Module-2

Virtual Work

16.1. INTRODUCTION

It has been observed that whenever a force acts on a body, and the body undergoes some displacement, some work is said to be done. Mathematically, if a force (P) acting on a body displaces it through a distance (s), then

Workdone = Force × Distance = $P \times s$

But, sometimes, the body does not move in the direction of force (or in other words, the force does not act in the direction of motion of the body). In such a case,

Workdone = Component of the force in the direction of motion × Distance

$$= P \cos \theta \times s$$

where θ is the inclination between the line of action of the force and the direction of the motion of the body.

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A little consideration will show that

1. If the value of θ is between $0°$ and $90°$, some work is done.
2. If the value of θ is $90°$, then no work is done (because $\cos 90° = 0$).
3. If the value of θ is between $90°$ and $180°$, the body will move in the opposite direction and work is called as *negative*.

16.2. CONCEPT OF VIRTUAL WORK

In the previous article, we have discussed that the work done by a force is equal to the force multiplied by the distance through which the body has moved in the direction of the force. But if the body is in equilibrium, under the action of a system of forces, the work done is zero. If we assume that the body, in equilibrium, undergoes an infinite small imaginary displacement (known as *virtual displacement*) some work will be *imagined* to be done. Such an imaginary work is called *virtual work*. This concept, of virtual work, is very useful in finding out the unknown forces in structures.

Note. The term 'virtual' is used to stress its purely hypothetical nature, as we do not actually displace the system. We only imagine, as to what would happen, if the system is displaced.

16.3. PRINCIPLE OF VIRTUAL WORK

It states, "*If a system of forces acting on a body or a system of bodies be in equilibrium, and the system be imagined to undergo a small displacement consistent with the geometrical conditions, then the algebraic sum of the virtual works done by all the forces of the system is zero.*"

Proof. Consider a body at O, subjected to a force P inclined at angle θ with X-X axis as shown in Fig. 16.1.

Let P_X = Component of the force along X-X axis, and

P_Y = Component of the force along Y-Y axis.

From the geometry of the figure, we find that

$$P_X = P\cos\theta \text{ and}$$
$$P_Y = P\sin\theta$$

Now consider the body to move from O to some other point C, under the action of the force P, such that the line OC makes an angle α with the direction of the force. Now draw CA and CB perpendiculars to OX and OY respectively as shown in Fig. 16.1.

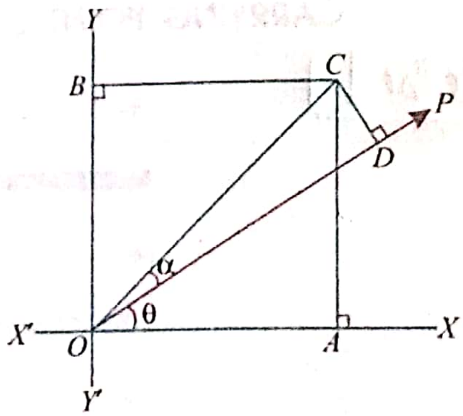

Fig. 16.1. Principle of virtual work.

From the geometry of the triangle OCA, we find that

$$\cos(\theta + \alpha) = \frac{OA}{OC}$$

$$\therefore \quad OA = OC \times \cos(\theta + \alpha)$$

Similarly, $\quad OB = AC = OC \times \sin(\theta + \alpha)$

We know that the sum of the works done by the components P_X and P_Y of the force P

$$= P_X \times OA + P_Y \times OB$$
$$= [P\cos\theta \times OC\cos(\theta + \alpha)] + [P\sin\theta \times OC\sin(\theta + \alpha)]$$
$$= P \times OC\,[\cos\theta \times \cos(\theta + \alpha) + \sin\theta \times \sin(\theta + \alpha)]$$
$$= P \times OC\cos(\theta - \theta - \alpha) \quad \text{... } (\because \cos A - B = \cos A\cos B + \sin A\sin B)$$
$$= P \times OC\cos(-\alpha)$$
$$= P \times OC\cos\alpha \quad \text{... } (\because \cos(-A) = \cos A) \quad \text{...}(i)$$

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We also know that the work done by the force P in moving the body from O to C.

$$= P \times OC \cos \alpha$$ ...(ii)

Since the equations (i) and (ii) are the same, therefore, work done by a force is equal to the sum of the works done by its resolved parts.

Note. For the sake of simplicity, we have considered only one force and its resolved parts. But it can be extended to any number of forces.

16.4. SIGN CONVENTIONS

Though there are different sign conventions for finding out the virtual works done in different books, yet we shall use the following sign conventions, which are internationally recognised.

1. Upward forces are considered as positive, whereas the downwards as negative.
2. Forces acting towards right are considered as positive, whereas those towards left as negative.
3. Forces acting in the clockwise direction are considered as positive, whereas the anticlockwise as negative.
4. Tensile forces are considered as positive whereas the compressive as negative.

16.5. APPLICATIONS OF THE PRINCIPLE OF VIRTUAL WORK

The principle of virtual work has very wide applications. But the following are important from the subject point of view:

1. Beams 2. Lifting machine. 3. Framed structures.

16.6. APPLICATION OF THE PRINCIPLE OF VIRTUAL WORK ON BEAMS CARRYING POINT LOAD

Method-1

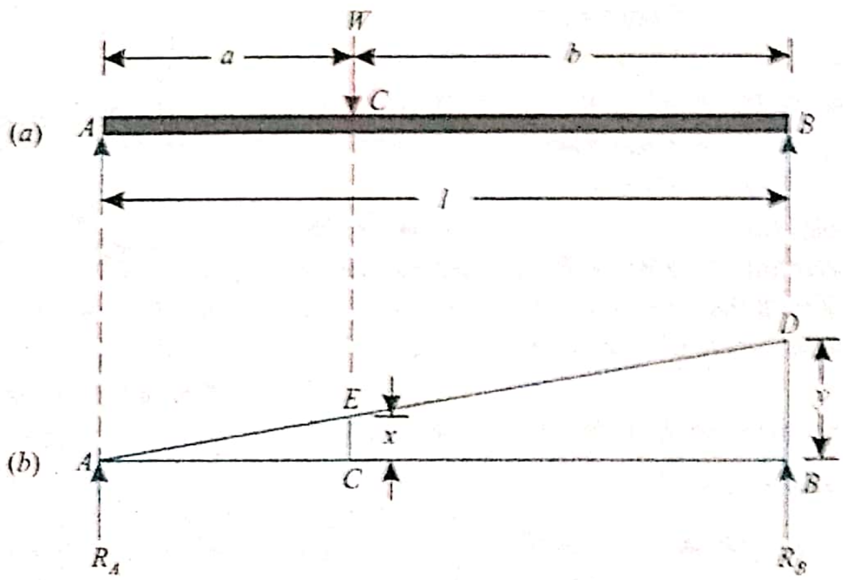

Fig. 16.2. Beam carrying point load.

Consider a beam AB, simply supported at its supports, and subjected to a point load W at C as shown in Fig. 16.2 (a)

Let R_A = Reaction at A, and

R_B = Reaction at B.

First of all, let us assume the beam to be hinged at A. Now consider an upward virtual displacement (y) of the beam at B. This is due to the reaction at B acting upwards as shown in Fig 16.2 (b). Let x be the upward virtual displacement of the beam at C due to the point load.

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Now in two similar triangles ABD and ACE,

$$\frac{x}{y} = \frac{a}{l} \quad \text{or} \quad x = \frac{a\,y}{l}$$

∴ Total virtual work done by the two reactions R_A and R_B

$$= + [(R_A \times 0) + (R_B \times y)] = + R_B \times y \quad ...(i)$$

... (Plus sign due to the reactions acting upwards)

and virtual work done by the point load*

$$= - W \times x \quad ...(ii)$$

... (Minus sign due to the load acting downwards)

We know that from the principle of virtual work, that algebraic sum of the virtual works done is zero. Therefore

$$R_B \times y - W \times x = 0$$

or

$$R_B = \frac{W \times x}{y} = \frac{W}{y} \times \frac{a \times y}{l} = \frac{W \times a}{l}$$

Similarly, it can be proved that the vertical reaction at A,

$$R_A = \frac{W \times b}{l}$$

Notes : 1. For the sake of simplicity, we have taken only one point load W at C. But this principle may be extended for any number of loads.

2. The value of reaction at A (*i.e.*, R_A) may also be obtained by subtracting the value of R_B from the downward load W. Mathematically,

$$R_A = W - \frac{Wa}{l} = W\left(1 - \frac{a}{l}\right) = W\left(\frac{l-a}{l}\right) = \frac{Wb}{l}$$

Example 16.1. *A beam AB of span 5 metres is carrying a point load of 2 kN at a distance 2 metres from A. Determine the beam reactions, by using the principle of the virtual work.*

Solution. Given: Span (l) = 5 m; Point load (W) = 2 kN and distance between the point load and support A = 2 m.

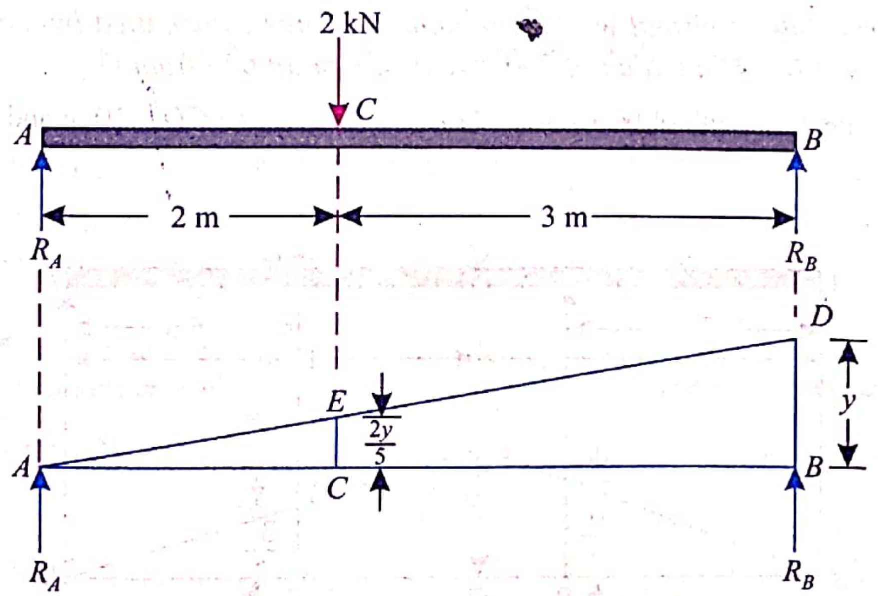

Fig. 16.3.

Let

R_A = Reaction at A,

R_B = Reaction at B, and

y = Virtual upward displacement of the beam at B.

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From the geometry of the figure, we find that when the virtual upward displacement of the beam at B is y, the virtual upward displacement of the beam at C is $\frac{2y}{5} = 0.4\,y$ as shown in Fig. 16.3.

$\therefore$ Total virtual work done by the two reactions R_A and R_B

$$= +[(R_A \times 0) + (R_B \times y)] = +R_B \times y \qquad \text{...}(i)$$

... (Plus sign due to the reactions acting upwards)

and virtual work done by the point load

$$= -Px = -2 \times 0.4 = -0.8\,y \qquad \text{...}(ii)$$

... (Minus sign due to the load acting downwards)

We know that from the principle of virtual work, that algebraic sum of the total virtual works done is zero. Therefore

$$R_B \times y - 0.8\,y = 0$$

or $\quad R_B = 0.8\,y/y = 0.8$ kN **Ans.**

and $\quad R_A = 2 - 0.8 = 1.2$ kN **Ans.**

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16.7. APPLICATION OF THE PRINCIPLE OF VIRTUAL WORK FOR BEAMS CARRYING UNIFORMLY DISTRIBUTED LOAD

Method-2

Consider a beam AB of length l simply supported at its both ends, and carrying a uniformly distributed load of w per unit length for the whole span from A to B as shown in Fig. 16.8 (a).

Let R_A = Reaction at A, and

R_B = Reaction at B.

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First of all, let us assume the beam to be hinged at A. Now consider an upward virtual displacement (y) of the beam at B. This is due to the reaction at B acting upwards as shown in Fig. 16.8 (b)

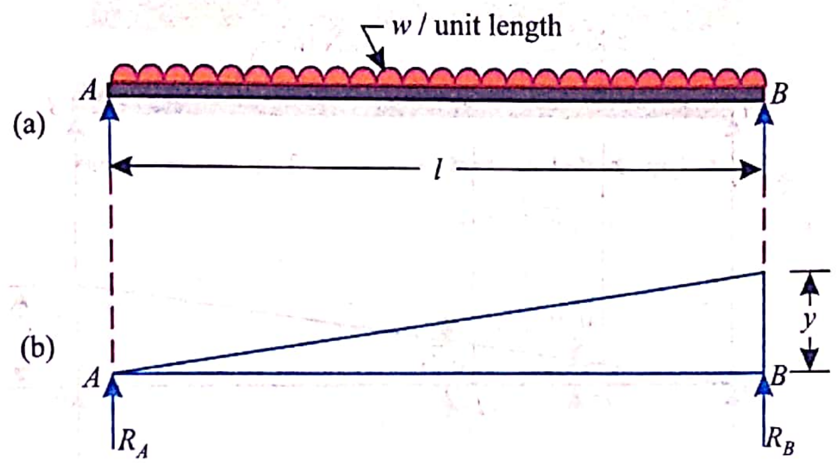

Fig. 16.8. Beam carrying uniformly distributed load.

∴ Total virtual work done by the two reactions R_A and R_B

$$= +[(R_A \times 0) + (R_B \times y)] = +R_B \times y \qquad ...(i)$$

...(Plus sign due to reaction acting upwards)

and virtual work done by the uniformly distributed load

$$= -w\left(\frac{0+y}{2} \times l\right) = -0.5\,wyl$$

...(Minus sign due to load acting downwards)

We know that from the principle of virtual work, that algebraic sum of the virtual works done is zero. Therefore

$$R_B \times y - 0.5wl \times y = 0$$

$$\therefore \qquad R_B \times y = 0.5\,wl \times y$$

$$R_B = 0.5\,wl$$

Note. For the sake of simplicity, we have taken the uniformly distributed load for the entire span from A to B. But this principle may be extended for any type of load on beam (*i.e.* simply supported or overhanging beam etc.)

Example 16.4. *A simply supported beam AB of span 5 m is loaded as shown in Fig. 16.9.*

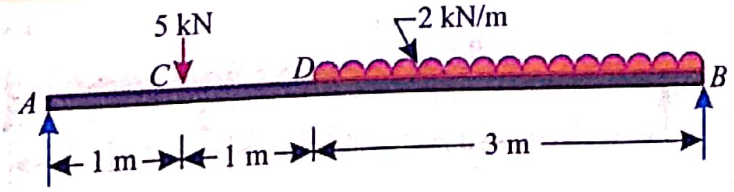

Fig. 16.9. Using the principle of virtual work, find the reactions at A and B.

Solution. Given : Length of beam $AB = 5$ m; Point Load at $C = 5$ kN and uniformly distributed load between D and $B = 2$ kN/m

Let R_A = Reaction at A,

R_B = Reaction at B, and

y = Virtual upward displacement of the beam at B.

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From the geometry of the figure, we find that when the virtual upward displacement of the beam at B is y, then the virtual upward displacement of the beam at C and D is $0.2\ y$ and $0.4\ y$ respectively as shown in Fig. 16.10.

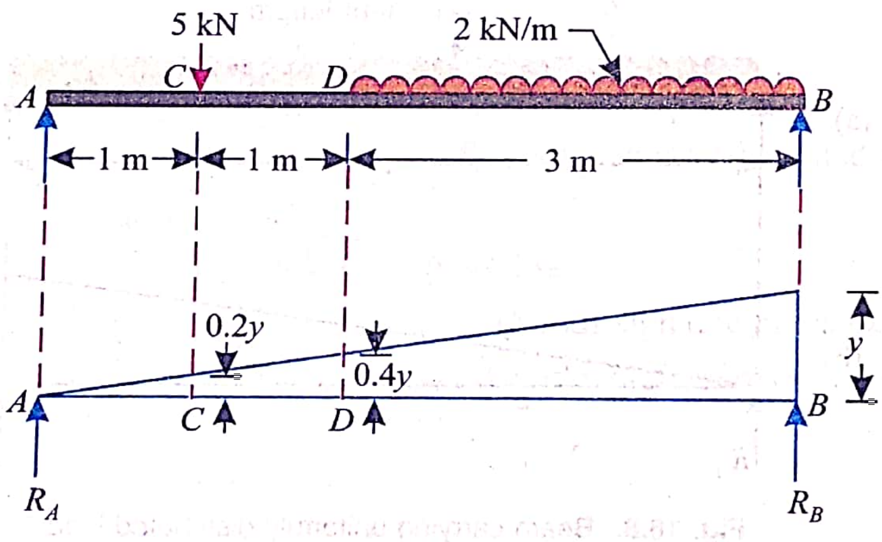

Fig. 16.10.

∴ Total virtual work done by the two reactions R_A and R_B

$$= +[(R_A \times 0) + (R_B \times y)] = +R_B \times y \quad \text{...}(i)$$

...(Plus sign due to reactions acting upwards)

and total virtual work done by the point load at C and uniformly distributed load between D and B.

$$= -\left[(5 \times 0.2y) + 2\left(\frac{0.4y + y}{2}\right) \times 3\right] = -5.2\ y \quad \text{...}(ii)$$

...(Minus sign due to loads acting downwards)

We know that from the principle of virtual work, that algebraic sum of the total virtual works done is zero. Therefore

$$R_B \times y - 5.2\ y = 0$$

or $\quad R_B = 5.2$ kN **Ans.**

and $\quad R_A = 5 + (2 \times 3) - 5.2 = 5.8$ kN **Ans.**

3. A simply supported beam of span 4 m is carrying a uniformly distributed load of 5 kN/m as shown in Fig. 16.14.

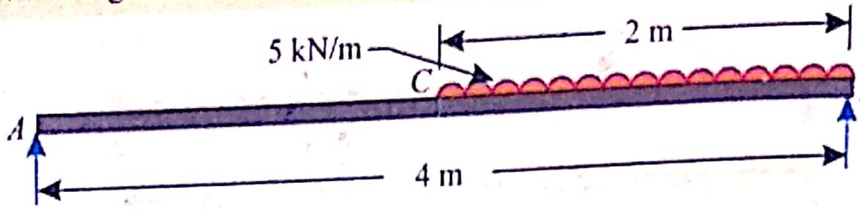

Fig. 16.14.

Using the principle of virtual work, find the reactions at A and B.

(Ans. 2.5 kN; 7.5 kN)

4. A beam of span 5 m is supported at A and B. It is subjected to a load system as shown in Fig. 16.15.

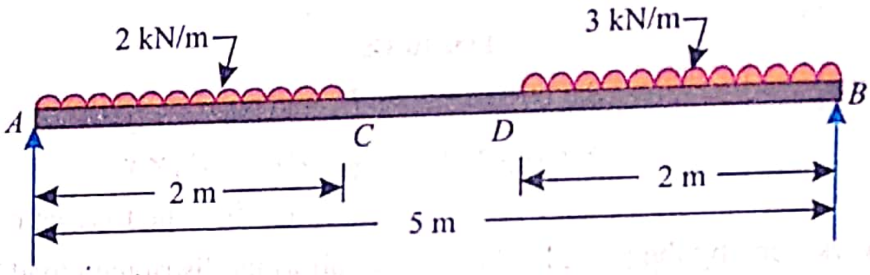

Fig. 16.15.

With the help of principle of virtual work, find the reactions at A and B.

(Ans. 4.4 kN; 5.6 kN)

ethod-3

16.8. APPLICATION OF THE PRINCIPLE OF VIRTUAL WORK ON LADDERS

We have already discussed in Art. 9.16 that in case of a ladder, its foot moves on the floor towards or away from the wall. It is thus obvious, that no work is done by the normal reaction (R_f) at the foot of the ladder. However, some work is done by the frictional force (F_f) at the foot of the ladder. Similarly, top of the ladder moves up or down along the wall. Thus no work is done by the normal reaction (R_w) at the top of the ladder. However, some work is done by the frictional force at the top of the ladder. This happens when the wall is not smooth, or in other words, the wall has some coefficient of friction.

Now the virtual works done by the frictional forces at the foot and top of the ladder are found out, and the principle of virtual work is applied as usual.

Note. If the vertical wall is smooth, then there is no frictional force at the top of the ladder. Thus no work is done at the top of the ladder.

Example 16.6. *A uniform ladder of weight 250 N rests against a smooth vertical wall and a rough horizontal floor making an angle of 45° with the horizontal. Find the force of friction at the floor using the method of virtual work.*

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Solution. Given : Weight of the ladder (W) = 250 N and inclination of the ladder with the horizontal (θ) = 45°

The ladder AB weighing 250 N and making an angle of 45° with the horizontal as shown in Fig. 16.16.

Let x = Virtual displacement of the foot of the ladder, and

y = Virtual displacement of the mid of the ladder at D.

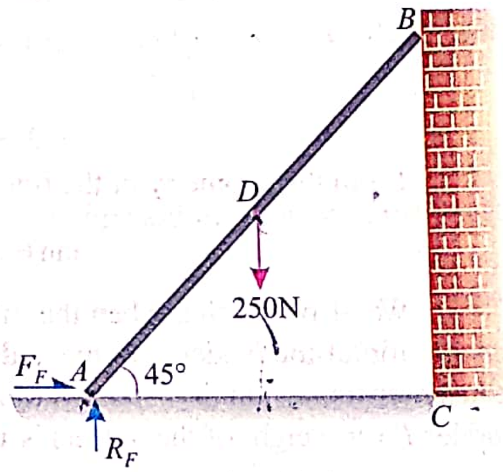

Fig. 16.16.

From the geometry of the figure, we find that when mid point D of the ladder moves downwards (due to its weight) then bottom A of the ladder moves towards left, which is prevented by the force of friction. Or in other words, the virtual displacement of the foot of the ladder A, due to force of friction (F_f) will be towards right.

Moreover, when the virtual displacement of the ladder at A due to frictional force towards right is x. Then the virtual displacement of the mid of the ladder,

$$y = \frac{x}{2 \tan 45°} = \frac{x}{2} = 0.5\,x$$

$\therefore$ Virtual work done by the frictional force

$$= + F_f \times x = F_f x$$

...(Plus sign due to movement of force towards right)

and virtual work done by the 250 N weight of the ladder

$$= -(250 \times y) = -(250 \times 0.5\,x) = -125\,x$$

...(Minus sign due to downward movement of the weight)

We know that from the principle of virtual work, that algebraic sum of the total virtual works done is zero. Therefore

$$F_f x - 125\,x = 0$$

or $$F_f = 125 \text{ N} \quad \textbf{Ans.}$$

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or $\qquad x \qquad x$

16.9. APPLICATION OF PRINCIPLE OF VIRTUAL WORK ON LIFTING MACHINES (Method-4)

We know that in the case of lifting machines, the effort moves downwards, whereas the load moves upwards. In such cases, the virtual works done by the effort and that by the load are found out. Now apply the principle of virtual work as usual.

Example 16.8. *A weight (W) of 5 kN is raised by a system of pulleys as shown in Fig. 16.18*

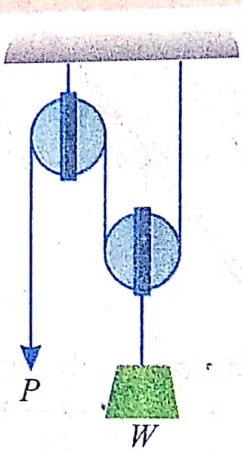

Fig. 16.18.

Using the method of virtual work, find the force P, which can hold the weight in equilibrium.

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Solution. Given : Weight $(W) = 5$ kN

Let P = Force which can hold the weight in equilibrium, and

y = Virtual upward displacement of the weight.

From the geometry of the system of pulleys, we find that when the virtual upward displacement of the weight is y, the virtual downward displacement of the force is $2y$.

$\therefore$ Virtual work done by the load

$$= + Wy = 5y \qquad \ldots(i)$$

...(Plus sign due to upward movement of the load)

and virtual work done by the effort

$$= - P \times 2y = -2\,Py \qquad \ldots(ii)$$

... (Minus sign due to downward movement of the effort)

We know that from the principle of virtual work, that algebraic sum of the virtual works done is zero. Therefore

$$5y - 2Py = 0$$

or
$$P = \frac{5y}{2y} = 2.5 \text{ kN} \quad \textbf{Ans.}$$

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