MODULE IV
CENTROID AND MOMENT OF INERTIA

1. Determine the centroid of semi-circular lamina of radius 'R' by method of integration.

(Dec2014 /Jan 2015)

$\overline{Y} = 0$

The area of the element is dA

$= ½ (r) (r\, d\theta)$

$A = \int_{-\pi/2}^{\pi/2} ½\, r^2\, d\theta$

$A = \pi r^2/2$

Using

$\overline{X} = (\int x.\, dA) / (A)$

$\overline{X} = 4r/3\pi$

2. Determine the moment of inertia of the section shown in fig about its centroidal axes. Calculate the least radius of gyration for the section as well. **(Dec2014 /Jan 2015)**

Solution.

Component No.	Component area A (mm^2)	Y (mm)	I_G		Rx= Y - y
			Ix	Iy	
1	180x10	5	15000	4.86×10^6	47.143
2	120x10	60	1000000	14400	-7.857
3	120x10	115	10000	1.44×10^6	-62.857

$$x = \frac{A1X1 + A2X2 + A3X3}{A1 + A2 + A3}$$

X = 57.42mm

Y = 52.14mm

Ixx $\sum(Ix + Arx2)$ =

Iyy $\sum(Iy + Ary2)$ =

Ixx = 9.8407 X 10^6

mm4

Iyy = 11.4058 X 10^6 mm4

As Ixx< Iyy

Least radius of gyration $K_{min} = \sqrt{\Sigma\frac{\bar{x}}{A}}$

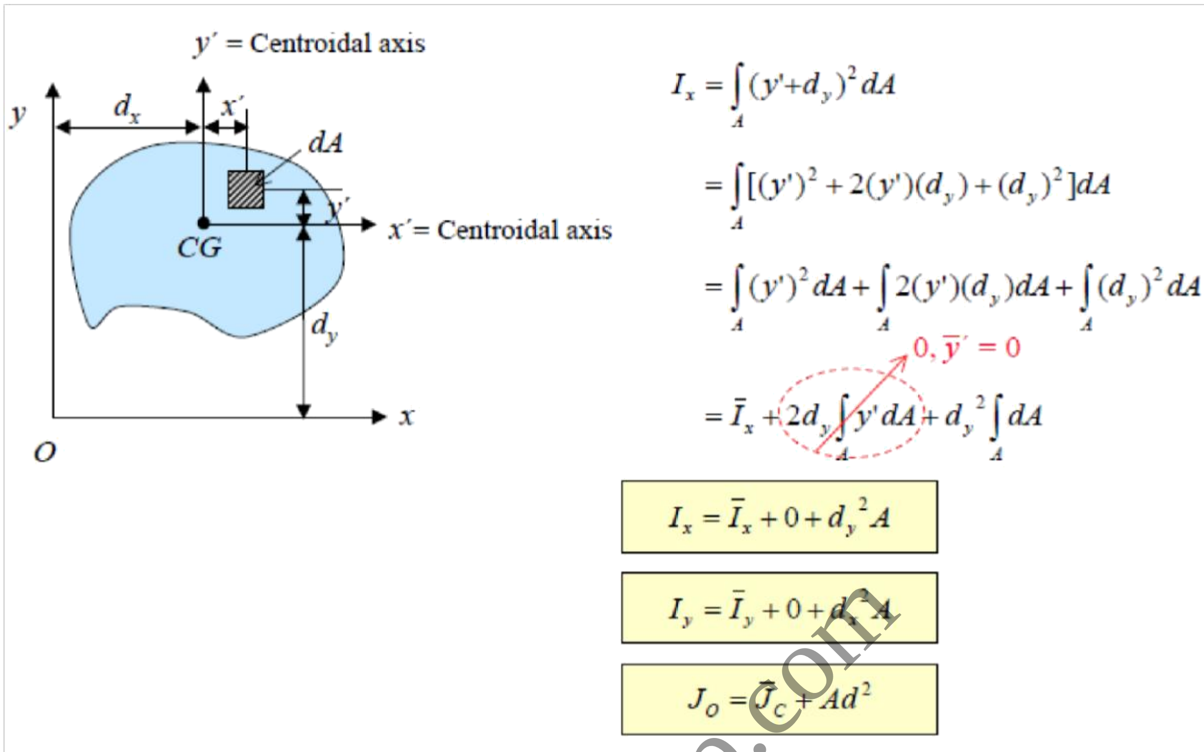
K_{min} = 48.405mm

3. State and prove parallel axis theorem (Dec2014 /Jan 2015)

Solution

Statement: "if the M.I of a plane area about an axis through its C.G is denoted by Ixx, then M.I. of the area about any reference axis (1)-(1) parallel to X-axis and at a distance of Y from C.G. is given by

$$I_{1\text{-}1} = Ixx + Ay_c^2$$



4. **Derive an expression for moment of inertia of a triangle with respect to horizontal centroidal axis (Dec2014 /Jan 2015)**

Solution.

By similarity of triangles

$$\frac{}{b} = \frac{h-y}{h}$$

$$\frac{b'}{b} = \left(\frac{h-y}{h}\right) b$$

M.I of strip about the base

dIAB = y

dIAB)y2. $= b\left(\frac{h-y}{h}\right.$ dy

$\frac{bh3}{12}$ IAB =

Using parallel axis theorem,

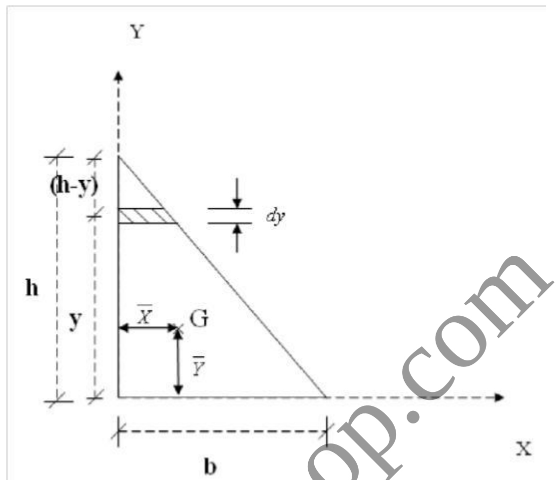
$$I_{AB} = I_G + Ad^2$$

Hence, $I_G = \frac{bh3}{36}$

5. Determine the centroid of a triangle by method of integration

(June/July2013, Jan 2013)

Centroid of a triangle



Let us consider a right angled triangle with a base b and height h as shown in figure. Let G be the centroid of the triangle. Let us consider the X- axis and Y- axis as shown in figure. Let us consider an elemental area dA of width b1 and thickness dy, lying at a distance y from X-axis. W.K.T

$$\bar{Y} = \frac{\int_0^h y.dA}{A}$$

$$A = \frac{b.h}{2}$$

$dA = b_1 \, . \, dy$

$$\bar{Y} = \frac{\int_0^h y.(b_1 \, . \, dy)}{\frac{b.h}{2}} \text{ [as x varies b1 also varies]}$$

$$\bar{Y} = \frac{2}{h} . \int_0^h \left(y - \frac{y^2}{h} \right) dy$$

$$\bar{Y} = \frac{2}{h} \left[\frac{y^2}{2} - \frac{y^3}{3.h} \right]_0^h$$

$$\bar{Y} = \frac{2}{h} \left[\frac{h^2}{2} - \frac{h^3}{3.h} \right]$$

$$\bar{Y} = \frac{2}{h} \left[\frac{h^2}{2} - \frac{h^2}{3} \right]$$

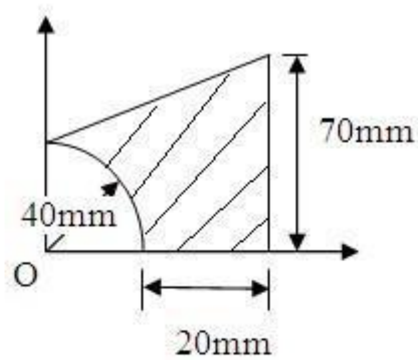
$$\bar{Y} = 2h \left[\frac{1}{2} - \frac{1}{3} \right]$$

$$\bar{Y} = \frac{2.h}{6}$$

$$\bar{Y} = \frac{h}{3} \text{ similarly } \bar{X} = \frac{b}{3}$$

6. Determine the centroid of the lamina shown in fig. wrt O.

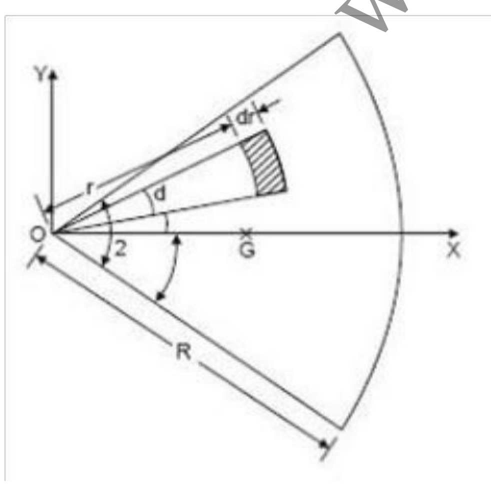
(June/July2012, June/July2013)



Component	Area (mm^2)	X (mm)	Y (mm)	aX	aY
Quarter circle	-1256.64	16.97	16.97	-21325.2	-21325.2
Triangle	900	40	50	36000	45000
Rectangle	2400	30	20	72000	48000
	∑a= 2043.36			∑ aX = 86674.82	∑ aY = 71674.82

$\overline{X}$ = 42.42 mm; $\overline{Y}$ = 35.08 mm

7. Determine the centroid of a sector of radius r by by method of integration. **(Jan 2011)**



Centroid of Sector of a Circle

Consider the sector of a circle of angle 2α as shown in Fig. Due to symmetry, centroid lies on x axis. To find its distance from the centre O, consider the elemental area shown.

Area of the element $= rd\theta\, dr$

Its moment about y axis

$$= rd\theta \times dr \times r\cos\theta$$
$$= r^2 \cos\theta\, drd\theta$$

$\therefore$ Total moment of area about y axis

$$= \int_{-\alpha}^{\alpha}\int_{0}^{R} r^2 \cos\theta\, drd\theta$$

$$= \left[\frac{r^3}{3}\right]_0^R [\sin\theta]_{-\alpha}^{\alpha}$$

$$= \frac{R^3}{3}\, 2\sin\alpha$$

Total area of the sector

$$= \int_{-\alpha}^{\alpha}\int_{0}^{R} rdrd\theta$$

$$= \int_{-\alpha}^{\alpha}\left[\frac{r^2}{2}\right]_0^R d\theta$$

$$= \frac{R^2}{2}[\theta]_{-\alpha}^{\alpha}$$

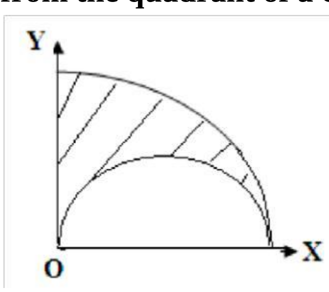
$$= R^2\alpha$$

$\therefore$ The distance of centroid from centre O

$$= \frac{\text{Moment of area about } y \text{ axis}}{\text{Area of the figure}}$$

$$= \frac{\frac{2R^3}{3}\sin\alpha}{R^2\alpha} = \frac{2R}{3\alpha}\sin\alpha$$

7. Find the centroid of the shaded area shown in fig, obtained by cutting a semicircle of diameter 100mm from the quadrant of a circle of radius 100mm. (Jan 2011, June 2014)



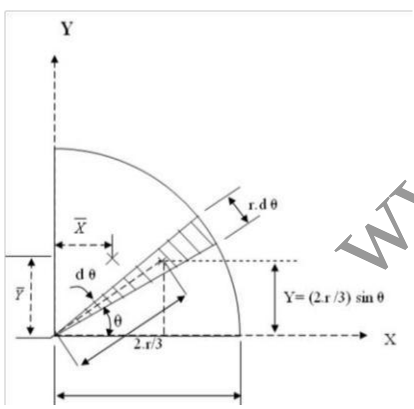
Component	Area (mm^2)	X (mm)	Y (mm)	aX	aY
Quarter circle	7853.98	42.44	42.44	333322.9	333322.9
Semi circle	-3926.99	50	21.22	-196350	-83330.7
	∑a= 3926.99			∑ aX = 136973.4	∑ aY = 249992.2

$\overline{X}$ = 34.88 mm; $\overline{Y}$ = 63.66 mm

8. Locate the centroid of quadrant of a circular lamina from first principle.

(June/July 2012, May/June2011)

Centroid of a quarter circle



Let us consider a quarter circle with radius r. Let O be the centre and G be the centroid of the quarter circle. Let us consider the x and y axes as shown in figure. Let us consider an elemental area dA with centroid g as shown in fig.

Let y be the distance of centroid g from x axis. Neglecting the curvature, the elemental area becomes an isosceles triangle with base r.dθ and height r.

Here $y = \frac{2r}{3}.\sin\theta$

W K T

$$\overline{Y} = \frac{\int y\,dA}{A}$$

$$A = \frac{\pi.r^2}{2}$$

$$\overline{Y} = \frac{\int y\,dA}{A}$$

$$\overline{Y} = \frac{\int \frac{2r}{3}.\sin\theta\,dA}{A}$$

$$dA = \frac{1}{2}.r.d\theta.r$$

$$dA = \frac{r^2}{2}.d\theta$$

$$\overline{Y} = \frac{\int \frac{2r}{3}.\sin\theta.\frac{r^2}{2}.d\theta}{\frac{\pi.r^2}{4}}$$

$$= \frac{4r}{3\pi}\int_0^{\pi/2}\sin\theta.d\theta$$

$$= \frac{2r}{3\pi}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{4r}{3\pi}[0+1]$$

$$\overline{Y} = \frac{4r}{3\pi}$$

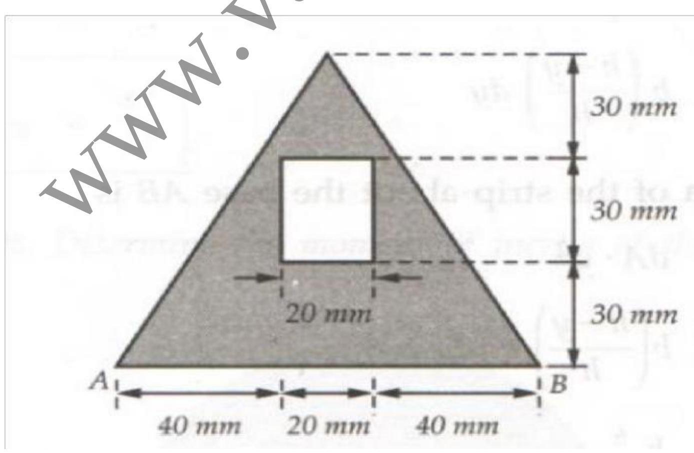
Similarly

$$\overline{X} = \frac{4r}{3\pi}$$

$\int$

9. Determine the moment of inertia and radii of gyration of the area shown in the fig about the base AB and the centroidal axis parallel to AB.

(Jan / Feb 2014 & 2012)



The M.I of triangle about its base is $bh^3/12$

$I_{AB} = 100X90^3/12 - (20X30^3/12 + (20X30)X45^2)$

$I_{AB} = 4.85X10^6\ mm^4$

Radius of gyration about AB is

$K_{AB} = \sqrt{\frac{I_{AB}}{\Sigma A}} = 35.137mm$

$Y = A_1y_1 + A_2y_2/ A_1 + A_2$

$Y= 27.692mm$

Using parallel axes theorem for the complete area,

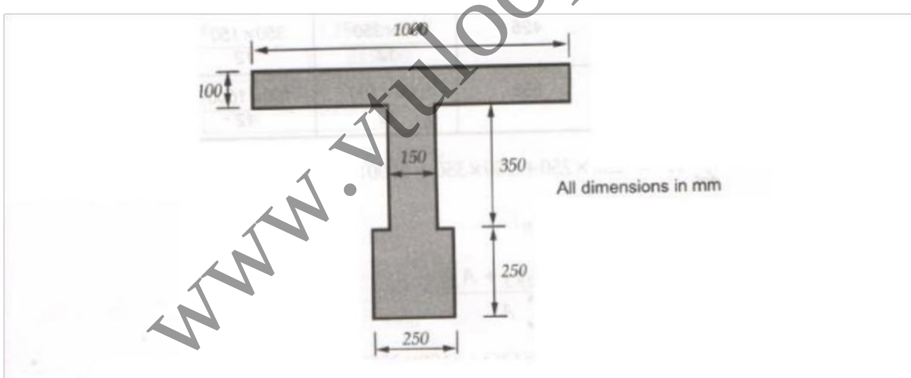
$I_{AB}= I_{XX} + AY^2$

$I_{XX} = I_{AB} - AY^2= 4.815x\ 10^6 – (½ x\ 100x\ 90 – 20x30) x\ 27.692^2$

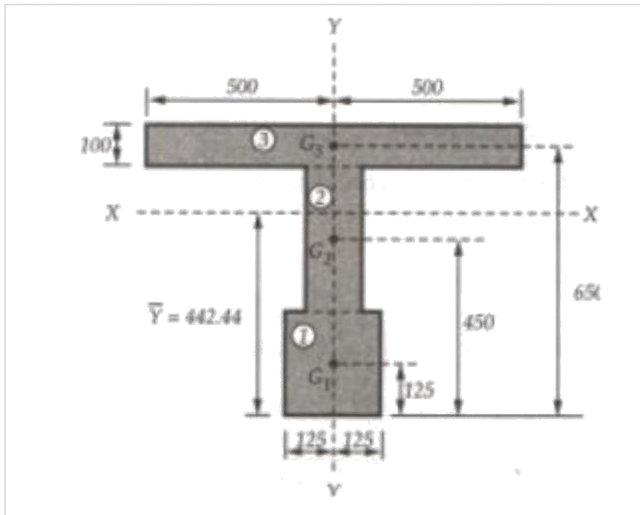
$I_{XX} = 1.824\ x10^6$

$K_{AB} = \sqrt{\frac{I_{xx}}{\Sigma A}} = 21.626mm$

10. The cross section of the prestressed concrete beam is as shown in the fig. Calculate the moment of inertia of this section about the centroidal axes parallel to the top edge and perpendicular to the plane of cross section. Also determine the radius of gyration.



The given area is symmetric about a vertical line passing through the centre. That vertical line is the centroidal Y axis. Divide area into three rectangles as shown in figure. Take base of the given figure as origin.



Component No.	Component area A (mm^2)	Y (mm)	I_G		Rx= Y - y
			Ix	Iy	
1	250x250	125	$250x250^3/ 12$	$250x250^3/ 12$	317.44
2	150x350	425	$150x350^3/ 12$	$350x150^3/ 12$	17.44
3	100x100	650	$1000x100^3/$ 12	$100x1000^3/$ 12	-207.56

$\sum A = 250x250 + 150x350 + 1000x100$

$\sum A = 215000mm^2$

$Y = A_1y_1 + A_2y_2/ A_1 + A_2$

$= 442.44mm$

$Ixx= \sum (Ix+ A\ r_x^{\ 2})$

$Ixx= 1.1567x\ 10^{10}\ mm^4$

The centroidal axis perpendicular to the plane of the cross section is the Z- Z axis

The M.I about Z- Z axis can be obtained using, Izz = Ixx + Iyy

$Iyy= 250x250^3/ 12 + 350x150^3/ 12 + 100x1000^3/$

$12\ Iyy= 8.7573x\ 10^9\ mm^4$

Izz = Ixx+ Iyy

= $1.1567 \times 10^{10} + 8.7573 \times 10^{9}$

= 2.03243×10^{10} mm^4

The radius of Gyration for X- X axis is

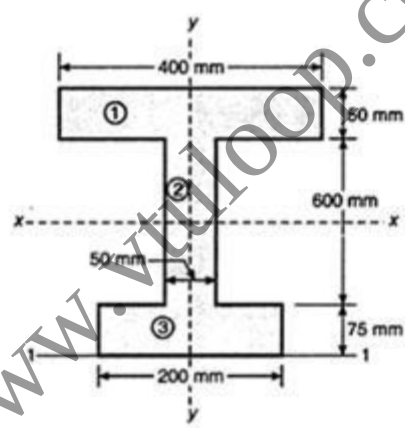
Kxx= = $\sqrt{\frac{Ixx}{\Sigma A}}$ = 231.95mm

The radius of Gyration for Z- Z axis is

Kzz = $\sqrt{\frac{Izz}{\Sigma A}}$ = 307.46mm

11. Find the moment of inertia along the horizontal and vertical axis passing through the centroid of a section shown in fig.

(June 2012)



Solution The given figure is symmetrical about the *y*-axis. Therefore, the centroidal *y*-axis coincides with the reference *y*-axis. Hence $\bar{x} = 0$.

Moment of inertia about the centroidal *x–x* axis

$$I_{1-1} = \bar{I}_x + A\,\bar{y}^2 = \Sigma \bar{I}_x + \Sigma A y^2$$

or

$$I_{1-1} - A\,\bar{y}^2 = \bar{I}_x$$

$$I_{2-2} = \bar{I}_y + A\,\bar{x}^2$$

Comp.	Area (mm²)	y	Ay (mm³)	Ay² (mm⁴)	$\overline{I}_x$ (mm⁴)	$\overline{I}_y$ (mm⁴)
1.	400×50 $= 20{,}000$	$75 + 600 + \frac{50}{2}$ $= 700$	14,000,000 $= 14 \times 10^6$	9.8×10^9	$\frac{400(50)^3}{12}$ $= 4.167 \times 10^6$	$\frac{50(400)^3}{12}$ $= 266{,}666{,}666.7$
2.	50×600 $= 30{,}000$	$75 + \frac{600}{2}$ $= 375$	11,250,000 $= 11.25 \times 10^6$	4.219×10^9	$\frac{50 \times 600^3}{12}$ $= 900 \times 10^6$	$\frac{600 \times 50^3}{12}$ $= 6{,}250{,}000$
3.	200×75 $= 15{,}000$	$\frac{75}{2} = 37.5$	562,500	21.09×10^6	$\frac{200 \times 75^3}{12}$ $= 7.03 \times 10^6$	$\frac{75 \times 200^3}{12}$ $= 5{,}00{,}00{,}000$
Σ	65,000		25.812×10^6	1.404×10^{10}	911.197×10^6	322,916,666.7

$$\overline{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{25.812 \times 10^6}{65{,}000} = 397.108 \text{ mm}$$

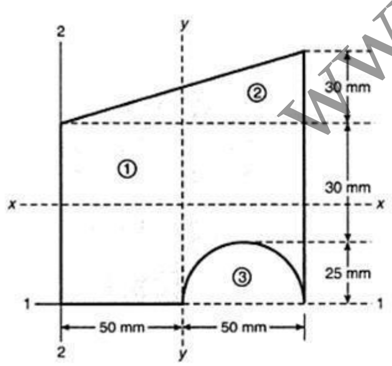
$$\overline{I}_{1-1} = \overline{I}_x + A\,\overline{y}^2 = \Sigma \overline{I}_x + \Sigma A y^2 = 1.495 \times 10^{10} \text{ mm}^4$$

$$\overline{I}_x = I_{1-1} - A\,\overline{y}^2 = 1.495 \times 10^{10} - 65000 \times (397.108)^2 = 4.691 \times 10^9 \text{ mm}^4 \qquad \textbf{Ans.}$$

When the moment of inertia is required on a symmetrical axis, then

$$\overline{I}_y = \Sigma \overline{I}_y = 329{,}216{,}666.7 \text{ mm}^4 \qquad \textbf{Ans.}$$

12. Find the least radius of gyration about X-axis and Y-axis shown in fig. (Jan 2013)



Solution

Component	*Area* (mm²)	*x* (mm)	*y* (mm)	*Ax*	*Ay*	*Ax²*	*Ay²*
Rectangle 1	$55 \times 100 = 5500$	50	$\frac{55}{2} = 27.5$	275,000	151,250	13,750,000	4,159,375
Triangle 2	$\frac{1}{2} \times 30 \times 100 = 1500$	$\frac{2}{3} \times 100 = 66.667$	$55 + 10 = 65$	100,000	97,500	6,666,667	6,337,500
Semicircle 3	$-\frac{\pi \times (25)^2}{2} = -981.748$	$50 + 25 = 75$	$\frac{4 \times 25}{3\pi} = 10.61$	−73,631	−10,416.7	−5,522,326	−110,524
Sum	6018.252			301,369	238,333.3	14,894,340	10,386,351

Component	$\bar{I}_x$ (mm⁴)	$\bar{I}_y$ (mm⁴)
Rectangle 1	$\frac{100 \times 55^3}{12} = 1,386,458$	$\frac{55 \times 100^3}{12} = 4,583,333$
Triangle 2	$\frac{100 \times 30^3}{36} = 75,000$	$\frac{30 \times 100^3}{36} = 833,333.3$
Semicircle 3	$-0.11(25)^4 = -42,968.8$	$\frac{-\pi(25)^4}{8} = -153,398$
Sum	1,418,490	5,263,269

$$\bar{x} = \frac{\Sigma Ax}{\Sigma A} = \frac{3.014 \times 10^5}{6018.252} = 50.07582 \text{ mm}$$

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{2.384 \times 10^5}{6018.252} = 39.60175 \text{ mm}$$

$$I_{x-x} = I_{1-1} - A\,\bar{y}^2$$

$$I_{1-1} = \Sigma \bar{I}_x + \Sigma Ay^2 = 1{,}418{,}490 + 10{,}386{,}351 = 11{,}804{,}840 \text{ mm}^4$$

$$I_{2-2} = \Sigma \bar{I}_y + Ax^2 = 5{,}263{,}269 + 14{,}894{,}340 = 20{,}157{,}609 \text{ mm}^4$$

$$\bar{I}_x = I_{1-1} - A\,\bar{y}^2 = 11{,}804{,}840 - 6018.253 \times (39.602)^2 = 2{,}366{,}424 \text{ mm}^4$$

$$\bar{I}_Y = I_{2-2} - A\,\bar{x}^2 = 20{,}157{,}609 - 6018.253 \times (50.076)^2 = 5{,}066{,}309 \text{ mm}^4$$

$$k_x = \sqrt{\frac{\bar{I}_x}{A}} = \sqrt{\frac{23{,}66{,}424}{6018.252}} = 19.829 \text{ mm}$$

$$k_y = \sqrt{\frac{\bar{I}_y}{A}} = \sqrt{\frac{50{,}66{,}309}{6018.253}} = 29.014 \text{ mm}$$

13. Calculate the polar moment of inertia of the area shaded in fig **(June 2014)**

Sol: Ixx = 10573000 mm

Iyy = 4960000 mm