A SIMPLE GROUP GENERATED BY INVOLUTIONS INTERCHANGING RESIDUE CLASSES OF THE INTEGERS

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ABSTRACT. We present a countable simple group which arises in a natural way from the arithmetical structure of the ring of integers.

1. INTRODUCTION

Various types of infinite simple groups are treated in the literature so far: We refer to Carter [4] for the simple groups of Lie type, to Higman [9] and Stein [21] for finitely presented simple groups, to Kegel, Wehrfritz [11] for locally finite simple groups, to Baer [1] for composition factors of infinite symmetric groups, and to Ol'shanskii [20] and Chehata [5] for constructions of simple groups with certain given special properties.

Here we present and investigate an infinite simple group, which emerges in a natural way from the arithmetical structure of the ring of integers:

Definition 1.1. Let $\mathrm{CT}(\mathbb{Z})$ be the group generated by the set of all *class transpositions*: Given disjoint residue classes $r_1(m_1)$ and $r_2(m_2)$ of $\mathbb{Z}$, we define the *class transposition* $\tau_{r_1(m_1),r_2(m_2)} \in \mathrm{Sym}(\mathbb{Z})$ as the permutation which interchanges $r_1 + km_1$ and $r_2 + km_2$ for each integer k and which fixes all other points. Here we assume that $0 \leqslant r_1 < m_1$ and that $0 \leqslant r_2 < m_2$. For convenience, we set $\tau := \tau_{0(2),1(2)} : n \mapsto n + (-1)^n$.

The theorems given below list various properties of the group $\mathrm{CT}(\mathbb{Z})$ and provide information on the class of groups which embed into it, respectively. Their proof is the main subject of this article.

Theorem 1.2 (Properties of $\mathrm{CT}(\mathbb{Z})$)**.**

(1) The group $\mathrm{CT}(\mathbb{Z})$ is simple.
(2) The group $\mathrm{CT}(\mathbb{Z})$ is countable, but it has an uncountable series of simple subgroups which is parametrized by the sets of odd primes.
(3) The group $\mathrm{CT}(\mathbb{Z})$ is not finitely generated.
(4) The torsion elements of $\mathrm{CT}(\mathbb{Z})$ are divisible.
(5) The group $\mathrm{CT}(\mathbb{Z})$ acts highly transitively on $\mathbb{N}_0$, and it has a locally finite simple subgroup which does so as well.

Theorem 1.3 (Richness of the class of subgroups of $\mathrm{CT}(\mathbb{Z})$)**.**

(1) Every finite group embeds into $\mathrm{CT}(\mathbb{Z})$.
(2) Every free group of finite rank embeds into $\mathrm{CT}(\mathbb{Z})$.
(3) The modular group $\mathrm{PSL}(2,\mathbb{Z})$ embeds into $\mathrm{CT}(\mathbb{Z})$.
(4) Every free product of finitely many finite groups embeds into $\mathrm{CT}(\mathbb{Z})$.
(5) The class of subgroups of $\mathrm{CT}(\mathbb{Z})$ is closed under taking
  (a) direct products,

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(b) wreath products with finite groups, and
(c) restricted wreath products with $(\mathbb{Z}, +)$.
(6) The group $\mathrm{CT}(\mathbb{Z})$ *has*
(a) finitely generated subgroups which do not have finite presentations, and
(b) finitely generated subgroups with unsolvable membership problem.

So far, research in computational group theory focussed mainly on finite permutation groups, matrix groups, finitely presented groups, polycyclically presented groups and automata groups. For details, we refer to [10].

This article describes another large class of groups which are accessible to computational methods. This class includes the subgroups of $\mathrm{CT}(\mathbb{Z})$. Algorithms to compute with such groups are described in [16] and implemented in the package RCWA [14] for the computer algebra system GAP [8]. Many of the results proved in this article have first been discovered during extensive experiments with the RCWA package.

As a little example of how to compute in the group $\mathrm{CT}(\mathbb{Z})$, we factor the permutation

$$\alpha \in \mathrm{Sym}(\mathbb{Z}): \quad n \longmapsto \begin{cases} 2n/3 & \text{if } n \in 0(3), \\ (4n-1)/3 & \text{if } n \in 1(3), \\ (4n+1)/3 & \text{if } n \in 2(3) \end{cases}$$

into class transpositions, which shows that $\alpha \in \mathrm{CT}(\mathbb{Z})$. This permutation has already been investigated by Lothar Collatz in 1932, and its cycle structure is unknown so far (cf. Keller [12], Wirsching [22]).

In addition to the results given in Theorem 1.2 and 1.3, in the last section we determine two simple supergroups of $\mathrm{CT}(\mathbb{Z})$ which are in a certain sense 'canonical'.

2. Basic Properties of $\mathrm{CT}(\mathbb{Z})$

In this section we prove that the group $\mathrm{CT}(\mathbb{Z})$ is not finitely generated, that every finite group embeds into it, that its torsion elements are divisible and that it acts highly transitively on $\mathbb{N}_0$. This covers Theorem 1.2, Assertion (3), (4) and the first part of (5), as well as Theorem 1.3, Assertion (1). However, first we need to introduce some basic terms:

Definition 2.1. We call a mapping $f : \mathbb{Z} \to \mathbb{Z}$ *residue-class-wise affine* if there is a positive integer m such that the restrictions of f to the residue classes $r(m) \in \mathbb{Z}/m\mathbb{Z}$ are all affine, i.e. given by $f|_{r(m)} : r(m) \to \mathbb{Z},\ n \mapsto (a_{r(m)} \cdot n + b_{r(m)})/c_{r(m)}$ for certain coefficients $a_{r(m)}, b_{r(m)}, c_{r(m)} \in \mathbb{Z}$ depending on $r(m)$. We call the least possible m the *modulus* of f, written $\mathrm{Mod}(f)$. For reasons of uniqueness, we assume that $\gcd(a_{r(m)}, b_{r(m)}, c_{r(m)}) = 1$ and that $c_{r(m)} > 0$. We define the *multiplier* of f by $\mathrm{lcm}_{r(m) \in \mathbb{Z}/m\mathbb{Z}}\, a_{r(m)}$, and the *divisor* of f by $\mathrm{lcm}_{r(m) \in \mathbb{Z}/m\mathbb{Z}}\, c_{r(m)}$. We call the mapping f *integral* if its divisor is 1. We call f *class-wise order-preserving* if all $a_{r(m)}$ are positive.

It is easy to see that the permutations of this kind form a countable supergroup of $\mathrm{CT}(\mathbb{Z})$.

Definition 2.2. We denote the group which is formed by all residue-class-wise affine permutations of $\mathbb{Z}$ by $\mathrm{RCWA}(\mathbb{Z})$, and call its subgroups *residue-class-wise affine* groups.

The notation '$\mathrm{CT}(\mathbb{Z})$' respectively '$\mathrm{RCWA}(\mathbb{Z})$' reflects that generalizations to suitable rings other than $\mathbb{Z}$ make perfect sense. For the sake of simplicity and to keep the article easy to read, we refrain from following this possibly fruitful direction of research here.

Theorem 2.3. *The group* $\mathrm{CT}(\mathbb{Z})$ *is not finitely generated.*

Proof. It is easy to see that the multiplier of a product of residue-class-wise affine permutations divides the product of the multipliers of the factors, and that the corresponding assertion about divisors holds as well. Further, inversion obviously interchanges multiplier and divisor. Therefore as there are infinitely many primes and as for any $n \in \mathbb{N}$ there is a class transposition $\tau_{1(2),0(2n)}$ with multiplier and divisor n, the assertion follows. □

Finite symmetric groups embed into $\mathrm{CT}(\mathbb{Z})$:

Definition 2.4. Let $m \in \mathbb{N}$, and let S_m be the symmetric group of degree m. We define the monomorphism $\varphi_m : \mathrm{S}_m \hookrightarrow \mathrm{CT}(\mathbb{Z})$ by $\sigma \mapsto (\sigma^{\varphi_m} : n \mapsto n + (n \bmod m)^\sigma - n \bmod m)$, where we assume that S_m acts naturally on the set $\{0, 1, \ldots, m-1\}$.

Theorem 2.5. *Any finite group embeds into* $\mathrm{CT}(\mathbb{Z})$*, and the group* $\mathrm{CT}(\mathbb{Z})$ *acts highly transitively on* $\mathbb{N}_0$.

Proof. The first assertion is immediate.

Let $m \in \mathbb{N}$. Just like the group S_m itself, its image under φ_m acts m-transitively on the set $\{0, 1, \ldots, m-1\}$. The second assertion follows since m can be chosen arbitrary large and since class transpositions map nonnegative integers to nonnegative integers. □

Theorem 2.6. *The torsion elements of* $\mathrm{CT}(\mathbb{Z})$ *are divisible.*

Proof. We show that given an element $g \in \mathrm{CT}(\mathbb{Z})$ of finite order and a positive integer k, there is always an $h \in \mathrm{CT}(\mathbb{Z})$ such that $h^k = g$: Since g has finite order, it permutes a partition $\mathcal{P}$ of $\mathbb{Z}$ into finitely many residue classes on all of which it is affine. A k-th root h can be constructed from g by 'slicing' cycles $\prod_{i=2}^{l} \tau_{r_1(m_1),r_i(m_i)}$ on $\mathcal{P}$ into cycles $\prod_{i=1}^{l} \prod_{j=\max(2-i,0)}^{k-1} \tau_{r_1(km_1),r_i+jm_i(km_i)}$ of the k-fold length on the refined partition obtained from $\mathcal{P}$ by decomposing any $r_i(m_i) \in \mathcal{P}$ into residue classes (mod km_i). □

3. The Simplicity of $\mathrm{CT}(\mathbb{Z})$

The aim of this section is to show that the group $\mathrm{CT}(\mathbb{Z})$ is simple, and that it has an uncountable series of simple subgroups which is parametrized by the sets of odd primes. This covers Theorem 1.2, Assertion (1) and (2). First we need some lemmata:

Lemma 3.1. *Given two class transpositions* $\tau_{r_1(m_1),r_2(m_2)}, \tau_{r_3(m_3),r_4(m_4)}$ *not equal to* τ*, there is a product* π *of 6 class transpositions such that* $\tau^{\pi}_{r_1(m_1),r_2(m_2)} = \tau_{r_3(m_3),r_4(m_4)}$.

Proof. Let $r_5(m_5), r_6(m_6) \subset \mathbb{Z} \setminus (r_1(m_1) \cup r_2(m_2))$ be disjoint residue classes such that $\cup_{i=3}^{6} r_i(m_i) \neq \mathbb{Z}$, and let $r_7(m_7), r_8(m_8) \subset \mathbb{Z} \setminus \cup_{i=3}^{6} r_i(m_i)$ be disjoint residue classes. Then the following hold:

(1) $\tau_{r_1(m_1),r_2(m_2)}{}^{\tau_{r_1(m_1),r_5(m_5)}} \cdot \tau_{r_2(m_2),r_6(m_6)} = \tau_{r_5(m_5),r_6(m_6)}$.
(2) $\tau_{r_5(m_5),r_6(m_6)}{}^{\tau_{r_5(m_5),r_7(m_7)}} \cdot \tau_{r_6(m_6),r_8(m_8)} = \tau_{r_7(m_7),r_8(m_8)}$.
(3) $\tau_{r_7(m_7),r_8(m_8)}{}^{\tau_{r_3(m_3),r_7(m_7)}} \cdot \tau_{r_4(m_4),r_8(m_8)} = \tau_{r_3(m_3),r_4(m_4)}$.

The assertion follows. □

Lemma 3.2. *Let* $\sigma, \upsilon \in \mathrm{RCWA}(\mathbb{Z})$*, and put* $m := \mathrm{Mod}(\sigma)$*. If* υ *is integral and fixes all residue classes (mod* m*) setwise, then the commutator* $[\sigma, \upsilon]$ *is integral as well.*

Proof. Since υ fixes all residue classes (mod m), an affine partial mapping α of $[\sigma, \upsilon]$ is given by $\alpha_{\upsilon^{-1}}^{\alpha_\sigma} \cdot \alpha_\upsilon$ for certain affine partial mappings α_σ, α_υ and $\alpha_{\upsilon^{-1}}$ of σ, υ and υ^{-1}, respectively. The assertion follows, since the translations and reflections generate a normal subgroup of the affine group of the rationals. □

Lemma 3.3. *Let G be a subgroup of $\mathrm{RCWA}(\mathbb{Z})$ which contains $\mathrm{CT}(\mathbb{Z})$. Then any nontrivial normal subgroup $N \trianglelefteq G$ has an integral element $\iota \neq 1$.*

Proof. Let $\sigma \in N \setminus \{1\}$, and let $m := \mathrm{Mod}(\sigma)$. Without loss of generality we can assume that there is a residue class $r(m)$ such that $r(m)^\sigma \neq r(m)$. By Lemma 3.2, the mapping $\iota := [\sigma, \tau_{r(2m),r+m(2m)}] \in N \setminus \{1\}$ is integral. □

Now we can prove our theorem:

Theorem 3.4. *The group $\mathrm{CT}(\mathbb{Z})$ is simple.*

Proof. Let N be a nontrivial normal subgroup of $\mathrm{CT}(\mathbb{Z})$. We have to show that N contains all class transpositions.

By Lemma 3.1, all class transpositions except of τ are conjugate in $\mathrm{CT}(\mathbb{Z})$. Furthermore we have $\tau = \tau_{0(4),1(4)} \cdot \tau_{2(4),3(4)}$. Therefore it is already sufficient to show that N contains one class transposition which is not equal to τ.

By Lemma 3.3, the normal subgroup N has an integral element $\iota_1 \neq 1$. Let $m \geqslant 3$ be a multiple of the modulus of ι_1, and choose a residue class $r(m)$ which is moved by ι_1. Then put $\iota_2 := \tau_{r(2m),r+m(2m)} \cdot \tau_{r(2m)^{\iota_1},(r+m(2m))^{\iota_1}} = [\tau_{r(2m),r+m(2m)}, \iota_1] \in N$.

By the choice of m, we can now choose two distinct residue classes $r_1(2m)$ and $r_2(2m)$ in the complement of the support of ι_2. Then we have

$$\begin{aligned} \tau_{r_1(2m),r_2(2m)} \;=\; & \iota_2^{\tau_{r(2m),r_1(4m)} \cdot \tau_{r+m(2m),r_2(4m)}} \\ & \cdot \iota_2^{\tau_{r(2m),r_1+2m(4m)} \cdot \tau_{r+m(2m),r_2+2m(4m)}} \;\in\; N, \end{aligned}$$

which completes the proof of the theorem. □

Remark 3.5. Assume $\mathrm{CT}(\mathbb{Z}) \leqslant G \leqslant \mathrm{RCWA}(\mathbb{Z})$, and let N be a nontrivial normal subgroup of G. Then the proof of Theorem 3.4 shows in fact that N contains $\mathrm{CT}(\mathbb{Z})$, if we additionally take care that in the third paragraph we choose m sufficiently large such that there is indeed a residue class $r(m)$ which is not mapped to itself under ι_1.

Definition 3.6. Given a set $\mathbb{P}$ of odd primes, let $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z}) \leqslant \mathrm{CT}(\mathbb{Z})$ denote the subgroup which is generated by all class transpositions $\tau_{r_1(m_1),r_2(m_2)}$ for which all odd prime factors of m_1 and m_2 lie in $\mathbb{P}$.

Corollary 3.7. *The groups $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ are simple. Therefore the group $\mathrm{CT}(\mathbb{Z})$ has an uncountable series of simple subgroups, which is parametrized by the sets of odd primes.*

Proof. All of our arguments in this section apply to the groups $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ as well: In the proof of Lemma 3.1, we can choose the four residue classes $r_5(m_5), \ldots, r_8(m_8)$ in such a way that all prime factors of their moduli already divide $m_1 m_2 m_3 m_4$. The proofs of Lemma 3.2, Lemma 3.3 and Theorem 3.4 likewise do not require the presence of class transpositions whose moduli have certain odd factors. □

Problem 3.8 (Isomorphism Problem)**.** *Are there are distinct sets $\mathbb{P}_1$ and $\mathbb{P}_2$ of odd primes such that $\mathrm{CT}_{\mathbb{P}_1}(\mathbb{Z}) \cong \mathrm{CT}_{\mathbb{P}_2}(\mathbb{Z})$?*

4. Richness of the Class of Subgroups of $\mathrm{CT}(\mathbb{Z})$

In this section we prove Theorem 1.3, Assertion (2) – (6), as well as the second part of Theorem 1.2, Assertion (5).

Theorem 4.1. *Free groups of finite rank and* $\mathrm{PSL}(2,\mathbb{Z})$ *embed into* $\mathrm{CT}(\mathbb{Z})$.

Proof. To prove the assertion concerning free groups, it suffices to show that the free group of rank 2 embeds. An example of an embedding is

$$\varphi_{\mathrm{F}_2}:\ \mathrm{F}_2=\langle a,b\rangle\ \hookrightarrow\ \mathrm{CT}(\mathbb{Z}),\ \ a\ \mapsto\ (\tau\cdot\tau_{0(2),1(4)})^2,\ \ b\ \mapsto\ (\tau\cdot\tau_{0(2),3(4)})^2.$$

This can be seen by applying the Table-Tennis Lemma (see for example la Harpe [6], Section II.B.) to the cyclic groups generated by the images of a and b under φ_{F_2} and the sets $0(4)\cup 1(4)$ and $2(4)\cup 3(4)$. Likewise it follows from the Table-Tennis Lemma that

$$\varphi_{\mathrm{PSL}(2,\mathbb{Z})}:\ \mathrm{PSL}(2,\mathbb{Z})\ \cong\ \mathrm{C}_2\star\mathrm{C}_3\ \cong\ \langle a,b\mid a^2=b^3=1\rangle\ \hookrightarrow\ \mathrm{CT}(\mathbb{Z}),$$
$$a\ \mapsto\ \tau,\ \ b\ \mapsto\ \tau_{0(4),2(4)}\cdot\tau_{1(2),0(4)}$$

is an embedding of the modular group $\mathrm{PSL}(2,\mathbb{Z})$. This time one can use the sets $1(2)$ and $0(2)$ in place of $0(4)\cup 1(4)$ and $2(4)\cup 3(4)$. □

Theorem 4.2. *Every free product of finitely many finite groups embeds into* $\mathrm{CT}(\mathbb{Z})$.

Proof. Let $G_0,\ldots,G_{m-1}$ be finite groups. To see that their free product embeds into $\mathrm{CT}(\mathbb{Z})$, proceed as follows: First consider regular permutation representations φ_r of the groups G_r on the residue classes (mod $|G_r|$). Then take conjugates $H_r:=(\operatorname{im}\varphi_r)^{\sigma_r}$ of the images of these representations under mappings $\sigma_r\in\mathrm{CT}(\mathbb{Z})$ which map $0(|G_r|)$ to $\mathbb{Z}\setminus r(m)$. Finally use the fact that point stabilizers in regular permutation groups are trivial and apply the Table-Tennis Lemma to the groups H_r and the residue classes $r(m)$ to see that the group generated by the H_r is isomorphic to their free product. □

The group $\mathrm{RCWA}(\mathbb{Z})$ is not co-Hopfian. We need the following monomorphisms:

Definition 4.3. Let f be an injective residue-class-wise affine mapping, and let further $\pi_f:\mathrm{RCWA}(\mathbb{Z})\hookrightarrow\mathrm{RCWA}(\mathbb{Z}),\sigma\mapsto\sigma_f$ be the monomorphism defined by the properties $\forall\sigma\in\mathrm{RCWA}(\mathbb{Z})\ f\sigma_f=\sigma f$ and $\operatorname{supp}(\operatorname{im}\pi_f)\subseteq\operatorname{im}f$. Then we call π_f the *restriction monomorphism* associated with f.

Theorem 4.4. *The class of subgroups of* $\mathrm{CT}(\mathbb{Z})$ *is closed under taking direct products, under taking wreath products with finite groups and under taking restricted wreath products with* $(\mathbb{Z},+)$. *It is also closed under taking upwards extensions by finite groups.*

Proof. Given any two subgroups $G,H\leqslant\mathrm{RCWA}(\mathbb{Z})$, the group generated by $\pi_{n\mapsto 2n}(G)$ and $\pi_{n\mapsto 2n+1}(H)$ is clearly isomorphic to $G\times H$. This argument applies to subgroups of our group $\mathrm{CT}(\mathbb{Z})$ as well, since the image of a class transposition $\tau_{r_1(m_1),r_2(m_2)}$ under a restriction monomorphism $\pi_{n\mapsto mn+r}$ is $\tau_{mr_1+r(mm_1),mr_2+r(mm_2)}$.

Looking at the monomorphisms $\pi_{n\mapsto mn+r}$ and φ_m, it is immediate to see that the classes of subgroups of $\mathrm{RCWA}(\mathbb{Z})$ and $\mathrm{CT}(\mathbb{Z})$ are also closed under taking wreath products with finite groups. The assertion on upwards extensions by finite groups follows now from the Universal Embedding Theorem (see e.g. Theorem 2.6A in Dixon, Mortimer [7]).

Given a subgroup $G\leqslant\mathrm{CT}(\mathbb{Z})$, the group generated by $\pi_{n\mapsto 4n+3}(G)$ and $\tau\cdot\tau_{0(2),1(4)}$ is isomorphic to the restricted wreath product $G\wr(\mathbb{Z},+)$. This holds since the orbit of the residue class 3(4) under the action of the cyclic group $\langle\tau\cdot\tau_{0(2),1(4)}\rangle$ consists of pairwise disjoint residue classes, which means that the conjugates of $\pi_{n\mapsto 4n+3}(G)$ under powers of $\tau\cdot\tau_{0(2),1(4)}$ have pairwise disjoint supports. □

Corollary 4.5. *The group* $\mathrm{CT}(\mathbb{Z})$ *has*

(1) finitely generated subgroups which do not have finite presentations, and
(2) finitely generated subgroups with unsolvable membership problem.

Proof.

(1) By Theorem 4.4, the group $\mathrm{CT}(\mathbb{Z})$ contains nontrivial restricted wreath products $G \wr (\mathbb{Z}, +)$. By Baumslag [2], these do not have finite presentations.
(2) Let $\mathrm{F}_2 = \langle a, b \rangle$ be the free group of rank 2. Further let $r_1, \ldots, r_k \in \mathrm{F}_2$ be the relators of a finitely presented group with unsolvable word problem – by Novikov [19] and Boone [3], such groups exist. Then the membership problem for the group $\langle (a,a), (b,b), (1,r_1), \ldots, (1,r_k) \rangle < \mathrm{F}_2 \times \mathrm{F}_2$ is unsolvable (cf. Mihailova [18]; see also Lyndon, Schupp [17], Chapter IV.4). Therefore as by Theorem 4.1 and Theorem 4.4 the group $\mathrm{F}_2 \times \mathrm{F}_2$ embeds into $\mathrm{CT}(\mathbb{Z})$, there exist finitely generated subgroups $G < \mathrm{CT}(\mathbb{Z})$ with unsolvable membership problem. □

The class transpositions which interchange two residue classes with the same modulus generate a proper subgroup of $\mathrm{CT}(\mathbb{Z})$, which acts highly transitively on $\mathbb{N}_0$ as well:

Definition 4.6. Let $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ denote the subgroup of $\mathrm{CT}(\mathbb{Z})$ which is generated by all integral class transpositions.

Theorem 4.7. *The group $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ is locally finite and simple.*

Proof. Finitely generated subgroups of $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ act faithfully on the set of residue classes modulo the lcm of the moduli of the generators. Therefore they are finite. Hence the group $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ is locally finite.

Let N be a nontrivial normal subgroup of $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$. In order to prove that $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ is simple, we have to show that N contains any class transposition of the form $\tau_{r_1(m),r_2(m)}$.

Let $\iota \in N \backslash \{1\}$, let $m > 2$ be a multiple of $\mathrm{Mod}(\iota)$ and choose a residue class $r(m)$ in the support of ι. Then we have $\tau_{r(2m),r+m(2m)} \cdot \tau_{r(2m)^\iota,(r+m(2m))^\iota} = [\iota, \tau_{r(2m),r+m(2m)}] \in N$.

All such products of two integral class transpositions with modulus $2m$ and disjoint supports are conjugate in $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$. The reason for this is that their preimages under the monomorphism φ_{2m} have the same cycle structure, and are therefore conjugate in S_{2m}.

Let $\tau_1 = \tau_{r_1(m),r_2(m)}$ be an integral class transposition with modulus m. Then for any integral class transposition τ_2 with modulus $2m$ whose support intersects trivially with the one of τ_1, we have $\tau_1 = (\tau_2 \cdot \tau_{r_1(2m),r_2(2m)}) \cdot (\tau_2 \cdot \tau_{r_1+m(2m),r_2+m(2m)}) \in N$.

Given a divisor d of m and an integral class transposition $\tau_{r_1(d),r_2(d)}$ with modulus d, we have $\tau_{r_1(d),r_2(d)} = \prod_{k=0}^{m/d-1} \tau_{r_1+kd(m),r_2+kd(m)} \in N$.

In the second paragraph of the proof, we can choose m to be a multiple of any given positive integer. Therefore we can conclude that N contains indeed any integral class transposition. Hence the group $\mathrm{CT}_{\mathrm{int}}(\mathbb{Z})$ is simple, as claimed. □

5. Collatz' Permutation Lies in $\mathrm{CT}(\mathbb{Z})$

In this section, as a small but illustrative example we show that Collatz' permutation

$$\alpha \in \mathrm{RCWA}(\mathbb{Z}): \quad n \longmapsto \begin{cases} 2n/3 & \text{if } n \in 0(3), \\ (4n-1)/3 & \text{if } n \in 1(3), \\ (4n+1)/3 & \text{if } n \in 2(3) \end{cases}$$

lies in $\mathrm{CT}(\mathbb{Z})$.

In Keller [12] it is shown that α has at most finitely many cycles of any given finite length. However according to Wirsching [22], it is for example not yet known whether the cycle $(\ \ldots\ 34\ 45\ 30\ 20\ 27\ 18\ 12\ 8\ 11\ 15\ 10\ 13\ 17\ 23\ 31\ \ldots\ )$ of α is finite or infinite.

In spite of this we can show that the permutation α lies in $\mathrm{CT}(\mathbb{Z})$ by determining an explicit factorization into generators.

The major obstacle we are confronted with when trying to obtain such a factorization is the fact that multiplier and divisor of α are coprime, whereas multiplier and divisor of a class transposition are always the same. We even need to form a product of class transpositions in such a way that one prime divisor gets eliminated from the multiplier of the product, but appears in the denominators of *all* of its affine partial mappings.

As a first step towards a solution of the factorization problem, we hence attempt to determine some product of class transpositions which has coprime multiplier and divisor. We find that given an odd prime p, the permutation

$$\begin{aligned}\sigma_p \ :=\ & \tau_{0(8),1(2p)} \cdot \tau_{4(8),2p-1(2p)} \\ & \cdot \tau_{0(4),1(2p)} \cdot \tau_{2(4),2p-1(2p)} \\ & \cdot \tau_{2(2p),1(4p)} \cdot \tau_{4(2p),2p+1(4p)} \ \in\ \mathrm{CT}(\mathbb{Z})\end{aligned}$$

has multiplier p and divisor 2. Indeed, evaluating this product yields

$$\sigma_p :\ n \longmapsto \begin{cases} (pn+2p-2)/2 & \text{if } n \in 2(4), \\ n/2 & \text{if } n \in 0(4) \setminus (4(4p) \cup 8(4p)), \\ n+2p-7 & \text{if } n \in 8(4p), \\ n-2p+5 & \text{if } n \in 2p-1(2p), \\ n+1 & \text{if } n \in 1(2p), \\ n-3 & \text{if } n \in 4(4p), \\ n & \text{if } n \in 1(2) \setminus (1(2p) \cup 2p-1(2p)). \end{cases}$$

The GAP [8] package RCWA [14] provides a factorization routine for residue-class-wise affine permutations, which uses certain elaborate heuristics. The permutations σ_p and their images under restriction monomorphisms $\pi_{n \mapsto mn+r}$ play a key role in this routine. It has been used to obtain the following factorization of α:

$$\begin{aligned}\alpha \ =\ & \tau_{2(3),3(6)} \cdot \tau_{1(3),0(6)} \cdot \tau_{0(3),1(3)} \cdot \tau \cdot \tau_{0(36),1(36)} \\ & \cdot \tau_{0(36),35(36)} \cdot \tau_{0(36),31(36)} \cdot \tau_{0(36),23(36)} \cdot \tau_{0(36),18(36)} \cdot \tau_{0(36),19(36)} \\ & \cdot \tau_{0(36),17(36)} \cdot \tau_{0(36),13(36)} \cdot \tau_{0(36),5(36)} \cdot \tau_{2(36),10(36)} \cdot \tau_{2(36),11(36)} \\ & \cdot \tau_{2(36),15(36)} \cdot \tau_{2(36),20(36)} \cdot \tau_{2(36),28(36)} \cdot \tau_{2(36),26(36)} \cdot \tau_{2(36),25(36)} \\ & \cdot \tau_{2(36),21(36)} \cdot \tau_{2(36),4(36)} \cdot \tau_{3(36),8(36)} \cdot \tau_{3(36),7(36)} \cdot \tau_{9(36),16(36)} \\ & \cdot \tau_{9(36),14(36)} \cdot \tau_{9(36),12(36)} \cdot \tau_{22(36),34(36)} \cdot \tau_{27(36),32(36)} \cdot \tau_{27(36),30(36)} \\ & \cdot \tau_{29(36),33(36)} \cdot \tau_{10(18),35(36)} \cdot \tau_{5(18),35(36)} \cdot \tau_{10(18),17(36)} \cdot \tau_{5(18),17(36)} \\ & \cdot \tau_{8(12),14(24)} \cdot \tau_{6(9),17(18)} \cdot \tau_{3(9),17(18)} \cdot \tau_{0(9),17(18)} \cdot \tau_{6(9),16(18)} \\ & \cdot \tau_{3(9),16(18)} \cdot \tau_{0(9),16(18)} \cdot \tau_{6(9),11(18)} \cdot \tau_{3(9),11(18)} \cdot \tau_{0(9),11(18)} \\ & \cdot \tau_{6(9),4(18)} \cdot \tau_{3(9),4(18)} \cdot \tau_{0(9),4(18)} \cdot \tau_{0(6),14(24)} \cdot \tau_{0(6),2(24)} \\ & \cdot \tau_{8(12),17(18)} \cdot \tau_{7(12),17(18)} \cdot \tau_{8(12),11(18)} \cdot \tau_{7(12),11(18)} \cdot \sigma_3^{-1} \\ & \cdot \tau_{7(12),17(18)} \cdot \tau_{2(6),17(18)} \cdot \tau_{0(3),17(18)} \cdot \sigma_3^{-3}.\end{aligned}$$

This shows constructively that $\alpha \in \mathrm{CT}(\mathbb{Z})$.

6. Two Simple Supergroups of $\mathrm{CT}(\mathbb{Z})$

In this section we present two simple subgroups of $\mathrm{RCWA}(\mathbb{Z})$ which properly contain $\mathrm{CT}(\mathbb{Z})$ and which act highly transitively on $\mathbb{Z}$.

We find them in the kernels of certain epimorphisms $\pi^+ : \mathrm{RCWA}^+(\mathbb{Z}) \to (\mathbb{Z}, +)$ and $\pi^- : \mathrm{RCWA}(\mathbb{Z}) \to \mathbb{Z}^\times \cong \mathrm{C}_2$, where $\mathrm{RCWA}^+(\mathbb{Z}) < \mathrm{RCWA}(\mathbb{Z})$ denotes the subgroup consisting of all class-wise order-preserving elements.

Using the notation $\sigma|_{r(m)} : n \mapsto (a_{r(m)} \cdot n + b_{r(m)})/c_{r(m)}$ for the affine partial mappings of an rcwa permutation σ with modulus m, these epimorphisms are given by

$$\pi^+ : \ \sigma \ \mapsto \ \frac{1}{m} \sum_{r(m) \in \mathbb{Z}/m\mathbb{Z}} \frac{b_{r(m)}}{|a_{r(m)}|}$$

and

$$\pi^- : \ \sigma \ \mapsto \ (-1)^{\pi^+(\sigma) + \sum\limits_{r(m):\ a_{r(m)} < 0} \frac{m - 2r}{m}},$$

respectively (see Sections 2.11 and 2.12 in Kohl [13]).

Definition 6.1. We denote the kernels of π^+ and π^- by K^+ and K^-, respectively.

It is easy to see that $\mathrm{CT}(\mathbb{Z}) < K^+ < K^- < \mathrm{RCWA}(\mathbb{Z})$.

Our simple groups will be the subgroups of K^+ and K^-, respectively, which are generated by the elements which are *tame* in the following sense:

Definition 6.2. We call an element $\sigma \in \mathrm{RCWA}(\mathbb{Z})$ *tame* if it permutes a partition of $\mathbb{Z}$ into finitely many residue classes on each of which it is affine, and *wild* otherwise. We call a group $G < \mathrm{RCWA}(\mathbb{Z})$ *tame* if there is a common such partition for all elements of G, and *wild* otherwise. We call the specified partitions *respected partitions* of σ respectively G.

For an alternative characterization of this notion of tameness and a generalization of it to not necessarily bijective residue-class-wise affine mappings, see Kohl [13], [15].

Obviously, finite residue-class-wise affine groups and integral residue-class-wise affine permutations are tame.

Tameness is invariant under conjugation: If $\alpha \in \mathrm{RCWA}(\mathbb{Z})$ respects a partition $\mathcal{P}$, then a conjugate α^β respects the partition consisting of the images of the intersections of the residue classes in $\mathcal{P}$ with the sources of the affine partial mappings of β under β.

The product of two tame permutations is in general not tame. Tameness of products also does not induce an equivalence relation on the set of tame permutations: Let for example $a := \tau_{1(6),4(6)}$, $b := \tau_{0(5),2(5)}$ and $c := \tau_{3(4),4(6)}$. Then ab and bc are tame, but ac is not.

If a tame group does not act faithfully on a respected partition, then the kernel of the action clearly does not act on $\mathbb{N}_0$. Thus as the group $\mathrm{CT}(\mathbb{Z})$ acts on $\mathbb{N}_0$, its tame subgroups are finite.

Definition 6.3. We denote the normal subgroups of K^+ and K^- which are generated by the tame elements by $\tilde{K}^+$ and $\tilde{K}^-$, respectively.

It is easy to see that all tame elements of $\mathrm{RCWA}(\mathbb{Z})$ can be factored into class transpositions and members of the following two series:

Definition 6.4. Let $r(m) \subseteq \mathbb{Z}$ be a residue class.

(1) We define the *class shift* $\nu_{r(m)} \in \mathrm{RCWA}(\mathbb{Z})$ by

$$\nu_{r(m)} : \ n \ \mapsto \begin{cases} n + m & \text{if } \ n \in r(m), \\ n & \text{otherwise.} \end{cases}$$

(2) We define the *class reflection* $\varsigma_{r(m)} \in \mathrm{RCWA}(\mathbb{Z})$ by

$$\varsigma_{r(m)} : \ n \mapsto \begin{cases} -n+2r & \text{if } n \in r(m), \\ n & \text{otherwise,} \end{cases}$$

where we assume that $0 \leqslant r < m$.

For convenience, we set $\nu := \nu_{\mathbb{Z}} : n \mapsto n+1$ and $\varsigma := \varsigma_{\mathbb{Z}} : n \mapsto -n$.

Obviously, class shifts and class reflections do not lie in K^-.

Theorem 6.5. *We have*

(1) $\tilde{K}^+ = \langle \mathrm{CT}(\mathbb{Z}), \nu_{1(3)} \cdot \nu_{2(3)}^{-1} \rangle$, *and*

(2) $\tilde{K}^- = \langle \mathrm{CT}(\mathbb{Z}), \nu_{1(3)} \cdot \nu_{2(3)}, \varsigma_{0(2)} \cdot \nu_{0(2)} \rangle$.

Proof. We determine series of generators:

(1) Considering respected partitions, we check that $\tilde{K}^+$ is generated by
   (a) all class transpositions and
   (b) all quotients of two class shifts with disjoint supports whose union has a nontrivial complement in $\mathbb{Z}$.

   For this we look at the process of factoring a given tame $\vartheta \in K^+$ into these elements:

   ad (a) Let $\mathcal{P}$ be a respected partition of ϑ. Divide ϑ by a product of class transpositions which respects $\mathcal{P}$ as well and which induces on $\mathcal{P}$ the same permutation as ϑ does. Now ϑ is integral and fixes $\mathcal{P}$.

   ad (b) Finally factor ϑ into quotients of two class shifts whose supports are distinct residue classes in $\mathcal{P}$. This is possible since the lattice in $\mathbb{Z}^{|\mathcal{P}|}$ which consists of all vectors with zero coordinate sum is spanned by the differences of two distinct canonical basis vectors.

(2) Considering respected partitions, we check that $\tilde{K}^-$ is generated by
   (a) all products of a class reflection and a class shift with the same support which has a nontrivial complement in $\mathbb{Z}$,
   (b) all class transpositions and
   (c) all products of two class shifts with disjoint supports whose union has a nontrivial complement in $\mathbb{Z}$.

   For this we look at the process of factoring a given tame $\vartheta \in K^-$ into these elements:

   ad (a) Let $\mathcal{P}$ be a respected partition of ϑ of length at least 3. Divide ϑ from the left by products $\varsigma_{r(m)} \cdot \nu_{r(m)}$, where $r(m)$ runs over all residue classes in $\mathcal{P}$ on which ϑ is order-reversing. Now ϑ is class-wise order-preserving.

   ad (b) Divide ϑ by a product of class transpositions which respects the partition $\mathcal{P}$ as well, and which also induces the same permutation on it. Now ϑ is integral and fixes $\mathcal{P}$.

   ad (c) Finally factor ϑ into products of two class shifts whose supports are distinct residue classes in $\mathcal{P}$ and inverses of such products. This is possible since the lattice in $\mathbb{Z}^{|\mathcal{P}|}$ which consists of all vectors with even coordinate sum is spanned by the sums of two distinct canonical basis vectors.

Now we collapse series 1.(b), 2.(a) and 2.(c) by taking orbit representatives under the conjugation action of the group $\mathrm{CT}(\mathbb{Z})$ to obtain the indicated single generators. □

Theorem 6.6. *The groups $\tilde{K}^+$ and $\tilde{K}^-$ are simple.*

Proof. By Remark 3.5, nontrivial normal subgroups of $\tilde{K}^+$ and $\tilde{K}^-$ contain $\mathrm{CT}(\mathbb{Z})$.

(1) Let N be a nontrivial normal subgroup of $\tilde{K}^+$. Given disjoint residue classes $r_1(m_1)$ and $r_2(m_2)$ whose union has a nontrivial complement in $\mathbb{Z}$, for an arbitrary residue class $r_3(m_3) \subseteq \mathbb{Z} \setminus (r_1(m_1) \cup r_2(m_2))$ we have

$$\nu_{r_1(m_1)} \cdot \nu_{r_2(m_2)}^{-1} \;=\; [\tau_{r_1(m_1),r_2(m_2)}, \nu_{r_3(m_3)} \cdot \nu_{r_2(m_2)}^{-1}] \;\in\; N.$$

Putting $r_1(m_1) := 1(3)$ and $r_2(m_2) := 2(3)$, the simplicity of $\tilde{K}^+$ follows from Theorem 6.5, Assertion (1).

(2) Let N be a nontrivial normal subgroup of $\tilde{K}^-$. Given disjoint residue classes $r_1(m_1)$ and $r_2(m_2)$ whose union has a nontrivial complement in $\mathbb{Z}$, for an arbitrary residue class $r_3(m_3) \subseteq \mathbb{Z} \setminus (r_1(m_1) \cup r_2(m_2))$ we have

$$\begin{aligned}\nu_{r_1(m_1)} \cdot \nu_{r_2(m_2)} \;&=\; [\tau_{r_1(m_1),r_2(m_2)}, \nu_{r_3(m_3)} \cdot \varsigma_{r_1(m_1)}] \\ &\quad \cdot [\tau_{r_1(m_1),r_2(m_2)}, \varsigma_{r_1(m_1)} \cdot \nu_{r_1(m_1)}] \;\in\; N.\end{aligned}$$

This shows in particular that N contains $\nu_{1(3)} \cdot \nu_{2(3)}$. Let $r(m) \subset \mathbb{Z}$ be a residue class. Then for any residue class $\tilde{r}(\tilde{m}) \subseteq \mathbb{Z} \setminus r(m)$ we have

$$\begin{aligned}\varsigma_{r(m)} \cdot \nu_{r(m)} \;&=\; [\tau_{r(m),\tilde{r}(\tilde{m})}, \varsigma_{r(m)} \cdot \nu_{\tilde{r}(\tilde{m})}] \cdot [\tau_{\tilde{r}(2\tilde{m}),\tilde{r}+\tilde{m}(2\tilde{m})}, \varsigma_{\tilde{r}(2\tilde{m})} \cdot \nu_{\tilde{r}(2\tilde{m})}] \\ &\quad \cdot (\nu_{\tilde{r}(2\tilde{m})} \cdot \nu_{\tilde{r}+\tilde{m}(2\tilde{m})} \cdot \tau_{\tilde{r}(2\tilde{m}),\tilde{r}+\tilde{m}(2\tilde{m})})^{-1} \;\in\; N.\end{aligned}$$

This shows in particular that N contains also $\varsigma_{0(2)} \cdot \nu_{0(2)}$, and the simplicity of the group $\tilde{K}^-$ follows from Theorem 6.5, Assertion (2). □

Theorem 6.7. *The groups $\tilde{K}^+$ and $\tilde{K}^-$ act highly transitively on $\mathbb{Z}$.*

Proof. Since $\tilde{K}^+ < \tilde{K}^-$ it is sufficient to prove the assertion for $\tilde{K}^+$. Let k be a positive integer, and let $(n_1, \ldots, n_k)$ and $(\tilde{n}_1, \ldots, \tilde{n}_k)$ be two k-tuples of pairwise distinct integers. We have to show that there is an element $\sigma \in \tilde{K}^+$ such that $(n_1^\sigma, \ldots, n_k^\sigma) = (\tilde{n}_1, \ldots, \tilde{n}_k)$.

Let $m := 2k + 1$, and choose a residue class $r(m)$ which does not contain one of the points n_i or $\tilde{n}_i$. Define $\sigma_1, \tilde{\sigma}_1 \in \tilde{K}^+$ by

$$\sigma_1 \;:=\; \prod_{i:n_i<0} (\nu_{r(m)} \cdot \nu_{n_i(m)}^{-1})^{\lfloor \frac{n_i}{m} \rfloor} \quad \text{and} \quad \tilde{\sigma}_1 \;:=\; \prod_{i:\tilde{n}_i<0} (\nu_{r(m)} \cdot \nu_{\tilde{n}_i(m)}^{-1})^{\lfloor \frac{\tilde{n}_i}{m} \rfloor},$$

respectively. Then the images of all points n_i under σ_1 are nonnegative, and the same holds for the images of the points $\tilde{n}_i$ under $\tilde{\sigma}_1$. Since $\mathrm{CT}(\mathbb{Z})$ acts highly transitively on $\mathbb{N}_0$, we can choose a $\sigma_2 \in \mathrm{CT}(\mathbb{Z}) < \tilde{K}^+$ which maps the images of the n_i under σ_1 to the images of the $\tilde{n}_i$ under $\tilde{\sigma}_1$. Now the permutation $\sigma := \sigma_1 \cdot \sigma_2 \cdot \tilde{\sigma}_1^{-1}$ serves our purposes. □

Conjecture 6.8. *The group* $\mathrm{RCWA}(\mathbb{Z})$ *is generated by its tame elements.*

Remark 6.9. Conjecture 6.8 is equivalent to the assertion that $\tilde{K}^+ = K^+$ and $\tilde{K}^- = K^-$. If it holds, we have $\mathrm{RCWA}(\mathbb{Z}) = \langle \mathrm{CT}(\mathbb{Z}), \varsigma_{0(2)} \rangle$:

(1) It is $\nu = \varsigma_{0(2)} \cdot \tau \cdot (\varsigma_{0(2)}^{\tau_{1(4),2(4)}} \cdot \varsigma_{0(2)}^{\tau_{1(2),0(4)}})^{\tau_{0(2),1(4)}} \in \langle \mathrm{CT}(\mathbb{Z}), \varsigma_{0(2)} \rangle$.

(2) It is $\nu_{0(2)} = \tau\nu$, $\nu_{1(2)} = \nu_{0(2)}^\tau$, $\varsigma_{1(2)} = \varsigma_{0(2)}^\tau$ and $\varsigma = \varsigma_{0(2)} \cdot \nu_{1(2)} \cdot \varsigma_{1(2)}$. Therefore we know that $\{\nu_{0(2)}, \nu_{1(2)}, \varsigma_{1(2)}, \varsigma\} \subset \langle \mathrm{CT}(\mathbb{Z}), \varsigma_{0(2)} \rangle$.

(3) Let $r(m) \subset \mathbb{Z}$ be a residue class $\neq 1(2)$. We choose an arbitrary residue class $\tilde{r}(\tilde{m}) \subseteq \mathbb{Z} \setminus (0(2) \cup r(m))$, and put $\vartheta := \tau_{0(2),\tilde{r}(\tilde{m})} \cdot \tau_{\tilde{r}(\tilde{m}),r(m)} \in \mathrm{CT}(\mathbb{Z})$. Then we have $\{\nu_{r(m)}, \varsigma_{r(m)}\} = \{\nu_{0(2)}^\vartheta, \varsigma_{0(2)}^\vartheta\} \subset \langle \mathrm{CT}(\mathbb{Z}), \varsigma_{0(2)} \rangle$.

The factorization routine in RCWA [14] provides some evidence for Conjecture 6.8.

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