a_2,\ \ldots\ldots\ldots\ldots a_n$ are constants.
Putting $x = a,\ a+h,\ a+2h,\ \ldots\ldots\ldots\ldots a+nh$ in to equation (1) respectively, we get

$$
\begin{aligned}
a_0 &= f(a) \\
a_0 + h &= f(a+h) \\
\text{or} \qquad a_1 h &= f(a+h) - f(a) \\
\implies a_1 &= \frac{f(a+h) - f(a)}{h} = \frac{\Delta f(a)}{h} \\
\text{and} \qquad a_0 + 2ha_1 + 2h^2 a_2 &= f(a+2h) \\
\implies 2h^2 a_2 &= f(a+2h) - a_0 - 2ha_1 \\
&= f(a+2h) - a_0 - 2\Delta f(a) \\
&= f(a+2h) - f(a+h) + f(a) \\
&= \Delta^2 f(a) \\
\implies a_2 &= \frac{\Delta^2 f(a)}{2!\, h^2}
\end{aligned}
$$

Continuing in this way, we get

$$a_3 = \frac{\Delta^3 f(a)}{3!\, h^3} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; a_n = \frac{\Delta^n f(a)}{n!\, h^n}$$

Now substituting these values of $a_0,\ a_1,\ a_2,\ \ldots\ldots\ldots\ a_n$ into equation (1), we get

$$\begin{aligned} f(x) =& f(a) + \frac{\Delta f(a)}{h}(x-a) + \frac{\Delta^2 f(a)}{2!\, h^2}(x-a)(x-a-h) \\ &+ \frac{\Delta^3 f(a)}{3!\, h^3}(x-a)(x-a-h)(x-a-2h) + \ldots\ldots\ldots\ldots \\ &+ \frac{\Delta^n f(a)}{n!\, h^n}(x-a)(x-a-h)\ldots\ldots(x-a-(n-1)h) \end{aligned} \tag{2}$$

Further put $x = a + Uh$, then
$x-a = Uh,\ x-a-h = (U-1)h,\ x-a-2h = (u-2)h,\ \ldots\ldots x-a-(n-1)h = U-(n-1)h$
Using these values in equation (2)

$$f(a+hU) = f(a) + U\Delta f(a) + \frac{U(U-1)}{2!}\Delta^2 f(a) + \ldots\ldots\ldots + \frac{U(U-1)\ldots\ldots(U-n+1)}{n!}\Delta^n f(a)$$

This formula is known as Newton's froward interpolation with equal intervals.

Example 1

From the following table find the number of students who obtain less than 45 marks

Range of Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No of Students	31	42	51	35	31

Using Newton's forward interpolation formula

Solution :

The difference table for the given data is as follows

Marks x	No of students $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
Less 40	31				
		42			
Less 50	73		9		
		51		-25	
Less 60	124		-16		37
		35		12	
Less 70	159		-4		
		31			
Less 80	190				

Here $h = 10$, $a = 40$, and $x = 45$

$$U = \frac{x - a}{h} = \frac{45 - 40}{10} = \frac{1}{2}$$

By Newton forward interpolation formula

$$
\begin{aligned}
f(a+hU) =& f(a) + U\Delta f(a) + \frac{U(U-1)}{2!}\Delta^2 f(a) + \frac{U(U-1)(U-2)}{3!}\Delta^3 f(a) \\
&+ \frac{U(U-1)(U-2)(U-3)}{4!}\Delta^4 f(a)
\end{aligned}
$$

$$
\begin{aligned}
\implies f(45) =& f(40) + \frac{1}{2}\Delta f(40) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\Delta^2 f(40) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\Delta^3 f(4 \\
&+ \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}\Delta^4 f(40) \\
=& 31 + \frac{1}{2}\times 42 - \frac{1}{8}\times 9 - \frac{1}{16}\times 25 - \frac{5}{128}\times 37 \\
=& 47.867
\end{aligned}
$$

Thus the number of students who obtain less than 45 marks are 48.

Example 2

For the following data calculate the differences and obtain Newton forward interpolating polynomial

x	0	1	2	3	4
$f(x)$	3	6	11	18	27

Solution :

The difference table is given by

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	3				
		3			
1	6		2		
		5		0	
2	11		2		0
		7		0	
3	18		2		
		9			
4	27				

Here $a = 0, \quad h = 1$

$$U = \frac{x-a}{h} = \frac{x-0}{1} = x$$

The Newton forward interpolation formula is

$$\begin{aligned}
f(a+hU) =& f(a) + U\Delta f(a) + \frac{U(U-1)}{2!}\Delta^2 f(a) + \frac{U(U-1)(U-2)}{3!}\Delta^3 f(a) \\
&+ \frac{U(U-1)(U-2)(U-3)}{4!}\Delta^4 f(a) \\
\implies f(x) =& f(0) + x\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(0) \\
&+ \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 f(0) \\
=& 3 + 3x + \frac{x(x-1)}{2}(2) + 0 + 0 \\
=& 3 + 3x + x^2 - x \\
=& x^2 + 2x + 3
\end{aligned}$$

Any Questions?

Thank You