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9.1 Relations and Their Properties(PG-608)

3. For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, whether it is symmetric, whether it is anti symmetric, and whether it is transitive.
a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
c) {(2, 4), (4, 2)}
d) {(1, 2), (2, 3), (3, 4)}
e) {(1, 1), (2, 2), (3, 3), (4, 4)}
f){(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

Solution :

Definitions

A relation ***R*** on a set ***A*** is **reflexive** if (a, a) ∈ ***R*** for every element **a** ∈ ***A***.

A relation ***R*** on a set ***A*** is **symmetric** if (b, a) ∈ ***R*** whenever (a, b) ∈ ***R***.

A relation ***R*** on a set ***A*** is **antisymmetric** if (b, a) ∈ ***R*** and (a, b) ∈ ***R*** implies a = b.

A relation ***R*** on a set ***A*** is **transitive** if (a, b) ∈ ***R*** and (b, c) ∈ ***R*** implies (a , c) ∈ ***R*** .

<u>SOLUTION :</u>

$$A = \{ 1, 2, 3, 4\}$$

(a)

$$\boldsymbol{R} = \{ (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$$

The relation $\boldsymbol{R}$ is **not reflexive** , because $\boldsymbol{R}$ does not contain $(1 , 1)$ and $(4 , 4)$.

The relation $\boldsymbol{R}$ is **not symmetric** , because $(2 , 4) \in \boldsymbol{R}$ $(4 , 2) \notin \boldsymbol{R}$.

The relation $\boldsymbol{R}$ is **not antisymmetric** , because $(2 , 3) \in \boldsymbol{R}$ and $(3 , 2) \in \boldsymbol{R}$ while $2 \neq 3$.

The relation $\boldsymbol{R}$ is **transitive** , because if $(a, b) \in \boldsymbol{R}$ and $(b , c) \in \boldsymbol{R}$ then we also note that $(a, c) \in \boldsymbol{R}$.

$(2, 2) \in \boldsymbol{R}$ and $(2 , 3) \in \boldsymbol{R} \Rightarrow (2 , 3) \in \boldsymbol{R}$

$(2, 2) \in \boldsymbol{R}$ and $(2 , 4) \in \boldsymbol{R} \Rightarrow (2 , 4) \in \boldsymbol{R}$

$(2, 3) \in \boldsymbol{R}$ and $(3 , 2) \in \boldsymbol{R} \Rightarrow (2 , 2) \in \boldsymbol{R}$

$(2, 3) \in \boldsymbol{R}$ and $(3 , 3) \in \boldsymbol{R} \Rightarrow (2 , 3) \in \boldsymbol{R}$

$(2, 3) \in \boldsymbol{R}$ and $(3 , 4) \in \boldsymbol{R} \Rightarrow (2 , 4) \in \boldsymbol{R}$

$(3, 2) \in \boldsymbol{R}$ and $(2 , 3) \in \boldsymbol{R} \Rightarrow (3 , 3) \in \boldsymbol{R}$

$(3, 2) \in \boldsymbol{R}$ and $(2 , 4) \in \boldsymbol{R} \Rightarrow (3 , 4) \in \boldsymbol{R}$

$(3, 3) \in \boldsymbol{R}$ and $(3 , 2) \in \boldsymbol{R} \Rightarrow (3 , 2) \in \boldsymbol{R}$

$(3, 3) \in \boldsymbol{R}$ and $(3 , 3) \in \boldsymbol{R} \Rightarrow (3 , 3) \in \boldsymbol{R}$

$(3, 3) \in \boldsymbol{R}$ and $(3 , 4) \in \boldsymbol{R} \Rightarrow (3 , 4) \in \boldsymbol{R}$

(b)

$$\mathbf{R} = \{ (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$$

The relation $\boldsymbol{R}$ is **reflexive** , because $\boldsymbol{R}$ contains (1 , 1), (2,2),(3,3) and (4 ,4).

The relation $\boldsymbol{R}$ is **symmetric** , because $(1 , 2) \in \boldsymbol{R}$ $(2,1) \in \boldsymbol{R}$.

The relation $\boldsymbol{R}$ is not **antisymmetric** , because $(1 , 2) \in \boldsymbol{R}$ and $(2 , 1) \in \boldsymbol{R}$ while $2 \neq 1$.

The relation $\boldsymbol{R}$ is **transitive** , because if $(a , b) \in \boldsymbol{R}$ and $(b, c) \in \boldsymbol{R}$ then we also note that $(a, c) \in \boldsymbol{R}$.

$$(1, 1) \in \boldsymbol{R} \text{ and } (1 , 2) \in \boldsymbol{R} \Rightarrow (1 , 2) \in \boldsymbol{R}$$

$$(2, 1) \in \boldsymbol{R} \text{ and } (1 , 2) \in \boldsymbol{R} \Rightarrow (2 , 2) \in \boldsymbol{R}$$

$$(1, 2) \in \boldsymbol{R} \text{ and } (2 , 1) \in \boldsymbol{R} \Rightarrow (1 , 1) \in \boldsymbol{R}$$

$$(1, 2) \in \boldsymbol{R} \text{ and } (2 , 2) \in \boldsymbol{R} \Rightarrow (1 , 2) \in \boldsymbol{R}$$

(c)

$$\mathbf{R} = \{ (2, 4), (4, 2)\}$$

The relation $\boldsymbol{R}$ is **not reflexive** , because $\boldsymbol{R}$ does not contain (1 ,1), (2, 2), (3 ,3)and (4 , 4).

The relation $\boldsymbol{R}$ is **symmetric** , because $(2 , 4) \in \boldsymbol{R}$ $(4,2) \in \boldsymbol{R}$.

The relation $\boldsymbol{R}$ is not **antisymmetric** , because $(2 , 4) \in \boldsymbol{R}$ and $(4 , 2) \in \boldsymbol{R}$ while $4 \neq 2$.

The relation $\boldsymbol{R}$ is **transitive** , because if $(2 , 4) \in \boldsymbol{R}$ and $(4, 2) \in \boldsymbol{R}$ while $(2, 2) \notin \boldsymbol{R}$.

(d)

$\mathbf{R} = \{ (1, 2), (2, 3), (3, 4)\}$

The relation ***R*** is **not reflexive** , because ***R*** does not contain (1 ,1), (2, 2), (3 ,3)and (4 , 4).

The relation ***R*** is **not symmetric** , because (1 , 2) ∈ ***R*** (2,1) ∉ ***R***.

The relation ***R*** is **antisymmetric** , because when (a , b) ∈ ***R***, then (b , a) ∉ ***R*** (in this case).

The relation ***R*** is **not transitive** , because (1 , 2) ∈ ***R*** and (2, 3) ∈ ***R*** while (1, 3) ∉ ***R***.

(e)

$\mathbf{R} = \{ (1, 1), (2, 2), (3, 3), (4, 4)\}$

The relation ***R*** is **reflexive** , because ***R*** contains (1 ,1), (2, 2), (3 ,3)and (4 , 4).

The relation ***R*** is **symmetric** , because if (a ,b) ∈ ***R***, then a =b(in this case) and thus

(b, a) = (a, a) ∈ ***R***.

The relation ***R*** is **antisymmetric** , because (a , b) ∈ ***R***, then (b , a) ∈ ***R***, then a = b.

The relation ***R*** is **transitive** , because if (a , b) ∈ ***R*** and (b, c) ∈ ***R***, then we also note that (a, c) ∈ ***R***.

$$(1, 1) \in \boldsymbol{R} \text{ and } (1 , 1) \in \boldsymbol{R} \Rightarrow (1 , 1) \in \boldsymbol{R}$$

$$(2 , 2) \in \boldsymbol{R} \text{ and } (2 , 2) \in \boldsymbol{R} \Rightarrow (2 , 2) \in \boldsymbol{R}$$

$$(3, 3) \in \boldsymbol{R} \text{ and } (3 , 3) \in \boldsymbol{R} \Rightarrow (3 , 3) \in \boldsymbol{R}$$

$$(4, 4) \in \boldsymbol{R} \text{ and } (4 , 4) \in \boldsymbol{R} \Rightarrow (4 , 4) \in \boldsymbol{R}$$

(f)

R = {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

The relation ***R*** is **not reflexive** , because ***R*** does not contain (1 ,1), (2, 2), (3 ,3)and (4 , 4).

The relation ***R*** is **not symmetric**, because (1 ,4) $\in$ ***R*** and (4, 1) $\notin$ ***R***.

The relation ***R*** is **not antisymmetric** , because (1 ,3) $\in$ ***R***, and (3 , 1) $\in$ ***R***, while, 1 $\neq$ 3.

The relation ***R*** is **not transitive** , because if (1 ,3) $\in$ ***R*** and (3, 1) $\in$ ***R***, then we also note that (1, 1) $\notin$ ***R***.

Result

(a) Transitive

(b) Reflexive, Symmetric and Transitive.

(c) Symmetric

(d) Antisymmetric

(e) Reflexive, Symmetric , Antisymmetric and Transitive.

(f) None

4. Determine whether the relation *R* on the set of all peopleis reflexive, symmetric, antisymmetric, and/or transitive,where (*a, b*) ∈*R* if and only if

a) *a* is taller than *b*.

b) *a* and *b* were born on the same day.

c) *a* has the same first name as *b*.

d) *a* and *b* have a common grandparent.

Solution:

a) a is taller than b.
A: Antisymmetric and transitive

b) a and b were born on the same day.
A: Reflexive, symmetric and transitive.

c) a has the same first name as b.
A: Reflexive, symmetric and transitive.

d) a and b have a common grandparent.
A: Reflexive and symmetric.

5. Determine whether the relation *R* on the set of all Web pages is reflexive, symmetric, antisymmetric, and/ortransitive, where (*a, b*) ∈*R* if and only if

a) everyone who has visited Web page *a* has also visited Web page *b*.

b) there are no common links found on both Web page *a* and Web page *b*.

c) there is at least one common link on Web page *a* and Web page *b*.

d) there is a Web page that includes links to both Webpage *a* and Web page *b*.

Solution:

$\boldsymbol{A}$ = Set of all Web pages.

(a)

$$\boldsymbol{R} = \{\, (\mathrm{a}\,,\, b) \mid \text{everyone who has visited webpage a has also visited webpage } b \}$$

The relation $\boldsymbol{R}$ is **reflexive,** because if you visit webpage A, then you have also visited webpage B.

The relation $\boldsymbol{R}$ is **not symmetric,** because if everyone who visited webpage A, has also visited webpage B, then it is possible that some people visited webpage B without visited webpage A.

The relation $\boldsymbol{R}$ is **not antisymmetric,** because if everyone who visited webpage A, has also visited webpage B, and if everyone who visited webpage B has also visited webpage A, then these web pages are not necessarily the same web page.

The relation $\boldsymbol{R}$ is **transitive,** because if everyone who visited webpage A, has also visited webpage B, and if everyone who visited webpage B has also visited webpage C, then everybody who visited web page A has also visited web page C.

(b)

$$\boldsymbol{R} = \{\, (\mathrm{a}\,,\, b) \mid \text{a and b have no common links} \}$$

The relation $\boldsymbol{R}$ is **not reflexive,** because a webpage has always common links with itself (if the web page has links).

The relation $\boldsymbol{R}$ is **symmetric,** because if webpage A and webpage B have no common links, then webpage B and webpage A have no common links.

The relation $\boldsymbol{R}$ is **not antisymmetric,** because if webpage A and webpage B have no common links and if webpage B and webpage A have no common links, then these two web pages are not necessarily the same web page(moreover, it is not possible they are the same webpage, as a webpage has always common links with itself ,if the web page has links).

The relation ***R*** is **not transitive,** because if webpage A and webpage B have no common links and if webpage B and webpage C have no common links, then it is possible that webpage A and webpage C have a common link (that is not common with web page B).

(c)

$\boldsymbol{R} = \{ (a , b) \mid$ there is at least one common link on web page a web page b $\}$

The relation ***R*** is **not reflexive,** because a webpage with no links will not have a common links with itself .

The relation ***R*** is **symmetric,** because if webpage A and B have a common link, then webpage B and A also have a common link.

The relation ***R*** is **not antisymmetric,** because if webpage A and webpage B have a common link and if webpage B and webpage A have a common link, then these two web pages are not necessarily the same web page.

The relation ***R*** is **not transitive,** because if webpage A and webpage B have a common link and if webpage B and webpage C have a common link, then webpage A and webpage C do not necessarily have a common link (if the common link between A and B is different than the common link between B and C).

(d)

$\boldsymbol{R} = \{(a , b) \mid$ there is a web that included links to both Web page a and web page b $\}$

The relation ***R*** is **not reflexive,** because there does no necessarily exist a webpage that includes links to a webpage A.

The relation ***R*** is **symmetric,** because if there is a webpage that includes links to web page A and web page B then the same webpage also includes links to webpage B and webpage A.

The relation ***R*** is **not antisymmetric,** because if there is a webpage that includes links to web page A and web page B and there is a webpage that includes links to web page B and web page A, then these two web pages are not necessarily the same web page.

The relation ***R*** is **not transitive,** because if there is a webpage that includes links to web page A and web page B and there is a webpage that includes links to web page B and web page C, then web pages A and C do not necessarily have links on the same web page(if the webpage with links of A and B is different than the webpage with links between B and C).

Result

(a) Reflexive and Transitive.

(b) Symmetric.

(c) Symmetric

(d) Symmetric

6. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where (x, y) ∈ R if and only if

Solution:

a) x + y = 0.

A:The only existing example of an ordered pair (x,y) where y = x and x + y = 0 is (0,0). Therefore, the set R does not represent all values of (x,x) where x is any real number, so the relation isn't reflexive. The relation is symmetric (and by the following logic, antisymmetric), because if x + y = 0, making (x,y) an element of R, and y + x = 0, then (y,x) must also be an element of R. The relation is not transitive, since, for example (-3,3) and (3,-3) are elements of the set R, but (-3,-3) is not.

c) $x - y$ is a rational number.

A:The relation is reflexive because all number minus themselves equals 0, meaning all ordered pairs of (x,x) are represented. The relation is symmetric, because if x-y = a rational number then y - x = a rational number. This also means that that the relation is not antisymmetric. The relation is transitive. If x-y = a rational number, and y - z = a rational number, the x - z must also equal a rational number.

e) $xy \geq 0$.

A:This relation is reflexive because any two equal numbers will result in either a zero or a positive number. This means that all values of (x,x) are included in R. The relation is also symmetric, because if xy is greater than or equal to 0, then yx is also greater than or equal to 0. Therefore, for every (x,y) ordered pair, there is a (y,x) pair in R. By this logic, the relation in not antisymmetric. This relation is not transitive. Should y equal 0 and x and z have different signs, then (x,y) and (y,z) will be in R, but (x,z) will not be. Look at this example: The ordered pair (2,0) is an element of R. So, it (0, -3). (2, -3) is not however, because $(2)(-3) < 0$

10. Give an example of a relation on a set that is

a) both symmetric and antisymmetric.

b) neither symmetric nor antisymmetric.

A relation R on the set A is **irreflexive** if for every $a \in A$, $(a, a) \notin R$. That is, R is irreflexive if no element in A is related to itself.

Solution:

a) A relation on a set is symmetric and antisymmetric, if the relation contain only elements of the form (a , a) ∈ ***R***.

For example :

$$\boldsymbol{R} = \{(1 , 1)\}$$

$$\boldsymbol{R} = \{(2 , 2), (3 , 3), (4 , 4)\}$$

$$\boldsymbol{R} = \{(0 , 0), (5 , 5), (10 , 10), (15 , 15), (20 , 20), \ldots\ldots \}$$

b) A relation not symmetric nor antisymmetric, if it contains an element (a , b) while it does not contain (b , a) and if it contains an element (c , d) while it also contains (d , c) with $c \neq d$.

For example :

$$\boldsymbol{R} = \{(0 , 1), (1 , 2), (2 , 1)\}$$

$$\boldsymbol{R} = \{(0 , 0), (1 , 5), (2 , 10), (10 , 2)\}$$

Results :

Answers may vary.

26. Let R be the relation $R=\{(a,b) \mid a<b\}$ on the set of integers.

Find

a) R^{-1}

b) $\bar{R}$

a) R^{-1}

A: $R=\{(a,b) \mid a<b\}$

Use the definition of the universe relation:

$$R^{-1}=\{(b,a) \mid (a,b)\in R\}$$
$$=\{(b,a) \mid a<b\}$$
$$=\{(a,b) \mid b<a\}$$
$$=\{(a,b) \mid a>b\}$$

b) $\bar{R}$

A: Use the definition of the complementary relation:

$$\bar{R}=\{(a,b) \mid (a,b)\notin R\}$$
$$=\{(a,b) \mid a\nless b\}$$
$$=\{(a,b) \mid a\geq b\}$$

52.

Let R be the relation $\{(1,2),(1,3),(2,3),(2,4),(3,1)\}$ and let S be the relation $\{(2,1),(3,1),(3,2),(4,2)\}$ Find $S \circ R$.

Solution;

Definitions

The composite $S \circ R$ consists of all ordered pairs (a,c) for which there exists an element b such that $(a,b) \in R$ and $(b,c) \in S$.

SOLUTION

$$R = \{(1,2),(1,3),(2,3),(2,4),(3,1)\}$$
$$S = \{(2,1),(3,1),(3,2),(4,2)\}$$

For every pair $(a,b) \in R$ and $(b,c) \in S$, the corresponding element (a,c) belongs to the composition

$(1,2) \in R$ and $(2,1) \in S \Rightarrow (1,1) \in S \circ R$

$(1,3) \in R$ and $(3,1) \in S \Rightarrow (1,1) \in S \circ R$

$(1,3) \in R$ and $(3,2) \in S \Rightarrow (1,2) \in S \circ R$

$(2,3) \in R$ and $(3,1) \in S \Rightarrow (2,1) \in S \circ R$

$(2,3) \in R$ and $(3,2) \in S \Rightarrow (2,2) \in S \circ R$

$(2,4) \in R$ and $(4,2) \in S \Rightarrow (2,2) \in S \circ R$

SoR then contains all the resulting ordered pairs (while we can ignore the repetitions of ordered pairs):

$$S \circ R = \{(1,1), (1,2), (2,1), (2,2)\}$$

RESULT:

$$S \circ R = \{(1,1), (1,2), (2,1), (2,2)\}$$

33) Let R be the relation on the set of people consist of pairs (a,b) where a is a parent of b. Let S be the relation on the set of people consisting of pairs (a,b), where a, and b are siblings (brother & sister). What are $S \circ R$ and $R \circ S$?

SOLUTION:

Consider $(a,b) \in S \circ R$, this means there is c such that $(a,c) \in R$ and $(c,b) \in S$, or in other words, a is a parent of c as well as c & b are siblings. This implies a is a parent of b too.

$S \circ R = \{(a,b) \mid a$ is a parent of $b\}$

Similarly, if $(a,b) \in R \circ S$, then there is c, so that $(a,c) \in S$, i.e., a & c are siblings & $(c,b) \in R$, or c is a parent of b. Thus a is an uncle or aunt of b.

$R \circ S = \{(a,b) \mid a$ is an uncle or aunt of $b\}$.

9.3 Representing Relations(PG-626)

4. List the ordered pairs in the relations on {1, 2, 3, 4} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order)

a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Ans:

A)We are interested in the relations on {1,2, 3, 4}. Thus a1 = b1 = 1, a2 = b2 = 2, a3 = b3 = 3, a4 = b4 = 4. Note that only the 1-values in the matrix will represent a tuple that is in the relation. We note that the element m11 is 1, thus (a1, b1) = (1, 1) € R. We note that the element m12 is 1, thus (a1, b2) = (1, 2) € R Similarly, we obtain for all 1-values in the given matrix:

(a1, b4) = (1, 4) € R
(a2, b1) = (2, 1) € R
(a2, b3) = (2, 3) € R
(a3, b2) = (3, 2) € R
(a3, b3) = (3, 3) € R
(a3, b4) = (3, 4) € R
(a4, b1) = (4, 1) €R
(a4, b3) = (4, 3) € R
(a4, b4) = (4, 4) € R

Thus, the relation R then contains all previously found tuples:
R = {(1,1), (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)}

B) We are interested in the relations on {1,2, 3,4}. Thus a1 = b1 = 1, a2 = b2 = 2, a3 = b3 = 3, a4 = b4 = 4. Note that only the 1-values in the matrix will represent a tuple that is in the relation. We note that the element m11 is 1, thus (a1, b1) = (1, 1) € R We note that the element m12 is 1, thus (a1, b2) = (1, 2) € R Similarly, we obtain for all 1-values in the given matrix:

(a1, b3) = (1, 3) € R
(a2, b2) = (2, 2) €R
(a3, b3) = (3, 3) € R
(a3, b4) = (3, 4) €R
(a4, b1) = (4, 1) €R
(a4, b4) = (4, 4) €R

Thus, the relation R then contains all previously found tuples:
R = {(1,1), (1, 2), (1,3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)}

C) We are interested in the relations on {1, 2, 3, 4}. Thus a1 = b1 = 1, a2 = b2 = 2, a3 = b3 = 3, a4 = b4 = 4. Note that only the 1-values in the matrix will represent a tuple that is in the relation. We note that the element m12 is 1, thus (a1, b2) = (1, 2) € R We note that the element m14 is 1, thus (a1, b4) = (1, 4) € R Similarly, we obtain for all 1-values in the given matrix:

(a2, b1) = (2, 1) € R
(a2, b3) = (2, 3) € R
(a3, b2) = (3, 2) € R
(a3, b4) = (3, 4) € R
(a4, b1) = (4, 1) € R
(a4, b3) = (4, 3) € R

Thus, the relation R then contains all previously found tuples:
R = {(1,2). (1,4), (2,1), (2, 3), (3, 2), (3,4), (4, 1). (4, 3)}

8. Determine whether the relations represented by the matrices in Exercise 4 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive

Solution:

14. Let $R1$ and $R2$ be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the matrices that represent

a) $R_1 \cup R_2$. b) $R_1 \cap R_2$. c) $R_2 \circ R_1$.
d) $R_1 \circ R_1$. e) $R_1 \oplus R_2$.

Ans:

(a) The matrix corresponding to the union of two relations is the **join** of the matrices representing each of the relations:

$$\begin{aligned} \mathbf{M}_{R_1 \cup R_2} &= \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \\ &= \begin{bmatrix} 0 \vee 0 & 1 \vee 1 & 0 \vee 0 \\ 1 \vee 0 & 1 \vee 1 & 1 \vee 1 \\ 1 \vee 1 & 0 \vee 1 & 0 \vee 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

(b) The matrix corresponding to the intersection of two relations is the **meet** of the matrices representing each of the relations:

$$
\begin{aligned}
\mathbf{M}_{R_1 \cap R_2} &= \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} \\
&= \begin{bmatrix} 0 \wedge 0 & 1 \wedge 1 & 0 \wedge 0 \\ 1 \wedge 0 & 1 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 1 & 0 \wedge 1 & 0 \wedge 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

(c) The matrix corresponding to the composite of two matrices is the Boolean product of two matrices.

$$
\begin{aligned}
\mathbf{M}_{R_2 \circ R_1} &= \mathbf{M}_{R_1} \odot \mathbf{M}_{R_2} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) \\ (1 \wedge 0) \vee (1 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) \\ (1 \wedge 0) \vee (0 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 1) \end{bmatrix} \\
&= \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

(d) The matrix corresponding to the composite of two matrices is the Boolean product of two matrices.

$$
\begin{aligned}
\mathbf{M}_{R_1 \circ R_1} &= \mathbf{M}_{R_1} \odot \mathbf{M}_{R_1} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 0) \\ (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 0) \\ (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 0) \end{bmatrix} \\
&= \begin{bmatrix} 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 1 \vee 1 & 1 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

(e) The matrix corresponding to the union of two relations is the **join** of the matrices representing each of the relations:

$$
\begin{aligned}
\mathbf{M}_{R_1 \oplus R_2} &= \mathbf{M}_{R_1} \oplus \mathbf{M}_{R_2} \\
&= \begin{bmatrix} 0 \oplus 0 & 1 \oplus 1 & 0 \oplus 0 \\ 1 \oplus 0 & 1 \oplus 1 & 1 \oplus 1 \\ 1 \oplus 1 & 0 \oplus 1 & 0 \oplus 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

15. Let R be the relation represented by the matrix

$$
\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
$$

Find the matrices that represent

a) R^2. **b)** R^3. **c)** R^4.

Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find the matrix representing

a) R^2

b) R^3

c) R^4

DEFINITIONS:

The composite $S \circ R$ consists of all ordered pairs (a,c) for which there exists an element b such that $(a,b) \in R$ & $(b,c) \in S$

A relation R can be represented by the matrix $M_R = [m_{ij}]$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

SOLUTION

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

a) The matrix corresponding to the composite of two relations is the Boolean product of two corresponding matrices.

$$M_{R^2} = M_R \odot M_R$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (0\wedge 0)\vee(1\wedge 0)\vee(0\wedge 1) & (0\wedge 1)\vee(1\wedge 0)\vee(0\wedge 1) & (0\wedge 0)\vee(1\wedge 1)\vee(0\wedge 0) \\ (0\wedge 0)\vee(0\wedge 0)\vee(1\wedge 1) & (0\wedge 1)\vee(0\wedge 0)\vee(1\wedge 1) & (0\wedge 0)\vee(0\wedge 1)\vee(1\wedge 0) \\ (1\wedge 0)\vee(1\wedge 0)\vee(0\wedge 1) & (1\wedge 1)\vee(1\wedge 0)\vee(0\wedge 1) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 0\vee 0\vee 0 & 0\vee 0\vee 0 & 0\vee 1\vee 0 \\ 0\vee 0\vee 1 & 0\vee 0\vee 1 & 0\vee 0\vee 0 \\ 0\vee 0\vee 0 & 0\vee 0\vee 0 & 0\vee 1\vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(b) The matrix corresponding to the composite of two relations is the Boolean product of two corresponding matrices.

$$M_{R^3} = M_{R^2} \odot M_R$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (0\wedge 0)\vee(0\wedge 0)\vee(1\wedge 1) & (0\wedge 1)\vee(0\wedge 0)\vee(1\wedge 1) & (0\wedge 0)\vee(0\wedge 1)\vee(1\wedge 0) \\ (1\wedge 0)\vee(1\wedge 0)\vee(0\wedge 1) & (1\wedge 1)\vee(1\wedge 0)\vee(0\wedge 1) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 0) \\ (0\wedge 0)\vee(1\wedge 0)\vee(1\wedge) & (0\wedge 1)\vee(1\wedge 0)\vee(1\wedge) & (0\wedge 0)\vee(1\wedge 1)\vee(1\wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 0\vee 0 \; 1 & 0\vee 0\vee 1 & 0\vee 0\vee 0 \\ 0\vee 0 \; 0 & 1\vee 0\vee 0 & 0\vee 1\vee 0 \\ 0\vee 0 \; 1 & 0\vee 0\vee 1 & 0\vee 1\vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

c) The matrix corresponding to the composite of two relations is the Boolean product of two corresponding matrices.

$$M_{R^3} = M_{R^2} \odot M_R$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1\wedge 0)\vee(1\wedge 0)\vee(0\wedge 1) & (1\wedge 1)\vee(1\wedge 0)\vee(0\wedge 1) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 0) \\ (0\wedge 0)\vee(1\wedge 0)\vee(1\wedge 1) & (0\wedge 1)\vee(1\wedge 0)\vee(1\wedge 1) & (1\wedge 0)\vee(1\wedge 1)\vee(1\wedge 0) \\ (1\wedge 0)\vee(1\wedge 0)\vee(1\wedge 1) & (1\wedge 1)\vee(1\wedge 0)\vee(1\wedge 1) & (1\wedge 0)\vee(1\wedge 1)\vee(1\wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 0\vee 0\vee 0 & 1\vee 0\vee 0 & 0\vee 1\vee 0 \\ 0\vee 0\vee 1 & 0\vee 0\vee 1 & 0\vee 1\vee 0 \\ 0\vee 0\vee 1 & 1\vee 0\vee 1 & 0\vee 1\vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

RESULT:

(a)

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

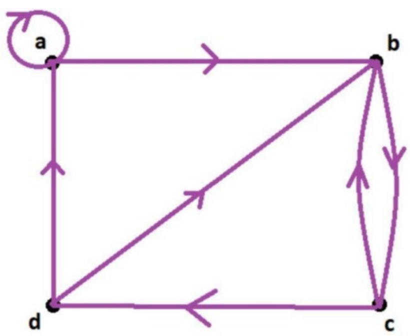
(c)

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

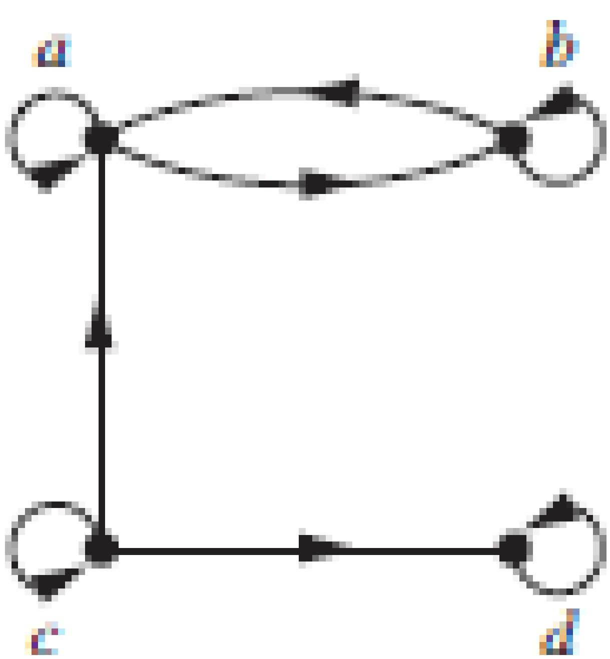
22. Draw the directed graph that represents the relation
$\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$.

<u>**Ans:**</u>

{(a, a), (a, b), (b, c), (c, b), (e, d), (d, a), (d, b)} We note 4 different values in the ordered pairs: a, b, c, d. Draw 4 points for a, b, c, d respectively. (a, a) is contained in the relation, thus we need to draw an arrow from point a to point a (which forms a loop). (a, b) is contained in the relation, thus we need to draw an arrow from point a to point b. (b, c) is contained in the relation, thus we need to draw an arrow from point b to point c. (c, b) is contained in the relation, thus we need to draw an arrow from point e to point b. (c, d) is contained in the relation, thus we need to draw an arrow from point c to point d. (d, a) is contained in the relation, thus we need to draw an arrow from point d to point a. (d, b) is contained in the relation, thus we need to draw an arrow from point d to point b



26.



Answer:

Solution:

1. {(a, c), (a, b), (b, c), (c, b) }
2. {(a, a), (a, b), (b, b), (b, a), (c, c) , (c, a), (c, d), (d, d) }

	reflexive	**irreflexive**	**symmetric**	**antisymmetric**
a	false	**true**	false	false
b	**true**	false	false	false

31. Determine whether the relations represented by the directed graphs shown in Exercises 23-25 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

a)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

c)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

<u>Answer:</u>

Definitions

A relation can be represented by the matrix $M_R = [m_{ij}]$

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$$

A relation R on a set A is reflexive if $(a,a) \in R$ for every element $a \in A$

A relation R on a set A is symmetric if $(b,a) \in R$ whenever $(a,b) \in R$

A relation R on a set A is antisymmetric if $(b,a) \in R$ & $(a,b) \in R$, implies $a=b$

A relation R on a set A is transitive if $(a,b) \in R$ & $(b,c) \in R$, implies $(a,c) \in R$

A relation R on a set A is irreflexive if $(a,a) \notin R$ for every element $a \in A$.

SOLUTION:

(a) Given

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The relation is reflexive, because all elements on the main diagonal are 1

The relation is not irreflexive, because there are some elements on the main diagonal are not 0.

The relation is symmetric, because the matrix is symmetric (about its main diagonal).

The relation is not antisymmetric, because $m_{13} = 1$ & $m_{31} = 1$, which implies $(1,3) \in R$ & $(3,1) \in R$, while $1 \neq 3$

The relation is transitive, because whenever $m_{ij} = 1$, then m_{ik} as well (Note this by calculating the square of the matrix & noticing that the matrix has the same nonzero elements as the original matrix).

(c) Given

$$M=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The relation is not reflexive, because there are some elements on the main diagonal are not 1.

The relation is not irreflexive, because there are some elements on the main diagonal are not 0.

The relation is symmetric, because the matrix is symmetric (about its main diagonal)

The relation is not antisymmetric, because $m_{12}=1$ & $m_{21}=1$, which implies $(1,2)\in R$ & $(2,1)\in R$, while $1\neq 2$

The relation is not transitive, because $m_{21}=1$, $m_{12}=1$ and $m_{22}=0$, which implies $(2,1)\in R$ & $(1,2)\in R$, while $(2,2)\notin R$

RESULT

a) Reflexive, Symmetric, Transitive

b) Antisymmetric, Transitive

(c) symmetric.

32. Determine whether the relations represented by the directed graphs shown in Exercises 26–28 are reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive.

Answer:

For 26. It's reflexive since there's a loop on every vertex. It's not symmetric since there's an arrow from c to d, but there isn't one back. It's not antisymmetric since there are arrows both ways between a and b. Neither is it asymmetric. It's not transitive since $c \longrightarrow a$ and $a \longrightarrow b$, but not $c \longrightarrow b$.

For 27. It's not reflexive since there's no loop at c. It is symmetric since for every arrow, there's an arrow back. It's not transitive since $c \longrightarrow a$ and $a \longrightarrow c$, but not $c \longrightarrow c$.

For 28. It's reflexive, symmetric, and transitive. So, it's an equivalence relation.