Fundamentals of Electric Circuit 2020.4

Chapter 9
Sinusoids and Phasors

Chapter 9 Sinusoids and Phasors

9.1 introduction

AC Analysis

DC Analysis: voltage and current are constant with respect to time.

AC Analysis: voltage and current vary with time.

AC can be sinusoidal, square waves, or arbitrary periodic waveforms.

Sinusoidal is particularly important

- ✓ Commonly used, e.g., power systems, communications, etc.
- ✓ Simple periodic function (e.g., derivative and anti-derivative of a sinusoidal is also a sinusoidal)
- ✓ Any periodic function can be represented as the sum of sinusoidal function

=> Fourier Series

- **We now begin the analysis of circuits in which the source voltage or current is sinusoid.**

> A sinusoid is a signal that has the form of the sine or cosine function.

- **Circuits driven by sinusoidal current or voltage sources are called *ac circuits*.**

9.2 Sinusoids

$$v(t) = V_m \sin \omega t$$

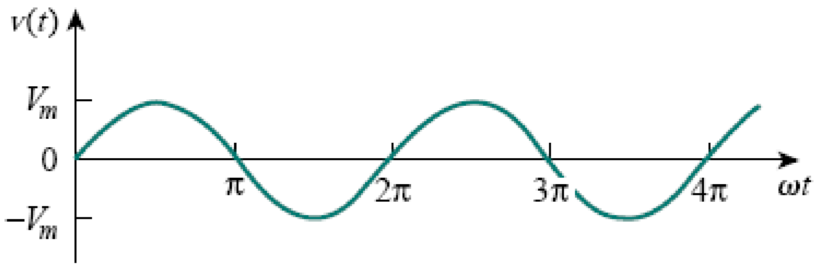
$$\boxed{T = \frac{2\pi}{\omega}}$$

V_m = the *amplitude* of the sinusoid

ω = the *angular frequency* in radians/s

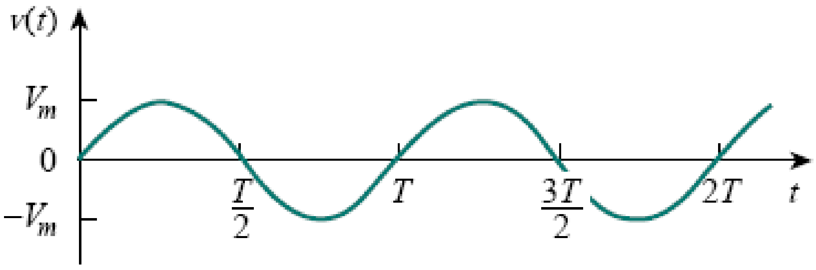
ωt = the *argument* of the sinusoid

$$\omega T = 2\pi$$



(a)

as a function of ωt



(b)

as a function of t

It is evident that the sinusoid repeats itself every ***T*** seconds; thus, *T* is called the ***period*** of the sinusoid.

period

$$v(t+T) = V_m \sin\omega(t+T) = V_m \sin\omega\left(t+\frac{2\pi}{\omega}\right)$$

$$= V_m \sin(\omega t + 2\pi) = V_m \sin\omega t = v(t)$$

$$\boxed{T = \frac{2\pi}{\omega}}$$

$$\boxed{v(t+T) = v(t)}$$

A periodic function is one that satisfies $f(t) = f(t + nT)$, for all t and for all integers n.

The reciprocal of this quantity is the number of cycles per second, known as the *cyclic frequency f* of the sinusoid.

$$\boxed{T = \frac{2\pi}{\omega}}$$

$$\boxed{f = \frac{1}{T}}$$

$$\omega = 2\pi f$$

Let us now consider a more general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \varphi)$$

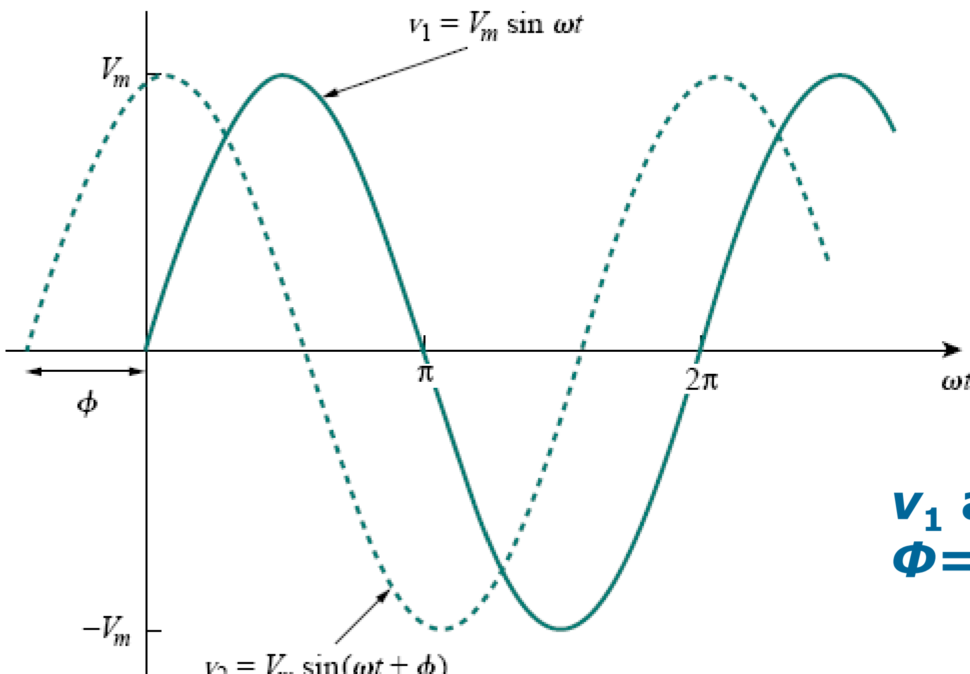
Three factors: V_m = the *amplitude* of the sinusoid

ω = the *angular frequency* in radians/s

φ is the *phase*.

Let us examine the two sinusoids

$$v_1(t) = V_m \sin\omega t \quad \text{and} \quad v_2(t) = V_m \sin(\omega t + \varphi)$$



v_2 leads v_1 by φ

v_1 lags v_2 by φ

v_1 and v_2 are in phase: $\Phi=0$

A sinusoid can be expressed in either *sine* or *cosine* form.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t$$

Using these relationships, we can transform a sinusoid from *sine* form to *cosine* form or vice versa.

EXAMPLE 9.1

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10^\circ)$$

Solution:

The amplitude is $V_m = 12$ V.

The phase is $\phi = 10^\circ$.

The angular frequency is $\omega = 50$ rad/s.

The period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{50} = 0.1257$ s.

The frequency is $f = \dfrac{1}{T} = 7.958$ Hz.

PRACTICE PROBLEM 9.1

Given the sinusoid $5\sin(4\pi t - 60^\circ)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Answer: 5, -60°, 12.57 rad/s, 0.5 s, 2 Hz.

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \cdots\cdots(1)$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \cdots\cdots(2)$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \cdots\cdots(3)$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \cdots\cdots(4)$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$$

9.3 Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than *sine* and *cosine* functions.

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

A complex number *z* can be written in rectangular form as

$$z = x + jy$$

$j = \sqrt{-1}$ **x is the real part of z ; y is the imaginary part of z.**

The complex number *z* can also be written in polar or exponential form as

$$z = r\angle\phi = re^{j\phi}$$

r is the magnitude of z

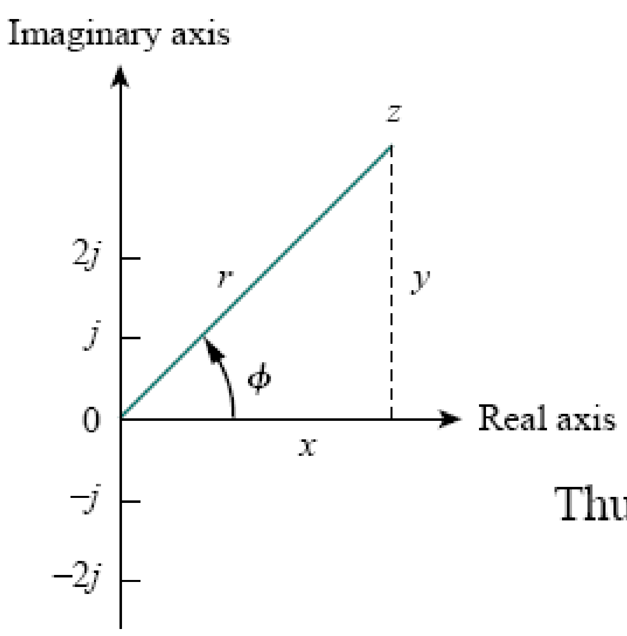
φ is the phase of z

Z can be represented in three ways:

$z = x + jy$ Rectangular form

$z = r\angle\phi$ Polar form

$z = re^{j\phi}$ Exponential form



$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi, \qquad y = r\sin\phi$$

Thus, z may be written as

$$\boxed{z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)}$$

The idea of phasor representation is based on Euler's identity.

$$\boxed{e^{\pm j\phi} = \cos\phi \pm j\sin\phi}$$

which shows that we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$; we may write

$$\cos\phi = \mathrm{Re}(e^{j\phi})$$

$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

where Re and Im stand for the *real part of* and the *imaginary part of*.

Given a sinusoid ***$v(t)=V_m\cos(\omega t + \varphi)$***

$$v(t) = V_m\cos(\omega t + \phi) = \mathrm{Re}(V_m e^{j(\omega t+\phi)}) \quad \text{or} \quad v(t) = \mathrm{Re}(V_m e^{j\phi} e^{j\omega t})$$

$$\boxed{v(t) = \mathrm{Re}(\mathbf{V} e^{j\omega t})} \qquad \mathbf{V} = V_m e^{j\phi} = V_m \underline{/\phi}$$

V is thus the *phasor representation* of the sinusoid *v(t)*

A phasor is a complex representation of the magnitude and phase of a sinusoid.

By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$\boxed{v(t) = \mathrm{Re}(\mathbf{V}e^{j\omega t})}$$

$$v(t) = V_m \cos(\omega t + \phi) \quad \Longleftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)

Note that in Eq. (9.25) the frequency (or time) factor $e^{j\omega t}$ is suppressed, and the frequency is not explicitly shown in the phasor-domain representation because ω is constant. However, the response depends on ω. For this reason, the phasor domain is also known as the *frequency domain.*

TABLE 9.1 Sinusoid-phasor transformation.

Time-domain representation	Phasor-domain representation
$V_m \cos(\omega t + \phi)$	$V_m \underline{/\phi}$
$V_m \sin(\omega t + \phi)$	$V_m \underline{/\phi - 90^\circ}$
$I_m \cos(\omega t + \theta)$	$I_m \underline{/\theta}$
$I_m \sin(\omega t + \theta)$	$I_m \underline{/\theta - 90^\circ}$

EXAMPLE 9.4

Transform these sinusoids to phasors:

(a) $v = -4\sin(30t + 50^\circ)$

(b) $i = 6\cos(50t - 40^\circ)$

$$v(t) = V_m \cos(\omega t + \phi) \iff \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

Solution:

(a) Since $-\sin A = \cos(A + 90^\circ)$,

$$v = -4\sin(30t + 50^\circ) = 4\cos(30t + 50^\circ + 90^\circ)$$

$$= 4\cos(30t + 140^\circ)$$

The phasor form of v is $\mathbf{V} = 4\angle 140^\circ$

(b) $i = 6\cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\angle -40^\circ$$

$$v(t) = V_m \cos(\omega t + \phi) \quad \Longleftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)

PRACTICE PROBLEM 9.4

Express these sinusoids as phasors:

(a) $v = -7\cos(2t + 40^\circ)$

(b) $i = 4\sin(10t + 10^\circ)$

Answer: (a) $\mathbf{V} = 7\underline{/220^\circ}$, (b) $\mathbf{I} = 4\underline{/-80^\circ}$.

EXAMPLE 9.5

Find the sinusoids represented by these phasors:

(a) $\mathbf{V} = j8e^{-j20°}$

(b) $\mathbf{I} = -3 + j4$

Solution:

(a) Since $j = 1\angle 90°$,

$$\mathbf{V} = j8\angle -20° = (1\angle 90°)(8\angle -20°)$$
$$= 8\angle 90° - 20° = 8\angle 70° \text{ V}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70°) \text{ V}$$

(b) $\mathbf{I} = -3 + j4 = 5\angle 126.87°$.

$$i(t) = 5\cos(\omega t + 126.87°) \text{ A}$$

EXAMPLE 9.6

Given $i_1(t) = 4\cos(\omega t + 30^\circ)$ and $i_2(t) = 5\sin(\omega t - 20^\circ)$, find their sum.

Solution:

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\angle 30^\circ$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20^\circ - 90^\circ) = 5\cos(\omega t - 110^\circ)$$

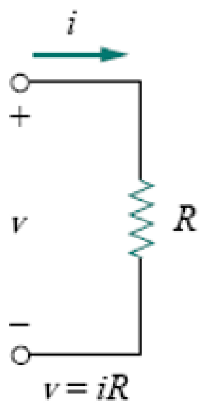
and its phasor is $\mathbf{I}_2 = 5\angle -110^\circ$

If we let $i = i_1 + i_2$, then $\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 4\angle 30^\circ + 5\angle -110^\circ$

$$= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698$$

$i(t) = 3.218\cos(\omega t - 56.97^\circ)$ A $\quad = 3.218\angle -56.97^\circ$ A

9.4 Phasors relationships for circuits elements



(a)

For the resistor R.

If the current through a resistor R is $i = I_m \cos(\omega t + \varphi)$,

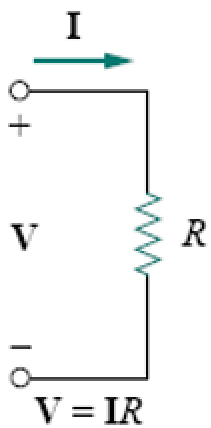
the voltage across it is given by Ohm's law as

$$v = iR = RI_m \cos(\omega t + \varphi)$$

Since $\mathbf{I} = I_m \angle \phi$ $\quad \mathbf{V} = RI_m \angle \phi$

So $\mathbf{V} = R\mathbf{I}$ the voltage-current relation for the resistor in the phasor domain

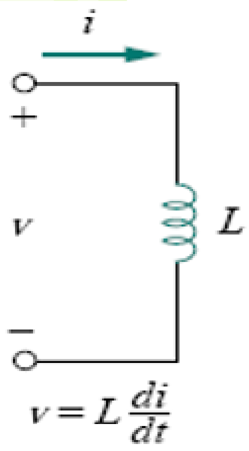
voltage and current are in phase



(b)

$$\mathbf{V} = R\mathbf{I}$$

For the inductor L



(a)

Assume the current through it is $i = I_m \cos(\omega t + \varphi)$.

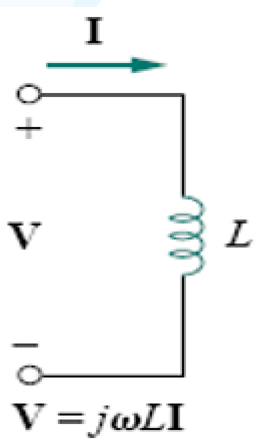
$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

Since $\mathbf{I} = I_m \angle \phi$ $e^{j90^\circ} = j$

$$\mathbf{V} = \omega L I_m e^{j(\phi + 90^\circ)} = \omega L I_m e^{j\phi} e^{j90^\circ} = \omega L I_m \angle \phi \, e^{j90^\circ}$$

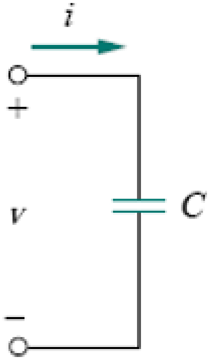
So $\mathbf{V} = j\omega L \mathbf{I}$

the current lags the voltage by 90°.



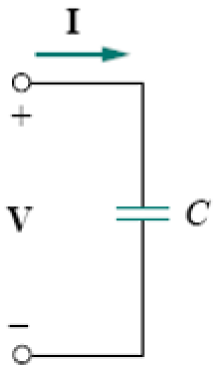
(b)

For the capacitor C



$i = C\frac{dv}{dt}$

(a)



$\mathbf{I} = j\omega C\mathbf{V}$

(b)

Aassume the voltage across it is $v = V_m \cos(\omega t + \varphi)$.

The current through the capacitor is

$$i = C\frac{dv}{dt}$$

$$\mathbf{I} = j\omega C\mathbf{V}$$

$$\Longrightarrow \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

the current leads the voltage by 90°.

Table 9.2 summarizes the time-domain and phasor-domain representations of the circuit elements.

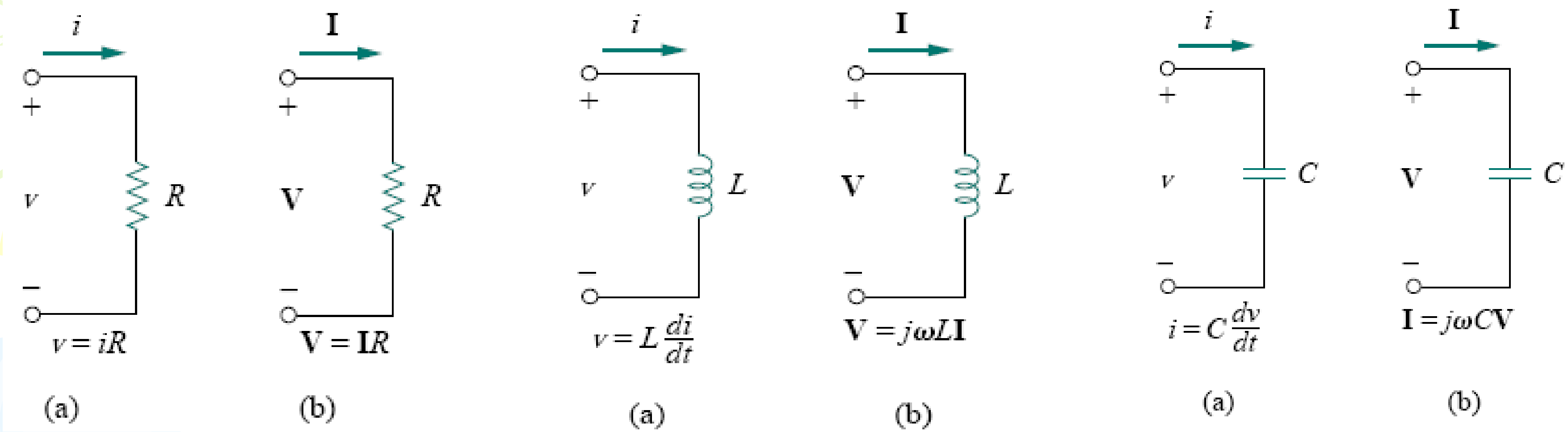


TABLE 9.2 Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

EXAMPLE 9.8

The voltage $v = 12\cos(60t + 45^\circ)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution:

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and

$$\mathbf{V} = 12\underline{/45^\circ}\text{ V.}$$

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\underline{/45^\circ}}{j60 \times 0.1} = \frac{12\underline{/45^\circ}}{6\underline{/90^\circ}} = 2\underline{/-45^\circ}\text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45^\circ)\text{ A}$$

PRACTICE PROBLEM 9.8

If voltage $v = 6\cos(100t - 30^\circ)$ is applied to a 50 μF capacitor, calculate the current through the capacitor.

Answer: $30\cos(100t + 60^\circ)$ **mA.**

9.5 Impedance and Admittance

In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$\boxed{\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \qquad \text{or} \qquad \mathbf{V} = \mathbf{Z}\mathbf{I}}$$

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \qquad \text{or} \qquad \mathbf{V} = \mathbf{Z}\mathbf{I}$$

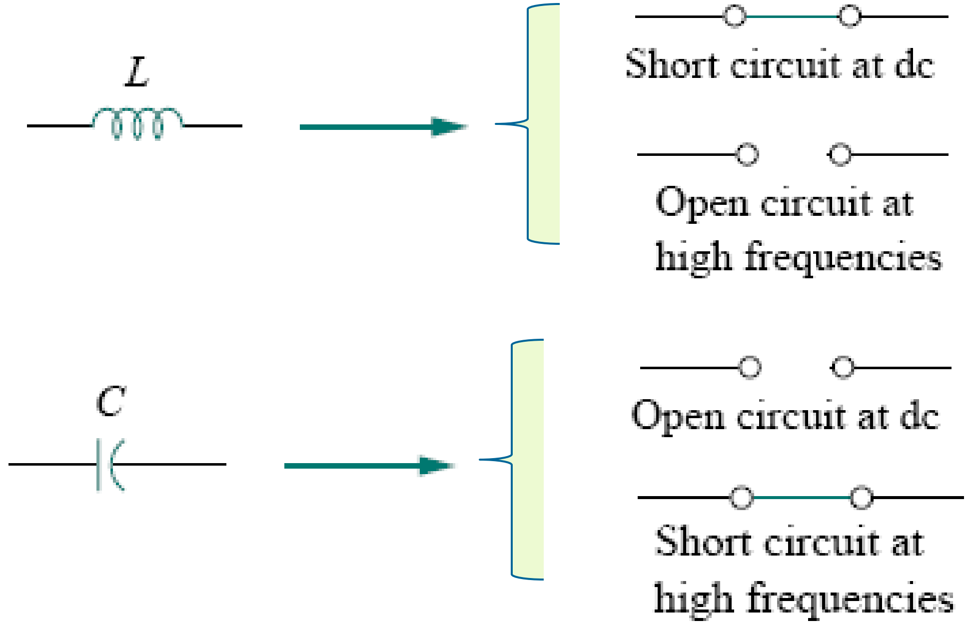
where **Z** is a frequency-dependent quantity known as *impedance*, measured in ohms.

The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω).

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

Element	Impedance
R	$\mathbf{Z} = R$
L	$\mathbf{Z} = j\omega L$
C	$\mathbf{Z} = \frac{1}{j\omega C}$

Element	Impedance
R	$\mathbf{Z} = R$
L	$\mathbf{Z} = j\omega L$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$



Consider two extreme cases of angular frequency.

1. L: When $\omega = 0$, $\mathbf{Z}_L = 0$ (short circuit)
   When $\omega \to \infty$, $\mathbf{Z}_L \to \infty$(open circuit)
2. C: When $\omega = 0$, $\mathbf{Z}_C \to \infty$ (open circuit)
   When $\omega \to \infty$, $\mathbf{Z}_C = 0$ (short circuit)

As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R + jX$$

R = Re $\mathbf{Z}$ is the *resistance* and X = Im $\mathbf{Z}$ is the *reactance*.

The impedance may also be expressed in polar form as

$$\mathbf{Z} = |\mathbf{Z}|\angle\theta$$

$$\boxed{\mathbf{Z} = R + jX = |\mathbf{Z}|\angle\theta}$$

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R}$$

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta$$

It is sometimes convenient to work with the reciprocal of impedance, known as *admittance*.

The admittance **Y** is the reciprocal of impedance, measured in siemens (S).

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}$$

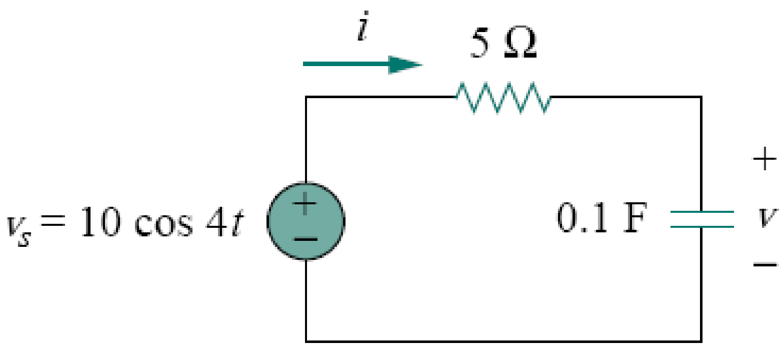
$$\mathbf{Y} = G + jB$$

G = Re**Y** is called the *conductance*

B = Im**Y** is called the *susceptance*.

EXAMPLE 9.9

Find *v(t)* and *i(t)* in the circuit shown in Fig. 9.16.



Solution:

From the voltage source 10cos4*t* , *ω* = 4, $\mathbf{V}_s = 10\angle 0^\circ \text{ V}$

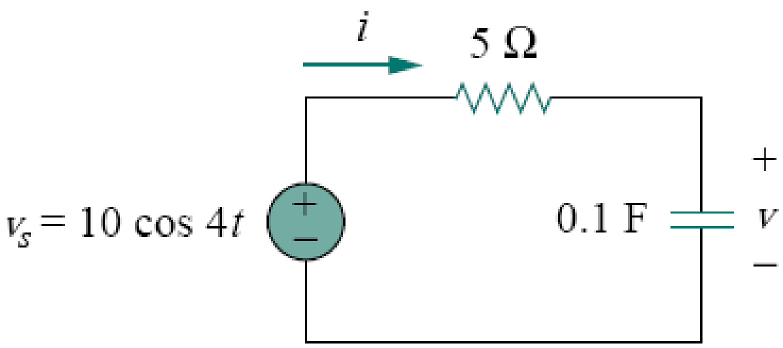
The impedance is $\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$

Hence the current $\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$

$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

EXAMPLE 9.9

Find *v(t)* and *i(t)* in the circuit shown in Fig. 9.16.



Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\underline{/0^\circ}}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$
$$= 1.6 + j0.8 = 1.789\underline{/26.57^\circ}\text{ A}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\underline{/26.57^\circ}}{j4 \times 0.1}$$
$$= \frac{1.789\underline{/26.57^\circ}}{0.4\underline{/90^\circ}} = 4.47\underline{/-63.43^\circ}\text{ V}$$

$$i(t) = 1.789\cos(4t + 26.57^\circ)\text{ A}$$

$$v(t) = 4.47\cos(4t - 63.43^\circ)\text{ V}$$

9.6 Kirchhoff's Laws in the frequency domain

We cannot do circuit analysis in the frequency domain without Kirchhoff's current and voltage laws. Therefore, we need to express them in the frequency domain.

For KVL, $$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

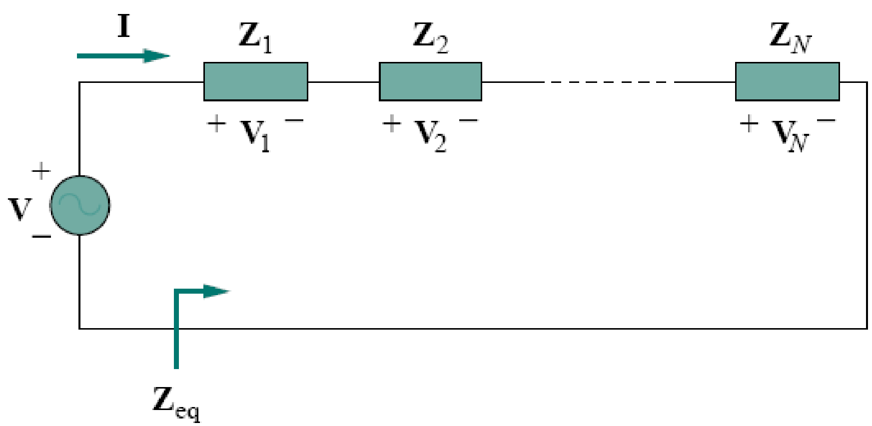
Kirchhoff's voltage law holds for phasors.

For KCL, $$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0$$

Kirchhoff's current law holds for phasors.

9.7 Impedance Combination

Consider the N series-connected impedances



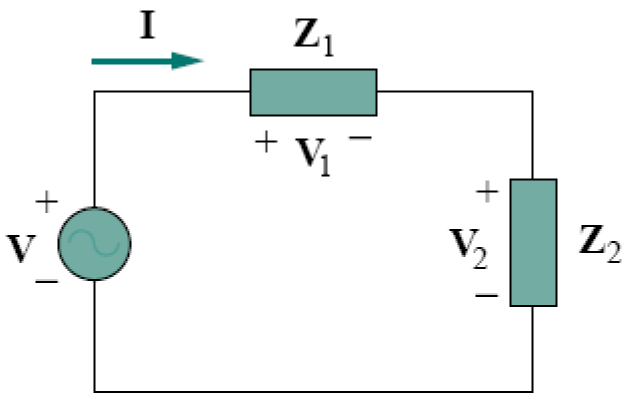
$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdot \quad \cdot \quad \cdot + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdot \quad \cdot \quad \cdot + \mathbf{Z}_N)$$

The equivalent impedance at the input terminals is

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

$$\boxed{\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N}$$

The total or equivalent impedance of series-connected impedances is the sum of the individual impedances. This is similar to the series connection of resistances.



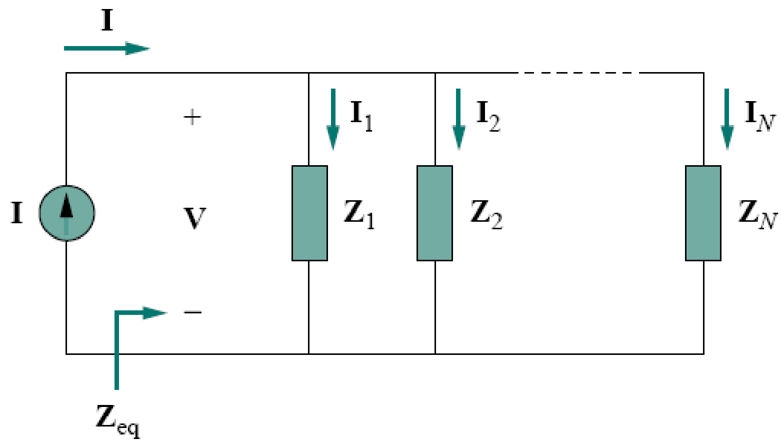
If N = 2, the current through the impedances is $\mathbf{I} = \dfrac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}$

Since $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$, then

$$\boxed{\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}}$$

which is the ***voltage-division*** relationship.

Consider the *N* parallel-connected impedances

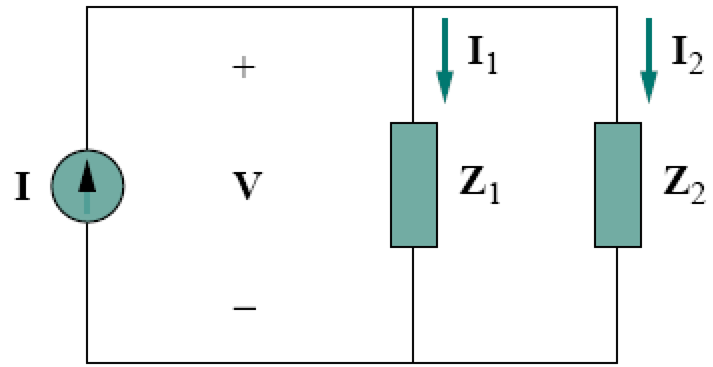


$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right)$$

The equivalent impedance is $$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

The equivalent admittance is $$\boxed{\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N}$$

the equivalent admittance of a parallel connection of admittances is the sum of the individual admittances. This is similar to the parallel connection of resistances.



When N = 2, the equivalent impedance becomes

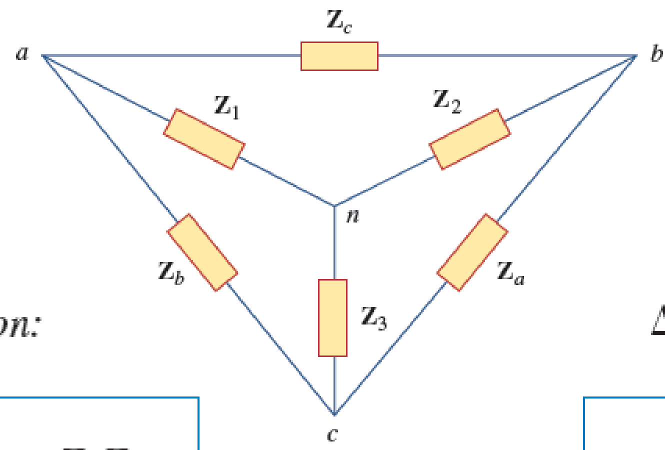
$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2$$

the currents in the impedances are

$$\boxed{\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}}$$

which is the ***current-division*** principle.



Y-Δ Conversion:

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2}$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

Δ-Y Conversion:

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

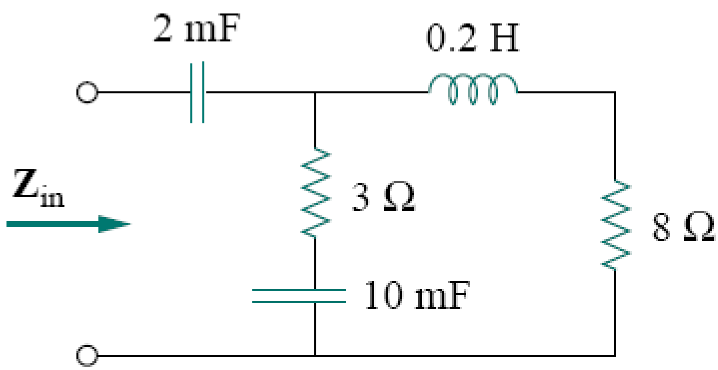
A delta or wye circuit is said to be **balanced** if it has equal impedances in all three branches.

$$\mathbf{Z}_Y = \mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 \text{ and } \mathbf{Z}_\Delta = \mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c.$$

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \qquad \text{or} \qquad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta$$

EXAMPLE 9.10

Find the input impedance of the circuit in this Fig. Assume that the circuit operates at ω = 50 rad/s.



The input impedance is

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3$$

$$= 3.22 - j11.07\ \Omega$$

Solution:

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3- Ω resistor in series with the 10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

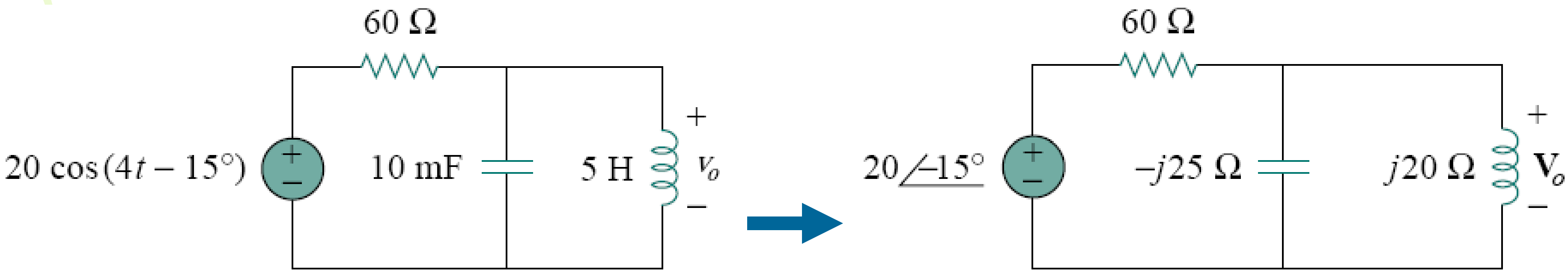
$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

EXAMPLE 9.11

Determine $v_o(t)$ in the circuit in Fig.25



Solution:

To do the analysis in the frequency domain, we must first transform the time-domain circuit in left Fig to the phasor-domain equivalent in right Fig. The transformation produces

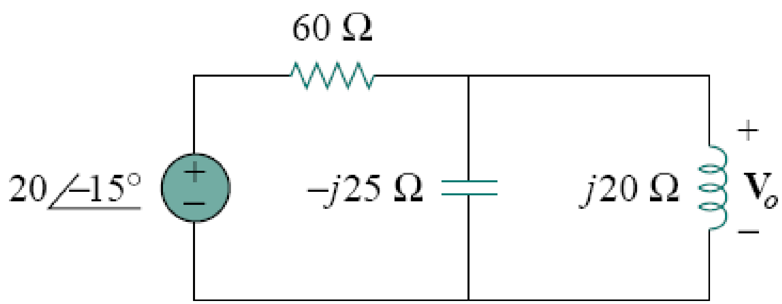
$$v_s = 20\cos(4t - 15^\circ) \quad \Longrightarrow \quad \mathbf{V}_s = 20\angle -15^\circ \ \text{V}, \qquad \omega = 4$$

$$10 \text{ mF} \quad \Longrightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5 \text{ H} \quad \Longrightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

EXAMPLE 9.11

Determine $v_o(t)$ in the circuit in Fig.25



$\mathbf{Z}_1$ = Impedance of the 60- resistor $\quad \mathbf{Z}_1 = 60\ \Omega$

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

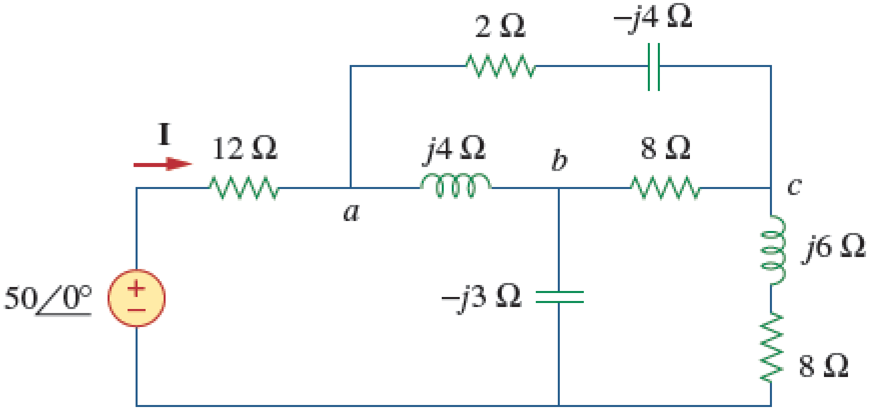
$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = 17.15\underline{/15.96^\circ}\ \text{V}.$$

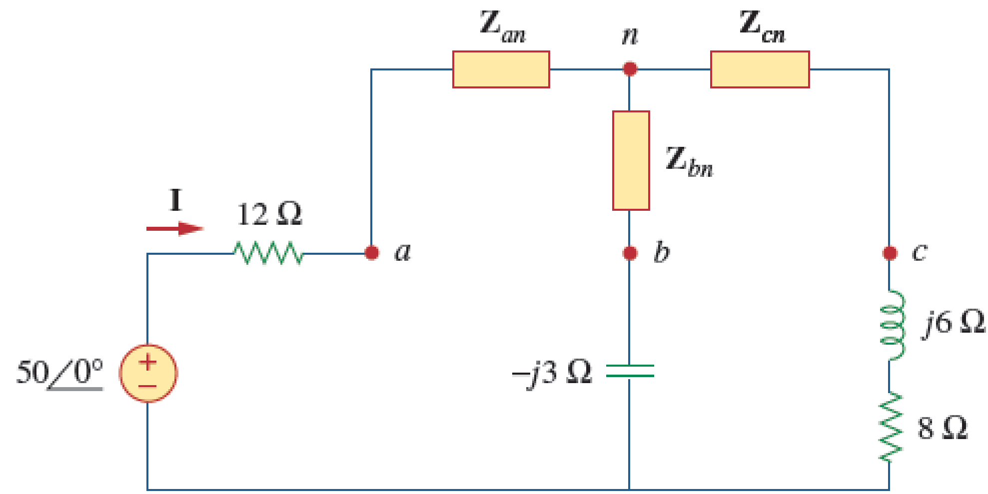
We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96^\circ)\text{V}$$

Example 9.12

Find current **I** in the circuit of Fig. 9.28.





Solution:

The delta network connected to nodes *a*, *b*, and *c* can be converted to the *Y* network of Fig. 9.29. We obtain the *Y* impedances as follows using Eq. (9.68):

$$\mathbf{Z}_{an} = \frac{j4(2 - j4)}{j4 + 2 - j4 + 8} = \frac{4(4 + j2)}{10} = (1.6 + j0.8)\ \Omega$$

$$\mathbf{Z}_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad \mathbf{Z}_{cn} = \frac{8(2 - j4)}{10} = (1.6 - j3.2)\ \Omega$$

The total impedance at the source terminals is

$$\mathbf{Z} = 12 + \mathbf{Z}_{an} + (\mathbf{Z}_{bn} - j3) \,\|\, (\mathbf{Z}_{cn} + j6 + 8) = 13.6 + j1 = 13.64\underline{/4.204^\circ}\ \Omega$$

The desired current is $\mathbf{I} = \dfrac{\mathbf{V}}{\mathbf{Z}} = \dfrac{50\underline{/0^\circ}}{13.64\underline{/4.204^\circ}} = 3.666\underline{/-4.204^\circ}\ \text{A}$

Summary and Review

1. A sinusoid is a signal in the form of the sine or cosine function. It has the general form

$$v(t) = V_m \cos(\omega t + \phi)$$

where V_m is the amplitude, $\omega = 2\pi f$ is the angular frequency, $(\omega t + \phi)$ is the argument, and ϕ is the phase.

2. A phasor is a complex quantity that represents both the magnitude and the phase of a sinusoid. Given the sinusoid $v(t) = V_m \cos(\omega t + \phi)$, its phasor $\mathbf{V}$ is

$$\mathbf{V} = V_m \underline{/\phi}$$

3. In ac circuits, voltage and current phasors always have a fixed relation to one another at any moment of time. If $v(t) = V_m \cos(\omega t + \phi_v)$ represents the voltage through an element and $i(t) = I_m \cos(\omega t + \phi_i)$ represents the current through the element, then $\phi_i = \phi_v$ if the element is a resistor, ϕ_i leads ϕ_v by 90° if the element is a capacitor, and ϕ_i lags ϕ_v by 90° if the element is an inductor.

4. The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage across it to the phasor current through it:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R(\omega) + jX(\omega)$$

The admittance $\mathbf{Y}$ is the reciprocal of impedance:

$$\mathbf{Z} = \frac{1}{\mathbf{Y}} = G(\omega) + jB(\omega)$$

Impedances are combined in series or in parallel the same way as resistances in series or parallel; that is, impedances in series add while admittances in parallel add.

5. For a resistor $\mathbf{Z} = R$, for an inductor $\mathbf{Z} = jX = j\omega L$, and for a capacitor $\mathbf{Z} = -jX = 1/j\omega C$.

6. Basic circuit laws (Ohm's and Kirchhoff's) apply to ac circuits in the same manner as they do for dc circuits; that is,

$$\mathbf{V} = \mathbf{ZI}$$

$$\sum \mathbf{I}_k = 0 \quad \text{(KCL)}$$

$$\sum \mathbf{V}_k = 0 \quad \text{(KVL)}$$

7. The techniques of voltage/current division, series/parallel combination of impedance/admittance, circuit reduction, and Y-Δ transformation all apply to ac circuit analysis.

8. AC circuits are applied in phase-shifters and bridges.

First time homework

9.11 Find the phasors corresponding to the following signals:

(a) $v(t) = 21\cos(4t - 15^\circ)$ V

(b) $i(t) = -8\sin(10t + 70^\circ)$ mA

(c) $v(t) = 120\sin(10t - 50^\circ)$ V

(d) $i(t) = -60\cos(30t + 10^\circ)$ mA

9.16 Transform the following sinusoids to phasors:

(a) $-20\cos(4t + 135^\circ)$ (b) $8\sin(20t + 30^\circ)$

(c) $20\cos(2t) + 15\sin(2t)$

9.18 Obtain the sinusoids corresponding to each of the following phasors:

(a) $\mathbf{V}_1 = 60\underline{/15^\circ}$ V, $\omega = 1$

(b) $\mathbf{V}_2 = 6 + j8$ V, $\omega = 40$

(c) $\mathbf{I}_1 = 2.8e^{-j\pi/3}$ A, $\omega = 377$

(d) $\mathbf{I}_2 = -0.5 - j1.2$ A, $\omega = 10^3$

Second time homework

9.30 A voltage $v(t) = 100\cos(60t + 20°)$ V is applied to a parallel combination of a 40-kΩ resistor and a 50-μF capacitor. Find the steady-state currents through the resistor and the capacitor.

9.35 Find current i in the circuit of Fig. 9.42, when $v_s(t) = 50\cos 200t$ V.

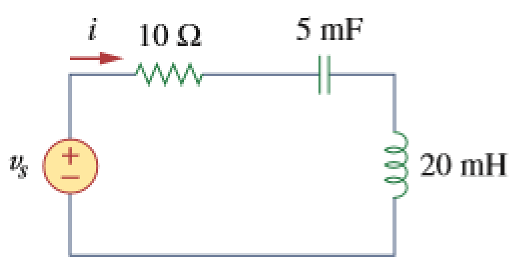


Figure 9.42
For Prob. 9.35.

9.34 What value of ω will cause the forced response, v_o, in Fig. 9.41 to be zero?

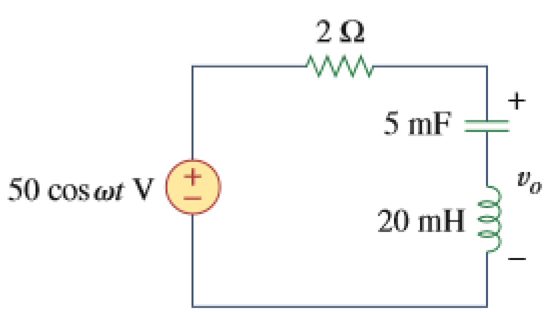


Figure 9.41
For Prob. 9.34.

Second time homework

9.42 Calculate $v_o(t)$ in the circuit of Fig. 9.49.

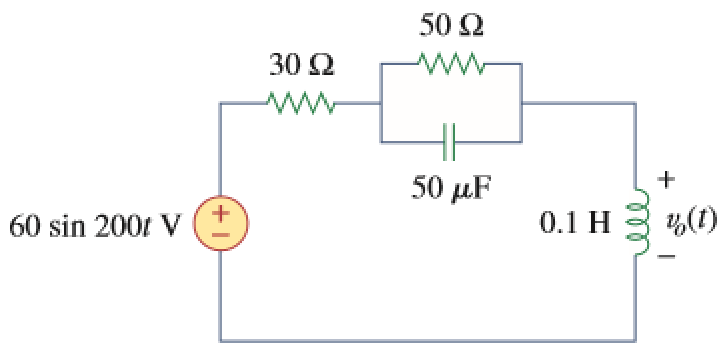


Figure 9.49
For Prob. 9.42.

9.56 At $\omega = 377$ rad/s, find the input impedance of the circuit shown in Fig. 9.63.

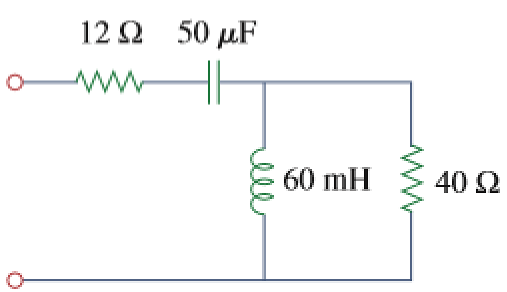


Figure 9.63
For Prob. 9.56.

Second time homework

9.61 Find $\mathbf{Z}_{eq}$ in the circuit of Fig. 9.68.

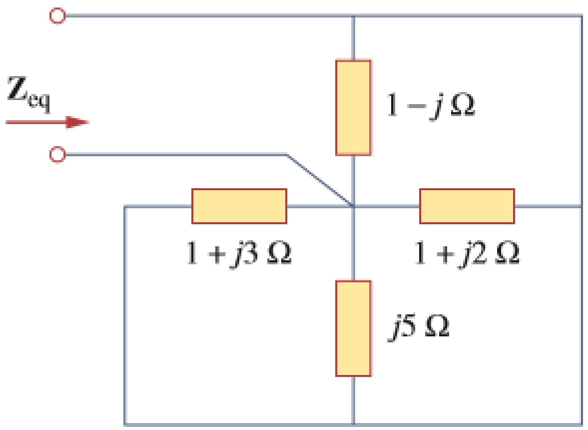


Figure 9.68
For Prob. 9.61.

9.63 For the circuit in Fig. 9.70, find the value of $\mathbf{Z}_T$.

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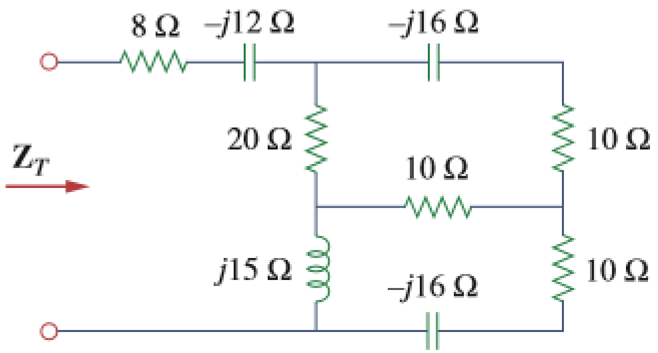

Figure 9.70
For Prob. 9.63.