MODULE IV

INFINITE SERIES AND POWER SERIES SOLUTIONS

Learning Objectives: After reading this chapter, you will be able to understand the following

- Series of positive terms
- Convergence and Divergence of a positive term series
- Cauchy's Root Test
- D'Alembert's Ratio Test.
- Series Solution of Bessel's differential equation.
- Bessel's function of first kind and orthogonal property.
- Series Solution of Legendre's differential equation.
- Legendre polynomial and Rodrigue's formula

INTRODUCTION: In many of the problems in engineering infinite series occur frequently. The adequate knowledge on the nature of the series is necessary to solve such problems and hence studying the convergence and divergence of an infinite series is very important..

INFINITE SERIES: An infinite series is a sum with infinitely many terms. If $\{u_n\}$ is a real valued sequence then an expression of the form $u_1 + u_2 + \cdots \ldots + u_n + \cdots to\ \infty$ is called an infinite series. In symbols it is written as $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

PARTIAL SUM:

Let $\sum u_n = u_1 + u_2 + u_3 + \cdots \ldots + u_n + \cdots \ldots \ldots to\ \infty$ be an infinite series.

Let S_n be the sum of first n terms of the series.

That is, $S_n = u_1 + u_2 + u_3 + \cdots \ldots + u_n$.

Then S_n is called the n^{th} partial sum of the series $\sum u_n$.

CONVERGENCE, DIVERGENCE AND OSCILLATION OF A SERIES:

Let $\sum u_n$ be an infinite series and

Let S_n be its n^{th} partial sum.

We say that the series $\sum u_n$ is convergent, divergent or oscillatory depending on the limit of S_n as $n \to \infty$.

The three Possibilities are:

S_n may have a finite sum, it may approach $\pm\infty$ or it may have more than one limit.

If $\lim_{n\to\infty} S_n = s$, $a\ finite\ number$ then we say that the series $\sum u_n$ is convergent and has the sum s.

If $\lim_{n\to\infty} S_n = \pm\infty$, then we say that the series $\sum u_n$ is divergent.

If $\lim_{n\to\infty} S_n = s\ or\ s^1$ (That is, S_n has more than one limit as $n \to \infty$, then we say that the series $\sum u_n$ is oscillatory.

For example, consider the following series

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$i)$ $1 + \frac{1}{2} + \frac{1}{4} + \cdots \ldots \ldots \ldots . + \frac{1}{2^{n-1}} + \cdots \ldots \ldots . to\infty$

$ii)$ $1 + 2 + 3 + \cdots \ \ldots \ldots \ldots \ldots to\ \infty$

$iii)$ $2 - 2 + 2 - \cdots \ldots \ldots \ldots \ldots .. + (-1)^{n+1} + \cdots \ldots \ldots . to\ \infty$

For the series $i)$ $1 + \frac{1}{2} + \frac{1}{4} + \cdots \ldots \ldots \ldots . + \frac{1}{2^{n-1}} + \cdots \ldots \ldots . to\infty$

This is a geometric series in the form

$a + ar + ar^2 + \cdots .. + ar^{n-1} + \cdots \ldots to\ \infty$ and hence has the sum $\frac{a}{1-r}$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 + \frac{1}{2} + \frac{1}{4} + \cdots \ldots \ldots \ldots . + \frac{1}{2^{n-1}})$

$$= \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$

Thus, $\lim_{n\to\infty} S_n = 2$, $a\ finite\ number$. Hence by the definition of convergence the given series (i) is convergent.

Now for the series $ii)$ $1 + 2 + 3 + \cdots \ \ldots \ldots \ldots \ldots to\ \infty$,

$$S_n = 1 + 2 + \cdots \ldots . + n = \frac{n(n+1)}{2}$$

$\therefore \lim_{n\to\infty} S_n = \infty$ and hence the series is divergent.

Next the series $iii)$ $2 - 2 + 2 - \cdots \ldots \ldots \ldots \ldots .. + (-1)^{n+1} 2 + \cdots \ldots \ldots . to\ \infty$

Here $S_n = 2 - 2 + 2 - \cdots \ldots \ldots \ldots \ldots .. + (-1)^{n+1} 2$ and has two sums.

$\lim_{n\to\infty} S_n = 0$ if n is even and

$\lim_{n\to\infty} S_n = 2$ if n is odd . Therefore this series is oscillating.

Note: If an infinite series is not convergent then it will be a divergent series.

NECCESSARY CONDITION FOR THE CONVERGENCE OF A POSITIVE TERM SERIES:

Condition: If a series of positive terms is convergent then the limit of its n^{th} term is equal to zero.

Proof:

Let $\sum u_n$ be a series of positive terms and let S_n be the n^{th} partial sum.

If the series $\sum u_n$ is convergent then by definition $\lim_{n\to\infty} S_n = s,\ a\ finite\ number$.

Also we have $\lim_{n\to\infty} S_{n-1} = s$.

Now, $\lim_{n\to\infty}[S_n - S_{n-1}] = \lim_{n\to\infty}[(u_1 + u_2 + \cdots \dots + u_n) - (u_1 + u_2 + \cdots \dots + u_{n-1})]$

That is, $[s - s] = \lim_{n\to\infty} u_n \Longrightarrow \lim_{n\to\infty} u_n = 0$

Note: The above result can be used to test the non convergence of a given positive term series. If a given series of positive terms $\sum u_n$ is convergent then by the necessary condition for convergence $\lim_{n\to\infty} u_n = 0$. Hence, if $\lim_{n\to\infty} u_n \neq 0$ for a given series $\sum u_n$ then the series $\sum u_n$ will be divergent.

Example 4.1: Test for convergence the series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots \dots \dots + \frac{n}{n+1} + \cdots \dots . . to\ \infty$

Solution: Let $\sum u_n = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots \dots \dots + \frac{n}{n+1} + \cdots \dots . . to\ \infty$

Then $u_n = \frac{n}{(n+1)}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n}{(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{n\left(1+\frac{1}{n}\right)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{\left(1+\frac{1}{n}\right)}\right) = \frac{1}{1+0} = 1 \neq 0$$

$\therefore \lim_{n\to\infty} u_n \neq 0$. Hence by the necessary condition for the convergence of a series , the given series is not convergent.

Example 4.2: Test for convergence the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$

Solution: Let $\sum u_n = \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$

Then $u_n = \left(1+\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e \neq 0$$

$\therefore \lim_{n\to\infty} u_n \neq 0$. Hence by the necessary condition for the convergence of a series , the given series is not convergent.

$\Longrightarrow$ The given series $\sum u_n$ is divergent.

Example 4.3: Test for convergence the series

$$\frac{1^1}{1+1^2}+\frac{2^2}{1+2^2}+\frac{3^2}{1+3^2}+\cdots\ldots\ldots+\frac{n^2}{n^2+1}+\cdots\ldots..to\ \infty$$

Solution: Let $\sum u_n = \frac{1^1}{1+1^2}+\frac{2^2}{1+2^2}+\frac{3^2}{1+3^2}+\cdots\ldots\ldots+\frac{n^2}{n^2+1}+\cdots\ldots..to\ \infty$

Then $u_n = \frac{n^2}{n^2+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n^2}\right)}$$

$$= 1 \neq 0$$

$\therefore \lim_{n\to\infty} u_n \neq 0$. Hence by the necessary condition for the convergence of a series , the given series is not convergent.

$\Longrightarrow$ The given series $\sum u_n$ is divergent. \

Observations on a positive terms series:

1. Addition of a finite number of terms to a series and deletion of a finite number of terms from a series does not alter the nature of the series. That is, if we add (or delete) a finite number of terms to a divergent series, the series again will be divergent. Similarly if add (or delete) some terms from a convergent series, the series again will be convergent.

2. If $\lim_{n\to\infty} u_n \neq 0$, the the series $\sum u_n$ will be a divergent series. The converse of this

   Result is not true. That is, If $\lim_{n\to\infty} u_n = 0$, then the series $\sum u_n$ need not be convergent.

3. The geometric series $1 + r + r^2 + r^3 + \cdots \ldots \ldots \ldots \ldots to\ \infty$ converges for $|r| < 1$ and diverges for $|r| \geq 1$.

4. The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \ldots \ldots . to\ \infty$ converges for $p > 1$ and diverges for $p \leq 1$

The nature of a positive term series can be found using the following two popular tests

(i) De'Alemberts Ratio Test and
(ii) Cauchy's Root Test.

D' Alembert's Ratio Test:

Statement: Let $\sum u_n$ be a series of positive terms such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \boldsymbol{\lambda}$.

Then the series is convergent for $\boldsymbol{\lambda} < \mathbf{1}$ and is divergent for $\boldsymbol{\lambda} > 1$

Note : The ratio test fail when $\boldsymbol{\lambda} = \mathbf{1}$

Example 1: Test for convergence the series $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots \ldots \ldots \ldots \ldots to\ \infty$

Solution: Let $\sum u_n = 1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots \ldots \ldots \ldots \ldots to\ \infty$

Here the n th term $u_n = \frac{n!}{n^n} \quad \Rightarrow u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{\left(\frac{(n+1)!}{(n+1)^{n+1}}\right)}{\left(\frac{n!}{n^n}\right)} = \frac{(n+1)!}{(n+1)^{n+1} +} \times \frac{n^n}{n!}$$

$$= \frac{n!(n+1)}{n^n\left(1+\frac{1}{n}\right)^n (n+1)} \times \frac{n^n}{n!}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\text{Now, } \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\right)$$

$$= \frac{1}{e} < 1 \quad \left[\text{since } \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \right]$$

Hence by D' Alembert's Ratio Test , the series $\sum u_n$ is convergent.

Example 2: Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$

Solution: Let $\sum u_n = \sum_{n=1}^{\infty}\frac{n^2}{3^n}$

Here $u_n = \frac{n^2}{3^n} \implies u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{\left(\frac{(n+1)^2}{3^{n+1}}\right)}{\left(\frac{n^2}{3^n}\right)} = \frac{(n+1)^2}{3^{n+1}} \times \frac{3^n}{n^2}$$

$$\text{Now, } \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{n^2(1+\frac{1}{n})^2}{3^n(3)} \times \frac{3^n}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(1+\frac{1}{n})^2}{(3)}\right)$$

$$= \frac{1}{3} < 1$$

Hence by D' Alembert's Ratio Test, the series $\sum u_n$ is convergent

Example 3: Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{n!3^n}{n^n}$

Solution: Let $\sum u_n = \sum_{n=1}^{\infty}\frac{n!3^n}{n^n}$

Here $u_n = \frac{n!3^n}{n^n} \implies u_{n+1} = \frac{(n+1)!\,3^{n+1}}{(n+1)^{n+1}}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{\left(\frac{(n+1)!\;3^{n+1}}{(n+1)^{n+1}}\right)}{\left(\frac{n!\;3^n}{n^n}\right)} = \frac{(n+1)!\;3^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{n!\;3^n}$$

$$= \frac{n!(n+1)3^n 3}{((n+1)^n(n+1))} \times \frac{n^n}{n!3^n}$$

$$= \frac{3}{n^n\left(1+\frac{1}{n}\right)^n} \times \frac{n^n}{1} = \frac{3}{\left(1+\frac{1}{n}\right)^n}$$

Now, $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\left(\frac{3}{\left(1+\frac{1}{n}\right)^n}\right) = \frac{3}{e} > 1$ [since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$]

Hence by D' Alembert's Ratio Test, the series $\sum u_n$ is divergent.

Example 4: Test for convergence the series $1 + \frac{1.3}{1.5} + \frac{1.3.5}{1.5.9.} + \frac{1.3.5.7}{1.5.9.13} + \cdots \ldots\ldots\ldots to\ \infty$

Let $\sum u_n = 1 + \frac{1.3}{1.5} + \frac{1.3.5}{1.5.9.} + \frac{1.3.5.7}{1.5.9.13} + \cdots \ldots\ldots\ldots to\ \infty$

Here $u_n = \frac{1.3.5.7\ldots\ldots(2n-1)}{1.5.9.13\ldots\ldots(4n-3)} \implies u_{n+1} = \frac{1.3.5.7\ldots\ldots(2n-1)(2n+1)}{1.5.9.13\ldots\ldots(4n-3)\ (4n+1)}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{1.3.5.7\ldots\ldots(2n-1)(2n+1)}{1.5.9.13\ldots\ldots(4n-3)\ (4n+1)} \times \frac{1.5.9.13\ldots\ldots(4n-3)}{1.3.5.7\ldots\ldots(2n-1)} = \frac{2n+1}{4n+1}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2n+1}{4n+1}$$

$$= \lim_{n\to\infty} \frac{n\left(2+\frac{1}{n}\right)}{n\left(4+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(2+\frac{1}{n}\right)}{\left(4+\frac{1}{n}\right)} = \frac{2}{4} = \frac{1}{2}$$

$$\implies \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$$

$\therefore$ By D' Alembert's Ratio Test, the series $\sum u_n$ is convergent.

Example 5: Test for convergence the series $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \cdots \ldots\ldots\ldots to\ \infty$

Solution: Let $\sum u_n = \frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \cdots \ldots\ldots\ldots to\ \infty$

Here $u_n = \frac{2.5.8\ldots\ldots(3n-1)}{1.5.9\ldots\ldots(4n-1)} \implies u_{n+1} = \frac{2.5.8\ldots\ldots(3n-1)(3n+2)}{1.5.9\ldots\ldots(4n-1)(4n+3)}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{2.5.8\ldots\ldots(3n-1)(3n+2)}{1.5.9\ldots\ldots(4n-1)(4n+3)} \times \frac{1.5.9\ldots\ldots(4n-1)}{2.5.8\ldots\ldots(3n-1)} = \frac{3n+2}{4n+3}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3n+2}{4n+3}$$

$$= \lim_{n\to\infty} \frac{n\left(3+\frac{2}{n}\right)}{n\left(4+\frac{3}{n}\right)}$$

Example 6: Test for convergence the series

$$\frac{1+\propto}{1+\beta}+\frac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)}+\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)}{(1+\beta)(1+2\beta)(1+3\beta)}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{to}\ \infty$$

Solution: Let $\sum u_n = \frac{1+\propto}{1+\beta}+\frac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)}+\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)}{(1+\beta)(1+2\beta)(1+3\beta)}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{to}\ \infty$

Here $u_n = \frac{(1+\alpha)(1+2\alpha)(1+3\alpha)\ldots\ldots(1+n\alpha)}{(1+\beta)(1+2\beta)(1+3\beta)\ldots\ldots\ldots(1+n\beta)}$

$$\Rightarrow \quad u_{n+1} = \frac{(1+\alpha)(1+2\alpha)(1+3\alpha)\ldots\ldots(1+n\alpha)[1+(n+1)\alpha}{(1+\beta)(1+2\beta)(1+3\beta)\ldots\ldots\ldots(1+n\beta)[1+(n+1)\beta]}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(1+\alpha)(1+2\alpha)(1+3\alpha)\ldots\ldots(1+n\alpha)[1+(n+1)\alpha]}{(1+\beta)(1+2\beta)(1+3\beta)\ldots\ldots\ldots(1+n\beta)[1+(n+1)\beta]}\times\frac{(1+\beta)(1+2\beta)(1+3\beta)\ldots\ldots\ldots(1+n\beta)}{(1+\alpha)(1+2\alpha)(1+3\alpha)\ldots\ldots(1+n\alpha)}$$

$$= \frac{[1+(n+1)\alpha]}{[1+(n+1)\beta]}$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{[1+(n+1)\alpha]}{[1+(n+1)\beta]}$$

$$= \lim_{n\to\infty}\frac{n\left(\frac{1}{n}+(1+\frac{1}{n}\right)\alpha}{n\left(\frac{1}{n}+(1+\frac{1}{n}\right)\beta}$$

$$= \frac{\alpha}{\beta}$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{\alpha}{\beta}$$

$\therefore$ By D' Alembert's Ratio Test , the series

$\sum u_n$ is convergent for $\frac{\alpha}{\beta} < 1$ [(That is, $\beta > \alpha > 0$]

and is divergent for $\frac{\alpha}{\beta} > 1$ [(That is, $\alpha > \beta > 0$]

and the ratio test fails when $\frac{\alpha}{\beta} = 1$ [That is when $\alpha = \beta$]

But, we have $u_n = 1$ when $\alpha = \beta$

$\Longrightarrow \lim\limits_{n\to\infty} u_n = 1 \neq 0$.

Hence by necessary condition for the convergence of a positive term series , $\sum u_n$ is divergent when $\alpha = \beta$.

Thus, the given series is convergent for $\beta > \alpha > 0$ and is divergent for $\beta \geq \alpha > 0$

Example 7: Test for convergence the series $\sum_{n=1}^{\infty} \frac{3.6.9.........3n}{4.7.10.........(3n+1)} \frac{5^n}{3n+2}$

Solution: Let $\sum u_n = \sum_{n=1}^{\infty} \frac{3.6.9.........3n}{4.7.10.........(3n+1)} \frac{5^n}{3n+2}$

Here $u_n = \frac{3.6.9.........3n}{4.7.10.........(3n+1)} \frac{5^n}{3n+2}$

$\Longrightarrow \quad u_{n+1} = \frac{3.6.9.........3n(3n+3)}{4.7.10.........(3n+1)(3n+4)} \frac{5^{n+1}}{3n+5}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{3.6.9.........3n(3n+3)}{4.7.10.........(3n+1)(3n+4)} \frac{5^{n+1}}{3n+5} \times \frac{4.7.10......(3n-1)}{3.6.9.........3n} \frac{(3n+2)}{5^n} = \frac{(3n+3)}{(3n+4)} \frac{(3n+2)}{(3n+5)} \frac{5^n 5}{5^n}$

$= \frac{5(3n+3)(3n+2)}{(3n+4)(3n+5)}$

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 5 \lim\limits_{n\to\infty} \frac{\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)}{\left(3+\frac{4}{n}\right)\left(3+\frac{5}{n}\right)} = 5 \frac{(3+0)(3+0)}{(3+0)(3+0)} = 5$

$\Longrightarrow \lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 5 > 1$

By D' Alembert's Ratio Test , the series $\sum u_n$ is divergent

Example 8: Examine the convergence of the series

$\frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \cdots \ldots \ldots \ldots \ldots to\ \infty$, where $x > 0$

Solution: Let $\sum u_n = \frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \cdots \ldots \ldots \ldots \ldots to$

Here $u_n = \frac{x^n}{1+x^n}$

$\Longrightarrow \quad u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{1+x^{n+1}} \times \frac{1+x^n}{x^n}$

$= \frac{x^n x}{x^n\left(\frac{1}{x^n}+x\right)} \times \frac{x^n\left(\frac{1}{x^n}+1\right)}{x^n} = x$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}(x) = x$$

By D' Alembert's Ratio Test, the series $\sum u_n$ is convergent for $x < 1$ and is divergent for $x > 1$, and the ratio test fails when $x = 1$

But, when $x = 1$, $u_n = \frac{1}{1+1} = \frac{1}{2}$

$\therefore \lim_{n\to\infty}(u_n) = \frac{1}{2} \neq 0$. Hence by necessary condition for the convergence of a positive term series, $\sum u_n$ is divergent when x = 1.

Thus, the given series is convergent for $x < 1$ and is divergent for $x \geq 1$

Example 9: Test the convergence of the series

$$\sum_{n=1}^{\infty}\frac{4.7.10.........(3n+1)}{1.2.3.........n}x^n \quad , \quad \text{where } x > 0$$

Solution: Let $\sum u_n = \sum_{n=1}^{\infty}\frac{4.7.10.........(3n+1)}{1.2.3.........n}x^n$

Here $u_n = \frac{4.7.10.........(3n+1)}{1.2.3.........n}x^n$

$$\Longrightarrow \quad u_{n+1} = \frac{4.7.10.........(3n+1)(3n+4)}{1.2.3.........n(n+1)}x^{n+1}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{4.7.10.........(3n+1)(3n+4)}{1.2.3.........n(n+1)}x^{n+1} \times \frac{(1.2.3.........n)}{4.7.10.........(3n+1)}\frac{1}{x^n}$$

$$= \frac{(3n+4)}{(n+1)}\frac{x}{1}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{n\left(3+\frac{4}{n}\right)}{n\left(1+\frac{1}{n}\right)}\frac{x}{1}\right) = \lim_{n\to\infty}\left(\frac{\left(3+\frac{4}{n}\right)}{\left(1+\frac{1}{n}\right)}\frac{x}{1}\right) = 3x$$

By D' Alembert's Ratio Test, the series $\sum u_n$ is convergent for $3x < 1$ (That is, $x < \frac{1}{3}$) and is divergent for $3x > 1$,(that is, $x > \frac{1}{3}$) and the ratio test fails when $3x = 1$ (that is , $x = \frac{1}{3}$).

Example 10: Test the convergence of the series $\sum_{n=1}^{\infty}\frac{n^3+\alpha}{2^n+\alpha}$

Solution: Let $\sum u_n = \sum_{n=1}^{\infty}\frac{n^3+\alpha}{2^n+\alpha}$

Here $u_n = \frac{n^3+\alpha}{2^n+\alpha}$

$$\Longrightarrow \quad u_{n+1} = \frac{(n+1)^3+\alpha}{2^{n+1}+\alpha}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^3+\alpha}{2^{n+1}+\alpha} \times \frac{2^n+\alpha}{n^3+\alpha}$$

$$= \frac{n^3[(1+\frac{1}{n^3})^3+\frac{\alpha}{n^3}]}{2^n[2+\frac{\alpha}{2^n}]} \times \frac{2^n[1+\frac{\alpha}{2^n}]}{n^3[1+\frac{\alpha}{n^3}]}$$

$$= \frac{[(1+\frac{1}{n^3})^3+\frac{\alpha}{n^3}]}{[2+\frac{\alpha}{2^n}]} \times \frac{[1+\frac{\alpha}{2^n}]}{[1+\frac{\alpha}{n^3}]}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{[(1+\frac{1}{n^3})^3+\frac{\alpha}{n^3}]}{[2+\frac{\alpha}{2^n}]} \times \frac{[1+\frac{\alpha}{2^n}]}{[1+\frac{\alpha}{n^3}]} \right) = \frac{[(1+0)+0]}{[2+0]} \times \frac{[1+0]}{[1+0]}$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1.$$

$\therefore$ By D' Alembert's Ratio Test, the series $\sum u_n$ is convergent.

Example 11: Test the convergence of the series $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots\ldots\ldots$ to ∞

Solution: Let $\sum u_n = 1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots\ldots\ldots$ to ∞

Here $u_n = \frac{n^p}{n!}$

$\Longrightarrow \quad u_{n+1} = \frac{(n+1)^p}{(n+1)!}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p} = \frac{n^p\left(1+\frac{1}{n}\right)^p}{n!\,(n+1)} \times \frac{n!}{n^p} = \frac{\left(1+\frac{1}{n}\right)^p}{(n+1)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^p}{(n+1)} = \frac{1}{\infty} = 0 < 1$$

$\therefore$ By D' Alembert's Ratio Test, the series $\sum u_n$ is convergent.

Example 12: Discuss the convergence of the series

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \left(\frac{1.2.3.4}{3.5.7.9}\right)^2 + \cdots \ldots \ldots \ldots \ldots to\ \infty$$

Solution: Let $\sum u_n = \left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \left(\frac{1.2.3.4}{3.5.7.9}\right)^2 + \cdots \ldots \ldots \ldots \ldots to\ \infty$

Here $u_n = \left(\frac{1.2.3.4\ldots\ldots\ldots\ldots n}{3.5.7.9\ldots\ldots\ldots(2n+1)}\right)^2$

$\Longrightarrow \quad u_{n+1} = \left(\frac{1.2.3.4\ldots\ldots\ldots\ldots n(n+1)}{3.5.7.9\ldots\ldots\ldots(2n+1)(2n+3)}\right)^2$

$$\therefore \frac{u_{n+1}}{u_n} = \left(\frac{1.2.3.4\ldots\ldots\ldots n(n+1)}{3.5.7.9\ldots\ldots(2n+1)(2n+3)}\right)^2 \times \left(\frac{3.5.7.9\ldots\ldots(2n+1)}{1.2.3.4\ldots\ldots\ldots n}\right)^2$$

$$= \left(\frac{n+1}{2n+1}\right)^2$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{n+1}{2n+1}\right)^2 = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right)^2 = \left(\frac{1+0}{2+0}\right)^2 = \frac{1}{4}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{1}{4} < 1$$

$\therefore$ By D' Alembert's Ratio Test, the series $\sum u_n$ is convergent.

Example 13: Test for convergence the series

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \cdots \ldots\ldots\ldots to\ \infty\ ,\ x > 0$$

Solution: Let $\sum u_n = 1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \cdots \ldots\ldots\ldots to\ \infty$

Here $u_n = \frac{2^{n+1}-2}{2^{n+1}+1}\, x^n$, [Neglecting the first term]

$$\Rightarrow \quad u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1}\, x^{n+1}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{2^{n+2}-2}{2^{n+2}+1}\, x^{n+1} \times \frac{2^{n+1}+1}{2^{n+1}-2}\,\frac{1}{x^n}$$

$$= \frac{2^{n+2}\left(1-\frac{2}{2^{n+2}}\right)}{2^{n+2}\left(1-\frac{1}{2^{n+2}}\right)}\, x^n x \times \frac{2^{n+1}\left(1-\frac{1}{2^{n+1}}\right)}{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}\,\frac{1}{x^n}$$

$$= \frac{\left(1-\frac{2}{2^{n+2}}\right)}{\left(1-\frac{1}{2^{n+2}}\right)}\, x \times \frac{\left(1-\frac{1}{2^{n+1}}\right)}{\left(1-\frac{2}{2^{n+1}}\right)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{\left(1-\frac{2}{2^{n+2}}\right)}{\left(1-\frac{1}{2^{n+2}}\right)}\, x \times \frac{\left(1-\frac{1}{2^{n+1}}\right)}{\left(1-\frac{2}{2^{n+1}}\right)}\right] = \frac{(1-0)}{(1-0)}x \times \frac{(1-0)}{(1-0)} = x$$

By D' Alembert's Ratio Test, the series $\sum u_n$ is convergent for $x < 1$ and is divergent for $x > 1$, and the ratio test fails when $x = 1$

But, when $x = 1$ we have , $u_n = \frac{2^{n+1}-2}{2^{n+1}+1}$

$\therefore \lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{2^{n+1}-2}{2^{n+1}+1}\right) = \lim_{n\to\infty}\frac{\left(1-\frac{2}{2^{n+2}}\right)}{\left(1-\frac{1}{2^{n+2}}\right)} = 1 \neq 0$. Hence by necessary condition for the convergence of a positive term series , $\sum u_n$ is divergent when x = 1.

Example 14: Test for convergence the series $\sum_{n=1}^{\infty} \frac{1.3.5.7.........(2n-1)}{4.7.10.........(3n+1)}$

Solution: Let $\sum u_n = \sum_{n=1}^{\infty} \frac{1.3.5.7.........(2n-1)}{4.7.10.........(3n+1)}$

Here $u_n = \frac{1.3.5.7.........(2n-1)}{4.7.10.........(3n+1)}$

$\Longrightarrow \quad u_{n+1} = \frac{1.3.5.7.........(2n-1)(2n+1)}{4.7.10.........(3n+1)(3n+4)}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{1.3.5.7\ldots\ldots\ldots(2n-1)(2n+1)}{4.7.10\ldots\ldots\ldots(3n+1)(3n+4)} \times \frac{4.7.10\ldots\ldots\ldots(3n+1)}{1.3.5.7\ldots\ldots\ldots(2n-1)}$$

$\frac{u_{n+1}}{u_n} = \frac{2n+1}{3n+4}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2n+1}{3n+4} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{3+\frac{4}{n}} = \frac{2+0}{3+0} = \frac{2}{3}$$

$\Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{2}{3} < 1$

$\therefore$ By D' Alembert's Ratio Test, the series $\sum u_n$ is convergent.

Example 15: Test for convergence the series $\frac{1}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \cdots \ldots\ldots\ldots to\ \infty$

Solution: Let $\sum u_n =$ series $\frac{1}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \cdots \ldots\ldots\ldots to\ \infty$

Here $u_n = \frac{n^2}{n!}$

$\Longrightarrow \quad u_{n+1} = \frac{(n+1)^2}{(n+1)!}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(n+1)!} \times \frac{n!}{n^2}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!} \times \frac{n!}{n^2}$$

$$= \lim_{n\to\infty} \frac{n^2(1+\frac{1}{n})^2}{n!(n+1)} \times \frac{n!}{n^2} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{(n+1)} = \frac{(1+0)^2}{\infty} = 0$$

$\Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 < 1$

$\therefore$ By D' Alembert's Ratio test, the series $\sum u_n$ is convergent.

Cauchy's Root Test:

Statement: Let $\sum u_n$ be a series of positive terms such that $\lim_{n\to\infty} u_n^{\frac{1}{n}} = \boldsymbol{\lambda}$.

Then the series is convergent for $\boldsymbol{\lambda} < \mathbf{1}$ and is divergent for $\boldsymbol{\lambda} > 1$

Note : The root test fails when $\boldsymbol{\lambda} = \mathbf{1}$

Example 1: Test for convergence of the series $\frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots \dots \dots \dots \dots to\infty$

Solution: Let $\sum u_n = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots \dots \dots \dots \dots to\infty$

Here, $u_n = \frac{1}{n^n}$

$$\therefore \lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0 < 1$$

By Cauchy's Root test the series $\sum u_n$ is convergent .

Example 2: Test for convergence the series $\sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

Solution: Let $\sum u_n = \sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

Here, $u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{\frac{3}{2}}}}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}\right) = \frac{1}{e}$$

$$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = \frac{1}{e} < 1$$

By Cauchy's Root test the series $\sum u_n$ is convergent .

Example 3: Test for convergence the series

$$1 + \frac{x}{1} + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \cdots \ldots \ldots \ldots to\infty,\ x > 0$$

Solution: Let $\sum u_n = 1 + \frac{x}{1} + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \cdots \ldots \ldots \ldots to\infty,$

Here, $u_n = \frac{x^n}{n^n} = \left(\frac{x}{n}\right)^n$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{x}{n}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{x}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = 0 < 1$$

$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = 0 < 1$. By Cauchy's Root test the series $\sum u_n$ is convergent

Example 4: Test for convergence the series the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

Solution: Let $\sum u_n = \sum_{n=1}^{\infty} \frac{n^3}{3^n}$

Here, $u_n = \frac{n^3}{3^n}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n^3}{3^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{\frac{3}{n}}}{3} = \frac{1}{3} \qquad [\text{since, } \lim_{n\to\infty}(n)^{\frac{1}{n}} = 1\,]$$

$$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = \frac{1}{3} < 1$$

By Cauchy's Root test the series $\sum u_n$ is convergent

Example 5: Discuss the nature of the series

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots \ldots \ldots \ldots t0\ \infty\ , x > 0$$

Solution: Let $\sum u_n = \frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots \ldots \ldots \ldots t0\ \infty$

Here, $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$ [Neglecting the first term]

$$u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{n+1}{n+2}\right)^n x^n\right)^{\frac{1}{n}} =$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)x == \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)x = \left(\frac{1+0}{1+0}\right)x = x$$

$$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = x > 0$$

By Cauchy's Root test the series $\sum u_n$ is convergent for $x < 1$

and is divergent for $x > 1$, and the ratio test fails at $x = 1$.

When $x = 1$, we have $u_n = \left(\frac{n+1}{n+2}\right)^n$ and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{\left(\frac{1}{e}\right)}{\left(\frac{1}{e^2}\right)} = e$$

[Since, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$]

$$\lim_{n\to\infty} u_n = e \neq 1$$

By necessary condition for convergence of positive term series, the series $\sum u_n$ is divergent.

Thus, the given series is converges for $x < 1$ and diverges for $x \geq 1$.

Example 6: Discuss the nature of the series

$$\left[\frac{2^2}{1^2}-\frac{2}{1}\right]^{-1} + \left[\frac{3^3}{2^3}-\frac{3}{2}\right]^{-2} + \left[\frac{4^4}{3^4}-\frac{4}{3}\right]^{-3} + \cdots \ldots\ldots\ldots\ldots\ldots to\infty$$

Solution: Let $\sum u_n = \left[\frac{2^2}{1^2}-\frac{2}{1}\right]^{-1} + \left[\frac{3^3}{2^3}-\frac{3}{2}\right]^{-2} + \left[\frac{4^4}{3^4}-\frac{4}{3}\right]^{-3} + \cdots \ldots\ldots\ldots\ldots\ldots to\infty$

Here, $u_n = \left[\frac{(n+1)^{(n+1)}}{n^{(n+1)}} - \frac{(n+1)}{n}\right]^{-n}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left[\frac{(n+1)^{(n+1)}}{n^{(n+1)}} - \frac{(n+1)}{n}\right]^{-n}\right)^{\frac{1}{n}} =$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^{(n+1)}}{n^{(n+1)}} - \frac{(n+1)}{n}\right]^{-1}$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^n(n+1)}{n^n n} - \frac{n(1+\frac{1}{n})}{n}\right]^{-1}$$

$$= \lim_{n\to\infty}\left[\frac{n^n\left(1+\frac{1}{n}\right)^n n\left(1+\frac{1}{n}\right)}{n^n n} - \frac{n\left(1+\frac{1}{n}\right)}{n}\right]^{-1}$$

$$= \lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}{1} - \frac{\left(1+\frac{1}{n}\right)}{1}\right]^{-1}$$

$$\Rightarrow \lim_{n\to\infty} {u_n}^{\frac{1}{n}} = (e-1)^{-1} = \frac{1}{e-1} = 0.582072 < 1$$

By Cauchy's Root test the series $\sum u_n$ is convergent.

Example 7: Discuss the nature of the series

$$\frac{3}{4}x + \left(\frac{4}{5}\right)^2 x^2 + \left(\frac{5}{6}\right)^3 x^3 + \cdots \ldots \ldots \ldots t0\ \infty,\ \ x > 0$$

Solution: Let $\sum u_n = \frac{3}{4}x + \left(\frac{4}{5}\right)^2 x^2 + \left(\frac{5}{6}\right)^3 x^3 + \cdots \ldots \ldots \ldots t0\ \infty$

Here, $u_n = \left(\frac{n+2}{n+3}\right)^n x^n$

$$\lim_{n\to\infty} {u_n}^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{n+2}{n+3}\right)^n x^n\right)^{\frac{1}{n}} =$$

$$= \lim_{n\to\infty}\left(\frac{n+2}{n+3}\right)x = = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x = \left(\frac{1+0}{1+0}\right)x = x$$

$$\Rightarrow \lim_{n\to\infty} {u_n}^{\frac{1}{n}} = x > 0$$

By Cauchy's Root test the series $\sum u_n$ is convergent for $x < 1$

and is divergent for $x > 1$, and the ratio test fails at $x = 1$.

When $x = 1$, we have $u_n = \left(\frac{n+2}{n+3}\right)^n$ and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n+2}{n+3}\right)^n = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)^n = \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = \frac{\left(\frac{1}{e^2}\right)}{\left(\frac{1}{e^3}\right)} = e$$

$$\left[\text{ Since, } \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x\right]$$

$$\lim_{n\to\infty} u_n = e \neq 1$$

By necessary condition for convergence of positive term series, the series $\sum u_n$ is divergent.

Thus, the given series is converges for $x < 1$ and diverges for $x \geq 1$.

Example8: Test for convergence the series

$1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\cdots\ldots\ldots\ldots to\infty \; x>0$

Solution: Let $\sum u_n = 1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\cdots\ldots\ldots\ldots to\infty$, ,

Here, $u_n = \frac{x^n}{(n+1)^n} = \left(\frac{x}{n+1}\right)^n$ [neglecting he first term]

$$u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{x}{n+1}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{x}{n+1} = 0$$

$$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = 0 < 1$$

$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = 0 < 1$. By Cauchy's Root test the series $\sum u_n$ is convergent

Example 9: Test for convergence the series $\sum \frac{1}{\left(1-\frac{1}{n}\right)^{n^2}}$

Solution: Let $\sum u_n = \sum \frac{1}{\left(1-\frac{1}{n}\right)^{n^2}}$

Here, $u_n = \frac{1}{\left(1-\frac{1}{n}\right)^{n^2}}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{\left(1-\frac{1}{n}\right)^{n^2}}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1-\frac{1}{n}\right)^n} = \frac{1}{e^{-1}} = e > 1$$

$$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = e > 1$$

By Cauchy's Root test the series $\sum u_n$ is divergent .

Example 10: Test for convergence the series $\sum\left(1-\frac{3}{n}\right)^{n^2}$

Solution: Let $\sum u_n = \sum\left(1-\frac{3}{n}\right)^{n^2}$

Here, $u_n = \left(1-\frac{3}{n}\right)^{n^2}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(1-\frac{3}{n}\right)^{n^2}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(1-\frac{3}{n}\right)^{n} = e^{-3}$$

$$\Rightarrow \lim_{n\to\infty} u_n^{\frac{1}{n}} = \frac{1}{e^3} < 1$$

By Cauchy's Root test the series $\sum u_n$ is convergent.

SERIES SOLUTION OF DIFFERENTIAL EQUATIONS:

In module 2, we discussed the solution of linear differential equations with constant coefficients. Many differential equations which arise from physical problems are linear but have variable coefficients and do not permit general solution in terms of standard well known functions. For such a problem a solution can be obtained in the form of an infinite convergent series. The series solution of certain differential equations give rise to special functions such as Bessel's function and Legendre Polynomial. These special functions have many applications in engineering.

VALIDITY OF SERIES SOLUTION:

Every differential equation of the form

$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$ ... ...(1) , where

where $P_0(x), P_1(x)$ and $P_2(x)$ are polynomials in x, does not have series solution. Here we discuss the conditions, under which the above equation admits series solution.

Dividing by $P_0(x)$, the above differential can be expressed as

$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0$. If $P_0(0) = 0$, then $x = 0$ is called a singular point of Eqn.(1). If $P_0(0) \neq 0$, then $x = 0$ is called an ordinary point of Eqn.(1). If

$If\ x\frac{P_1(x)}{P_0(x)}$ and $x^2\frac{P_2(x)}{P_0(x)}$ possess derivatives of all orders in the neighbourhood of $x = 0$ then $x = 0$ is called a regular singular point of (1).

When $x = 0$ is an ordinary point of (1) , its every solution can be expressed as a convergent series of the form

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \dots \dots \dots . to\ \infty$$

When $x = 0$ is a regular singular point of (1) , at least one of its solution can be expressed as

$$y = \sum_{r=0}^{\infty} a_r x^{m+r} = x^m(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \ldots \ldots \ldots . to\ \infty)$$

When $x = 0$ is an irregular singular point of (1), then the differential equation (1) has no series solution of the above type.

BESSEL'S DIFFERENTIAL EQUATION:

A differential equation in the form $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$ is known as Bessel's differential equation of order n. Its particular solutions are called Bessel's functions of order n. Many physical problems involving vibrations or heat conduction in cylindrical regions give rise to this equation.

SERIES SOLUTION OF BESSEL'S EQUATION:

Bessel's equation is $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \ldots \ldots \ldots \ldots (1)$

Since $x = 0$ is a regular singular point of the given differential equation, let its solution be $y = \sum_{r=0}^{\infty} a_r x^{m+r} = x^m(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \ldots \ldots \ldots . to\ \infty)$

$\Rightarrow \frac{dy}{dx} = \sum_{r=0}^{\infty}(m+r)a_r x^{m+r-1}$

And $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-2}$

Substituting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1), we get

$$x^2\left(\sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-2}\right) + x\left(\sum_{r=0}^{\infty}(m+r)a_r x^{m+r-1}\right)$$

$+(x^2 - \ n^2)(\sum_{r=0}^{\infty} a_r x^{m+r}) = 0$

$\Rightarrow \sum_{r=0}^{\infty}[(m+r)^2 - (m+r) + (m+r) - n^2]a_r x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$

$\Rightarrow \sum_{r=0}^{\infty}[(m+r)^2 - n^2]a_r x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$ ....(2)

Equating to zero the coefficient of x^m (that is , corresponding to $r = 0$), we get

$[m^2 - n^2]a_0 = 0 \Rightarrow [m^2 - n^2] = 0$, Since Since, $a_0 \neq 0$

$\Rightarrow\ m = \pm n$

Equating to zero the coefficient of x^{m+1} (that is , corresponding to $r = 1$) in (2), we get

$[(m+1)^2 - n^2]a_1 = 0 \Rightarrow a_1 = 0$, $[(m+1)^2 - n^2] \neq 0\ for\ m = \pm n.$

$[(m+r+2)^2-n^2]a_{r+2}+a_r=0$

$\Rightarrow a_{r+2}=\frac{-a_r}{(m+r+2)^2-n^2}=-\frac{a_r}{(m-n+r+2)(m+n+r+2)}$

Putting r = 1, 3, 5,....., we get $a_3=a_5=a_7=\cdots\ldots=0$ and

Putting r = 0, 2, 4,....., we get

$a_2=-\frac{a_0}{(m-n+2)(m+n+2)}$

$a_4=-\frac{a_2}{(m-n+4)(m+n+4)}$

$=\frac{a_0}{(m-n+4)(m+n+4)(m-n+2)(m+n+2)}$ and so on.

$$\therefore\ y=\sum_{r=0}^{\infty}a_rx^{m+r}=x^m(a_0+a_1x+a_2x^2+a_3x^3+\cdots\ldots\ldots\ldots.to\ \infty)$$

$$\Rightarrow\ y=a_0x^m\left[1-\frac{x^2}{(m+2)^2-n^2}+\frac{x^4}{[(m+2)^2-n^2][(m+4)^2-n^2]}-\cdots\ldots\ldots.\right]$$

We get different solutions, depending upon the values of n.

For $m=n$, we get

$$y_1=a_0x^n\left[1-\frac{x^2}{2^2(n+1)}+\frac{x^4}{2^4 2(n+1)(n+2)}-\cdots\ldots\ldots\right]$$

$$=a_0x^n\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{2^{2r}(r!)(n+1)(n+2)\ldots\ldots(n+r)}x^{2r}\right]$$

$$y_1=a_0x^n\sum_{r=0}^{\infty}\frac{(-1)^r\gamma(n+1)}{r!\,2^{2r}\,\gamma(n+r+1)}x^{2r}$$

The second independent solution corresponding to $m=-n$ is

$$y_1=a_0x^{-n}\sum_{r=0}^{\infty}\frac{(-1)^r\gamma(-n+1)}{r!\,2^{2r}\,\gamma(-n+r+1)}x^{2r}\ .$$

The complete solution of (1) is $y=c_1y_1+c_2y_2$

Since, a_0 is arbitrary constant, we can choose it as we like. Choosing $a_0=\frac{1}{2^n\,\gamma(n+1)}$

$$y_1=\frac{x^n}{2^n\,\gamma(n+1)}\sum_{r=0}^{\infty}\frac{(-1)^r\gamma(n+1)}{r!\,2^{2r}\,\gamma(n+r+1)}x^{2r}=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\ \gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

This is called Bessel function of the first kind of order n and is denoted by $J_n(x)$.

Thus, Bessel function $J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{(r)!\ \gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

Note: The solution corresponding to $m=-n$ is

$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!\ \gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}$ which is called Bessel function of the first kind of order $(-n)$. The complete solution of the Bessel equation is given by

$y = c_1 J_n(x) + c_2 J_{-n}(x)$

ORTHOGONALITY OF BESSEL FUNCTION:

If α and β are the roots of Bessel equation $J_n(x) = 0$,

Then $\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = 0$, for $\alpha \neq \beta$

Proof: Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$, then u *and* v are the solutions of the Bessel equations

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad \text{......... (1)}$$

$$and \quad x^2v'' + xv' + (\alpha^2x^2 - n^2)v = 0 \qquad \text{......... (2)}$$

Multiplying (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$ and subtracting, we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)xuv = 0$$

$$\Rightarrow \frac{d}{dx}[x(u'v - uv')] = (\beta^2 - \alpha^2)xuv$$

Now, integrating b0th sides with respect to x, from 0 to 1, we get

$$(\beta^2 - \alpha^2)\int_0^1 xuv\ dx\ 0 = (u'v - uv') \text{ at } x = 1. \qquad \text{......... (3)}$$

Since $u = J_n(\alpha x)$ and $v = J_n(\beta x)$;

$$u' = \alpha\, J_n{}'(\alpha x) \text{ and } v' = \beta\, J_n{}'(\beta x) \ \ and$$

$at\ x = 1$

$$u' = \alpha\, J_n{}'(\alpha) \text{ and } v' = \beta\, J_n{}'(\beta) \ .$$

Substituting these values in (2)

$$\int_0^1 xJ_n(\alpha x)\, J_n(\beta x)dx\ 0 = \frac{\alpha\, J_n{}'(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n{}'(\beta)}{(\beta^2 - \alpha^2)} \qquad \text{.......... (4)}$$

Since α and β are the roots of $J_n(x) = 0$, we have $J_n(\alpha) = 0$ and $J_n(\beta) = 0$

And (4) reduces to the form

$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = 0$, for $\alpha \neq \beta$

Example1: Show that $\boldsymbol{J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots \ldots \ldots \ldots \ldots}$

Answer: We have $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!\ \gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Putting n = 0 , we get

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!\ \gamma(r+1)} \left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!(r!)} \left(\frac{x}{2}\right)^{2r}, \qquad [\text{Since } \gamma(r+1) = r!]$$

$$= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots \ldots\ldots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots \ldots\ldots\ldots$$

$$\Rightarrow \qquad \boldsymbol{J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots \ldots\ldots\ldots}$$

Example 2: Show that $\boldsymbol{J_1(x) = \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} + \cdots \ldots}$

Answer: **We have** $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!\ \gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Putting $n = 1$, we get

$$J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!\ \gamma(r+2)} \left(\frac{x}{2}\right)^{1+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!\ (r+1)!} \left(\frac{x}{2}\right)^{1+2r}$$

$$= \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 3 + \frac{1}{(2!)(3!)}\left(\frac{x}{2}\right)^5 - \cdots \ldots\ldots\ldots$$

$$\boldsymbol{J_1(x) = \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} + \cdots \ldots}$$

Example3: Show that $\boldsymbol{J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\ sinx}$

We have $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!\ \gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Putting $n = \frac{1}{2}$, we get

$$J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!\ \gamma\left(\frac{1}{2}+r+1\right)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2r}$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \left[\sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!\ \gamma\left(\frac{1}{2}+r+1\right)} \left(\frac{x}{2}\right)^{2r}\right]$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}}\left[\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots \ldots \ldots\right]$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}}\left[\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots \ldots \ldots\right]$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}}\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{1}{\frac{1}{2}} - \frac{1}{\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\frac{5}{2}\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \cdots \ldots \ldots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}}\frac{1}{\sqrt{\pi}}\left[\frac{2}{1!} - \frac{2}{3!}x^2 + \frac{2}{5!}x^4 - \cdots \ldots \ldots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}}\frac{1}{\sqrt{\pi}}\frac{2}{x}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \ldots \ldots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \ldots \ldots\right]$$

$$\boldsymbol{J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, sinx}$$

Example 4: show that $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, cosx$

Answer: We have $J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{(r)!\,\gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

Now putting $n = \frac{-1}{2}$, we get

$$J_{-\frac{1}{2}}(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{(r)!\,\gamma\left(-\frac{1}{2}+r+1\right)}\left(\frac{x}{2}\right)^{-\frac{1}{2}+2r}$$

$$= \left(\frac{x}{2}\right)^{\frac{-1}{2}}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{(r)!\,\gamma\left(\frac{1}{2}+r\right)}\left(\frac{x}{2}\right)^{2r}\right]$$

$$= \left(\frac{2}{x}\right)^{\frac{1}{2}}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots \ldots \ldots\right]$$

$$= \left(\frac{2}{x}\right)^{\frac{1}{2}}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots \ldots \ldots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}}\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{1}{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \ldots \ldots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}}\frac{1}{\sqrt{\pi}}\left[\frac{1}{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \ldots \ldots\right]$$

$$= \sqrt{\frac{2}{\pi x}} cosx$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, cosx$$

LEGENDRE'S EQUATION: A differential equation in the form

$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$, where n is a real number, is called a Legendre differential equation. This equation is of considerable importance in applied mathematics, particularly in boundary value problems involving spherical configurations. The series solution of this equation can be obtained in ascending or descending powers of x. The solution in descending powers of x is more important than the one in ascending powers.

SERIES SOLUTION OF LEGENDER'S EQUATION:

Legendre's equation is $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \ldots\ldots\ldots\ldots (1)$

Let $y = \sum_{r=0}^{\infty} a_r x^{m-r}$, $where\ a_0 \neq 0$, be the solution of the Legendre equation.

Then $\frac{dy}{dx} = \sum_{r=0}^{\infty}(m-r)a_r x^{m-r-1}$

And $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty}(m-r)(m-r-1)a_r x^{m-r-2}$

Substituting the values of $y, \frac{dy}{dx}\ and \frac{d^2y}{dx^2}$ in equation (1), we get

$$(1-x^2)\left(\sum_{r=0}^{\infty}(m-r)(m-r-1)a_r x^{m-r-2}\right) - 2x\left(\sum_{r=0}^{\infty}(m-r)a_r x^{m-r-1}\right)$$

$$+n(n+1)(\textstyle\sum_{r=0}^{\infty} a_r x^{m-r}) = 0$$

$$\Rightarrow \sum_{r=0}^{\infty}[(m-r)(m-r-1]a_r x^{m-r-2}$$

$$-\sum_{r=0}^{\infty}[(m-r)(m-r-1) + 2(m-r) - n(n+1)]a_r x^{m-r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty}[(m-r)(m-r-1]a_r x^{m-r-2}$$

$$-\sum_{r=0}^{\infty}[(m-r)^2 - n^2 + (m-r) - n]a_r x^{m-r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty}[(m-r)(m-r-1]a_r x^{m-r-2}$$

$$-\sum_{r=0}^{\infty}[(m-r-n)(m-r+n+1]a_r x^{m-r} = 0$$

Equating to zero the co-efficient of the highest power of x.(i.e. corresponding to r = 0 which is of x^m, *we get*

$a_0(m-n)(m+n+1)=0 \implies m=n,\ or\ m=-(n+1)$, Since $a_0 \neq 0$

Equating to zero the co-efficient of x^{m-1}, *we get*

$(m+n)(m-n+1)a_1=0 \Rightarrow a_1=0$.)Since $(m+n)$and $(m-n+1)\neq 0$ for $m=n,\ or\ m=-(n+1)$.

Equating zero the co-efficient of x^{m-r} , we get

$$[m-(r-2)][m-(r-2)-1]a_{r-2}-(m-r-n)(m-r+n+1)a_r=0$$

$$a_r=-\frac{(m-r+2)(m-r+1)}{(n-m+r)(n+m-r+1)}a_{r-2}\ldots\ldots(2)$$

Since$a_1=0$ we get $a_3=a_5=a_7=\cdots\ldots=0$

When m = n, Eqn.(2) reduces to

$$a_r=-\frac{(n-r+2)(n-r+1)}{(r)(2n-r+1)}a_{r-2}$$

Putting r = 2, 4, 6,....... we get

$$a_2=-\frac{n(n-1)}{2(2n-1)}a_0$$
$$a_4=-\frac{(n-2)(n-3)}{4(2n-3)}a_2 = \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}a_0 \text{ , etc.}$$

Therefore a solution of Legendre's equation is given by

$$y_1=a_0\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}+\cdots\ldots\ldots\ldots\right]$$

When $m=-(n+1)$, eqn.(2) reduces to

$$a_r=-\frac{(n+r-1)(n+r)}{(r)(2n+r+1)}a_{r-2}$$

Putting r =2,4, 6........, we get

$$a_2=\frac{(n+1)(n+2)}{2(2n+3)}a_0$$

$$a_4=-\frac{(n+3)(n+4)}{4(2n+5)}a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}a_0 \text{ , etc.}$$

Therefore the second solution of Legendre's equation is given by

$$y_2=a_0\left[x^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5}+\cdots\ldots\ldots\ldots\right]$$

Hence $y=y_1+y_2$ is the general solution the Legendre equation.

If n is a positive even integer, the series solution y_1 terminates at term in x^n and it becomes a polynomial degree **n**. Thus, when ever n is a positive integer, the general solution of the Legendre equation consists of a polynomial solution and an infinite series solution. These polynomial solution, with a_0 or a_1 so chosen that, the value of the polynomial is 1 for x = 1 ((That is, $P_n(1) = 1$) are called Legendre polynomials of order n and are denoted by $P_n(x)$. The infinite series solution with a_0 or a_1 properly chosen is called Legendre function of the second kind and is denoted by $Q_n(x)$. If we choose $a_0 = \frac{1.3.5\ldots\ldots(2n-1)}{n!}$

$$P_n(x) = \frac{1.3.5\ldots\ldots(2n-1)}{n!}\Big[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} + \cdots\ldots\ldots\Big]$$

RODRIGUE'S FORMULA:

The Legendre polynomial of order n is also is given by

$$\boldsymbol{P_n(x) = \frac{1}{2^n.n!}\frac{d}{dx^n}^{n}((x^2-1)^n)}\ .$$

This result is known as Rodrigue's formula

ILLUSTRATIVE EXAMPLES

Example 1: Using Rodrigue's formula find $P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)$ and $P_5(x)$

We have

$$P_n(x) = \frac{1}{2^n.n!}\frac{d}{dx^n}^{n}((x^2-1)^n)$$

Putting n = 0,1,2,3,......., we get

$$P_0(x) = \frac{1}{2^0.0!}((x^2-1)^0) = 1$$

$$P_1(x) = \frac{1}{2^1.1!}\frac{d}{dx}((x^2-1)^1) = x$$

$$\boldsymbol{P_2(x) = \frac{1}{2^2.2!}\frac{d}{dx^2}^{2}((x^2-1)^2) = \frac{1}{2}(3x^2-1)}$$

$$P_3(x) = \frac{1}{2^3.3!}\frac{d}{dx^3}^{3}((x^2-1)^3 = \frac{1}{8.6}\frac{d^2}{dx^2}(3(x^2-1)^2)2x$$

$$= \frac{1}{8.6}\frac{d^2}{dx^2}(3(x^2-1)^2)2x = \frac{1}{8}\frac{d^2}{dx^2}(x^5-2x^3+x)$$

$$= \frac{1}{8}\frac{d}{dx}(5x^4-6x^2+1) = \frac{1}{8}(20x^3-12x) = \frac{1}{2}(5x^3-3x)$$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

$P_4(x) = \frac{1}{2^4.4!}\frac{d}{dx^4}^4((x^2-1)^4 = \frac{1}{16.\,24}\frac{d^4}{dx^4}(x^8 - 4x^6 + 6x^4 - 4x^2 + 1)$

$= \frac{1}{16.\,24}\frac{d^3}{dx^3}(8x^7 - 24x^5 + 24x^3 - 8x) = \frac{1}{2.24}\frac{d^3}{dx^3}(x^7 - 3x^5 + 3x^3 - x)$

$= \frac{1}{2.24}\frac{d^2}{dx^2}(7x^6 - 15x^4 + 9x^2 - 1) = \frac{1}{2.24}\frac{d}{dx}(42x^5 - 60x^3 + 18x)$

$= \frac{1}{2.4}\frac{d}{dx}(7x^5 - 10x^3 + 3x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{2^5.5!}\frac{d}{dx^5}^5((x^2-1)^5 = \frac{1}{32.120.}\frac{d^5}{dx^5}(x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1)$

$= \frac{1}{32.12.}\frac{d^4}{dx^4}(x^9 - 4x^7 + 6x^5 - 4x^3 + x)$

$= \frac{1}{32.12}\frac{d^3}{dx^3}(9x^8 - 28x^6 + 30x^4 + 12x^2 + 1)$

$= \frac{1}{32.12}\frac{d^2}{dx^2}(72x^7 - 168x^5 + 120x^3 + 24x)$

$= \frac{1}{16}\frac{d^2}{dx^2}(3x^7 - 7x^5 + 5x^3 + x) = \frac{1}{16}\frac{d}{dx}(21x^6 - 35x^4 + 15x^2 + 1)$

$\frac{1}{16}\frac{d}{dx}(126x^5 - 140x^3 + 30x)$

$P_5(x) = \frac{1}{8}[63x^5 - 70x^3 + 15x]$

Note: In general we have $P_n(x) = \sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n\, r!(n-r)!(n-2r)!}x^{n-2r}$,

Where $N = \begin{cases} \frac{n}{2}, & if\ n\ is\ even \\ \frac{(n-1)}{2}, & if\ n\ is\ odd \end{cases}$

Example 2: Show that $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$

Solution: We have $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\Rightarrow \quad 3x^2 - 1 = 2P_2(x)$

$\Rightarrow \quad 3x^2 = 2P_2(x) + 1$

$\Rightarrow \quad x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$

Example3: Show that $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$

Solution: We have, $P_1(x) = x$, and

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\Rightarrow \quad (5x^3 - 3x) = 2P_3(x)$$

$$\Rightarrow \quad 5x^3 = 2P_3(x) + 3x$$

$$\Rightarrow \quad x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

Example 4: Express $x^3 - 5x^2 + x + 2$ in terms of Legendre's polynomials.

Solution: We have, $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$,

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x) \text{ and } x = P_1(x)$$

$$\therefore\ x^3 - 5x^2 + x + 2 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) - 5\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] + P_1(x) + 2P_0(x)$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) + \frac{1}{3}P_0(x)$$

Example 5: Express $4x^3 + 6x^2 + 7x + 2$ in terms of Legendre's polynomials.

Solution: We have $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$ and

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x) \text{ and } x = P_1(x)$$

$$\therefore\ 4x^3 + 6x^2 + 7x + 2$$

$$= 4\left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] + 6\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] + 7[P_1(x)] + 2P_0(x)$$

$$= \frac{8}{5}P_3(x) + 4P_2(x) + \left(\frac{12}{5} + 7\right)P_1(x) + (2 + 2)P_0(x)$$

$$= \frac{8}{5}P_3(x) + 4P_2(x) + \frac{47}{5}P_1(x) + 4P_0(x)$$

$$\Rightarrow 4x^3 + 6x^2 + 7x + 2 = \frac{8}{5}P_3(x) + 4P_2(x) + \frac{47}{5}P_1(x) + 4P_0(x)$$

Example 6: Express $x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre's polynomials.

Solution: We have, $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$,

$$x^2 = \tfrac{2}{3}P_2(x) + \tfrac{1}{3}P_0(x) \;, \quad x = P_1(x) \text{ and}$$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$\Rightarrow 35x^4 - 30x^2 + 3 = 8P_4(x)$

$\Rightarrow 35x^4 = 8P_4(x) + 30x^2 - 3$

$$x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$$

$$= \tfrac{8}{35}P_4(x) + \tfrac{6}{7}\left(\tfrac{2}{3}P_2(x) + \tfrac{1}{3}P_0(x)\right) - \tfrac{3}{35}$$

$$x^4 = \tfrac{8}{35}P_4(x) + \tfrac{4}{7}P_2(x) + \tfrac{1}{5}P_0(x)$$

$$\therefore\; x^4 + 3x^3 - x^2 + 5x - 2 = \left(\frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)\right)$$

$$+\; 3\left(\tfrac{2}{5}P_3(x) + \tfrac{3}{5}P_1(x)\right) - \left(\tfrac{2}{3}P_2(x) + \tfrac{1}{3}P_0(x)\right)$$

$$+5(P_1(x)) - 2P_0(x)$$

$$= \tfrac{8}{35}P_4(x) + \tfrac{6}{5}\,P_3(x) + \left(\tfrac{4}{7} - \tfrac{2}{3}\right)P_2(x) + \left(\tfrac{9}{5} + 5\right)P_1(x) + \left(\tfrac{1}{5} - \tfrac{1}{3} - 2\right)P_0(x)$$

$$\therefore\; x^4 + 3x^3 - x^2 + 5x - 2 = \frac{8}{35}P_4(x) + \frac{6}{5}\,P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1(x) - \frac{32}{15}P_0(x)$$

Example 7: Express $2x^2 - 4x + 2$ in terms of Legendre's polynomials.

Solution: We have, $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$ and $x = P_1(x)$

$$\therefore\; 2x^2 - 4x + 2 = 2\left(\tfrac{2}{3}P_2(x) + \tfrac{1}{3}P_0(x)\right) - 4P_1(x) + 2P_0(x)$$

$$= \tfrac{4}{3}P_2(x) - 4\,P_1(x) + \tfrac{8}{3}P_0(x)$$

Example 8: If $x^3 + 2x^2 - 4x + 5 = aP_3(x) + bP_2(x) + cP_1(x) + dP_0(x)$, find $a, b, c \text{ and } d$

Solution: We have, $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$,

$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$ and

$x = P_1(x)$

$\therefore\ x^3 + 2x^2 - 4x + 5$

$$= \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) + 2\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] - 4P_1(x) + 5P_0(x)$$

$$= \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{17}{5}P_1(x) + \frac{17}{3}P_0(x)$$

If $x^3 + 2x^2 - 4x + 5 = aP_3(x) + bP_2(x) + cP_1(x) + dP_0(x)$ then

$$aP_3(x) + bP_2(x) + cP_1(x) + dP_0(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{17}{5}P_1(x) + \frac{17}{3}P_0(x)$$

$\Rightarrow a = \frac{2}{5},\ b = \frac{4}{3}, c = \frac{-17}{5}$ *and* $d = \frac{17}{3}$

Example 9: If $x^4 - 3x^2 + x = aP_4(x) + bP_3(x) + cP_2(x) + dP_1(x) + eP_0(x)$, find a, b, c, d *and* e

Solution: We have, $x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x),$$

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x),$$

$$x = P_1(x)$$

$$\therefore\ x^4 - 3x^2 + x = \left(\frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)\right)$$

$$-3\left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right)$$

$$+ P_1(x)$$

$$= \frac{8}{35}P_4(x) + \left(\frac{4}{7} - 2\right)P_2(x) + P_1(x) + \left(\frac{1}{5} - 1\right)P_0(x)$$

$$\therefore\ x^4 - 3x^2 + x = \frac{8}{35}P_4(x) - \frac{10}{7}P_2(x) + P_1(x) - \frac{4}{5}P_0(x)$$

If $x^4 - 3x^2 + x = aP_4(x) + bP_3(x) + cP_2(x) + dP_1(x) + eP_0(x)$ then

$$aP_4(x) + bP_3(x) + cP_2(x) + dP_1(x) + eP_0(x)$$
$$= \frac{8}{35}P_4(x) - \frac{10}{7}P_2(x) + P_1(x) - \frac{4}{5}P_0(x)$$

$\Rightarrow a = \frac{8}{35}, b = 0,\ c = \frac{-10}{7}, d = 1$ *and* $e = \frac{-4}{5}$

Example 10: Show that $P_n(-x) = (-1)^n P_n(x)$

Solution: We have $P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n\, r!(n-r)!(n-2r)!} x^{n-2r}$,

Where $N = \begin{cases} \frac{n}{2}, & if\ n\ is\ even \\ \frac{(n-1)}{2}, & if\ n\ is\ odd \end{cases}$

Replacing x by (-x), we get

$$P_n(-x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n\, r!(n-r)!(n-2r)!} (-x)^{n-2r}$$

$$= \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n\, r!(n-r)!(n-2r)!} (x)^{n-2r} (-1)^{n-2r}$$

$$= (-1)^{n-2r} \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n\, r!(n-r)!(n-2r)!} (x)^{n-2r}$$

$$= (-1)^n P_n(x) \quad [\text{ Since, } (-1)^{-2r} = 1]$$

$\therefore P_n(-x) = (-1)^n P_n(x)$

Example 11: Show that $P_2(cos\theta) = \frac{1+3\ cos\theta}{4}$

Solution: we have $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\Longrightarrow\ P_2(cos\theta) = \frac{1}{2}(3cos^2\theta - 1)$

$$= \frac{1}{2}\left[3\left(\frac{1+cos2\theta}{2}\right) - 1\right]$$

$$= \frac{1}{2}\left[\frac{1+3\ cos2\theta}{2}\right]$$

$\therefore P_2(cos\theta) = \frac{1+3\ cos\theta}{4}$