

Online Supplement: MSD applied to the construction of the British Library basement: a multi-stage excavation in London Clay

Authors: J.J. Crispin, A.H. Bateman, E. Voyagaki, A. Campbell, G. Mylonakis, M.D. Bolton and P.J. Vardanega

This online supplement details the equations used in the calculation of the mobilisable strength design (MSD) predictions in the main text. They are used within the accompanying spreadsheet and python code for the example detailed in the main text. Section 1 of this supplement details the displacements from Stage 1 (pure rotation). Section 2 of this supplement details the displacements from the bulging stages (stages 2 onwards). This supplement is based on previous internal reports at the University of Bristol (Voyagaki *et al.* 2018a; b).

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NOTATION

Latin

A = Normalised (dimensional) parameter accounting for the potential energy loss at each stage (Eq. A6)
 B = Normalised (dimensional) parameter accounting for the change in internal elastoplastic work (Eq. A7a)
 $B_{max} = B$ normalised by the mobilisation factor (Eq. A13b)
 b = Soil non-linearity exponent
 b_0 = Contribution to B of the constant component of s_u (Eq. A7b)
 b_{var} = Contribution to B of the depth varying component of s_u (Eq. A7c)
 C_1, C_2 = Normalised (dimensional) parameters used to describe the strain energy stored in the wall (Eq. A8)
 EI = Plane-strain bending stiffness per meter length of the wall
 H_m = Excavated depth at stage m
 $H_{p,m}$ = Depth to the prop installed prior to the excavation stage m
 $h_{p,m}$ = Distance between the excavated depth and the last installed prop [$h_{p,m} = H_m - H_{p,m}$]
 L = Length of the wall
 M_c = Similarity factor
 m = Excavation stage index
 s_m = Distance between the last installed prop and the base of the wall [$s_m = L - H_{p,m}$]
 s_u = Soil undrained shear strength
 $s_{u,0}$ = Soil undrained shear strength at the top of the wall
 $s_{u,var}$ = Change of the soil undrained shear strength per meter depth
 w_m = Wall displacement at excavation stage m
 y = Depth below the top of the wall

Greek

α_λ = Modification factor to scale the displacement mechanism for different wall base conditions
 β_m = Soil strength mobilisation factor (τ/c_u) (Eq. A10)
 γ = Soil engineering shear strain
 γ_{50} = Soil shear strain when 50% of the soil shear strength is mobilised
 $\gamma_{ave,m}$ = Average mobilised shear strain in the mechanism for each stage m (Eq. A11)
 γ_{sat} = Soil saturated unit weight
 ΔU_m = Elastic strain energy stored in the wall at stage m
 ΔP_m = Potential energy loss at each stage m
 ΔW_m = Internal elastoplastic work at stage m
 $\delta u_{y,m}$ = Incremental vertical soil displacement at stage m
 δw_m = Incremental wall displacement at stage m
 $\delta w_{max,m}$ = Maximum incremental wall displacement at stage m (Eq. A9)
 $\delta\theta$ = Angle of wall rotation during stage 1
 λ_m = Wavelength of the deformation mechanism
 τ = Soil engineering shear stress
 χ_1 = Contribution of the previous stages to β_m when $b = 0.5$ (Eq. A12b)
 χ_2 = Contribution of the current stage to β_m when $b = 0.5$ (Eq. A12c)

1. RIGID WALL ROTATION

Stage 1 of the excavation is modelled as pure rotation around the base of the wall (see Fig. A1). Triangular displacement mechanisms are modelled on both sides of the wall (Fig. A1). These deform in uniform shear (Osman and Bolton 2004b) and energy equilibrium is considered (potential energy loss equals internal plastic work) to determine the wall rotation.

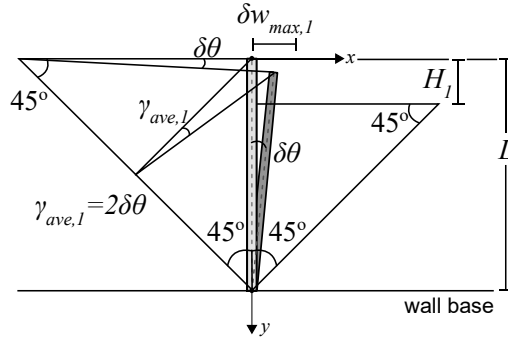


Figure A1. Illustration of the wall rotation around the base of the wall during stage 1 (modified from Osman and Bolton 2006b).

By assuming a power-law soil constitutive model (Eq. 1, see Fig. A2a) and a linear variation of the shear strength, s_u , with depth below the top of the wall, y , ($s_u = s_{u,0} + s_{u,var}y$, see Fig. A2b), the following solution is obtained for the wall displacement at stage 1, δw_1 (modified from Osman and Bolton 2006b).

$$\delta w_1 = \delta w_{max,1} \left(1 - \frac{y}{L}\right) \quad (A1a)$$

$$\delta w_{max,1} = \frac{L\gamma_{50}}{2} \left[\frac{\gamma_{sat}H_1 \left[3 - 3\left(\frac{H_1}{L}\right) + \left(\frac{H_1}{L}\right)^2 \right]}{3s_{u,0} \left[2 - 2\left(\frac{H_1}{L}\right) + \left(\frac{H_1}{L}\right)^2 \right] + s_{u,var}L \left[2 - 3\left(\frac{H_1}{L}\right)^2 + 2\left(\frac{H_1}{L}\right)^3 \right]} \right]^{\frac{1}{b}} \quad (A1b)$$

where $\delta w_{max,1}$ is the maximum wall displacement at stage 1, L denotes the length of the wall, H_1 is the excavated depth at stage 1, γ_{sat} is the saturated unit weight of the soil, γ_{50} is the shear strain when 50% of the soil shear strength is mobilised, b is a soil non-linearity exponent, $s_{u,0}$ is the undrained shear strength at the surface of the soil and $s_{u,var}$ is the variation of the undrained shear strength with depth. The average mobilised shear strain in the mechanism $\gamma_{ave,1}$ is approximately twice the angle of rotation of the wall $\delta\theta$ [$\delta\theta = \delta w_{max,1}/L = \gamma_{ave,1}/2$].

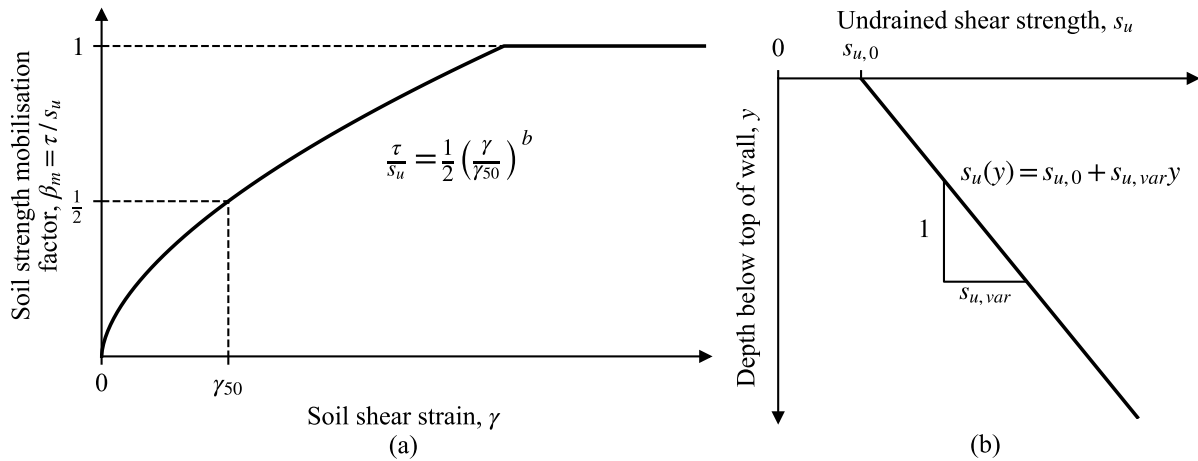


Figure A2. (a) Soil constitutive behaviour modelled using a power-law function (Eq. 1), and (b) variation of soil undrained shear strength with depth

2. BULGING OF A BRACED EXCAVATION

An elastoplastic deformation mechanism, used by Osman and Bolton (2006b), is assumed (depicted in Fig. A3). The wall, of length L , has a plane-strain bending stiffness per meter length of the wall EI . The wall has an assumed incremental displacement for each stage (for $m \geq 2$) according to the shape function (O'Rourke 1993):

$$\delta w_m(y) = \frac{1}{2} \left[1 - \cos \left(\frac{2\pi(y-H_{p,m})}{\lambda_m} \right) \right] \delta w_{max,m} \quad (A2)$$

where $\delta w_m(y)$ is the wall displacement as a function of the depth measured from the top of the wall y and $\delta w_{max,m}$ is the maximum wall displacement. Stage m refers to the current excavation stage, where H_m is the excavated depth, $H_{p,m}$ is the depth to the prop installed prior to this excavation stage. λ_m refers to the size of the deformation mechanism and is dependent on wall fixity at the base α_λ (shown in Fig. A3), given by:

$$\lambda_m = s_m \alpha_\lambda \quad (A3)$$

where s_m is the distance between the previously installed prop and the base of the wall [$s_m = L - H_{p,m}$] and α_λ is parameter used to model the boundary condition at the base of the wall (suggested by Osman and Bolton 2006b; values of this parameter are discussed in the main text).

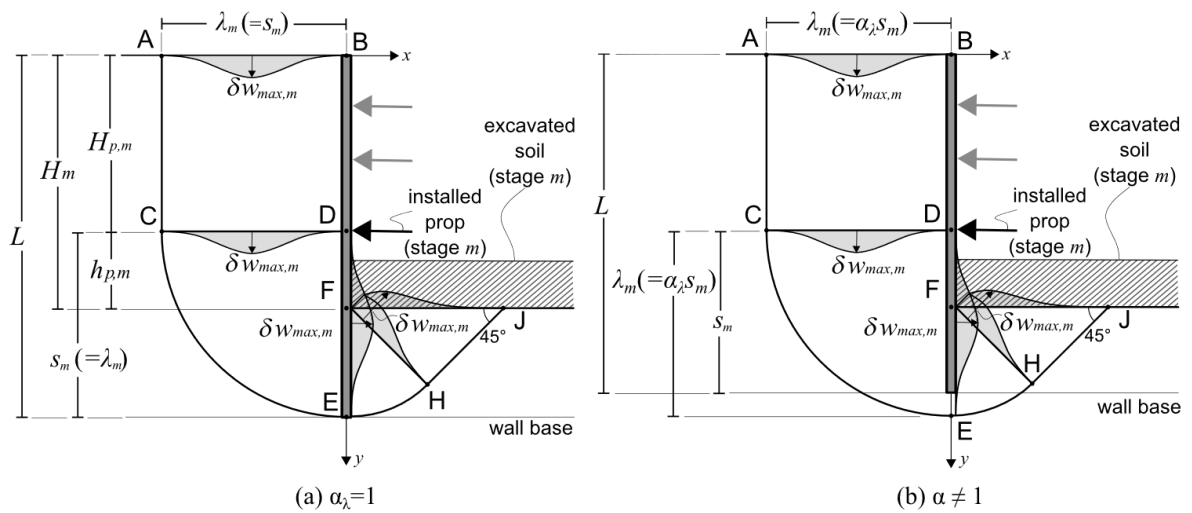


Figure A3. Incremental wall bulging at excavation stage m , plus plastic deformation mechanism considered for propped wall, for (a) $\alpha_\lambda = 1$, and (b) $\alpha_\lambda \approx 1.1$ (modified from Osman and Bolton 2006b).

2.1 General Solution

Considering the above elastoplastic mechanism for each stage, the potential energy loss ΔP_m equals the sum of the internal elastoplastic work ΔW_m and the elastic strain energy stored in the wall ΔU_m (Lam and Bolton 2011):

$$\Delta P_m = \Delta W_m + \Delta U_m \quad (A4)$$

each term of which can be expressed as (modified from Lam and Bolton 2011):

$$\Delta P_m = \int_{area} \gamma_{sat} \delta u_{y,m} dA \quad (A5a)$$

$$\Delta W_m = \int_{area} |\beta_m (s_{u,0} + s_{u,var} y) \delta y| dA \quad (A5b)$$

$$\Delta U_m = \frac{1}{2} EI \left[\int_{H_{p,m}}^L (\delta w_m'')^2 + 2 \delta w_m'' \sum_{i=2}^{m-1} \delta w_i'' dy \right] \quad (A5c)$$

where $\delta u_{y,m}$ is the incremental vertical soil displacement within the mechanism at each specific stage, β_m is the average mobilised stress within the mechanism (τ/c_u) defined by Vardanega and Bolton (2011b), and y is the depth measured from the top of the wall. These equations are simplified from those provided in Lam and Bolton (2011) by removing the discretisation of the mechanism. Note these equations are for the bulging stages only (the sum in Eq. A5c starts at stage 2). Calculations for stage 1 (the rotation stage) are discussed in Section 1.

The potential energy loss ΔP_m (Eq. A5a) within the mechanism is given by:

$$\Delta P_m = A \delta w_{max,m} = \left[\left(\frac{1}{2} H_{p,m} + \frac{1}{4} h_{p,m} \right) \gamma_{sat} \lambda_m \right] \delta w_{max,m} \quad (\text{A6})$$

where A is a normalised (dimensional) parameter used to describe the change in potential energy. The contribution to A for each zone within the mechanism is given in Table A1.

Similarly, the internal elastoplastic work ΔW_m (Eq. A5b) within the mechanism is given by:

$$\Delta W_m = B \delta w_{max,m} = [\beta_m \lambda_m (b_0 s_{u,0} + b_{var} s_{u,var})] \delta w_{max,m} \quad (\text{A7a})$$

$$b_0 = \frac{2H_{p,m}}{\lambda_m} + \frac{3\pi}{8} + 1 \quad (\text{A7b})$$

$$b_{var} = H_{p,m} \left(\frac{H_{p,m}}{\lambda_m} + \frac{\pi}{4} \right) + \sqrt{2} (\lambda_m - h_{p,m}) \left(\frac{5\pi^2 - 12}{16\pi^2} \right) + \lambda_m \left(\frac{4\pi^2 - 16}{16\pi^2} \right) + H_m \left(1 + \frac{\pi}{8} \right) \quad (\text{A7c})$$

where b_0 and b_{var} are the contributions due to the constant and depth varying components of s_u , respectively. B is a normalised (dimensional) parameter used to describe the change in internal elastoplastic work. The contribution to B for each zone within the mechanism is given in Table A1.

Table A1: Potential energy loss and internal elastoplastic work calculated for each zone within the mechanism

Zone	A (Eq. A6)	B (Eq. A7a)
ABCD (rectangular)	$\frac{1}{2} \gamma_{sat} H_{p,m} \lambda_m$	$2\beta_m \left(s_{u,0} + \frac{s_{u,var}}{2} H_{p,m} \right) H_{p,m}$
CDE (circular segment)	$\frac{1}{4} \gamma_{sat} \lambda_m^2$	$\frac{\pi}{4} \beta_m \lambda_m \left(s_{u,0} + s_{u,var} \left(H_{p,m} + \lambda_m \frac{\pi^2 - 4}{\pi^3} \right) \right)$
FHE (circular segment)	$-\frac{\sqrt{2}-1}{4\sqrt{2}} \gamma_{sat} \lambda_m (\lambda_m - h_{p,m})$	$\frac{\pi}{8} \beta_m \lambda_m \left(s_{u,0} + s_{u,var} \left(H_m + \sqrt{2} (\lambda_m - h_{p,m}) \frac{\pi^2 - 4}{\pi^3} \right) \right)$
FHJ (triangular)	$-\frac{\sqrt{2}}{8} \gamma_{sat} \lambda_m (\lambda_m - h_{p,m})$	$\beta_m \lambda_m \left(s_{u,0} + s_{u,var} \left(H_m + \sqrt{2} (\lambda_m - h_{p,m}) \frac{3\pi^2 - 4}{16\pi^2} \right) \right)$
Total	$\left(\frac{1}{2} H_{p,m} + \frac{1}{4} h_{p,m} \right) \gamma_{sat} \lambda_m$	$\beta_m \lambda_m (b_0 s_{u,0} + b_{var} s_{u,var})$

Finally, the elastic strain energy stored in the wall ΔU_m (Eq. A5c) can be calculated as (note Eqs. A8b and A8c simplify when $\alpha_\lambda = 1$):

$$\Delta U_m = C_1 \delta w_{max,m}^2 + C_2 \delta w_{max,m} \quad (\text{A8a})$$

$$C_1 = \pi^4 \frac{EI}{\lambda_m^3} \left[\frac{1}{\alpha_\lambda} + \frac{1}{4\pi} \sin \left(\frac{4\pi}{\alpha_\lambda} \right) \right] \quad (\text{A8b})$$

$$C_2 = \pi^3 EI \sum_{i=2}^{m-1} \frac{\delta w_{max,i}}{\lambda_i^3 \left(1 + \frac{\lambda_m}{\lambda_i} \right)} \left[\frac{2}{\lambda_i} \sin \left(\frac{2\pi}{\alpha_\lambda} \left(\frac{\lambda_m}{\lambda_i} - 1 \right) \right) + \frac{\lambda_i}{\lambda_m} \sin \left(\frac{4\pi}{\alpha_\lambda} \right) \right] \quad (\text{A8c})$$

where C_1 and C_2 are normalised (dimensional) parameters used to describe the strain energy stored in the wall.

Substituting Eqs. A6, A7 and A8 into Eq. A4 yields:

$$\delta w_{max,m} = \frac{A-B-C_2}{C_1} \quad (A9)$$

The above solution requires assuming a value for the mobilisation factor β_m contributing to coefficient B . The soil shear stress mobilisation factor β_m (τ/c_u) can be found using the assumed soil constitutive model (Eq. 1) and setting the shear strain to average shear strain in the mechanism for each stage, $\gamma_{ave,m}$:

$$\beta_m = \frac{\tau}{s_u} = \frac{1}{2} \left(\frac{\gamma_{ave,m}}{\gamma_{50}} \right)^b \quad (A10a)$$

The average shear strain in the mechanism can be calculated as (Lam and Bolton 2011):

$$\gamma_{ave,m} = M_c \sum_{i=2}^m \frac{\delta w_{max,i}}{\lambda_i} \quad (A10b)$$

where M_c is a similarity factor suggested by Osman and Bolton (2006b) to be 2.0 ($\pm 10\%$).

Eq. A9 is a polynomial of $\delta w_{max,m}$. When $b = 0.58$ (as per the example in the main text) an iterative approach is suggested to find a solution. In the supplied spreadsheet, a “guess” can be changed manually until it matches the predicted (“result”) incremental maximum displacement within a tolerable error. This is automated in the provided python script.

The overall displacements (for $m \geq 2$) can then be found by superimposing the displacements along the wall for each stage, including the rotation stage, Eq. A1a (Clough *et al.* 1989):

$$w_m = \delta w_{max,1} \left(1 - \frac{y}{L} \right) + \sum_{i=2}^m \frac{1}{2} \left[1 - \cos \left(\frac{2\pi(y-H_{p,m})}{\lambda_m} \right) \right] \delta w_{max,m} \quad (A11)$$

2.2 Closed Form Solution ($b = 0.5$)

For the special case where $b = 0.5$, a closed-form solution can be derived that allows solving directly for $\delta w_{max,m}$ without iteration (as is required for the above general solution). Using this assumption, β_m can be written as:

$$\beta_m = \frac{1}{2} \sqrt{\chi_1 + \chi_2 \delta w_{max,m}} \quad (A12a)$$

$$\chi_1 = \frac{M_c}{4\gamma_{50}} \sum_{i=1}^{m-1} \frac{\delta w_{max,i}}{\lambda_i} \quad (A12b)$$

$$\chi_2 = \frac{M_c}{4\lambda_m \gamma_{50}} \quad (A12c)$$

where χ_1 and χ_2 are the contributions of the previous and current stages, respectively, to β_m .

Then, substituting Eqs. A6, A7, A8 and A12 into Eq. A4 yields:

$$\delta w_{max,m} = \frac{1}{2C_1^2} \left[B_{max}^2 \chi_2 + 2C_1(A - C_2) - B_{max} \sqrt{B_{max}^2 \chi_2^2 + 4\chi_2 C_1 A + 4\chi_1 C_1^2 - 4\chi_2 C_1 C_2} \right] \quad (A13a)$$

$$B_{max} = \frac{B}{\beta_m} = \lambda_m (b_0 s_{u,0} + b_{var} s_{u,var}) \quad (A13b)$$

where B_{max} is B normalised by the mobilisation factor.

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