MODULE-5

NUMERICAL METHODS

Finite Differences

Let y = f(x) be represented by a table

x :	x_0	x_1	x_2	x_3	….	x_n
y :	y_0	y_1	y_2	y_3	…	y_n

where x_0, x_1,x_2….x_n are equidistant. ($x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = h$)
We now define the following operators called the difference operators.

Forward difference operator (Δ)

Forward difference operator (Δ)

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta y_r = y_{r+1} - y_r, \quad r = 0,1,2,\ldots,n-1$$

$$\left.\begin{aligned} &\Delta y_0 = y_1 - y_0 \\ &\Delta y_1 = y_2 - y_1 \\ &. \\ &. \\ &\Delta y_{n-1} = y_n - y_{n-1} \end{aligned}\right\} \text{first forward differences}$$

$\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2, \ldots$, are called the second differences

$$\text{Now } \Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$$

$$= \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

$|||^{ly}$ $\quad \Delta^2 y_1 = y_3 - 2y_2 + y_1$

$$\Delta^2 y_r = y_{r+2} - 2y_{r+1} + y_r$$

Note : $\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$

$$\therefore \quad \Delta^k y_r = y_{r+k} - {}^kC_1 y_{r+k-1} + {}^kC_2\, y_{r+k-2} - + \ldots + (-1)^k\, {}^kC_r$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_0$		
x_3	y_3		$\Delta^2 y_2$			
		Δy_3				
x_4	y_4					

Δy_0 , $\Delta^2 y_0$, $\Delta^3 y_0$,... are called the leading differences.

Ex: The following table gives a set of values of x and the corresponding values of y = f(x)

x :	**10**	**15**	**20**	**25**	**30**	**35**
y :	**19.97**	**21.51**	**22.47**	**23.52**	**24.65**	**25.89**

Form the difference table and find $\Delta f(10)$, $\Delta^2 f(10)$, $\Delta^3 f(20)$, $\Delta^4 f(15)$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
10	19.97					
		1.54				
15	21.51		-0.58			
		0.96		0.67		
20	22.47		0.09		-0.68	
		1.05		-0.01		0.72
25	23.52		0.08		0.04	
		1.13		0.03		
30	24.65		0.11			
		1.24				
35	25.89					

$\Delta f(10) = 1.54$, $\Delta^2 f(10) = -0.58$, $\Delta^3 f(20) = 0.03$, $\Delta^4 f(15) = 0.04$

Note: The nth differences of a polynomial of n the degree are constant

Backward difference operator (∇)

Let $y = f(x)$

We define $\nabla f(x) = f(x) - f(x - h)$

i.e. $\nabla y_1 = y_1 - y_0 = \Delta y_0$

$\nabla y_2 = y_2 - y_1 = \Delta y_1$

$\nabla y_3 = y_3 - y_2 = \Delta y_2$

,

,

$\nabla y_n = y_n - y_{n-1} = \Delta y_{n-1}$

$\therefore \quad \nabla y_r = y_r - y_{r-1} = \Delta y_{r-1}$

Note:

1. $\nabla f(x + h) = f(x + h) - f(x) = \Delta f(x)$
2. $\nabla^2 f(x + 2h) = \nabla(\nabla f(x + 2h))$

$= \nabla \{f(x + 2h) - f(x + h)\}$

$= \nabla f(x + 2h) - \nabla f(x + h)$

$= f(x + 2h) - f(x$ http://www.vtucs.com

$= f(x + 2h) - 2f(x + h) + f(x)$

$= \Delta^2 f(x)$

$|||^{ly}\ \nabla^n f(x + nh) = \Delta^n f(x)$

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
X_0	y_0			
		∇y_1		
X_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\nabla^3 y_3$
X_2	y_2		$\nabla^2 y_3$	
		∇y_3		
X_3	y_3			

1. Form the difference table for

x	**40**	**50**	**60**	**70**	**80**	**90**
y	**184**	**204**	**226**	**250**	**276**	**304**

find ∇y (30), $\nabla^2 y$ (70), $\nabla^5 y$ (90)

Soln:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

∇y (30) = 26, $\nabla^2 y$ (70) = 2, $\nabla^5 y$ (90) = 0

2. Given Construct the difference table and write the values of $\nabla f(4)$, $\nabla^2 f(4)$, $\nabla^3 f(3)$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	4			
		8		
1	12		12	
		20		**12**
2	32		24	
		44		12
3	76		**36**	
		80		
4	156			

3) Find the missing term from the table:

x	**0**	**1**	**2**	**3**	**4**
y	**1**	**3**	**9**	**-**	**81**

Explain why the value obtained is different by putting x = 3 in 3^x.

Denoting the missing value as a, b, c ..etc. Construct a difference table and solve.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	2			
1	3	6	4		
2	9	a - 9	a - 15	a - 19	-4a + 124
3	a	81 - a	81 - a	-3a +105	
4	81				

Put $\Delta^4 y = 0$ (assuming f(x) its be a polynomial of degree 3) i.e., $-4a + 124 = 0$

$a = 31$

4) Given $u_1 = 8$, $u_3 = 64$, $u_5 = 216$ find u_2 and u_4

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$
x_1	8			
x_2	a	a - 8	-2a + 72	b + 3a - 200
x_3	64	64 - a	b + a - 128	-3b - a + 408
x_4	b	b - 64	-2b + 280	
x_5	216	216 -b		

We carryout upto the stage where we get two entries ($\because$ 2 unknowns) and equate each of those entries to zero. (Assuming) to be a polynomial of degree 2.

$b + 3a - 200 = 0$

$-3b - a + 408 = 0$ We get $a = 24$ $b = 128$

Interpolation:

The word interpolation denotes the method of computing the value of the function $y = f(x)$ for any given value of x when a set $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$ are given.

Note:

Since in most of the cases the exact form of the function is not known. In such cases the function f(x) is replaced by a simpler function $\varphi(x)$ which has the same values as f(x) for $x_0, x_1, x_2 \ldots, x_n$.

$$\varphi(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n y_0$$

is called the Newton Gregory forward difference formula

Note :

1. Newton forward interpolation is used to interpolate the values of y near the beginning of a set of tabular values.
2. y_0 may be taken as any point of the table but the formula contains those values of y which come after the value chosen as y_0.

Problems:

1) The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earths surface.

2)

x = height	**100**	**150**	**200**	**250**	**300**	**350**	**400**
y = distance	**10.63**	**13.03**	**15.04**	**16.81**	**18.42**	**19.90**	**21.27**

Find the values of y when i) x = 120, ii) y =218

Solution:

x	**y**	**Δ**	**Δ^2**	**Δ^3**	**Δ^4**	**Δ^5**	**Δ^6**
100	10.63						
		2.40					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.02			
350	19.90		-0.11				
		1.37					
400	21.27						

Choose $x_0 = 100$

i) $x = 120,\ u = \dfrac{120-100}{50} = 0.4$

$$f(120) = 10.63 + \frac{0.4}{1!}(2.40) + \frac{(0.4)(0.4-1)}{2!}(-0.39)$$

$$+\frac{(0.4)(0.4-1)(0.4-2)}{3!}(0.15)$$

$$+\frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}(-0.07)$$

$$+\frac{(0.4)(0.4-1)(0.4-2)(0.4-3)(0.4-4)}{5!}(0.02)$$

$$+\frac{(0.4)(0.4-1)(0.4-2)(0.4-3)(0.4-4)(0.4-5)}{6!}(0.02) = 11.649$$

ii) Let $x = 218,\ x_0 = 200,\ u = \dfrac{218-200}{50} = \dfrac{18}{50} = 0.36$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16)$$

$$+\frac{0.36(-0.64)(-1.64)}{6}(0.03) + \ldots$$

$$= 15.7$$

3) Find the value of f(1.85).

x	y	Δy	Δ^2y	Δ^3y	Δ^4y	Δ^5y	Δ^6y
1.7	5.474						
		0.575					
1.8	6.049		0.062				
		0.637		0.004			
1.9	6.686		0.066		0.004		
		0.703		0.008		-0.004	
2.0	7.389		0.074		0		0.004
		0.777		0.008		0	
2.1	8.166		0.082		0		
		0.859		0.008			
2.2	9.025		0.090				
		0.949					
23	9.974						

Choose $x_0 = 1.8$, $x = 1.85$ $u = \dfrac{x - x_0}{h} = \dfrac{1.85 - 1.8}{0.1} = 0.5$

$$f(1.85) = 6.049 + (0.5)\,(0.637) + \frac{(0.5)(-0.5)}{2}(0.066)$$

$$+ \frac{(0.5)(-0.5)\,(-1.5)}{6}(0.008)$$

$$= 6.049 + 0.3185 - 0.0008 + 0.0005$$

$$= 6.359$$

4) Given sin 45° = 0.7071, sin 50° = 0.7660, sin 55° =0.8192, sin 60° = 0.8660. Find sin 48°.

x	y	Δ	Δ^2	Δ^3
45	0.7071			
		0.589		
50	0.7660		-0.0057	
		0.0532		0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

$x = 48,\ x_0 = 45;\ h = 5\ u = \dfrac{x - x_0}{h} = 0.6$

$\sin 48^\circ = 0.7071 + (0.6)\,(0.0589)$

$$+ \frac{(0.6)(-0.4)}{2}(-0.0057) + \frac{(0.6)(-0.4)\,(-1.4)}{6}(0.0007) = 0.7431$$

5) From the following data find the number of students who have obtained ≤ 45 marks. Also find the number of students who have scored between 41 and 45 marks.

Marks	**0 - 40**	**41 - 50**	**51 - 60**	**61 -70**	**71 - 80**
No. of students	**31**	**42**	**51**	**35**	**31**
x	**y**	□	**□2**	**□3**	**□4**

40	**31**				
		42			
50	**73**		**9**		
		51		**-25**	
60	**124**		**-16**		**37**
		35		**12**	
70	**159**		**-4**		
		31			
80	**190**				

$$f(45) = 31 + (0.5)\,(42) + \frac{(0.5)\,(-0.5)\,9}{2} + \frac{(0.5)\,(-0.5)\,(-1.5)(-25)}{3!}$$
$$+ \frac{(0.5)\,(-0.5)\,(-1.5)\,(-2.5)}{4!}(37) = 47.8672 \approx 48$$

f(45) - f(40) = 70 = Number of students who have scored between 41 and 45.

6) Find the interpolating polynomial for the following data: f(0) = 1, f(1) = 0, f(2) = 1, f(3) = 10. Hence evaluate f(0.5)

x	**y**	**Δ**	**Δ^2**	**Δ^3**
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

$$f(x) = 1 + x(-1) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)\,(x-2)}{3!}6 = x^3 - 2x^2 + 1$$

7) Find the interpolating polynomial for the following data:

x:	0	1	2	3	4
f(x) :	3	6	11	18	27

x	y	Δ	Δ^2	Δ^3	Δ^4
0	3				
		3			
1	6		2		
		5		0	
2	11		2		0
		7		0	
3	18		2		
		9			
4	27				

$$u = \frac{x-0}{1} = x$$

$$f(x) = 3 + x(3) + \frac{x(x-1)}{2}(2) + \frac{x(x-1)}{x!}(0) \quad = 3 + 2x + x^2$$

Newton Gregory Backward Interpolation formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots$$

where $u = \dfrac{x - x_n}{h}$

1) **The values of tan x are given for values of x in the following table. Estimate tan (0.26)**

x	0.10	0.15	0.20	0.25	0.30
y	0.1003	0.1511	0.2027	0.2553	0.3093

x	y	∇	∇^2	∇^3	∇^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

$$u = \frac{0.26-0.3}{0.05} = -0.8$$

$$f(0.26) = 0.3093 + (-0.8)(0.054) + \frac{(-0.8)}{2}(0.2)(0.0014) + \frac{(-0.8)(0.2)(1.2)}{6}(0.0004) = 0.2659$$

2) The deflection d measured at various distances x from one end of a cantilever is given by the following table. Find d when x = 0.95

$u = \frac{0.95-1}{0.2} = -0.25$ $\quad d = 0.3308$ when $x = 0.95$

x	d	∇	∇^2	∇^3	∇^4	∇^5
0	0					
		0.0347				
0.2	0.0347		0.0479			
		0.0826		-0.0318		
0.4	0.1173		0.0161		0.0003	
		0.0987		-0.0321		-0.0003
0.6	0.2160		-0.016		0	
		0.0827		-0.032		
0.8	0.2987		-0.0481			
		0.0346				
1.0	0.3333					

3)The area y of circles for different diameters x are given below:

x :	**80**	**85**	**90**	**95**	**100**
y :	**5026**	**5674**	**6362**	**7088**	**7854**

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

Answer:

$$u = \frac{x - x_n}{n} = -0.4$$

$$y = 7542$$

1) Find the interpolating polynomial which approximates the following data.

x	0	1	2	3	4
y	-5	-10	-9	4	35

x	y	∇	∇^2	∇^3	∇^4
0	-5				
		-5			
1	-10		6		
		1		6	
2	-9		12		0
		13		6	
3	4		18		
		31			
4	35				

$$u = \frac{x-4}{1}$$

$$f(x) = 35 + (x-4)(31) + (x-4)(x-3)\frac{18}{2!} + \frac{(x-4)(x-3)(x-2)(6)}{3!}$$

$$\boxed{f(x) = x^3 + 2x^2 + 6x - 5}$$

Interpolation with unequal intervals

Newton backward and forward interpolation is applicable only when x_0, $x_1, \ldots, x_{n-1}$ are equally spaced. Now we use two interpolation formulae for unequally spaced values of x.

i) Lagranges formula for unequal intervals:

If $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$ then

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} f(x_2)+\ldots$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})} f(x_n)$$ is known as the lagrange's interpolation formula

ii) Divided differences (Δ)

$$\Delta f(x_0) = \Delta y_0 = \frac{y_1 - y_0}{x_1 - x_0} = [x_0, x_1]$$

$$\Delta y_1 = \frac{y_2 - y_1}{x_2 - x_1} = [x_2, x_1]$$

$$\Delta y_{n-1} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = [x_{n-1}, x_n]$$

second divided difference

$$\Delta^2 f(x_0) = \Delta^2 y_0 = \frac{\Delta y_1 - \Delta y_0}{x_2 - x_0}$$

$$= \frac{[x_2, x_1] - [x_1, x_0]}{x_2 - x_0} = [x_0, x_1, x_2]$$

$$||^{ly}\ \Delta^2 y_1 = \frac{\Delta y_2 - \Delta y_1}{x_3 - x_1} = \frac{[x_3, x_2] - [x_2, x_1]}{x_3 - x_1} = [x_1, x_2, x_3]$$

similarly $\Delta^3 y_0, \ldots$ can be defined

Newton's divided difference interpolation formula

$$y = f(x) = y_0 + (x - x_0)\,\Delta y_0 + (x - x_0)\,(x - x_1)\,\Delta^2 y_0 + (x - x_0)\,(x - x_1)(x - x_2)\,\Delta^3 y_0$$

$$+ \dots + (x - x_0)\,(x - x_1)\,\dots(x - x_n)\,\Delta^n y_0$$

is called the Newton's divided difference formula.

Note: Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients need to be recalculated.

Inverse interpolation: Finding the value of y given the value of x is called interpolation where as finding the value of x for a given y is called inverse interpolation.

Since Lagrange's formula is only a relation between x and y we can obtain the inverse interpolation formula just by interchanging x and y.

$$\therefore x = \frac{(y - y_1)\,(y - y_2)\dots(y - y_n)}{(y_0 - y_1)\,(y_0 - y_2)\dots(y_0 = y_n)}\,x$$

$$+ \frac{(y - y_0)\,(y - y_0)\,(y - y_2)\,(y - y_3)\dots(y - y_n)}{(y_1 - y_0)\,(y_1 - y_2)\,(y_1 - y_3)\dots(y_1 - y_n)}\,x_1 + \dots$$

$$+ \dots + \frac{(y - y_0)\,(y - y_1)\dots(y - y_{n-1})}{(y_n - y_0)\,(y_n - y_1)\dots(y_n - y_{n-1})}\,x_n$$

is the Lagranges formula for inverse interpolation

1) The following table gives the values of x and y

x :	**1.2**	**2.1**	**2.8**	**4.1**	**4.9**	**6.2**
y :	**4.2**	**6.8**	**9.8**	**13.4**	**15.5**	**19.6**

Find x when y = 12 using Lagranges inverse interpolation formula. Using Langrages formula

$$x = \frac{(y - y_1)\,(y - y_2)\,(y - y_3)\,(y - y_4)\,(y - y_5)}{(y_0 - y_1)\,(y_0 - y_2)\,(y_0 - y_3)\,(y_0 - y_4)\,(y_0 - y_5)}\,x_0$$

$$+\ldots+\frac{(y-y_0)\,(y-y_1)\,(y-y_2)\,(y-y_3)\,(y-y_4)}{(y_5-y_0)\,(y_5-y_1)\,(y_5-y_2)\ldots(y_5-y_4)}\,x_4$$

$$=0.022-0.234+1.252+3.419-0.964+0.055$$

$$=3.55$$

2) Given the values

x :	**5**	**7**	**11**	**13**	**17**
f(x) :	**150**	**392**	**1452**	**2366**	**5202**

Evaluate f(9) using (i) Lagrange's formula (ii) Newton's divided difference formula.

i) Lagranges formula

$$f(9)=\frac{(9-7)\,(9-11)\,(9-13)\,(9-17)}{(5-7)\,(5-11)\,(5-13)\,(5-17)}(150)+\frac{(9-5)\,(9-11)\,(9-13)\,(9-17)}{(7-5)\,(7-11)\,(7-13)\,(7-17)}.392$$

$$+\frac{(9-5)\,(9-7)\,(9-13)\,(9-17)}{(11-5)\,(11-7)\,(11-13)\,(11-17)}(1452)+\frac{(9-5)\,(9-7)\,(9-11)\,(9-17)}{(13-5)\,(13-7)\,(13-11)\,(13-17)}(2366)$$

$$+\frac{(9-5)\,(9-7)\,(9-13)(9-13)}{(17-5)\,(17-7)\,(17-11)\,(17-13)}(5202)=810$$

$f(9)=810$

5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

f(9) = 150 + 121 (9 - 5) + 24 (9 - 5) (9 - 7) + 1(9 - 5) (9 - 7) (9 - 11) = 810

3) Using i) Langranges interpolation and ii) divided difference formula. Find the value of y when x = 10.

x :	**5**	**6**	**9**	**11**
y :	**12**	**13**	**14**	**16**

i) Lagranges formula

$$y=f(10)=\frac{(10-6)\ (10-9)\ (10-11)}{(5-6)\ (5-9)\ (5-11)}\times 12+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\times 13$$

$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}\times 14+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times 16$$

$$=\frac{44}{3}$$

Divided difference

x	y	Δ	Δ^2	Δ^3
5	12			
		1		
6	13		$\frac{-2/3}{4}=\frac{-1}{6}$	
		$\frac{1}{3}$		$\frac{\frac{2}{15}+\frac{1}{6}}{11-5}=\frac{\frac{27}{90}}{6}=\frac{3/10}{6}=\frac{1}{20}$
9	14		$\frac{2/3}{5}=\frac{2}{15}$	
		$\frac{2}{2}=1$		
11	16			

$$f(10)=12+(10-5)+(10-5)(10-6)\left(-\frac{1}{6}\right)+(10-5)(10-6)(10-9)\left(\frac{1}{20}\right)$$

$$=\frac{44}{3}$$

4) **If y(1) = -3, y(3) = 9, y(4) =30, y(6) = 132 find the lagranges interpolating polynomial that takes the same values as y at the given points.**

x	1	3	4	6
y	-3	9	30	132

$$f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}.(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}.9$$

$$+\frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}.30 + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}.132$$

$$= x^3 - 3x^2 + 5x - 6$$

5) Find the interpolating polynomial using Newton divided difference formula for the following data

x	**0**	**1**	**2**	**5**
y	**2**	**3**	**12**	**147**

x	y	Δ	Δ^2	Δ^3
0	2			
		1		
1	3		4	
		9		1
2	12		9	
		45		
5	147			

$$F(x) = 2 + (x-0)(1) + (x-0)(x-1)(4) + (x-0)(x-1)(x-2)\,1$$

$$= x^3 + x^2 - x + 2$$

Numerical Integration:-

Numerical Integration:-

To find the value of $I=\int_a^b y\,dx$ numerically given the set of values (x_i, y_i), $i=0,1,2,\ldots,n$ at regular intervals.

(i) **Simpson's one third rule:-**

$$I=\frac{h}{3}[(y_0+y_n)+4(y_1+y_3+\ldots\ldots+y_{n-1})+2(y_2+y_4+\ldots\ldots+y_{n-2})]$$

when n is even.

(ii) **Simpson's three-eighth rule:-**

$$I=\frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\ldots.+y_{n-1})+2(y_3+y_6+\ldots\ldots+y_{n-3})]$$

when n is a multiple of 3.

(iii) **Weddle's rule:-**

$$I=\frac{3h}{10}[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6+\ldots\ldots]$$

when n is a multiple of 6.

Problems:

1) Using Simpson's $\frac{1}{3}^{rd}$ rule evaluate $\int_0^1 \frac{dx}{1+x^2}$ by dividing the interval $(0, 1)$ into 4 equal sub intervals and hence find the value of π correct to four decimal places.

Solution: Let us divide [0,1] into 4 equal strips (n = 4)

$\therefore$ length of each strip: $h=\frac{1-0}{4}=\frac{1}{4}$

The points of division are $x=0,\frac{1}{4},\frac{2}{4}=\frac{1}{2},\frac{3}{4},\frac{4}{4}=1$

$$\text{By data } y=\frac{1}{1+x^2}$$

Now we have the following table.

x	0	1/4	1/2	3/4	1
$y=\frac{1}{1+x^2}$	1	16/17	4/5	16/25	1/2
	y_0	y_1	y_2	y_3	y_4

Simpson's $\frac{1}{3}^{rd}$ rule for n = 4 is given by

$$\int_a^b y\,dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right]$$

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \frac{1/4}{3}\left[\left(1+\frac{1}{2}\right)+4\left(\frac{16}{17}+\frac{16}{25}\right)+2.\frac{4}{5}\right]=0.7854$$

$$\text{Thus } \int_0^1 \frac{1}{1+x^2}dx = 0.7854$$

To deduce the value of π: We perform theoretical integration and equate the resulting value to the numerical value obtained.

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1)-\tan^{-1}(0)=\frac{\pi}{4}$$

$$\text{We must have, } \frac{\pi}{4} = 0.7854 \Rightarrow \pi = 4(0.7854) = 3.1416$$

$$\text{Thus } \boxed{\pi = 3.1416}$$

2) Given that

x	**4**	**4.2**	**4.4**	**4.6**	**4.8**	**5**	**5.2**
log x	**1.3863**	**1.4351**	**1.4816**	**1.5261**	**1.5686**	**1.6094**	**1.6487**

Evaluate $\int_4^{5.2} \log x\,dx$ using Simpson's $\frac{3}{8}^{th}$ rule

Solution: Simpson's $\frac{3}{8}^{th}$ rule for n = 6 is given by

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$\int_4^{5.2} \log_e x\,dx = \frac{3(0.2)}{8}\left[(1.3863+1.6487)+3(1.4351+1.4816+1.5686+1.6094)+2(1.5261)\right]$$

$$\int_4^{5.2} \log_e x\,dx = 1.8279$$

3) Using Weddle's rule evaluate $\int_0^1 \frac{xdx}{1+x^2}$ **by taking seven ordinate and hence find** $\log_e 2$

Solution: Let us divide [0,1] into 6 equal strips (since seven ordinates)

$\therefore$ length of each strip: $h = \frac{1-0}{6} = \frac{1}{6}$

The points of division are $x = 0, \frac{1}{6}, \frac{2}{6} = \frac{1}{3}, \frac{3}{6} = \frac{1}{2}, \frac{4}{6} = \frac{2}{3}, \frac{5}{6}, \frac{6}{6} = 1$

By data $y = \frac{1}{1+x^2}$

Now we have the following table

x	0	1/6	1/3	1/2	2/3	5/6	1
$y = \frac{x}{1+x^2}$	0	6/37	3/10	2/5	6/13	30/61	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Weddle's rule for n = 6 is given by

$$\int_a^b ydx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$\int_0^1 \frac{x}{1+x^2}dx = \frac{3(1/6)}{10}\left[0 + 5(6/37) + 3/10 + 6(2/5) + 6/13 + 5(30/61) + 1/2\right]$$

$$\int_0^1 \frac{x}{1+x^2}dx = 0.3466$$

To deduce the value of $\log_e 2$: We perform theoretical integration and equate the resulting value to the numerical value obtained.

$$\therefore \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\log_e(1+x^2)\Big]_0^1 = \frac{1}{2}\log_e 2 - \frac{1}{2}\log_e 1$$

$$Hence \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\log_e 2$$

$$We\ must\ have,\ \frac{1}{2}\log_e 2 = 0.3466 \Rightarrow \log_e 2 = 2(0.3466) = 0.6932$$

$$Thus\ \boxed{\log_e 2 = 0.6932}$$