1.2Applications of Propositional Logic

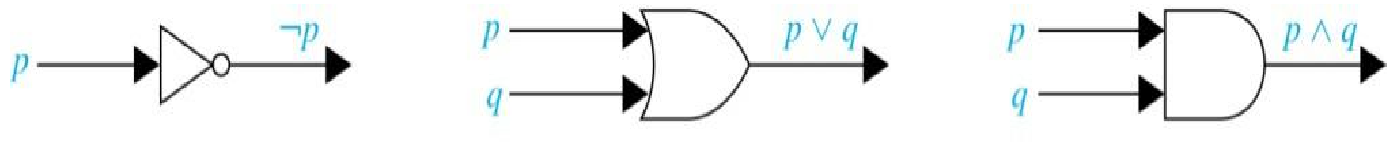
- Translating English to Propositional Logic
- System Specifications 系统规范
- Boolean Searching
- Logic Puzzles
- Logic Circuits

Translating sentences

- ‘If I go to school or go home, I will not go shopping.’
  - P: I go to school
  - Q: I go home
  - R: I will go shopping
- If......P......or.....Q.....then....not.....R
  - $(P \vee Q) \rightarrow \neg R$
- You can access the Internet from campus only if you are a computer science major or you are not a freshman.
- $a \rightarrow (c \vee \neg f)$

Logic Circuits

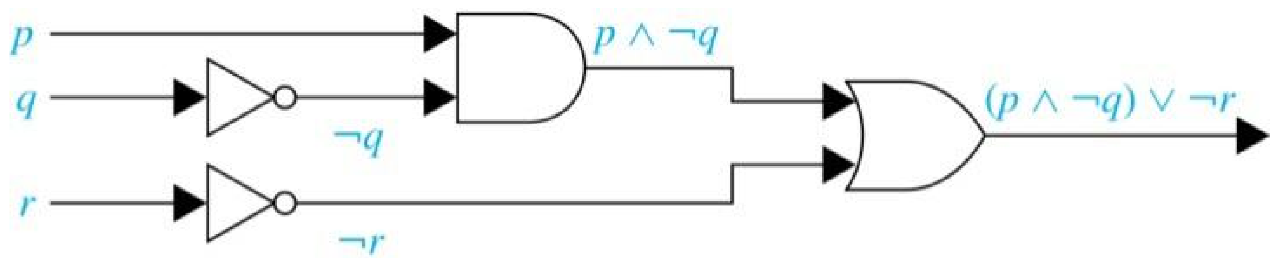
- Propositional logic can be applied to the design of computer hardware.
- Complicated circuits are constructed from three basic circuits called gates.



Inverter OR gate AND gate

  - The inverter (**NOT gate**)takes an input bit and produces the negation of that bit.
  - The **OR gate** takes two input bits and produces the value equivalent to the disjunction of the two bits.
  - The **AND gate** takes two input bits and produces the value equivalent to the conjunction of the two bits.

- Given a circuit, we determine the output by tracing through the circuit
- If we know the output, we can build a digital circuit using basic gates.



1.3 Propositional Equivalences
(逻辑等价式)

Tautologies, Contradictions, and Contingencies

- A tautology is a proposition which is always true, *no matter what* the truth values of its atomic propositions are!
  - Example: $p \vee \neg p$
- A *contradiction* is a proposition which is always false.
  - Example: $p \wedge \neg p$

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
0	1	1	0
1	0	1	0

- A *contingency* is a proposition which is neither a tautology nor a contradiction, such as p

Which of these are tautologies?

1. $p \rightarrow (q \rightarrow p)$
2. $p \rightarrow (\neg p \rightarrow p)$
3. $(q \rightarrow p) \rightarrow (p \rightarrow q)$
4. $(q \rightarrow p) \vee (p \rightarrow q)$
5. $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$

Please prove your claims, using truth tables.

Which of these are tautologies?

1. $p \rightarrow (q \rightarrow p)$ *Tautologous*
2. $p \rightarrow (\neg p \rightarrow p)$ *Tautologous*
3. $(q \rightarrow p) \rightarrow (p \rightarrow q)$ *Contingent*
4. $(q \rightarrow p) \vee (p \rightarrow q)$ *Tautologous*
5. $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$ *Tautologous*

Logical Equivalence

Compound proposition p is *logically equivalent* to compound proposition q, denoted by $p \Leftrightarrow q$, **if** $p \leftrightarrow q$ is tautology.

- We write this as $p \Leftrightarrow q$ or $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.

What's the difference between $\leftrightarrow$ and $\Leftrightarrow$?

- p$\Leftrightarrow$ q means p $\leftrightarrow$ q is a tautology. p and q must be the same truth values.
- $\leftrightarrow$ is a logical connective, and its truth value can be fasle. The truth values of p and q can be different.

Example

- This truth table show $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

- Identity Laws:

$$p \wedge T \equiv p \qquad p \vee F \equiv p$$

- Domination Laws:

$$p \vee T \equiv T \qquad p \wedge F \equiv F$$

- Idempotent laws: $p \vee p \equiv p \qquad p \wedge p \equiv p$

- Double Negation Law: $\neg(\neg p) \equiv p$

- Negation Laws: $p \vee \neg p \equiv T \qquad p \wedge \neg p \equiv F$

Key Logical Equivalences (*cont*)

- Commutative Laws:

$$p \vee q \equiv q \vee p \qquad p \wedge q \equiv q \wedge p$$

- Associative Laws:

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

- Distributive Laws:

$$(p \vee (q \wedge r) \equiv (p \vee q)) \wedge (p \vee r)$$

$$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$$

- Absorption Laws:

$$p \vee (p \wedge q) \equiv p \qquad p \wedge (p \vee q) \equiv p$$

More Logical Equivalences

$p \to q \Leftrightarrow \neg p \vee q$ Implication

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$

$p \to q \equiv \neg q \to \neg p$

$p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \wedge \neg q$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

How to judge Logical Equivalences

- 1. Truth table
- 2、Calculation of proposition formula Basic Logical Equivalences
  - Substitution rule
  - Replacement rule

Substitution rule

- In a tautology, if we replace proposition variable R with another proposition formula, the new formula is still a tautology.

  - Ex: show $(p \rightarrow q) \lor \neg (p \rightarrow q)$ is tautology
  - $R \lor \neg R \Leftrightarrow T$
  - replace R with $(p \rightarrow q)$

Replacement rule

- In a given formula A， the sub proposition areA_1， A_2， …A_n ， if $A_i \Leftrightarrow B_i$, after replace A_i with B_i, get a new formula B， then $A \Leftrightarrow B$。
- Ex： Show
- $(P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\neg Q \vee R)$ is a tautology
- *because* $(Q \rightarrow R) \Leftrightarrow (\neg Q \vee R)$
- Replace $(Q \rightarrow R)$ with$(\neg Q \vee R)$ in original formula.

Example

Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative law} \\
& && \text{for disjunction} \\
&\equiv (\neg p \wedge \neg q) && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Example

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$
\begin{array}{llll}
(p \wedge q) \rightarrow (p \vee q) & \equiv & \neg(p \wedge q) \vee (p \vee q) & \text{by truth table for } \rightarrow \\
& \equiv & (\neg p \vee \neg q) \vee (p \vee q) & \text{by the first De Morgan law} \\
& \equiv & (\neg p \vee p) \vee (\neg p \vee \neg q) & \text{by associative and} \\
& & & \text{commutative laws} \\
& & & \text{laws for disjunction} \\
& \equiv & T \vee T & \text{by truth tables} \\
& \equiv & T & \text{by the domination law}
\end{array}
$$

Logical equivalence application

- 1.Judge if two formula are equivalence
- 2.Judge tautology , Contradictions
- 3. Simplify proposition formula

EXAMPLE

- $P \leftrightarrow Q$ 与 $P \wedge Q \vee \neg P \wedge \neg Q$
- $(\mathbf{P} \rightarrow Q) \wedge (Q \rightarrow P) = (\neg \mathbf{P} \vee Q) \wedge (P \vee \neg Q)$

$= \neg \mathbf{P} \wedge (P \vee \neg Q) \vee Q \wedge (P \vee \neg Q)$

$= (\neg \mathbf{P} \wedge P \vee \neg \mathbf{P} \wedge \neg Q) \vee (Q \wedge P \vee Q \wedge \neg Q)$

$= P \wedge Q \vee \neg P \wedge \neg Q$

EXAMPLE

- Show that $(p \land q) \rightarrow (p \lor q)$ is a tautology.

$$
\begin{aligned}
&(p \land q) \rightarrow (p \lor q) \\
&\quad \Leftrightarrow \neg(p \land q) \lor (P \lor q) \\
&\quad \Leftrightarrow (\neg p \lor \neg q) \lor (p \lor q) \\
&\quad \Leftrightarrow (\neg p \lor p) \lor (\neg q \lor q) \\
&\quad \Leftrightarrow T \lor T \\
&\quad \Leftrightarrow T
\end{aligned}
$$

Exercise

- Show $Q \vee \neg((\neg P \vee Q) \wedge P)$ is a tautology

first : truth table

second: Calculation by means of basic Equivalences

Simplification

- It is not the case that if he does not come, I will not go.
- P: he comes. Q: I will go

$$\neg(\neg P \rightarrow \neg Q)$$
$$\Leftrightarrow \neg(\neg\neg P \vee \neg \mathbf{Q}) \Leftrightarrow \neg\neg\neg \mathbf{P} \wedge \neg\neg \mathbf{Q}$$
$$\Leftrightarrow \neg \mathbf{P} \wedge \mathbf{Q}$$

Homework

- § 1.2 2,4,36,40
- § 1.3 12, 20, 21, 23, 24, 27, 30, 31