FREQUENCY FILTRATION OF MEDICAL IMAGES IN MATLAB

Frequency filtering of medical images in MATLAB

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Digital Image Processing

The guidelines set out the basic methods for processing biomedical images in the frequency domain using the capabilities of the mathematical software package MATLAB. Application of standard methods of frequency filtering of initial biomedical images is considered. The order of the work and the requirements for the report are given.

CONTENT

GENERAL INFORMATION

Objective: to study methods of frequency filtering of biomedical images using the capabilities of the MATLAB package of computer calculations.

THEORETICAL BASES OF WORK

➢ 1.1 Introduction to Fourier Analysis

Frequency-based image enhancement methods are based on performing the Fourier transform on a function of two variables — a discrete image function. The direct discrete Fourier transform of the function f (x, y) of an image of size M × N is given by the equality:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-i2\pi(\frac{ux}{M}+\frac{vy}{N})},$$

Where : u = 0, 1, 2,…, M – 1; v = 0, 1, 2,…, N – 1.

The inverse Fourier transform is determined by the expression:

$$f(x,y) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(u,v)e^{i2\pi(\frac{ux}{M}+\frac{vy}{N})}.$$

The variables u and v are called transform variables or frequency variables, and the x and y variables are called spatial variables or image variables. As a rule, the numbers M and N are even to simplify the computer implementation, and the center of the Fourier transform is at the point with coordinates: $u = (M/2) + 1$, $v = (N/2) + 1$. Fourier transform value at a point $(u, v) = (0, 0)$ is equal:

$$F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \, .$$

Thus, if f (x, y) is the image, then the Fourier value is the transformation at the origin equal to the average brightness in the image. The values of F (0, 0) are often called the constant component of the spectrum.

Due to the fact that the function f (x, y) is real, the spectrum of the Fourier transform of the image has the property of symmetry. The following relationships between the samples in the spatial and frequency domains are valid:

$$\Delta u = \frac{1}{M\Delta x}, \quad \Delta v = \frac{1}{N\Delta y}.$$

The Fourier spectrum of the image consists of pixels having a large dynamic range of brightness. The image reproduction system, as a rule, is not able to correctly display such a large range of intensity values, which leads to the fact that in the usual display of the Fourier spectrum a significant number of details are lost. In this regard, in order to improve the visual perception of halftones, the image of the spectrum is subjected to a logarithmic transformation.

On the img.1 The image is a white rectangle with dimensions of 20 × 40 pixels superimposed on a black background with dimensions of 512 × 512 pixels, and the centered Fourier spectrum of the image is also presented. In order to center the spectrum, it is necessary to multiply the original image by (–1) x + y before calculating the Fourier transform.

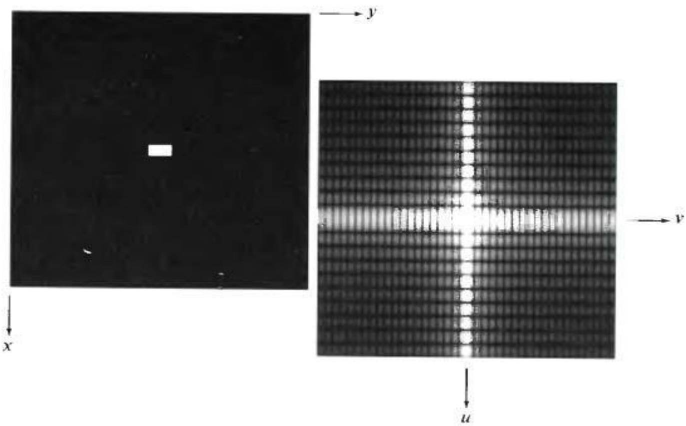

Fig.1 On the left is an image of white rectangle with dimensions of 20 × 40 pixels ,on a black background with the dimension of 512 × 512 pixels, on the right is centered Fourier spectrum, presented after applying the logarithmic transformation.

1.2 BASICS OF FREQUENCY FILTERING

The frequency domain of a digital image is nothing but the space in which the variables (u, v) of the Fourier transform take values. As you know, the frequency of a signal is directly related to the rate of change of the signal, so it is intuitively clear that the frequencies in the Fourier transform are associated with variations in brightness in the image. The most slowly changing (constant) frequency component (u = 0, v = 0) coincides with the average image brightness. The low frequencies corresponding to the points near the origin of the Fourier transform determine the slowly changing image components. As you move away from the origin, higher frequencies begin to correspond to more and more rapid changes in brightness, which are the boundaries of objects and other image details characterized by sharp changes in brightness, for example, noise in the image.

The image filtering procedure in the frequency domain consists of the following steps:

1. The original image is multiplied by (–1) x + y so that its Fourier transform is centered.

2. The direct discrete Fourier transform (DFT) is calculated F (u, v) of the original image.

3. The function F (u, v) is multiplied by the filter function H (u, v).

4. The inverse DFT from the result of step 3 is calculated

5. The real part of the result of step 4 is highlighted.

6. The result of step 5 is multiplied by (–1) x + y.

The filter function H (u, v) or the transfer characteristic of the filter suppresses some conversion frequencies, while leaving others unchanged. In fig. 2 is a structural diagram including the main stages of image filtering in the frequency domain. At the pre-processing stage, in addition to multiplying the image by (-1) x + y, luminance scaling operations, normalizing the size of the original image, converting the input data format to a floating-point format, and a number of others can be applied.

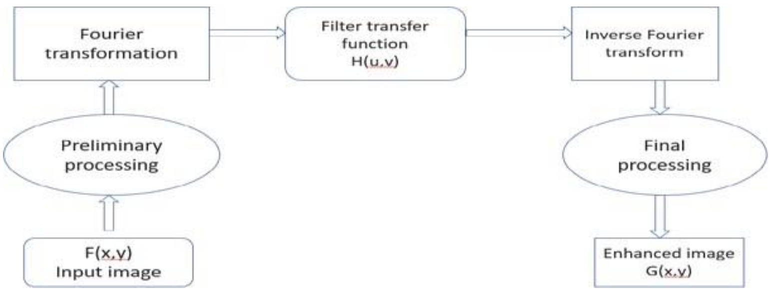

Fig. 2. The main stages of image filtering in the frequency domain

➢ 1.3 Basic frequency filters and their properties

The main types of filters used are: low-pass filter, attenuating high frequencies, while skipping low; high-pass filter with opposite properties. Low Fourier transform frequencies are responsible for the appearance of prevailing brightness values on smooth parts of the image, while high frequencies are mainly responsible for contours and noise. After applying low-pass filtering, the image contains fewer sharp details than the original image. After applying high-pass filtering, brightness changes within large smooth areas are reduced in the image, and transition zones of rapid brightness changes, that is, image contours, are highlighted. As a rule, such an image is sharper than the original.

Due to the fact that high-frequency filters almost completely suppress the constant component F (0, 0), which determines the average brightness of the image, the output image after processing with such a filter it looks very dark, to eliminate this drawback, a constant equal to half the height of the filter is added to the transfer function of the filter.

Another type of frequency filter is a filter plug, or notch filter that cuts out a specific brightness value from an image. As a rule, this brightness value at the origin is the average brightness of the image. The average value of the image brightness cannot be strictly equal to zero, since for this some elements of the image must contain negative values, and the means for displaying graphic information cannot operate with negative values of brightness. To eliminate this contradiction, the smallest negative value is equal to zero (black level), and the remaining values are proportionally increased.

➢ 1.4 Smoothing frequency filters

As noted earlier, contours and other sharp changes in brightness in the image, including noise, make a significant contribution to the high-frequency components of its Fourier transform. Image smoothing in the frequency domain is achieved by attenuating the high-frequency components of a certain Fourier range of the image.

The model of filtering the image in the frequency domain in a generalized form can be described by the following equality:

$$G(u, v)=H(u, v)\cdot F(u, v),$$

where F (u, v) is the Fourier transform of the image to be filtered, H (u, v) is the transfer function of the filter, which weakens the high-frequency components F (u, v) and forms the function G (u, v)

Consider three types of low-pass filters: an ideal filter, a Butterworth filter, and a Gaussian filter. These filters by the type of transfer function cover the range from very sharp filters (ideal filter) to very smooth filters (Gaussian filter).

The Butterworth filter is the only one considered, which is characterized by the order of the filter - a parameter that determines the slope of the transfer function of the filter. For small filter order values, the transfer function has a smooth shape that is close in shape to the transfer characteristic of a Gaussian filter; at high values, the filter is characterized by a steep transfer function and is close in shape to the characteristics of an ideal filter.

➢ 1.4.1 PERFECT LOW PASS FILTERS

An ideal low-pass filter is such a low-pass filter that cuts off all the high-frequency components of the Fourier transform located at a greater distance from the origin of the centered image than some given distance D_0. This type of filter is also called a two-dimensional ideal low-pass filter and has a transfer function:

$$H(u,v) = \begin{cases} 1, \text{at } D(u,v) \leq D_0 \\ 0, \text{at} D(u,v) > D_0 \end{cases},$$

where D_0 – is the given non-negative quantity, the point of the filter profile, at which the transition from the values of H (u, v) = 1 to the values of H (u, v) = 0, called the cutoff frequency, is made; D (u, v) is the distance from the point (u, v) to the origin - the center of the frequency rectangle. Due to the

fact that the Fourier transform is centered, the distance D (u, v) is determined by the formula:

$D(u,v) = [(u - m/2)^2 + (v - n/2)^2]^{0,5}$, where m и n –source image size.

In fig. 3 shows an ideal low-pass filter in the form of isofermentations, as well as the radial profile of the filter:

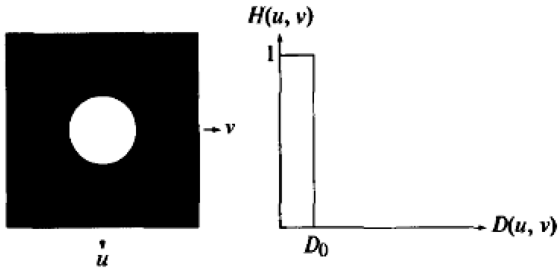

Fig. 3. On the left is a representation of an ideal low-pass filter as an image; on the right is a radial filter profile

The ideality of the filter is emphasized by the fact that all frequencies inside a circle of radius D_0 pass unchanged, while frequencies outside the circle are completely suppressed. An ideal low-pass filter has the property of symmetry with respect to the origin, so in order to uniquely define a filter, it is enough to construct one radial profile - a function of the distance from the origin. The transfer function of the filter in the coordinates H (u, v) is obtained by rotating the profile by 3600 around the origin. One way to select a reference set of cutoff frequency positions is to determine the circles in which a given portion of the total image energy PT is enclosed. The total energy will be defined as the sum of the components of the energy spectrum at all points (u, v); $u = 0, 1, 1 \ldots M - 1$; $v = 0, 1, 1 \ldots N - 1$.

$$P_T = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} P(u, v) ,$$

where the value $P(u, v)$defined as :

$$P(u, v) = |F(u, v)|^2,$$

where $F(u, v)$ –Fourier transform of the original $f(x, y)$.

The frequency r (α) is defined as the radius of the circle centered in the center of the frequency rectangle containing α percent of the energy of the image spectrum, i.e.:

$$\alpha = 100\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\frac{P(u,v)}{P_T}.$$

α

Choosing a sub-optimal filter cut-off frequency can lead to significant distortion of the processed image: selecting a cut-off frequency too low will eliminate most of the image details. While the selection of a relatively large value of the cutoff frequency will lead to the fact that the output image will slightly differ from the original, the noise level will also not change.

When using an ideal low-pass filter, undesirable effects occur, for example, blurring effects and the appearance of false contours, while reducing the width of the used filter in the frequency domain, the blurring effects are enhanced.

The resulting undesirable effects when using an ideal low-pass filter are called “ringing” (or the Gibbs effect), which is expressed in the appearance of false contours around real ones. The structure of the false loops becomes thinner as the energy of the cut-off high-frequency component decreases.

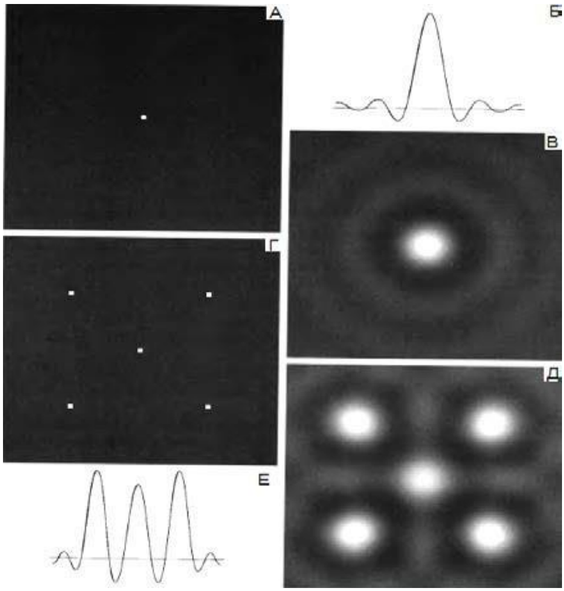

Fig. 4. A is the image of an ideal low-pass filter, Б is the brightness profile in the spatial domain, B is the image of the ideal low-pass filter in the spatial domain, Г is a conditional image consisting of 5 bright points in the spatial domain, Д is the convolution of images B and Д, E - brightness profile filtered image Д

The nature of the appearance of false contours can be explained using the convolution theorem, according to which there is a spatial analogue of an ideal low-pass filter. The spatial function of such a filter can be determined using the inverse Fourier transform of the transfer function of the low-pass filter and is a sequence of circular concentric rings of different brightness, leading mainly to the appearance of false contours:

Fig. 4 illustrates the manifestation of the Gibbs effect using an ideal low-pass filter.

By the concept of brightness profile is meant the dependence of the change in brightness of pixels on the number of pixels located on a horizontal line, usually passing through the center of the image; pixel counting in a straight line starts from the leftmost pixel and ends with the rightmost pixel.

Thus, the presented filtering results using an ideal low-pass filter show that ideal low-pass filters do not have much practical application.

- ### *1.4.2BUTTERWORTH LOW PASS FILTERING*

Butterworth low pass filter transfer function

n with a cutoff frequency at a distance D0 from the origin is given by the formula:

$$H(u, v) = \frac{1}{1 + [D(u, v) / D_0]^{2n}}.$$

The radial profiles of the transfer function of the Butterworth low-pass filter, depending on the order of the filter, are shown in Fig. five.

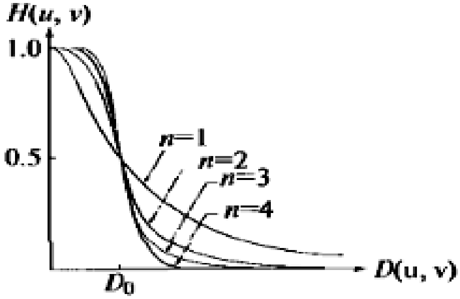

Fig. 5. Radial profiles of the transfer function of the low-frequency Butterworth filter depending on the order of the filter

Unlike an ideal low-pass filter, the transfer function of the Butterworth low-pass filter does not have a gap that sets the exact boundary between the transmitted and cut-off frequencies. The advantages of Butterworth low-pass filters include a much smaller manifestation of undesirable blurring effects and the appearance of false loops compared to ideal low-pass filters. With an increase in the order of the low-frequency Butterworth filter, the manifestation of blur effects increases. It is generally accepted that the second-order Butterworth low-pass filter is optimal from the point of view of a compromise between the low-pass filtering efficiency and the acceptable level of manifestation of false contours and the general blurring of the image.

In fig. Figure 6 shows the brightness profiles of images of low-frequency Butterworth filters with the same cutoff frequency of the order of 1, 2, 5, and 20, respectively.

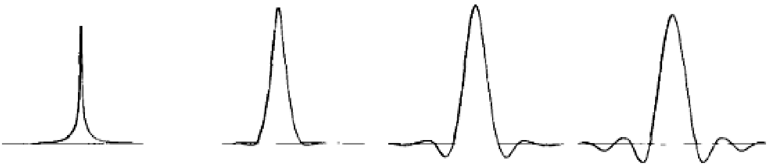

Fig. 6. From left to right: brightness profiles of low-frequency Butterworth filters with the same cutoff frequency of the order of 1, 2, 5, and 20, respectively

➢ 1.4.3 GAUSSIAN LOW-PASS FILTERS

In the two-dimensional case, Gaussian low-pass filters are given by the formula:

$$H(u,v) = e^{-\frac{D^2(u,v)}{2\sigma^2}}.$$

When D (u, v) = D_0 ($D_0$is the filter cutoff frequency), the filter transfer function is 0.667 of its maximum value. In fig. Figure 7 shows the radial profiles of a Gaussian filter for various values of D_0.

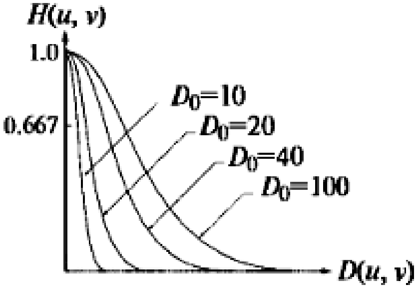

Fig. 7. Radial profiles of the transfer function of the low-frequency Gaussian filter at various values of the cutoff frequency D_0

As is known, the inverse Fourier transform of a Gaussian function is also a Gaussian function. The spatial Gaussian low-pass filter obtained using the inverse Fourier transform of the transfer function of the frequency Gaussian filter will be positive and will not have concentric rings, which will lead to the complete absence of manifestations of the Gibbs effect in the processed image, which is the main advantage of the Gaussian low-pass filters compared to Butterworth filters. Nevertheless, the use of a low-frequency Gaussian filter often provides less smoothing at the same cutoff frequency than Butterworth filters, therefore, in those cases where tight control of the transition zone from low frequencies to high frequencies is required, Butterworth filters seem to be a more suitable choice. In the tasks of processing biomedical images, the absence of false contours and the inadmissibility of the manifestation of artifacts of any kind is preferable to use Gaussian filters.

➢ 1.5 Frequency sharpening filters

Contours and other abrupt changes in brightness in the image are associated with high-frequency components of the Fourier transform of the image. The sharpening of the image can be achieved in the frequency domain using the high-pass filtering procedure, which suppresses the low-frequency components and does not affect the high-frequency part of the Fourier transform.

The transfer function of high-pass filters can be obtained using the following relationship:

$H_{hp}(u, v) = 1 - H_{lp}(u, v)$,

Where $H_{lp}(u, v)$ – transfer function of the corresponding low-pass filter.

Consider the following types of high-pass filters: ideal high-pass filters, Butterworth high-pass filters and Gaussian high-pass filters.

➢ 1.5.1 Ideal high pass filters

Two-dimensional ideal high-pass filters are determined by the formula:

$$H(u,v) = \begin{cases} 0, \text{при } D(u,v) \le D_0 \\ 1, \text{при } D(u,v) > D_0 \end{cases},$$

where D_0 –cutoff frequency, $D(u, v)$ –distance from point with the coordinates(u, v) to the center of the frequency rectangle (origin).

In fig. Figure 8 shows the radial profile of the transfer function of the ideal high-pass filter, the function of the ideal high-pass filter in the spatial domain, and the corresponding brightness profile.

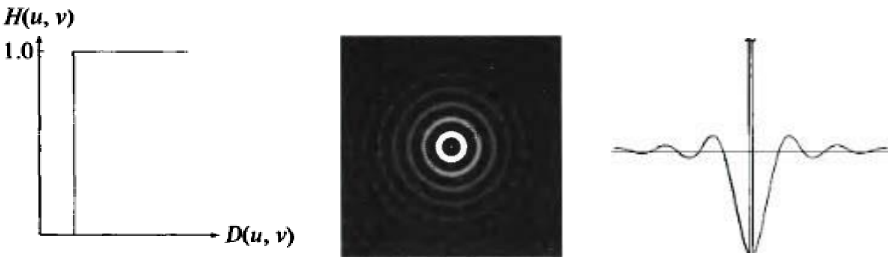

Fig. 8. From left to right: radial profile of the transfer function of an ideal high-pass filter; perfect high pass filter function in the spatial domain; corresponding brightness profile

The ideal high-pass filters, as well as the ideal low-pass filters, are characterized by the Gibbs effect, which leads to the appearance of “ringing” in the processed image.

By analogy with ideal low-pass filters, ideal high-pass filters have almost no real practical application

1.5.2 Butterworth High Pass Filters

The transfer function of the Butterworth high-pass filter of order n with a cutoff frequency at a distance D_0 from the origin is determined by the expression:

$$H(u, v) = \frac{1}{1 + [D_0 / D(u, v)]^{2n}}.$$

In fig. Figure 9 shows the radial profile of the transfer function of the second-order high-frequency Butterworth filter, the function of the high-frequency Butterworth filter in the spatial domain, as well as the brightness profile of the spatial filter function.

High-frequency Butterworth filters result in much less distortion of the boundaries of objects than ideal high-frequency Butterworth filters. With an increase in the order of the Butterworth high-pass filter, the distortions of the boundaries of objects noticeably increase.

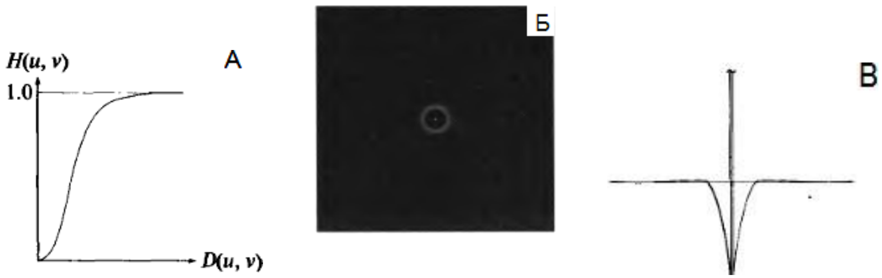

Fig. 9. A is the radial profile of the transfer function of the high-frequency Butterworth filter of the second order, Б is the function of the high-frequency Butterworth filter in the spatial domain, В is the brightness profile of the spatial filter function

1.5.3 Gaussian High Pass Filters

The transfer function of a Gaussian high-pass filter with a cutoff frequency located at a distance D_0 from the origin is specified by the expression:

$$H(u,v) = 1 - e^{-\frac{D^2(u,v)}{2D_0^2}}$$

Figure 10 shows the radial profile of the transfer function of the Gaussian high-pass filter, the function of the high-frequency Gaussian filter in the spatial domain, and the brightness profile of the spatial function of the Gaussian filter. The high-frequency Gaussian filter provides high quality filtering even for small objects and thin bands, characterized by a complete absence of distortion.

Fig. 10. A is the radial profile of the transfer function of the high-frequency Gaussian filter, B is the function of the high-frequency Gaussian filter in the spatial domain, B is the brightness profile of the spatial function of the filter

1.6 Unsharp masking. High pass filtering with

INCREASING FREQUENCY RESPONSE

Images obtained as a result of high-pass filtering have an average background brightness value close to zero, since the high-frequency filters used to process image data destroy the constant component (zero component) of their Fourier transform. To eliminate this drawback, you can add to the filtering result a certain fraction of the original image as called “image underlay”. In some cases, it seems necessary to increase the contribution made by the original image to the final filtering result. This approach is called the high-pass filtering method with an increase in the frequency response and is a generalization of the unsharp masking method.

Unsharp masking consists in forming a sharp image by subtracting from the original its smoothed copy. In terms of frequency transformations, this means that there is high-pass filtering, which is achieved by subtracting from the original image f (x, y) the result of its low-pass filtering $f_{lp}(x, y)$:

$$f_{hp}(x, y) = f(x, y) - f_{lp}(x, y).$$

A generalization of the last expression is the following expression describing high-pass filtering with an increase in the frequency response:

$$f_{hb}(x, y) = A \cdot f(x, y) - f_{lp}(x, y).$$

Filtering with an increase in the frequency response allows you to change the contribution made by the original image to the final processing result.

➢ 1.7 High Pass Filtering

When solving some problems of image processing, it is advantageous to enhance its high-frequency component. The transfer function of the filter with high-frequency amplification is given by:

$H_{hfe}(u, v) = a + b \cdot H_{hp}(u, v)$,

Where $H_{lp}(u, v)$ – transfer function of the low-pass filter.

Typical values of a are in the range from 0.25 to 0.5; characteristic values of b are in the range from 1.5 to 2.5 (it is important to note that b> a). For values of b> 1, high frequencies are amplified, hence the name of the method.

The main field of application of filtering with amplification of high frequencies in the tasks of processing biomedical images is to sharpen radiographic images. X-rays cannot be focused with lenses like light rays, so most x-rays look blurry. Often, the brightness of the X-ray images is shifted to the dark region, so it is also necessary to use image processing methods to increase the brightness and contrast of the image.

➢ 1.8 Implementation of frequency filtering methods in the system of computer computing MATLAB

To perform a direct Fourier, transform in the MATLAB package, the fft2function is used, which has the following syntax:

F = fft2(f)

where **f** –source image, **F** – Fourier transform of a two-dimensional function f(x,y)

To obtain the spectrum of the image, it is necessary to execute the following command:

S = abs(F)

The command **abs** calculate the absolute value (module) of a complex function **F**.

To shift the origin of the Fourier, transform to the center of the frequency domain, use the fftshift function with the following syntax:

Fc = fftshift(F)

where **Fc** – centered Fourier transform.

Due to the fact that the dynamic range of the Fourier transform image is very wide, it is necessary to perform the operations of converting the brightness range to correctly display the Fourier transform on the display. This operation can be performed using a logarithmic transformation as follows:

S2 = log(1 + abs(Fc))

The inverse Fourier transform is performed using the function ifft2, which has the following syntax:

F = ifft2(F)

where **F** – Fourier transform, **f** – corresponding image.

After performing the inverse Fourier transform, it is necessary to select the real part of the result using the command of the following form:

F = real(ifft2(F))

MATLAB has special functions for creating matrices and vectors. The simplest way to specify a one-dimensional array (vector) is to write a command of the following form:

[name] = X1:dX:Xk

where **name** – the name of the variable, which will be assigned to the generated array, **X1** – value of the first element of the array, **Xk** – the value of the last element, **dX** – step, with the help of which each next element is formed (by default it is 1). This entry forms a row vector. In order to form a column vector, you must use the following entry:

[name] = [X1:dX:Xk]'

Where the symbol' means the transpose of the matrix.

When solving the problems of digital image processing, it is necessary to form two-dimensional arrays of the coordinate grid. For this purpose, it is convenient to use the meshgrid function in the MATLAB system, which generates output arrays that determine the coordinates of the nodes of the rectangle specified by the input vectors. The formed rectangle defines the domain of the function of two variables. The meshgrid function has the following syntax :

[X, Y] = meshgrid(*x*, *y*)

The rows of the matrix X are copies of the vector x, and the columns of the matrix Y are copies of the vector y. The formation of such arrays simplifies the calculation of functions of two variables, allowing you to apply operations on arrays.

Example: let $x = (0, 1, 2)$ $y = (0, 1)$. Then

$$X = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

The formation of such arrays simplifies the calculation of functions of two variables, allowing the use of operations on arrays.

Thus, to implement the frequency filtering of the original image f (x, y), it is necessary to perform the following sequence of operations:

1) Determine the size of the image f (x, y) using the size command; generate vectors of frequency coordinates u and v, the sizes of which are equal to the sizes of the vectors of the corresponding spatial coordinates x and y;

2) Perform a direct Fourier transform of the original image using the fft2 command and obtain the frequency representation F (u, v);

3) Set the value of the cutoff frequency in pixels (as a rule, determined by units of percent of the maximum horizontal or vertical image size); generate the transfer characteristic of the required filter H (u, v), which is a two-dimensional matrix, the dimensions of which coincide with the dimensions of the matrix of the original image;

4) Obtain the frequency representation of the filtered image G (u, v) by performing the elementwise multiplication of the transfer characteristic of the filter H (u, v) and the Fourier transform of the original image F (u, v);

5) Obtain a spatial representation of the filtered image g (x, y) by performing the inverse Fourier transform procedure of the frequency representation G (u,

v) using the ifft2 command;

6) Select the real part of the result obtained at the previous stage using the real command.

➢ 2 PROCEDURE FOR PERFORMANCE OF WORK

1. Upload chest X-ray image **xraychest.jpg**. Perform Fourier transform of the original image, get the spectrum image in the frequency domain. To improve visual perception use graduation logarithmic transformation.

2. Apply a high pass filtering technique with a raising the frequency response of the original chest x-ray image **xraychest.jpg**.

2.1. Perform low-pass filtering of the original chest X-ray image **xraychest.jpg**.

2.2. By selecting a parameter A, achieve the best result sharpening by raising the frequency response

2.3 Image obtained as a result of step 2.2, subject to the histogram equalization procedure to get MATLAB system help to perform this procedure use the command **doc histeq**.

3. Apply high-pass filtering techniques to the original chest x-ray image xraychest.jpg to sharpen the image.

3.1. Perform high-pass filtering of the original chest x-ray image xraychest.jpg.

3.2. By selecting parameters, a and b, achieve high frequency amplification.

3.3. The image obtained as a result of step 3.2 is subjected to the histogram equalization procedure.

➢ 2 CONTENTS OF THE REPORT

1. Purpose of the work
2. Listings of written programs (M-files) in MATLAB for each task.
3. Conclusion about the results, comparison with the theory.

➢ 3 CHECK-UP QUESTIONS

1. Two -dimensional Fourier transform of images.
2. The basics of image frequency filtering.
3. Smoothing frequency filtering.
4. Frequency sharpening filters.
5. Unsharp masking high-pass filtering with rising frequency characteristics.
6. High pass filtering.

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