Integral calculus

1. Reduction formula for $\int \sin^n x \, dx$, $n$ is a positive integer.

Let $I_n = \int \sin^n x \, dx \longrightarrow (1)$

$$= \int \sin^{n-1} x \cdot \sin x \, dx$$

Applying the integration by parts in R.H.S

$$= -\sin^{n-1} x \cdot \cos x + \int \cos x \cdot (n-1) \sin^{n-2} x \cdot \cos x \, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \, (1-\sin^2 x) \, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int (\sin^{n-2} x - \sin^n x) \, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

i.e., $I_n = -\sin^{n-1} x \cdot \cos x + (n-1) I_{n-2} - (n-1) I_n$

$\Rightarrow I_n + (n-1) I_n = -\sin^{n-1} x \cdot \cos x + (n-1) I_{n-2}$ [using eqn (1)

$$n I_n = -\sin^{n-1} x \cdot \cos x + (n-1) I_{n-2}$$

$$\therefore I_n = -\frac{\sin^{n-1} x \cdot \cos x}{n} + \left[\frac{n-1}{n}\right] I_{n-2} \longrightarrow (2)$$

This is the required reduction formula.

2. Reduction formula for $\int_0^{\pi/2} \sin^n x \, dx$

From eqn (2),

$$I_n = \left[-\frac{\sin^{n-1} x \cdot \cos x}{n}\right]_0^{\pi/2} + \left[\frac{n-1}{n}\right] I_{n-2}$$

$$= \frac{n-1}{n} \cdot I_{n-2} \longrightarrow (3)$$

Replace $n$ by $n-2$ in eq$^n$ (3),

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

Hence eq$^n$ (3) becomes,

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} I_{n-4} \longrightarrow (4)$$

Similarly from eq$^n$ (3),

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$$\therefore (4) \Rightarrow \quad I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} I_{n-6}$$

Continuing like this we get,

$$I_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3} I_1 & \text{if } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{1}{2} I_0 & \text{if } n \text{ is even} \end{cases} \longrightarrow (5)$$

But ~~From~~ $I_1 = \int_0^{\pi/2} \sin x \, dx$

$$= -\cos x \Big]_0^{\pi/2}$$

$$= 1$$

and $I_0 = \int_0^{\pi/2} 1 \cdot dx$

$$= \frac{\pi}{2}$$

Thus (5) becomes,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3} & \text{if } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{1}{2} \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

1. Evaluate $\int_0^{\pi/2} \sin^7 x \, dx$

Sol$^n$ :- Here $n = 7$

$$\therefore \int_0^{\pi/2} \sin^7 x \, dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{35}$$

[redacted]. Evaluate $\int_0^{\pi/2} \sin^8 x \, dx$

<u>Soln</u> :- $I = \int_0^{\pi/2} \sin^8 x \, dx$

$$= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{105\pi}{512}$$

3. Evaluate $\int_0^{\pi} \sin^5(x/2) \, dx$

<u>Soln</u> :- $I = \int_0^{\pi} \sin^5(x/2) \, dx$

put $t = x/2 \Rightarrow 2\,dt = dx$

and $t$ varies from $0$ t

$$\therefore I = 2\int_0^{\pi/2} \sin^5 t \, dt$$

$$= 2 \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$= \frac{16}{15}$$

5. Evaluate $\int_0^{\pi} x \sin^8 x \, dx$

Sol'n :- We have the property,

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\therefore I = \int_0^{\pi} x \sin^8 x \, dx \longrightarrow (1)$$

$$= \int_0^{\pi} (\pi - x) \sin^8(\pi - x) \, dx$$

$$= \int_0^{\pi} (\pi - x) \sin^8 x \, dx$$

$$= \pi \int_0^{\pi} \sin^8 x \, dx - \int_0^{\pi} x \sin^8 x \, dx$$

$$= \pi \int_0^{\pi} \sin^8 x \, dx \; - I$$

$$\therefore \quad 2I = 2\pi \int_0^{\pi/2} \sin^8 x \, dx$$

$$I = \pi \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{35\pi^2}{256}$$

6. Evaluate $\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2} \, d\theta$

Sol'n :- $I = \int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2} \, d\theta$

$$= \int_0^{\pi} \frac{[2\sin\theta/2 \cdot \cos\theta/2]^4}{[2\cos^2\theta/2]^2} \, d\theta$$

$$= \int_0^{\pi} \frac{16 \cdot \sin^4\theta/2 \cdot \cos^4\theta/2}{4 \cdot \cos^4\theta/2} \, d\theta$$

$$= 4\int_0^{\pi} \sin^4\theta/_2 \, d\theta$$

put $\frac{\theta}{2} = t$

$\Rightarrow d\theta = 2dt$ and $t$ varies from $0$ to $\pi/2$

$$\therefore \quad I = 4\int_0^{\pi/2} \sin^4 t \cdot 2dt$$

$$= 8 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi}{2}$$

7. Evaluate $\int_0^2 \frac{x^4}{\sqrt{4-x^2}} dx$

Soln :- Let $I = \int_0^2 \frac{x^4}{\sqrt{4-x^2}} dx$

put $x^2 = 4\sin^2\theta$

(or) $x = 2\sin\theta$

$\Rightarrow dx = 2\cos\theta \, d\theta$

and $\theta : 0$ to $\pi/2$

$$\therefore \quad I = \int_0^{\pi/2} \frac{16 \cdot \sin^4\theta}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta \, d\theta$$

$$= 16\int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$= 16 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 3\pi$$

8. Evaluate $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx$

Sol'n: Let $I = \int_0^1 x^{3/2}(1-x)^{3/2}\,dx$

put $x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

and $\theta$ : 0 to $\pi/2$

$$\therefore I = \int_0^{\pi/2} \sin^3\theta \cdot \cos^3\theta \cdot \underset{2\sin\theta}{\cos\theta}\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^4\theta \cdot \cos^4\theta\,d\theta$$

$$= 2 \cdot \frac{3}{8} \cdot \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi}{128}$$

4. Evaluate $\int_0^{\pi} \sin^4 x\,dx$

Sol'n: Here $f(x) = \sin^4 x$. and if $2a = \pi$

$\Rightarrow a = \pi/2$

$$f(2a-x) = \sin^4(2a-x)$$
$$= \sin^4(\pi - x)$$
$$= \sin^4 x$$
$$= f(x)$$

i.e., $f(2a-x) = f(x)$

Thus by property, $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ when n is Even

$$\Rightarrow I = 2\int_0^{\pi/2} \sin^4 x\,dx = 2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{8}$$

**Reduction formula for $\int \cos^n x \, dx$ and $\int_0^{\pi/2} \cos^n x \, dx$**

Let $I_n = \int \cos^n x \, dx \longrightarrow (1)$

$$= \int \cos^{n-1} x \cdot \cos x \, dx$$

$$= \cos^{n-1} x \cdot \sin x - \int \sin x \,(n-1) \cos^{n-2} x \cdot (-\sin x) \, dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \cdot \sin^2 x \, dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \cdot (1 - \cos^2 x) \, dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore I_n + (n-1) I_n = \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2}$$

$$\Rightarrow I_n = \frac{\cos^{n-1} x \cdot \sin x}{n} + \frac{n-1}{n} \cdot I_{n-2} \longrightarrow (2)$$

Thus $$\int_0^{\pi/2} \cos^n x \, dx = \left[ \frac{\cos^{n-1} x \cdot \sin x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \cdot I_{n-2}$$

$$\Rightarrow \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} I_{n-2} \longrightarrow (3)$$

Similar to the previous derivation we can obtain,

$$\int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3} & \text{if } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{1}{2} \, \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

1. Evaluate $\int_0^{\pi/2} \cos^6 x \, dx$

Let $I = \int_0^{\pi/2} \cos^6 x \, dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{15\pi}{96}$

2. Evaluate $\int_0^{\pi/2} \cos^7 x \, dx$

Soln: Let $I = \int_0^{\pi/2} \cos^7 x \, dx$

$$= \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$= \frac{16}{35}$$

3. Evaluate $\int_0^{\pi} x \cos^6 x \, dx$

Soln: Let $I = \int_0^{\pi} x \cos^6 x \, dx \rightarrow (1)$

$$= \int_0^{\pi} (\pi - x) \cos^6 (\pi - x) \, dx$$

$$= \int_0^{\pi} (\pi - x) \cos^6 x \, dx$$

$$= \int_0^{\pi} \pi \cos^6 x \, dx - \int_0^{\pi} x \cos^6 x \, dx$$

$$= \int_0^{\pi} \pi \cos^6 x \, dx - I$$

$$\Rightarrow \quad 2I = 2\pi \int_0^{\pi/2} \cos^6 x \, dx$$

$$\therefore I = \pi \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{15\pi^2}{96}$$

4. Evaluate $\int_{-\pi/2}^{\pi/2} \cos^8 x \, dx$

Soln: Let $I = \int_{-\pi/2}^{\pi/2} \cos^8 x \, dx$

$$= 2\int_0^{\pi/2} \cos^8 x \, dx$$

$$= 2 \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{35\pi}{128}$$

5. Evaluate $\int_0^{\pi/4} \cos^6 2x \, dx$

Soln :- Let $I = \int_0^{\pi/4} \cos^6 2x \, dx$

put $t = 2x$

$\Rightarrow \frac{1}{2} dt = dx$

and $t$ : $0$ to $\pi/2$

$$\therefore \quad I = \int_0^{\pi/2} \cos^6 t \cdot \frac{1}{2} \, dt$$

$$= \frac{1}{2} \int_0^{\pi/2} \cos^6 t \, dt$$

$$= \frac{1}{2} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{15\pi}{192}$$

Note :-

1) If $a-x$ is involved, use $x = a\sin^2\theta$ as substitution

2) If $a+x$ is " ———— " $x = a\tan^2\theta$ as " ———— "

Properties :-

1) $\int_0^{2a} f(x)\, dx = 2\int_0^a f(2a-x)\, dx$

2) $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$.

3) $\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx & \text{if } f(-x) = f(x) \\ 0 & \text{if } f(-x) = -f(x) \end{cases}$

4) $\int_0^a f(x)\, dx = \int_0^{a/2} f(x)\, dx + \int_0^{a/2} f(a-x)\, dx$.

Scanned by CamScanner

**Reduction formula for $\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx$**

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = \frac{[(m-1)(m-3)\cdots\cdots][(n-1)(n-3)\cdots\cdots]}{(m+n)(m+n-2)(m+n-4)\cdots\cdots} \times K$$

Where $K = \frac{\pi}{2}$ when $m$ and $n$ are even

and $K = 1$ otherwise.

1. Evaluate $\int_0^{\pi/2} \sin^5 x \cdot \cos^4 x \, dx$

Sol'n :- $m = 5, n = 4$

$$\therefore \int_0^{\pi/2} \sin^5 x \cdot \cos^4 x \, dx = \frac{[(5-1)(5-3)][(4-1)(4-3)]}{(5+4)(5+4-2)(5+4-4)(5+4-6)(5+4-8)}$$

$$= \frac{[4 \cdot 2][3 \cdot 1]}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}$$

$$= \frac{8}{315}$$

2. Evaluate $\int_0^{\pi/2} \sin^7 x \cdot \cos^5 x \, dx$

Sol'n :- Here $m = 7, n = 5$

$$\therefore \int_0^{\pi/2} \sin^7 x \cdot \cos^5 x \, dx = \frac{[6 \cdot 4 \cdot 2][4 \cdot 2]}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2}$$

$$= \frac{1}{120}$$

3. Evaluate $\int_0^{\pi/2} \sin^6 x \cdot \cos^5 x \, dx$

Sol'n :- Here $m = 6, n = 5$

$$\int_0^{\pi/2} \sin^6 x \cdot \cos^5 x \, dx = \frac{[5 \cdot 3 \cdot 1][4 \cdot 2]}{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} = \frac{8}{693}$$

4. Evaluate $\int_0^{\pi/2} \sin^8 x \cdot \cos^6 x \, dx$

Sol'n :- $m = 8$, $n = 6$ $\Rightarrow$ both are even

$\therefore K = \frac{\pi}{2}$

$$\int_0^{\pi/2} \sin^8 x \cdot \cos^6 x \, dx = \frac{[7 \cdot 5 \cdot 3 \cdot 1][5 \cdot 3 \cdot 1]}{14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{5\pi}{4096}$$

5. Evaluate $\int_0^{\pi} \sin^6 x \cdot \cos^4 x \, dx$

Sol'n :- $m = 6$, $n = 4$ $\Rightarrow$ $K = \frac{\pi}{2}$

$$\int_0^{\pi} \sin^6 x \cdot \cos^4 x \, dx = 2 \int_0^{\pi/2} \sin^6 x \cdot \cos^4 x \, dx$$

$$= 2 \cdot \frac{[5 \cdot 3 \cdot 1][3 \cdot 1]}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \frac{\pi}{2}$$

$$= \frac{3\pi}{256}$$

6. Evaluate $\int_0^{\pi/6} \cos^4 3x \sin^2 6x \, dx$

Sol'n :- we can write,

$$\sin 6x = 2 \sin 3x \cos 3x$$

$$\therefore \int_0^{\pi/6} \cos^4 3x \, (2 \sin 3x \cos 3x)^2 \, dx$$

$$= 4 \int_0^{\pi/6} \sin^2 3x \cdot \cos^6 3x \, dx$$

put $3x = t$ $\Rightarrow$ $dx = \frac{1}{3} dt$

and $t$ : 0 to $\pi/2$

$\therefore \int_0^{\pi/6} \cos^4 3x \sin^2 6x \, dx = 4 \int_0^{\pi/2} \sin^2 t \cdot \cos^6 t \cdot \frac{1}{3} \, dt$

$= \frac{4}{3} \frac{[1][5.3.1]}{8.6.4.2} \cdot \frac{\pi}{2}$

$= \frac{5\pi}{192}$

7. Evaluate $\int_0^1 x^2 (1-x^2)^{3/2} \, dx$

HW

Sol'n :- Let $x = \sin\theta$

$dx = \cos\theta \, d\theta$

and $\theta : 0$ to $\pi/2$

$\therefore \int_0^1 x^2(1-x^2)^{3/2} dx = \int_0^{\pi/2} \sin^2\theta \, [1-\sin^2\theta]^{3/2} \cdot \cos\theta \, d\theta$

$= \int_0^{\pi/2} \sin^2\theta \cdot \cos^4\theta \, d\theta$

$= \frac{[1][3.1]}{6.4.2} \cdot \frac{\pi}{2}$

$= \frac{\pi}{32}$

Do

7. $\int_0^1 x^6 \sqrt{1-x^2} \, dx = 5\pi/256$

8. Evaluate $\int_0^1 x^{3/2} (1-x)^{3/2} \, dx$

Sol'n :- Let $x = \sin^2\theta$

$dx = 2\sin\theta \cos\theta \, d\theta$

and $\theta : 0$ to $\pi/2$

$\Rightarrow \int_0^1 x^{3/2}(1-x)^{3/2} dx = \int_0^{\pi/2} \sin^3\theta \cdot \cos^3\theta \cdot 2\sin\theta\cos\theta \, d\theta$

$= 2\int_0^{\pi/2} \sin^4\theta \cdot \cos^4\theta \, d\theta$

$$= 2 \cdot \frac{[3\cdot1][3\cdot1]}{8\cdot6\cdot4\cdot2} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi}{128}$$

9. Evaluate $\int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx$

Soln :- put $x = 2a\sin^2\theta$

$dx = 4a \sin\theta \cos\theta \, d\theta$

and $\theta : 0$ to $\pi/2$

$$\therefore \int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx = \int_0^{\pi/2} 4a^2 \sin^4\theta \cdot \sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta} \cdot 4a \sin\theta \cos\theta$$

$$= \int_0^{\pi/2} 4a^2 \sin^4\theta \cdot 2a\sin\theta \cos\theta \cdot 4a \sin\theta \cos\theta \, d\theta$$

$$= 32a^4 \int_0^{\pi/2} \sin^6\theta \cdot \cos^2\theta \, d\theta$$

$$= 32a^4 \frac{[5\cdot3\cdot1][1]}{8\cdot6\cdot4\cdot2} \cdot \frac{\pi}{2}$$

$$= \frac{5\pi a^4}{8}$$

10. Evaluate $\int_0^{\infty} \frac{x^4}{(1+x^2)^4}\, dx$

Soln :- Let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

and $\theta : 0$ to $\pi/2$

$$\int_0^{\infty} \frac{x^4}{(1+x^2)^4}\, dx = \int_0^{\pi/2} \frac{\tan^4\theta}{(1+\tan^2\theta)^4} \cdot \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^4\theta}{\sec^8\theta} \cdot \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^4\theta}{\sec^6\theta} \, d\theta$$

$$= \int_0^{\pi/2} \sin^4\theta \cdot \cos^2\theta \, d\theta$$

$$= \frac{[3 \cdot 1][1]}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{32}$$

11. Evaluate $\int_0^{\pi} x \sin^2 x \cos^4 x \, dx$

Sol'n :- $I = \int_0^{\pi} x \sin^2 x \cos^4 x \, dx \rightarrow (1)$

$$= \int_0^{\pi} (\pi - x) \sin^2(\pi - x) \cos^4(\pi - x) \, dx$$

$$= \int_0^{\pi} (\pi - x) \sin^2 x \cos^4 x \, dx$$

$$= \pi \int_0^{\pi} \sin^2 x \cos^4 x \, dx - \int_0^{\pi} x \sin^2 x \cos^4 x \, dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \sin^2 x \cos^4 x \, dx$$

$$\therefore I = \pi \int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$$

$$= \pi \frac{[1][3 \cdot 1]}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{32}$$

## DE (OR) ODE :-

If $y=f(x)$ is an unknown fn, an eqn which involves atleast one derivative of $y$ w.r.t $x$ is called an ODE (or) simply DE.

**Note** :- 1) The order of the DE is the order of the highest derivative present in the eqn.

2) The Degree of the DE is the degree of the highest order derivative after clearing the fractional powers.

3) Finding $y$ as a fn of $x$ explicitly $[y=f(x)]$ (or) a relationship in $x$ & $y$ satisfying the DE $[f(x,y)=c]$ constitutes the Solution of the DE.

Ex :- 1) $\left(\frac{dy}{dx}\right)^2 + 3\left(\frac{dy}{dx}\right) + 2 = 0$ order $=1$, degree $=2$

2) $\frac{d^3y}{dx^3} + 5\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^3 = \sin x$ (order $=3$, degree $=1$)

3) $(r^2+r_1^2)^{3/2} = r^2+2r_1^2 - rr_2$ where $r_1 = \frac{dr}{d\theta}$, $r_2 = \frac{d^2r}{d\theta^2}$

clearing the fractional powers, we square on b.s.

$\Rightarrow (r^2+r_1^2)^3 = (r^2+2r_1^2-rr_2)^2$

we have $r_2^2$ in RHS.

$\Rightarrow$ (order $=2$, degree $=2$)

* Equations of first order and first degree :-

* Linear Equations : Bernoulli's Equation

A DE is said to be linear if the dependent variable and its derivative occur in first degree only and they are not multiplied together.

* Standard forms of linear D.E and its Solution :-

1) A DE of the form $\frac{dy}{dx} + Py = Q \rightarrow (1)$ is called a linear Eqn where P, Q are fns of $x$ only, in $y$.

To solve (1), find Integrating factor (IF) given by

$$IF = e^{\int P\,dx}$$

Solution is $y\,(IF) = \int Q\,(IF)\,dx + c$.

2) A DE of the form $\frac{dx}{dy} + Px = Q \longrightarrow (2)$ is called a linear eqn where P, Q are fns of $y$ only, in $x$.

To solve (2), find IF given by $IF = e^{\int P\,dy}$

Solution is $x\,(IF) = \int Q\,(IF)\,dy + c$.

Illustrative Examples :- Solve :-

1) $\frac{dy}{dx} + y\cot x = \cos x$.

Soln:- Given eqn is of the form

$\frac{dy}{dx} + Py = Q$ where $P = \cot x$, $Q = \cos x$. (fns of $x$)

$$IF = e^{\int P\,dx} = e^{\int \cot x\,dx}$$

$$= e^{\log(\sin x)} = \sin x.$$

General Soln is

$$y\,(IF) = \int Q\,(IF)\,dx + c$$

$$\Rightarrow \quad y \sin x = \int \cos x \cdot \sin x \, dx + c$$

$$= \int \frac{\sin 2x}{2} \, dx + c$$

$$= \frac{1}{2}\left[ -\frac{\cos 2x}{2} \right] + c$$

$$\therefore \quad y \sin x = -\frac{\cos 2x}{4} + c$$

2) $(1+x^2)(dy - dx) = 2xy \, dx$.

$$\Rightarrow \quad dy - dx = \frac{2xy}{1+x^2} \, dx.$$

$\div$ by $dx$.

$$\Rightarrow \quad \frac{dy}{dx} - 1 = \frac{2xy}{1+x^2}$$

(or) $$\frac{dy}{dx} - \left(\frac{2x}{1+x^2}\right) y = 1 \rightarrow (1)$$

Eqn (1) is of the form $\frac{dy}{dx} + Py = Q$

where $P = \frac{-2x}{1+x^2}$, $Q = 1$.

$$IF = e^{\int P\,dx} = e^{-\int \frac{2x}{1+x^2} dx} = e^{-\log(1+x^2)} \cancel{= 1+x^2}.$$

$$= e^{\log(1+x^2)^{-1}}$$

$$= (1+x^2)^{-1} = \frac{1}{1+x^2}.$$

Soln is $$y\,(IF) = \int Q\,(IF)\,dx + c.$$

$$\Rightarrow \quad y \cdot \frac{1}{1+x^2} = \int \frac{1}{1+x^2} \, dx + c$$

$$\Rightarrow \quad \frac{y}{1+x^2} = \tan^{-1} x + c$$

3) $(x+y+1)\frac{dy}{dx} = 1$

Soln :- $\frac{dy}{dx} = \frac{1}{x+y+1}$

(or) $\frac{dx}{dy} = x+y+1$

$\Rightarrow \frac{dx}{dy} - x = y+1 \rightarrow (1)$

Eq$^n$ (1) is of the form $\frac{dx}{dy} + Px = Q$

where $P = -1, \quad Q = y+1$

$IF = e^{\int P\,dy} = e^{\int -dy} = e^{-y}.$

G. soln is

$$x(IF) = \int Q(IF)\,dy + c.$$

$$xe^{-y} = \int \underset{u}{(y+1)}\,\underset{v}{e^{-y}}\,dy + c$$

$$= (y+1)\frac{e^{-y}}{-1} - \left[\int \frac{e^{-y}}{-1} \cdot 1\,dy + c\right]$$

$$= -(y+1)e^{-y} + \frac{e^{-y}}{-1} + c$$

$$= -(y+1)e^{-y} - e^{-y} + c.$$

$$\Rightarrow \quad x = -(y+1) - 1 + ce^{y}$$

(or) $\quad x+y+2 = ce^{y}$

## * Bernoulli's DE [ $Eq^{ns}$ reducible to Linear $Eq^{n}$ ]

The DE of the form $\frac{dy}{dx} + Py = Qy^n$, where P & Q are fns of $x$, is called as Bernoulli's DE in $y$.

$\div$ the above $eq^n$ by $y^n$,

$$\frac{1}{y^n}\frac{dy}{dx} + P\,y^{1-n} = Q. \rightarrow (1)$$

Put $y^{1-n} = t$

$$\therefore (1-n)\,y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$$

(or) $$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dt}{dx}.$$

(1) $\Rightarrow$ $$\frac{1}{(1-n)}\frac{dt}{dx} + Pt = Q.$$

$\Rightarrow$ $$\frac{dt}{dx} + P(1-n)\,t = Q(1-n),$$ which is a linear $Eq^n$ in $t$.

$||^{ly}$ $\frac{dx}{dy} + Px = Qx^n$, where P & Q are fns of $y$ is called Bernoulli's $eq^n$ in $x$.

Divide by $x^n$ & later put $x^{1-n} = t$ to obtain a linear $eq^n$ in $t$.

Problems

1) $\frac{dy}{dx} - \frac{y}{x} = y^2 x$.

Soln :- The given Eqn is of the form

$\frac{dy}{dx} + Py = Qy^n$ , which is a Bernoulli's eqn.

$\div$ by $y^2$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{x}\cdot\frac{1}{y} = x.$$

(or) $y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = x \rightarrow (1)$

Put $y^{-1} = t$.

$\Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dt}{dx}$ (or) $y^{-2}\frac{dy}{dx} = -\frac{dt}{dx}$

$(1) \Rightarrow -\frac{dt}{dx} - \frac{1}{x}\cdot t = x$

(or) $\frac{dt}{dx} + \frac{1}{x}t = -x. \rightarrow (2)$

This is a linear Eqn in $t$, of the form

$\frac{dt}{dx} + Pt = Q$ where $P = 1/x$, $Q = -x$.

$$If = e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x.$$

Soln of (2) is

$$t\,(If) = \int Q\,(If)\,dx + c$$

$$\Rightarrow \frac{x}{y} = \int -x^2\,dx + c$$

$$\Rightarrow \frac{x}{y} = -\frac{x^3}{3} + c$$

(or) $\frac{x}{y} + \frac{x^3}{3} = c$

2) $x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$

Soln:- $\div$ by $x^3 y^4$

$\Rightarrow \frac{1}{y^4}\frac{dy}{dx} - \frac{1}{x}\cdot\frac{1}{y^3} = \frac{-\cos x}{x^3} \rightarrow (1)$

Put $\frac{1}{y^3} = t$

$\Rightarrow -3y^{-4}\frac{dy}{dx} = \frac{dt}{dx}$ (or) $\frac{1}{y^4}\frac{dy}{dx} = -\frac{1}{3}\frac{dt}{dx}$

$(1) \Rightarrow -\frac{1}{3}\frac{dt}{dx} - \frac{1}{x}\cdot t = -\frac{\cos x}{x^3}$

$\Rightarrow \frac{dt}{dx} + \frac{3}{x}t = \frac{3\cos x}{x^3} \rightarrow (2)$

Eqn (2) is linear in $t$, $\rightarrow \frac{dt}{dx} + Pt = Q$

where $P = 3/x$, $Q = \frac{3\cos x}{x^3}$

$If = e^{\int P dx} = e^{\int \frac{3}{x}dx} = e^{3\log x} = x^3$

Soln is $t(If) = \int Q(If)\, dx + c$

$\Rightarrow \frac{1}{y^3}\cdot x^3 = \int \frac{3\cos x}{x^3}\cdot x^3\, dx + c$

$\Rightarrow \frac{x^3}{y^3} = 3\sin x + c$

3) $(y\log x - 2)y\,dx = x\,dy$.

Soln:- $\frac{dy}{dx} = \frac{y^2\log x - 2y}{x}$

$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = \frac{\log x}{x}\cdot y^2$

$\div$ by $y^2$

$\frac{1}{y^2}\frac{dy}{dx} + \frac{2}{x}\cdot\frac{1}{y} = \frac{\log x}{x} \rightarrow (1)$

Put $\frac{1}{y} = t \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dt}{dx}$.

(1) $\Rightarrow$ $-\frac{dt}{dx} + \frac{2}{x} t = \frac{1}{x} \log x$.

(OR) $\frac{dt}{dx} - \frac{2}{x} t = -\frac{1}{x} \log x \longrightarrow$ (2)

Eqn (2) is linear in t, $\rightarrow \frac{dt}{dx} + Pt = Q$

where $P = -2/x$, $Q = -\frac{\log x}{x}$.

$$\text{If} = e^{\int P\,dx} = e^{-\int 2/x\,dx} = e^{-2\log x} = x^{-2} = 1/x^2$$

Soln is

$$t\,(\text{If}) = \int Q\,(\text{If})\,dx + c$$

$$\Rightarrow t \cdot \frac{1}{x^2} = \int -\frac{\log x}{x} \cdot \frac{1}{x^2}\,dx + c.$$

$$\Rightarrow \frac{t}{x^2} = -\int \underset{\text{I}}{\underline{\log x}} \cdot \underset{\text{II}}{\underline{\frac{1}{x^3}}}\,dx + c$$

$$= -\left[\log x \cdot \int x^{-3}\,dx - \int \int x^{-3} dx \cdot \frac{1}{x}\,dx\right] + c$$

$$= -\left[\log x \cdot \frac{x^{-2}}{-2} - \int \frac{x^{-2}}{-2} \cdot \frac{1}{x}\,dx\right] + c$$

$$= +\frac{\log x}{2x^2} - \frac{1}{2}\int x^{-3}\,dx + c$$

$$= \frac{\log x}{2x^2} - \frac{1}{2} \cdot \frac{x^{-2}}{(-2)} + c$$

$$\frac{1}{x^2 y} = \frac{\log x}{2x^2} + \frac{1}{4x^2} + c.$$

(or) $$\frac{1}{y} = \frac{\log x}{2} + \frac{1}{4} + cx^2$$

4) PTO

Scanned by CamScanner

Q) $\frac{dy}{dx} - \frac{1}{2}\left(1+\frac{1}{x}\right)y + \frac{3y^3}{x} = 0.$

$\Rightarrow \frac{dy}{dx} - \frac{1}{2}\left(1+\frac{1}{x}\right)y = -\frac{3y^3}{x}$

$\div$ by $y^3$

$$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{2}\left(1+\frac{1}{x}\right)\cdot\frac{1}{y^2} = -\frac{3}{x} \rightarrow (1)$$

$\left(f'(y)\frac{dy}{dx} + P f(y) = Q\right) \times$

~~where $f(y) = -1/y^2$~~

Put $\boxed{-1/y^2 = t}$ $\Rightarrow \frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$ (or) $\frac{1}{y^3}\frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx}$

$(1) \Rightarrow \frac{1}{2}\frac{dt}{dx} + \frac{1}{2}\left(1+\frac{1}{x}\right)t = -\frac{3}{x}$

$\Rightarrow \frac{dt}{dx} + \left(1+\frac{1}{x}\right)t = -\frac{6}{x} \rightarrow (2)$ linear in t.

$P = (1+1/x)$ ; $Q = -6/x$

$$If = e^{\int P\,dx} = e^{\int(1+1/x)\,dx} = e^{x+\log x} = e^x \cdot e^{\log x} = \boxed{xe^x}$$

Soln to (2) is

$t\,(If) = \int Q\,(If)\,dx + c$

$\Rightarrow t \cdot xe^x = \int\left(-\frac{6}{x}\right)(xe^x)\,dx + c$

$= -6\int e^x\,dx + c$

$t\,xe^x = -6e^x + c$

$\Rightarrow -\frac{xe^x}{y^2} = -6e^x + c$

(or) $\frac{x}{y^2} = 6e^x - ce^{-x}$

Ex $\frac{dy}{dx} - y\tan x = \frac{\sin x . \cos^2 x}{y^2}$

5 xply by $y^2$

$$y^2 \frac{dy}{dx} - (\tan x)\, y^3 = \sin x . \cos^2 x \rightarrow (1)$$

$$\left(f'(y)\frac{dy}{dx} + P\, f(y) = Q\right) \times$$

where $f(y) = -y^3$.

Put $\boxed{-y^3 = t}$

$$\therefore -3y^2 \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow y^2\frac{dy}{dx} = -\frac{1}{3}\frac{dt}{dx}$$

$$① \Rightarrow -\frac{1}{3}\frac{dt}{dx} + (\tan x)\, t = \sin x \cos^2 x$$

$$\Rightarrow \frac{dt}{dx} - (3\tan x)\, t = -3\sin x \cos^2 x \rightarrow (2)$$

linear in t

$P = -3\tan x$ ; $Q = -3\sin x . \cos^2 x$.

$$IF = e^{\int P\,dx} = e^{\int -3\tan x\, dx} = e^{-3\log(\sec x)} = (\sec x)^{-3} = \cos^3 x.$$

Soln to ② is

$$t\,(IF) = \int Q\,(IF)\,dx + c$$

$$t . \cos^3 x = \int -3\sin x \cos^2 x . \cos^3 x\, dx + c$$

$$= -3\int \sin x . \cos^5 x\, dx + c$$

Put $\cos x = u$

$-\sin x\, dx = du$

$$\therefore t\cos^3 x = 3\int u^5\, du + c$$

$$= 3\frac{u^6}{6} + c = \frac{u^6}{2} + c$$

$$\therefore -y^3 . \cos^3 x = \frac{\cos^6 x}{2} + c$$

6) $6y^2\, dx - x(x^3+2y)\, dy = 0$.

Soln :- $\dfrac{dx}{dy} = \dfrac{x^4+2xy}{6y^2} = \dfrac{x^4}{6y^2} + \dfrac{x}{3y}$.

$\Rightarrow \dfrac{dx}{dy} - \dfrac{x}{3y} = \dfrac{x^4}{6y^2}$

$\div$ by $x^4$

$$\frac{1}{x^4}\frac{dx}{dy} - \frac{1}{3y}\cdot\frac{1}{x^3} = \frac{1}{6y^2} \rightarrow (1)$$

Put $\dfrac{1}{x^3} = t \quad \Rightarrow \quad -\dfrac{3}{x^4}\dfrac{dx}{dy} = \dfrac{dt}{dy}$

(or) $\dfrac{1}{x^4}\dfrac{dx}{dy} = -\dfrac{1}{3}\dfrac{dt}{dy}$

(1) $\Rightarrow -\dfrac{1}{3}\dfrac{dt}{dy} - \dfrac{1}{3y}\cdot t = \dfrac{1}{6y^2}$

$\times$ by $-3$

$$\frac{dt}{dy} + \frac{1}{y}t = -\frac{1}{2y^2} \rightarrow (2)$$

This is a linear eqn in $t \rightarrow \dfrac{dt}{dy} + Pt = Q$

where $P = 1/y$, $Q = -1/2y^2$

$IF = e^{\int P\,dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$.

Soln is

$t\,(IF) = \int Q\,(IF)\,dy + c$

$\Rightarrow \dfrac{1}{x^3}\cdot y = \int -\dfrac{1}{2y^2}\cdot y\, dy + c$

$= -\dfrac{1}{2}\int \dfrac{1}{y}\, dy + c$

$$\frac{y}{x^3} = -\frac{1}{2}\log y + c.$$

7) $(x^2y^3 + xy)\frac{dy}{dx} = 1$

Soln:. $\frac{dx}{dy} = x^2y^3 + xy$

$\Rightarrow \frac{dx}{dy} - xy = x^2y^3$

$\div$ by $x^2$

$\frac{1}{x^2}\frac{dx}{dy} - y\cdot\frac{1}{x} = y^3 \cdot \rightarrow (1)$

Put $\frac{1}{x} = t \Rightarrow -1/x^2 \frac{dx}{dy} = \frac{dt}{dy}$

(or) $\frac{1}{x^2}\frac{dx}{dy} = -dt/dy$

(1) $\Rightarrow -\frac{dt}{dy} - y\cdot t = y^3$

(or) $\frac{dt}{dy} + y\cdot t = -y^3 \longrightarrow (2)$ linear in 't'.

$P = y$ ; $Q = -y^3$

$If = e^{\int y\, dy} = e^{y^2/2}$.

Soln is $t\,(If) = \int Q\,(If)\,dy + c$

$\Rightarrow \frac{1}{x}\cdot e^{y^2/2} = \int -y^3 \cdot e^{y^2/2}\,dy + c \rightarrow (3)$

Put $y^2/2 = u \Rightarrow y\,dy = du$.

Also $y^3 \cancel{dy} = y^2 \cdot y\ dy = 2u \cdot du$.

(3) $\Rightarrow \frac{1}{x}e^{y^2/2} = -\int 2u\, e^u\, du + c$

$= -2\left[ue^u - e^u\right] + c$

$= -2ue^u + 2e^u + c$.

(OR) $\frac{e^{y^2/2}}{x} = -2\frac{y^2}{2}\cdot e^{y^2/2} + 2e^{y^2/2} + c$

$\Rightarrow \frac{1}{x} = -y^2 + 2 + ce^{-y^2/2}$

8) $r\sin\theta - \cos\theta \dfrac{dr}{d\theta} = r^2$

Soln:- $\cos\theta \dfrac{dr}{d\theta} = r\sin\theta - r^2$

$\Rightarrow \cos\theta \dfrac{dr}{d\theta} - r\sin\theta = -r^2$

$\div$ by $r^2 \cos\theta$.

$$\frac{1}{r^2}\frac{dr}{d\theta} - \tan\theta \cdot \frac{1}{r} = -\frac{1}{\cos\theta}$$

(or) $\dfrac{1}{r^2}\dfrac{dr}{d\theta} - \tan\theta \cdot \dfrac{1}{r} = -\sec\theta \rightarrow (1)$.

Put $\dfrac{1}{r} = t \Rightarrow \dfrac{1}{r^2}\dfrac{dr}{d\theta} = -\dfrac{dt}{d\theta}$

(1) $\Rightarrow -\dfrac{dt}{d\theta} - \tan\theta \cdot t = -\sec\theta$

(or) $\dfrac{dt}{d\theta} + (\tan\theta)\, t = \sec\theta \rightarrow (2)$

linear in 't'. where $P = \tan\theta$, $Q = \sec\theta$.

$If = e^{\int P\,d\theta} = e^{\int \tan\theta\, d\theta} = e^{\log(\sec\theta)} = \sec\theta$.

Soln is

$t\,(If) = \int Q\,(If)\, d\theta + c$.

$\Rightarrow \dfrac{1}{r}\cdot \sec\theta = \int \sec\theta \cdot \sec\theta\, d\theta + c$

$= \int \sec^2\theta\, d\theta + c$

$\dfrac{\sec\theta}{r} = \tan\theta + c$.

Hw

1) $\dfrac{dy}{dx} + xy = xy^3$ Ans:- $\dfrac{1}{y^2} = 1 + ce^{x^2}$

2) $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2 x$ Ans:- $\dfrac{1}{xy} + x = c$.

3) $(x^2+y^2+x)\,dx + xy\,dy = 0$ Ans:- $x^2y^2 = -2\left[\dfrac{x^4}{4} + \dfrac{x^3}{3}\right] + c$

# Exact Differential Equations

**Defn:-** A DE of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ — ①, where M & N are fns of $x, y$.

Eqn ① is the necessary & sufficient condn for the DE to be exact.

The solution of the exact eqn is

$$\int_{y\text{-const}} M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = c.$$

**Problems:-**

**Solve:-**

1) $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0.$

**Soln:-**

$$M = 5x^4 + 3x^2y^2 - 2xy^3$$

$$N = 2x^3y - 3x^2y^2 - 5y^4$$

$$\frac{\partial M}{\partial y} = 0 + 6x^2y - 6xy^2 \;;\quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Given DE is exact.}$$

Soln is

$$\int_{y\text{-const}} M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = c$$

$$\Rightarrow \int_{y\text{-const}} (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int (-5y^4)\,dy = c.$$

$$\Rightarrow 5\frac{x^5}{5} + 3y^2\cdot\frac{x^3}{3} - 2y^3\cdot\frac{x^2}{2} - \frac{5y^5}{5} = c$$

$$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = c$$

2) $\left[\cos x \cdot \tan y + \cos(x+y)\right] dx + \left[\sin x \cdot \sec^2 y + \cos(x+y)\right] dy = 0$

Soln:- $\quad M = \cos x \cdot \tan y + \cos(x+y)$

$$\frac{\partial M}{\partial y} = \cos x \cdot \sec^2 y - \sin(x+y)$$

$$N = \sin x \cdot \sec^2 y + \cos(x+y)$$

$$\frac{\partial N}{\partial x} = \cos x \cdot \sec^2 y - \sin(x+y)$$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ Given DE is exact.

Soln is

$$\int_{y\text{-const}} \left[\cos x \cdot \tan y + \cos(x+y)\right] dx + \int (0)\, dy = C$$

$$\Rightarrow \quad \tan y \int \cos x\, dx + \int \cos(x+y)\, dx = C$$

$$\Rightarrow \quad \sin x \cdot \tan y + \sin(x+y) = C$$

$x\, dx + y\, dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$ given $y=1$ when $x=1$

S:- $\quad x\, dx + y\, dy + \frac{x}{x^2+y^2} dy - \frac{y}{x^2+y^2} dx = 0.$

$$\Rightarrow \left(x - \frac{y}{x^2+y^2}\right) dx + \left(y + \frac{x}{x^2+y^2}\right) dy = 0.$$

$M = x - \frac{y}{x^2+y^2}$ ; $\quad N = y + \frac{x}{x^2+y^2}$

$\frac{\partial M}{\partial y} = -\left\{\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}\right\}$ ; $\quad \frac{\partial N}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$

$= \frac{-x^2+y^2}{(x^2+y^2)^2}$ $\qquad = \frac{-x^2+y^2}{(x^2+y^2)^2}$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ given eqn is exact

Soln is

$$\int_{y\text{-const}} \left(x - \frac{y}{x^2+y^2}\right) dx + \int y \, dy = c.$$

$$\Rightarrow \frac{x^2}{2} - y \cdot \tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2}{2} = c \rightarrow (1)$$ is the General Soln.

$$\Rightarrow \cancel{x^2 - 2y \tan^{-1}\left(\frac{x}{y}\right) + y^2 = 2c} \rightarrow (1)$$

continued at the back

4) $(y^2 e^{xy^2} + 4x^3)\, dx + (2xy\, e^{xy^2} - 3y^2)\, dy = 0.$

$M = y^2 e^{xy^2} + 4x^3$ ; $N = 2xy\, e^{xy^2} - 3y^2.$

$$\frac{\partial M}{\partial y} = y^2 \cdot e^{xy^2} \cdot 2xy + e^{xy^2} \cdot 2y$$

$$= e^{xy^2} \cdot 2y\,(xy^2 + 1)$$

$$\frac{\partial N}{\partial x} = 2y\left[x e^{xy^2} \cdot y^2 + e^{xy^2} \cdot 1\right]$$

$$= 2y\, e^{xy^2}(xy^2 + 1)$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ given eqn is exact.

Soln is

$$\int_{y\text{-const}} \left[y^2 e^{xy^2} + 4x^3\right] dx + \int (-3y^2)\, dy = c$$

$$\Rightarrow y^2 \cdot \frac{e^{xy^2}}{y^2} + 4\frac{x^4}{4} - \frac{3y^3}{3} = c$$

$$\Rightarrow e^{xy^2} + x^4 - y^3 = c$$

5) $(1 + e^{x/y})\, dx + e^{x/y}(1 - x/y)\, dy = 0.$

Soln:- $M = 1 + e^{x/y}$ ; $N = e^{x/y}(1 - x/y)$

$$\frac{\partial M}{\partial y} = e^{x/y}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(-\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right) e^{x/y} \cdot \frac{1}{y}$$

$$= -\frac{1}{y} e^{x/y} + \frac{e^{x/y}}{y} - \frac{x e^{x/y}}{y^2}$$

$$= -\frac{x e^{x/y}}{y^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Soln is

$$\int_{y\text{-const}} (1+e^{x/y})\,dx + \int (0)\,dy = c.$$

$$\Rightarrow \quad x + \frac{e^{x/y}}{1/y} = c.$$

$$\Rightarrow \quad x + y\,e^{x/y} = c$$

Hw

1) $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + \cos y + x} = 0$. Ans :- $y\sin x + x\sin y + xy =$

2) $\left[y\left(1+\frac{1}{x}\right) + \cos y\right] dx + \left[x + \log x - x\sin y\right] dy = 0.$

Ans :- $y(x+\log x) + x\cos y = c.$

3) $(2 + 2x^2\sqrt{y})\, y\, dx + (x^2\sqrt{y} + 2)\, x\, dy = 0.$

Ans :- $2xy + \frac{2}{3}x^3 y^{3/2} = c.$

4) $y e^{xy}\, dx + (xe^{xy} + 2y)\, dy = 0$

Ans :- $e^{xy} + y^2 = c.$

5) $y \sin 2x\, dx - (1+y^2+\cos^2 x)\, dy = 0$

Ans:- $3y\cos 2x + 6y + 2y^3 = c.$

Continued Given when $x=1$, $y=1$.

Sub in ①, $\frac{1}{2} - \cancel{2}\tan^{-1}(1) + \frac{1}{2} = \cancel{2c}.$

$1\cancel{2} - \cancel{2}\cdot\frac{\pi}{4} = \cancel{2c}$ $\cancel{\Rightarrow 2c = 2 - \frac{\pi}{2}}$.

(or) $\boxed{c = 1 - \pi/4}$

$\therefore$ ① $\Rightarrow \frac{x^2}{2} - \tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2}{2} = 1 - \frac{\pi}{4}$ is the particular Soln

## Equations reducible to exact form :-

If the given DE is not exact, then it can be transformed into an exact eqⁿ by multiplying some fⁿ (factor) known as the IF.

### Type 1 :-

If the eqⁿ is homogeneous*, then $\frac{1}{Mx+Ny}$ is the IF, provided $Mx+Ny \neq 0$.

* Homogeneous means - degree of each term is same throughout the eqⁿ.

### Type 2 :-

If the difference $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is close to the expression of N, then compute $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ which will be a fⁿ of $x$ alone. IF is $e^{\int f(x)\,dx}$

### Type 3 :-

If the difference $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is close to M, then

Compute $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$ which will be a fⁿ of $y$ alone.

IF is $e^{-\int g(y)\,dy}$.

3) $(xy+y^2)\,dx + (x+2y-1)\,dy = 0. \rightarrow (1)$

<u>Soln</u>:- $M = xy + y^2$ ; $N = x+2y-1.$

$$\frac{\partial M}{\partial y} = x+2y \quad ; \quad \frac{\partial N}{\partial x} = 1.$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Eq}^n \ (1) \text{ is not exact.}$$

$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x+2y-1$ .... close to N.

$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x+2y-1}{x+2y-1} = 1$ (a constant) $= f(x)$ (say)

$IF = e^{\int f(x)\,dx} = e^{\int 1 . dx} = e^x$.

x$^{ply}$ Eq$^n$ ① by $IF = e^x$,

$(xy+y^2)e^x\,dx + (x+2y-1)e^x\,dy = 0 \longrightarrow ②$

$M = (xy+y^2)e^x$ ; $N = (x+2y-1)e^x$

$\frac{\partial M}{\partial y} = e^x(x+2y)$ ; $\frac{\partial N}{\partial x} = (x+2y-1)e^x + e^x . 1$

$= (x+2y)e^x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ Eq$^n$ ② is exact.

Soln is

$\int_{y\text{-const}} (xy+y^2)e^x\,dx + \int (0)\,dy = c$.

$\Rightarrow y\int xe^x\,dx + y^2\int e^x\,dx = c$

$\Rightarrow y[xe^x - e^x] + y^2e^x = c$

$\Rightarrow \underline{\underline{xy - y + y^2 = ce^{-x}}}$

4) $(x^2+y^2+x)\,dx + xy\,dy = 0. \to ①$

Soln:- $M = x^2+y^2+x$ ; $N = xy$.

$\frac{\partial M}{\partial y} = 2y$ ; $\frac{\partial N}{\partial x} = y$.

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Eq$^n$ ① is not exact.

$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y$ .... close to N.

$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y}{xy} = \frac{1}{x} = f(x)$.

$$\text{If} = e^{\int f(x)\,dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x.$$

$x^{ply}$ Eqⁿ ① by If $= x$.

$$(x^3 + xy^2 + x^2)\,dx + x^2y\,dy = 0 \longrightarrow ②$$

$M = x^3 + xy^2 + x^2$ ; $N = x^2y$.

$\frac{\partial M}{\partial y} = 2xy$ ; $\frac{\partial N}{\partial x} = 2xy$

Equal $\Rightarrow$ Eqⁿ ② is exact.

Soln is

$$\int_{y\text{-const}} (x^3 + xy^2 + x^2)\,dx + \int (0)\,dy = c$$

$$\Rightarrow \frac{x^4}{4} + y^2 \cdot \frac{x^2}{2} + \frac{x^3}{3} = c.$$

$$\Rightarrow \underline{3x^4 + 6x^2y^2 + 4x^3 = 12c}$$

5) $(3x^2y^4 + \underbrace{2xy}_{}) \,dx + (2x^3y^3 - x^2)\,dy = 0 \rightarrow ①$

$xy(3xy^3 + 2)$

Soln:- $M = 3x^2y^4 + 2xy$ ; $N = 2x^3y^3 - x^2$

$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$ ; $\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$.

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Eqⁿ ① is not exact

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 12x^2y^3 + 2x - 6x^2y^3 + 2x$$

$$= 6x^2y^3 + 4x$$

$$= 2x(3xy^3 + 2) \quad \ldots \text{ Close to } M.$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{2x(3xy^3 + 2)}{xy(3xy^3 + 2)} = \frac{2}{y} = g(y) \text{ (say)}$$

$$\text{If} = e^{-\int g(y)\,dy} = e^{-\int 2/y\,dy} = e^{-2\log y} = y^{-2} = 1/y^2$$

$x^{ply}$ Eqⁿ ① by $\frac{1}{y^2}$.

$$\Rightarrow \left(\frac{3x^2y^4}{y^2} + \frac{2xy}{y^2}\right) dx + \left(\frac{2x^3y^3}{y^2} - \frac{x^2}{y^2}\right) dy = 0$$

$$\Rightarrow \left(3x^2y^2 + \frac{2x}{y}\right) dx + \left(2x^3y - \frac{x^2}{y^2}\right) dy = 0 \rightarrow (2)$$

$$M = 3x^2y^2 + \frac{2x}{y} \quad ; \quad N = 2x^3y - \frac{x^2}{y^2}$$

$$\frac{\partial M}{\partial y} = 6x^2y - \frac{2x}{y^2} \quad ; \quad \frac{\partial N}{\partial x} = 6x^2y - \frac{2x}{y^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Eq}^n \ (2) \text{ is exact}$$

Soln is

$$\int_{y\text{-const}} \left(3x^2y^2 + \frac{2x}{y}\right) dx + \int (0)\, dy = c$$

$$\Rightarrow 3y^2 \cdot \frac{x^3}{3} + \frac{2}{y} \cdot \frac{x^2}{2} = c$$

$$\Rightarrow \underline{\underline{x^3y^2 + \frac{x^2}{y} = c}}$$

6) $y(x+y+1)\, dx + x(x+3y+2)\, dy = 0 \rightarrow (1)$

HW

Soln :- $M = xy + y^2 + y \quad ; \quad N = x^2 + 3xy + 2x$.

$$\frac{\partial M}{\partial y} = x + 2y + 1 \quad ; \quad \frac{\partial N}{\partial x} = 2x + 3y + 2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Eq}^n \ (1) \text{ is not exact.}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -x - y - 1 = -(x+y+1) \ \ldots \text{close to } M$$

$$\therefore \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-(x+y+1)}{y(x+y+1)} = -\frac{1}{y} = g(y)$$

$$\text{IF} = e^{-\int g(y) dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

$x^{ply}$ Eq$^n$ (1) by $y$.

$$(xy^2 + y^3 + y^2)\, dx + (x^2y + 3xy^2 + 2xy)\, dy = 0 \rightarrow (2)$$

$\frac{\partial M}{\partial y} = 2xy + 3y^2 + 2y$ ; $\frac{\partial N}{\partial x} = 2xy + 3y^2 + 2y$.

Equal $\Rightarrow$ Eqn ② is exact.

Soln is $\int\limits_{y\text{-const}} (xy^2 + y^3 + y^2)\,dx + \int (0)\,dy = c$

$\Rightarrow \quad y^2\frac{x^2}{2} + (y^3+y^2)x = c$

7) $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3)\,dy = 0 \rightarrow ①$

Soln:- $M = xy^2 - x^2$ ; $N = 3x^2y^2 + x^2y - 2x^3$

$\frac{\partial M}{\partial y} = 2xy$ ; $\frac{\partial N}{\partial x} = 6xy^2 + 2xy - 6x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Eqn ① is not exact.

$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \cancel{xy^2 - x^2 - 3x^2y^2 - x^2y + 2x^3}$

$= 2\cancel{x}y - 6xy^2 - 2\cancel{x}y + 6x^2$

$\cancel{= 6x^2(x} \quad = 6(x^2 - xy^2)$ ... close to M

$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-6(xy^2 - x^2)}{(xy^2 - x^2)} = -6 = g(y)$ (say)

$\text{If} = e^{-\int g(y)dy} = e^{\int 6\,dy} = e^{6y}$

$x^{ply}$ ① by $e^{6y}$

$(xy^2 - x^2)e^{6y}\,dx + (3x^2y^2 + x^2y - 2x^3)e^{6y}\,dy = 0 \rightarrow ②$

$\frac{\partial M}{\partial y} = (xy^2 - x^2).\,6e^{6y} + e^{6y}(2xy)$ | $\frac{\partial N}{\partial x} = e^{6y}(6xy^2 + 2xy - 6x^2)$

$= e^{6y}(6xy^2 - 6x^2 + 2xy)$

$\Rightarrow$ Eqn ② is exact

Soln is $\int\limits_{y\text{-const}} (xy^2 - x^2)e^{6y}\,dx + \int (0)\,dy = c$

$\Rightarrow \left(y^2\frac{x^2}{2} - \frac{x^3}{3}\right)e^{6y} = c$.

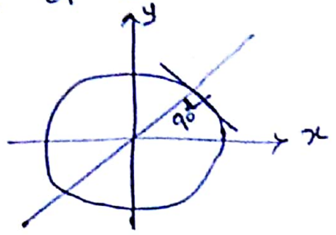
Scanned by CamScanner

# Applications of Differential Equations

## Orthogonal Trajectories :-

Defn :- If 2 family of curves are such that every member of one family intersects every member of the other family at right angles, then they are said to be orthogonal trajectories of each other.

Ex :-




## O.T in Cartesian form :-

→ Given $f(x,y,c)=0$ (or) $f(x,y)=c$, differentiate wrt $x$ and eliminate $c$.

→ Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and solve the Eqn.

Self orthogonal :- If the DE of the given family remains unaltered after replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, then the given family of curves is said to be self orthogonal.

## O.T in Polar form :-

→ Given $f(r,\theta,c)=0$ (or) $f(r,\theta)=c$, take log first, then diff wrt $\theta$ and eliminate $c$.

→ Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ and solve the Eqn.

Problems :-

1) Given $y = Ke^{-2x} + 3x$, find the member of the O.T passing through $(0,3)$

Soln :- Consider $y = Ke^{-2x} + 3x \rightarrow (1)$

diff wrt $x$

$$\frac{dy}{dx} = -2Ke^{-2x} + 3$$

(or) $$2Ke^{-2x} = 3 - \frac{dy}{dx}$$

$$\Rightarrow \quad K = \frac{e^{2x}}{2}\left(3 - \frac{dy}{dx}\right)$$

(1) $\Rightarrow$ $$y = \frac{e^{2x}}{2}\left(3 - \frac{dy}{dx}\right)e^{-2x} + 3x$$

$$y = \frac{3}{2} - \frac{1}{2}\frac{dy}{dx} + 3x$$

$$\Rightarrow \quad 2y = 3 - \frac{dy}{dx} + 6x$$

(or) $$\frac{dy}{dx} = 6x - 2y + 3 \rightarrow (2)$$

$\Rightarrow$ $\cancel{\frac{dy}{dx} + 2y = (6x+3)}$ Eqn (2) is the DE of the given family.

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = 6x - 2y + 3 \Rightarrow \frac{dx}{dy} = -6x + 2y - 3$$

(or) $$\frac{dx}{dy} + 6x = 2y - 3 \rightarrow (3)$$

Eqn (3) is the DE of the orthogonal family and it is a linear Eqn in $x$, where $P = 6$, $Q = 2y - 3$.

$$\text{If} = e^{\int P\,dy} = e^{6y}$$

Soln is $$x\,(\text{If}) = \int Q\,(\text{If})\,dy + c$$

$$\Rightarrow \quad xe^{6y} = \int (2y-3)e^{6y}\,dy + c.$$

$$\Rightarrow x e^{6y} = (2y-3)\frac{e^{6y}}{6} - \int \frac{e^{6y}}{6} \cdot 2\, dy + c$$

$$\Rightarrow x e^{6y} = \frac{(2y-3)}{6} e^{6y} - \frac{1}{3} \cdot \frac{e^{6y}}{6} + c$$

$$\Rightarrow x e^{6y} = \frac{(2y-3)}{6} e^{6y} - \frac{1}{18} e^{6y} + c \rightarrow (4)$$

This is the O.T of the given family.

At $(0,3)$, (4) $\Rightarrow$

$$0 = \frac{3}{6} \cdot e^{18} - \frac{1}{18} e^{18} + c$$

$$\Rightarrow c = \frac{1}{18} e^{18} - \frac{1}{2} e^{18} = e^{18}\left(\frac{1}{18} - \frac{1}{2}\right)$$

$$= \frac{1-9}{18} e^{18} = -\frac{8}{18} e^{18}$$

$$\boxed{c = -\frac{4}{9} e^{18}}$$

$$(4) \Rightarrow x e^{6y} = \frac{(2y-3)}{6} e^{6y} - \frac{1}{18} e^{6y} - \frac{4}{9} e^{18}$$

2) Find the O.T of the family of curves $x^2+y^2+2\lambda x+c=0$, where $\lambda$ is a parameter.

Soln:- Consider $x^2+y^2+2\lambda x+c=0 \rightarrow (1)$

diff wrt $x$.

$$2x + 2y\frac{dy}{dx} + 2\lambda = 0$$

$$\Rightarrow x + y\frac{dy}{dx} + \lambda = 0 \Rightarrow \lambda = -\left(x + y\frac{dy}{dx}\right)$$

Sub in (1), $x^2+y^2+2x\left[-\left(x+y\frac{dy}{dx}\right)\right]+c=0$

$$\Rightarrow x^2+y^2-2x^2-2xy\frac{dy}{dx}+c=0.$$

$$\Rightarrow y^2-x^2+c = 2xy\frac{dy}{dx} \rightarrow (2)$$ is the DE of given family.

Replace (or) $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eqn (II)

$$y^2 - x^2 + c = -2xy\frac{dx}{dy}$$

$$\Rightarrow \quad 2xy\frac{dx}{dy} = x^2 - y^2 - c$$

$$\Rightarrow \quad 2xy\frac{dx}{dy} - x^2 = -(c+y^2)$$

$\div$ by $y$

$$\Rightarrow \quad 2x\frac{dx}{dy} - \frac{1}{y}x^2 = -\frac{(c+y^2)}{y} \longrightarrow (1)$$

Put $x^2 = t \Rightarrow 2x\frac{dx}{dy} = \frac{dt}{dy}$

(3) $\Rightarrow \frac{dt}{dy} - \frac{1}{y}\cdot t = -\frac{(c+y^2)}{y}$ is a linear Eqn in $t$.

$$P = -1/y, \quad Q = -\frac{(c+y^2)}{y}$$

$$If = e^{\int P\,dy} = e^{\int -\frac{1}{y}\,dy} = e^{-\log y} = y^{-1} = 1/y.$$

Soln is $t\,(If) = \int Q\,(If)\,dy + k$

$$\Rightarrow \quad x^2\cdot\frac{1}{y} = \int -\frac{(c+y^2)}{y}\cdot\frac{1}{y}\,dy + k$$

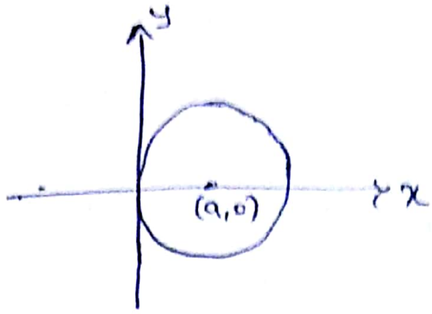
$$= -\int\left(\frac{c}{y^2} + 1\right)dy + k$$

$$= -c\cdot\frac{y^{-1}}{-1} \mp y + k$$

$$\frac{x^2}{y} = \frac{c}{y} \mp y + k.$$

PTO.

3) Show that the O.T of a family of circles passing thru' the origin having their centres on the x-axis, is a family of circles passing thru' the origin having their centres on the y-axis.




Soln:- Since the circle passing thru' the origin & centre is on the x-axis,

we have centre = (a,0), radius = a.

∴ Eqn of given family of circles is

$$(x-a)^2+(y-0)^2 = a^2$$

i.e $x^2+a^2-2ax+y^2-a^2=0.$

$\Rightarrow$ $x^2+y^2-2ax=0. \rightarrow (1)$

diff wrt $x$.

$$2x+2y\cdot\frac{dy}{dx}-2a=0$$

$\Rightarrow$ $x+y\frac{dy}{dx}-a=0$ (or) $a = x+y\frac{dy}{dx}$

$\Rightarrow$ (1) $\Rightarrow$ $x^2+y^2-2\left(x+y\frac{dy}{dx}\right)x=0.$

(or) $x^2+y^2-2x^2-2yx\frac{dy}{dx}=0.$

$\Rightarrow$ $y^2-x^2-2xy\frac{dy}{dx}=0 \rightarrow (2)$

Eqn (2) is the DE of the given family.

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$y^2-x^2+2xy\frac{dx}{dy}=0. \rightarrow (3)$ is the DE of the O.T

$\Rightarrow$ $y^2-x^2=-2xy\frac{dx}{dy}$

(or) $2xy\frac{dx}{dy}=x^2-y^2 \Rightarrow 2xy\frac{dx}{dy}-x^2=-y^2$

÷ by $y$

$$2x\frac{dx}{dy} - \frac{1}{y}\cdot x^2 = -y \rightarrow (4)$$

Put $x^2 = t \Rightarrow 2x\frac{dx}{dy} = \frac{dt}{dy}$

(4) $\Rightarrow \frac{dt}{dy} - \frac{1}{y}\cdot t = -y$ is a linear Eqn in 't'.

$P = -1/y$, $Q = -y$.

$$\text{If} = e^{\int P\,dy} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = y^{-1} = 1/y.$$

Soln is $t\,(\text{If}) = \int Q\,(\text{If})\,dy + c$

$$\Rightarrow t\cdot\frac{1}{y} = \int(-y)\cdot\frac{1}{y}\,dy + c$$

$$\Rightarrow \frac{x^2}{y} = -y + c.$$

$$\Rightarrow x^2 = -y^2 + cy.$$

(or) $x^2 + y^2 - cy = 0$.

for convenient, let $c = 2b$.

$x^2 + y^2 - 2by = 0$.

add & sub $b^2$

$x^2 + (y^2 + b^2 - 2by) - b^2 = 0$.

$\Rightarrow x^2 + (y-b)^2 = b^2$

This represents the eqn of the family of circles passing thru' the origin having their centres on the y-axis.

5) Show ~~that~~ the curve $y^2 = 4a(x+a)$ is self orthogonal.

Soln:- Consider $y^2 = 4a(x+a) \rightarrow (1)$.

diff wrt $x$.

$2y\cdot\frac{dy}{dx} = 4a. \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$.

(or) $a = \frac{y}{2}\cdot\frac{dy}{dx}$

Sub the value of 'a' in ①.

$$y^2 = 4\left(\frac{y}{2}\cdot\frac{dy}{dx}\right)\left[x + \frac{y}{2}\cdot\frac{dy}{dx}\right]$$

for convenience let $\frac{dy}{dx} = p$ (use $\frac{dy}{dx} = y_1$ instead of $p$)

$$y^2 = 2yp\left[x + \frac{py}{2}\right]$$

(or) $y = 2p\left(x + \frac{py}{2}\right)$

$$y = 2px + p^2 y.$$

$\Rightarrow \; yp^2 + 2xp - y = 0 \rightarrow$ ②

This is the DE of the given family.

Replace $\frac{dy}{dx} = p$ by $-\frac{dx}{dy} = -\frac{1}{p}$, in Eqn ②,

$$y\cdot\left(-\frac{1}{p}\right)^2 + 2x\left(-\frac{1}{p}\right) - y = 0.$$

$$\frac{y}{p^2} - \frac{2x}{p} - y = 0.$$

$$y - 2xp - p^2 y = 0$$

(or) $p^2 y + 2xp - y = 0 \rightarrow$ ③ is the DE of O.T

Since Eqn ② & ③ are same, the given system is self orthogonal.

4) Find the O.T of the family of curves

$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, ① where $\lambda$ is a parameter.

Soln:- $(b^2+\lambda)x^2 + a^2y^2 = a^2(b^2+\lambda)$

diff wrt $x$.

$$(b^2+\lambda)\,2x + 2a^2 y\frac{dy}{dx} = 0.$$

$$\Rightarrow \quad (b^2+\lambda) = -\frac{a^2 y}{x}\frac{dy}{dx}.$$

sub in Eqn ①,

$$\frac{x^2}{a^2} + \frac{y^2}{-\frac{a^2 y}{x}\frac{dy}{dx}} = 1$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{xy}{a^2}\frac{dx}{dy} = 1$$

(or) $$x^2 - xy\frac{dx}{dy} = a^2 \rightarrow (2)$$

Eq$^n$ (2) is the DE of given family of curves.

Replace $\frac{dx}{dy}$ by $-\frac{dy}{dx}$

$$x^2 + xy\frac{dy}{dx} = a^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{a^2 - x^2}{xy}$$

$$\Rightarrow y\,dy = \frac{a^2 - x^2}{x}\,dx.$$ (by separating the variables)

on Integration

$$\int y\,dy = a^2\int \frac{1}{x}\,dx - \int x\,dx + c$$

$$\Rightarrow \frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c$$

6) Show that the family of curves $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ where $\lambda$ is a parameter, is self orthogonal.

Soln :- Consider $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \rightarrow (1).$

$$\Rightarrow (b^2+\lambda)x^2 + (a^2+\lambda)y^2 = (a^2+\lambda)(b^2+\lambda).$$

diff wrt $x$.

$$(b^2+\lambda)2x + (a^2+\lambda)2y\frac{dy}{dx} = 0.$$

$$\Rightarrow (b^2+\lambda)x + (a^2+\lambda)yp = 0$$ where $p = \frac{dy}{dx}$.

$$\Rightarrow b^2x + \lambda x + a^2yp + \lambda yp = 0.$$

Scanned by CamScanner

$$\Rightarrow \lambda(x+yp) = -(b^2x+a^2yp)$$

$$\Rightarrow \lambda = -\frac{(b^2x+a^2yp)}{(x+yp)}$$

$$\therefore a^2+\lambda = a^2 - \frac{(b^2x+a^2yp)}{x+yp} = \frac{a^2x+\cancel{a^2yp}-b^2x-\cancel{a^2yp}}{x+yp}$$

$$= \frac{(a^2-b^2)x}{x+yp}$$

III$^{ly}$ $$b^2+\lambda = b^2 - \frac{(b^2x+a^2yp)}{x+yp} = \frac{\cancel{b^2x}+b^2yp-\cancel{b^2x}-a^2yp}{x+yp}$$

$$= \frac{(b^2-a^2)yp}{x+yp} = \frac{-(a^2-b^2)yp}{(x+yp)}$$

sub in ①,

$$\frac{x^2}{\frac{(a^2-b^2)x}{x+yp}} - \frac{y^2}{\frac{(a^2-b^2)yp}{x+yp}} = 1$$

$$\Rightarrow \frac{x(x+yp)}{a^2-b^2} - \frac{y(x+yp)}{p(a^2-b^2)} = 1.$$

$$\Rightarrow x(x+yp) - \frac{y}{p}(x+yp) = (a^2-b^2)$$

$$\Rightarrow (x+yp)\left(x-\frac{y}{p}\right) = a^2-b^2 \rightarrow ②.$$

This is the DE of the given family.

Replace $p$ by $-1/p$.

$$(x-y/p)(x+yp) = a^2-b^2 \rightarrow ③$$ is the DE of O.T

Since ② & ③ are same, the given family of curves is self orthogonal.

# O.T of polar curves

1) Find the O.T of the family of curves $r^n = a \sin n\theta$

<u>Soln</u>:- Consider $r^n = a \sin n\theta$

taking log on b.s

$$n \log r = \log a + \log(\sin n\theta)$$

diff wrt $\theta$.

$$n \cdot \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\sin n\theta} \times n \cos n\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta \rightarrow (1)$$ is the DE of the given family.

Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$.

$$(1) \Rightarrow -\frac{1}{r} \cdot r^2 \frac{d\theta}{dr} = \cot n\theta \rightarrow (2)$$ is the DE of O.T

$$\Rightarrow -r\frac{d\theta}{dr} = \cot n\theta \Rightarrow \frac{1}{\cot n\theta} d\theta = -\frac{1}{r} dr.$$

$$\Rightarrow \frac{1}{r} dr + \tan n\theta \, d\theta = 0.$$

$$\Rightarrow \int \frac{1}{r} dr + \int \tan n\theta \, d\theta = c$$

$$\Rightarrow \log r - \frac{\log(\cos n\theta)}{n} = \log c.$$

Note :-
$\int \tan\theta \, d\theta = -\log(\cos\theta)$
(OR)
$\int \tan\theta \, d\theta = \log(\sec\theta)$

$$\Rightarrow n \log r = \log(\cos n\theta) + \log c^n$$

$$\text{i.e. } \log r^n = \log[c^n \cos n\theta]$$

$$\Rightarrow r^n = c^n \cos n\theta \quad \text{(or) if } c^n = b.$$

$$\boxed{r^n = b \cos n\theta}$$

2) Find the O.T of the cardioide $r = a(1-\cos\theta)$.

<u>Soln</u>:- $\log r = \log a + \log(1-\cos\theta)$

diff wrt $\theta$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta} \sin\theta = \frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\sin^2(\theta/2)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot(\theta/2) \rightarrow (1)$$

This is the DE of the given family.

Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$.

$$\frac{1}{r}\left\{-r^2 \frac{d\theta}{dr}\right\} = \cot(\theta/2)$$

$\Rightarrow$ $-r \frac{d\theta}{dr} = \cot(\theta/2)$ is the DE of O.T

$$\Rightarrow \quad -\frac{dr}{r} = \tan(\theta/2)\, d\theta$$

$$\Rightarrow \quad \int \frac{dr}{r} + \int \tan(\theta/2)\, d\theta = c.$$

$$\Rightarrow \quad \log r \mp \frac{\log(\cos(\theta/2))}{1/2} = \log c.$$

$$\Rightarrow \quad \log r - 2\log \cos(\theta/2) = \log c.$$

$$\Rightarrow \quad \log r = \log \cos^2(\theta/2) + \log c$$

$$= \log\left[c \cos^2(\theta/2)\right]$$

$$r = c.\cos^2(\theta/2)$$

$$= c\left(\frac{1+\cos\theta}{2}\right)$$

Let $c/2 = b$

$\therefore$ $\boxed{r = b(1+\cos\theta)}$ is the ~~soln~~ O.T of given family.

Scanned by CamScanner

3) Find the O.T of $r = a(1+\sin\theta)$ (26)

Soln :- $\log r = \log a + \log(1+\sin\theta)$

diff wrt $\theta$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+\sin\theta}\cdot\cos\theta$$

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$-r\frac{d\theta}{dr} = \frac{\cos\theta}{1+\sin\theta}$$

$$\Rightarrow \frac{1+\sin\theta}{\cos\theta}\,d\theta = -\frac{1}{r}\,dr.$$

$$\Rightarrow \int\frac{1}{r}\,dr + \int\frac{1+\sin\theta}{\cos\theta}\,d\theta = c$$

$$\Rightarrow \log r + \int\sec\theta\,d\theta + \int\tan\theta\,d\theta = c.$$

$$\Rightarrow \log r + \log(\sec\theta+\tan\theta) + \log(\sec\theta) = \log b\ (\text{say})$$

$$\Rightarrow \log\left[r\sec\theta(\sec\theta+\tan\theta)\right] = \log b$$

$$\Rightarrow r\sec\theta(\sec\theta+\tan\theta) = b$$

$$\Rightarrow r\cdot\frac{1}{\cos\theta}\left[\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right] = b$$

$$\Rightarrow \frac{r(1+\sin\theta)}{\cos^2\theta} = b\,\cancel{\cos^2\theta}$$

$$\Rightarrow r\frac{(1+\sin\theta)}{(1-\sin^2\theta)} = b$$

$$\Rightarrow \frac{r(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} = b$$

(or) $\boxed{r = b(1-\sin\theta)}$ is the required O.T

P TO

4) Find the O.T of the family of curves $r = 4a\sec\theta\tan\theta$.

Soln:-

$$r = 4a\sec\theta\tan\theta.$$

$$\log r = \log(4a) + \log(\sec\theta) + \log(\tan\theta)$$

diff wrt $\theta$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\sec\theta}\cdot\sec\theta.\tan\theta + \frac{1}{\tan\theta}\cdot\sec^2\theta$$

$$= \tan\theta + \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos^2\theta}$$

$$= \tan\theta + \frac{1}{\sin\theta.\cos\theta}$$

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\frac{1}{r}\cdot\left\{-r^2\frac{d\theta}{dr}\right\} = \tan\theta + \frac{1}{\sin\theta.\cos\theta}$$

$$\Rightarrow \quad -r\frac{d\theta}{dr} = \tan\theta + \frac{1}{\sin\theta.\cos\theta}$$

$$(\text{or}) \quad \frac{1}{\tan\theta + \frac{1}{\sin\theta.\cos\theta}}\, d\theta = -\frac{1}{r}dr$$

$$\Rightarrow \quad \frac{1}{\frac{\sin\theta}{\cos\theta} + \frac{1}{\sin\theta.\cos\theta}}\, d\theta = -\frac{1}{r}dr.$$

$$\Rightarrow \quad \frac{1}{\frac{\sin^2\theta+1}{\sin\theta\cos\theta}}\, d\theta = -\frac{1}{r}dr.$$

$$\Rightarrow \quad \int\frac{\sin\theta.\cos\theta}{1+\sin^2\theta}\, d\theta = -\int\frac{1}{r}dr$$

$$\Rightarrow \quad \log r + \frac{1}{2}\log(1+\sin^2\theta) = \log c.$$

$$\Rightarrow \quad 2\log r + \log(1+\sin^2\theta) = 2\log c = \log c^2$$

$$\Rightarrow \quad \log(r^2(1+\sin^2\theta)) = \log k.$$

$$\Rightarrow \quad r^2(1+\sin^2\theta) = K$$ is the required O.T.

5) Find the O.T of family of curves $\left(r+\frac{K^2}{r}\right)\cos\theta = a$, where 'a' is an arbitrary constant.

Soln:- $\left(r+\frac{K^2}{r}\right)\cos\theta = a$.

$$\log\left(r+\frac{K^2}{r}\right)+\log(\cos\theta) = \log a.$$

diff wrt $\theta$

$$\frac{1}{r+\frac{K^2}{r}}\left(1-\frac{K^2}{r^2}\right)\frac{dr}{d\theta}+\frac{1}{\cos\theta}(-\sin\theta) = 0.$$

$$\Rightarrow \frac{r}{r^2+K^2}\cdot\frac{r^2-K^2}{r^2}\frac{dr}{d\theta}-\tan\theta = 0.$$

$$\Rightarrow \frac{(r^2-K^2)}{r(r^2+K^2)}\frac{dr}{d\theta}-\tan\theta = 0.$$

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\frac{(r^2-K^2)}{r(r^2+K^2)}\left\{-r^2\frac{d\theta}{dr}\right\}-\tan\theta = 0.$$

$$\Rightarrow -\frac{r(r^2-K^2)}{(r^2+K^2)}\frac{d\theta}{dr}-\tan\theta = 0.$$

$$\Rightarrow \frac{1}{\tan\theta}d\theta+\frac{(r^2+K^2)}{r(r^2-K^2)}dr = 0.$$

$$\Rightarrow \int\cot\theta\, d\theta+\int\frac{(r^2+K^2)}{r(r+K)(r-K)}dr = c \rightarrow (1)$$

Let $\frac{r^2+K^2}{r(r+K)(r-K)} = \frac{A}{r}+\frac{B}{r+K}+\frac{C}{r-K}$

$$\Rightarrow r^2+K^2 = A(r^2-K^2)+Br(r-K)+Cr(r+K)$$

Put $r=K$; $2K^2 = CK(2K) \Rightarrow \boxed{C=1}$

Coeft of $r^2$ on b.s

$1 = A+B+C \Rightarrow A+B=0$

coeff of $r$ on b.s

$0 = -BK+CK \Rightarrow -B+C=0$

$B=C \Rightarrow \boxed{B=1}$

$\Rightarrow \boxed{A=-1}$

$\therefore \quad \dfrac{r^2+k^2}{r(r^2-k^2)} = -\dfrac{1}{r} + \dfrac{1}{r+k} + \dfrac{1}{r-k}.$

Sub in ①,

$$\int \cot\theta \, d\theta - \int \frac{1}{r}\,dr + \int \frac{1}{r+k}\,dr + \int \frac{1}{r-k}\,dr = c$$

$$\Rightarrow \log(\sin\theta) - \log r + \log(r+k) + \log(r-k) = \log b.$$

$$\Rightarrow \log\left[\frac{\sin\theta\,(r^2-k^2)}{r}\right] = \log b.$$

$$\Rightarrow \frac{(r^2-k^2)}{r}\sin\theta = b.$$

(or) $$\left(r - \frac{k^2}{r}\right)\sin\theta = b$$

6) Prove that the curves $r = a(\sin\theta + \cos\theta)$ and $r = b(\sin\theta - \cos\theta)$ intersect eachother orthogonally.

Soln :- Consider $r = a(\sin\theta + \cos\theta)$

$$\log r = \log a + \log(\sin\theta + \cos\theta).$$

diff wrt $\theta$.

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\sin\theta + \cos\theta}(\cos\theta - \sin\theta)$$

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$.

$$-r\frac{d\theta}{dr} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}.$$

$$\Rightarrow \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}\,d\theta = -\frac{1}{r}\,dr.$$

$$\Rightarrow \int \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}\,d\theta + \int \frac{1}{r}\,dr = c.$$

Put $\cos\theta - \sin\theta = t$

$$\Rightarrow (-\sin\theta - \cos\theta)\,d\theta = dt$$

$\Rightarrow (\sin\theta + \cos\theta)\, d\theta = -dt$

$\therefore \int \frac{-dt}{t} + \int \frac{1}{r}\, dr = c$

$\Rightarrow -\log t + \log r = \log k.$

$\Rightarrow \log\left(\frac{r}{t}\right) = \log k.$

$\Rightarrow r = tk$

$\Rightarrow r = k(\cos\theta - \sin\theta)$

(or) $r = -c(\sin\theta - \cos\theta)$

Let $-c = b$.

$$\boxed{r = b(\sin\theta - \cos\theta)}$$

Hw

1) Find the O.T of $r^n = a^n \cos(n\theta)$, where 'a' is the parameter.
$\rightarrow r^n = b^n \sin(n\theta)$

2) Find the O.T of $r = a(1+\cos\theta)$
$\rightarrow r = b(1-\cos\theta)$.

3) Find the O.T of $r^2 = a^2 \cos 2\theta$.
$\rightarrow r^2 = b^2 \sin 2\theta$

4) Find the O.T of $\frac{2a}{r} = 1-\cos\theta$ $\rightarrow r\cos^2\left(\frac{\theta}{2}\right) = k.$

5) Find the O.T of $r = e^{a\theta}$ $\rightarrow (\log r)^2 + \theta^2 = k.$

6) S.T the O.T of the family of cardioides $r = a\cos^2(\theta/2)$ is another family of cardioides $r = b\sin^2(\theta/2)$.

7) Test for self orthogonality $r^n = a\sin n\theta$.
$\rightarrow$ (Not self orthogonal, $r^n = b\cos n\theta$)

# Applications of DE

## Newton's law of Cooling :

The law states that, "the change of temperature of a body is proportional to the difference b/w the temperature of a body & that of the surrounding medium".

Let $t_1°c$ be the initial temp of the body & $t_2°c$ be the constant temp of the medium.

Let $T°c$ be the temp of the body at any time 't', then

By Newton's law of cooling,

$$\frac{dT}{dt} \propto (T-t_2)$$

(or) $\frac{dT}{dt} = -k(T-t_2)$ where $k$ is the constant of proportionality & $-$ve sign indicates the cooling of the body with the increase in time.

Since $t_1°c$ is the initial temp of the body,

we write $T=t_1$ when $t=0$ (or) $T(0)=t_1$

Now to solve the first order DE,

$\frac{dT}{dt} = -k(T-t_2)$ with the condn $T(0)=t_1$

$$\Rightarrow \int \frac{dT}{(T-t_2)} = \int -k\,dt + \alpha.$$

$$\Rightarrow \log(T-t_2) = -kt+\alpha$$

$$\Rightarrow T-t_2 = e^{-kt+\alpha} = e^{-kt}\cdot e^{\alpha}$$

say $e^{\alpha} = c.$

$$\therefore T-t_2 = c\,e^{-kt} \rightarrow ①$$

Condn: $T=t_1$ when $t=0$

① $\Rightarrow t_1-t_2 = c\,e^0 = c \Rightarrow \boxed{c=t_1-t_2}$

sub 'c' in ①,

$$T-t_2 = (t_1-t_2)e^{-kt} \Rightarrow \boxed{T = t_2+(t_1-t_2)e^{-kt}}$$

This is the expression for the temperature function.

Problems:

1) If a substance cools from 370°c to 330°c in 10 minutes, when the temp of the surrounding air is 290°c. Find the temp of the substance after 40 minutes.

Soln:- According to Newton's law of cooling, the expression for temp fn at any time t is

$$\boxed{T = t_2 + (t_1 - t_2) e^{-kt}} \rightarrow (1).$$

By data, $t_1 = 370$, $t_2 = 290$,

~~(and T = 330°c when t = 10).~~

(1) $\Rightarrow$ $T = 290 + 80 e^{-kt}$ $\rightarrow$ (2)

Given $T = 330$ when $t = 10$.

$$330 = 290 + 80 e^{-10k}$$

$\Rightarrow$ $80 e^{-10k} = 40$. $\Rightarrow$ $e^{-10k} = 0.5$

$\Rightarrow$ $-10k = \log_e 0.5$

$\Rightarrow$ $-10k = -0.6931$

$\Rightarrow$ $\boxed{k \simeq 0.0693}$

Sub k in (2),

$$T = 290 + 80 e^{-0.0693t}$$

Now to find T when t = 40min

$$T = 290 + 80 e^{-0.0693(40)}$$

$$T = 295^\circ c$$

Thus the temp of the body at the end of 40 min is 295°c.

PTO

2) If the temp of the air is 30°c and a metal ball cools from 100°c to 70°c in 15min. Find how long will it take for the metal ball to reach a temp of 40°c. (30)

Soln:- By Newton's law of cooling, the expression for temp fn at any time 't' is

$$T = t_2 + (t_1 - t_2)e^{-kt} \rightarrow (1)$$

By data, $t_1 = 100$, $t_2 = 30$ & $T = 70$ ~~at~~ when $t = 15$

$\therefore \quad T = 30 + 70e^{-kt}$

By applying the initial condn,

$$70 = 30 + 70e^{-15k}$$

$$\Rightarrow e^{-15k} = \frac{40}{70} = \frac{4}{7} \Rightarrow e^{15k} = 7/4$$

$$\Rightarrow 15k = \log_e(7/4)$$

$$\Rightarrow \boxed{k \simeq 0.0373}$$

Thus we have $T = 30 + 70e^{-0.0373t}$

Now to find 't' when $T = 40$

$$\therefore 40 = 30 + 70e^{-0.0373t}$$

$$\Rightarrow e^{-0.0373t} = \frac{10}{70} = \frac{1}{7}$$

$$\Rightarrow e^{0.0373t} = 7 \Rightarrow t = \frac{\log_e 7}{0.0373}$$

$$\Rightarrow \boxed{t \simeq 52.17} \approx 52.2$$

Thus it takes 52.2 min for the metal ball to reach a temp of 40°c.

3) A cake is removed from an oven at 210° F & left to cool at room temp which is 70° F. After 30 min, the temp of the cake is 150° F. Find the temp of the cake at 't' min (after it is removed from the oven). When will it be at 120° F ?

Soln:- By Newton's law of cooling, the expression for temp fn at any time 't' is

$$T = t_2 + (t_1 - t_2) e^{-kt} \rightarrow ①$$

By data, $t_1 = 210$, $t_2 = 70$, and $T = 150$ when $t = 30$

$$\therefore \quad T = 70 + (140) e^{-kt}$$

Applying the initial condn.

$$150 = 70 + 140 e^{-30k}$$

$$\Rightarrow \quad e^{-30k} = \frac{80}{140} = \frac{8}{14} = \frac{4}{7}$$

$$\Rightarrow \quad e^{30k} = 7/4$$

$$k = \frac{\log_e(7/4)}{30} \Rightarrow \boxed{k \simeq 0.01865}$$

Thus we have $T = 70 + 140 e^{-0.01865t}$

This is the expression for temp of the cake at 't' min (after it is removed from the oven).

Now to find 't' when $T = 120°$ F.

$$120 = 70 + 140 e^{-0.01865t}$$

$$\Rightarrow \quad e^{-0.01865t} = \frac{50}{140} = \frac{5}{14}$$

$$\Rightarrow \quad e^{0.01865t} = 14/5 \Rightarrow t = \frac{\log_e(14/5)}{0.01865}$$

$$\Rightarrow \boxed{t \simeq 55.2}$$

Thus the cake will ~~take~~ be at 120° F after about 55.2 min.

4) A bottle of mineral water at a room temp of 72° F is kept in a refrigerator where the temp is 44° F. After half an hour, water cooled to 61° F. (31)

(i) what is the temp of the mineral water in another half an hour?

(ii) How long will it take to cool to 50° F?

Soln:- Acc to Newton's law of Cooling, the expression for temp fn is $T = t_2 + (t_1 - t_2)e^{-kt}$

By data, $t_1 = 72$, $t_2 = 44$ & $T = 61$ when $t = 30$

$\therefore T = 44 + 28e^{-kt} \longrightarrow ①$

(i) By applying the initial condn,

$61 = 44 + 28e^{-k(30)} \Rightarrow e^{-30k} = \frac{17}{28}$

$\Rightarrow e^{30k} = 28/17 \Rightarrow k = \frac{\log_e(28/17)}{30}$

$\Rightarrow \boxed{k \simeq 0.0166}$

Now to find T when $t = 30 + 30 = 60$ min

$\therefore (T)_{t=60} = 44 + 28e^{-0.0166(60)} \simeq 54.3$

Thus the temp of the mineral water after another half an hour (1 hr = 60 min) is 54.3° F.

(ii) To find 't' when T = 50.

$① \Rightarrow 50 = 44 + 28e^{-0.0166t}$

$\Rightarrow e^{-0.0166t} = \frac{6}{28} = 3/14$

$\Rightarrow e^{0.0166t} = 14/3 \Rightarrow t = \frac{\log_e(14/3)}{0.0166} \Rightarrow \boxed{t \simeq 92.8}$

Thus it takes 92.8 min (about 1½ hrs) for mineral water to cool to 50° F.