概率统计

# Section 3 Probability of events

- 1. The definition and properties of frequency
- 2. Statistical definition of probability
- 3. Classical probability
- 4. Geometric probability
- 5. Axiomatic definition of probability

# 一、The definition and properties of frequency

**1. Definition**

**Experiment n times in the same condition. Event A happens $n_A$ times. Then**

$\frac{n_A}{n}$ **is the frequency of A, denoted by** $f_n(\mathrm{A})$ **.**

## 2. Properties

**If** ***A is any event from E*** **, then**

**(1)** $0 \le f_n(A) \le 1$;

(2) $f(\Omega) = 1$, $f(\varnothing) = 0$;

(3) *if* $A_1, A_2, \cdots, A_k$ **are mutually exclusive, then**

$$f(A_1 \cup A_2 \cup \cdots \cup A_k) = f_n(A_1) + f_n(A_2) + \cdots + f_n(A_k).$$

**e. g.** Toss a coin 5 times、50 times、500 times, and repeat the experiment 7 times. Observe the number and calculate the frequency of event A, which denotes the number of the head appears. ***n close to infinity, f*** **changes very slowly.**

| Experiment | $n=5$ | | $n=50$ | | $n=500$ | |
|---|---|---|---|---|---|---|
| | $n_H$ | $f$ | $n_H$ | $f$ | $n_H$ | $f$ |
| 1 | 2 | 0.4 | 22 | 0.44 | 251 | 0.502 |
| 2 | 3 | 0.6 | 25 | 0.50 | 249 | 0.498 |
| 3 | 1 | 0.2 | 21 | 0.42 | 256 | 0.512 |
| 4 | 5 | 1.0 | 25 | 0.50 | 247 | 0.494 |
| 5 | 1 | 0.2 | 24 | 0.48 | 251 | 0.502 |
| 6 | 2 | 0.4 | 18 | 0.36 | 262 | 0.524 |
| 7 | 4 | 0.8 | 27 | 0.54 | 258 | 0.516 |

**Based on the data, the following laws can be obtained.**

**(1)** $f$ **has volatility(波动性). For different** $n$**,**

***$f$ is different;***

***(2) For the experiment of tossing a coin, if n is small,*** $f$ ***changes drastically*, but given *n is large,* $f$ *is close to a fixed number*, i.e., when** $n$ **is close to infinity,** $f$ ***swings slowly at* 0.5.**

| experimenter | $n$ | $n_H$ | $f$ |
|---|---|---|---|
| De Morgan | 2048 | 1061 | 0.5181 |
| Buffon | 4040 | 2048 | 0.5069 |
| K. Pearson | 12000 | 6019 | 0.5016 |
| K.Pearson | 24000 | 12012 | 0.5005 |

$$f_n(A) \xrightarrow{n \quad large} \frac{1}{2}.$$

## Conclusion

**Given small** ***n, then the f of event A swings large*****.**

**Given large n , then the f is close to a fixed number** $p$ **,**

**where** $0 \le p \le 1$.

**This number shows the possibility of event to happen.**

**It is called probability of the event A, denoted by**

$$Prob(A) = p \qquad or \qquad P(A) = p.$$

**Note.**

**1°** $f_n(A)=\dfrac{n_A}{n}$ **and *P*(A) are different.**

$f_n(A)=\dfrac{n_A}{n}$ **is a random number, which is depended on E;** **_P_(A)is a fixed number!**

**2° *if n is* close to infinity， then** $P(A)\approx f_n(A)=\dfrac{n_A}{n}$

**3° Drawbacks of classical probability**

**(1) Difficult to study, i.e.**

**needing to experiment many times, to check if** $f_n(A)=\dfrac{n_A}{n}$ **is closed to a fixed number of not.**

## Question ?

One doctor tells you the following information: “you got serious sickness, and one of ten can survive.” “But you are lucky, because nine person who are before you are dead because of this sickness . Thus, you can survive”

Is the doctor’s conclusion true?

# 二、Statistical Definition Of Probability

## 1.Definition 1.2

In E, if n is close to infinity, then $f_n(\mathrm{A})$ is close to a fixed number p .

***Here p is called the probability of A,*** denoted by $\boldsymbol{P(A)=p}$ .

## Properties 1.1

**(1) For any** $A$ , $0 \le p(A) \le 1;$

(2) $P(\Omega)=1, P(\varnothing)=0;$

**(3)If for any i, j** $\in N$ **,** $A_i \cap A_j = \varnothing$ **, then**

$$A_1, A_2, \cdots, A_m,$$

$$P(A_1 + A_2 + \cdots + A_m) = P(A_1) + P(A_2) + \cdots P(A_m)$$

**Note: It is difficult to determine the probability of some events.**

# 三、Classical probability

## 1.definition:

**If *E has the following* two properties:**

**1) finiteness**

$\Omega$ only has finite sample points: $\omega_1, \omega_2, \cdots, \omega_n$

$i.e., \quad \Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$

**2) Equal probability**

$$p(\omega_1) = p(\omega_2) = \cdots = p(\omega_n)$$

**Then, E is called classical probability.**

## 2. Determine the classical probability(1.3)

**For *E* , *sample space of E is* $\Omega$ composing by *n* sample points, *A is any event, the cardinality of which* is m, i.e.,** $A \subset \Omega, |A| = m$

**, then:**

$$P(A) = \frac{m}{n} = \frac{|A|}{|\Omega|}.$$

**This is the classical probability of an event .**

**e.g., 1** The number for a company is 200, composed by 160 female.

A: choosing one person from the company, and the person is male. P(A)=?

**Solution:** Number of $\Omega$ : $n = 200$

$A$ =“the person chose is male”

The number of sample points in A (the number of male):

$m = 200\text{-}160\text{=}40$

$$\therefore \quad P(A) = \frac{m}{n} = \frac{40}{200} = \frac{1}{5} = 0.2$$

## 3. Three classical experiments

(1) Taking ball model；

(2) Allocating room model；

(3) Choosing Number model.

## e.g.,5 Taking ball model

**(1) Taking ball without replacement**

**Question1:** **Given a bag containing *M white ball* and *N black balls*, draw out m+n balls without replacement. A: the m+n balls are composed by m white balls, *n black balls*. *P(A)*=?**

**Solution:**

The number of sample point in $\Omega$: $C_{M+N}^{m+n}$

The number of sample point in *A*: $C_M^m \cdot C_N^n$

$$Thus: \quad P(A) = \frac{C_M^m \cdot C_N^n}{C_{M+N}^{m+n}}.$$

(2) Taking ball with replacement

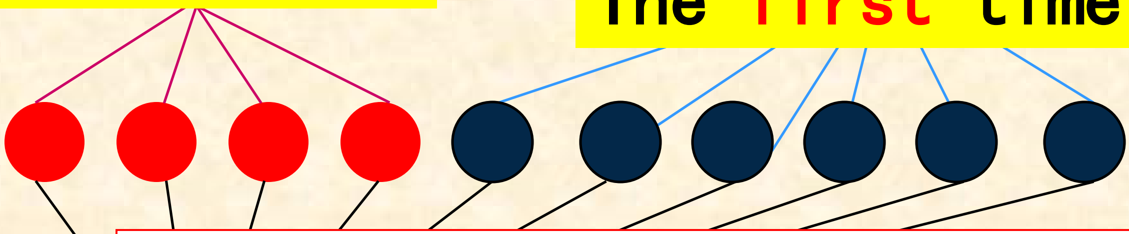
**Question 2:** Given a bag containing 4 red balls and 6 black balls, draw out balls 3 times with replacement. A:the color of the balls took out are black, black and red. P(A)=?

**Solution:**

The second time is black

The third time is red

The first time is black 6



No. of basic event: $10 \times 10 \times 10 = 10^3$

The first time $\longrightarrow$ 10

The third time

The second time

**Solution**:

The no. of sample point in $\Omega$ : $\mathbf{10\times10\times10=10^3}$,

***The no. of sample points in A:*** $\mathbf{6\times6\times4}$,

$$Thus: \quad P(A)=\frac{6\times6\times4}{10^3}=\mathbf{0.144}.$$

# Allocating Room model

***Given the*** **probability for a certain person(total is n person) is allocated to a room(total is N(n≤N) ) is 1/*N*.**

**Question: determine the probability of the following events.**

**(1) The certain *n rooms have only 1 person in each room*；**

**(2) There exist *n* rooms, in which there are 1 person in each room ；**

**(3) One fixed room has *m*(*m* ≤*n*)person.**

**Solution:**

**1 º First, calculate the no. of sample points in Ω.**

**Analysis:**

Allocate *n person into N rooms*. For every allocation,

it corresponds one sample point. Because each person can come into any N rooms, the total no. to allocate one person into one room is N. Since there are n person, there are $N \times N \times \cdots \times N = N^n$ ways to allocate

the n person.

No. of Sample points in $\Omega$ : $N^n$

2 º (1) suppose *A*="The certain n rooms have only 1 person in each room"

No. of Sample points in *A*: $P_n^n = n!$ $\therefore$ $P(A) = \dfrac{n!}{N^n}$

(factorial n)

(2) Suppose $B$="There exist n rooms, in which there are 1 person in each room"

analysis For event $B$, the *n rooms are not chose*, thus, the *n rooms can be chose from N total rooms. The total no. of ways* to choose the rooms is $C_N^n$

*Once the n* person is chose, the no. of ways to allocate the people is $n!$, thus, *B contains* $C_N^n \cdot n!$

sample points.

$$\therefore \quad P(B) = \frac{C_N^n \cdot n!}{N^n}$$

**(3) Suppose $C$ =“One fixed room has m(m $\le$n)person” .**

**analysis“the ways choosing $m(m \le n)$ person from n person and put him in the fixed room is $C_n^m$ ” ,**

***The other n-m person can be put in the remaining N-1 rooms. The No. of ways is*** $(N-1)^{n-m}$

**Thus, the No. of sample points in *C is*:**

$$C_n^m \cdot (N-1)^{n-m}$$

$$\therefore \quad P(C) = \frac{C_n^m \cdot (N-1)^{n-m}}{N^n}$$

# Choosing Number model

choose one no. from 0, 1, 2, $\cdots$;9 with replacement.

Suppose $P(\omega)=\dfrac{1}{10}$ .Choose 7 numbers，

Determine the probability：

(1) 7 number are totally different；

(2) do not have 4 and 7；

(3) 9 appears 2 times；

(4)9 appears at least two times

**Solution:** No. of the sample point in $\Omega$ ： $10^7$

(1) A="7 number are totally different"

The no. of sample points in A:

$$P_{10}^{7} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$$

$$\therefore \quad P(A) = \frac{P_{10}^{7}}{10^7} = \frac{10!}{10^7 \cdot 3!}$$

(2) B="do not have 4 and 7"

$$P(B) = \frac{8^7}{10^7} \approx 0.2097$$

**(3) C="9 appears only 2 times"**

$$P(C) = \frac{C_7^2 \cdot 9^5}{10^7}$$

**(4) D="9 appears at least two times".**

$D_k (k \leq 7) =$ **9 appears only k times**

$$P(D_k) = \frac{C_7^k \cdot 9^{7-k}}{10^7}$$

**(solution1)** $D = D_2 + D_3 + \cdots + D_7$

$$P(D) = P(D_2) + P(D_3) + \cdots + P(D_7)$$

$$\therefore \quad P(D)=\sum_{k=2}^{7}\frac{C_7^k\cdot 9^{7-k}}{10^7}$$

**(solution2)** $\overline{D}=D_0+D_1$

$$P(D)=1-P(\overline{D})$$

$$=1-P(D_0)-P(D_1)$$

$$=1-\frac{9^7}{10^7}-\frac{C_7^1\cdot 9^6}{10^7}\approx 0.1497$$

# 四、Geometric probability

**1. Definition**

If E has the following properties:

1) infinite: **the Ω of the E is a geometry area, including infinite sample points. Every point in the area corresponds one sample points;**

2) Equally likely possibility:

**If** $|A|=|B|$ **, then** $P(\mathrm{A})=\mathrm{P}(\mathrm{B})$**, where** $|\ |$ **is the geometric metric of the event. i.e., in one dimensional space,** $|\ |$ **is the length.**

**Then the model described by E is Geometric probability model.**

**Note.**

| | | | | |
|---|---|---|---|---|
| geometric space | One dimension | two | three | … |
| metric | length | area | volume | … |

## 2. Geometric probability

**Definition.** For E， $m$(A) is the geometric metric of event A， Ω is the sample space. If 0< $m(\Omega)<+\infty$, then

$$P(A)=\frac{m(A)}{m(\Omega)}$$

**Note.**

**One dimension** : $P(A) = \frac{|A|}{|\Omega|};$

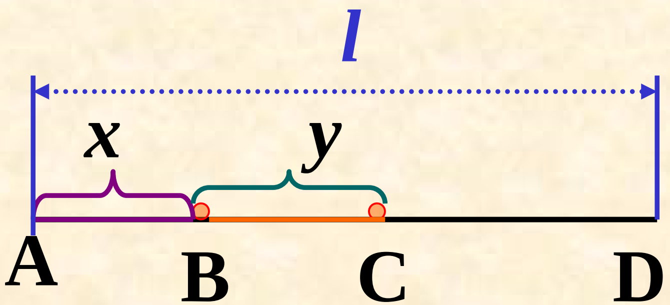
**Two dimension** : $P(A) = \frac{areaA}{area\Omega};$

**Three dimension** : $P(A) = \frac{volumeA}{volume\Omega}.$

**e.g.8** B,C are two points in the line segment AD. Cut AD into three segments AB, BC and CD.

**Question?** What is the probability that these three segment could form a triangle?

**Solution:**

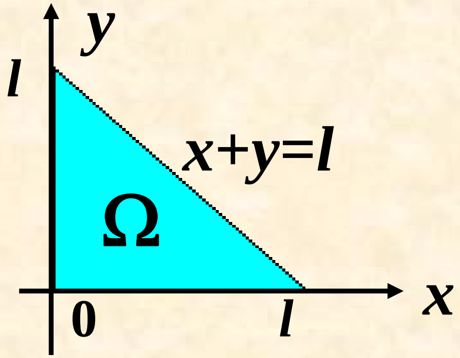



$$\begin{cases} 0 < x < l, 0 < y < l \\ 0 < l - (x + y) < l \end{cases}$$

**Sample space Ω:**

$$0 < x < l, \quad 0 < y < l$$

$$0 < x + y < l$$

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**AB, BC and CD can form a triangle**

$\Leftrightarrow$ **The length of any one segment is smaller than the sum of the length of the other two segments.**

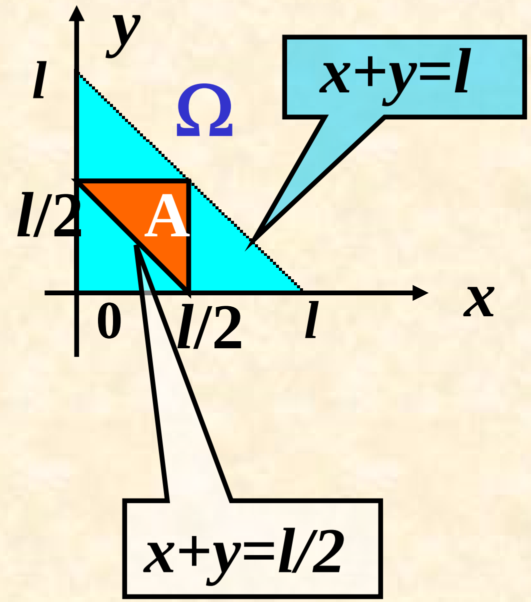
$$\therefore \quad 0 < x < l - x, \ 0 < y < l - y$$

$$and \quad 0 < l - (x + y) < x + y$$

*denote* $A =$ "**AB, BC, CD can from a triangle** "

$$then\ A: 0 < x < \frac{l}{2},\ 0 < y < \frac{l}{2},$$

$$\frac{l}{2} < x + y < l$$




$$\therefore P(A) = \frac{S(A)}{S(\Omega)} = \frac{\frac{1}{2}(\frac{l}{2})^2}{\frac{1}{2}l^2} = \frac{1}{4}$$

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# 五、Axiomatic definition of probability

**1.definition.** **Given Ω is the sample space of E，for** $A \subset \Omega$**,**

**There exists a real number *P*(A) corresponding to A.**

***If P*(A) satisfied the following properties：**

**(1) non negative：for and** $A \subset \Omega$**，** $P(A) \geq 0$ **holds;**

**(2) normalization：** $P(\Omega)=1$**;**

**(3) additivity：**

$if \quad i \neq j, \quad A_i A_j = \varnothing \quad (i, j = 1,2, \cdots)$ **,it holds**

$$P(A_1 + A_2 + \cdots + A_m + \cdots)$$
$$= P(A_1) + P(A_2) + \cdots + P(A_m) + \cdots$$

**Then $P(A)$ is the probability of event $A$.**

## 2. Properties of probability

**(1) $P(\varnothing)=0$**

**Proof:** $\Omega = \Omega + \varnothing + \varnothing + \cdots$

$$P(\Omega)=P(\Omega)+P(\varnothing)+P(\varnothing)+\cdots$$

$$P(\Omega)=1 \qquad \therefore\ P(\varnothing)=0$$

(2) **additivity:**

if $A_1, A_2, \cdots, A_m$ **are mutually exclusive with each other, then**

$$P(\sum_{i=1}^{m} A_i) = \sum_{i=1}^{m} P(A_i)$$

**Proof:** $A_1 + A_2 + \cdots + A_m = A_1 + A_2 + \cdots + A_m + \varnothing +$

$+\varnothing + \varnothing + \cdots$

$$P(A_1 + A_2 + \cdots + A_m)$$

$$= P(A_1 + A_2 + \cdots + A_m + \varnothing + \varnothing + \cdots)$$

$$= P(A_1) + P(A_2) + \cdots + P(A_m) + P(\varnothing) + P(\varnothing) + \cdots$$

$$= P(A_1) + P(A_2) + \cdots + P(A_m)$$

**(3) For any event A,**

$$P(\overline{A}) = 1 - P(A)$$

**Proof:** $\because A + \overline{A} = \Omega,\ A\overline{A} = \varnothing$

$$\therefore P(A) + P(\overline{A}) = P(\Omega) = 1$$

*Thus,* $P(\overline{A}) = 1 - P(A)$

**(4).** $if B \subset A,$ then $P(A-B) = P(A) - P(B)$

**Proof** $\because B \subset A \quad \therefore A = A \cup B = B + (A - B)$

$$\because \quad B(A-B) = BA\overline{B} = \varnothing$$

$$\therefore \quad P(A) = P(B) + P(A - B)$$

*Thus,* $$P(A-B) = P(A) - P(B)$$

**Conclusion1(monotonicity)** *if* $B \subset A$, then $P(B) \le P(A)$

**Proof: By (4)，and $P(A-B) \ge 0$,the conclusion holds.**

**(5) additivity:**

***For any two events A, B*，it holds**

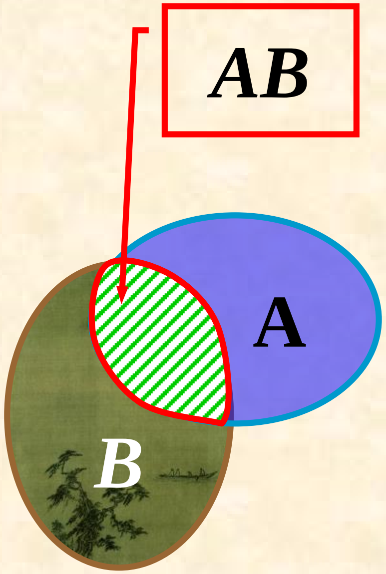
$$P(A \cup B) = P(A) + P(B) - P(AB)$$

$$\textbf{Proof} \because \quad A \cup B = A + (B - A)$$

$$B - AB = B\overline{AB} = B(\overline{A} \cup \overline{B})$$

$$= B\overline{A} \cup B\overline{B} = B\overline{A} \cup \varnothing$$

$$= B\overline{A} = B - A$$




$$\therefore \quad A \cup B = A + (B - AB)$$

$$\because \quad A(B - AB) = AB\overline{A} = \varnothing$$

$$\therefore \quad P(A \cup B) = P(A) + P(B - AB)$$

$$\because AB \subset B$$

$$\therefore \quad P(B - AB) = P(B) - P(AB)$$

$$\textit{Thus,} \quad P(A \cup B) = P(A) + P(B) - P(AB)$$

**Conclusion2.** $P(A \cup B) \leq P(A) + P(B)$

**Generally,** $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$

**Conclusion3.** **For any events,** $A_1, A_2, \cdots, A_n$,

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i A_j) +$$

$$\sum_{1 \leq i < j < k \leq n} P(A_i A_j A_k) + \cdots + (-1)^{n+1} P(A_1 A_2 \cdots A_n)$$

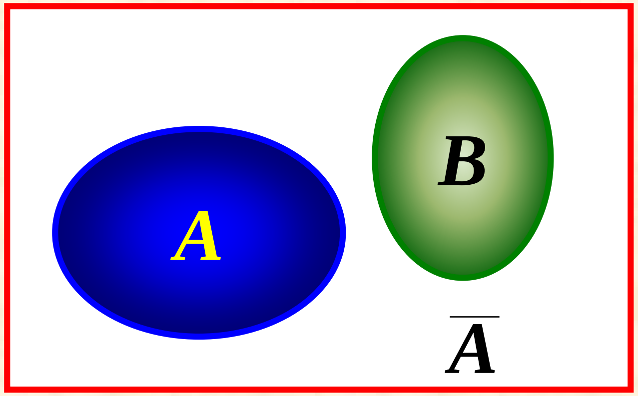
**E.g.10** **If** $P(A)=\frac{1}{3}, P(B)=\frac{1}{2},$

$$P(B\bar{A})=?$$

**(1) If A and B are M.E.**

**(2)** $\boldsymbol{A \subset B}$;

**(3)** $\boldsymbol{P(AB)=\frac{1}{8}}$.




**Solution (1)** $\boldsymbol{AB=\varnothing \ , B \subset \bar{A} \quad \therefore B\bar{A}=B}$

$$Thus, \quad P(B\bar{A})=P(B)=\frac{1}{2}$$

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(2) $A \subset B$;

$$P(B\overline{A}) = P(B - A)$$

$$= P(B) - P(A) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

(3) $P(AB) = \frac{1}{8}$.

$\because B\overline{A} = B - A = B - AB \quad AB \subset B$

$$\therefore \quad P(B\overline{A}) = P(B - AB) = P(B) - P(AB)$$

$$= \frac{1}{2} - \frac{1}{8} = \frac{3}{8}.$$