Vector

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Contents

Chapter 1

Vector

1.1 Basic Formula

$$\overrightarrow{AB} \neq \overrightarrow{BA} \tag{1.1}$$

$$\overrightarrow{AB} = \overrightarrow{BA} \tag{1.2}$$

$$|\overrightarrow{AB} = |\overrightarrow{BA}| \tag{1.3}$$

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2} \tag{1.4}$$

1.2 Types of Vector

1.2.1 Null Vector (Important)

Vector having magnitude 0

$$|\overrightarrow{AA}| = |\overrightarrow{BB}| = 0 \tag{1.5}$$

$$|\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}| = 0 \tag{1.6}$$

1.2.2 Unit Vector (Important)

Vector whose magniute is 1. It is denoted by a cap as $\cap$

$$|\overrightarrow{AB}| = 1 \implies \overrightarrow{AB} \text{ is unit vecotr} \tag{1.7}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \tag{1.8}$$

1.2.3 Unit Vector in Orthogonal System

$$|\hat{i}| + |\hat{j}| + |\hat{k}| = 1 \tag{1.9}$$

1.2.4 Like and Unlike Vectors

Same direction are like otherwise unlike.

1.2.5 Equal Vectors

Two vectors having equal magniute and same direction

1.2.6 Collinear or Parallel Vectors

Vectos having same direction. No need to have same magniute.

$$\vec{a} = \lambda\vec{b}, \text{ where } \lambda = \text{Postive real number} \tag{1.10}$$

Unlike vectors are not Collinear.

1.2.7 Coplanar Vector

Vector lie in the same plane or parallel to the same plane.

1.2.8 Coinitial Vectors

Vectors having the same initial point.

1.2.9 Proper Vector

Any non zero vector.

1.2.10 Position Vector

O be the poin of origin.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \tag{1.11}$$

1.3 Section Formula

Let A and B be two points with position vector $\vec{a}$ and $\vec{b}$ and P divides in the radio $m : n$ then position vecotr of

$$P = \overrightarrow{OP} = \vec{r} = \frac{m_1\vec{b} + m_2\vec{a}}{m_1 + m_2} \tag{1.12}$$

This works same as in coordinate (door wala).
If P is the midpoint then

$$\overrightarrow{OP} = \vec{r} = \frac{\vec{a} + \vec{b}}{2} \tag{1.13}$$

External Division

$$P = \overrightarrow{OP} = \vec{r} = \frac{m_1\vec{b} - m_2\vec{a}}{m_1 - m_2} \tag{1.14}$$

1.4 Operations on Vectors

1.4.1 Addition of two vectors

Two types of Addition

1. Triangle Law

2. Parallelogram Law

Add correspoinding coefficients of i j and k.

Cancellation Trick

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PR} + \overrightarrow{RQ} \\ &= \overrightarrow{P\cancel{R}} + \overrightarrow{\cancel{R}Q}\end{aligned}$$

1.5 Component of vectors

Magniute of Vector $a\hat{i} + y\hat{j} + z\hat{k}$ is

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2} \tag{1.15}$$

1.5.1 Along Coordinate Axes

For point $A(x_1, y_1)$ and $B(x_2, y_2)$

$$\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} \tag{1.16}$$
$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.17}$$

1.6 Product of Vectors(Important)

1.6.1 Scalar Product

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \tag{1.18}$$
$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \tag{1.19}$$

Use this to get angle between vectors.

Properties

1. Dot product of same vector is its magniute squared
$$(\vec{a})^2 = |\vec{a}|^2 \tag{1.20}$$
2. Product of like vectors is product of magniutes
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \tag{1.21}$$
3. Product of unlike vectors is negative product of magniutes
$$\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}| \tag{1.22}$$
4. Perpendicular Vector
$$\vec{a} \cdot \vec{b} = 0 \tag{1.23}$$

Orthogonal System

$$\vec{i} \cdot \vec{i} = |\vec{i}||\vec{i}| \cos 0^\circ = 1$$

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \tag{1.24}$$

$$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0 \tag{1.25}$$

Parallel and Perpendicular

Parallel condition

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \tag{1.26}$$

Perpendicular condition

$$\vec{a} \cdot \vec{b} = 0 \tag{1.27}$$

Memorize Answers

$\vec{a}$, $\vec{b}$,$\vec{c}$ are unit vectors.

1. $\vec{a} + \vec{b} + \vec{c} = 0$, then the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \frac{-3}{2}$
2. $\vec{a} + \vec{b} + \vec{c} = \lambda$, then the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \frac{\lambda^2-3}{2}$

Important Formulae

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2[ab + bc + ca]$$

1.6.2 Vector Product

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n} \tag{1.28}$$

$$\vec{b} \times \vec{a} = |\vec{a}||\vec{b}| \sin\theta(-\hat{n})$$

$$\sin\theta = \frac{\vec{a} \times \vec{b}}{|\vec{a}||\vec{b}|} \tag{1.29}$$

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} = -\vec{b} \times \vec{a}$$

For Vectors

$$\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$$
$$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Orthogonal System

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \tag{1.30}$$

Cyclic Order of i j k:

$$\hat{i} \times \hat{j} = \hat{k} \tag{1.31}$$
$$\hat{j} \times \hat{k} = \hat{i} \tag{1.32}$$
$$\hat{k} \times \hat{i} = \hat{j} \tag{1.33}$$

Reverse Order of i j k:

$$\hat{j} \times \hat{i} = -\hat{k} \tag{1.34}$$
$$\hat{k} \times \hat{j} = -\hat{i} \tag{1.35}$$
$$\hat{i} \times \hat{k} = -\hat{j} \tag{1.36}$$

Parallel

For Parallel/Colinear vectors($\vec{a} = \lambda\vec{b}$), cross product is 0

$$\vec{a} \times \vec{a} = \vec{b} \times \vec{b} = \vec{c} \times \vec{c} = 0$$

Geometrical Interpretation

For given adjecent sides $\vec{a}$ and $\vec{b}$

$$\text{Area of Parallelogram} = |\vec{a} \times \vec{b}| \tag{1.37}$$
$$\text{Area of Triangle} = \frac{1}{2}|\vec{a} \times \vec{b}| \tag{1.38}$$

For given diagonals of Parallelogram

$$\text{Area of Parallelogram} = \frac{1}{2}|\vec{d_1} \times \vec{d_2}| \text{Area of Triangles} = \frac{1}{4}|\vec{d_1} \times \vec{d_2}| \quad (1.39)$$

1.6.3 Relation Between dot and cross product

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 + |\vec{b}|^2 \quad (1.40)$$

1.7 Projection

1.7.1 Scalar Projection

$$\text{Projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \quad (1.41)$$

1.7.2 Vector Projection

$$\text{Vector Projection of } \vec{a} \text{ on } \vec{b} = \frac{(\vec{a} \cdot \vec{b})\vec{b}}{|\vec{b}|^2} \quad (1.42)$$

1.8 Triple Product

1.8.1 Important Observations

$$\begin{aligned}
\vec{a} \cdot \vec{b} &\rightarrow \text{scalar} \\
\vec{a} \times \vec{b} &\rightarrow \text{vector} \\
\vec{a} \cdot \vec{b} \cdot \vec{c} &\rightarrow \text{Not Definted} \\
3\vec{a} &\rightarrow \text{Constant times vector is defined} \\
3 \cdot \vec{a} &\rightarrow \text{Not Defined} \\
3 \times \vec{a} &\rightarrow \text{Not Defined}
\end{aligned}$$

Two Vectors must have either dot or cross product sign in between them. Vector and scalar must not have a sign in between

1.8.2 Scalar Triple Product

$$\vec{a} \cdot (\vec{b} \times \vec{b}) = [\vec{a} \quad \vec{b} \quad \vec{c}] \tag{1.43}$$

For given vector

$$\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$$
$$\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$
$$\vec{c} = a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$$

$$[\vec{a} \quad \vec{b} \quad \vec{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{1.44}$$

Exchange while preserving order is ok, else change sign.

$$\begin{aligned} [\vec{a} \quad \vec{b} \quad \vec{c}] &= [\vec{b} \quad \vec{c} \quad \vec{a}] = [\vec{c} \quad \vec{b} \quad \vec{a}] \\ &= -[\vec{a} \quad \vec{c} \quad \vec{b}] = -[\vec{b} \quad \vec{a} \quad \vec{c}] = -[\vec{c} \quad \vec{b} \quad \vec{a}] \end{aligned}$$

This is because

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} = -\vec{b} \times \vec{a}$$

Geometrical Interpretation

Volume of a cube/cuboid/parallelopiped

$$V = [\vec{a} \quad \vec{b} \quad \vec{c}] \tag{1.45}$$

Coplaner

If vectors $\vec{a}$,$\vec{b}$ and $\vec{c}$ are Coplanar then

$$V = 0 \implies [\vec{a} \quad \vec{b} \quad \vec{c}] = 0 \tag{1.46}$$

Properties

1. **Important!** Cyclic i j k

$$[\hat{i} \quad \hat{j} \quad \hat{k}] = 1 \tag{1.47}$$

2. Non cyclic i j k
$$[\hat{i} \quad \hat{k} \quad \hat{j}] = -1$$

3. **Important!** The scalar tripple product 0 if any two vectors are equal or proportional
$$[\vec{a} \quad \vec{b} \quad \vec{a}] = 0 \quad [\because \vec{a} \times \vec{a} = 0]$$
$$[\vec{a} \quad \vec{b} \quad \lambda\vec{a}] = 0$$
Usage
$$\vec{b} \cdot (\vec{c} \times 2\vec{b}) = 0 \,[\text{Since } \vec{b} \text{ Proportinoal}]$$

4. Take common from one row of determinant
$$[\lambda\vec{a} \quad \vec{b} \quad \vec{c}] = \lambda[\vec{a} \quad \vec{b} \quad \vec{c}] \tag{1.48}$$
Take 3 common from all three rows of determinant
$$[\lambda\vec{a} \quad \lambda\vec{b} \quad \lambda\vec{c}] = \lambda^3[\vec{a} \quad \vec{b} \quad \vec{c}] \tag{1.49}$$
Usage
$$3\vec{a} \cdot (4\vec{b} \times 5\vec{c}) = 60[\vec{a} \quad \vec{b} \quad \vec{c}]$$

5. Determinant property
$$[\vec{a} + \vec{b} \quad \vec{c} \quad \vec{d}] = [\vec{a} \quad \vec{c} \quad \vec{d}] + [\vec{b} \quad \vec{c} \quad \vec{d}] \tag{1.50}$$

6. Sum and products in Vector triple product
$$[\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] = 2[\vec{a} \quad \vec{b} \quad \vec{c}] \tag{1.51}$$
$$[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = [\vec{a} \quad \vec{b} \quad \vec{c}]^2 \tag{1.52}$$

1.8.3 Vector Triple Product

Pehle Door phir paas beech mei minus technique is used
$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \tag{1.53}$$
$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a} \tag{1.54}$$