# Coordinate Geometry

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# Contents

# Chapter 1

# Coordinate System

## 1.1 Distance Formulae

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

## 1.2 Basic Shapes

### 1.2.1 Square

4 sides equal, Diagonal equal
$d_1 = d_2 = a\sqrt{2}$

### 1.2.2 Rhombus

4 sides equal, diagonal not equal
$d_1 \neq d_2$

### 1.2.3 Rectangle

Opposite Sides are equal, diagonal equal
$d_1 = d_2 = a\sqrt{l^2 + b^2}$

### 1.2.4 Parallelogram

Opposite sides equal, diagonal not equal
$d_1 \neq d_2$

### 1.2.5 Triangle

- If the vertices of a triangle have integral coordinates then the triangle cannot be an equilateral triangle.

## 1.3 Area

### 1.3.1 Area of $\triangle$

$$\triangle = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

### 1.3.2 Area of any Polygon

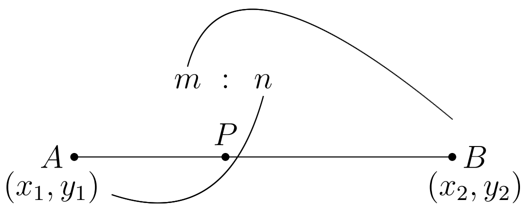
$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \dots & \dots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$$

### 1.3.3 Condition for Colinearity of 3 poitns

Area of polygon $= 0$

# 1.4 Section Formula

## 1.4.1 Internal Division




$$
\begin{aligned}
x &= \frac{mx_2 + mx_1}{m+n} \\
y &= \frac{my_2 + ny_1}{m+n}
\end{aligned}
$$

For Given eq of 2 Lines and point use:

$$
\begin{aligned}
&= -\frac{l_1}{l_2} \\
\frac{m}{n} &= -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}
\end{aligned}
$$

## 1.4.2 External Division

$$
\begin{aligned}
x &= \frac{mx_2 - nx_1}{m-n} \\
y &= \frac{my_2 - ny_1}{m-n}
\end{aligned}
$$

### 1.4.3 Mid Point Formula

$$x = \frac{x_1 + x_2}{2}$$
$$y = \frac{y_1 + y_2}{2}$$

## 1.5 Locus

1. Assume the coordinates of the points $(h, k)$ whose locus is to be found
2. Write the given condition in mathematical form involving $(h, k)$
3. Eliminate variables if any except $(h, k)$
4. Replace $h$ by $x$ and $k$ by $y$ : $(h, k) \rightarrow (x, y)$ in the result obtained in step 3.

# Chapter 2

# Straight Line

## 2.1 Slope or Gradient of a Line

$$\text{slope} = m = \tan\theta = \left.\frac{dy}{dx}\right|_{(x,y)}$$

| Slope($m$) | $\theta$ | Line |
|---|---|---|
| $m = 0$ | $0^\circ$ | Line parallel to x-axis |
| $m = \infty$ | $90^\circ$ | perpendicular to x-axis/paralle to y-axis |
| $m > 0$ | $0^\circ - 90^\circ$ | acute +ve slope |
| $m < 0$ | $90^\circ - 180^\circ$ | obtuse -ve slope |

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

**Angle Between two lines**

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

**Condition for Parallelism of lines**

$$m_1 = m_2$$

**Condition for Perpendicularity of Lines**

$$m_1 = -\frac{1}{m_2}\text{(Negative Reciprocal)}$$

### 2.1.1 Condition for 3 Lines being concurrent

Area of triangle $= 0$ Then the lines are concurrent.

## 2.2 Forms

### 2.2.1 Slope Intercept Form

$$y = mx + c$$

### 2.2.2 Point Slope Form

$$(y - y_1) = m(x - x_1)$$

### 2.2.3 Two Point Form

$$\frac{y - y_1}{x - x_1} = \frac{x_2 - x_1}{y_2 - y_1}$$

### 2.2.4 Intercept Form

$$\frac{x}{a} + \frac{y}{b} = 1$$

### 2.2.5 Normal Form

$$x \cos\alpha + y \sin\alpha = p$$

### 2.2.6 Distance Form

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

$$P = (x_1 + r\cos\theta, y_1 + r\sin\theta)$$

## 2.3 Transformations

For Line

$$ax + by + c = 0$$

### 2.3.1 To Slope intercept form

Isolate y on L.H.S.:

$$\begin{aligned}
&\text{Form: } y = mx + c \\
&y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right) \\
&m = -\frac{a}{b} \text{ and } c = -\frac{c}{b}
\end{aligned}$$

### 2.3.2 To Intercept form

Make R.H.S 1:

$$\begin{aligned}
&\text{Form: } \frac{x}{a} + \frac{y}{b} = 1 \\
&\text{or, } \frac{x}{-c/a} + \frac{y}{-c/b} = 1 \\
&\therefore \text{x-int} = -\frac{c}{a} \text{ and y-int} = -\frac{c}{b}
\end{aligned}$$

### 2.3.3 To Normal form

Divide by $\frac{1}{\sqrt{a^2+b^2}}$ and make +ve constant on R.H.S.

$$-\frac{ax}{\sqrt{a^2+b^2}} - \frac{by}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}}$$

## 2.4 Point Of intersetion of Lines

$$\begin{aligned}
a_1x + b_1y + c_1 = 0 \\
a_2x + b_2y + c_2 = 0
\end{aligned}$$

$$\text{P.O.I.} = \left(\frac{b_1c_2 - b_2c_1}{a_1 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

## 2.5 Perpendicular & Parallel Lines

### 2.5.1 Formation

For a given Line $ax + by + c = 0$
Paralled Line: $ax + by + k = 0$
Perpendicular Line: $bx - ay + k = 0$
$k$ can be obtained by using given condition

### 2.5.2 Condition

For given lines:

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

Lines are:

1. Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c+1}{c_2}$
2. Parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$
3. Intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$
4. Perpendicular, if $\left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right) = -1$

## 2.6 Distance of Point From a Line

Line: $ax + by + c = 0$ point: $(x_1, y_1)$

$$D = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

Line: $ax + by + c = 0$ Origin: $(0, 0)$

$$D = \frac{|c|}{\sqrt{a^2 + b^2}}$$

## 2.7 Distance between Parallel Lines

For given lines:

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

Distance is:

$$D = \left| \frac{c_2 - c_1}{\sqrt{a^2 + b^2}} \right|$$

## 2.8 Area of Parallelogram

For a parallelogram formed by the following parallel Lines:

$$a_1x + b_1y + c_1 = 0$$
$$a_1x + b_1y + c_2 = 0$$
$$a_2x + b_2y + d_1 = 0$$
$$a_2x + b_2y + d_2 = 0$$

Area of Parallelogram is given by:

$$\left| \frac{(c_1 - d_1)(c_2 - d_2)}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \right|$$

## 2.9 Equation of Lines passing through a given point and making a given angle with the line

Point: $(x_1, y_2)$ Angle: $\alpha$ with the line $y = mx + c$

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

## 2.10 Family of Lines through the intersection of two given lines

$$L_1 + \lambda L_2 = 0$$

# Chapter 3

# Pair Of Straight Lines

## 3.1 Definition

For two given lines

$$L_1 : a_1x + b_1y + c_1 = 0$$
$$L_2 : a_2x + b_2y + c_2 = 0$$

The joint equations is

$$L_1 \times L_2 = 0$$
$$\text{or, } (a_1x + b_1y + c_1)(L_2 : a_2x + b_2y + c_2) = 0$$
$$\text{or, } ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

This is the equation of pair of Straight Lines

### 3.1.1 Identification of 2nd Degree Equation

The Homogenous Equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight line if,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Thus,

$$\Delta = 0 \implies \text{Pair of Staright Line}$$

$$\Delta \neq 0 \implies \begin{cases} a = b \ \& \ h = 0 & \implies \text{Circle} \\ h^2 = ab & \implies \text{Parabola} \\ h^2 < ab & \implies \text{Ellipse} \\ h^2 > ab & \implies \text{Hyperbola} \\ h^2 > ab \ \& \ a + b = 0 & \implies \text{Rectangular Hyperbola} \end{cases}$$

Hint: **H**yperbola has h larger, while Ellipse has it smaller.

### 3.1.2 To get the point of intersection

For a given equation of pair of straight line

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Partially differentiate w.r.t. $x$ and $y$

$$\frac{\partial S}{\partial x} = 0 \quad (3.1)$$

$$\frac{\partial S}{\partial y} = 0 \quad (3.2)$$

Solve (1) and (2) to get $x$ and $y$
Remeber: Point of Intersection statifies all these 3 Equations.

## 3.2 Homogenous Equations: Pair of Straight Line through origin

For two given lines through origin

$$y = m_1 x$$
$$y = m_2 x$$

The joint equation is

$$(m_1 x - y)(m_2 - y) = 0$$
$$\text{or, } m_1 m_2 x^2 - (m_1 + m_2)xy + y^2 = 0$$
$$\text{or, } ax^2 + 2hxy + by^2 = 0$$

This is the equation of a pair of Straight lines through origin.

### 3.2.1 Component Lines

For a given equation of pair of straight line through origin

$$ax^2 + 2hxy + by^2 = 0$$

Consider the eq. to be a quadratic in $x$ and use quadratic formula.

$$a^2 + (2hy)x + by^2 = 0$$

### 3.2.2 Slopes

For a given eq of pair of straight line through origin

$$ax^2 + 2hxy + by^2 = 0$$

Divide both sides by $x^2$

$$a + 2h\frac{y}{x} + b\left(\frac{y}{x}\right)^2 = 0$$
$$\text{or, } b\left(\frac{y}{x}\right)^2 + 2h\frac{y}{x} + a = 0$$

Solving as a quadratic in $\frac{y}{x}$

$$\therefore \frac{y}{x} = m = \frac{-h \pm \sqrt{h^2 - ab}}{2b}$$

Which given two slopes $m_1$ and $m_2$ for $\pm$.

#### Sum and Product of slopes

The sum and product of the slopes are

$$m_1 m_2 = \frac{a}{b}$$
$$m_1 + m_2 = -\frac{2h}{b}$$

#### Observation from slope

From the term under square root in formula for slop:

1. $h^2 - ab > 0 \implies h^2 > ab \implies$ lines are real and distinct
2. $h^2 - ab = 0 \implies h^2 = ab \implies$ lines are coincident
3. $h^2 - ab < 0 \implies h^2 < ab \implies$ lines are imaginary

### 3.2.3 Eq of pair of Straight Lines Perpendicular to the given pair of lines

The combined eq of pair of staright lines through origin and perpendicular the the pair of staright lines

$$ax^2 + 2hxy + by^2 = 0$$

is

$$bx^2 - 2hxy + ay^2 = 0$$

Hint: exchange the coefficient of $x^2$ and $y^2$ and change the sign of $xy$.

### 3.2.4 Angle b/w the lines

For a given eq of pair of straight line through origin

$$ax^2 + 2hxy + by^2 = 0$$

The angle $\theta$ b/w the two lines is obtained by

$$\tan\theta = \left|\frac{2\sqrt{h^2 + ab}}{b}\right|$$

### 3.2.5 Eq of Angle Biscectors

For a given eq of pair of straight line through origin

$$ax^2 + 2hxy + by^2 = 0$$

The Eq of biscectors of the angles between the lines is given by

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

## 3.3 Non Homogenous Equation:

For two given lines

$$y = m_1x + c_1$$
$$y = m_2x + c_2$$

The joint equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

### 3.3.1 Pont of Intersection

For a given equation of pair of straight line

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

The point of intersection is

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

Alternatively we can use Partial Derivative method used in homogenous equation.

## 3.4 Condition for Parallel lines

For a given equation of pair of straight line

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

The lines are parallel If

$$\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$$

## 3.5 Distance between Parallel Lines

For a given equation of pair of straight line

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Distance between parllel lines is

$$d = 2\sqrt{\frac{g^2 - ac}{a(a+b)}} = 2\sqrt{\frac{f^2 - bc}{c(a+b)}}$$

## 3.6 Angle biscector for known point

For a given equation of pair of straight line

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

If the point of intersection is $(x_1, y_1)$, then the equation for angle biscector is

$$\frac{(x-x_1)^2-(y-y_1)^2}{a-b}=\frac{(x-x_1)(y-y_1)}{h}$$

This looks similar to homogenous formula added with a shift of origin.

## 3.7 Product of Perpendiculars

For a given equation of pair of straight line

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

If a perpendicular distances from a point $(x_1, y_1)$ be $P_1$ and $P_2$, then the product of perpenducular distances is given by

$$P_1P_2=\left|\frac{ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c}{\sqrt{(a-b)^2+(2h)^2}}\right|$$

## 3.8 Homogenization

For a given equation of a curve

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and a stragiht line

$$lx+my+n=0$$

that cuts the given curve at two points. A pair of straight line can be found that passes through the point of their intersection and has its own intersection at the origin. This produces a Homogenous Equation. This process is called Homogenization.

### 3.8.1 Process

1. Make each term in the given curve to have 2 degrees by multiplying powers of 1

$$ax^2+2hxy+by^2+2gx(1)+2fy(1)+c(1)^2=0$$

2. Get the value of 1 from the given eq of line

$$lx + my + n = 0$$
$$\text{or, } lx + my = -n$$
$$\text{or, } \frac{lx + my}{-n} = 1$$

3. Put the value of 1 from setp 2 into the eq obtained in setp 1.

$$ax^2+2hxy+by^2+2gx\left(\frac{lx+my}{-n}\right)+2fy\left(\frac{lx+my}{-n}\right)+c\left(\frac{lx+my}{-n}\right) = 0$$

4. On simplification

$$Ax^2 + 2Hxy + by^2 = 0$$