Closures

Example

A={1,2,3}

R={<1,1>,<1,2>,<2,1>}

R_1={<1,1>,<1,2>,<2,1>,<2,2>,<3,3>}

unique， the smallest reflexive relation containing R

R_2={<1,1>,<1,2>,<2,1>,<2,2>,<3,3>,<1,3>}

R_3={<1,1>,<1,2>,<2,1>,<2,2>,<3,3>,<2,3>}

Closure: the Concept

- Let R be a relation on A, $\mathcal{P}$ is some property, S is called $\mathcal{P}$ **closure** if:
  - S has property $\mathcal{P}$
  - $R \subseteq S$
  - If there is some relation $S1$ on A has property $\mathcal{P}$ and includes R as well, then $S \subseteq S1$

Reflexive Closure

- Reflexive closure of R is $R \cup I_A$

Example

$A=\{a,b,c\}$; $R=\{<a,b>,<b,c>,<c,a>\}$

$r(R)=R \cup I=\{<a,b>,<b,c>,<c,a>,<a,a>,<b,b>,<c,c>\}$

Symmetric Closure

- Symmetric closure of R is $R \cup R^{-1}$
- R^{-1} *is the inverse relation from B to A, if R is relation from A to B*

$$\{(b,a) \mid (a,b) \in R\}$$

- A={a,b,c}; R={<a,b>,<b,c>,<c,a>}

$s(R)=R \cup R^{-1}=$
$\{\langle a,b\rangle,\langle b,c\rangle,\langle c,a\rangle,\langle b,a\rangle,\langle c,b\rangle,\langle a,c\rangle\}$

Transitive Closure

Transitive closure of R is $R \cup R^2 \cup R^3 \cup \ldots$

A={a,b,c}; R={<a,b>,<b,c>,<c,a>}

R^2= {<a,c>,<b,a>,<c,b>}

R*={<a,b>,<b,c>, <c,a> <a,c>, <b,a>,<c,b>...}

Transitive Closure

- If $|A|=n$, then the transitive closure of R is

$$\bigcup_{i=1}^{n} R^i = R \cup R^2 \cup \cdots \cup R^n$$

A={a,b,c}; R={<a,b>,<b,c>,<c,a>}

R^2= {<a,c>,<b,a>,<c,b>}

R^3= {<a,a>,<b,b>,<c,c>}=I

$t(R)=R \cup R^2 \cup R^3=$

{<a,b>,<b,c>,<c,a>,<a,c>,<b,a>,<c,b>,<a,a>,<b,b>,<c,c>}

Graphs

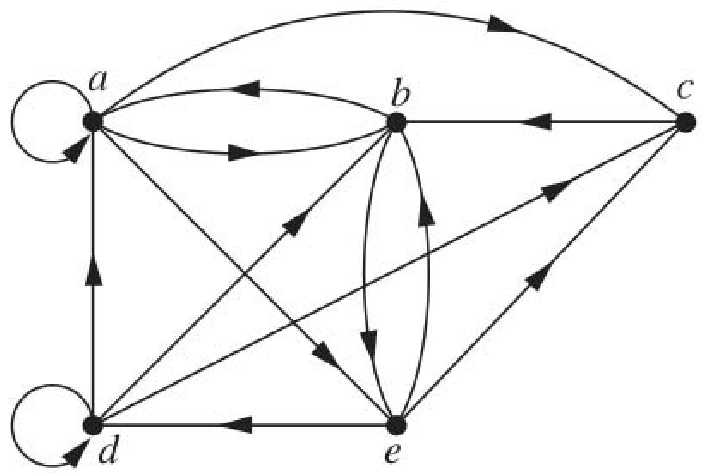
Definition :

A path from a to b in the directed graph G is a sequence of edges (x_0, x_1), (x_1, x_2),(x_2, x_3), … , (x_{n-1}, x_n), in G, where n is a nonnegative integer, and x_0 = a and x_n = b, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1 \ldots\ldots x_{n-1}, x_n$ and has length n.We view the empty set of edges as a path of length zero from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a circuit or cycle.

Graphs

Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if (a, b) $\in$ R^n.

The number of length is the same as the number of composition (power)



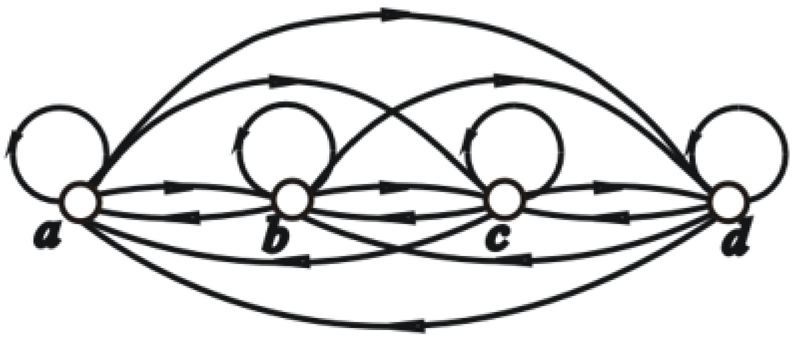
LEMMA

Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least

one in R from a to b, then there is such a path with length not exceeding n. Moreover, when

$a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with

length not exceeding $n - 1$.



Example

Program calls

$P=\{P_1,P_2,P_3,P_4\}, R=\{<P_1,P_2>,<P_1,P_3>,<P_2,P_3>,<P_2,P_4>\}$

$r(R)=R\cup I=\{<P_1,P_2>,<P_1,P_3>,<P_2,P_3>,<P_2,P_4>,$
$<P_1,P_1>,<P_2,P_2>,<P_3,P_3>,<P_4,P_4>\}$

$s(R)=R\cup R^{-1}=\{<P_1,P_2>,<P_1,P_3>,<P_2,P_3>,<P_2,P_4>,$
$\{<P_2,P_1>,<P_3,P_1>,<P_3,P_2>,<P_4,P_2>\}$

$t(R)=R\cup R^2\cup R^3\cup R^4$

$=\{<P_1,P_2>,<P_1,P_3>,<P_2,P_3>,<P_2,P_4>,<P_1,P_4>\}$

Where $R^2=\{<P_1,P_3>,<P_1,P_4>\}$ $R^3=R^4=\emptyset$

Example

$A=\{a,b,c\}$; $R=\{\langle a,b\rangle,\langle b,c\rangle,\langle c,a\rangle\}$

$r(R)=R\cup I=\{\langle a,b\rangle,\langle b,c\rangle,\langle c,a\rangle,\langle a,a\rangle,\langle b,b\rangle,\langle c,c\rangle\}$

$s(R)=R\cup R^{-1}=\{\langle a,b\rangle,\langle b,c\rangle,\langle c,a\rangle,\langle b,a\rangle,\langle c,b\rangle,\langle a,c\rangle\}$

$R^2=\{\langle a,c\rangle,\langle b,a\rangle,\langle c,b\rangle\}$

$R^3=\{\langle a,a\rangle,\langle b,b\rangle,\langle c,c\rangle\}=I$

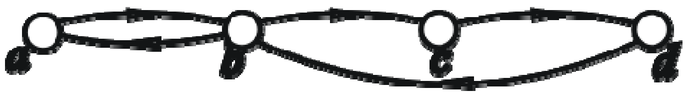
$t(R)=R\cup R^2\cup R^3=$

$\{\langle a,b\rangle,\langle b,c\rangle,\langle c,a\rangle,\langle a,c\rangle,\langle b,a\rangle,\langle c,b\rangle,\langle a,a\rangle,\langle b,b\rangle,\langle c,c\rangle\}$

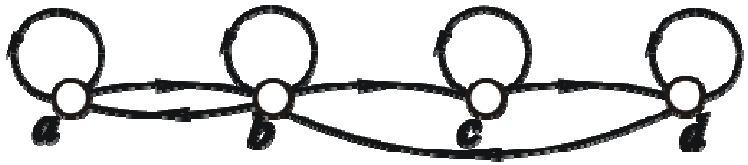
Examples

$A=\{a,b,c,d\}$,$R=\{\langle a,b\rangle,\langle b,a\rangle,\langle b,c\rangle,\langle c,d\rangle,\langle d,b\rangle\}$,

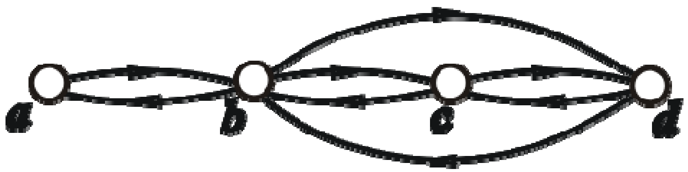
$r(R)$, $s(R)$, $t(R)$ are as follows



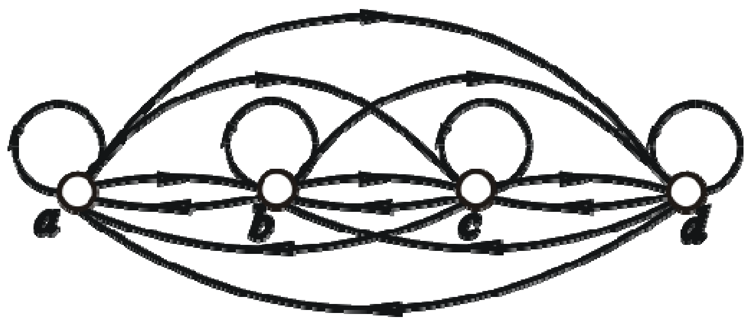
R



***r*(*R*)**



***s*(*R*)**



***t*(*R*)**

Conclusions

(1) R is reflexive if and only if $r(R)=R$.

(2) R is symmetric if and only if $s(R)=R$.

(3) R is transitive if and only if $t(R)=R$.

Eg.

$<$ is a relation on integers.

$r(<)=\leq$,

$S(<)=\neq$,

$t(<)=<$

homework

- P637

1, 2, 3, 5, 6, 9

Additional question: Find the directed graphs of the transitive closures of the relations with directed graphs shown in Exercises 5–7.