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# Assignment 8.1

**7.5** Prove each of the following assertions:
**a**. $\alpha$ is valid if and only if True $\models \alpha$.
**b**. For any $\alpha$, False $\models \alpha$.
**c**. $\alpha \models \beta$ if and only if the sentence $(\alpha \Rightarrow \beta)$ is valid.
**d**. $\alpha \equiv \beta$ if and only if the sentence $(\alpha \Leftrightarrow \beta)$ is valid.
**e**. $\alpha \models \beta$ if and only if the sentence $(\alpha \wedge \neg\beta)$ is unsatisfiable.

## Solution:

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# 7.5

a) $\alpha$ is valid and only if True $\models \alpha$
Here, True $\models \alpha$ means True entails $\alpha$ if and only if $\alpha$ is true in each model where True is true. For a statement to be valid, it must be true in every model. Hence $\alpha$ is valid if and only if True $\models \alpha$.

b) For any $\alpha$, False $\models \alpha$
Here, False $\models \alpha$ means False entails $\alpha$ if and only if in every model where False is true, $\alpha$ must be true. Since false is false in every model, for any $\alpha$, False $\models \alpha$.

c) $\alpha \models \beta$ if and only if sentence $\alpha \Rightarrow \beta$ is valid.

Now, $\alpha \Rightarrow \beta$ = true (given)
$\neg(\alpha) \cup \beta$ = true (implication elimination)
$\neg(\text{True}) \cup \beta$ = true
False $\cup \beta$ = true
$\beta$ is true in $m$ (semantics of $\cup$)

Thus, if $\alpha \Rightarrow \beta$ then $\alpha \models \beta$

Now,
$\alpha \models \beta$ is given
$\alpha \models \beta$ means $\alpha$ entails $\beta$ if only if in every model where $\alpha$ is true, $\beta$ must be true.

Let $M = M1 \cup M2$
For all $m \in M$,
Where for $M2$, $\alpha$ = False and for $M1$, $\alpha$ = true
For every $m \in M1$, $\alpha$ is true in $m$, $\beta$ is true in $m$

$\alpha \Rightarrow \beta \equiv \neg\alpha \cup \beta$
$\equiv$ False $\cup$ True
$\equiv$ True (semantics of $\cup$)

For every $m \in M2$, $\alpha$ is False in $m$, $\beta$ is False in $m$
$\alpha \Rightarrow \beta \equiv \neg\alpha \cup \beta$

$\equiv$ True $\cup$ False

$\equiv$ True (semantics of $\cup$)

Therefore, if $\alpha \models \beta$, then $\alpha \Rightarrow \beta$ is true

Thus, $\alpha \models \beta$ if and only if sentence $\alpha \Rightarrow \beta$ is valid.

d) $\alpha \equiv \beta$ if and only if the sentence $(\alpha \Leftrightarrow \beta)$ is valid.

$\alpha \equiv \beta$ (given)

According to defination of logical equivalence, $\alpha \equiv \beta$, if they are true in the same set of models.

i.e, $\alpha \equiv \beta$ then $\alpha \models \beta$ and $\beta \models \alpha$

now if $\alpha \models \beta$, $\alpha \Rightarrow \beta$

and if $\beta \models \alpha$, $\beta \Rightarrow \alpha$

as $\alpha \Rightarrow \beta$ is true and $\beta \Rightarrow \alpha$ is true by defination, $\alpha \Leftrightarrow \beta$ is true $(\alpha \Leftrightarrow \beta)$ is valid.

Thus if $\alpha \equiv \beta$ $(\alpha \Leftrightarrow \beta)$ is valid.

$(\alpha \Leftrightarrow \beta)$ (given)

$(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$ (Biconditional elimination)

Thus, $\alpha | \beta$ and $\beta \models \alpha$ i.e, $\alpha \equiv \beta$

So, we can say that $\alpha \equiv \beta$ if and only if the sentence $(\alpha \Leftrightarrow \beta)$ is valid

e) $\alpha \models \beta$ if and only if the sentence $(\alpha \wedge \neg\beta)$ is valid.

$\alpha \models \beta$ is true if and only if $\alpha \Rightarrow \beta$

$\alpha \models \beta$ (given)

$\alpha \Rightarrow \beta = \text{true}$

$(\neg\alpha \vee \beta) = \text{true}$ (implication elimination)

$\neg(\neg\alpha \vee \beta) = \neg(\text{true})$ ($\neg$ on both side)

$(\alpha \wedge \neg\beta) = \text{false}$ (De Morgan)

Thus, if $\alpha \models \beta$ then $(\alpha \wedge \neg\beta)$ is unsatisfiable

Now, $(\alpha \wedge \neg\beta) = \text{false}$ (given)

$\neg(\alpha \wedge \neg\beta) = \neg(\text{false})$

$(\neg\alpha \vee \beta) = \text{true}$

$\alpha \Rightarrow \beta = \text{true}$

i.e. $\alpha \models \beta$

Thus, if $(\alpha \wedge \neg\beta)$ is unsatisfiable, $\alpha \models \beta$

Thus, $\alpha \models \beta$ is unsatisfiable if and only if $(\alpha \wedge \neg\beta)$ is unsatisfiable.

**7.10** Decide whether each of the following sentences is valid, unsatisfiable, or neither. Verify your decisions using truth tables or the equivalence rules of Figure 7.11 (page 249).

**a**. Smoke ⇒ Smoke

**b**. Smoke ⇒ Fire

**c**. (Smoke ⇒ Fire) ⇒ (¬Smoke ⇒ ¬Fire)

**d**. Smoke ∨ Fire ∨ ¬Fire

**e**. ((Smoke ∧ Heat ) ⇒ Fire) ⇔ ((Smoke ⇒ Fire) ∨ (Heat ⇒ Fire))

**f**. (Smoke ⇒ Fire) ⇒ ((Smoke ∧ Heat ) ⇒ Fire)

**g**. Big ∨ Dumb ∨ (Big ⇒ Dumb)

## Solution:

# 7.10

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a) smoke => smoke

| smoke | smoke=>smoke |
|---|---|
| True | True |
| False | True |

Thus, statement is valid as it is true in all models.

b) smoke => fire

| smoke | fire | smoke => fire |
|---|---|---|
| True | True | True |
| True | False | False |
| False | True | True |
| False | False | True |

Hence, the statement can be true and can be false, it is neither satisfiable nor unsatisfiable.

c) (smoke => fire) => (¬smoke => ¬fire)
(¬smoke V fire) => (smoke V ¬fire) (implication elimination)
¬(smoke V fire) V (smoke V ¬fire) (implication elimination)
(¬smoke ^ ¬fire) V (smoke V ¬fire) (De Morgan
(¬smoke V smoke V ¬fire) ^ (¬fire V smoke V ¬fire) (Distribution of V over ^)
TRUE ^ (¬fire V smoke) (semantics of V)
¬fire V smoke (semantics of ^)

| fire | smoke | ¬fire | ¬fire V smoke |
|---|---|---|---|
| True | True | False | True |
| True | False | False | False |
| False | True | True | True |
| False | False | True | True |

Hence, the statement can be true and can be false, it is neither satisfiable nor unsatisfiable.

d) $smoke \lor fire \lor \neg fire$

$smoke \lor True$ (semantics of $\lor$)

$True$ (semantics of $\lor$)

Hence, as the statement is true for all models, it is <u>valid</u>.

e) $((smoke \land heat) \Rightarrow fire) \Leftrightarrow ((smoke \Rightarrow fire) \lor (heat \Rightarrow fire))$

$(\neg(smoke \land heat) \lor fire) \Leftrightarrow ((\neg smoke \lor fire) \lor (\neg heat \lor fire))$ (implication elimination)

$(\neg smoke \lor \neg heat \lor fire) \Leftrightarrow (\neg smoke \lor fire \lor \neg heat \lor fire)$ (De Morgan)

$(\neg smoke \lor \neg heat \lor fire) \Leftrightarrow (\neg smoke \lor \neg heat \lor fire)$ (semantics of $\lor$)

$((\neg smoke \lor \neg heat \lor fire) \cancel{\Leftrightarrow} \Rightarrow (\neg smoke \lor \neg heat \lor fire)) \land$ (Biconditional elimination)

$((\neg smoke \lor \neg heat \lor fire) \Rightarrow (\neg smoke \lor \neg heat \lor fire))$

$(True) \land (True)$ ($a \Rightarrow a$ is True)

$True$ (semantics of $\land$)

Hence, as the statement is true for all models, it is <u>valid</u>.

f) $(smoke \Rightarrow fire) \Rightarrow ((smoke \land heat) \Rightarrow fire)$ (implication elimination)

$(\neg smoke \lor fire) \Rightarrow (\neg(smoke \land heat) \lor fire)$ (De Morgan)

$(\neg smoke \lor fire) \Rightarrow (\neg smoke \lor \neg heat \lor fire)$ (implication elimination)

$\neg(\neg smoke \lor fire) \lor (\neg smoke \lor \neg heat \lor fire)$ (De Morgan)

$(smoke \land \neg fire) \lor (\neg smoke \lor \neg heat \lor fire)$ (Distribution of $\lor$)

$(smoke \lor (\neg smoke \lor \neg heat \lor fire)) \land (\neg fire \lor (\neg smoke \lor \neg heat \lor fire)$ (Associativity of $\lor$)

$((smoke \lor \neg smoke) \lor \neg heat \lor fire) \land ((\neg fire \lor fire) \lor (\neg smoke \lor \neg heat)$ (semantics of $\lor$)

$(True \lor \neg heat \lor fire) \land (True \lor \neg smoke \lor \neg heat)$ (semantics of $\lor$)

$(True) \land (True)$ (semantics of $\lor$)

$True$

Hence, as the statement is true for all models, it is <u>valid</u>.

9) $big \lor dumb \lor (big \Rightarrow dumb)$

$big \lor dumb \lor (\neg big \lor dumb)$ (implication elimination)

$big \lor \neg big \lor (dumb \lor dumb)$ (Associativity of $\lor$)

$True \lor dumb$ (semantics of $\lor$)

$True$ (semantics of $\lor$)

$True$

Hence, as the statement is true for all models, it is <u>valid</u>.