|| Jai Shri Gurudev ||

S.J.C Institute of Technology, Chickballapur

Department of Mathematics

Lecture Notes

Complex Analysis, Probability and Statistical Methods

[18MAT41]

Prepared By:

Purushotham.P phone: 8197481658

Assistant Professor

SJC institute of techology

Email id: ppurushothamp48@gmail.com

COMPLEX ANALYSIS, PROBABILITY AND STATISTICAL METHOD

MODULE:01 - CALCULUS OF COMPLEX FUNCTIONS AND CONSTRUCTION OF ANALYTIC FUNCTIONS

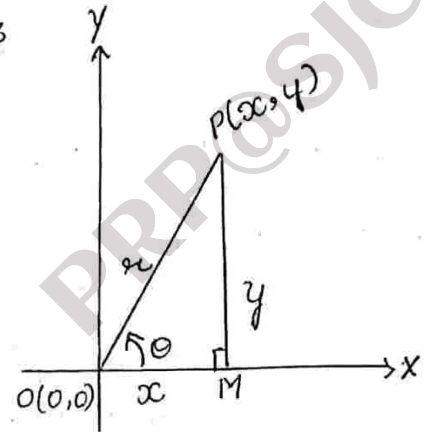
Complex variable

Introduction:

Let x and y be the two real values, the variable is of the form $x+iy$ is called the complex variable in the cartesian form and which is denoted by 'z', $z=x+iy$. The complex variable has two parts, they are called real part and imaginary part i.e, (x,y). The modulus of the complex variable $z=x+iy$ is defined as $|z|=\sqrt{x^2+y^2}$ and vector function is denoted as $\vec{V}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and the modulus of $|\vec{V}|$ is $|\vec{V}|=\sqrt{a_1^2+a_2^2+a_3^2}$ and the conjugate of the complex variable is defined as $\bar{z}=x-iy$, therefore $|\bar{z}| = \sqrt{x^2-y^2}$.

Geometrical representation:

Let $\overrightarrow{OX}, \overrightarrow{OY}$ be the Co-ordinate axis and let $P(x,y)$ be any point on the plane, then from ΔOPM, we have,



$$\sin\theta = \frac{PM}{OP} = \frac{y}{r}$$

$$\frac{y}{r} = \sin\theta$$

$$\boxed{y = r\sin\theta}$$

III^{ly} $$\cos\theta = \frac{OM}{OP} = \frac{x}{r}$$

$$\frac{x}{r} = \cos\theta$$

$$\boxed{x = r\cos\theta}$$

W.K.T the complex variable in the cartesian form

$$z = x+iy$$

$$\Rightarrow z = r\cos\theta + ir\sin\theta$$

$$\Rightarrow z = r(\cos\theta + i\sin\theta)$$

$\Rightarrow \boxed{z = r.e^{i\theta}}$ This is called the complex variable in polar form.

Some important results:

1. The function $w = f(z) = u(x,y) + iv(x,y)$ is called the complex valued function for complex variable $z = x+iy$ in the cartesian form where $u(x,y)$ and $v(x,y)$ are the two real valued functions in x and y.

2. The function $w = f(z) = u(r,\theta) + iV(r,\theta)$ is called the complex valued function in polar form for complex variable $z = r \cdot e^{i\theta}$ in the polar form, where $u(r,\theta)$ and $V(r,\theta)$ are the two real valued functions.

3. W.K.T

$$e^{ix} = \cos x + i\sin x \rightarrow (1)$$

$$e^{-ix} = \cos x - i\sin x \rightarrow (2)$$

$$(1) + (2) \Rightarrow e^{ix} + e^{-ix} = 2\cos x$$

$$\Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2} \rightarrow (3)$$

Put $x \rightarrow ix$

$$(3) \Rightarrow \cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2}$$

$$\Rightarrow \cos ix = \frac{e^{i^2 x} + e^{-i^2 x}}{2}$$

$$\Rightarrow \cos ix = \frac{e^{-x} + e^{x}}{2}$$

$$\Rightarrow \boxed{\cos(ix) = \cosh x}$$

$$(1) - (2) \Rightarrow 2i\sin x = e^{ix} - e^{-ix}$$

$$\Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i} \rightarrow (4)$$

Put $x \rightarrow ix$

$$(4) \Rightarrow \sin(ix) = \frac{e^{i^2 x} - e^{-i^2 x}}{2i}$$

$$\sin(ix) = \frac{1}{i}\left[\frac{e^{-x} - e^{x}}{2}\right]$$

$$\Rightarrow \sin(ix) = \frac{-i}{i^2}\left[\frac{e^{x}-e^{-x}}{2}\right]$$

$$\Rightarrow \sin(ix) = \frac{-i}{-1} \sinh x$$

$$\boxed{\sin ix = i \sinh x}$$

Definitions:

1) Limit of complex variable: Let z be a complex variable for the complex value z_0, such that for any positive quantity of $\delta > 0$, then the limit of z at z_0 is defined as $|z - z_0| < \delta$.

2) Limit of complex valued function: Let $w = f(z)$ be a complex valued function for the complex variable z, then the limit of $f(x)$ at z tends to z_0 is defined as $|f(z) - f(z_0)| < \varepsilon$ (or) $\lim\limits_{z \to z_0} f(z) = l$, where ε is a small positive quantity.

3) Continuity of a function: Let $w = f(z)$ be a complex valued function for a complex variable z, then if $\lim\limits_{z \to z_0} f(z) = f(z_0)$ is said to be continuous at $z = z_0$.

4) Differentiable of a function: Let $w = f(z)$ is a complex valued function for a complex variable z, then the differentiable of $f(z)$ at $z = z_0$ is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

(or)

The derivative of $f(z)$ at any point of z is defined as $f'(z) = \lim\limits_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$, where $\delta z \neq 0$.

5) **Analytic function**: The complex valued function $w = f(z)$ is differentiable at any point of z i.e.,

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}, \text{ where } \delta z \neq 0$$ is

called an analytic function (or) regular function (or) holomorphic function.

Cauchy Reimann equation in cartesian form:

Statement:

A necessary condition that a complex valued function $w = f(z) = u(x,y) + iv(x,y)$ is analytic at the complex variable $z = x+iy$, then the first four first order partial derivative $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ can satisfy the following $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$

Proof:

Given $w = f(z) = u(x,y) + iv(x,y) \rightarrow (1)$

is analytic at $z = x+iy$

$\therefore w = f(z)$ is differentiable

$(1) \Rightarrow f(x+iy) = u(x,y) + iv(x,y) \rightarrow (2)$

Diff (2) partially w.r.t. x

$$(2) \Rightarrow f'(x+iy) \cdot \frac{\partial}{\partial x}(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow f'(x+iy)(1) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\boxed{\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \rightarrow (3)$$

$|||^{ly}$ diff ② partially w.r.t y

$$② \Rightarrow f'(x+iy)\cdot\frac{\partial}{\partial y}(x+iy) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

$$f'(x+iy)(0+i) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

$$f'(x+iy) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{i}{i}\frac{\partial v}{\partial y}$$

$$f'(x+iy) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = \frac{i}{i^2}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{\Rightarrow f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}} \rightarrow ④$$

From eqⁿ ③ and ④ we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\boxed{\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Cauchy Reimann equation in Polar form

Statement:

A necessary condition that $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic at $z = r.e^{i\theta}$, then the first four first order partial derivative $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ can satisfy the following

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

Proof:

Given $\omega = f(z) = u(r,\theta) + i\,v(r,\theta) \rightarrow (1)$ is analytic at $z = r\cdot e^{i\theta}$

$\therefore f(z)$ is differentiable at $z = r\cdot e^{i\theta}$

$(1) \Rightarrow f(r\cdot e^{i\theta}) = u(r,\theta) + i\,v(r,\theta) \rightarrow (2)$

differentiable (2) partially w.r.t 'r'

$$(2) \Rightarrow f'(r\cdot e^{i\theta})\cdot \frac{\partial}{\partial r}(r\cdot e^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

$$\Rightarrow f'(r\cdot e^{i\theta})\, e^{i\theta}(1) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

$$\Rightarrow f'(r\cdot e^{i\theta}) = \frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$\boxed{\Rightarrow f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)} \rightarrow (3)$$

|||ly diff (2) partially w.r.t. 'θ'

$$(2) \Rightarrow f'(r\cdot e^{i\theta})\cdot\frac{\partial}{\partial\theta}(r\cdot e^{i\theta}) = \frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}$$

$$\Rightarrow f'(r\cdot e^{i\theta})\, r\cdot i e^{i\theta} = \frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}$$

$$\Rightarrow f'(r\cdot e^{i\theta}) = \frac{1}{r\cdot i e^{i\theta}}\frac{\partial u}{\partial\theta} + \frac{i}{r\cdot i\cdot e^{i\theta}}\frac{\partial v}{\partial\theta}$$

$$\Rightarrow f'(r\cdot e^{i\theta}) = \frac{1}{e^{i\theta}}\left[\frac{1}{r}\,\frac{i}{i^2}\,\frac{\partial u}{\partial\theta} + \frac{1}{r}\frac{\partial v}{\partial\theta}\right]$$

$$\boxed{\Rightarrow f'(z) = e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial\theta} - i\frac{1}{r}\frac{\partial u}{\partial\theta}\right]} \rightarrow (4)$$

$\therefore$ From eqⁿ (3) and (4) we get.

$$\cancel{e^{-i\theta}}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \cancel{e^{-i\theta}}\left[\frac{1}{r}\frac{\partial v}{\partial\theta} - i\frac{1}{r}\frac{\partial u}{\partial\theta}\right]$$

$$\therefore \frac{\partial u}{\partial r} + i\frac{\partial V}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\boxed{\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial \theta}, \quad \frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

Derivative of analytic function

1. Cartesian form:

a) Write the given complex valued function as $\omega = f(z)$ and let the complex variable in the cartesian form $z = x+iy$, then the given complex valued function can be written as $f(z) = u(x,y) + iV(x,y)$.

b) Verify the CR equation in the cartesian form, they are $\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y}, \frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$, hence we say that the given $f(z)$ is analytic function.

c) We know that $f'(z) = u_x + iV_x$ and substitute $z = x+iy$ in $f'(z)$ finally.

2. Polar form:

a) Let $\omega = f(z)$ be a complex valued function in the polar form, let $z = r \cdot e^{i\theta}$, then $\omega = f(z)$ can be written as $f(z) = u(r,\theta) + iV(r,\theta)$

b) Verify the CR equation in the polar form i.e, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial \theta}, \frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$, hence the given $f(z)$ is called analytic function.

c) Find $f'(z)$ by using $f(z) = e^{-i\theta}(u_r + iV_r)$ and replace $z = r \cdot e^{i\theta}$ finally.

Problems :

1) Show that $f(z)=e^z$ is analytic function and hence find its derivative.

Given,

$$f(z)=e^z \rightarrow (1)$$

Let $z=x+iy$

$$(1) \Rightarrow f(z)=e^{x+iy}$$

$$=e^x \cdot e^{iy}$$

$$=e^x(\cos y+i\sin y)$$

$$=e^x\cos y+i \cdot e^x \sin y$$

$$\Rightarrow f(z)=u(x,y)+iv(x,y)$$

$u=e^x\cos y \qquad v=e^x \sin y$

$$\frac{\partial u}{\partial x}=e^x\cos y \qquad \frac{\partial v}{\partial x}=e^x\sin y$$

$$\frac{\partial u}{\partial y}=-e^x\sin y \qquad \frac{\partial v}{\partial y}=e^x\cos y$$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \; ; \; \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

$\therefore \omega=f(z)=u(x,y)+iv(x,y)$ is analytic

W.K.T $\quad f'(z)=u_x+iv_x$

$$=e^x\cos y+ie^x\sin y$$

$$=e^x(\cos y+i\sin y)$$

$$=e^x \cdot e^{iy}$$

$$=e^{x+iy}$$

$$\boxed{\Rightarrow f'(z)=e^z}$$

2) Verify $f(z)=z^2$ is analytic or not, hence find $f'(z)$.

Given,

$$f(z)=z^2 \longrightarrow ①$$

Let $z=x+iy$

$$① \Rightarrow f(z)=(x+iy)^2$$

$$=x^2+(iy)^2+2(x)(iy)$$

$$=x^2+i^2y^2+2(x)(iy)$$

$$=x^2-y^2+i\,2xy$$

$$f(z)=u(x,y)+iv(x,y)$$

$u=x^2-y^2$ $\quad$ $v=2xy$

$\frac{\partial u}{\partial x}=2x$ $\quad$ $\frac{\partial v}{\partial x}=2y$

$\frac{\partial u}{\partial y}=-2y$ $\quad$ $\frac{\partial v}{\partial y}=2x$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

$\therefore$ $W=f(z)=u(x,y)+iv(x,y)$ is analytic

W.K.T $\quad f'(z)=u_x+iv_x$

$$=2x+i\,2y$$

$$=2(x+iy)$$

$$\boxed{\Rightarrow f'(z)=2z}$$

3) Verify $f(z)=z^3$ is analytic or not, hence find $f'(z)$.

Given,

$$f(z)=z^3 \longrightarrow ①$$

Let $z=x+iy$

$$① \Rightarrow f(z)=(x+iy)^3$$

$$f(z) = x^3 + (iy)^3 + 3(x^2)(iy) + 3(x)(iy)^2$$

$$f(z) = x^3 + i \cdot i^2 y^3 + 3ix^2y + 3 \cdot x \cdot i^2 y^2$$

$$f(z) = x^3 - iy^3 + 3ix^2y - 3xy^2$$

$$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$$

$$w = f(z) = u(x,y) + iv(x,y)$$

$\therefore \quad u = x^3 - 3xy^2 \qquad v = 3x^2y - y^3$

$\dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \dfrac{\partial v}{\partial x} = 6xy$

$\dfrac{\partial u}{\partial y} = -6xy \qquad \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\therefore \quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

$\therefore w = f(z) = z^3$ is analytic

W.K.T $\quad f'(z) = u_x + iv_x$

$$= 3x^2 - 3y^2 + i6xy$$

$$= 3(x^2 - y^2 + 2ixy)$$

$$f'(z) = 3(x^2 + iy^2 + 2(x)(iy))$$

$$f'(z) = 3(x+iy)$$

$$\boxed{f'(z) = 3z^2}$$

4) Show that $f(z) = \sin z$ is analytic, hence find its derivative.

Given, $\qquad f(z) = \sin z \rightarrow (1)$

$$z = x + iy$$

$$f(z) = \sin(x+iy)$$

$f(z) = \sin x \cdot \cos (iy) + \cos x \cdot \sin (iy)$

$f(z) = \sin x \cdot \cosh y + i \cos x \cdot \sinh y$

$f(z) = u(x,y) + iv(x,y)$

$\therefore \; u = \sin x \cdot \cosh y \qquad v = \cos x \cdot \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cdot \cosh y \qquad \frac{\partial v}{\partial x} = -\sin x \cdot \sinh y$

$\frac{\partial u}{\partial y} = \sin x \cdot \sinh y \qquad \frac{\partial v}{\partial y} = \cos x \cdot \cosh y$

$\therefore \; \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore \; w = f(z) = \sin z$ is analytic

W.K.T $\quad f'(z) = u_x + i v_x$

$= \cos x \cdot \cosh y + i(-\sin x \sinh y)$

$= \cos x \cdot \cos(iy) - \sin x . \sin(iy)$

$f'(z) = \cos(x+iy)$

$\boxed{f'(z) = \cos z}$

5) Show that $f(z) = \cosh z$ is analytic, hence find its derivative.

Given, $\quad f(z) = \cosh z \rightarrow (1)$

Let $z = x+iy$

$(1) \Rightarrow f(z) = \cosh(x+iy) \qquad (\cosh x = \cos ix)$

$f(z) = \cos i(x+iy)$

$f(z) = \cos(ix + i^2 y)$

$f(z) = \cos(ix - y)$

$$f(z) = \cos ix \cdot \cos y + \sin ix \cdot \sin y$$
$$= \cosh x \cdot \cos y + i \sinh x \cdot \sin y$$
$$\Rightarrow f(z) = u(x,y) + iv(x,y)$$

$u = \cosh x \cdot \cos y$ $\qquad$ $v = \sinh x \cdot \sin y$

$\frac{\partial u}{\partial x} = \sinh x \cdot \cos y$ $\qquad$ $\frac{\partial v}{\partial x} = \cosh x \cdot \sin y$

$\frac{\partial u}{\partial y} = -\cosh x \cdot \sin y$ $\qquad$ $\frac{\partial v}{\partial y} = \sinh x \cdot \cos y$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore f(z) = \cosh z$ is analytic

W.K.T $\quad f'(z) = u_x + i v_x$

$$= \sinh x \cdot \cos y + i(\cosh x \cdot \sin y)$$
$$f'(z) = \frac{1}{i}(\sin ix \cos y + i^2 \cos ix \cdot \sin y)$$
$$f'(z) = \frac{1}{i}(\sin ix \cos y - \cos ix \cdot \sin y)$$
$$f'(z) = \frac{1}{i}\sin(ix - y)$$
$$f'(z) = \frac{1}{i}\sin\left[i\left(x - \frac{y}{i}\right)\right]$$
$$f'(z) = \sinh(x + iy)$$
$$\boxed{f'(z) = \sinh z}$$

6) Show that $f(z) = \log z$ is analytic, hence find its derivative

Given, $\quad f(z) = \log z \rightarrow (1)$

let $z = r \cdot e^{i\theta}$

$$f(z) = \log(r \cdot e^{i\theta})$$

$$= \log r + \log . e^{i\theta}$$

$$= \log r + i.\theta \log_e^e$$

$$f(z) = \log r + i \cdot \theta$$

$$f(z) = u(r,\theta) + iV(r,\theta)$$

$u = \log r$ $\qquad\qquad$ $V = \theta$

$\dfrac{\partial u}{\partial r} = \dfrac{1}{r}$ $\qquad\qquad$ $\dfrac{\partial V}{\partial r} = 0$

$\dfrac{\partial u}{\partial \theta} = 0$ $\qquad\qquad$ $\dfrac{\partial V}{\partial \theta} = 1$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial \theta}, \quad \frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$\therefore$ $f(z) = \log z$ is analytic

W.K.T. $f'(z) = e^{-i\theta}(u_r + iV_r)$

$$= e^{-i\theta}\left(\frac{1}{r} + i(0)\right)$$

$$= \frac{1}{r \cdot e^{i\theta}}$$

$$\boxed{f'(z) = \frac{1}{z}}$$

7) Show that $w = f(z) = z + e^z$ is analytic and hence find $\dfrac{dw}{dz}$.

Given,

$$w = f(z) = z + e^z \rightarrow (1)$$

Let $z = x + iy$

$$(1) \Rightarrow f(z) = (x+iy) + e^{(x+iy)}$$

$$f(z) = x + iy + e^x . e^{iy}$$

$$f(z) = x + iy + e^x(\cos y + i\sin y)$$

$$f(z) = u(x,y) + iv(x,y)$$

$$f(z) = (x + e^{x}\cos y) + i(y + e^{x}\sin y)$$

$u = x + e^{x}\cos y$ $\qquad$ $v = y + e^{x}\sin y$

$\dfrac{\partial u}{\partial x} = 1 + e^{x}\cos y$ $\qquad$ $\dfrac{\partial v}{\partial y} = 1 + e^{x}\cos y$

$\dfrac{\partial u}{\partial y} = -e^{x}\sin y$ $\qquad$ $\dfrac{\partial v}{\partial x} = e^{x}\sin y$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Hence the function is analytic

W.K.T $\quad f'(z) = u_x + iv_x$

$$= 1 + e^{x}\cos y + i.e^{x}\sin y$$

$$= 1 + e^{x}(\cos y + i\sin y)$$

$$= 1 + e^{x}.e^{iy}$$

$$= 1 + e^{x+iy}$$

$$\boxed{f'(z) = 1 + e^{z}}$$

8) Verify $f(z) = \sin 2z$ is analytic or not, hence find its derivative.

Given, $\quad f(z) = \sin 2z \rightarrow (1)$

Let $z = x + iy$

$(1) \Rightarrow f(z) = \sin 2(x+iy)$

$$= \sin(2x + 2iy)$$

$$f(z) = \sin 2x . \cos(2iy) + \cos(2x).\sin(2iy)$$

$$f(z) = \sin(2x).\cosh(2y) + \cos 2x . \sinh(2y)$$

$$f(z) = u(x,y) + iv(x,y)$$

$u = \sin(2x).\cosh(2y)$ $\qquad$ $v = \cos 2x . \sinh(2y)$

$\frac{\partial u}{\partial x} = 2\cos(2x)\cdot\cosh(2y)$ $\quad$ $\frac{\partial v}{\partial x} = -2\sin(2x).\sinh(2y)$

$\frac{\partial u}{\partial y} = 2\sin(2x)\cdot\sinh(2y)$ $\quad$ $\frac{\partial v}{\partial y} = 2\cos(2x)\cdot\cosh(2y)$

$\therefore$ f(z) is an analytic function

$$f'(z) = u_x + iv_x$$
$$= 2\cos(2x)\cdot\cosh(2y) - i2\sin(2x)\cdot\sinh(2y)$$
$$f'(z) = 2\cos 2x\cdot\cos 2(iy) - i\,2\sin 2x\cdot\sin 2(iy)$$
$$f'(z) = 2\,[\cos 2x\cdot\cos(2iy) - \sin(2x)\cdot\sin(2iy)]$$
$$f'(z) = 2\cos(2x+2iy)$$
$$f'(z) = 2\cos 2(x+iy)$$
$$\boxed{f'(z) = 2\cos 2z}$$

9) Verify $f(z) = z^n$ is an analytic function or not, hence find its derivative for any positive integer 'n'.

Given, $\quad f(z) = z^n \rightarrow (1)$

Let $z = r\cdot e^{i\theta}$

$$(1) \Rightarrow f(z) = (r\cdot e^{i\theta})^n$$
$$= r^n\cdot e^{ni\theta}$$
$$f(z) = r^n(\cos n\theta + i\sin n\theta)$$
$$f(z) = r^n\cos n\theta + i r^n\sin n\theta$$
$$f(z) = u(r,\theta) + iv(r,\theta)$$

$u = r^n\cos n\theta$ $\quad$ $v = r^n\sin n\theta$

$\frac{\partial u}{\partial r} = n.r^{n-1}\cos n\theta$ $\quad$ $\frac{\partial v}{\partial r} = n.r^{n-1}\sin n\theta$

$\frac{\partial u}{\partial \theta} = -n.r^n\sin n\theta$ $\quad$ $\frac{\partial v}{\partial \theta} = n\cdot r^n\cos n\theta$

$\therefore \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ $\qquad \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

$\therefore f(z) = z^n$ is a analytic function

$\therefore$ W.K.T, $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$f'(z) = e^{-i\theta}(n.r^{n-1}\cos n\theta + i.n.r^{n-1}\sin n\theta)$$

Let $r = z$ $\theta = 0$

$$\boxed{f'(z) = n.z^{n-1}}$$

Construction of analytic function by Milne Thomson method.

1. Consider $w = f(z) = u + iv$ be the analytic function to be Constructed

2. Find u_x, u_y or v_x, v_y in the cartesian form or find u_r, u_θ or v_r, v_θ in the polar form.

3. Write $f'(z) = u_x + iv_x$ in Cartesian form and $f'(z) = e^{-i\theta}(u_r + iv_r)$ in polar form and simplify the same by using CR equation, also substitute $x = z, y = 0$ in Cartesian form, and $r = z, \theta = 0$ in polar form in $f'(z)$.

4. Integrate $f'(z)$ w.r.t. z on both sides and get $f(z) = u + iv$ in terms of z and it is called the required analytic function.

Problems:

1) Find the analytic function $f(z) = u + iv$, whose imaginary part $v = e^x(x\sin y + y\cos y)$

Let $w = f(z) = u + iv$ be the analytic function to be Constructed.

Given, $V = e^{x}(x\sin y + y\cos y)$

$$\frac{\partial v}{\partial x} = e^{x}(x\sin y + y\cos y) + e^{x}(1\sin y + 0)$$

$$\frac{\partial v}{\partial x} = e^{x}(x\sin y + y\cos y) + e^{x}(\sin y)$$

when $x = z,\ y = 0$

$$\frac{\partial v}{\partial x} = e^{z}(0+0) + e^{z}(0)$$

$$\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial y} = 0(x\sin y + y\cos y) + e^{x}(x\cos y + 1\cdot\cos y + y(-\sin y))$$

$$\frac{\partial v}{\partial y} = e^{x}(x\cos y + \cos y - y\sin y)$$

when $x = z\quad y = 0$

$$\frac{\partial v}{\partial y} = e^{z}(z\cos 0 + \cos 0 - 0)$$

$$\frac{\partial v}{\partial y} = e^{z}(z+1)$$

$$\frac{\partial v}{\partial y} = z\cdot e^{z} + e^{z}$$

$$f'(z) = u_x + iV_x$$

But $u_x = V_y$

$$f'(z) = V_y + iV_x$$

$$= z\cdot e^{z} + e^{z} + i(0)$$

$$f'(z) = ze^{z} + e^{z}$$

$$\frac{df}{dz} = ze^{z} + e^{z}$$

$$df = ze^{z} + e^{z}\, dz$$

$$\int df = \int z e^z dz + \int e^z dz$$

$$f(z) = (z-1)e^z + e^z + C$$

$$f(z) = ze^z - e^z + e^z + C$$

$\boxed{f(z) = z \cdot e^z + C}$ is the required analytic function

2. Find the analytic function $f(z) = u + iv$, whose real part is $\dfrac{x^4 - y^4 - 2x}{x^2 + y^2}$

Let $w = f(z) = u + iv$ be the analytic function to be constructed.

Given, $$u = \frac{x^4 - y^4 - 2x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(4x^3 - 2) - (x^4 - y^4 - 2x)(2x)}{(x^2+y^2)^2}$$

when $x = z$, $y = 0$

$$\frac{\partial u}{\partial x} = \frac{z^2(4z^3 - 2) - 2z(z^4 - 2z)}{z^4}$$

$$\frac{\partial u}{\partial x} = \frac{4z^5 - 2z^2 - 2z^5 + 4z^2}{z^4}$$

$$\frac{\partial u}{\partial x} = \frac{2z^5 + 2z^2}{z^4}$$

$$\frac{\partial u}{\partial x} = 2z + \frac{2}{z^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(-4y^3) - (x^4 - y^4 - 2x)(2y)}{(x^2+y^2)^2}$$

when $x = z,\ y = 0$

$$\frac{\partial u}{\partial y} = 0$$

$\therefore\ f'(z) = u_x + i v_x$

But $v_x = -u_y$

$$f'(z) = u_x - i v_y$$

$$= 2z + \frac{2}{z^2} - i(0)$$

$$\frac{df}{dz} = 2z + \frac{2}{z^2}$$

$$\int df = 2\int z\,dz + 2\int \frac{1}{z^2}\,dz$$

$$f(z) = 2\cdot\frac{z^2}{2} + 2\left(-\frac{1}{z}\right) + C$$

$$\boxed{f(z) = z^2 - \frac{2}{z} + C}$$

3) Find the analytic function $f(z) = u + iv$ whose real part is $u = \dfrac{x^4 - y^4 - x}{x^2 + y^2}$

Given,

$$u = \frac{x^4 - y^4 - x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(4x^3-1) - (x^4-y^4-x)(2x)}{(x^2+y^2)^2}$$

when $x = z,\ y = 0$

$$\frac{\partial u}{\partial x} = \frac{z^2(4z^3-1) - (z^4-z)(2z)}{z^4}$$

$$\frac{\partial u}{\partial x} = \frac{4z^5 - z^2 - 2z^5 + 2z^2}{z^4}$$

$$\frac{\partial u}{\partial x} = \frac{2z^5 + z^2}{z^4}$$

$$\frac{\partial u}{\partial x} = 2z + \frac{1}{z^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(-4y^3) - (x^4-y^4-x)(2y)}{(x^2+y^2)^2}$$

when $x = z,\ y = 0$

$$\frac{\partial u}{\partial y} = 0$$

$$f'(z) = u_x + i v_x$$

$$f'(z) = u_x - i u_y$$

$(\because\ v_x = -u_y)$

$$f'(z) = 2z + \frac{1}{z^2} - i(0)$$

$$\frac{df}{dz} = 2z + \frac{1}{z^2}$$

$$\int df = 2\int z\,dz + \int \frac{1}{z^2}\,dz$$

$$f(z) = 2\cdot\frac{z^2}{2} + \left(-\frac{1}{z}\right) + c$$

$$\boxed{f(z) = z^2 - \frac{1}{z} + c}$$

4) Find the analytic function $f(z) = u + iv$ whose real part $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Let $w = f(z) = u + iv$ be the analytic function to be constructed

Given, $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(0 + 2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2x\,(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

when $x = z$, $y = 0$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2z\,(1 - \cos 2z) - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1 - \cos 2z)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$\frac{\partial u}{\partial x} = \frac{-2}{1 - \cos 2z}$$

$$\frac{\partial u}{\partial x} = \frac{-2}{2\sin^2 z}$$

$$\frac{\partial u}{\partial x} = -\operatorname{cosec}^2 z$$

$$\frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

when $x = z,\ y = 0$

$$\frac{\partial u}{\partial y} = 0$$

$\therefore f'(z) = u_x + i v_x$

But $v_x = -u_y$

$$f'(z) = u_x - i u_y$$

$$f'(z) = -\operatorname{cosec}^2 z - i(0)$$

$$\frac{df}{dz} = -\operatorname{cosec}^2 z$$

$$\int df = \int -\operatorname{cosec}^2 z \, dz$$

$$\boxed{f(z) = \cot z + C}$$

5) Find the analytic function $f(z) = u + iv$ given $v = e^{-x}(x\cos y + y\sin y)$

Let $w = f(z) = u + iv$ be the analytic function to be constructed.

Given, $v = e^{-x}(x\cos y + y\sin y)$

$$\frac{\partial v}{\partial x} = -e^{-x}(x\cos y + y\sin y) + e^{-x}(1\cos y + 0)$$

$$\frac{\partial v}{\partial x} = -e^{-x}(x\cos y + y\sin y) + e^{-x}(\cos y)$$

when $x = z,\ y = 0$

$$\frac{\partial v}{\partial x} = -e^{-z}(z(1)+0)+e^{-z}(1)$$

$$\frac{\partial v}{\partial x} = -ze^{-z}+e^{-z}$$

$$\therefore \frac{\partial v}{\partial y} = 0\,(x\cos y + y\sin y) + e^{-x}(-x\sin y + 1(\sin y) + y\cos y)$$

$$\frac{\partial v}{\partial y} = e^{-x}(-x\sin y + \sin y + y\cos y)$$

when $x = z,\ y = 0$

$$\frac{\partial v}{\partial y} = 0$$

$$\therefore \quad f'(z) = u_x + iv_x$$

But $u_x = v_y$

$$f'(z) = v_y + iv_x$$

$$= 0 + i(-ze^{-z}+e^{-z})$$

$$f'(z) = i(e^{-z} - ze^{-z})$$

$$\int df = i\int e^{-z}\,dz - \int z \cdot e^{-z}\,dz$$

$$f(z) = i \cdot \frac{e^{-z}}{-1} - (z+1)e^{-z}$$

$$\boxed{f(z) = i\left[-e^{-z} - (z+1)e^{-z}\right]}$$

6) Find the analytic function $w = f(z) = u+iv$, whose real part is $u = e^{2x}(x\cos 2y - y\sin 2y)$

Given,

$$u = e^{2x}(x\cos 2y - y\sin 2y) \rightarrow (1)$$

$$\frac{\partial u}{\partial x} = 2e^{2x}(x\cos 2y - y\sin 2y) + e^{2x}(\cos 2y - 0)$$

$$\frac{\partial u}{\partial x} = 2e^{2x}(x\cos y - y\sin 2y) + e^{2x}(\cos 2y)$$

when $x = z,\ y = 0$

$$\frac{\partial u}{\partial x} = 2e^{2z}(z-0) + e^{2z}(1)$$

$$\frac{\partial u}{\partial x} = 2.e^{2z}.z + e^{2z}$$

$$\frac{\partial u}{\partial y} = e^{2x}(-2x\sin 2y - 1.\sin 2y - 2y\cos 2y)$$

when $x = z, y = 0$

$$\frac{\partial u}{\partial y} = 0$$

W.K.T $\quad f'(z) = u_x + iV_x$

But $V_x = -u_y$

$$f'(z) = u_x - iu_y$$
$$= 2e^{2z}.z + e^{2z} - i(0)$$
$$f'(z) = e^{2z}(2z+1)$$
$$\int df = \int e^{2z}(2z+1)\,dz$$
$$= (2z+1)\int e^{2z}\,dz - \int\left(2.\int e^{2z}\,dz\right).dz$$
$$f(z) = z.e^{2z} + \frac{1}{2}e^{2z} - \frac{1}{2}e^{2z} + c$$
$$\boxed{f(z) = z.e^{2z} + c}$$

7) Find the analytic function whose real part is

$$u = \frac{x^4.y^4 - 2x}{x^2+y^2}$$

Given,

$$u = \frac{x^4.y^4 - 2x}{x^2+y^2} \rightarrow (1)$$

diff (1) partially w.r.t 'x'

$$(1) \Rightarrow \frac{\partial u}{\partial x} = \frac{(x^2+y^2)(4x^3y^4 - 2) - (x^4y^4 - 2x)(2x)}{(x^2+y^2)^2}$$

when $x = z, y = 0$

$$\frac{\partial u}{\partial x} = \frac{(z^2+0)\ (4z^3(0)-2) - (z^4(0)-2(z))(2z)}{z^4}$$

$$\frac{\partial u}{\partial x} = \frac{z^2(-2)+4z^2}{z^4}$$

$$\frac{\partial u}{\partial x} = \frac{-2z^2+4z^2}{z^4}$$

$$\frac{\partial u}{\partial x} = \frac{2}{z^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)\ 4x^4 \cdot y^3 - (x^4 \cdot y^4 - 2x)\ (2y)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y\ (x^4 \cdot y^4 - 2x) - (x^2+y^2)(4x^4 y^3)}{(x^2+y^2)^2}$$

when $x = z, y = 0$

$$\frac{\partial u}{\partial y} = 0$$

$$f'(z) = u_x + i V_x$$

$$V_x = -u_y$$

$$f'(z) = \frac{2}{z^2} - i(0)$$

$$f'(z) = \frac{2}{z^2}$$

$$\int df = 2\int \frac{1}{z^2}\, dz$$

$$\boxed{f(z) = -\frac{2}{z} + c}$$

8) Find the analytic function $f(z) = u + iv$, whose imaginary part is $\dfrac{y}{x^2+y^2}$.

Given,

$$V = \frac{y}{x^2+y^2}$$

$$V_x = \frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2}$$

$$V_x = \frac{-2xy}{(x^2+y^2)^2}$$

when $x = z, y = 0$

$$V_x = 0$$

$$\frac{\partial V}{\partial y} = \frac{(x^2+y^2)\,1 - y(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial V}{\partial y} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial V}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

when $x = z, y = 0$

$$\frac{\partial V}{\partial y} = \frac{z^2}{z^4} = \frac{1}{z^2}$$

W.K.T $\quad f'(z) = u_x + iV_x$

But $u_x = V_y$

$$f'(z) = V_y + iV_x$$

$$f'(z) = \frac{1}{z^2} + i(0)$$

$$\int df = \int \frac{1}{z^2}\, dz$$

$$\boxed{f(z) = -\frac{1}{z} + C}$$

9) Construct the analytic function, whose real part is $\dfrac{\sin 2x}{\cosh 2y+\cos 2x}$

Given,
$$u=\frac{\sin 2x}{\cosh 2y+\cos 2x} \rightarrow (1)$$

$$\frac{\partial u}{\partial x}=\frac{(\cosh 2y+\cos 2x)(2\cos 2x)-\sin 2x(0+(-2\sin 2x))}{(\cosh 2y+\cos 2x)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2\cos 2x(\cosh 2y+\cos 2x)+2\sin^2 2x}{(\cosh 2y+\cos 2x)^2}$$

when $x=z, y=0$

$$\frac{\partial u}{\partial x}=\frac{2\cos 2z(1+\cos 2z)+2\sin^2 2z}{(1+\cos 2z)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2\cos 2z+2\cos^2 2z+2\sin^2 2z}{(1+\cos 2z)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2\cos 2z+2(\cos^2 2z+\sin^2 2z)}{(1+\cos 2z)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2\cos 2z+2}{(1+\cos 2z)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2(1+\cos 2z)}{(1+\cos 2z)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2}{1+\cos 2z}$$

$$\frac{\partial u}{\partial y}=\frac{(\cosh 2y+\cos 2x)(0)-\sin 2x(2\sinh 2y+0)}{(\cosh 2y+\cos 2x)^2}$$

when $x = z,\ y = 0$

$$\frac{\partial u}{\partial y} = 0$$

W.K.T $\quad f'(z) = u_x + i v_x$

$$v_x = -u_y$$

$$f'(z) = u_x - i u_y$$

$$f'(z) = \frac{2}{1+\cos 2x} - i(0)$$

$$f'(z) = \frac{2}{2\cos^2 z}$$

$$\int df = \int \frac{1}{\cos^2 z}\, dz$$

$$\boxed{f(z) = \tan z + c}$$

10) Find the analytic function $f(z) = u + iv$ whose imaginary part is $v = \left(r - \frac{1}{r}\right)\sin\theta,\ r \neq 0$.

Given,

$$v = \left(r - \frac{1}{r}\right)\sin\theta, \quad r \neq 0$$

$$\frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right)\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta$$

W.K.T $\quad f'(z) = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}\left(\frac{1}{r} v_\theta + i v_r\right)$$

$$= e^{-i\theta}\left(\frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta + i\left(1 + \frac{1}{r^2}\right)\sin\theta\right)$$

when $z = r,\ \theta = 0$

$$f'(z) = \left[\frac{1}{z}\left(z - \frac{1}{z}\right)\right]$$

$$f'(z) = 1 - \frac{1}{z^2}$$

$$\int df = \int 1 - \frac{1}{z^2}\, dz$$

$$\boxed{f(z) = z + \frac{1}{z} + c}$$

11) Find the analytic function $f(z) = u + iv$, where $u - v = (x-y)(x^2+4xy+y^2)$

Let $w = f(z) = u + iv$ be the analytic function to be constructed.

Given, $u - v = (x-y)(x^2+4xy+y^2)$

$$u - v = x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3$$

$$u - v = x^3 + 3x^2y - 3xy^2 - y^3 \rightarrow (1)$$

Diff (1) partially w.r.t. 'x' on both sides

$$(1) \Rightarrow u_x - v_x = 3x^2 + 6xy - 3y^2$$

$\therefore$ when $x = z, y = 0$

$$u_x - v_x = 3z^2 \rightarrow (2)$$

Diff (1) partially w.r.t. 'y'

$$(1) \Rightarrow u_y - v_y = 3x^2 - 6xy - 3y^2$$

when $x = z, y = 0$

$$u_y - v_y = 3z^2 \rightarrow (3)$$

W.K.T $u_y = -v_x$

$v_y = u_x$

$$(3) \Rightarrow -v_x - u_x = 3z^2$$

$$u_x + v_x = -3z^2 \rightarrow (4)$$

$$(2) + (4) \Rightarrow 2u_x = 0$$

$$u_x = 0$$

② - ④ $\Rightarrow$ $-2V_x = 6z^2$

$V_x = -3z^2$

$\therefore f'(z) = u_x + iV_x$

$f'(z) = 0 + i(-3z^2)$

$\frac{df}{dz} = -3iz^2$

$\int df = -3i\int z^2\,dz$

$\boxed{f(z) = -iz^3 + c}$

12) Find the analytic function $f(z) = u + iv$, where $u - v = e^x(\cos y - \sin y)$

Given, $f(z) = u + iv \rightarrow ①$

$u - v = e^x(\cos y - \sin y) \rightarrow ②$

Diff ② partially w.r.t. x

② $\Rightarrow u_x - v_x = e^x(\cos y - \sin y) \rightarrow ③$

Diff ② partially w.r.t. y

② $\Rightarrow u_y - v_y = -e^x(\cos y + \sin y) \rightarrow ④$

W.K.T $u_y = -v_x$, $v_y = u_x$

④ $\Rightarrow -v_x - u_x = -e^x(\cos y + \sin y) \rightarrow ⑤$

③ + ⑤ $\Rightarrow$

$u_x - v_x - v_x - u_x = e^x(\cos y - \sin y) - e^x(\cos y + \sin y)$

$-2v_x = e^x(\cos y - \sin y - \cos y + \sin y)$

$v_x = 0$

③ - ⑤ $\Rightarrow$

$2u_x = e^x(\cos y - \sin y + \cos y + \sin y)$

$2u_x = 2e^x\cos y$

$u_x = e^x\cos y$

when $x = z,\ y = 0$

$u_x = e^z$

W.K.T, $f'(z) = u_x + iv_x$

$f'(z) = e^z + i(0)$

$\int df = \int e^z\, dz$

$$\boxed{f(z) = e^z + c}$$

13) Find the analytic function $w = f(z) = u(x,y) + iv(x,y)$ given $u + v = (x+y) + e^x(\cos y + \sin y)$

Given,

$u + v = (x+y) + e^x(\cos y + \sin y) \rightarrow (1)$

Diff eq^n (1) partially w.r.t. 'x'

$u_x + v_x = 1 + e^x(\cos y + \sin y)$

when $x = z,\ y = 0$

$u_x + v_x = 1 + e^z \rightarrow (2)$

Diff eq^n (1) partially w.r.t. 'y'

$u_y + v_y = 1 + e^x(-\sin y + \cos y)$

when $x = z,\ y = 0$

$u_y + v_y = 1 + e^z \rightarrow (3)$

But $u_y = -v_x,\ v_y = u_x$

(3) $\Rightarrow$ $-v_x + u_x = 1 + e^z \rightarrow (4)$

(2) + (4) $\Rightarrow 2u_x = 2 + 2e^z$

$u_x = 1 + e^z$

(2) − (4) $\Rightarrow 2v_x = 0$

$v_x = 0$

$f'(z) = u_x + iv_x$

$\int df = \int 1 + e^z\, dz$

$$\boxed{f(z) = (z + e^z) + c}$$

Harmonic Property for the Complex valued Function

1) Let $\phi(x,y)$ be the real valued function in the Cartesian form then it is said to be a harmonic when it can satisfy the laplace in equation $\nabla^2\phi = 0$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

2) Suppose $\phi(r,\theta)$ be the real valued function in the polar form and it is said to be harmonic when it can satisfy the equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Theorem:

1) Show that the analytic function $f(z) = u(x,y) + iv(x,y)$ in the Cartesian form whose real and imaginary parts are harmonic

Sol^n Given, $f(z) = u(x,y) + iv(x,y)$ be an analytic function with $u(x,y)$, $v(x,y)$ are the real and imaginary parts

Case ①: W.K.T the CR equation for analytic function $f(z) = u + iv$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow ①$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \rightarrow ②$$

Diff eq^n ① partially w.r.t 'x'

$$① \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \rightarrow ③$$

Diff eqⁿ ② partially w.r.t. 'y'

$$② \Rightarrow \frac{\partial^2 v}{\partial x} = \frac{-\partial^2 u}{\partial y^2}$$

$$\Rightarrow -\frac{\partial^2 v}{\partial y^2} = \frac{\partial^3 v}{\partial x \partial y} \rightarrow ④$$

From ③ and ④

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$\therefore u(x,y)$ is harmonic

Case ② : diff eqⁿ ① w.r.t. 'y'

$$① \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \rightarrow ⑤$$

diff eqⁿ ② w.r.t. 'x'

$$② \Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{-\partial^2 u}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \rightarrow ⑥$$

From eqⁿ ⑤ and ⑥

$$\Rightarrow \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0}$$

$\therefore v(x,y)$ is harmonic

Theorem:

2) Show the real & imaginary parts are analytic function $w = f(z) = u(r,\theta) + iv(r,\theta)$ are harmonic.

Sol^n

Given, $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic

W.K.T

The CR equation $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \rightarrow (1)$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \rightarrow (2)$$

diff eq^n (1) partially w.r.t. 'r'

$$(1) \Rightarrow \frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} \rightarrow (3)$$

diff eq^n (2) partially w.r.t. 'θ'

$$(2) \Rightarrow \frac{\partial^2 v}{\partial r \partial \theta} = -\left[\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}\right]$$

$$\frac{\partial^2 v}{\partial r \partial \theta} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \rightarrow (4)$$

substitute eq^n (4) in eq^n (3)

$$(3) \Rightarrow \frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right) + \frac{1}{r}\left[-\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}\right]$$

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\left(r \cdot \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$\therefore u(r,\theta)$ is harmonic

III^{ly} differentiate eq^n (1) w.r.t. 'θ'

$$(1) \Rightarrow \frac{\partial^2 u}{\partial r \partial \theta} = \frac{1}{r}\frac{\partial^2 v}{\partial \theta^2} \rightarrow (5)$$

diff eqⁿ ② w.r.t. 'r'

$$② \Rightarrow \frac{\partial^2 V}{\partial r^2} = -\left[-\frac{1}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta \partial r}\right]$$

$$\Rightarrow \frac{\partial^2 V}{\partial r^2} = \frac{1}{r^2}\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\partial^2 u}{\partial \theta \partial r} \rightarrow ⑥$$

substitute eqⁿ ⑤ in eqⁿ ⑥

$$⑥ \Rightarrow \frac{\partial^2 V}{\partial r^2} = \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right) - \frac{1}{r}\left[\frac{1}{r}\frac{\partial^2 V}{\partial \theta^2}\right]$$

$$\Rightarrow \frac{\partial^2 V}{\partial r^2} = \frac{1}{r^2}\left(-r \cdot \frac{\partial V}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$

$$\Rightarrow \frac{\partial^2 V}{\partial r^2} = -\frac{1}{r}\left(\frac{\partial V}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$

$$\boxed{\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0}$$

$\therefore$ $V(r,\theta)$ is harmonic

Theorem:-

If $w = f(z)$ is regular function of z, show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(f(|z|^2)) = 4(f'|z|^2)$$

Given, $f(z) = u(x,y) + iV(x,y) = u + iV$ is regular

$\Rightarrow f(z)$ is differentiable at any point of $z = x+iy$

$\Rightarrow f'(z) = u_x + iV_x$

and WKT, $f|z| = \sqrt{u^2+V^2}$

$$f(|z|^2) = u^2 + V^2 = \phi \rightarrow ①$$

$$\therefore f'|z| = \sqrt{u_x^2 + V_x^2}$$

$$f'(|z|^2) = u_x^2 + V_x^2 \longrightarrow ②$$

$$LHS = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot f(|z|^2)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot \phi$$

$$LHS = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$\therefore \frac{\partial \phi}{\partial x} = 2u \cdot u_x + 2V \cdot V_x$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2\left(u_x \cdot u_x + u \cdot U_{xx}\right) + 2\left(V_x \cdot V_x + V \cdot V_{xx}\right)$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = 2\left(u_x^2 + u \cdot U_{xx} + V_x^2 + V \cdot V_{xx}\right) \rightarrow (3)$$

$$\therefore \frac{\partial \phi}{\partial y} = 2\, U \cdot U_y + 2V \cdot V_y$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = 2 \cdot \left(U_y \cdot U_y + U \cdot U_{yy}\right) + 2\left(V_y \cdot V_y + V \cdot V_{yy}\right)$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2\left(U_y^2 + U \cdot U_{yy} + V_y^2 + V \cdot V_{yy}\right) \rightarrow (4)$$

From (3) & (4) (or) LHS (Consider)

$$LHS = 2\left(u_x^2 + u \cdot U_{xx} + V_x^2 + V \cdot V_{xx} + U_y^2 + U \cdot U_{yy} + V_y^2 + V \cdot V_{yy}\right)$$

$$= 2\left[U_x^2 + V_x^2 + U_y^2 + V_y^2 + u\left(u_{xx} + U_{yy}\right) + V\left(V_{xx} + V_{yy}\right)\right]$$

$$= 2\left[U_x^2 + V_x^2 + U_y^2 + V_y^2 + u(0) + V(0)\right]$$

$$= 2\left[U_x^2 + V_x^2 + (-V_x)^2 + (u_x)^2\right]$$

$$= 2\left(u_x^2 + V_x^2 + V_x^2 + u_x^2\right)$$

$$= 2\left[2\left(u_x^2 + V_x^2\right)\right]$$

$$= 4\left(u_x^2 + V_x^2\right)$$

$$= 4\left(|f'(z)|^2\right)$$

$$= RHS.$$

Theorem: If $\omega = f(z)$ is regular, then show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 = \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

Proof:

Given $\omega = f(z) = u + iv$ is regular or analytic

$\therefore f(z)$ is differentiable at any point of $z = x+iy$

$\Rightarrow f'(z) = u_x + i v_x$

and w.k.t $|f(z)| = \sqrt{u^2+v^2} = \phi$

$\Rightarrow u^2 + v^2 = \phi^2 \rightarrow (1)$

and $|f'(z)| = \sqrt{u_x^2 + v_x^2}$

$\Rightarrow |f'(z)|^2 = u_x^2 + v_x^2 \rightarrow (2)$

diff (1) w.r.t. 'x' partially

$\therefore (1) \Rightarrow 2u \cdot u_x + 2v \cdot v_x = 2\phi\phi_x$

$\Rightarrow u \cdot u_x + v \cdot v_x = \phi \cdot \phi_x \rightarrow (3)$

$|||^{ly}$ diff (1) w.r.t 'y' partially

$(1) \Rightarrow 2u u_y + 2v v_y = 2\phi\phi_y$

$u u_y + v v_y = \phi\phi_y \rightarrow (4)$

$\therefore$ function $f(z)$ is analytic

$\therefore$ WKT $u_x = v_y$, $v_x = -u_y$

$u_y = -v_x$

$(4) \Rightarrow -u \cdot v_x + v \cdot u_x = \phi\phi_y$

$v \cdot u_x - u \cdot v_x = \phi\phi_5 \rightarrow (5)$

$\therefore (3)^2 + (5)^2 \Rightarrow$

$\Rightarrow (u u_x + v \cdot v_x)^2 + (v u_x - u v_x)^2 = \phi^2 \phi_x^2 + \phi^2 \phi_y^2$

$\Rightarrow u^2 u_x^2 + v^2 \cdot v_x^2 + 2uv \cdot u_x v_x + v^2 u_x^2 + u^2 v_x^2 - 2uv u_x v_x = \phi^2\phi_x^2 + \phi^2\phi_y^2$

$\Rightarrow u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x^2) = \phi^2(\phi_x^2+\phi_y^2)$

$\Rightarrow (u^2+v^2)(u_x^2+v_x^2) = \phi^2(\phi_x^2+\phi_y^2)$

$\Rightarrow \cancel{\phi^2}(u_x^2+v_x^2) = \cancel{\phi^2}(\phi_x^2+\phi_y^2)$

$\Rightarrow \phi_x^2+\phi_y^2 = u_x^2+v_x^2$

$\Rightarrow \left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2 = |f'(z)|^2$

$\Rightarrow \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$

Theorem: If $f(z)$ is regular function with constant modulus. Show that $f(z)$ is also a constant function.

Given, $w=f(z)=u+iv$ is regular function

$\therefore f(z)$ is differentiable

$\therefore f'(z)=u_x+iv_x$

and $|f'(z)| = \sqrt{u_x^2+v_x^2}$

$|f'(z)|^2 = u_x^2+v_x^2 \rightarrow (1)$

and given $|f(z)|=k$

$\Rightarrow \sqrt{u^2+v^2}=k$

$\Rightarrow u^2+v^2=k^2 \rightarrow (2)$

diff eq^n (2) partially w.r.t. 'x'

(2) $\Rightarrow 2uu_x+2vv_x=0$

$uu_x+vv_x=0 \rightarrow (3)$

III^{ly} diff eq^n (2) partially w.r.t. 'y'

(2) $\Rightarrow 2uu_y+2vv_y=0$

$\Rightarrow uu_y+vv_y=0 \rightarrow (4)$

$\therefore$ But $f(z)$ is analytic

$\therefore u_x = v_y$ and $v_x = -u_y$

$\Rightarrow u_y = -v_x$

(4) $\Rightarrow u(-v_x) + v u_x = 0$

$v u_x - u v_x = 0 \rightarrow$ (5)

$(3)^2 + (5)^2 \Rightarrow$

$\Rightarrow (u u_x + v v_x)^2 + (v u_x - u v_x)^2 = 0$

$\Rightarrow u^2 u_x^2 + v^2 v_x^2 + 2vu \cdot \cancel{v_x u_x} + v^2 u_x^2 + u^2 v_x^2 - 2uv \cdot \cancel{u_x v_x} = 0$

$\Rightarrow u^2 (u_x^2 + v_x^2) + v^2 (u_x^2 + v_x^2) = 0$

$\Rightarrow (u^2 + v^2)(u_x^2 + v_x^2) = 0$

$\Rightarrow k^2 (u_x^2 + v_x^2) = 0$

$\Rightarrow k^2 \neq 0, \; u_x^2 + v_x^2 = 0$

$\Rightarrow |f'(z)|^2 = 0$

$|f'(z)| = 0$

$\therefore f(z) = c$

$\Rightarrow f(z)$ is constant.

If $w = f(z) = \phi(x,y) + i\psi(x,y)$ represents the complex potential of an electrostatic field where $\psi = (x^2 - y^2) + \dfrac{x}{x^2+y^2}$, find $f(z)$ and hence determine ϕ.

Sol^n

Given the complex potential of an electrostatic field

$$W = f(z) = \phi(x,y) + i\psi(x,y)$$

where the imaginary part is

$$\psi = (x^2 - y^2) + \frac{x}{x^2+y^2}$$

$$\psi = \frac{(x^2-y^2)(x^2+y^2) + x}{x^2+y^2}$$

$$\psi = \frac{x^4 - y^4 + x}{x^2 + y^2} \rightarrow (1)$$

Diff (1) partially w.r.t. x we get

$$(1) \Rightarrow \psi = \frac{(x^2+y^2)(4x^3+1) - 2x(x^4 - y^4 + x)}{(x^2+y^2)^2}$$

when $x = z, y = 0$ we get

$$\Rightarrow \psi_x = \frac{z^2(4z^3+1) - 2z(z^4+z)}{z^4}$$

$$\psi_x = \frac{4z^5 + z^2 - 2z^5 - 2z^2}{z^4}$$

$$\psi_x = \frac{2z^5 - z^2}{z^4}$$

$$\psi_x = \frac{2z^3 - 1}{z^2} = 2z - \frac{1}{z^2}$$

Diff (1) partially w.r.t 'y' we get

$$(1) \Rightarrow \psi_y = \frac{-4y^3(x^2+y^2) - 2y(x^4 - y^4 + x)}{(x^2+y^2)^2}$$

when $x = z, y = 0$ we get

$$\psi_y = 0 = \phi_x$$

W.K.T $\quad f'(z) = \phi_x + i\psi_x$

$$\Rightarrow f'(z) = 0 + i\left[2z - \frac{1}{z^2}\right]$$

$$f'(z) = i\left[2z - \frac{1}{z^2}\right]$$

$$\int df = i\int 2z - \frac{1}{z^2}\, dz$$

$$f(z) = i\left[z^2 + \frac{1}{z}\right]$$

when $z = x + iy$

$$f(z) = i\left[(x+iy)^2 + \frac{1}{x+iy}\right]$$

$$f(z) = i\left[(x^2 - y^2 + 2ixy) + \frac{x - iy}{x^2+y^2}\right]$$

$$f(z) = i\left[\frac{x^4 - y^4 + 2ix^3y + 2ixy^3 + x - iy}{x^2+y^2}\right]$$

$$f(z) = i\left[\frac{x^4 - y^4 + x}{x^2+y^2} + i\,\frac{2x^3y + 2xy^3 - y}{x^2+y^2}\right]$$

$$\Rightarrow f(z) = \phi(x,y) + i\psi(x,y) = -\left[\frac{2x^3y + 2xy^3 - y}{x^2+y^2}\right] + i\left[\frac{x^4 - y^4 + x}{x^2+y^2}\right]$$

$$\therefore \phi(x,y) = -\left[\frac{2x^3y + 2xy^3 - y}{x^2+y^2}\right]$$