MODULE 1- VECTOR CALCULUS

Syllabus

- Vector Differentiation
  - Velocity and Acceleration
  - Gradient
  - Divergence
  - Curl
  - Laplacian
  - Solenoidal and Irrotational Vectors

Recap

- Scalar product : $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

  $= |\vec{a}||\vec{b}| \cos\theta$, θ is the angle between $\vec{a}$ and $\vec{b}$

- Projection : projection of $\vec{a}$ on $\vec{b}$ is $\vec{a} \cdot \hat{b}$.

- Work done : If a force $\vec{F}$ displaces are provide from A to B then

  the work done is $\vec{F} \cdot \vec{AB}$

- Vector product : $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

  $= |\vec{a}||\vec{b}| \sin\theta\ \hat{n}$ where $\hat{n}$ is unit vector perpendicular to both

  $\vec{a}$ and $\vec{b}$

- Moment: If the vector moment of a force $\vec{F}$ about a point A

  through a point P is $\vec{M} = \vec{AP} \times \vec{F}$

- Scalar triple product : $[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$

  $$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & c_2 & b_3 \\ c_1 & c_3 & c_3 \end{vmatrix}$$

Equation of line :

- Equations of a line passing through (x_1, y_1, z_1) and direction ratios

a, b, c are $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

- Equation of a plane : is of the form

$ax + by + cz + d = 0$ where a,b,c are the DR's of the normal.

Introduction

In this chapter the basic concepts of Differential Calculus of scalar functions are extended to vector functions.

Here we study vector differentiation, gradient, divergence, curl, solenoidal and irrational fields.

Also we study about vector integration and verification of the integral theorems.

Vector Function of a Scalar Variable

Let $\vec{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$ where f_1, f_2 and f_3 are functions of a variable t is called a vector function.

If f_1, f_2 and f_3 are differentiable then we define

$$\frac{d\vec{f}}{dt} = \lim_{\delta t \to 0} \frac{\vec{f}(t + \delta t) - \vec{f}(t)}{\delta t} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

Similarly higher order derivatives can also be defined.

Note:

$\frac{d\vec{f}}{dt} = \vec{0}$ iff $\vec{f}$ is a constant vector.

Unit Normal to a Space Curve

Let A be a fixed point on the curve and s be the length of the arc AP where P(x,y,z) is a point on the space curve.

Let $\hat{t}$ be the unit tangent vector at P.

Consider $\hat{t}.\hat{t} = 1$

d.w.r.t s

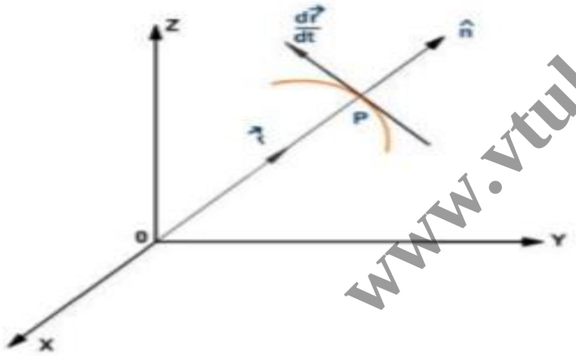
$$\hat{t}.\frac{d\hat{t}}{ds} + \frac{d\hat{t}}{ds}.\hat{t} = 0$$

$$\Rightarrow \quad 2\hat{t}.\frac{d\hat{t}}{ds} = 0$$

$$\Rightarrow \quad \hat{t}.\frac{d\hat{t}}{ds} = 0$$

$\frac{d\hat{t}}{ds}$ is a normal to the space curve and $\hat{n} = \dfrac{\frac{d\hat{t}}{ds}}{\left|\frac{d\hat{t}}{ds}\right|}$

is called the unit normal to the curve.



Scalar and Vector Fields

Scalar Valued Function

Let P(x,y,z) which is a scalar is called a scalar point function.

Note: A scalar point function is also called a scalar fixed . $\phi = \phi(x,y,z)$

Ex.: $\phi(x,y,z) = x^2 + y^2 + z^2$

Vector Valued Function

Let P(x,y,z) be a point. A vector point function is a vector whose components are real valued functions of x,y,z. A vector point function is called a vector field $\vec{F}$.

$$\vec{f} = \vec{f}\,(x,y,z) = f_1\,\hat{i} + f_2\,\hat{j} + f_3\,\hat{k}$$

Ex:

$$\vec{F} = xyz\,\hat{i} + x^2y\,\hat{j} + yz\,\hat{k}$$

Differential operator ∇ :

∇ denotes the operation $\frac{\partial}{\partial x}\partial\,\hat{i} + \frac{\partial}{\partial y}\,\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$= \sum \frac{\partial}{\partial x}\,\hat{i}$$

∇ does not represent a vector if only defines the differential operator

Gradient of a Scalar Field

Let $\phi = \phi(x,y,z)$ be a scalar field, then $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial z}\,\hat{k} + \sum \frac{\partial\phi}{\partial x}\,\hat{i}$ is called the gradient of ϕ denoted by $\nabla\phi$.

The following are some results obtained by the definition of $\nabla\phi$.

1. $\nabla(\alpha\phi) = \alpha\nabla\phi$, α is a scalar constant.
2. $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$, ϕ and ψ are scalar fields.
3. $\nabla(\alpha\phi+\beta\psi) = \alpha\nabla\phi + \beta\nabla\psi$, α and β are scalar constants
4. $\nabla(\phi+\psi) = \phi\nabla\psi + \psi\nabla\phi$
5. $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$
6. $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$= \left(i\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\left(i dx + \hat{j}dy + \hat{k}dz\right)$$

$$d\phi = \nabla\phi \,.\, d\vec{r}$$

Note 1: Since $\phi = c$ on the surface.

$\therefore d\phi = \nabla\phi \cdot d\vec{r} = 0$

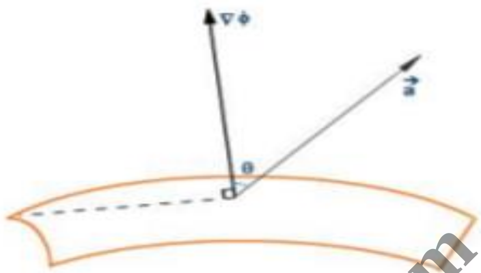
which shows that $\nabla\phi$ is perpendicular to the tangent plane at any point.

Note 2 : Unit vector along $\nabla\phi$ is denoted by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$ is called the unit normal to the surface $\phi(x,y,z) = c$.

Directional Derivative

Let $\vec{a}$ be a vector inclined at an angle θ with $\nabla\phi$ then $\nabla\phi$.

$\hat{a}$ is called the directional derivative along $\vec{a}$.



Note : Maximum value of directional derivative is $\nabla\phi = \hat{n} = |\nabla\phi|$

Divergence of a Vector Field

Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector field then divergence of $\vec{f}$ denoted by div $\vec{f}$ or $\nabla.\vec{f}$ is defined as

$$\text{div}.\ \vec{f} = \nabla.\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \sum \frac{\partial f_i}{\partial x}$$

Physical Interpretation

If $\vec{v}$ the velocity of a moving fluid at a point P at time t, then div $\vec{v}$ represents the rate at which the fluid moves out of a unit volume enclosing the point P.

Solenoidal Vector

A vector field $\vec{f}$ is said to be solenoidal if div. $\vec{f} = 0$.

Note 1:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Note 2:

A scalar field ϕ is called a harmonic function if $\nabla^2 \phi = 0$

Note 3:

$\nabla^2(\alpha\phi \pm \beta\psi) = \alpha\nabla^2\phi \pm \beta\nabla^2\psi$, where α and β are constants,

ϕ and ψ are scalar fields.

Curl of a Vector field

Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ then curl of $\vec{f}$ denoted by curl $\hat{f}$ or

$\nabla \times \vec{f}$ is defined as $\nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$= \sum\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i}$$

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Irrational Vector

$\vec{f}$ is said to irrational if curl $\vec{f} = \vec{0}$

A particle moves along the curve whose parametric coordinates are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$. Here t is time i) Determine its velocity and acceleration at time t ii) find the magnitude of velocity and acceleration at t = 0.

Suggested answer:

i) Equation to the curve is

$\vec{r} = e^{-t}\,\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$

$$\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\,\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = e^{-t}\,\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$$

ii) At t = 0, $\vec{v} = -\hat{i} + 6\hat{k}$, $\vec{a} = \hat{i} - 18\hat{j}$ and

$|\vec{v}| = \sqrt{37}$, $|\vec{a}| = \sqrt{325}$

04. Find the unit tangent vectors at any point on the curve $x = t^2+1$, $y = 4t - 3$, $z = 2t^2 - 6t$. Determine the unit tangent at the point $t = 2$.

Suggested answer:

Equation to the space curve is

$$\vec{r} = (t^2 + 1)\,\hat{i} + (4t - 3)\,\hat{j} + (2t^2 - 6t)\,\hat{k}$$

$$\frac{d\vec{r}}{dt} = 2t\,\hat{i} + 4\,\hat{j} + (4t - 6)\,\hat{k}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{t=2} = 4\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}$$

unit tangent $\hat{t} = \dfrac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$

$$= \frac{4\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}}{\sqrt{36}}$$

$$= \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

If $x = t^2+1$, $y = 4t - 3$, $z = 2t^2 - 6t$ represents the parametric equations of a space curve. Find the angle between the unit tangents at $t = 1$ and $t = 2$.

$$\vec{r} = (t^2+1)\hat{i} + (4t-3)\hat{j} + (2t^2-6t)\hat{k}$$

$$\frac{d\vec{r}}{dt} = 2t\,\hat{i} + 4\,\hat{j} + (4t-6)\hat{k}$$

$$\therefore \hat{t} = \frac{2t\,\hat{i} + 4\,\hat{j} + (4t-6)\hat{k}}{\sqrt{4t^2+16+16t^2+36-48t}}$$

$$= \frac{2t\,\hat{i} + 4\,\hat{j} + (4t-6)\hat{k}}{\sqrt{20t^2-48t+52}}$$

$$\hat{t}|_{t=1} = \frac{2\,\hat{i} + 4\,\hat{j} - 2\hat{k}}{\sqrt{24}} = \vec{a} \text{ (say)}$$

$$\hat{t}|_{t=2} = \frac{4\,\hat{i} + 4\,\hat{j} + 2\hat{k}}{\sqrt{36}}$$

$$= \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k} = \vec{b} \text{ (say)}$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$

$$\theta = \cos^{-1}\left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right\}$$

$$\theta = \cos^{-1}\left\{\frac{5}{3\sqrt{6}}\right\}$$

The position vectors of a moving particle at time t is $\vec{r} = t^2\,\hat{i} - t^3\,\hat{j} + t^4\,\hat{k}$. Find the tangential and normal components of its acceleration at t = 1.

Suggested answer:

$$\vec{r} = t^2\,\hat{i} - t^3\,\hat{j} + t^4\,\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 2t\,\hat{i} - 3t^2\,\hat{j} + 4t^4\,\hat{k}$$

$$\vec{v}|_{t=1} = 2\,\hat{i} - 3\,\hat{j} + 4\hat{k}$$

$$\hat{v} = \frac{1}{\sqrt{29}}(2\,\hat{i} - 3\,\hat{j} + 4\hat{k})$$

$$\vec{a}|_{t=1} = 2\,\hat{i} - 6\,\hat{j} + 12\hat{k}$$

tangential component acceleration is $\vec{a} \cdot \hat{v} = \frac{70}{\sqrt{29}}$

Normal component of acceleration $= |\vec{a} - (\vec{a} \cdot \hat{v})\hat{v}|$

$$= \left| \frac{-82\hat{i} + 36\hat{j} + 68\hat{k}}{29} \right|$$

$$= \frac{1}{29}\sqrt{82^2 + 36^2 + 68^2}$$

$$= 3.8774$$

Find the equation of the tangent plane to the surface xyz = 6 at the point (1, 2, 3).

Suggested answer:

$\phi = xyz$

$\nabla\phi = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$(\nabla\phi)_{(1,2,3)} = 6\hat{i} + 3\hat{j} + 2\hat{k}$

unit normal to the surface

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{6\hat{i} + 3\hat{j} + 2\hat{k}}{7}$$

Equation of the tangent plane in vector form is $(\vec{r} - \vec{a}).\hat{n} = 0$

Where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

ie $6(x-1) + 3(y-2) + 2(z-3) = 0$

i.e., $6x + 3y + 2z = 18$

Find the constants a and b such that the surfaces $ax^2 - byz = (a + 2)x$ and $4x^2y+z^3$ are orthogonal at (1, -1, 2).

Suggested answer:

Let $\phi_1 = ax^2 - byz - (a+2)x \quad \ldots(1)$

$$\phi_2 = 4x^2y + z^3 - 4 \qquad \ldots(2)$$

$$\nabla\phi_1 = [2ax - (a+2)]\hat{i} - bz\,\hat{j} - by\,\hat{k}$$

$$\nabla\phi_2 = 8xy\,\hat{i} + 4x^2\,\hat{j} + 3z^2\,\hat{k}$$

$$(\nabla\phi_1)_{(1,-1,2)} = (a-2)\hat{i} - 2b\,\hat{j} + b\,\hat{k},\ (\nabla\phi_2)_{(1,-1,2)} = -8\,\hat{i} + 4\,\hat{j} + 12\,\hat{k}$$

given ϕ_1 and ϕ_2 are orthogonal

$\therefore \nabla\phi_1 . \nabla\phi_2 = 0$

i.e., $-8(a - 2) + 4(-2b) + 12b = 0$

$\Rightarrow 2a - b = 4 \quad \ldots(3)$

Also $\phi_1 = \phi_2$ at (1, -1, 2)

$\Rightarrow a + 2b - (a+2) = -4 + 8 - 4$

i.e., $2b - 2 = 0$

$\Rightarrow b = 1$

substituting in (3), we get

$a = 5/2$.

. Find the directional derivative of $\phi = xyz$ along the direction of the normal to the surface $x^2z + y^2x + yz^2 = 3$, at $(1,1,1)$.

Suggested answer:

Let $\phi = xyz$, $\psi = x^2z + y^2x + yz^2$

$$\nabla\phi = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$$

$$\nabla\phi \text{ at } (1, 1, 1) = \hat{i} + \hat{j} + \hat{k}$$

$$\nabla\phi = (2xz + y^2)\,\hat{i} + (2yx + z^2)\,\hat{j} + (x^2 + 2zy)\hat{k}$$

$$\nabla\psi \text{ at } (1, 1, 1) = 3\,\hat{i} + 3\,\hat{j} + 3\hat{k}$$

$$\hat{n} = \frac{\nabla\psi}{|\nabla\psi|} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$$

Directional directive of ϕ along $\hat{n}$ is

$$\nabla\phi.\hat{n} = \frac{1(1) + 1(1) + (1)}{\sqrt{3}} = \sqrt{3}$$

If $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$ find div $\vec{F}$ and curl $\vec{F}$.

Suggested answer:

$$\vec{F} = (3x^2 - 3yz)\,\hat{i} + (3y^2 - 3xz)\,\hat{j} + (3z^2 - 3xy)\hat{k}$$

$$\text{div}\,\vec{F} = 6x + 6y + 6z$$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= \hat{i}(-3x + 3x) - \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z)$$

$$= \vec{0}$$

If $\vec{f} = (x+2y-z)\hat{i} + (x-y+z)\hat{j} + (-x+y+2z)\hat{k}$ find div $\vec{f}$.

Suggested answer:

$$\text{div } \vec{f} = \frac{\partial}{\partial x}(x+2y-z) + \frac{\partial}{\partial y}(x-y+z) + \frac{\partial}{\partial z}(-x+y+2z)$$

$$= 1 - 1 + 2$$

$$= 2$$

. If $\vec{f} = x^2\hat{i} + y^2\hat{j} + z\hat{k}$ and $\vec{g} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ show that $\vec{f} \times \vec{g}$ is a solenoidal vector.

Suggested answer:

$$\vec{f} \times \vec{g} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$$

$$\vec{f} \times \vec{g} = \hat{i}\{xy^3 - xz^3\} - \hat{j}\{x^3y - yz^3\} + \hat{k}\{zx^3 - zy^3\}$$

$$\text{div } (\vec{f} \times \vec{g}) = (y^3 - z^3) + (z^3 - x^3) + (x^3 - y^3)$$

$$= 0$$

$\therefore \vec{f} \times \vec{g}$ is solenoidal.

If $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ show that $\vec{F}.\text{Curl}\,\vec{F} = 0$.

Suggested answer:

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$= \hat{i}(-1) - \hat{j}(-1) + \hat{k}(-1)$$

$$\text{Curl}\,\vec{F} = -\hat{i} + \hat{j} - \hat{k}$$

$$\vec{F}.\text{Curl}\,\vec{F} = (x+y+1)(-1) + 1(1) - (x+y)(-1)$$

$$= -x - y - 1 + 1 + x + y$$

$$= 0$$

If $\vec{F} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$ find $\nabla x(\nabla x\,\vec{F})$.

Suggested answer:

$$\nabla x\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \hat{i}(2z+2x) - \hat{j}(0) + \hat{k}(-2z-x^2)$$

$$\nabla x(\nabla x\,\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+2x & 0 & -2z-x^2 \end{vmatrix}$$

$$= (2x+2)\,\hat{j}$$

Find the directional derivative of $\phi = x^2yz + 4xz^2$ at (1, -2, -1) in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.

Suggested answer:

$$\nabla\phi = (2xyz + 4z^2)\hat{i} + x^2z\hat{j} + (x^2y + 8xz)\hat{k}$$

$\nabla\phi$ at $(1, -2, -1)$ is $8\hat{i} - \hat{j} - 10\hat{k}$

Let $\vec{c} = 2\hat{i} - \hat{j} - 2\hat{k}$

Directional derivative in the direction of $\vec{c}$ is

$$\nabla\phi.\hat{c} = (8\hat{i} - \hat{j} - 10\hat{k}).\frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3}$$

$$= \frac{1}{3}(16 + 1 + 20)$$

$$= \frac{37}{3}$$

Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at (2, -1, 2) in the direction of $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Suggested answer:

$$\nabla\phi = (4x^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

$\nabla\phi$ at $(2, -1, 2) = 8\hat{i} + 48\hat{j} + 84\hat{k}$

$\vec{c} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$$\nabla\phi.\hat{c} = \frac{1}{7}[8(2) + 48(-3) + (84)(6)]$$

$$= \frac{376}{7}$$

Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, -1, 2).

Suggested answer:

Let $\phi_1 = x^2 + y^2 + z^2 - 9$

$\phi_2 = x^2 + y^2 - z - 3$

$\nabla\phi_1 = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$

$\nabla\phi_1$ at $(2, -1, 2) = 4\,\hat{i} - 2\,\hat{j} + 4\,\hat{k}$

$\nabla\phi_2 = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$

$\nabla\phi_2$ at $(2, -1, 2) = 4\,\hat{i} - 2\,\hat{j} - \hat{k}$

Angle between the surfaces is the angle between $\nabla\phi_1$ and $\nabla\phi_2$.

$$= \cos^{-1}\left\{\frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}\right\}$$

$$= \cos^{-1}\left\{\frac{4(4) + (-2)(-2) + 4(-1)}{\sqrt{36}\sqrt{21}}\right\}$$

$$= \cos^{-1}\left\{\frac{8}{3\sqrt{21}}\right\}$$

Find the constant a, b, c so that the vector

$\vec{f} = (x + 2y + az)\,\hat{i} + (bx - 3y - z)\,\hat{j} + (4x + cy + 2z)\,\hat{k}$ is irrotational.

Suggested answer:

$\vec{F}$ is irrotational if Curl $\vec{F} = \vec{0}$

$$\text{Curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$\therefore c+1=0, 4-a=0, b-2=0$

$\Rightarrow a=4, b=2, c=-1$

Find the value of a if $\vec{f} = (axy - x^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ is irrotational.

Suggested answer:

Since $\vec{f}$ is irrotational

Curl $\vec{f} = \vec{0}$

$$\text{i.e.,} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$$\hat{i}\{0-0\} - \hat{j}\{(1-a)z^2 + 3z^2\} + \hat{k}\{(a-2)2x - ax\} = \vec{0}$$

$\therefore z^2\{1-a+3\} = 0$ and $x\{(a-2)2-a\} = 0$

$\Rightarrow a = 4$

Vector Integration

This chapter treats integration in vector fields. It is the mathematics that engineers and physicists use to describe fluid flow, design underwater transmission cable, explain the flow of heat in stars, and put satellites in orbit. In particular, we define line integral, which are used to find the work done by a force field in moving an object along a path through the field. We also define surface integrals so we can find the rate that a fluid flows across a surface. Along the way we develop key concepts and result, such as conservative force fields and Green's theorem, to simplify our calculations of these new integrals by connecting them to the single, double, and triple integrals we have already studied.

Line integral: Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.

Line integral of a vector function $\vec{F}$ along a curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

Component of $\vec{F}$ along a tangent PT at P.

$$= \text{Dot product of } \vec{F} \text{ and unit vector along PT}$$

$$= \vec{F} \bullet \frac{\vec{dr}}{ds} \left(\frac{\vec{dr}}{ds} \textit{ is a unit vector along PT} \right)$$

Line Integral $= \sum \vec{F} \bullet \frac{\vec{dr}}{ds}$ *from A to B along the curve*

Therefore Line integral $= \int_c \left(\vec{F} \bullet \frac{\vec{dr}}{ds} \right) ds = \int_c \vec{F} \bullet \vec{dr}$

Note:

(1) **Work**. If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done

$$= \int_c \vec{F} \bullet \vec{dr}$$

(2) **Circulation:** If $\vec{V}$ represents the velocity of a liquid then $\oint_c \vec{V} \bullet \vec{dr}$ is called the circulation of V round the closed curve c.
If the circulation of V round every closed curve is zero then V is said to be irrotational there.

(3) When the path of integration is a closed curve then the notation is $\oint$ in place of $\int$.

Examples:

1. If a force $\vec{F} = 2x^2 y\, \hat{i} + 3xy\hat{j}$ displaces a particle in the xy-plane from (0,0) to (1,4) along a curve $y = 4x^2$. Find the work done.

Solution:

$$\textit{work done} = \int_c \vec{F} \bullet \vec{dr}$$

$$\vec{r} = x\hat{i} + y\hat{j} \quad \therefore \vec{dr} = dx\hat{i} + dy\hat{j}$$

$$= \int_c \left(2x^2 y\hat{i} + 3xy\hat{j}\right) \bullet \left(dx\hat{i} + dy\hat{j}\right)$$

$$= \int_c \left(2x^2 y dx + 3xy dy\right)$$

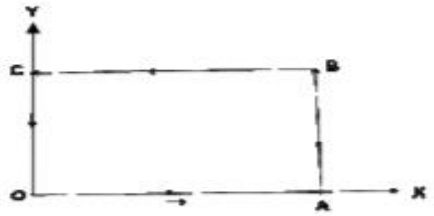
putting the value $y = 4x^2$ *and* $dy = 8x dx$, *we get*

$$= \int_0^1 \left[2x^2\left(4x^2\right)dx + 3x\left(4x^2\right)8x dx\right]$$

$$= 104 \int_0^1 x^4 dx = 104 \left(\frac{x^5}{5} \right)_0^1 = \frac{104}{5}$$

Evaluate $\int_c F \bullet dr$ *where* $F = x^2\hat{i} + xy\hat{j}$ and C is the boundary of the square in the plane z=0 and bounded by the lines x=0, y=0, x=a and y=a.

Solution: From the figure ,we have



$$\int_c \vec{F} \bullet \vec{dr} = \int_{OA} \vec{F} \bullet \vec{dr} + \int_{AB} \vec{F} \bullet \vec{dr} + \int_{BC} \vec{F} \bullet \vec{dr} + \int_{CO} \vec{F} \bullet \vec{dr}$$

Here $\vec{r} = x\hat{i} + y\hat{j} \quad \therefore \quad \vec{dr} = dx\hat{i} + dy\hat{j}, \quad \vec{F} = x^2\hat{i} + xy\hat{j}$

$$\vec{F} \bullet \vec{dr} = x^2dx + xydy$$

On OA, y=0, $\therefore \quad \vec{F} \bullet \vec{dr} = x^2dx$

$$\int_{OA} \vec{F} \bullet \vec{dr} = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

On AB, x=a, dx=0 $\therefore \quad \vec{F} \bullet \vec{dr} = aydy$

$$\therefore \quad \vec{F} \bullet \vec{dr} = \int_0^a aydy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

On BC , y=a, dy=0 $\therefore \quad \vec{F} \bullet \vec{dr} = x^2dx$

$$\int_{BC} \vec{F} \bullet \vec{dr} = \int_a^0 x^2dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3}$$

On CO , x=0, $\therefore \quad \vec{F} \bullet \vec{dr} = 0$

$$\int_{CO} \vec{F} \bullet \vec{dr} = 0$$

$$\int_c \vec{F} \bullet \vec{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}.$$

VOLUME INTEGRAL

Let $\vec{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral $= \iiint_v \vec{F}\, dv$

Example 11: If $\vec{F} = 2z\,\hat{i} - x\,\hat{j} + y\,\hat{k}$, Evaluate $\iiint_v \vec{F} dv$ where, v is the region bounded by the surfaces,

$$x = 0, \quad x = 2, \quad y = 0, \quad y = 4, \quad z = x^2, \quad z = 2.$$

Solution:

$$\iiint_v \vec{F} dv = \iiint \left(2z\,\hat{i} - x\,\hat{j} + y\,\hat{k}\right) dx dy dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^{2} \left(2z\,\hat{i} - x\,\hat{j} + y\,\hat{k}\right) dz$$

$$= \int_0^2 dx \int_0^4 dy \left[z^2\,\hat{i} - xz\,\hat{j} + yz\,\hat{k}\right]_{x^2}^{2}$$

$$= \int_0^2 dx \int_0^4 dy \left[4\hat{i} - 2x\,\hat{j} + 2y\,\hat{k} - x^4\,\hat{i} - x^3\,\hat{j} + x^2 y\,\hat{k}\right]$$

$$= \int_0^2 dx \left[4y\hat{i} - 2xy\,\hat{j} + y^2\,\hat{k} - x^4\,y\hat{i} - x^3 y\,\hat{j} + \frac{x^2 y^2}{2}\hat{k}\right]_0^4$$

$$= \int_0^2 \left[16\,\hat{i} - 8x\,\hat{j} + 16\,\hat{k} - 4x^4\,\hat{i} - 4x^3\,\hat{j} - 8\,x^2\hat{k}\right] dx$$

$$= \left[16x\hat{i} - 4x^2\,\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} - x^4\,\hat{j} - \frac{8\,x^3}{3}\hat{k}\right]_0^2$$

$$= \left[32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k}\right]$$

$$= \left[\frac{32}{5}\hat{i} + \frac{32}{3}\hat{k}\right] = \frac{32}{15}\left[3\hat{i} + 5\hat{k}\right]$$

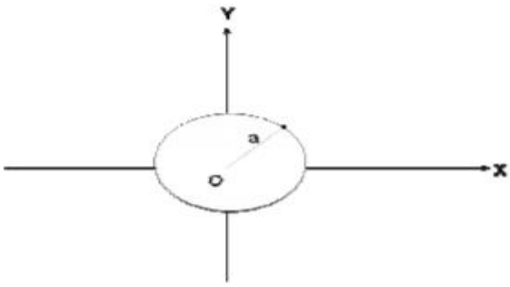
GREEN'S THEOREM

Statement: If $\phi(x, y), \varphi(x, y), \frac{\partial\phi}{\partial y}$ *and* $\frac{\partial\varphi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in xy-plane, then

$$\oint_c (\phi\, dx + \varphi\, dy) = \iint_R \left(\frac{\partial\varphi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\, dy$$

Example 12: A vector field $\vec{F}$ is given by $\vec{F} = \sin y\,\hat{i} + x(1 + \cos y)\hat{j}$. Evaluate the line integral $\int_c \vec{F} \bullet \vec{dr}$ where C is the circular path given by $x^2 + y^2 = a^2$.

Solution:



Given $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$

$$\int_C \vec{F} \bullet \vec{dr} = [\sin y\,\hat{i} + x(1+\cos y)\hat{j}] \bullet (\hat{i}\,dx + \hat{j}\,dy) = \int_C [\sin y\,dx + x(1+\cos y)dy]$$

On applying Green's Theorem, we have

$$\oint_c (\phi\,dx + \varphi\,dy) = \iint_s \left(\frac{\partial\varphi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\,dy$$

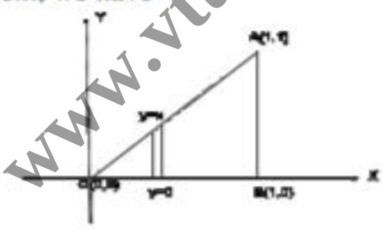
$$= \iint_S [(1+\cos y) - \cos y]\,dx\,dy$$

Where s is the circular plane surface of radius a.

$$= \iint_S dx\,dy = Area\ of\ the\ circle = \pi a^2$$

Using Green's theorem, evaluate $\int_C (x^2 y\,dx + x^2\,dy)$, where C is the boundary described counter clockwise of the triangle with vertices (0,0),(1,0),(1,1).

Solution: By Green's theorem, we have



$$\oint_c (\phi\,dx + \varphi\,dy) = \iint_s \left(\frac{\partial\varphi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\,dy$$

$$\int_C (x^2 y\,dx + x^2 dy) = \iint_R (2x - x^2)\,dx\,dy$$

$$\int_0^1 (2x - x^2)dx \int_0^x dy = \int_0^1 (2x - x^2)dx[y]_0^x$$

$$= \int_0^1 (2x - x^2)dx[x]$$

$$= \int_0^1 (2x^2 - x^3)dx$$

$$= \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^1 = \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{12}$$

STOKES THEOREM

Statement: Surface integral of the component of curl F along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

Therefore $$\oint_c \vec{F} \bullet d\vec{r} = \iint_S curl\ \vec{F} \cdot \hat{n}\ ds$$

Where $\hat{n} = \cos\alpha\ \hat{i} + \cos\beta\ \hat{j} + \cos\gamma\ \hat{k}$ is a unit external normal to any surface ds.

Evaluate by Stoke's theorem $\oint_c (yz\,dx + zx\,dy + xy\,dz)$, where C is the curve $x^2 + y^2 = 1, \quad z = y^2$.

Solution : Given $$\oint_c (yz\,dx + zx\,dy + xy\,dz)$$
$$= \int (yz\ \hat{i} + zx\ \hat{j} + xy\ \hat{k}) \cdot (\hat{i}\ dx + \hat{j}\ dy + \hat{k}\ dz)$$
$$= \oint_c \vec{F} \bullet d\vec{r}$$
$$= \iint_S curl\ \vec{F} \cdot \hat{n}\ ds$$

$$\therefore Curl\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

Here $\vec{F} = (yz\,\hat{i} + zx\ \hat{j} + xy\ \hat{k})$

$$= (x-x)\hat{i} + (y-y)\hat{j} + (z-z)\hat{k} = 0$$

$$\therefore \oint_c \vec{F} \bullet d\vec{r} = \iint_S curl\ \vec{F} \cdot \hat{n}\ ds = 0$$

Gauss Divergence Theorem

(Relation between surface integral and volume integral)

Statement: The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.

Mathematically $\iint_S \vec{F} \bullet \hat{n} ds = \iiint_V div\, \overline{F\, dv}$

Example 1. Evaluate $\iint_S \vec{F} \bullet \hat{n} ds$ using Gauss divergence theorem where S is the surface of the sphere $x^2 + y^2 + z^2 = 16$ and $\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$.

Solution: Given $\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$ and radius of the sphere r=4

Therefore $\nabla \bullet \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \bullet (3x\hat{i} + 4y\hat{j} + 5z\hat{k})$

$\nabla \bullet \vec{F} = 3 + 4 + 5 = 12$

Then by theorem, we have

$$\iint_S \vec{F} \bullet \hat{n} ds = \iiint_V div\, \overline{F\, dv} = \iiint_V 12 \bullet dv = 12v$$

Because v is the volume of the sphere

$$= 12\,\frac{4}{3}\pi(4)^3 = \frac{3072\pi}{3}$$