MODULE II-DIFFERENTIAL EQUATIONS OF HIGHER ORDER

INTRODUCTION:

We have studied methods of solving ordinary differential equations of first order and first degree, in chapter-7 (Ist semester). In this chapter, we study differential equations of second and higher orders. Differential equations of second order arise very often in physical problems, especially in connection with mechanical vibrations and electric circuits.

Linear Differential Equations of Second and Higher Order With Constant Coefficients

A differential equation of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = X \qquad \dots(1)$$

where X is a function of x and $a_1, a_2 \dots, a_n$ are constants is called a linear differential equation of n^{th} order with constant coefficients. Since the highest order of the derivative appearing in (1) is n, it is called a differential equation of n^{th} order and it is called linear.

Using the familiar notation of differential operators:

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad D^3 = \frac{d^3}{dx^3} \dots, D^n = \frac{d^n}{dx^n}$$

Then (1) can be written in the form

$$\{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots a_n\} y = X$$

i.e., $$f(D)\, y = X \qquad \dots(2)$$

where $$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots a_n.$$

Here $f(D)$ is a polynomial of degree n in D

If $x = 0$, the equation

$$f(D)\, y = 0$$

is called a homogeneous equation.

If $x \neq 0$ then the Eqn. (2) is called a non-homogeneous equation.

Solution of A Homogeneous Second Order Linear Differential equation

We consider the homogeneous equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

where p and q are constants

$$(D^2 + pD + q)\, y = 0$$

The Auxiliary equations (A.E.) put $D = m$

$$m^2 + pm + q = 0$$

Eqn. (3) is called auxiliary equation (A.E.) or characteristic equation of the D.E. eqn. (quadratic in m, will have two roots in general. There are three cases.

Case (*i*): Roots are real and distinct

The roots are real and distinct, say m_1 and m_2 *i.e.,* $m_1 \neq m_2$

Hence, the general solution of eqn. (1) is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

where C_1 and C_2 are arbitrary constant.

Case (*ii*): Roots are equal

The roots are equal *i.e.,* $m_1 = m_2 = m$.

Hence, the general solution of eqn. (1) is

$$y = (C_1 + C_2 x)\, e^{mx}$$

where C_1 and C_2 are arbitrary constant.

Case (*iii*): Roots are complex

The Roots are complex, say $\alpha \pm i\beta$

Hence, the general solution is

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

where C_1 and C_2 are arbitrary constants.

Note. Complementary Function (C.F.) which itself is the general solution of the D.E.

1. *Solve* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$

Solution. Given equation is $(D^2 - 5D + 6)\, y = 0$

A.E. is $m^2 - 5m + 6 = 0$

i.e., $(m-2)(m-3) = 0$

i.e., $m = 2, 3$

$\therefore$ $m_1 = 2, m_2 = 3$

$\therefore$ The roots are real and distinct.

$\therefore$ The general solution of the equation is

$$y = C_1 e^{2x} + C_2 e^{3x}.$$

2. *Solve* $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0.$

Solution. Given equation is $(D^3 - D^2 - 4D + 4)\, y$

A.E. is $m^3 - m^2 - 4m + 4 = 0$

$$m^2 (m - 1) - 4 (m - 1) = 0$$

$$(m - 1) (m^2 - 4) = 0$$

$$m = 1, m = \pm 2$$

$$m_1 = 1, m_2 = 2, m_3 = -2$$

$\therefore$ The general solution of the given equation is

$$y = C_1 e^x + C_2 e^{2x} + C_3$$

3. *Solve* $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0.$

Solution. The D.E. can be written as

$$(D^2 - D - 6)\, y = 0$$

A.E. is $m^2 - m - 6 = 0$

$\therefore$ $(m - 3) (m + 2) = 0$

$\therefore$ $m = 3, -2$

$\therefore$ The general solution is

$$y = C_1 e^{3x} + C_2 e^{-2x}.$$

4. *Solve* $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0.$

Solution. The D.E. can be written as

$$(D^2 + 8D + 16)\, y = 0$$

A.E. is $m^2 + 8m + 16 = 0$

$\therefore$ $(m + 4)^2 = 0$

$$(m + 4) (m + 4) = 0$$

$$m = -4, -4$$

$\therefore$ The general solution is

$$y = (C_1 + C_2 x)\, e^{-4x}.$$

5. *Solve* $\dfrac{d^2y}{dx^2} + w^2 y = 0.$

Solution. Equation can be written as

$$(D^2 + w^2)\, y = 0$$

A.E. is $m^2 + w^2 = 0$

$$m^2 = -w^2 = w^2 i^2 \quad (i^2 = -1)$$
$$m = \pm w\, i$$

This is the form $\alpha \pm i\beta$ where $\alpha = 0$, $\beta = w$.

$\therefore$ The general solution is

$$y = e^{0t}(C_1 \cos wt + C_2 \sin wt)$$

$\therefore$ $y = C_1 \cos wt + C_2 \sin wt$.

6. *Solve* $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13y = 0.$

Solution. The equation can be written as

$$(D^2 + 4D + 13)\, y = 0$$

A.E. is $m^2 + 4m + 13 = 0$

$$m = \frac{-4 \pm \sqrt{16-52}}{2}$$
$$= -2 \pm 3i \text{ (of the form } \alpha \pm i\beta)$$

$\therefore$ The general solution is

$$y = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x).$$

INVERSE DIFFERENTIAL OPERATOR AND PARTICULAR INTEGRAL

Consider a differential equation

$$f(D)\, y = x \qquad \text{...(1)}$$

Define $\dfrac{1}{f(D)}$ such that

$$f(D)\left\{\frac{1}{f(D)}\right\} x = x \qquad \text{...(2)}$$

Here $f(D)$ is called the inverse differential operator. Hence from Eqn. (1), we obtain

$$y = \frac{1}{f(D)}\, x \qquad \text{...(3)}$$

Since this satisfies the Eqn. (1) hence the particular integral of Eqn. (1) is given by Eqn. (3)

Thus, particular Integral (P.I.) = $\dfrac{1}{f(D)}\, x$

The inverse differential operator $\dfrac{1}{f(D)}$ is linear.

i.e., $\dfrac{1}{f(D)}\{ax_1 + bx_2\} = a\dfrac{1}{f(D)}x_1 + b\dfrac{1}{f(D)}x_2$

where a, b are constants and x_1 and x_2 are some functions of x.

Special Forms of The Particular Integral

Type 1: P.I. of the form $\dfrac{e^{ax}}{f(D)}$

We have the equation $f(D)\, y = e^{ax}$

Let $f(D) = D^2 + a_1 D + a_2$

We have $D(e^{ax}) = a\, e^{ax},\ D^2(e^{ax}) = a^2 e^{ax}$ and so on.

$$\therefore \quad f(D)\, e^{ax} = (D^2 + a_1 D + a_2)\, e^{ax}$$
$$= a^2 e^{ax} + a_1 . a e^{ax} + a_2 e^{ax}$$
$$= (a^2 + a_1 . a + a_2)\, e^{ax} = f(a)\, e^{ax}$$

Thus $f(b)\, e^{ax} = f(a)\, e^{ax}$

Operating with $\dfrac{1}{f(D)}$ on both sides

We get,
$$e^{ax} = f(a) . \frac{1}{f(D)} . e^{ax}$$

or
$$\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(D)}$$

In particular if $f(D) = D - a$, then using the general formula.

We get,
$$\frac{1}{D-a} e^{ax} = \frac{e^{ax}}{(D-a)\phi(D)} = \frac{1}{D-a} . \frac{e^{ax}}{\phi(a)}$$

i.e.,
$$\frac{e^{ax}}{f(D)} = \frac{1}{\phi(a)} e^{ax} \int 1 . d\, x = \frac{1}{\phi(a)} . x\, e^{ax} \qquad \ldots(1)$$

$$\therefore \quad f'(a) = 0 + \phi(a)$$

or
$$f'(a) = \phi(a)$$

Thus, Eqn. (1) becomes
$$\frac{e^{ax}}{f(D)} = x . \frac{e^{ax}}{f'(D)}$$

where $f(a) = 0$

and $f'(a) \neq 0$

This result can be extended further also if

$$f'(a) = 0, \ \frac{e^{ax}}{f(D)} = x^2 . \frac{e^{ax}}{f''(a)} \text{ and so on.}$$

Type 2: P.I. of the form $\dfrac{\sin ax}{f(D)}, \dfrac{\cos ax}{f(D)}$

We have $D(\sin ax) = a \cos ax$

$$D^2 (\sin ax) = -a^2 \sin ax$$
$$D^3 (\sin ax) = -a^3 \cos ax$$
$$D^4 (\sin ax) = a^4 \sin ax$$
$$= (-a^2)^2 \sin ax \text{ and so on.}$$

Therefore, if $f(D^2)$ is a rational integral function of D^2 then $f(D^2) \sin ax = f(-a^2) \sin ax$.

Hence $\frac{1}{f(D^2)}\left\{f(D^2)\sin ax\right\} = \frac{1}{f(D^2)} f(-a^2) \sin ax$

i.e., $\sin ax = f(-a^2) \frac{1}{f(D^2)} \sin ax$

i.e., $\frac{1}{f(D^2)} \sin ax = \frac{\sin ax}{f(-a^2)}$

Provided $f(-a^2) \neq 0$ ...(1)

Similarly, we can prove that

$$\frac{1}{f(D^2)} \cos ax = \frac{\cos ax}{f(-a^2)}$$

if $f(-a^2) \neq 0$

In general, $\frac{1}{f(D^2)} \cos ax = \frac{\cos ax}{f(-a^2)}$

if $f(-a^2) \neq 0$ ...(2)

$$\frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b)$$

and $\frac{1}{f(D^2)} \cos(ax+b) = \frac{1}{f(-a^2)} \cos(ax+b)$

These formula can be easily remembered as follows.

$$\frac{1}{D^2+a^2} \sin ax = \frac{x}{2}\int \sin ax\, dx = \frac{-x}{2a} \cos ax$$

$$\frac{1}{D^2+a^2} \cos ax = \frac{x}{2}\int \cos ax\, ax = \frac{x}{2a} \sin ax.$$

Type 3: P.I. of the form $\frac{\phi(x)}{f(D)}$ where $\phi(x)$ is a polynomial in x, we seeking the polynomial Eqn. as the particular solution of

$$f(D)y = \phi(x)$$

where $\phi(x) = a_0 x^n + a_1 x^{n-1} + \ldots a_{n-1} x + a_n$

Hence P.I. is found by divisor. By writing $\phi(x)$ in descending powers of x and $f(D)$ in ascending powers of D. The division get completed without any remainder. The quotient so obtained in the process of division will be particular integral.

Type 1

1. *Solve* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{5x}$.

Solution. We have

$$(D^2 - 5D + 6)\,y = e^{5x}$$

A.E. is $m^2 - 5m + 6 = 0$

i.e., $(m-2)(m-3) = 0$

$\Rightarrow$ $m = 2, 3$

Hence the complementary function is

$\therefore$ $\text{C.F.} = C_1 e^{2x} + C_2 e^{3x}$

Particular Integral (P.I.) is

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6} e^{5x} \qquad (D \to 5)$$

$$= \frac{1}{5^2 - 5\times 5 + 6} e^{5x} = \frac{e^{5x}}{6}.$$

$\therefore$ The general solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2x} + C_2 e^{3x} + \frac{e^{5x}}{6}.$$

2. *Solve* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10e^{3x}$.

Solution. We have

$$(D^2 - 3D + 2)\,y = 10\,e^{3x}$$

A.E. is $m^2 - 3m + 2 = 0$

i.e., $(m-2)(m-1) = 0$

$m = 2, 1$

$\text{C.F.} = C_1 e^{2x} + C_2 e^{x}$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} 10e^{3x} \qquad (D \to 3)$$

$$= \frac{1}{3^2 - 3\times 3 + 2} 10e^{3x}$$

$$\text{P.I.} = \frac{10\,e^{3x}}{2}$$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2x} + C_2 e^{x} + \frac{10\,e^{3x}}{2}.$$

3. *Solve* $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = e^{2x}$.

Solution. Given equation is

$$(D^2 - 4D + 4)y = e^{2x}$$

A.E. is $\quad m^2 - 4m + 4 = 0$

i.e., $\quad (m-2)(m-2) = 0$

$$m = 2, 2$$

$$\text{C.F.} = (C_1 + C_2)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4} e^{2x} \qquad (D = 2)$$

$$= \frac{1}{2^2 - 4(2) + 4} e^{2x} \qquad (Dr = 0)$$

Differentiate the denominator and multiply 'x'

$$= x \cdot \frac{1}{2D - 4} e^{2x} \qquad (D \to 2)$$

$$= x \cdot \frac{1}{2(2) - 4} e^{2x} \qquad (Dr = 0)$$

Again differentiate denominator and multiply 'x'

$$= x^2 \frac{1}{2} e^{2x}$$

$$\text{P.I.} = \frac{x^2 e^{2x}}{2}$$

$$y = \text{C.F.} + \text{P.I.} = (C_1 + C_2 x)\, e^{2x} + \frac{x^2 e^{2x}}{2}.$$

TYPE 2:

1. *Solve* $(D^3 + D^2 - D - 1)\, y = \cos 2x.$

Solution. The A.E. is

$$m^3 + m^2 - m - 1 = 0$$

$$i.e.,\ m^2(m+1) - 1\,(m+1) = 0$$

$$(m+1)(m^2-1) = 0$$

$$m = -1,\ m^2 = 1$$

$$m = -1,\ m = \pm 1$$

$$\therefore \quad m = -1, -1, 1$$

$$\text{C.F.} = C_1 e^x + (C_2 + C_3 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{D^3 + D^2 - D - 1} \cos 2x \qquad (D^2 \to -2^2)$$

$$= \frac{1}{(D+1)(D^2-1)} \cos 2x$$

$$= \frac{1}{(D+1)(-2^2-1)} \cos 2x$$

$$= \frac{-1}{5} \frac{1}{D+1} \cos 2x$$

$$= \frac{-1}{5} \frac{\cos 2x}{D+1} \times \frac{D-1}{D-1}$$

$$= \frac{-1}{5} \frac{(D-1)\cos 2x}{D^2-1} \qquad (D^2 \to -2^2)$$

$$= \frac{-1}{5}\left[\frac{-2\sin 2x - \cos 2x}{-2^2-1}\right]$$

$$= \frac{-1}{25}\,(2\sin 2x + \cos 2x)$$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^x + (C_2 + C_3 x)\, e^{-x} - \frac{1}{25}\,(2\sin 2x + \cos 2x).$$

2. *Solve* $(D^2 + D + 1)\, y = \sin 2x$.

Solution. The A.E. is

$$m^2 + m + 1 = 0$$

i.e.,
$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

Hence the C.F. is

$$\text{C.F.} = e^{-\frac{x}{2}}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1}\sin 2x \qquad (D^2 \to -2$$

$$= \frac{1}{-2^2 + D + 1}\sin 2x$$

$$= \frac{1}{D-3}\sin 2x$$

Multiplying and dividing by $(D + 3)$

$$= \frac{(D+3)\sin 2x}{D^2 - 9}$$

$$= \frac{(D+3)\sin 2x}{-2^2 - 9} = \frac{-1}{13}(2\cos 2x + 3\sin 2x)$$

$$\therefore\ y = \text{C.F.} + \text{P.I.} = e^{\frac{-x}{2}}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right] - \frac{1}{3}\ (2\cos 2x + 3\sin 2x).$$

3. *Solve* $(D^2 + 5D + 6)\, y = \cos x + e^{-2x}$.

Solution. The A.E. is

$$m^2 + 5m + 6 = 0$$

i.e.,
$$(m + 2)(m + 3) = 0$$

$$m = -2, -3$$

$$\text{C.F.} = C_1 e^{-2x} + C_2 e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2 + 5D + 6}\cdot [\cos x + e^{-2x}]$$

$$= \frac{\cos x}{D^2 + 5D + 6} + \frac{e^{-2x}}{D^2 + 5D + 6}$$

$$= \text{P.I.}_1 + \text{P.I.}_2$$

$$\text{P.I.}_1 = \frac{\cos x}{D^2 + 5D + 6} \qquad (D^2 = -1^2)$$

$$= \frac{\cos x}{-1^2 + 5D + 6} = \frac{\cos x}{5D + 5}$$

$$= \frac{1}{5}\ \frac{\cos x\,(D-1)}{(D+1)(D-1)}$$

$$= \frac{1}{5}\ \frac{(D-1)\cos x}{D^2-1}$$

$$= \frac{1}{5}\ \frac{-\sin x-\cos x}{-1^2-1}$$

$$= \frac{-1}{5}\ \frac{\sin x+\cos x}{-2}$$

$$= \frac{1}{10}(\sin x+\cos x)$$

$$\text{P.I.}_2 = \frac{e^{-2x}}{D^2+5D+6} \qquad (D \to -2)$$

$$= \frac{e^{-2x}}{(-2)^2+5\times-2+6} \qquad (Dr = 0)$$

Differential and multiply 'x'

$$= \frac{x\,e^{-2x}}{2D+5} \qquad (D \to -2)$$

$$= \frac{x\,e^{-2x}}{2(-2)+5} = \frac{x\,e^{-2x}}{1} = x\,e^{-2x}$$

$$\text{P.I.} = \frac{1}{10}\ (\sin x + \cos x) + x\ e^{-2x}$$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1\,e^{-2x} + C_2\,e^{-3x} + \frac{1}{10}\ (\sin x + \cos x) + x\,e^{-2x}.$$

Type 3

1. *Solve* $y'' + 3y' + 2y = 12x^2$.

Solution. We have $(D^2 + 3D + 2)\, y = 12x^2$

A.E. is $\quad m^2 + 3m + 2 = 0$

i.e., $\quad (m + 1)(m + 2) = 0$

$\Rightarrow \quad m = -1, -2$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^{-2x}$$

$$\text{P.I.} = \frac{12x^2}{D^2 + 3D + 2}$$

We need to divide for obtaining the P.I.

$$\begin{array}{r|l}
 & 6x^2 - 18x + 21 \\
\hline
2 + 3D + D^2 & 12x^2 \\
 & 12x^2 + 36x + 12 \\
\hline
 & \quad - 36x - 12 \\
 & \quad - 36x - 54 \\
\hline
 & \qquad\quad 42 \\
 & \qquad\quad 42 \\
\hline
 & \qquad\quad 0
\end{array}$$

Note:
$3D(6x^2) = 36x$
$D^2(6x^2) = 12$

Hence, P.I. $= 6x^2 - 18x + 21$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + 6x^2 - 18x + 21.$$

2. *Solve* $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 2x + x^2$.

Solution. We have $(D^2 + 2D + 1)\, y = 2x + x^2$

A.E. is $\quad m^2 + 2m + 1 = 0$

i.e., $\quad (m + 1)^2 = 0$

i.e., $\quad (m + 1)(m + 1) = 0$

$\Rightarrow \quad m = -1, -1$

$$\text{C.F.} = (C_1 + C_2 x)\, e^{-x}$$

$$\text{P.I.} = \frac{2x + x^2}{D^2 + 2D + 1} = \frac{x^2 + 2x}{1 + 2D + D^2}$$

$$
\begin{array}{r|l}
 & x^2 - 2x + 2 \\ \hline
1 + 2D + D^2 & x^2 + 2x \\
 & x^2 + 4x + 2 \\ \hline
 & -2x - 2 \\
 & -2x - 4 \\ \hline
 & 2 \\
 & 2 \\ \hline
 & 0
\end{array}
$$

$\therefore$ $\quad$ P.I. $= x^2 - 2x + 2$

$\therefore$ $\quad$ $y = $ C.F. + P.I.

$$= (C_1 + C_2 x)\, e^{-x} + (x^2 - 2x + 2).$$

Method of Variation of Parameters

Consider a linear differential equation of second order

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = \phi(x) \qquad ...(1)$$

where a_1, a_2 are functions of 'x'. If the complimentary function of this equation is known then we can find the particular integral by using the method known as the method of variation of parameters.

Suppose the complimentary function of the Eqn. (1) is

C.F. $= C_1 y_1 + C_2 y_2$ where C_1 and C_2 are constants and y_1 and y_2 are the complementary solutions of Eqn. (1)

The Eqn. (1) implies that

$$y_1'' + a_1 y_1' + a_2 y_1 = 0 \qquad ...(2)$$

$$y_2'' + a_1 y_2' + a_2 y_2 = 0 \qquad ...(3)$$

We replace the arbitrary constants C_1, C_2 present in C.F. by functions of x, say A, B respectively,

$$\therefore \qquad y = Ay_1 + By_2 \qquad ...(4)$$

is the complete solution of the given equation.

The procedure to determine A and B is as follows.

From Eqn. (4) $\qquad y' = (Ay_1' + By_2') + (A'y_1 + B'y_2) \qquad ...(5)$

We shall choose A and B such that

$$A'y_1 + B'y_2 = 0 \qquad ...(6)$$

Thus Eqn. (5) becomes $\quad y_1' = Ay_1' + By_2' \qquad ...(7)$

Differentiating Eqn. (7) w.r.t. 'x' again, we have

$$y'' = (Ay_1'' + Ay_2'') + (A'y_1' + B'y_2') \qquad ...(8)$$

Thus, Eqn. (1) as a consequence of (4), (7) and (8) becomes

$$A'y_1' + B'y_2' = \phi(x) \qquad ...(9)$$

Let us consider equations (6) and (9) for solving

$$A'y_1 + B'y_2 = 0 \qquad ...(6)$$

$$A'y_1' + B'y_2' = \phi(x) \qquad ...(9)$$

Solving A' and B' by cross multiplication, we get

$$A' = \frac{-y_2\,\phi(x)}{W}, B' = \frac{y_1\,\phi(x)}{W} \qquad ...(10)$$

Find A and B

Integrating, $$A = -\int \frac{y_2\,\phi(x)}{W}\,dx + k_1$$

$$B = \int \frac{y_1\,\phi(x)}{W}\,dx + k_2$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

Substituting the expressions of A and B

$y = Ay_1 + By_2$ is the complete solution.

1. *Solve by the method of variation of parameters*

$$\frac{d^2y}{dx^2} + y = cosec\ x.$$

Solution. We have

$$(D^2 + 1)\, y = \operatorname{cosec} x$$

A.E. is $m^2 + 1 = 0 \quad \Rightarrow \quad m^2 = -1 \quad \Rightarrow \quad m = \pm i$

Hence the C.F. is given by

$$\therefore \quad y_c = C_1 \cos x + C_2 \sin x \quad ...(1)$$

$$y = A \cos x + B \sin x \quad ...(2)$$

be the complete solution of the given equation where A and B are to be found.

The general solution is $y = Ay_1 + By_2$

We have $y_1 = \cos x$ and $y_2 = \sin x$

$$y_1' = -\sin x \text{ and } y_2' = \cos x$$

$$W = y_1 y_2' - y_2 y_1'$$

$$= \cos x \cdot \cos x + \sin x \cdot \sin x = \cos^2 x + \sin^2 x = 1$$

$$A' = \frac{-y_2\, \phi(x)}{W}, \qquad B' = \frac{y_1\, \phi(x)}{W}$$

$$= \frac{-\sin x \cdot \operatorname{cosec} x}{1}, \qquad B' = \frac{\cos x \cdot \operatorname{cosec} x}{1}$$

$$A' = -1, \qquad B' = \cot x$$

$$A = \int (-1)\, dx + C_1,\ i.e.,\ A = -x + C_1$$

$$B = \int \cot x\, dx + C_2,\ i.e.,\ B = \log \sin x + C_2$$

Hence the general solution of the given Eqn. (2) is

$$y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log \sin x.$$

2. *Solve by the method of variation of parameters*

$$\frac{d^2y}{dx^2}+4y = 4 \tan 2x.$$

Solution. We have

$$(D^2 + 4)\, y = 4 \tan 2x$$

A.E. is $m^2 + 4 = 0$

where $\phi(x) = 4 \tan 2x$.

i.e., $m = \pm 2i$

Hence the complementary function is given by

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

$$y = A \cos 2x + B \sin 2x \quad \text{...(1)}$$

be the complete solution of the given equation where A and B are to be found

We have $y_1 = \cos 2x$ and $y_2 = \sin 2x$

$y_1' = -2 \sin 2x$ and $y_2' = 2 \cos 2x$

Then
$$W = y_1y_2' - y_2y_1'$$
$$= \cos 2x \cdot 2 \cos 2x + 2 \sin 2x \cdot \sin 2x$$
$$= 2 (\cos^2 2x + \sin^2 2x)$$
$$= 2$$

Also, $\phi(x) = 4 \tan 2x$

$$A' = \frac{-y_2\,\phi(x)}{W} \quad \text{and} \quad B' = \frac{y_1\phi(x)}{W}$$

$$A' = \frac{-\sin 2x \cdot 4\tan 2x}{2}, \; B' = \frac{-\cos\, 2x \cdot 4\tan 2x}{2}$$

$$A' = \frac{-2\sin^2 2x}{\cos 2x}, \; B' = 2\, \sin\, 2x$$

On integrating, we get

$$A = -2\int \frac{\sin^2 2x}{\cos 2x}\,dx \,, B = 2\int \sin 2x\, dx$$

$$= -2\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$= -2\int \{\sec 2x - \cos 2x\}\, dx$$

$$= -2\left\{\frac{1}{2}\log(\sec 2x + \tan 2x) - \frac{1}{2}\sin 2x\right\}$$

$$A = -\log(\sec\, 2x + \tan\, 2x) + \sin\, 2x + C_1$$

$$B = 2\int \sin 2x\, dx$$

$$= \frac{2(-\cos 2x)}{2} + C_2$$

$$B = -\cos\, 2x + C_2$$

Substituting these values of A and B in Eqn. (1), we get

$$y = C_1 \cos\, 2x + C_2 \sin\, 2x - \cos\, 2x \log\, (\sec\, 2x + \tan\, 2x)$$

which is the required general solution.

Solution of Cauchy's homogeneous linear equation and Legendre's linear equation

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \cdot \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1}\, x \cdot \frac{dy}{dx} + a_n y = \phi(x) \qquad \ldots(1)$$

Where a_1, a_2, a_3 ...a_n are constants and $\phi(x)$ is a function of x is called a homogeneous linear differential equation of order n.

The equation can be transformed into an equation with constant coefficients by changing the independent variable x to z by using the substitution $x = e^z$ or $z = \log x$

Now $\quad z = \log x \Rightarrow \frac{dz}{dx} = \frac{1}{x}$

i.e., $\quad x \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} \cdot \frac{1}{x} - \frac{dy}{dx}$

$$= \frac{1}{x} \cdot \frac{d^2 y}{dz^2} - \frac{1}{x} \cdot \frac{dy}{dz}$$

i.e., $\quad x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz}$

i.e., $\quad x^2 \frac{d^2 y}{dx^2} = (D^2 - D)\, y = D\,(D - 1)\, y$

Similarly, $\quad x^3 \frac{d^3 y}{dx^3} = D\,(D - 1)\,(D - 2)\, y$

....

....

$$x^n \frac{d^n y}{dx^n} = D\,(D - 1) \ldots (D - n + 1)\, y$$

Substituting these values of $x \frac{dy}{dx}, x^2 \frac{d^2 y}{dx^2} \cdots\cdots\cdots x^n \frac{d^n y}{dx^n}$ in Eqn. (1), it reduces to a linear differential equation with constant coefficient can be solved by the method used earlier.

Also, an equation of the form,

$$(ax+b)^n \cdot \frac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \cdot \frac{d^{n-1} y}{dx^{n-1}} + \ldots any = (x) \qquad \ldots(2)$$

where a_1, a_2 .....a_n are constants and ϕ (x) is a function of x is called a homogeneous linear differential equation of order n. It is also called "Legendre's linear differential equation".

This equation can be reduced to a linear differential equation with constant coefficients by using the substitution.

$$ax + b = e^z \text{ or } z = \log (ax + b)$$

As above we can prove that

$$(ax+b) \cdot \frac{dy}{dx} = a\, Dy$$

$$(ax+b)^2 \cdot \frac{d^2y}{dx^2} = a^2 D(D-1)y$$

....

....

$$(ax+b)^n \cdot \frac{d^ny}{dx^n} = a^n D(D-1)(D-2) \ldots\ldots (D-n+1)y$$

The reduced equation can be solved by using the methods of the previous section.

PROBLEMS:

1. *Solve* $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4.$

Solution. The given equation is

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4 \qquad \ldots(1)$$

Substitute $\quad x = e^z \quad \text{or} \quad z = \log x$

So that $\quad x\dfrac{dy}{dx} = Dy, \quad x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

The given equation reduces to

$$D(D-1)y - 2Dy - 4y = (e^z)^4$$

$$[D(D-1) - 2D - 4]y = e^{4z}$$

i.e., $\quad (D^2 - 3D - 4)y = e^{4z} \qquad \ldots(2)$

which is an equation with constant coefficients

A.E. is $\quad m^2 - 3m - 4 = 0$

i.e., $\quad (m-4)(m+1) = 0$

$\therefore \quad m = 4, -1$

C.F. is $\quad \text{C.F.} = C_1e^{4z} + C_2e^{-z}$

$$\text{P.I.} = \frac{1}{D^2-3D-4}e^{4z} \qquad D \to 4$$

$$= \frac{1}{(4)^2-3(4)-4}e^{4z} \qquad Dr = 0$$

$$= \frac{1}{2D-3}ze^{4z} \qquad D \to 4$$

$$= \frac{1}{(2)(4)-3}ze^{4z}$$

$$= \frac{1}{5}ze^{4z}$$

$\therefore$ The general solution of (2) is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 e^{4z} + C_2 e^{-z} + \frac{1}{5} z e^{4z}$$

Substituting $e^z = x$ or $z = \log x$, we get

$$y = C_1 x^4 + C_2 x^{-1} + \frac{1}{5} \log x \left(x^4\right)$$

$$y = C_1 x^4 + \frac{C_2}{x} + \frac{x^4}{5} \log x$$

is the general solution of the Eqn. (1).

2. *Solve* $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 4y = (x + 1)^2$.

Solution. The given equation is

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = (x + 1)^2 \qquad \text{...(1)}$$

Substituting $x = e^z$ or $z = \log x$

Then $x \dfrac{dy}{dx} = Dy, \quad x^2 \dfrac{d^2y}{dx^2} = D(D - 1)\, y$

$\therefore$ Eqn. (1) reduces to

$$D(D - 1)\, y - 3\, Dy + 4y = (e^z + 1)^2$$

i.e., $(D^2 - 4D + 4)\, y = e^{2z} + 2e^z + 1$

which is a linear equation with constant coefficients.

A.E. is $m^2 - 4m + 4 = 0$

i.e., $(m - 2)^2 = 0$

$\therefore$ $m = 2, 2$

$$\text{C.F.} = (C_1 + C_2 z)\, e^{2z}$$

$$\text{P.I.} = \frac{1}{(D-2)^2}\left(e^{2z} + 2e^z + 1\right) \qquad \text{...(2)}$$

$$= \frac{e^{2z}}{(D-2)^2} + \frac{2e^z}{(D-2)^2} + \frac{e^{0z}}{(D-2)^2}$$

$$= \text{P.I.}_1 + \text{P.I.}_2 + \text{P.I.}_3$$

$$\text{P.I.}_1 = \frac{e^{2z}}{(D-2)^2} \qquad (D \to 2)$$

$$= \frac{e^{2z}}{(2-2)^2} \qquad (Dr = 0)$$

$$= \frac{ze^{2z}}{2(D-2)} \qquad (D \to 2)$$

$$= \frac{ze^{2z}}{2(2-2)} \qquad (Dr = 0)$$

$$\text{P.I.}_1 = \frac{z^2 e^{2z}}{2}$$

$$\text{P.I.}_2 = \frac{2e^z}{(D-2)^2} \qquad (D \to 1)$$

$$= \frac{2e^z}{(-1)^2}$$

$$\text{P.I.}_2 = 2e^z$$

$$\text{P.I.}_3 = \frac{e^{0z}}{(D-2)^2} \qquad (D \to 0)$$

$$= \frac{e^{0z}}{4} = \frac{1}{4}$$

$$\text{P.I.} = \frac{z^2}{2} e^{2z} + 2e^z + \frac{1}{4}$$

The general solution of Eqn. (2) is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = (C_1 + C_2 z) e^{2z} + \frac{z^2 e^{2z}}{2} + 2e^z + \frac{1}{4}$$

Substituting $e^z = x$ or $z = \log x$, we get

$$y = (C_1 + C_2 \log x)\, x^2 + \frac{x^2 (\log x)^2}{2} + 2x + \frac{1}{4}$$

is the general solution of the equation (1).

3. *Solve* $x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^2 \log x.$

Solution. The given Eqn. is

$$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^2 \log x \qquad ...(1)$$

Substituting $x = e^z$ or $z = \log x$, so that

$$x\frac{dy}{dx} = Dy, \quad \text{and} \quad x^2 \frac{d^2y}{dx^2} = D(D-1)\,y$$

which is the Linear differential equation with constant coefficients.

A.E. is $m^2 + m - 12 = 0$

i.e., $(m + 4)(m - 3) = 0$

$\therefore$ $m = -4, 3$

$$\text{C.F.} = C_1 e^{-4z} + C_2 e^{3z}$$

$$\text{P.I.} = \frac{1}{D^2 + D - 12} z e^{2z}$$

$$= e^{2z} \frac{z}{(D+2)^2 + (D+2) - 12} \qquad (D \to D + 2)$$

$$= e^{2z} \left[\frac{z}{D^2 + 5D - 6} \right]$$

$$\begin{array}{r|l}
 & -\frac{1}{6}z - \frac{5}{36} \\
\hline
-6 + 5D + D^2 & z \\
 & z - \frac{5}{6} \\
\hline
 & \frac{5}{6} \\
 & \frac{5}{6} \\
\hline
 & 0
\end{array}$$

$$\text{P.I.} = e^{2z} \left[-\frac{z}{6} - \frac{5}{36} \right] = \frac{-e^{2z}}{6} \left[z + \frac{5}{6} \right]$$

$\therefore$ General solution of Eqn. (2) is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 e^{-4z} + C_2 e^{3z} - \frac{e^{2z}}{6} \left(z + \frac{5}{6} \right)$$

Substituting $e^z = x$ or $z = \log x$, we get

$$y = C_1 x^{-4} + C_2 x^3 - \frac{x^2}{6} \left(\log x + \frac{5}{6} \right)$$

$$y = \frac{C_1}{x^4} + C_2 x^3 - \frac{x^2}{6} \left(\log x + \frac{5}{6} \right)$$

which is the general solution of Eqn. (1).