Equivalence Relations

Section 9.5

Section Summary

- Equivalence Relations
- Equivalence Classes
- Equivalence Classes and Partitions

Equivalence Relations

Definition 1: A relation on a set *A* is called an *equivalence relation*(等价关系) if it is reflexive, symmetric, and transitive.

Definition 2: Two elements *a*, and *b* that are related by an equivalence relation are called *equivalent.* The notation $a \sim b$ is often used to denote that *a* and *b* are equivalent elements with respect to a particular equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

– *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
– *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
– *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(a)$ also holds and aRc.

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation

$R = \{(a,b) \mid a \equiv b \pmod{m}\}$

is an equivalence relation on the set of integers.

Solution: $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

$$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

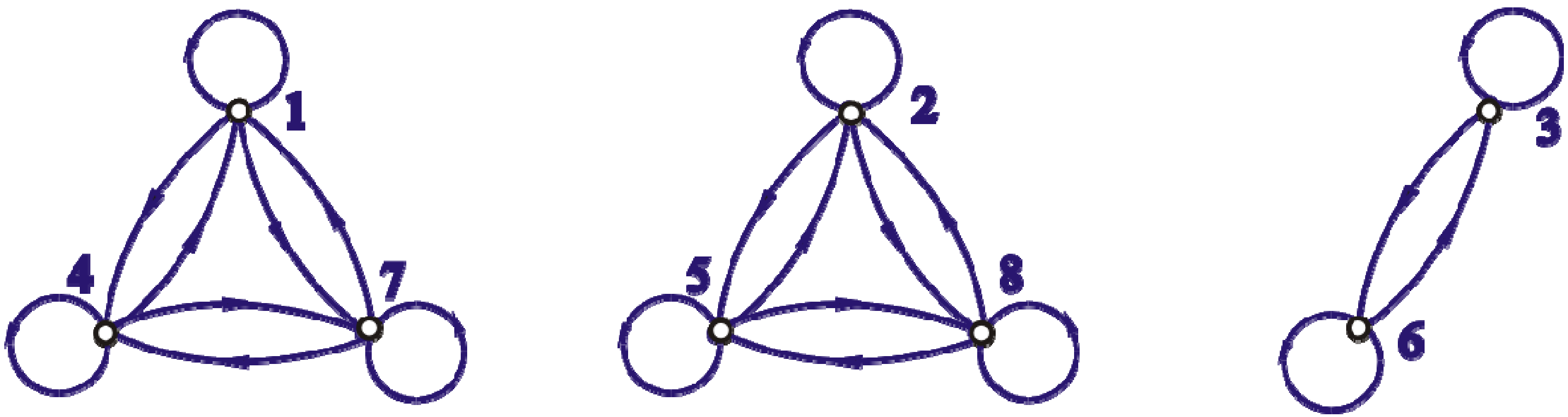
Therefore, $a \equiv c \pmod{m}$.

$A=\{1,2,3,4,5,6,7,8\}$

$R=\{<x,y>|\ x,y \in A \wedge x \equiv y \pmod 3\}=\{<x,y>|\ 3|(x-y)\}$

$R=\{<1,4>,<4,1>,<1,7>,<7,1>,<4,7>,<7,4>,<2,5>,<5,2>,<2,8>,<8,2>,<5,8>,<8,5>,<3,6>,<6,3>\} \cup I$

$\{1, 4, 7\}$ $\{3, 6\}$ $\{2, 5, 8\}$



Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not *Symmetric*. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class*(等价类) of a. The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

- Note that $[a]_R = \{s \mid (a,s) \in R\}$.
- If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.

- $A=\{a,b,c,d,e,f\}$, $R=\{\langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle a,b\rangle, \langle b,a\rangle, \langle a,c\rangle, \langle c,a\rangle, \langle b,c\rangle, \langle c,b\rangle, \langle d,d\rangle, \langle e,e\rangle, \langle d,e\rangle, \langle e,d\rangle, \langle f,f\rangle\}$,

equivalence classes are :

$$[a] = [b] = [c] = \{a,b,c\}$$

$$[d] = [e] = \{d,e\}$$

$$[f] = \{f\}.$$

Equivalence Classes and Partitions

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(*i*) aRb

(*ii*) $[a] = [b]$

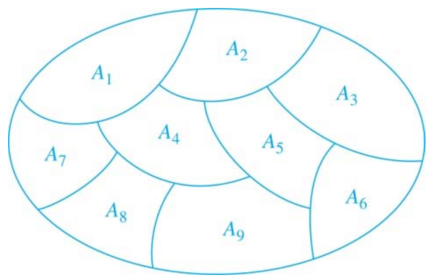
(*iii*) $[a] \cap [b] \neq \emptyset$

Different classes have no common elements.

Partition of a Set

Definition: A *partition*（划分） of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S$.



A Partition of a Set

Example

- $S=\{1,2,3\}$

$A=\{\{1,2\},\{2,3\}\}$ $B=\{\{1\},\{1,2\},\{1,3\}\}$

$C=\{\{1\},\{2,3\}\}$ $D=\{\{1,2,3\}\}$

$E=\{\{1\},\{2\},\{3\}\}$ $F=\{\{1\},\{1,2\}\}$

Equivalence relation and Partition

- Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Example

$A=\{a,b,c,d,e\}$, $R=\{\langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle b,b\rangle, \langle b,a\rangle, \langle b,c\rangle, \langle c,c\rangle, \langle c,a\rangle, \langle c,b\rangle, \langle d,d\rangle, \langle d,e\rangle, \langle e,e\rangle, \langle e,d\rangle\}$

the equivalence classes form a partition of A $\pi=\{\{a,b,c\},\{d,e\}\}$.

Conversely, given $\pi=\{\{a,b,c\},\{d,e\}\}$, we can get an equivalence relation $R=\{a,b,c\}\times\{a,b,c\}\cup\{d,e\}\times\{d,e\}=\{\langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle b,b\rangle, \langle b,a\rangle, \langle b,c\rangle, \langle c,c\rangle, \langle c,a\rangle, \langle c,b\rangle, \langle d,d\rangle, \langle d,e\rangle, \langle e,e\rangle, \langle e,d\rangle\}$

Homework

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