Technical Appendix II: MSD Solution for Rigid Wall Rotation

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This document has been superseded by the documentation and python code available at: https://github.com/AbiBateman/MSD_Walls.

This technical appendix should be used in conjunction with the calculation detailed in Technical Appendix I: Energy Solution for Embedded Walls.

Section 1 of this appendix details the elastoplastic solution. Section 2 of this appendix details the elastic solution.

Technical Appendix II

MSD Solution for Rigid Wall Rotation

1. Elastoplastic solution in Inhomogeneous Soil (linear variation of S_u with depth)

Energy Equilibrium

Considering the rotation mechanism depicted in Fig B1, the potential energy loss equals the sum of the internal elastoplastic work plus the elastic strain energy stored in the wall

$$\Delta P_1 = \Delta W_1 + \Delta U_1 \tag{B1}$$

In this solution the wall rotates as a rigid body and thus $\Delta U_1 = 0$, resulting in

$$\Delta P_1 = \Delta W_1 \tag{B2}$$

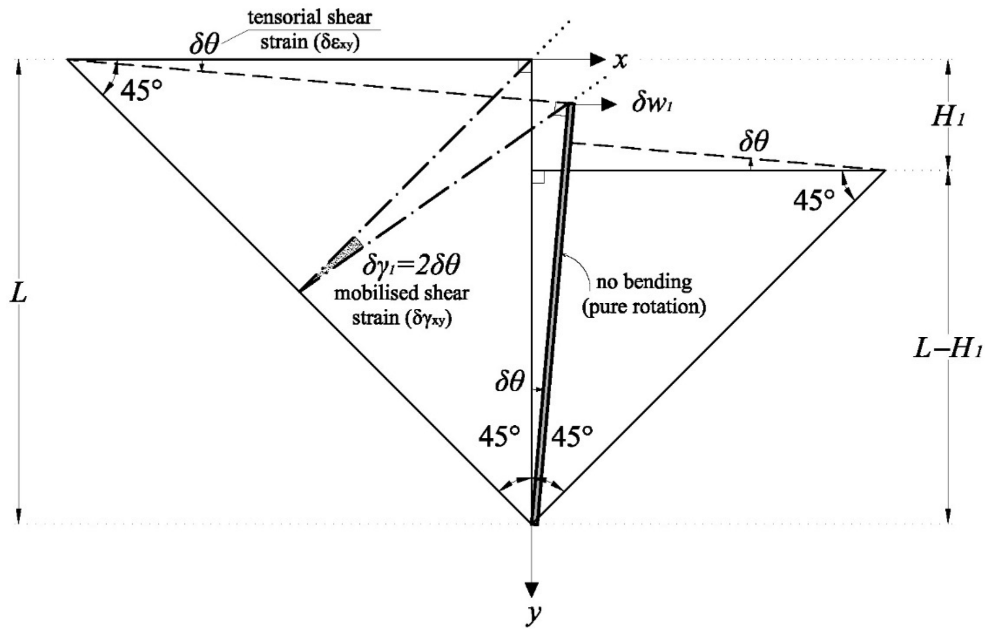

Figure B1. Pure rotation of the cantilever wall during first excavation stage and assumed plastic deformation mechanism (undrained conditions).

<u>*Region I (ABO)*</u>

The stresses in region I (*ABO*) (Figure B2) are calculated from the strains as:

$$\delta\varepsilon_h = -\delta\theta = -\frac{\partial \delta w}{\partial x} \Rightarrow \int dx \Rightarrow \delta w(x,y) = \delta\theta x + c_1 y + c_2 \tag{B3}$$

$$\delta\varepsilon_v = +\delta\theta = -\frac{\partial \delta v}{\partial x} \Rightarrow \int dy \Rightarrow \delta v(x, y) = -\delta\theta y + c_3 x + c_4 \tag{B4}$$

The boundary conditions along *AO*, which can be written as $x - y = -L$, are

$$\delta w_{AO}(x, y) = \delta w_{AO}(x, x + L) = 0 \tag{B5a}$$

$$\delta v_{AO}(x, y) = \delta v_{AO}(x, x + L) = 0 \tag{B5b}$$

Specifically, at Point *A* (–*L*, 0), $\delta w(-L, 0) = -\delta\theta L + 0 + c_2 = 0 \Rightarrow c_2 = \delta\theta L$

and at Point *O* (0, *L*), $\delta v(0, L) = -\delta\theta L + 0 + c_4 = 0 \Rightarrow c_4 = \delta\theta L$

Substituting c_2 and c_4 in Eqs. (3), (4) and (5) we get the values for c_1 and c_3

$$\delta w_{AO} = \delta\theta x + c_1(x + L) + \delta\theta L = 0 \Rightarrow c_1 = -\delta\theta$$

$$\delta v_{AO} = -\delta\theta(x + L) + c_3 x + \delta\theta L = 0 \Rightarrow c_3 = \delta\theta$$

Thus

$$\delta w(x, y) = \delta\theta(x - y + L) \tag{B6a}$$

$$\delta v(x, y) = \delta\theta(x - y + L) \tag{B6b}$$

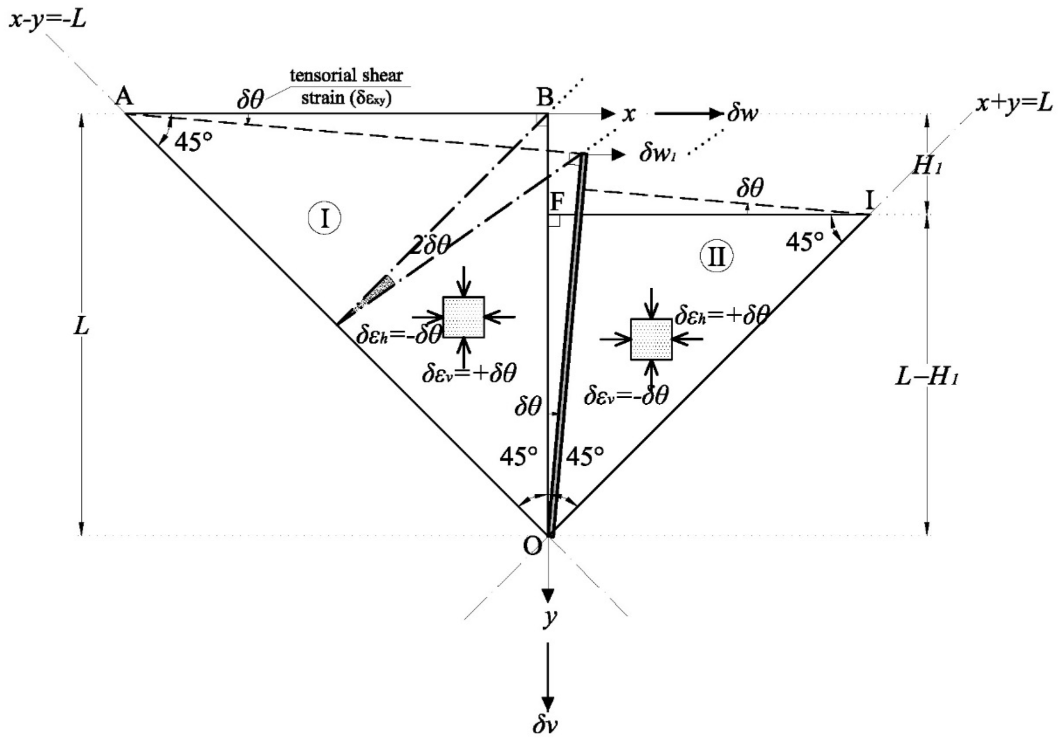

Figure B2. Strains and corresponding boundary conditions in the two regions of the considered mechanism.

Region II (FIO)

The stresses in region II (*FIO*) (Figure B2) are calculated from the strains as:

$$\delta\varepsilon_h = +\delta\theta = -\frac{\partial \delta w}{\partial x} \Rightarrow \int dx \Rightarrow \delta w(x,y) = -\delta\theta x + c_1 y + c_2 \qquad \text{(B7)}$$

$$\delta\varepsilon_v = -\delta\theta = -\frac{\partial \delta v}{\partial x} \Rightarrow \int dy \Rightarrow \delta v(x,y) = \delta\theta y + c_3 x + c_4 \qquad \text{(B8)}$$

The boundary conditions along *IO*, which can be written as $x + y = L$, are

$$\delta w_{IO}(x,y) = \delta w_{IO}(x, L-x) = 0 \qquad \text{(B9a)}$$

$$\delta v_{IO}(x,y) = \delta v_{IO}(x, L-x) = 0 \qquad \text{(B9b)}$$

Specifically,

at Point I $(L-H_1, H_1)$, $\delta w(L-H_1, H_1) = -\delta\theta\,(L-H_1) + c_1 H_1 + c_2 = 0 \Rightarrow c_2 = \delta\theta(L-H_1) - c_1 H_1$

and at Point O $(0, L)$, $\delta v(0, L) = \delta\theta L + 0 + c_4 = 0 \Rightarrow c_4 = -\delta\theta L$

Substituting c_2 and c_4 in Eqs. (7), (8) and (9) we get the values for c_1 and c_3

$$\delta w_{IO} = -\delta\theta x + c_1(L-x) + \delta\theta(L-H_1) - c_1 H_1 = 0 \Rightarrow c_1 = -\delta\theta$$

$$\delta v_{IO} = \delta\theta(L-x) + c_3 x - \delta\theta L = 0 \Rightarrow c_3 = \delta\theta$$

Thus

$$\delta w(x,y) = \delta\theta(-x - y + L) \qquad \text{(B10a)}$$

$$\delta v(x,y) = \delta\theta(x + y - L) \qquad \text{(B10b)}$$

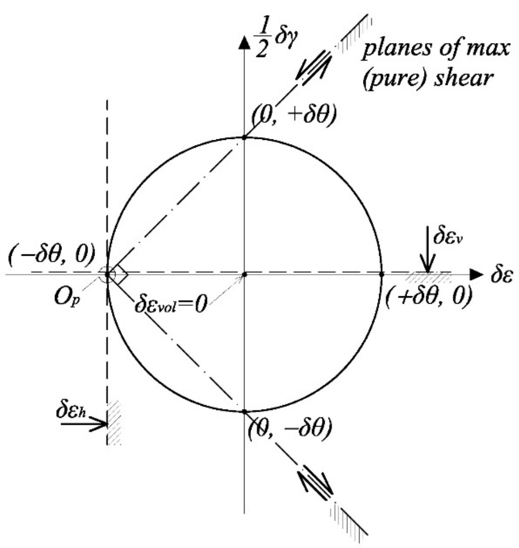

Figure B3. Mohr circle representation of plastic strains and stresses in the proposed mechanism.

The problem unknown is

$$\delta w_1 = \delta\theta \cdot L \tag{B11}$$

From the Mohr circle in Figure B3

$$\delta\gamma_{xy} = \pm\frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial x}\right) = \pm\frac{1}{2}\left(-\delta\theta + \delta\theta\right) = 0 \tag{B12a}$$

$$\delta\omega_{xy} = \pm\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial x}\right) = \pm\frac{1}{2}\left(-\delta\theta - \delta\theta\right) = -\delta\theta \tag{B12b}$$

The displacement gradient is written as

$$\nabla u = \begin{bmatrix} \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} \end{bmatrix} + \begin{bmatrix} 0 & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial x}\right) \\ -\dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial x}\right) & 0 \end{bmatrix} \tag{B13a}$$

which after substituting Eq. (12) yields

$$\nabla u = \underbrace{\begin{bmatrix} -\delta\theta & 0 \\ 0 & +\delta\theta \end{bmatrix}}_{\substack{\text{strain tensor} \\ \text{on horizontal} \\ \text{and vertical} \\ \text{planes}}} + \underbrace{\begin{bmatrix} 0 & -\delta\theta \\ +\delta\theta & 0 \end{bmatrix}}_{\text{rotation}} \tag{B13b}$$

1a. Potential Energy Loss Increment at excavation stage 1, ΔP_1

The potential energy loss can be written, in two dimensions, as

$$\Delta P_1 = \int_{area} \begin{Bmatrix} 0 \\ \gamma_{sat} \end{Bmatrix}^T \begin{Bmatrix} \delta w \\ \delta v \end{Bmatrix} dA = \int_{area} \gamma_{sat}\delta v dxdy \tag{B14}$$

Calculation of the above integral for both regions I, II of the considered mechanism (Fig B2) is given below:

$$\Delta P_1 = \gamma_{sat}\delta\theta\left[\int_0^L \int_{-L+y}^0 (x-y+L)dxdy + \int_{H_1}^L \int_0^{L-y} (x+y-L)dxdy\right] \tag{B15}$$

which, after substituting Eq. (11) and performing the integrations, equals

$$\Delta P_1 = \gamma_{sat}\frac{H_1}{6L}\left[3L^2 - 3H_1L + H_1^2\right]\delta w_1 \tag{B16}$$

1b. Internal Plastic Work Increment at excavation stage 1, ΔW_1

The internal plastic work increment in the 1st excavation stage, is calculated as

$$\Delta W_1 = \int_{area} |\beta S_u \delta\gamma| dA = \int_{area} |\beta S_u 2\delta\theta| dA = \int_{area} \left|\beta S_u \frac{2\delta w_1}{L}\right| dA \tag{B17}$$

For the case of inhomogeneous soil, and specifically undrained shear strength linearly increasing with depth as $S_u(y)=S_u+cy$, the calculation of the above integral for both regions of the considered mechanism is given below:

$$\Delta W_1 = \frac{2\delta w_1}{L}\beta\left[\int_{area_I} S_u(y) dA_I + \int_{area_{II}} S_u(y) dA_{II}\right] \tag{B18a}$$

which equals

$$\Delta W_1 = \frac{2\delta w_1}{L}\beta\left[\int_{-L}^{0}\int_{0}^{L+x}(S_u+cy)dydx + \int_{0}^{L-H_1}\int_{H_1}^{L-x}(S_u+cy)dydx\right] \tag{B18b}$$

and after performing the integration

$$\Delta W_1 = \frac{\beta}{L}\left[S_u(2L^2-2LH_1+H_1^2)+c(\frac{2}{3}L^3-H_1^2L+\frac{2}{3}H_1^3)\right]\delta w_1 \tag{B19}$$

Solution

For mobilization factor β equal to

$$\beta = \chi\left(\frac{\delta\gamma}{\gamma_{M=2}}\right)^b = \chi\left(\frac{2\delta w_1}{L\gamma_{M=2}}\right)^b \tag{B20}$$

and after substituting Eqs. (B16) and (B19) in the equilibrium equation Eq. (B2), we get the following solution for δw_1

$$\delta w_1 = \frac{L\gamma_{M=2}}{2}\left[\frac{\gamma_{sat}H_1}{2\chi}\cdot\frac{3-3\frac{H_1}{L}+\frac{H_1^2}{L^2}}{S_u(6-6\frac{H_1}{L}+3\frac{H_1^2}{L^2})+c(2L-3\frac{H_1^2}{L}+2\frac{H_1^3}{L^2})}\right]^{1/b} \tag{B21}$$

2. Elastic Solution in Inhomogeneous Soil (linear variation with depth)

For small values of the mobilisation factor β, elastic response may be assumed. The stress-strain behaviour in the mechanism is then described by Fig B4 and the internal elastic work ΔW_1 at the first excavation stage equals

$$\Delta W_{1,el} = \frac{1}{2} \int_{area} G \delta\gamma_1^2 dA \quad \text{(B22)}$$

with the assumption that G is proportional to the undrained shear strength S_u as described by Eq. (23)

$$G = k \cdot S_u(y) = k(S_{uo} + cy) \quad \text{(B23)}$$

and by substituting Eq. (B11) in Eq. (B22) one gets

$$\Delta W_{1,el} = 2\frac{\delta w_1^2}{L^2} k \left[\int_{area_I} S_u(y) dA_I + \int_{area_{II}} S_u(y) dA_{II} \right] \quad \text{(B24a)}$$

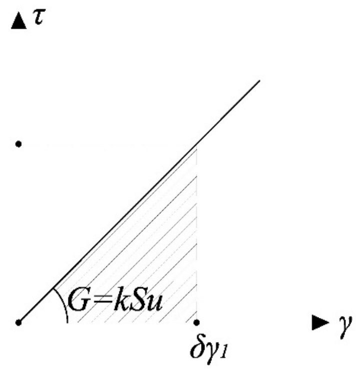

Figure B4. Stress -strain diagram for elastic response

which equals

$$\Delta W_{1,el} = 2\frac{\delta w_1^2}{L^2} k \left[\int_{-L}^{0} \int_{0}^{L+x} (S_u + cy) dy dx + \int_{0}^{L-H_1} \int_{H_1}^{L-x} (S_u + cy) dy dx \right] \quad \text{(B24b)}$$

and after performing the integration

$$\Delta W_{1,el} = \frac{k}{L^2} \left[S_u (2L^2 - 2LH_1 + H_1^2) + c(\frac{2}{3}L^3 - H_1^2 L + \frac{2}{3}H_1^3) \right] \delta w_1^2 \quad \text{(B25)}$$

Solution

After substituting Eqs. (B16) and (B25) in the equilibrium equation Eq. (B2), we get the following solution for δw_1

$$\delta w_{1,el} = \gamma_{sat} \frac{H_1 L}{6k} \frac{\left[3L^2 - 3H_1 L + H_1^2 \right]}{\left[S_u (2L^2 - 2LH_1 + H_1^2) + c(\frac{2}{3}L^3 - H_1^2 L + \frac{2}{3}H_1^3) \right]} \quad \text{(B26)}$$