MODULE – 3

PARTIAL DIFFERENTIAL EQUATIONS

Introduction:

Many problems in vibration of strings, heat conduction, electrostatics involve two or more variables. Analysis of these problems leads to partial derivatives and equations involving them. In this unit we first discuss the formation of PDE analogous to that of formation of ODE. Later we discuss some methods of solving PDE.

Definitions:

An equation involving one or more derivatives of a function of two or more variables is called a **partial differential equation.**

The **order** of a PDE is the order of the highest derivative and the **degree** of the PDE is the degree of highest order derivative after clearing the equation of fractional powers.

A PDE is said to be linear if it is of first degree in the dependent variable and its partial derivative.

In each term of the PDE contains either the dependent variable or one of its partial derivatives, the PDE is said to be **homogeneous.** Otherwise it is said to be a **nonhomogeneous** PDE.

- Formation of pde by eliminating the arbitrary constants
- Formation of pde by eliminating the arbitrary functions

Solutions to first order first degree pde of the type

$Pp + Qq = R$

Formationof pde by eliminating the *arbitrary constants:*

(1) $$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$ Solve:

Sol: Differentiating (i) partially with respect to x and y,

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \; or \; \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

$$\frac{2\partial z}{\partial y} = \frac{2y}{b^2} \; or \; \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial x} = \frac{q}{y}$$

Substituting these values of $1/a^2$ and $1/b^2$ in (**i**), we get

(2) $z = (x^2 + a)(y^2 + b)$

Sol: Differentiating the given relation partially

$(x-a)^2 + (y-b)^2 + z^2 = k^2$ ...(i)

Differentiating (i) partially w. r. t. x and y,

$$(x-a)+z\frac{\partial z}{\partial x}=0,(y-b)+z\frac{\partial z}{\partial y}=0$$

Substituting for (x- a) and (y- b) from these in (i), we get

$$z^2\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=k^2$$ This is the required partial differential equation.

(3) $z = ax + by + cxy$ ...(i)

Sol: Differentiating (i) partially w.r.t. xy, we get

$$\frac{\partial z}{\partial x}=a+cy..(ii)$$

$$\frac{\partial z}{\partial y}=b+cx..(iii)$$

It is not possible to eliminate a,b,c from relations (i)-(iii).

Partially differentiating (ii),

$$\frac{\partial^2 z}{\partial x\partial y}=c$$ Using this in (ii) and (iii)

$$a=\frac{\partial z}{\partial x}-y\frac{\partial^2 z}{\partial x\partial y}$$

$$b=\frac{\partial z}{\partial y}-x\frac{\partial^2 z}{\partial x\partial y}$$

Substituting for a, b, c in (i), we get

$$z = x\left[\frac{\partial z}{\partial x} - y\frac{\partial^2 z}{\partial x \partial y}\right] + y\left[\frac{\partial z}{\partial y} - x\frac{\partial^2 z}{\partial x \partial y}\right] + xy\frac{\partial^2 z}{\partial x \partial y}$$

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - xy\frac{\partial^2 z}{\partial x \partial y}$$

(5) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol: Differentiating partially w.r.t. x,

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0, \text{or } \frac{x}{a^2} = -\frac{z}{c^2}\frac{\partial z}{\partial x}$$

Differentiating this partially w.r.t. x, we get

$$\frac{1}{a^2} = -\frac{1}{c^2}\left\{\left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2}\right\} \quad \text{or} \quad \frac{c^2}{a^2} = -\left\{\left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2}\right\}$$

: Differentiating the given equation partially w.r.t. y twice we get

$$\frac{z}{y}\frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial y}\right)^2 + z\frac{\partial^2 z}{\partial y^2} \quad \frac{z}{x}\frac{\partial z}{\partial x} = \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2}$$

Is the required p. d. e..

Note:

As another required partial differential equation.

P.D.E. obtained by elimination of arbitrary constants need not be not unique

Formation of p d e by eliminating the *arbitrary functions:*

1) $z = f(x^2 + y^2)$

Sol: Differentiating z partially w.r.t. x and y,

$$p=\frac{\partial z}{\partial x}=f'(x^2+y^2).2x, q=\frac{\partial z}{\partial y}=f'(x^2+y^2).2y$$

p /q = x / y or y p –x q=0 is the required pde

(2) z = f (x +ct) + g (x -ct)

Sol: Differentiating *z* partially with respect to *x* and *t*,

$$\frac{\partial z}{\partial x}=f'(x+ct)+g'(x-ct), \frac{\partial^2 z}{\partial x^2}=f''(x+ct)+g''(x-ct)$$

Thus the pde is

$$\frac{\partial^2 z}{\partial t^2}+\frac{\partial^2 z}{\partial x^2}=0$$

(3) $x + y + z = f(x^2 + y^2 + z^2)$

Sol:Differentiating partially w.r.t. x and y

$$1+\frac{\partial z}{\partial x}=f'(x^2+y^2+z^2)\left[2x+2z\frac{\partial z}{\partial x}\right]$$

$$1+\frac{\partial z}{\partial y}=f'(x^2+y^2+z^2)\left[2y+2z\frac{\partial z}{\partial y}\right]$$

$$2f'(x^2+y^2+z^2)=\frac{1+(\partial z/\partial x)}{x+z(\partial z/\partial x)}=\frac{1+(\partial z/\partial y)}{y+z(\partial z/\partial y)}$$

$$(y-z)\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=x-y$$ is the required pde

(4) z = f (xy / z).

Sol: Differentiating partially w.r.t. x and y

$$\frac{\partial z}{\partial x}=f'\left(\frac{xy}{z}\right)\left\{\frac{y}{z}-\frac{xy}{z^2}\frac{\partial z}{\partial x}\right\}$$

$$\frac{\partial z}{\partial y}=f'\left(\frac{xy}{z}\right)\left\{\frac{x}{z}-\frac{xy}{z^2}\frac{\partial z}{\partial x}\right\}$$

$$f'\left(\frac{xy}{z}\right)=\frac{\partial z/\partial x}{(y/z)\{1-(x/z)(\partial z/\partial x\}}=\frac{\partial z/\partial y}{(x/z)\{1-(y/z)(\partial z/\partial y\}}$$

$$x\frac{\partial z}{\partial x}=y\frac{\partial z}{\partial y}$$

or xp = yq is the required pde.

(5) $z = y^2 + 2\, f(1/x + \log y)$

$$Sol:\ \frac{\partial z}{\partial y}=2y+2f'(1/x+\log y)\left\{\frac{1}{y}\right\}$$

$$\frac{\partial z}{\partial x}=2f'(1/x+\log y)\left\{-\frac{1}{x^2}\right\}$$

$$2f'(1/x+\log y)=-x^2\frac{\partial z}{\partial x}=y\left(\frac{\partial z}{\partial y}-2y\right)$$

Hence $x^2\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=2y^2$

(6) $Z = x\Phi(y) + y\,\psi(x)$

$$Sol:\ \frac{\partial z}{\partial x}=\varphi(y)+y\psi'(x);\ \frac{\partial z}{\partial y}=x\varphi'(y)+\psi(x)$$

Substituting $\phi'(y)$ *and* $\psi'(x)$

$$xy\frac{\partial^2 z}{\partial x\partial y}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-[x\phi(y)+y\psi(x)]$$

$$xy\frac{\partial^2 z}{\partial x\partial y}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-z$$ is the required pde.

7) Form the partial differential equation by eliminating the arbitrary functions from $z = f(y-2x) + g(2y-x)$ (Dec 2011)

Sol: By data, **$z = f(y-2x) + g(2y-x)$**

$$p = \frac{\partial z}{\partial x} = -2f'(y-2x) - g'(2y-x)$$

$$q = \frac{\partial z}{\partial y} = f'(y-2x) + 2g'(2y-x)$$

$$r = \frac{\partial^2 z}{\partial x^2} = 4f''(y-2x) + g''(2y-x) \ldots\ldots\ldots\ldots\ldots(1)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = -2f''(y-2x) - 2g''(2y-x) \ldots\ldots\ldots(2)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''(y-2x) + 4g''(2y-x) \ldots\ldots\ldots\ldots\ldots(3)$$

$$(1)\times 2 + (2) \Rightarrow 2r + s = 6f''(y-2x) \ldots\ldots\ldots\ldots(4)$$

$$(2)\times 2 + (3) \Rightarrow 2s + t = -3f''(y-2x) \ldots\ldots\ldots\ldots(5)$$

Now dividing (4) by (5) we get

$$\frac{2r+s}{2s+t} = -2 \quad or\, 2r + 5s + 2t = 0$$

$$Thus \quad 2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0 \text{ is the required PDE}$$

LAGRANGE'S FIRST ORDER FIRST DEGREE PDE: Pp+Qq=R

(1) Solve: $yzp + zxq = xy$.

$$Sol: \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Subsidiary equations are

From the first two and the last two terms, we get, respectively

$$\frac{dx}{y}=\frac{dy}{x} \; or \; xdx-ydy=0 \quad \text{and} \quad \frac{dy}{z}=\frac{dz}{y} \; or \; ydy-zdz=0.$$

Integrating we get $x^2 - y^2 = a$, $y^2 - z^2 = b$.

Hence, a general solution is

$\Phi(x^2-y^2, y^2-z^2) = 0$

(2) Solve: $y^2p - xyq = x(z-2y)$

$$Sol: \frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)}$$

From the first two ratios we get

$x^2 + y^2 = a$ from the last ratiostwo we get

$$\frac{dz}{dy}+\frac{z}{y}=2$$

from the last ratiostwo we get

$\frac{dz}{dy}+\frac{z}{y}=2$ ordinary linear differential equation hence

$yz - y^2 = b$

solution is $\Phi(x^2 + y^2, yz - y^2) = 0$

(3) Solve : $z(xp - yq) = y^2 - x^2$

$$Sol: \frac{dx}{zx}=\frac{dy}{-zy}=\frac{dz}{y^2-x^2}$$

$$\frac{dx}{x}=\frac{dy}{-y}, \quad \text{or} \quad xdy+ydx=0 \quad \text{or} \quad d(xy)=0,$$

on integration, yields $xy = a$

$xdx + ydy + zdz = 0 \quad x^2 + y^2 + z^2 = b$

Hence, a general solution of the given equation

Φ(xy,x2+y2+z2)=0

(4) Solve: $\frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$

$Sol: \frac{yz}{y-z}dx = \frac{zx}{z-x}dy = \frac{xy}{x-y}dz$

$x\,dx + y\,dy + z\,dz = 0$ …(i)

Integrating (i) we get

$x^2 + y^2 + z^2 = a$

$yz\,dx + zx\,dy + xy\,dz = 0$ …(ii)

Dividing (ii) throughout by xyz and then integrating,

we get $xyz = b$

$\Phi(x^2 + y^2 + z^2, xyz) = 0$

(5)$(x+2z)p + (4zx - y)q = 2x^2 + y$

$Sol: \frac{dx}{x+2z} = \frac{dy}{4zx-y} = \frac{dz}{2x^2+y}$..(i)

Using multipliers 2x, -1, -1 we obtain $2x\,dx - dy - dz = 0$

Using multipliers y, x, -2z in (i), we obtain

$y\,dx + x\,dy - 2z\,dz = 0$ which on integration yields

$xy - z^2 = b$ ….(iii)

5) Solve

$z_{xy} = \sin x \sin y$ *for which* $z_y = -2\sin y$ *when* $x = 0$ *and* $z = 0$ when y is an odd multiple of $\frac{\pi}{2}$.

Sol: Here we first find z by integration and apply the given conditions to determine the arbitrary functions occurring as constants of integration.

The given PDF can be written as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \sin y$

Integrating w.r.t x treating y as constant,

$$\frac{\partial z}{\partial y} = \sin y \int \sin x\, dx + f(y) = -\sin y \cos x + f(y)$$

Integrating w.r.t y treating x as constant

$$z = -\cos x \int \sin y\, dy + \int f(y)\, dy + g(x)$$

$$z = -\cos x\ (-\cos y) + F(y) + g(x),$$

$$where\ F(y) = \int f(y)\, dy.$$

$$Thus\ z = \cos x \cos y + F(y) + g(x)$$

$$Also\ by\ data, \frac{\partial z}{\partial y} = -2\sin y\ when\ x = 0.\ Using\ this\ in\ (1)$$

$$-2\sin y = (-\sin y).1 + f(y)\ (\cos 0 = 1)$$

$$Hence\ F(y) = \int f(y)\, dy = \int -\sin y\, dy = \cos y$$

$$With\ this, (2)\ becomes\ z = \cos x \cos y + \cos y + g(x)$$

$$Using\ the\ condition\ that\ z = 0\ if\ y = (2n+1)\frac{\pi}{2}\ in\ (3)\ we\ have$$

$$0 = \cos x \cos(2n+1)\frac{\pi}{2} + \cos x \cos(2n+1)\frac{\pi}{2} + g(x)$$

$$But\ \cos(2n+1)\frac{\pi}{2} = 0.\ and\ hence\ 0 = 0 + 0 + g(x)$$

Thus the solution of the PDE is given by

z=cos x cosy + cosy

Solution of non homogenious PDE by Direct Integration Method

Objective**:**

At the end of this section we will be able to:

- Know different types of solution of a P.D.E.
- Solve non homogeneous P.D.E by direct integration

The general form of a first order partial differential equation is $F(x, y, z, p, q) = 0$.....................(1) where x, y are two independent variables , z is the dependent variable and $p = z_x, q = z_y$.

A solution or integral of the p.d.e is the relation between the dependent and independent variable satisfying the equation.

Complete Solution

Any function $f(, y, z, a, b) = 0$..................................(2) involving two arbitrary constants a, b and satisfying p.d.e (1) is known as the complete solution or complete integral or primitive. Geometrically the complete solution represents a two parameter family of surfaces .

Eg: $(x + a)(y + b) = z$ is the complete solution of the p.d.e $z = pq$

Particular Solution

A solution obtained by giving particular value to the arbitrary constants in the complete solution is called a particular solution of the p.d.e. It represents a particular surface of a family of surfaces given by the complete solution.

Eg: $(x + 3)(y + 4) = z$ is the particular solution of the p.d.e $z = pq$

Genral Solution

In the complete solution if we put $b = \varphi(a)$ then we get a solution containing an arbitryary function φ, which is called a general solution. It represents the envelope of the family of surfaces

$f\big(x, y, z,\ \varphi(a)\big) = 0.$

Singular Solution

Differentiating the p.d.e (1) w.r.t the arbitrary constants a and b we get

$\frac{\partial F}{\partial a} = 0$ and $\frac{\partial F}{\partial b} = 0$

Suppose it is possible to eliminate a and b from the three equations then the relation so obtained is called the singular solution of the p.d.e.

Singular solution represents the envelope of the two parameter family of surfaces.

Eg: The complete solution of the p.d.e $z = pq$ is $(x+a)(y+b) = z$

Differentiating partially w.r.t a and b we get

$x + a = 0$ and $(y+b) = 0. \Rightarrow z = 0$ is the singular solution.

Solution by Direct Integration

Examples:

1) Solve the equation $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

Solution: The given equation can be written as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2 y$

Integrating w.r.t. x we get

$$\frac{\partial z}{\partial y} = \frac{x^3 y}{3} + f(y)$$

Where $f(y)$ is an arbitrary function of y

Integrating the above w.r.t. y we get

$$z = \frac{x^3 y^2}{6} + \int f(y)dy + g(x)$$

$$z = \frac{1}{6}(x^3 y^2) + F(y) + g(x)$$

Where $g(x)$ is an arbitrary function of x

This is the general solution of the given equation.

2) Solve the equation $xy\frac{\partial^2 z}{\partial x \partial y} - x\frac{\partial z}{\partial x} = y^2$

Solution: Diving throughout by x the equation may be written as

$$y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial z}{\partial x} = \frac{y^2}{x}$$

$$y\frac{\partial p}{\partial y} - p = \frac{y^2}{x} \text{ or } \frac{1}{y}\frac{\partial p}{\partial y} - \frac{p}{y^2} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{\partial}{\partial y}\left(\frac{p}{y}\right) = \frac{1}{x}$$

Integrating the above w.r.t. y we get

$$\frac{p}{y} = \frac{1}{x}y + f(x)$$

$$\frac{1}{y}\frac{\partial z}{\partial x} = \frac{y}{x} + f(x)$$

Integrating w.r.t. x we get

$$\frac{z}{y} = ylogx + \int f(x)dx + g(y)$$

$$\Rightarrow \quad z = y^2 logx + yF(x) + G(y)$$

3)Solve : $\frac{\partial^2 z}{\partial y^2} - x\frac{\partial z}{\partial y} = -siny - xcosy$

Solution: We know that q=$\frac{\partial z}{\partial y}$,the given equation may be written as

$$\frac{\partial q}{\partial y} - xq == -siny - xcosy \text{..................................(1)}$$

Since x is treated as constant , this equation is a first order ordinary linear d.e in which q is the dependent variable and y is the independent variable. For this equation

I.F=$e^{\int -xdy} = e^{-xy}$

∴ The solution of (1) is

$$qe^{-xy} = \int(-siny - xcosy)e^{-xy}dy + f(x)$$

$$= \int \frac{d}{dy}(cosy)e^{-xy}dy + f(x)$$

$$qe^{-xy} = e^{-xy}cosy + f(x)$$

or

$$\frac{\partial z}{\partial y} = cosy + e^{xy} f(x)$$

Integrating w.r.t y, we get

$$z = siny + \frac{e^{xy}}{x}f(x) + g(y)$$

Solution of Homogenious Equation

Objective

At the end of this Section we will be able to:

- Solve the homogeneous P.D.E
- Obtain the particular solution of a homogeneous P.D.E using a given set of conditions.

Solution of Homogenious Equation

Examples:

1) Solve the equation $\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$ under the condition z=0 and $\frac{\partial z}{\partial x} = a\sin y$ when x=0

Solution: Using the D-Operator, we write the above equation as

$(D^2 - a^2)z = 0$ where, $D = \frac{\partial}{\partial x}$

Here we treat y as constant. Then this equation is an ordinary second order linear homogeneous d.e. in which x is the independent variable and z is the dependent variable.

$\therefore$ The A.E is $m^2 - a^2 = 0 \quad \Rightarrow m = \pm a$

$\therefore G.S \text{ is } \; z = Ae^{ax} + Be^{-ax}$

Where A and B are arbitrary function of y

Now $\frac{\partial z}{\partial x} = Aae^{ax} - Bae^{-ax}$

Using the given condition i.e., z=0 and $\frac{\partial z}{\partial x} = a\sin y$ when x = 0

$a\sin y = Aa - Ba = a(A - B)$

0= A+B

$\Rightarrow A - B = \sin y$

$A + B = 0$

Solving the above simultaneous equations

$2A = \sin y$

$\Rightarrow A = \frac{1}{2}\sin y$

$B = -\frac{1}{2}\sin y$

The solution is

$z = \frac{1}{2}(e^{ax} - e^{-ax})\sin y$

$z = \sinh ax \sin y$

2) Solve the equation $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + 2z = 0$ Given that z=e^y and $\frac{\partial z}{\partial x} = 0$ when x=0

Solution: The given equation can be put in the form

$(D^2 - 2D + 2)z = 0$ where D=$\frac{\partial}{\partial x}$

If y is treated as constant , then the above equation is a O.D.E with the A.E

$(m^2 - 2m + 2)$=0

$\Rightarrow m = 1 \pm i$

∴The G.S is $z = e^x(Acosx + Bsinx)$

Where A and B are arbitrary functions of y

$\frac{\partial z}{\partial x} = e^x(Acosx + Bsinx) + e^x(-Asinx + Bcosx)$

$\frac{\partial z}{\partial x}$=$e^x[(A + B)cosx + (B - A)sinx]$

But $\frac{\partial z}{\partial x}$=0

⇒0=B+A

And z=e^y when x=0

⇒$e^y = A$

⇒B=-e^y

∴ z=$e^x[e^y cosx - e^y sinx]$

⇒ z=$e^{x+y}[cosx - sinx]$

3. Solve the equation $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$ using the substitution the $\frac{\partial u}{\partial x} = v.$

Sol: Since we have $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

given PDE can be rewritten as

$$\frac{\partial v}{\partial y} = v, \quad \text{where } v = \frac{\partial u}{\partial x}$$

$$\text{or} \quad \frac{\partial v}{\partial y} - v = 0$$

By considering it as a ODE we can write it as

$$\frac{dv}{dy} - v = 0 \Rightarrow (D-1)v = 0$$

A. E is m-1=0 $\Rightarrow m = 1$

$\therefore$ the solution is given by $v = f(x)e^y$

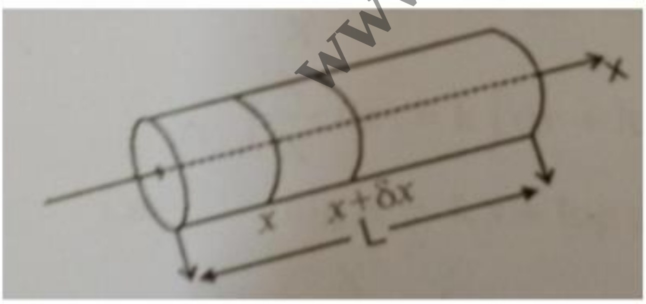
i.e, $\frac{\partial u}{\partial x} = f(x)e^y$

On integrating w. r. to x, we get

$u = F(x)e^y + g(y)$ where $F(x) = \int f(x)dx$

APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS:

ONE DIMENSIONAL HEAT EQUATION:



Consider a heat conducting homogeneous rod of length L placed along x-axis. One end of the rod at x=0(Origin) and the other end of the rod at x=L.

Assume that the rod as constant density ρ and uniform cross section A. Also assume that the rod is insulated laterally and therefore heat flows only in the x direction. The rod is sufficiently thin so that the temperature is same at all points of any cross sectional area of the rod.

Let u(x, t) be the temperature of the cross section at the point x at any time t.

The amount of heat crossing any section of the rod per second depends on the area A of the cross section, the thermal conductivity k of the material of rod and the temperature gradient $\frac{\partial u}{\partial x}$

i.e., the rate of change of temperature with respect to distance normal to the area.

Therfore q_1 the quatity of heat flowing into the cross section at a distance x in uint time is $q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x$ per second

Negative sign appears because heat flows in the direction of decreasing temperature (as x increases u decreases)

q_2 the quantity of heat flowing out of the cross section at a distance $x+\delta x$ (i.e, the rate of heat flow at cross section $x+\delta x$)

$$q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{ per second}$$

The rate of change of heat content in segment of the rod between x and x+δx must be equal to net heat flow into this segment of the rod is

$$q_1 - q_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{ per second.............(1)}$$

But the rate of increase of heat in the rod

$$s\rho A\delta x\frac{\partial u}{\partial t}\text{...............(2)}$$

Where S is the specific heat, ρ the density of material.

From (1) & (2)

$$s\rho A\delta x\frac{\partial u}{\partial t}=kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}\right]$$

$$or\ s\rho\frac{\partial u}{\partial t}=k\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}\right]$$

Taking limit as $\delta x\to 0$, we have

$$s\rho\frac{\partial}{\partial t}=k\frac{\partial^2 u}{\partial x^2}\ or\ \frac{\partial u}{\partial t}=\frac{k}{s\rho}\frac{\partial^2 u}{\partial x^2}$$

$$or\ \frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}\text{..........................(3)}\quad \text{where } c^2=\frac{k}{s\rho}$$

Is known as diffusivity constant.

Equation (3) is the one dimensional heat equation which is second order homogenous and parabolic type.

Various possible solutions of standard p.d.es by the method of separation of variables.

We need to obtain the solution of the ODEs by taking the constant k equal to

i) Zero ii) positive: $k=p^2$ iii) negative: $k=-p^2$

Thus we obtain three possible solutions for the associated p.d.e

<u>Various possible solutions of the one dimensional heat equation $u_t=c^2u_{xx}$ by the method of separation of variables.</u>

Consider $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$

Let u= XT where X=X(x),T=T(t) be the solution of the PDE

Hence the PDE becomes

$$\frac{\partial\ XT}{\partial t}=c^2\frac{\partial^2\ XT}{\partial x^2}\quad or\ X\frac{dT}{dt}=c^2\frac{d^2X}{dx^2}$$

Dividing by c^2XT we have $\frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$

Equating both sides to a common constant k we have

$\frac{1}{X}\frac{d^2X}{dx^2}$=k and $\frac{1}{c^2T}\frac{dT}{dt}$ =k

$\frac{d^2X}{dx^2} - kX = 0$ and $\frac{dT}{dt} - c^2kT = 0$

$D^2 - k \;\; X = 0$ and $D - c^2k \;\; T = 0$

Where $D^2 = \frac{d^2}{dx^2}$ in the first equation and $D = \frac{d}{dt}$ in the second equation

Case (i) : let k=0

AEs are m=0 amd m^2=0 amd m=0,0 are the roots

Solutions are given by

$T = c_1e^{0t} = c_1 \; and \; X = \; c_2x + c_3 \;\; e^{0x} \; = \; c_2x + c_3$

Hence the solution of the PDE is given by

U= XT= $c_1 \;\; c_2x + c_3$

Or u(x,t) =Ax+B where c_1c_2=A and c_1c_3=B

Case (ii) let k be positive say $k=+p^2$

AEs are $m - c^2p^2 = 0$ and $m^2 - p^2 = 0$

$m = c^2p^2$ and m=+p

Solutions are given by

$T = c'_1e^{c^2p^2t} \; and \; X = c'_2e^{px} \;\; + c'_3e^{-px}$

Hence the solution of the PDE is given by

$u = XT = c'_1e^{c^2p^2t}.(c'_2e^{px} \;\; + c'_3e^{-px})$

Or $u(x,t) = c'_1 e^{c^2p^2t}(A' e^{px} + B e^{-px})$ where $c_1'c_2'=A'$ and $c_1'c_3'=B'$

Case (iii): let k be negative say $k=-p^2$

AEs are $m+c^2p^2=0$ and $m^2+p^2=0$

$m=-c^2p^2$ and $m=\pm ip$

solutions are given by

$$T = c''_1 e^{-c^2p^2t} \quad and \quad X = c''_2 \cos px + c''_3 \sin px$$

Hence the solution of the PDE is given by

$$u = XT = c''_1 e^{-c^2p^2t} . (c''_2 \cos px + c''_3 \sin px)$$

$$u(x,t) = e^{-c^2p^2t}(A'' \cos px + B'' \sin px)$$

ONE DIMENSIONAL WAVE EQUATION:

Consider a tightly stretched elastic string of length l stretched between two points O and A and displaced slightly from its equilibrium position OA. Taking O as origin and OA as x axis and a perpendicular line through O as Y- axis. We shall find the displacement y a function of the distance x and the time t.

We shall obtain the equation of motion of string under the following assumptions.

i) The string is perfectly flexible and offers no resistance to bending
ii) Points on the string move only in the vertical direction, there is no motion in the horizontal direction. The motion takes place entirely in the X Y plane .
iii) Gravitational forces on the string is neglected.

Let m be the mass per unit length of the string. Consider the motion of an element PQ of length δs. Since the string does not offer resistance to bending, the tensions T_1

At P and T_2 at Q are tangential to the curve.

Since the is no motion in the horizontal direction, some of the forces in the horizontal direction must be zero.

i.e., $-T_1\cos\alpha + T_2\cos\beta=0$ or $T_1\cos\alpha = T_2\cos\beta = T = \text{constant}$.....(1)

Since gravitational force on the string is neglected , the only two forces acting on the string are the vertical components of tension - $T_1\sin\alpha$ at P and $T_2\sin\beta$ at Q with up[ward direction takes as positive.

Mass of an element PQ is $m\delta s$. By Newton's second law of motion , the equation of motion in the vertical direction is

Resultant of forces = mass *acceleration

$$T_2\sin\beta - T_1\sin\alpha = m\delta s\frac{\partial^2 y}{\partial t^2}\text{.............(2)}.$$

$$\frac{2}{1}\,gives\,\frac{T_2\sin\beta}{T_2\cos\beta}-\frac{T_1\sin\alpha}{T_1\cos\alpha}=\frac{m\delta s}{T}\frac{\partial^2 y}{\partial t^2}$$

$$or\tan\beta-\tan\alpha=\frac{m\delta s}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{m\delta s}\ \tan\beta-\tan\alpha$$

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{m\delta s}\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial y}{\partial x}\right)_{x}\right]$$

$$(\because \delta s=\delta x\, to\, a\, first\, approximation\, and\, \tan\alpha,\tan\beta\, are\, the\, slopes\, of\, the\, curve\, of\, the\, string\, at\, x\, and\, x+\delta x)$$

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial y}{\partial x}\right)_{x}}{\delta x}\right]$$

Taking Limit as $\delta x\to 0$

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{m}\frac{\partial^2 y}{\partial x^2}\qquad or\qquad \frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}\text{....................(3)}\quad \text{where } c^2=\frac{T}{m}$$

Which is the partial differential equation giving the transverse vibrations of the string .

Equation (3) is the one dimensional wave equation which is second order homogenous and parabolic type.

Various possible solutions of the one dimensional wave equation $u_{tt}=c^2u_{xx}$ by the method of separation of variables.

Consider $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$

Let u= XT where X=X(x),T=T(t) be the solution of the PDE

Hence the PDE becomes

$$\frac{\partial^2\ XT}{\partial t^2} = c^2 \frac{\partial^2\ XT}{\partial x^2} \quad or\ X\frac{d^2T}{dt^2} = c^2\frac{d^2X}{dx^2}$$

Dividing by c^2XT we have $\frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$

Equating both sides to a common constant k we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = k \quad \text{and} \quad \frac{1}{c^2T}\frac{d^2T}{dt^2} = k$$

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} - c^2kT = 0$$

$$D^2 - k\ \ X = 0 \quad \text{and} \quad D^2 - c^2k\ \ T = 0$$

Where $D^2 = \frac{d^2}{dx^2}$ in the first equation and $D^2 = \frac{d^2}{dt^2}$ in the second equation

__Case(i)__ : let k=0

AEs are m=0 amd m^2=0 amd m=0,0 are the roots

Solutions are given by

$T = c_1e^{0t} = c_1\ and\ X = \ c_2x + c_3\ \ e^{0x} = c_2x + c_3$

Hence the solution of the PDE is given by

U= XT= $c_1\ \ c_2x + c_3$

Or u(x,t) =Ax+B where c_1c_2=A and c_1c_3=B

Case (ii) let k be positive say $k=+p^2$

AEs are $m - c^2p^2 = 0$ and $m^2 - p^2 = 0$

$m = c^2p^2$ and $m=+p$

Solutions are given by

$T = c_1^{'} e^{c^2 p^2 t}$ and $X = c_2^{'} e^{px} + c_3^{'} e^{-px}$

Hence the solution of the PDE is given by

$u = XT = c_1^{'} e^{c^2 p^2 t} .(c_2^{'} e^{px} + c_3^{'} e^{-px})$

Or u(x,t) $= c_1^{'} e^{c^2 p^2 t}$(A' e^{px} +B e^{-px}) where $c_1'c_2'$=A' and $c_1'c_3'$=B'

Case (iii): let k be negative say k=-p^2

AEs are m+ c^2p^2=0 and m^2+p^2=0

m=- c^2p^2 and m=+ip

Solutions are given by

$T = c_1^{''} e^{-c^2 p^2 t}$ and $X = c_2^{''} \cos px + c_3^{''} \sin px$

Hence the solution of the PDE is given by

$u = XT = c_1^{''} e^{-c^2 p^2 t} .(c_2^{''} \cos px + c_3^{''} \sin px)$

$u(x,t) = e^{-c^2 p^2 t} (A^{''} \cos px + B^{''} \sin px)$