Module - II

Differential Calculus - II

I **Taylor's and Maclaurin's series expansions**

- The series expansions of a function $y=f(x)$ about a point $x=a$ is given by $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \rightarrow (1)$

is called a taylor's series expansions.

- If $a=0$ then eq (1) becomes $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$ is called a maclaurin's series expansion.

Q. Obtain the taylor's series of expansion of $\log(\cos x)$ about a point $a = \pi/4$

$\Rightarrow$ $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \rightarrow (1)$

$f(x) = \log(\cos x) \Rightarrow f(\pi/4) = \log(\cos \pi/4) = \log(1/\sqrt{2}) = -\log\sqrt{2}$

$f'(x) = \frac{(-\sin x)}{\cos x} \Rightarrow f'(x) = -\tan x \Rightarrow f(\pi/4) = -\tan \pi/4 = -1$

$f''(x) = -\sec^2 x \Rightarrow f''(\pi/4) = -\sec^2 \pi/4 = -2$

$f'''(x) = -2\sec^2 x \tan x \Rightarrow f'''(\pi/4) = -2\sec^2 \pi/4 \cdot \tan \pi/4 = -4.$

sub in (1)

$\log(\cos x) = -\log\sqrt{2} + (x - \pi/4)(-1) + \frac{(x-\pi/4)^2}{2}(-2) + \left[\frac{x-\pi/4}{6}\right)^3(-4) + \cdots$

classmate

Date
Page

13/09/2017

Q.2 Obtain the ~~Taylors~~ Power series expansion of $f(x)=\log x$ about a point of $x=1$. by considering the term up to 4th degrees.

W.k.t

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\frac{(x-a)^4}{4!}f^{iv}(a)+$$

$f(x)=\log x \Rightarrow f(1)=\log(1)=0$

$f'(x)=1/x \Rightarrow f'(1)=1/1=1$

$f''(x)=-1/x^2 \Rightarrow f''(1)=-1/1^2=-1$

$f'''(x)=2/x^3 \Rightarrow f'''(1)=2/1^3=2$

$f^{iv}(x)=-6/x^4 \Rightarrow f^{iv}(1)=-6/1=-6$

sub in eq^n (1) we get.

$$f(x)=0+(x-1)(1)+\frac{(x-1)^2}{2}(-1)+\frac{(x-1)^3}{\cancel{6}\,3}(\cancel{2})+\frac{(x-1)^4}{\cancel{24}\,4}(-\cancel{6})+\cdots$$

$$\log x=x-1-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}+\frac{(x-1)^4}{4}+\cdots$$

Q.3 Find the series expansion of $\tan^{-1}x$ in powers of $(x-1)$.

Given $f(x)=\tan^{-1}x$ and $a=1$

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3}f'''(a)+\cdots$$

$f(x)=\tan^{-1}x \Rightarrow f(1)=\tan^{-1}(1)=\pi/4$.

$f'(x)=\frac{1}{1+x^2} \Rightarrow f'(1)=\frac{1}{1+1^2}=\frac{1}{2}$.

$f''(x)=\frac{-1}{(1+x^2)^2}\times 2x=\frac{-2x}{(1+x^2)^2} \Rightarrow f''(1)=\frac{-2}{(1+1^2)^2}=-1/2$

$$f'''(x)=\frac{(1+x^2)^2(-2)-(-2x)\times 2(1+x^2)(2x)}{(1+x^2)^4}.$$

$f'''(1)=\frac{-8+16}{16}=8/16=1/2$

Scanned by CamScanner

classmate
Date
Page

sub in eq^n (1)

$$f(x) = \pi/4 + (x-1)(1/2) + \frac{(x-1)^2}{2} \times \left(-1/2\right) + \frac{(x-1)^3}{6}\left(1/2\right) + \cdots$$

$$\tan^{-1}x = \pi/4 + \left(\frac{x-1}{2}\right) - \left(\frac{x-1}{2}\right)^2 + \left(\frac{x-1)^3}{12}\right. + \cdots$$

* Maclaurin's series

1. Obtain the maclaurin's series of expression of $f(x) = \log(1+e^x)$.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

we have $f(x) = \log(1+e^x) = f(0) = \log(1+e^0) = \log(1+1) = \log 2$.

$$f'(x) = \frac{e^x}{1+e^x} \Rightarrow f'(0) = \frac{e^0}{1+e^0} = \frac{1}{(1+1)} = 1/2.$$

$$f''(x) = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \frac{e^x + \cancel{e^{2x}} - \cancel{e^{2x}}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}.$$

$$= f''(0) = \frac{e^0}{(1+e^0)^2} = \frac{1}{(1+1)^2} = 1/4.$$

$$f'''(x) = \frac{(1+e^x)^2(e^x) - e^x \cdot 2(1+e^x)e^x}{(1+e^x)^4} = f'''(0) = \frac{4-4}{24} = 0$$

sub in (1).

$$f(x) = \log 2 + x/2 + \frac{x^2}{2}(1/4) + \frac{x^3}{3!}(0) + \cdots$$

$$\log(1+e^x) = \log 2 + x/2 + \frac{x^2}{8} + \cdots$$

2. Find the power series expansion of $f(x) = \log(\sec x)$ in powers of x upto the terms containing x^4.

w.k.t.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) \cdots$$

we have $f(x) = \log(\sec x) = f(0) = \log(\sec 0) = \log 1 = 0$

$$f'(x) = \frac{1}{\cancel{\sec x}} \cancel{\sec x} \tan x \quad \tan x \quad f'(0) = \tan 0 = 0.$$

$$f''(x) = \sec^2 x \Rightarrow f''(0) = \sec^2(0) = 1$$

$$f'''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x = f'''(0) = 2(\sec^2(0) \cdot \tan(0)) = 0$$

$f^{IV}(x) = 2\sec^2 x(\sec^2 x) + (2\tan x)(2\sec^2 x \cdot \tan x)$.

$f^{IV}(0) = 2\sec^2(0)(\sec^2(0)) + 0 = 2$

sub in eq^n 1.

$f(x) = 0 + x \times 0 + \frac{x^2(1)}{2!} + \frac{x^3}{3!}(0) + \frac{x^2}{4!}(2) + \cdots$

$= \frac{x^2}{2} + \frac{x^4}{12} + \cdots$

03. obtain the Maclaurin's series of expansion $f(x) = \tan^{-1}x$

Given $f(x) = \tan^{-1}x$.

w.k.t

$f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots$

we have.

$f(x) = \tan^{-1}x \Rightarrow f(0) = \tan^{-1}(0) = 0$.

$f'(x) = \frac{1}{1+x^2} = f'(0) = \frac{1}{1+0^2} = 1$

$f''(x) = \frac{-1}{(1+x^2)^2} \times 2x = \frac{-2x}{(1+x^2)^2} = f(0) = \frac{(-2 \times 0)}{(1+0^2)^2} = 0$.

$f'''(x) = \frac{(1+x^2)^2(-2) - (-2x) \times 2(1+x^2)(2x)}{(1+x^2)^4}$.

$f'''(0) = \frac{-2+0}{(1+0)^4} = -2$

sub in eq 1

$f(x) = 0 + x \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times -2 + \cdots$

$\tan^{-1}x = x - \frac{x^3}{3} + \cdots$

14/09/2017

(04) Obtain the Maclurin's series expansion of $f(x) = \log(1+\cos x)$. in power of x upto the terms containg x^4.

w.k.t.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \dots$$

we have

$$f(x) = \log(1+\cos x) \Rightarrow f(0) = \log(1+\cos(0)) = \log 2.$$

$$f'(x) = \frac{-\sin x}{1+\cos x} = \frac{-\cancel{2}\sin x/2 \, \cancel{\cos x/2}}{\cancel{2}\cos^{\cancel{2}} x/2} = f'(x) = -\tan x/2$$

$$f'(0) = 0$$

$$f''(x) = -1/2 \sec^2 x/2 = f''(0) = -1/2 \sec^2(0) = -1/2.$$

$$f'''(x) = -1/\cancel{2} \times \cancel{2}\sec^2 x/2 \tan x/2 = f'''(0) = -\sec^2(0)\tan(0) = 0$$

$$f^{iv}(x) = -\sec^2 x/2 \cdot \sec^2 x/2 \times 1/2 - \tan x/2 \times 2\sec^2 x/2 \tan x/2$$

$$f^{iv}(0) = -1/2 \cdot \sec^4(0) - 0 = -1/2$$

sub in ①

$$f(x) = \log 2 + x \times 0 + \frac{x^2}{2} \times (-1/2) + \frac{x^3}{6} \times 0 + \frac{x^4}{24} \times \left(\frac{-1}{2}\right) + \dots$$

$$\log(1+\cos x) = \log 2 - \frac{x^2}{4} - \frac{x^4}{48} + \dots$$

05. Find the maclurin's series of $\sqrt{1+\sin 2x}$ by considering the terms up to 4th degree.

w.k.t.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \dots \rightarrow ①$$

we have.

$$f(x) = \sqrt{1+\sin 2x}$$

$$= \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x} = \sqrt{(\sin x + \cos x)^2}$$

$$f(x) = \sin x + \cos x \Rightarrow f(0) = 0+1 = 1$$

$$f'(x) = \cos x - \sin x \Rightarrow f(0) = 1-0 = 1$$

$$f''(x) = -\sin x - \cos x \Rightarrow f(0) = 0-1 = -1$$

$$f'''(x) = -\cos x + \sin x \Rightarrow f(0) = -1+0 = -1$$

$$f^{iv}(x) = \sin x + \cos x \Rightarrow f(0) = 0+1 = 1$$

sub in (1) we get.

$$f(x) = 1 + x \times 1 + \frac{x^2}{2!} \times -1 + \frac{x^3}{3!} + \frac{-\cancel{1}x^4}{4!} \times 1 + \cdots$$

$$\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

06. Q. Obtain the series expansion of $e^{\sin x}$ in power of x.

Given

$$f(x) = e^{\sin x}.$$

w.k.t.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \rightarrow (1).$$

we have.

$$f(x) = e^{\sin x} \qquad f(0) = e^{\sin(0)} = e^0 = 1.$$

$$f'(x) = e^{\sin x} \cdot \cos x = f'(x) = f(x) \cos x = f'(0) = f(0) \times \cos(0)$$

$$= 1 //.$$

$$f''(x) = f(x) \cdot -\sin x + \cos x \cdot f'(x) = f''(0) = -f(0) \cdot \sin(0) + \cos(0) \cdot f'(0)$$

$$1 - 0 + 1 \cdot 1$$

$$= 0 + 1$$

$$= 1$$

$$f'''(x) = -f(x) \cos x - \sin x\, f'(x) - \sin x\, f'(x) + \cos x \cdot f''(x).$$

$$f'''(0) = -f(0) \cos(0) - \sin(0) f'(0) - \sin(0) f'(0) + \cos(0) \cdot f''(0)$$

$$= -1 - 0 - 0 + 1$$

$$= 0$$

Sub in (1) we get.

$$f(x) = 1 + x \times 1 + \frac{x^2}{2!} \times 1 + \frac{x^3}{3!} \times 0 + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \cdots //.$$

classmate
Date
Page

14/09/2017

* **InDeterminate forms :-**

If the expansion $f(x)$ at $x=a$ assumes the forms like. $0/0$, ∞/∞, $0\times\infty$, 0^0, ∞^0, 1^∞ etc. which do not repersents any value are called the indeterminate forms.

$+\infty-\infty$

15/09/2017

* **L Hospital's rule**

If $f(x)$ and $g(x)$ are any two functions such that ~~lim~~

(i) $\lim_{x\to a} f(x) = f(a) = 0$

$\lim_{x\to 0} g(x) = g(a) = 0$

(ii) $f'(x)$ and $g'(x)$ exists then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

(iii) further if $f'(a) = 0 = g'(a)$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$$

Q.1 **Type one**

$0/0$ and ∞/∞ forms

1. Evalute the following.

(i) $\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$

(ii) $\lim_{x\to 0}\frac{1-\cos x}{x\log(1+x)}$

(iii) $\lim_{x\to \pi/2}\frac{\log(\sin x)}{(\pi/2 - x)^2}$

(iv) $\lim_{x\to 0}\frac{a^x - b^x}{x}$

(v) $\lim_{x\to \pi/4}\frac{\sec^2 x - 2\tan x}{1+\cos 4x}$

(vi) $\lim_{x\to \pi/2}\frac{\log(\cos x)}{\tan x}$.

(vii) $\lim_{x\to \pi/2}\frac{\log(x-\pi/2)}{\tan x}$.

(viii) $\lim_{x\to 0}\log_{\sin x}\sin 2x$.

(ix) $\lim_{x\to 0}\log_{\tan bx}\tan ax$

(x) $\lim_{x\to 0}\frac{\log x}{\operatorname{cosec} x}$.

classmate
Date
Page

01. let $L = \lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$ $\left(\frac{0}{0}\right)$

Appliing LHR, we get

$$L = \lim_{x\to 0} \frac{xe^x + e^x - \left(\frac{1}{1+x}\right)}{2x} = \left(\frac{0}{0}\right)$$

using LHR, we get

$$L = \lim_{x\to 0} \frac{xe^x + e^x + e^x + \frac{1}{(1+x^2)}}{2}$$

$$= \frac{0+1+1+1}{2} = 3/2.$$

02. let $L = \lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)}$ $\left(\frac{0}{0}\right)$

using LHR, we get

$$L = \lim_{x\to 0} \frac{\sin x}{\frac{x}{(1+x)} + \log(1+x)} = \left(\frac{0}{0}\right)$$

using LHR, we get

$$L = \lim_{x\to 0} = \frac{\cos x}{\frac{1+x-x}{(1+x)^2} + \frac{1}{1+x}}$$

$$= \lim_{x\to 0} \frac{\cos x}{\frac{1}{(1+x)^2} + \frac{1}{(1+x)}}$$

$$L = 1/1+1 = 1/2 .$$

03. let $L = \lim_{x\to \pi/2} \frac{\log(\sin x)}{(\pi/2 - x)^2}$ $\left(\frac{0}{0}\right)$

using LHR we get

$$L = \lim_{x\to \pi/2} \frac{\frac{1}{\sin x} \times \cos x}{2(\pi/2 - x)(-1)}$$

$$L = -\frac{1}{2} \lim_{x \to \pi/2} \frac{\cot x}{\pi/2 - x} \left(\frac{0}{0}\right)$$

using LHR, we get

$$L = \lim_{x \to 0} \frac{\cos x}{\frac{1+x-x}{(1+x)^2} + \frac{1}{1+x}}.$$

$$= \lim_{x \to 0} \frac{\cos x}{\frac{1}{(1+x)^2} + \frac{1}{(1+x)}}$$

$$L = \frac{1}{1+1} = 1/2.$$

03. let $L = \lim_{x \to \pi/2} \frac{\log(\sin x)}{(\pi/2 - x)^2} \left(\frac{0}{0}\right)$

using LHR we get

$$L = \lim_{x \to \pi/2} \frac{\frac{1}{\sin x} \times \cos x}{2(\pi/2 - x)(-1)}$$

$$L = -\frac{1}{2} \lim_{x \to \pi/2} \frac{\cot x}{\pi/2 - x} \left(\frac{0}{0}\right)$$

using LHR we get

$$L = -\frac{1}{2} \lim_{x \to \pi/2} \frac{\cancel{-}\operatorname{cosec}^2 x}{\cancel{-}1}$$

$$= -\frac{1}{2} \times \operatorname{cosec}^2 \pi/2 = -\frac{1}{2} \times 1 = -\frac{1}{2}$$

04. let $L = \lim_{x \to 0} \frac{a^x - b^x}{x} \left(\frac{0}{0}\right)$

using LHR we get

$$L = \lim_{x \to 0} = \frac{a^x \log a - b^x \log b}{1}$$

$$= a^0 \log a - b^0 \log b$$

$$\log a - \log b = \log(a/b)$$

5. $\lim_{x\to\pi/4} \frac{\sec^2x - 2\tan x}{1+\cos 4x}$.

$$\frac{\sec^2\pi/4 - 2\tan\pi/4}{1+\cos 4(\pi/4)} = \left(\frac{0}{0}\right)$$

using LHR, we get.

$$\frac{2\sec^2x \cdot \tan x - 2\sec^2x}{\cancel{0} - \sin 4x \cdot 4}$$

$$\cancel{\lim_{x\to\pi/4} = \frac{2\sec^2\pi/4 \cdot \tan\pi/4 - 2\sec^2\pi/4}{0 - \sin 4\pi/4 \times 4}}$$

$$\lim_{x\to\pi/4} \frac{\sec^2x \tan x - \sec^2x}{0 - 2\sin 4x} = \left(\frac{0}{0}\right).$$

using LHR we get

$$L = \lim_{x\to\pi/4} \frac{\sec^4x + (\tan x)(2\sec^2x \tan x) - (2\sec^2x \tan x)}{-2\cos 4x \times 4}.$$

$$= \frac{\sec^4\pi/4 + (\tan\pi/4)(2\sec^2\pi/4 \tan\pi/4) - (2\sec^2\pi/4 \tan\pi/4)}{-2\cos 4(\pi/4) \times 4}.$$

$$= \frac{4+4-4}{-8(-1)} = \frac{4}{8} = 1/2 //.$$

20/09/2017

6 let $L = \lim_{x\to\pi/2} \frac{\log(\cos x)}{\tan x} \left(\frac{\infty}{\infty}\right)$

Applying LHR.

$$L = \lim_{x\to\pi/2} \frac{\frac{1}{\cos x} \times -\sin x}{\sec^2x} = \lim_{x\to\pi/2} \frac{-\tan x}{\sec^2x} = \left(\frac{\infty}{\infty}\right)$$

$$L = -\lim_{x\to\pi/2} \frac{\cos^2x}{\cot x} \left(\frac{0}{0}\right)$$

Applying LHR.

Scanned by CamScanner

classmate
Date
Page

$$L = -\lim_{x\to\pi/2} \frac{\cancel{+}2\cos x \sin x}{\cancel{+}\cancel{2}\,\operatorname{cosec}^2 x}$$

$$= -\frac{2\cos\pi/2 \cdot \sin\pi/2}{\operatorname{cosec}^2 \pi/2} = \frac{2\times0\times1}{1} = 0$$

7. let $L = \lim_{x\to\pi/2} \frac{\log(x-\pi/2)}{\tan x} \left(\frac{\infty}{\infty}\right)$

Applying LHR

$$L = \lim_{x\to\pi/2} \frac{\frac{1}{x-\pi/2}}{\frac{\sec^2 x}{1}} \left(\frac{\infty}{\infty}\right)$$

$$L = \lim_{x\to\pi/2} \frac{1}{x-\pi/2} \times \frac{1}{\sec^2 x}$$

$$L = \lim_{x\to\pi/2} \frac{\cos^2 x}{x-\pi/2} \left(\frac{0}{0}\right)$$

$$L = \lim_{x\to\pi/2} -\frac{2\cos x \sin x}{1}$$

$$= -2\cos\pi/2 \cdot \sin\pi/2 = 0 //$$

8. let $L = \lim_{x\to0} \log_{\sin x} \sin 2x = \lim_{x\to0} \frac{\log \sin 2x}{\log \sin x} \left(\frac{\infty}{\infty}\right)$

Applying L'HR we get

$$L = \lim_{x\to0} \frac{\frac{1}{\sin 2x} \times \cos 2x \times 2}{\frac{1}{\sin x} \times \cos x} = 2\lim_{x\to0} \frac{\cot 2x}{\cot x} \left(\frac{\infty}{\infty}\right)$$

$$L = 2\lim_{x\to0} \frac{\tan x}{\tan 2x} \left(\frac{0}{0}\right)$$

using LHR

$$L = \cancel{2}\lim_{x\to0} \frac{\sec^2 x}{\sec^2 2x \times \cancel{2}} = \frac{1}{1} = 1$$

9. let

$$L = \lim_{x \to 0} \log_{\tan bx} \tan ax = \lim_{x \to 0} \frac{\log \tan ax}{\log \tan bx}$$

Applying LHR,

$$L = \lim_{x \to 0} \frac{\frac{1}{\tan ax} \times \sec^2 ax \times a}{\frac{1}{\tan bx} \times \sec^2 bx \times b}$$

$$\frac{a}{b} \lim_{x \to 0} \frac{\frac{\cos ax}{\sin bx} \times \frac{1}{\cos^2 ax}}{\frac{\cos bx}{\sin bx} \times \frac{1}{\cos^2 bx}}$$

$$\frac{a}{b} \lim_{x \to 0} \frac{2 \sin bx \cos bx}{2 \sin ax \cos ax} = \frac{a}{b} \lim_{x \to 0} \frac{\sin 2bx}{\sin 2ax} \left(\frac{0}{0}\right)$$

using LHR.

$$L = \cancel{\frac{a}{b}} \lim_{x \to 0} \frac{\cos 2bx \times 2\cancel{b}}{\cos 2ax \times 2\cancel{a}} = 1$$

10. let

$$L = \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x} \left(\frac{\infty}{\infty}\right)$$

Applying LHR.

$$L = \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cdot \cot x} = -\lim_{x \to 0} \frac{\sin x \times \tan x}{x} \left(\frac{0}{0}\right)$$

$$= -\lim_{x \to 0} \left(\frac{\sin x}{x}\right) \times \lim_{x \to 0} (\tan x)$$

$$= -1 \times \tan(0) = -1 \times 0$$

$$= 0 //$$

classmate
Date
Page

21/09/2017

Type 2 $0^0, \infty^0, 1^\infty$ forms.

Evalute the following:

01. $\lim_{x\to 1} x^{\frac{1}{1-x}}$

02. $\lim_{x\to \pi/2} (\sin x)^{\tan x}$.

03. $\lim_{x\to 1} (1-x^2)^{\frac{1}{\log(1-x)}}$

04. $\lim_{x\to 0} \left(2-x/a\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

05. $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$.

06. $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

07. $\lim_{x\to \infty} \left(\pi/2 - \tan^{-1}x\right)^{1/x}$

08. $\lim_{x\to 0} (\cot x)^{\tan x}$

09. $\lim_{x\to 0} x^{\sin x}$.

01. Let $L = \lim_{x\to 1} x^{\frac{1}{1-x}}$ (1^∞)

$\log L = \lim_{x\to 1} \log\left\{x^{\frac{1}{1-x}}\right\}$

$= \lim_{x\to 1} \frac{1}{1-x} \cdot \log x$.

$\log L = \lim_{x\to 1} \frac{\log x}{1-x} \left(\frac{0}{0}\right)$

Applying LHR.

$\log L = \lim_{x\to 1} \frac{1/x}{-1}$

$\log L = -1$

$L = e^{-1}$

02. Let $L = \lim_{x\to \pi/2} (\sin x)^{\tan x} (1^\infty)$

$\log L = \lim_{x\to \pi/2} \log(\sin x)^{\tan x}$

$= \lim_{x\to \pi/2} \tan x \cdot \log(\sin x)$

$\lim_{x\to \pi/2} \frac{\log(\sin x)}{\cot x} \left(\frac{0}{0}\right)$

using LHR

$\log L = \lim_{x\to \pi/2} \frac{\frac{1}{\sin x} \times \cos x}{-\operatorname{cosec}^2 x}$.

$= -\lim_{x\to \pi/2} \frac{\cot x}{\operatorname{cosec}^2 x}$

$\log L = \frac{-0}{1} = 0$

$L = e^0 = 1$

03. Let $L = \lim_{x\to 1} (1-x^2)^{\frac{1}{\log(1-x)}}$ (0^0)

$\log L = \lim_{x\to 1} \log(1-x^2)^{\frac{1}{\log(1-x)}}$

$= \lim_{x\to 1} \frac{1}{\log(1-x)} \cdot \log(1-x^2)$

$\log L = \lim_{x\to 1} \frac{\log(1-x^2)}{\log(1-x)} \left(\frac{\infty}{\infty}\right)$

applying LHR.

$$\log L = \lim_{x\to 1} \frac{\frac{1}{(1-x^2)} \times (-2x)}{\frac{1}{(1-x)} \times (-1)}$$

$$= 2\lim_{x\to 1} \frac{x}{(1-x)(1+x)} \times (1-x)$$

$$\log L = 2\times \frac{1}{(1+1)} = 2\times\frac{1}{2} = 1$$

$$L = e.$$

4. let $L = \lim_{x\to a} \left(2 - x/a\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ (1^∞)

$$\cancel{\tan} \log L = \lim_{x\to a} \log\left(2 - x/a\right)$$

$$= \lim_{x\to a} \tan\left(\frac{\pi x}{2a}\right) \cdot \log\left(2 - \frac{x}{a}\right)$$

$$\log L = \lim_{x\to a} \frac{\log\left(2 - x/a\right)}{\cot\left(\frac{\pi x}{2a}\right)} \quad \left(\frac{0}{0}\right)$$

Applying LHR, we get.

$$\log L = \lim_{x\to a} \frac{\frac{1}{(2 - x/a)} \times \left(-1/a\right)}{-\operatorname{cosec}^2\left(\pi x/2a\right) \times \pi/2a}$$

$$= 2/\pi \times \frac{1}{\operatorname{cosec}^2(\pi/2)} = 2/\pi \times 1$$

$$\log L = 2/\pi \Rightarrow L = e^{2/\pi}$$

05. let

$$L = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x} \quad (1^\infty)$$

$$\log L = \lim_{x\to 0} \log\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x} = \lim_{x\to 0} \frac{\log\left(\frac{a^x+b^x+c^x}{3}\right)}{x} = \left(\frac{0}{0}\right)$$

applying LHR we get.

$$\log L = \lim_{x\to 0} \frac{\frac{1}{\left(\frac{a^x+b^x+c^x}{3}\right)} \times \frac{1}{3}\left\{a^x\log a + b^x\log b + c^x\log c\right\}}{1}$$

$$\tfrac{1}{3}\left\{\frac{1}{\left(\frac{a^0+b^0+c^0}{3}\right)} \times \left[a^0\log a + b^0\log b + c^0\log c\right]\right\}.$$

$$\tfrac{1}{3}\left\{1/(\cancel{3}/\cancel{3})\left\{\log a + \log b + \log c\right\}\right\}$$

$$= \tfrac{1}{3}\log(abc).$$

$$\log L = \log(abc)^{1/3} =$$

$$L = (abc)^{1/3}$$

(6) let

$$L = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} (1^\infty)$$

$$\log L = \lim_{x\to 0} \log\left(\frac{\tan x}{x}\right)^{1/x}.$$

$$\lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x} \left(\frac{0}{0}\right)$$

applying LHR

$$\log L = \lim_{x\to 0} \frac{1}{\left(\frac{\tan x}{x}\right)}\left\{\frac{x(\sec^2 x) - \tan x}{x^2}\right\}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2} \left(\frac{0}{0}\right)$$

LHR

$$\log L = \lim_{x\to 0}\left\{\frac{\sec^{\cancel{2}}\cancel{x} + x(2\sec^2 x\tan x) - \sec^{\cancel{2}}\cancel{x}}{2x}\right\}$$

$$\log L = \lim_{x\to 0} \frac{\cancel{2x}\sec^2 x \tan x}{\cancel{2x}}$$

$$= \lim_{x\to 0} \sec^2 x \tan x.$$

$$= \sec^2(0) \times \tan(0)$$

$$= 1 \times 0$$

$$\log L = 0$$

$$L = e^0 = 1$$

22/09/2017

** 7 ** let

$$L = \lim_{x\to\infty} \left(\pi/2 - \tan^{-1}x\right)^{1/x} = (0^0)$$

$$\log L = \lim_{x\to\infty} \log\left[\left(\pi/2 - \tan^{-1}x\right)^{1/x}\right]$$

$$= \lim_{x\to\infty} \frac{\log\left(\pi/2 - \tan^{-1}x\right)}{x} \left(\frac{\infty}{\infty}\right)$$

applying LHR, we get

$$\log L = \lim_{x\to\infty} \frac{\frac{1}{(\pi/2 - \tan^{-1}x)} \left\{0 - \frac{1}{(1+x^2)}\right\}}{1}$$

$$= -\lim \frac{1/(1+x^2)}{\pi/2 - \tan^{-1}x} \left(\frac{0}{0}\right)$$

using LHR.

$$\log L = -\lim_{x\to\infty} \frac{\cancel{-}\frac{1}{(1+x^2)^2} \times 2x}{\cancel{-}\frac{1}{\cancel{(1+x^2)}}}$$

$$= -\lim_{x\to\infty} \frac{2x}{1+x^2} \left(\frac{\infty}{\infty}\right)$$

using LHR

$$\log L = -\lim_{x\to\infty} \frac{\cancel{2}}{\cancel{2}x}$$

$$= \frac{-1}{\infty} = 0$$

$$L = e^0 = 1$$

8. let $L = \lim_{x \to 0} (\cot x)^{\tan x} \quad (\infty^0)$

$$\log L = \lim_{x \to 0} \log (\cot x)^{\tan x}.$$

$$= \lim_{x \to 0} \tan x \times \log (\cot x)$$

$$= \lim_{x \to 0} \frac{\log (\cot x)}{\cot x} \left(\frac{\infty}{\infty}\right)$$

applying LHR

$$\log L = \lim_{x \to 0} \frac{\frac{1}{\cot x} \times - \cancel{\operatorname{cosec}^2 x}}{\cancel{-\operatorname{cosec}^2 x}}$$

$$= \lim_{x \to 0} \tan x = 0$$

$$L = e^0 = 1.$$

9. let $L = \lim_{x \to 0} x^{\sin x} \quad (0^0)$

$$\log L = \lim_{x \to 0} \log (x^{\sin x})$$

$$= \lim_{x \to 0} \sin x \times \log x$$

$$= \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x} \left(\frac{\infty}{\infty}\right)$$

applying LHR

$$\log L = \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cdot \cot x}.$$

$$= -\lim_{x \to 0} \frac{\sin x \times \tan x}{x}.$$

$$\log L = -\lim_{x \to 0} \left(\frac{\sin x}{x}\right) \times \tan x$$

$= -1 \times \tan(0) = -1 \times 0$

$\log L = 0 = L = e^0 = 1$

* Partial differentiation.

Partial derivatives:-

- If $u = f(x, y)$ is function of two independent variable x and y then the partial derivative of u with respect to x is defined as $\frac{\partial u}{\partial x}$ (or) $u_x = \lim\limits_{x \to 0} \frac{f(x + \delta x, y) - f(x-y)}{\delta x}$.

(Diff^n u w.r.t x, assuming y as constant).

- Similarly a partial derivative of u w.r.t y is defined as
$\frac{\partial u}{\partial y}$ (or) $u_y = \lim\limits_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$.

(Diff^n u w.r.t y, assuming x as constant).

* Higher order partial derivatives

$u = f(x, y)$

$\frac{\partial u}{\partial x}$ or u_x $\qquad$ $\frac{\partial u}{\partial y}$ or u_y

$\frac{\partial^2 u}{\partial x^2}$ or u_{xx} $\qquad$ $\underbrace{\frac{\partial^2 u}{\partial x \partial y} \text{ or } u_{xy} \qquad \frac{\partial^2 u}{\partial y \partial x} \text{ (or) } u_{yx}}_{\text{(mixed partial derivatives)}}$ $\qquad$ $\frac{\partial^2 u}{\partial y^2}$ or u_{yy} (second order)

Note $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ (or) $u_{xy} = u_{yx}$

are always equal.

03/10/2017

Q.1 if $q = u = x^3 - 3xy^2 + x + e^x (\cos y + 1) \rightarrow 1$ then proove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Given $u = x^3 - 3xy^2 + x + e^x \cos y + 1 \rightarrow ①$

Diffⁿ eqⁿ ① partially w.r.t x.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2(1) + 1 + (\cos y)e^x$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + (\cos y)e^x \rightarrow ②$$

Diffⁿ eqⁿ ① partially w.r.t y

$$\frac{\partial u}{\partial y} = -3x(2y) + e^x(-\sin y)$$

$$\frac{\partial^2 u}{\partial y^2} = -6x - e^x \cos y \rightarrow ③$$

② + ③ =

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{6x} + \cancel{e^x \cos y} - \cancel{6x} - \cancel{e^x} \cos y$$
$$= 0$$

**

Q.2. If $u = e^{-2\pi^2 t} \sin \pi x \sin \pi y$, then p.t $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

Given

$$u = e^{-2\pi^2 t} \sin \pi x \sin \pi y \rightarrow ①$$

Diffⁿ eqⁿ ① partially w.r.t x.

$$\frac{\partial u}{\partial x} = e^{-2\pi^2 t} \sin \pi y \cdot \cos \pi x \times \pi$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-2\pi^2 t} \sin \pi y \times \pi X - \sin \pi \times \pi$$

$$\frac{\partial^2 u}{\partial x^2} = -\pi^2 e^{-2\pi^2 t} \sin \pi y \sin \pi x \rightarrow ②$$

III^ly

$$\frac{\partial^2 u}{\partial y^2} = -\pi^2 e^{-2\pi^2 t} \sin \pi x \sin \pi y \rightarrow ③$$

Diffⁿ eqⁿ ① paratially w.r.t t.

$$\frac{\partial u}{\partial t} = \sin\pi x \cdot \sin\pi y \times e^{-2\pi^2 t} \times (-2\pi^2)$$

$$\frac{\partial u}{\partial t} = -2\pi^{2} e^{-2\pi^2 t} \sin\pi x \sin\pi y \rightarrow (4)$$

$(2) + (3) \Rightarrow$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2\pi^2 e^{-2\pi^2 t} \sin\pi x \sin\pi y.$$

using (4).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

3. If $u = e^{ax+by} f(ax-by)$ then S.T $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$.

$\Rightarrow$

$$u = e^{ax+by} f(ax-by) \rightarrow (1)$$

Diffⁿ eqⁿ (1) partially w.r.t x.

$$\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by)(a) + f(ax-by) \times e^{ax+by} \times a.$$

$$= a e^{ax+by} \{ f'(ax-by) + f(ax-by) \}.$$

$$b\frac{\partial u}{\partial x} = ab e^{ax+by} \{ f'(ax-by) + f(ax-by) \} \rightarrow (2)$$

Diffⁿ eqⁿ (1) partially w.r.t y.

$$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by) e^{ax+by} \times b$$

$$= b e^{ax+by} \{ -f'(ax-by) + f(ax-by) \}$$

$$a\frac{\partial u}{\partial y} = ab e^{ax+by} \{ -f'(ax-by) + f(ax-by) \} \rightarrow (3)$$

$(2) + (3) \Rightarrow$

Scanned by CamScanner

classmate

Date ______

Page ______

$$b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = abe^{ax+by}\{2f(ax-by)\} = 2abu$$

04/10/2017

4. If $v = e^{a\theta}\cos(a\log r)$ then P.T $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$.

$r = e^{a\theta}\cos(a\log r) \rightarrow ①$

Diff^n eq^n ① partially w.r.t r

$$\frac{\partial v}{\partial r} = e^{a\theta} \times -\sin(a\log r) \times a \times \frac{1}{r}$$

$$\frac{\partial v}{\partial r} = -ae^{a\theta}\cdot\frac{\sin(a\log r)}{r}$$

Diff^n again w.r.t r

$$\frac{\partial^2 v}{\partial r^2} = -ae^{a\theta}\frac{\{r\cdot\cos(a\log r)\times\frac{a}{r} - \sin(a\log r)\times 1\}}{r^2}$$

$$\frac{\partial^2 v}{\partial r^2} = -a^2e^{a\theta}\frac{\cos(a\log r)}{r^2} + ae^{a\theta}\frac{\sin(a\log r)}{r^2}$$

Diff^n eq^n ① partially w.r.t θ.

$$\frac{\partial v}{\partial \theta} = \cos(a\log r) \times e^{a\theta} \times a$$

$$\frac{\partial^2 v}{\partial \theta^2} = \cos(a\log r) \times e^{a\theta} \times a^2$$

Consider

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = \frac{-a^2e^{a\theta}\cos(a\log r)}{r^2} + \frac{ae^{a\theta}\sin(a\log r)}{r^2}$$

$$- \frac{ae^{a\theta}\sin(a\log r)}{r^2} + \frac{a^2e^{a\theta}\cos(a\log r)}{r^2}$$

$$= 0$$

*** Note**

Suppose $u = f(x, y, z)$ is a symmetric function then by finding any one partial ~~de~~ derivative we write the other partial derivative directly.

5. If $u = \log\sqrt{x^2+y^2+z^2}$ then prove that $(x^2+y^2+z^2)(u_{xx}+u_{yy}+u_{zz}) = 1$

$u = \log\sqrt{x^2+y^2+z^2}$

$u = \frac{1}{2}\log(x^2+y^2+z^2) \rightarrow (1)$

Diff^n eq^n (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}} \times \frac{1}{x^2+y^2+z^2} \times \cancel{2}x = \frac{x}{x^2+y^2+z^2}$$

Diff^n again partially w.r.t x.

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2)(1) - x(2x)}{(x^2+y^2+z^2)^2}$$

$$u_{xx} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

|||^{ly} $u_{yy} = \frac{-y^2+z^2+x^2}{(x^2+y^2+z^2)^2}$

$$u_{zz} = \frac{-z^2+x^2+y^2}{(x^2+y^2+z^2)^2}$$

$$u_{xx}+u_{yy}+u_{zz} = \frac{\cancel{-x^2}+\cancel{y^2}+\cancel{z^2} \cancel{-z^2}+\cancel{x^2}+\cancel{y^2} -\cancel{y^2}+z^2+x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\cancel{x^2+y^2+z^2}}{(x^2+y^2+z^2)^{\cancel{2}}} = \frac{1}{(x^2+y^2+z^2)}$$

$$(x^2+y^2+z^2)(u_{xx}+u_{yy}+u_{zz}) = 1$$

classmate
Date
Page

(6). If $u = \log(x^3+y^3+z^3-3xyz)$ then p.t

(a) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(b) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

$\Rightarrow$

Given $u = \log(x^3+y^3+z^3-3xyz) \rightarrow (1)$

Diff^n eq^n (1) partially w.r.t x.

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3x^2-3yz)$$

$$\frac{\partial u}{\partial x} = \frac{3(x^2-yz)}{(x^3+y^3+z^3-3xyz)}$$

$$\frac{\partial u}{\partial y} = \frac{3(y^2-xz)}{(x^3+y^3+z^3-3xyz)}$$

$$\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}.$$

(a) consider.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-yz-xz-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-yz-xz-xy)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$= \frac{3}{(x+y+z)}.$$

(b) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$ //

classmate

Date
Page

(07) If $u = \tan^{-1}(y/x)$ → then prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

$u = \tan^{-1}(y/x) \Rightarrow ①$

Diffⁿ eqⁿ ① partially w.r.t x and y.

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times y \times \frac{-1}{x^2}$$

$$= \frac{1}{\left(\frac{x^2+y^2}{\cancel{x^2}}\right)} \times \frac{-y}{\cancel{x^2}}$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2} \Rightarrow ②$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} \times 1$$

$$= \frac{1}{\left(\frac{x^2+y^2}{\cancel{x^2}}\right)} \times \frac{1}{\cancel{x}}$$

$$\frac{\partial u}{\partial y} = \frac{x}{x^2+y^2} \Rightarrow ③$$

Diffⁿ eqⁿ ② partially w.r.t y.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow ④$$

Diffⁿ eqⁿ ③ partially w.r.t x

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow ⑤$$

from ④ and ⑤ we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$ //.

05/10/2017

***
08 Verify that $U_{xy} = U_{yx}$ given that $u = x^y \rightarrow (1)$

Given $u = x^y \rightarrow (1)$

Diffⁿ eqⁿ (1) partially w.r.t x and y.

$$\frac{\partial u}{\partial x} = y\, x^{y-1} \rightarrow (2)$$

$$\frac{\partial u}{\partial y} = x^y \log x \rightarrow (3)$$

Diffⁿ eqⁿ (2) partially w.r.t y.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = x^{y-1}(1) + y.x^{y-1} \times \log x.$$

$$\frac{\partial^2 u}{\partial y \partial x} \text{ (or) } U_{yx} = x^{y-1}\left[1 + y\log x\right] \rightarrow (4)$$

Diffⁿ eqⁿ (3) partially w.r.t x.

$$\frac{\partial^2 u}{\partial x \partial y} \text{ (or) } U_{xy} = x^{y-1}\left[1 + y\log x\right] \rightarrow (5)$$

from (4) and (5)

$= U_{xy} = U_{yx}$ //

* Jacobian

$$J\left(\frac{u, v}{x, y}\right) \text{ or } \frac{\partial(u,v)}{\partial(x,y)} = \cancel{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{vmatrix}} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J\left(\frac{u, v, w}{x, y, z}\right) \text{ or } \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Date ____
Page ____

Q.1 find the Jacobiane of u, v, w with respect to x, y, z, Given that $u = x+y+z$, $v = y+z$, $w = z$

$\frac{\partial u}{\partial x} = 1$ $\quad \frac{\partial v}{\partial x} = 0$ $\quad \frac{\partial w}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 1$ $\quad \frac{\partial v}{\partial y} = 1$ $\quad \frac{\partial w}{\partial y} = 0$

$\frac{\partial u}{\partial z} = 1$ $\quad \frac{\partial v}{\partial z} = 1$ $\quad \frac{\partial w}{\partial z} = 1$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

(2) find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ where $= x^2+y^2+z^2$, $v = xy+yz+xz$, $w = x+y+z$

$u = x^2+y^2+z^2$, $v = xy+yz+xz$, $w = x+y+z$.

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial v}{\partial x} = y+z$ $\quad \frac{\partial w}{\partial x} = 1$

$\frac{\partial u}{\partial y} = 2y$ $\quad \frac{\partial v}{\partial y} = x+z$ $\quad \frac{\partial w}{\partial y} = 1$

$\frac{\partial u}{\partial z} = 2z$ $\quad \frac{\partial v}{\partial z} = y+x.$ $\quad \frac{\partial w}{\partial z} = 1$

$$\begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$2x(x+z - (y+x)) - 2y((y+z) - (y+x)) + 2z((y+z) - (x+z))$

$2x(\cancel{x}+\cancel{z}-y-\cancel{x}) - 2y(\cancel{y}+z-\cancel{y}-x) + 2z(y+\cancel{z}-x-\cancel{z})$

$\neq 0$

$\cancel{2xz} - \cancel{2yx} - \cancel{2yz} + \cancel{2yx} + \cancel{2zy} - \cancel{2xz} = 0$

classmate

Date

Page

(3) If $u = yz/x$, $v = zx/y$, $w = \frac{xy}{z}$ then ST. $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

$u = yz/x \quad v = zx/y \quad w = xy/z$

$\frac{\partial u}{\partial x} = \frac{-yz}{x^2} \quad \frac{\partial v}{\partial x} = z/y \quad \frac{\partial w}{\partial x} = y/z$

$\frac{\partial u}{\partial y} = z/x \quad \frac{\partial v}{\partial y} = \frac{-zx}{y^2} \quad \frac{\partial w}{\partial y} = x/z$

$\frac{\partial u}{\partial z} = y/x \quad \frac{\partial v}{\partial z} = x/y \quad \frac{\partial w}{\partial z} = -xy/z^2$

w.k.t

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & z/x & y/x \\ z/y & -zx/y^2 & x/y \\ y/z & x/z & -xy/z^2 \end{vmatrix}$$

$$= -yz/x^2\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - z/x\left[-x/z - x/z\right] + y/x\left[x/y + x/y\right]$$

$$= 0 - \cancel{z}/\cancel{x}\left(\frac{-2\cancel{x}}{\cancel{z}}\right) + \cancel{y}/\cancel{x}\left(\frac{2\cancel{x}}{\cancel{y}}\right)$$

$$= 2+2 = 4 //.$$

**4 If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ $z = r\cos\theta$.
prove that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

$x = r\sin\theta\cos\phi$	$y = r\sin\theta\sin\phi$	$z = r\cos\theta$
$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$	$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$	$\frac{\partial z}{\partial r} = \cos\theta$
$\frac{\partial x}{\partial \theta} = \cos\theta \cdot \cos\phi r$	$\frac{\partial y}{\partial \theta} = \cos\theta \cdot r \cdot \sin\phi$	$\frac{\partial z}{\partial \theta} = -\sin\theta \cdot r$
$\frac{\partial x}{\partial \phi} = -\sin\phi \cdot \sin\theta \cdot r$	$\frac{\partial y}{\partial \phi} = \cos\phi \cdot r \cdot \sin\theta$	$\frac{\partial z}{\partial \phi} = 0$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\phi\cos\theta & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\sin\theta\cos\phi\left[0 + r^2\sin^2\theta\cos\phi\right] - r\cos\phi\cos\theta\left[0 - r\sin\theta\cos\theta\cos\phi\right] - r\sin\theta\sin\phi\left[-r\sin^2\theta\sin\phi - r\sin\phi\cos^2\theta\right].$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\cos^2\phi\cos^2\theta\sin\theta + r^2\sin^2\phi\sin\theta(\sin^2\theta + \cos^2\theta)$$

$$= r^2\sin\theta(\sin^2\theta\cos^2\phi + \cos^2\phi\cos^2\theta + \sin^2\phi)$$

$$= r^2\sin\theta(\cos^2\phi(\sin^2\theta + \cos^2\theta) + \sin^2\phi)$$

$$= r^2\sin\theta(\cos^2\phi + \sin^2\phi)$$

$$= r^2\sin\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

06/10/2017

***5. If $x+y+z = u$ $y+z = v$ $z = uvw$ then find the value of $\frac{\partial(x,y,z)}{\partial(u,v,w)} =$

$x+y+z = u$ $y+z = v$ $z = uvw$.

$x = u - (y+z)$	$y = v - z$	$z = uvw$
$x = u - v$	$y = v - uvw$	$\frac{\partial z}{\partial u} = vw$
$\frac{\partial x}{\partial u} = 1$	$\frac{\partial y}{\partial u} = -1vw$	$\frac{\partial z}{\partial v} = uw$
$\frac{\partial x}{\partial v} = -1$	$\frac{\partial y}{\partial v} = 1 - uw$	
$\frac{\partial x}{\partial w} = 0$	$\frac{\partial y}{\partial w} = -uv$	$\frac{\partial z}{\partial w} = uv$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$\cancel{1(1-uw+uv)+1(-vw+uv)}$ $\quad 1((1-uw)uv+u^2vw)+1$

$\cancel{1-uw+uv-vw+uv}$ $\quad (-uv^2w+uv^2w]+0$

$$= uv - \cancel{u^2wv} + \cancel{u^2vw} - \cancel{uv^2w} + \cancel{uv^2w}$$

$$= uv$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = uv$$

* <u>Total differentiation.</u>

$$du = \frac{\partial u}{\partial x} \times dx + \frac{\partial u}{\partial y} + \partial y .$$

If $u = f(x,y)$, $x = x(t)$ and ~~get~~ $y = y(t)$. then the total derivative of u w.r.t t is $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

The total derivative is also called as excert derivative.

Q.1 Find the total or excert derivative of the following.

1. $u = x^3 + xy^2 + x^2y + y^3$
2. $u = x \sin\theta \cos\phi$
3. $z = xy^2 + x^2y$ where $x = at$ and $y = 2at$
4. $u = xy + yz + zx$, where $x = t\cos t$, $y = t\sin t$, $z = t$ at $\pi/4$.

1. $u = x^3 + xy^2 + x^2y + y^3$

Given $u = x^3 + xy^2 + x^2y + y^3$

Diffⁿ u partially w.r.t x and y

$$\frac{\partial u}{\partial x} = 3x^2+y^2+2xy$$

$$\frac{\partial u}{\partial y} = 2xy+x^2+3y^2$$

w.k.t

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$du = (3x^2+y^2+2xy)\,dx + (2xy+x^2+3y^2)\,dy$$

2. Given $x = r\sin\theta\cos\phi$

Diffⁿ x partially w.r.t r, θ and ϕ

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \times 1$$

$$\frac{\partial x}{\partial \theta} = r\cos\phi\cos\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

w.k.t

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi$$

$$dx = (\sin\theta\cos\phi)\,dr + (r\cos\phi\cos\theta)\,d\theta - (r\sin\theta\sin\phi)\,d\phi$$

3. Given $z = xy^2+x^2y \rightarrow (1)$ when $x = at$ and $y = 2at$

Diffⁿ eqⁿ (1) partially w.r.t x and y

$$\frac{\partial z}{\partial x} = y^2+2xy = (2at)^2+2(at)(2at) = 8a^2t^2$$

$$\frac{\partial z}{\partial y} = 2xy+x^2 = 2(at)(2at)+(at)^2 = 5a^2t^2$$

further we have

$x = at$ $\quad\quad$ $y = 2at$

$$\frac{dx}{dt} = a \quad \text{and} \quad \frac{dy}{dt} = 2a$$

classmate
Date ______
Page ______

w.k.t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (8a^2t^2)(a) + (5a^2t^2)(2a) = 18a^3t^2$$

(4) Given $u = xy + yz + zx$

$$\frac{\partial u}{\partial x} = y + z = (t\sin t + t)$$

$$\frac{\partial u}{\partial y} = x + z = (t\cos t + t)$$

$$\frac{\partial u}{\partial z} = y + x = (t\sin t + t\cos t)$$

$x = t\cos t$	$y = t\sin t$	$z = t$
$\frac{dx}{dt} = \cos t - t\sin t$	$\frac{dy}{dt} = \sin t + t\cos t$	$\frac{dz}{dt} = 1$

w.k.t

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

at $t = \pi/4$.

$$\frac{du}{dt} = \left(\frac{\pi}{4\sqrt{2}} + \pi/4\right)\left(1/\sqrt{2} - \cancel{\frac{\pi}{4\sqrt{2}}}\right) + \left(\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}} + \cancel{\pi/4\sqrt{2}}\right) +$$

$$\left(\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right)(1)$$

$$\frac{du}{dt} = \left(\pi/4\sqrt{2} + \pi/4\right)(\sqrt{2}) + \frac{\pi}{2\sqrt{2}}$$

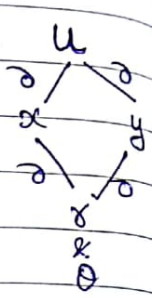
* <u>Differentiation of composite function.</u>

09/10/2017 * <u>chain rule of partial differentiation.</u>

If $u = f(x, y)$ where $x = f(r, \theta)$ and $y = f(r, \theta)$. that is u is function of x and y and x and y are the functions of r, θ then the partial derivative of u is with respect to r and θ

are defined as $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial r}$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial \theta}$$



are called chain rule of partial differentiation.

Problems:-

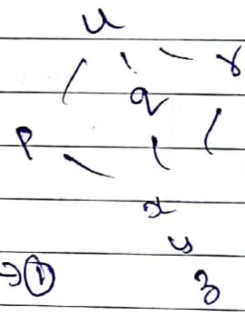
Q.1 If $u = f(x/y, y/z, z/x)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial y} = 0$

$u = f(x/y, y/z, z/x)$

$p = x/y \quad q = y/z \quad r = z/x.$

$u = f(p, q, r)$

By chain rule of partial diffⁿ we get.



$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}. \\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}.\end{aligned}\right\} \rightarrow (1)$$

$p = x/y$	$q = y/z$	$r = z/x$
$\frac{\partial p}{\partial x} = 1/y$	$\frac{\partial q}{\partial x} = 0$	$\frac{\partial r}{\partial x} = -z/x^2$
$\frac{\partial p}{\partial y} = -x/y^2$	$\frac{\partial q}{\partial y} = 1/z$	$\frac{\partial r}{\partial y} = 0$
$\frac{\partial p}{\partial z} = 0$	$\frac{\partial q}{\partial z} = -y/z^2$	$\frac{\partial r}{\partial z} = \frac{1}{x}$

Sub in $\rightarrow (1)$

$\frac{\partial u}{\partial x} = \frac{1}{y}\frac{\partial u}{\partial p} - \frac{z}{x^2}\frac{\partial u}{\partial r} \Rightarrow x \cdot \frac{\partial u}{\partial x} = x/y \frac{\partial u}{\partial p} - z/x \frac{\partial u}{\partial r} \rightarrow (2)$

$\frac{\partial u}{\partial y} = -x/y^2 \frac{\partial u}{\partial p} + \frac{1}{z}\frac{\partial u}{\partial q} \Rightarrow y \cdot \frac{\partial u}{\partial y} = -x/y \frac{\partial u}{\partial p} + y/z \frac{\partial u}{\partial q} \rightarrow (3)$

$$\frac{\partial u}{\partial t} = -y/z^2 \frac{\partial u}{\partial q} + \frac{1}{z}\frac{\partial u}{\partial r} \Rightarrow z\frac{\partial u}{\partial t} = -y/z \frac{\partial u}{\partial q} + z/x \frac{\partial u}{\partial r} \rightarrow (4)$$

Adding (2) (3) and (4), we get

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial t} = 0.$$

10/10/2017

Q2. If $z = (x,y)$, $x = u-v$ and $y = uv$ then P.T

(i) $(u+v)\frac{\partial z}{\partial x} = u\cdot\frac{\partial z}{\partial x} - v\frac{\partial z}{\partial v}$

(ii) $(u+v)\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

By chain rule of P.D

$$\left.\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}\right\} \rightarrow (1)$$

We have.

$x = u-v$	$y = uv$
$\frac{\partial x}{\partial u} = 1$	$\frac{\partial y}{\partial u} = v$
$\frac{\partial x}{\partial v} = -1$	$\frac{\partial y}{\partial v} = u$

Sub in (1).

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} \rightarrow (2)$$

$$\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y} \rightarrow (3).$$

(i) $(u \times eq^n(2)) - (v \times eq^n(3))$

$$u\cdot\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = u\frac{\partial z}{\partial x} + \cancel{uv\frac{\partial z}{\partial y}} + v\frac{\partial z}{\partial x} - \cancel{uv\frac{\partial z}{\partial y}}$$

$$= (u+v)\frac{\partial z}{\partial x}$$

classmate

Date
Page

ii) adding eq^n 2 and 3

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = v\frac{\partial z}{\partial y} + u\frac{\partial z}{\partial y}$$

$$= (v+u)\frac{\partial z}{\partial y} \text{ //.}$$

(3) If $z=f(x,y)$ where $x=u^2-v^2$ and $y=2uv$ then prove that

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 4(u^2+v^2)\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

$\Rightarrow$ By chain rule of PD.

$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \end{aligned}\right\} \rightarrow (1)$$

we have.

$x = u^2-v^2$	$y = 2uv$
$\frac{\partial x}{\partial u} = 2u$	$\frac{\partial y}{\partial u} = 2v$
$\frac{\partial x}{\partial v} = -2v$	$\frac{\partial y}{\partial v} = 2u$

Sub in (1).

$$\frac{\partial z}{\partial u} = 2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y} = 2\left(u\cdot\frac{\partial z}{\partial x} + v\cdot\frac{\partial z}{\partial y}\right) \rightarrow (2)$$

$$\frac{\partial z}{\partial v} = -2v\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y} = 2\left[u\frac{\partial z}{\partial x} - v\frac{\partial z}{\partial x}\right) \rightarrow (3)$$

squaring and adding eq^n 2 and 3

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 4\left[u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}\right]^2 + 4\left[u\frac{\partial z}{\partial y} - v\cdot\frac{\partial z}{\partial x}\right]^2$$

$$= 4\left\{u^2\left(\frac{\partial z}{\partial x}\right)^2 + v^2\left(\frac{\partial z}{\partial y}\right)^2 + \cancel{2uv\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}} + \right.$$

$$\left. u^2\left(\frac{\partial z}{\partial y}\right)^2 + v^2\left(\frac{\partial z}{\partial x}\right)^2 - \cancel{2uv\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}}\right]$$

$$= 4\left\{u^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] + v^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]\right\}$$

Scanned by CamScanner

$$= 4(u^2+v^2)\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}$$

** (4) If $z=f(x,y)$ where $x=e^u \sin v$ and $y = e^u \cos v$ then prove that.

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}.$$

By chain rule of P.D

$$\left.\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}\right\} \rightarrow (1)$$

we have.

$x = e^u \sin v$	$y = e^u \cos v$.
$\frac{\partial x}{\partial u} = e^u \sin v$	$\frac{\partial y}{\partial u} = e^u \cos v$
$\frac{\partial x}{\partial v} = e^u \cos v$	$\frac{\partial y}{\partial v} = -e^u \sin v$

Sub in (1).

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u \sin v + \frac{\partial z}{\partial y}\cdot e^u \cos v = e^u\left[\left(\sin v\frac{\partial z}{\partial x}\right)+\left(\cos v\frac{\partial z}{\partial y}\right)\right] \rightarrow (2)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot e^u \cos v - \frac{\partial z}{\partial y}\cdot e^u \sin v = e^u\left[\left(\cos v\frac{\partial z}{\partial x}\right)-\left(\sin v\frac{\partial z}{\partial y}\right)\right] \rightarrow (3)$$

squaring and adding eq^n (2) and (3)

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left\{\left[\left(\sin v\frac{\partial z}{\partial x}\right)+\left(\cos v\frac{\partial z}{\partial y}\right)\right]^2 + \left[\cos v\frac{\partial z}{\partial x} - \sin v\frac{\partial z}{\partial y}\right]^2\right\}$$

$$= e^{2u}\left\{\left(\sin^2 v\left(\frac{\partial z}{\partial x}\right)^2 + \cos^2 v\left(\frac{\partial z}{\partial y}\right)^2 + \cancel{2\sin v\cos v\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}}\right) + \left(\cos^2 v\left(\frac{\partial z}{\partial x}\right)^2 + \sin^2 v\left(\frac{\partial z}{\partial y}\right)^2 - \cancel{2\sin v\cos v\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}}\right)\right\}.$$

$$= e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2\left(\sin^2 v + \cos^2 v\right) + \left(\frac{\partial z}{\partial y}\right)^2\left(\sin^2 v + \cos^2 v\right)\right\}$$

$$= e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} //.$$

* Homogenous functions and Euler's theorm

* Homogenous function

- A function $u=f(x,y)$ is said to be homogenous function. if it can be expressed as $u=x^n f(y/x)$ (or) $u=y^n g(x/y)$
- where n is a degree of homogenous function

ex:-

$$u=\frac{x^3+y^3}{\sqrt{x+y}} = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{\sqrt{x}\left(\sqrt{1+y/x}\right)}$$

$$= x^{3-1/2}\left\{\frac{1+(y/x)^3}{\sqrt{1+(y/x)}}\right\}$$

$$= u = x^{5/2}\left\{\frac{1+(y/x)^3}{\sqrt{1+(y/x)}}\right\}$$

$=$ u is a HF of degree $n=5/2$.

~~By Euler's theorem / $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$ = 5/2 u~~

* Euler's theorem.

If $u=f(x,y)$ is a homogenous function of degree n then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

* Euler's extension theorem:

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=n(n-1)u.$$

(or)

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 u=n(n-1)u.$$

1: If $u=\frac{x^3+y^3}{\sqrt{x+y}}$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=5/2\,u$

$$u=\frac{x^3+y^3}{\sqrt{x+y}} = \frac{x^3\left(1+y^3/x^3\right)}{\sqrt{x}\left(\sqrt{1+y/x}\right)}$$

$$x^{3-1/2}\left\{\frac{1+(y/x)^3}{\sqrt{1+(y/x)}}\right\}$$

$$u = x^{5/2}\left\{\frac{1+(y/x)^3}{\sqrt{1+(y/x)}}\right\}$$

u is a H.F of degree $n = 5/2$.

By Euler's theorem.

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = nu$$

$$= 5/2\, u.$$

(2) If $u = \left(\frac{z^4}{x^3+y^3}\right)^{1/3}$ then prove that $3(xu_x + yu_y + zu_z) = u$.

$$u = \left(\frac{z^4}{x^3+y^3}\right)^{1/3}$$

$$= \left\{\frac{x^4(z^4/x^4)}{x^3(1+y^3/x^3)}\right\}^{1/3}$$

$$= \left\{\frac{x(z/x)^4}{1+(y/x)^3}\right\}^{1/3}.$$

$$u = x^{1/3}\left\{\frac{(z/x)^4}{1+(y/x)^3}\right\}^{1/3}$$

u is a H.F of degree $n = 1/3$.

By Euler's theorem.

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 1/3\, u$$

$$= 3(xu_x + yu_y + zu_z) = u. //$$

(3) If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{(x+y)}$ then prove that $xu_x + yu_y + zu_z = 0$

$$u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}.$$

$$= \frac{\cancel{x}}{\cancel{x}(y/x + z/x)} + \frac{\cancel{x}(y/x)}{\cancel{x}(z/x+1)} + \frac{\cancel{x}(z/x)}{\cancel{x}(1+y/x)}$$

$$u = x^0 \times \left[\frac{1}{(y/x+z/x)} + \frac{y/x}{(1+z/x)} + \frac{z/x}{(1+y/x)}\right\}$$

u is a HF of degree $n = 0$

By Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$$

$$=0\times 4.$$

$$x\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$$

(4) If $u=\log\left(\frac{x^4+y^4}{x+y}\right)$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$

Given $u=\log\left(\frac{x^4+y^4}{x+y}\right)$

$$e^u=\frac{x^4+y^4}{x+y}.$$

let $e^u=z$

$$\therefore\ z=\frac{x^4+y^4}{x+y}$$

$$z=\frac{x^4\left(1+y^4/x^4\right)}{x\left(1+y/x\right)}\Rightarrow x^3\left\{\frac{1+(y/x)^4}{1+(y/x)}\right\}$$

$\therefore$ z is a H.F of degree $n=3$

By Euler's theorem.

$$x\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial z}{\partial y}=nz$$

put $z=e^u$ and $n=3$.

$$x\cdot\frac{\partial}{\partial x}(e^u)+y\frac{\partial}{\partial y}(e^u)=3\times e^u$$

$$xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=3e^u$$

Dividing by e^u

$$x\cdot\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$$

(5) If $u=e^{\left(\frac{x^3y^3}{x^2+y^2}\right)}$ then p.t $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=4u\log u.$

$$u=e^{\frac{x^3y^3}{x^2+y^2}}$$

$$\log u=\frac{x^3y^3}{x^2+y^2}$$

let $\log u = z$

$$z = \frac{x^3y^3}{x^2+y^2}$$

$$z = \frac{x^6(y^3/x^3)}{x^2(1+y^2/x^2)} = x^4\left\{\frac{(y/x)^3}{1+(y/x)^2}\right\}$$

z is a HF of degree $n=4$.

By Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

put $z=\log u$ and $n=4$.

$$x \cdot \frac{\partial}{\partial x}(\log u) + y\frac{\partial}{\partial y}(\log u) = 4\log u.$$

$$x \times \frac{1}{u} \times \frac{\partial u}{\partial x} + y \times \frac{1}{u}\frac{\partial u}{\partial y} = 4\log u$$

$\otimes$ ly by u

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 4u\log u_{//}.$$

***(6) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that (i) $xU_x + yU_y = \sin 2u$.

(ii) $x^2U_{xx} + 2xyU_{xy} + y^2U_{yy} = \sin 4u - \sin 2u$.

given $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\tan u = \frac{x^3+y^3}{x-y}.$$

let $\tan u = z$

$$z = \frac{x^3+y^3}{x-y} = \frac{x^3(1+y^3/x^3)}{x(1-y/x)}$$

$$= x^2\left\{\frac{1+(y/x)^3}{1-(y/x)}\right\}$$

z is a HF of degree $n=2$

(i) By Euler's theorem, we get

$$x \cdot \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$$

put $z = \tan u$ and $n = 2$

$$x \cdot \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = 2 \tan u.$$

$$x \sec^2 u \cdot \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2 \tan u$$

Dividing by $\sec^2 u$

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u}$$

$$= \frac{2 \sin u}{\cos u} \times \cos^2 u$$

$$= 2 \sin u \cos u$$

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u \rightarrow (1)$$

(ii) Diffⁿ eqⁿ (1) partially w.r.t x

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = \cos 2u \times 2 \frac{\partial u}{\partial x}$$

$\times^{ly}$ by x.

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \partial y} = 2 \cos 2u \cdot x \frac{\partial u}{\partial x} \rightarrow (3)$$

$|||^{ly}$

$$y^2 \frac{\partial^2 u}{\partial y^2} + y \cdot \frac{\partial u}{\partial y} + xy \frac{\partial^2 u}{\partial x \partial y} = 2 \cos 2u \cdot y \frac{\partial u}{\partial y} \rightarrow (4)$$

adding (3) + (4)

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right) = 2 \cos 2u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

using eq (1)

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + \sin 2x = 2 \cos 2u \cdot \sin 2u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \sin 4u - \sin 2u$$

classmate
Date
Page

(7) If $u = cosec^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)$ then prove that $\left(x\cdot\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 u = \frac{tan u}{6}\left(1+\frac{sec^2 u}{6}\right)$

$$u = cosec^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)$$

$$cosec u = \frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}.$$

let $cosec u = z$

$$z = \frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}} = \frac{x^{1/2}\left(1+y^{1/2}/x^{1/2}\right)}{x^{1/3}\left(1+y^{1/3}/x^{1/3}\right)}$$

$$z = x^{1/6}\left\{\frac{1+(y/x)^{1/2}}{1+(y/x)^{1/3}}\right\}$$

z is a HF of degree $n = 1/6$

By Euler's theorem, we get.

$$x\cdot\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = nz$$

put $z = cosec u$ and $n = 1/6$

$$x\frac{\partial}{\partial x}(cosec u)+y\frac{\partial}{\partial y}(cosec u) = 1/6 \times cosec u$$

$$-x\, cosec u\cdot cot u\cdot\frac{\partial u}{\partial x} - y\, cosec u\cdot cot u\frac{\partial u}{\partial y} = 1/6\, cosec u.$$

Dividing by $(-cosec u\cdot cot u)$

$$x\cdot\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1/6\, \cancel{cosec u}}{-\cancel{cosec} u\cdot cot u}$$

$$x\cdot\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = -1/6\, tan u \rightarrow (1)$$

Diffn eqn (1) partially w.r.t x.

$$x\cdot\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y} = -1/6 \times sec^2 u \times \frac{\partial u}{\partial x}.$$

(X)ly by x

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y} = -1/6\, sec^2 u \times x\frac{\partial u}{\partial x} \rightarrow (3)$$

111'y.

$$y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} + xy\frac{\partial^2 u}{\partial x \partial y} = -1/6 \sec^2 u \cdot y\frac{\partial u}{\partial y} \rightarrow (4)$$

adding (3) and (4)

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + \left(x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y}\right) = -1/6 \sec^2 u \times \left[x\frac{\partial u}{\partial x} + y\right.$$

using eqn (1)

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} - 1/6 \tan u = -1/6 \sec^2 u \left(-1/6 \tan u\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{\sec^2 u}{6}\left(\frac{\tan u}{6}\right) + \frac{\tan u}{6}$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 u = \frac{\tan u}{6}\left(1 + \frac{\sec^2 u}{6}\right)$$