Mathematics III (RMA3A001)

Module I

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Lecture - 2

Bisection Method

- This method is based on the repeated application of the intermediate value theorem.
- If we know that a root of $f(x) = 0$ lies in the interval $I_0 = (a_0, b_0)$ we bisect I_0 at the point $m_1 = \frac{1}{2}(a_0 + b_0)$. Denote by I_1, the interval (a_0, m_1) if $f(a_0 f(m_1)) < 0$ or the interval (m_1, b_0) if $f(m_1 f(b_0)) < 0$.
- Therefore the interval I_1 also contain the root. We bisect the interval I_1 and get the sub-interval I_2 at whose end point $f(x)$ takes the values of opposite signs and therefore contains the root.
- Containing this procedure we obtain a sequence of nested set of sub-intervals $I_0 \supset I_1 \supset I_2 \supset \;.\;.\;.\;.\;..$ Such that each sub-interval contains the root.

- After repeating the bisection process n times we either find the root or find the interval I_n of length $\frac{b_0 - a_0}{2^n}$.
- We take the midpoint of the last sub interval as the desired approximation to the root.
- This method is simple but slowly convergent method.
- The bisection method is convergent linearly.
- This is a two point formula because two interval approximation are required for finding out the root of the equation.

Example 1

Find the real root of the equation

$$f(x) = x^3 - 5x + 1 = 0$$

correct up to three decimal places by using bisection method.

Solution : We have,

$$f(x) = x^3 - 5x + 1 = 0$$

Now

$$f(0) = 1 > 0$$
$$f(0) = -2 < 0$$

So

$$f(0.1) = 0.501 > 0$$
$$f(0.2) = 0.008 > 0$$
$$f(0.3) = -0.473 < 0$$

Thus the root of the equation lies in the interval $(0.2, 0.3)$. Let $x_0 = 0.2$ and $x_1 = 0.3$ be the initial approximation to the root of the equation.

First approximation:

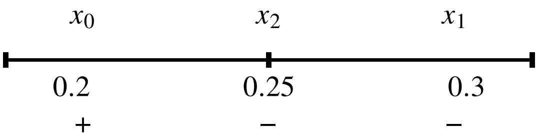
$$x_0 = 0.2$$

$$x_1 = 0.3$$

Let

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0.2 + 0.3}{2} = 0.25$$

$$f(x_2) = f(0.25) = -0.234 < 0$$



So the root of the equation lies in the interval $(x_0, x_2) = (0.2, 0.25)$

Second approximation:

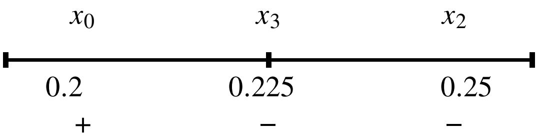
$$x_0 = 0.2$$

$$x_2 = 0.25$$

Let

$$x_3 = \frac{x_0 + x_2}{2} = \frac{0.2 + 0.25}{2} = 0.225$$

$$f(x_3) = f(0.225) = -0.113 < 0$$



So the root of the equation lies in the interval $(x_0, x_3) = (0.2, 0.225)$

Third approximation:

$$x_0 = 0.2$$

$$x_3 = 0.225$$

Let

$$x_4 = \frac{x_0 + x_3}{2} = \frac{0.2 + 0.225}{2} = 0.2125$$

$$f(x_4) = f(0.2125) = -0.052 < 0$$

$$\begin{array}{ccc} x_0 & x_4 & x_3 \\ 0.2 & 0.2125 & 0.225 \\ + & - & - \end{array}$$

So the root of the equation lies in the interval $(x_0, x_4) = (0.2, 0.2125)$

Fourth approximation:

$$x_0 = 0.2$$

$$x_4 = 0.2125$$

Let

$$x_5 = \frac{x_0 + x_4}{2} = \frac{0.2 + 0.2125}{2} = 0.20625$$

$$f(x_5) = f(0.20625) = -0.0224 < 0$$

$$\begin{array}{ccc} x_0 & x_5 & x_4 \\ 0.2 & 0.20625 & 0.2125 \\ + & - & - \end{array}$$

So the root of the equation lies in the interval $(x_0, x_5) = (0.2, 0.20625)$

Fifth approximation:

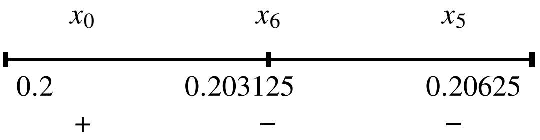
$$x_0 = 0.2$$

$$x_5 = 0.20625$$

Let

$$x_6 = \frac{x_0 + x_5}{2} = \frac{0.2 + 0.2125}{2} = 0.203125$$

$$f(x_6) = f(0.203125) = -0.0072 < 0$$



So the root of the equation lies in the interval $(x_0, x_6) = (0.2, 0.203125)$

Sixth approximation :

$$x_0 = 0.2$$

$$x_6 = 0.203125$$

Let

$$x_7 = \frac{x_0 + x_6}{2} = \frac{0.2 + 0.203125}{2} = 0.2015625$$

$$f(x_7) = f(0.2015625) = 0.00037 > 0$$

x_0	x_7	x_6
0.2	0.2015625	0.203125
+	+	−

So the root of the equation lies in the interval $(x_7, x_6) = (0.2015625, 0.203125)$

Seventh approximation :

$$x_7 = 0.2015625$$

$$x_5 = 0.203125$$

Let

$$x_8 = \frac{x_7 + x_6}{2} = \frac{0.2015625 + 0.203125}{2} = 0.20234375$$

$$f(x_8) = f(0.20234375) = -0.003 < 0$$

x_7	x_8	x_6
0.2015625	0.20234375	0.203125
$+$	$-$	$-$

So the root of the equation lies in the interval $(x_7, x_8) = (0.2015625, 0.20234375)$

Eights approximation :

$$x_7 = 0.2015625$$

$$x_8 = 0.20234375$$

Let

$$x_9 = \frac{x_7 + x_8}{2} = \frac{0.2015625 + 0.20234375}{2} = 0.201953125$$

$$f(x_9) = f(0.201953125) = -0.001 < 0$$

x_7	x_9	x_8
0.2015625	0.201953125	0.20234375
$+$	$-$	$-$

So the root of the equation lies in the interval $(x_7, x_9) = (0.2015625, 0.201953125)$
Thus the required root of the equation correct up to three decimal places by bisection method is 0.201

Example 2

Find the real root of the equation

$$f(x) = x - e^{-x} = 0$$

by using bisection method.

OR

Perform five step to find the real root of the equation

$$f(x) = x - e^{-x} = 0$$

by using bisection method.

Solution : We have

$$f(x) = x - e^{-x} = 0$$

$$f(0) = -1 < 0$$

$$f(1) = 0.6321$$

$$f(0.5) = -0.1065 < 0$$

$$f(0.6) = 0.0511 > 0$$

Thus the root of the equation lies in the interval $(0.5, 0.6)$.
Let $x_0 = 0.5$, $x_1 = 0.6$ be the initial approximation to the root of the equation.

First approximation:

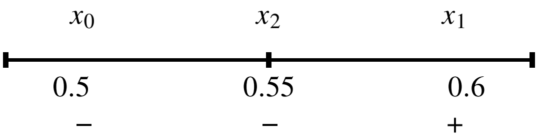
$$x_0 = 0.5$$

$$x_1 = 0.6$$

Let

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0.5 + 0.6}{2} = 0.55$$

$$f(x_2) = f(0.55) = -0.0269 < 0$$



So the root of the equation lies in the interval $(x_2, x_1) = (0.55, 0.6)$

Second approximation:

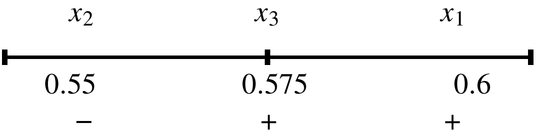
$$x_2 = 0.55$$

$$x_1 = 0.6$$

Let

$$x_3 = \frac{x_2 + x_1}{2} = \frac{0.55 + 0.6}{2} = 0.575$$

$$f(x_3) = f(0.575) = 0.012 > 0$$



So the root of the equation lies in the interval $(x_2, x_3) = (0.55, 0.575)$

Third approximation:

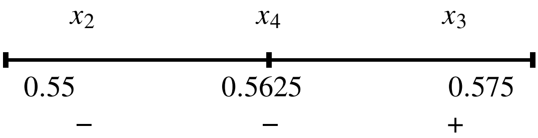
$$x_2 = 0.55$$

$$x_3 = 0.575$$

Let

$$x_4 = \frac{x_2 + x_3}{2} = \frac{0.55 + 0.575}{2} = 0.5625$$

$$f(x_4) = f(0.5625) = -0.007 < 0$$



So the root of the equation lies in the interval $(x_4, x_3) = (0.5625, 0.575)$

Fourth approximation:

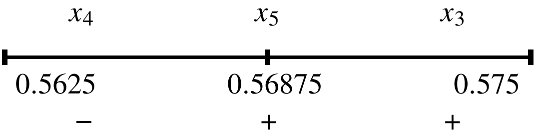
$$x_4 = 0.5625$$

$$x_3 = 0.575$$

Let

$$x_5 = \frac{x_4 + x_3}{2} = \frac{0.5625 + 0.575}{2} = 0.56875$$

$$f(x_5) = f(0.56875) = 0.002 > 0$$



So the root of the equation lies in the interval $(x_4, x_5) = (0.5625, 0.56875)$

Fifth approximation:

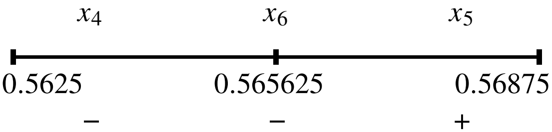
$$x_4 = 0.5625$$

$$x_5 = 0.56875$$

Let

$$x_6 = \frac{x_4 + x_5}{2} = \frac{0.5625 + 0.56875}{2} = 0.565625$$

$$f(x_6) = f(0.565625) = -0.002 < 0$$



So the root of the equation lies in the interval $(x_6, x_5) = (0.565625, 0.56875)$

Sixth approximation :

$$x_6 = 0.565625$$

$$x_5 = 0.56875$$

Let

$$x_7 = \frac{x_6 + x_5}{2} = \frac{0.565625 + 0.56875}{2} = 0.5671875$$

So the root of the given equation by bisection method after fifth step is 5671875

Any Questions?

Thank You