Graphs

Chapter 10

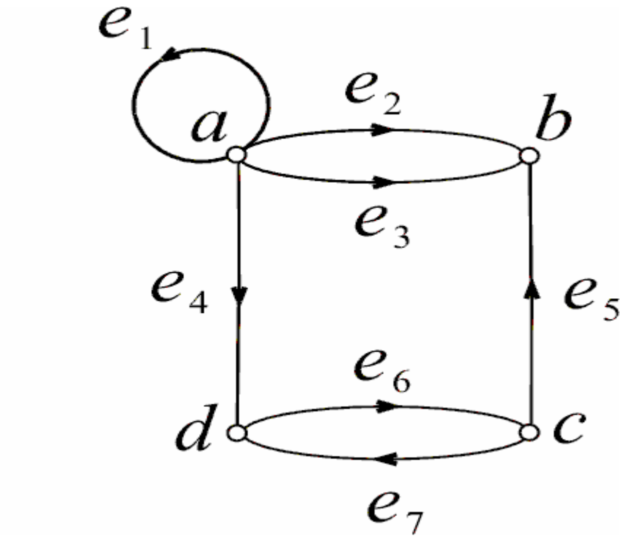
- Graph can be applied in many areas.
- Describe the webs, airline routes, call and so on
- Determine whether two computers are connected by a communications link
- Design the shortest water pipes for every house
- Get the shortest path from one city to another city
- Use graph to find the number of colors needed to color the regions of maps
- Design the shortest path from one server to another server

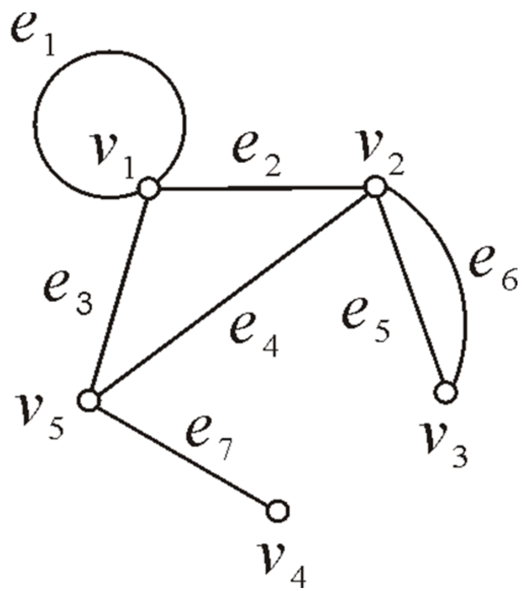
Graph

- A graph G consists of a nonempty set of vertices and a set of connections linking pairs of vertices. These pairs of vertices are called edges. G=(V,E)
- A directed graph (V,E) consists of a set of vertices V and a set of directed edges E. Each directed edge associated with the ordered pair (u,v) is said to start at u and end at v.

Types of graphs

- Directed graph
- Undirected graph
- Mixed graph



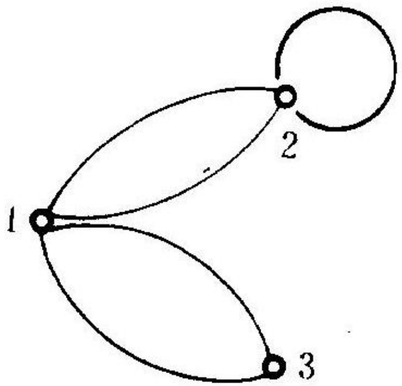


$V = \{v_1, v_2, \ldots, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

Types of graphs

- Simple graph: each edge connects two different vertices and no two edges connect the same pair of vertices.
- Multigraph:



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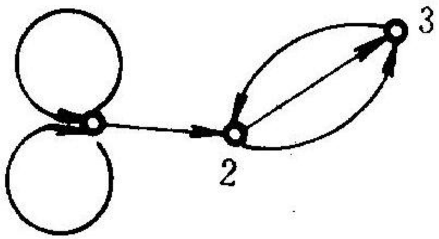

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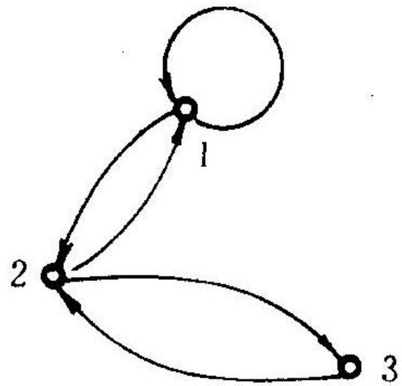

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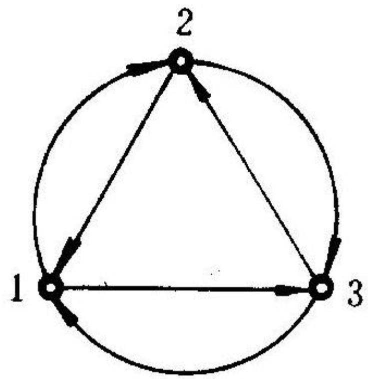
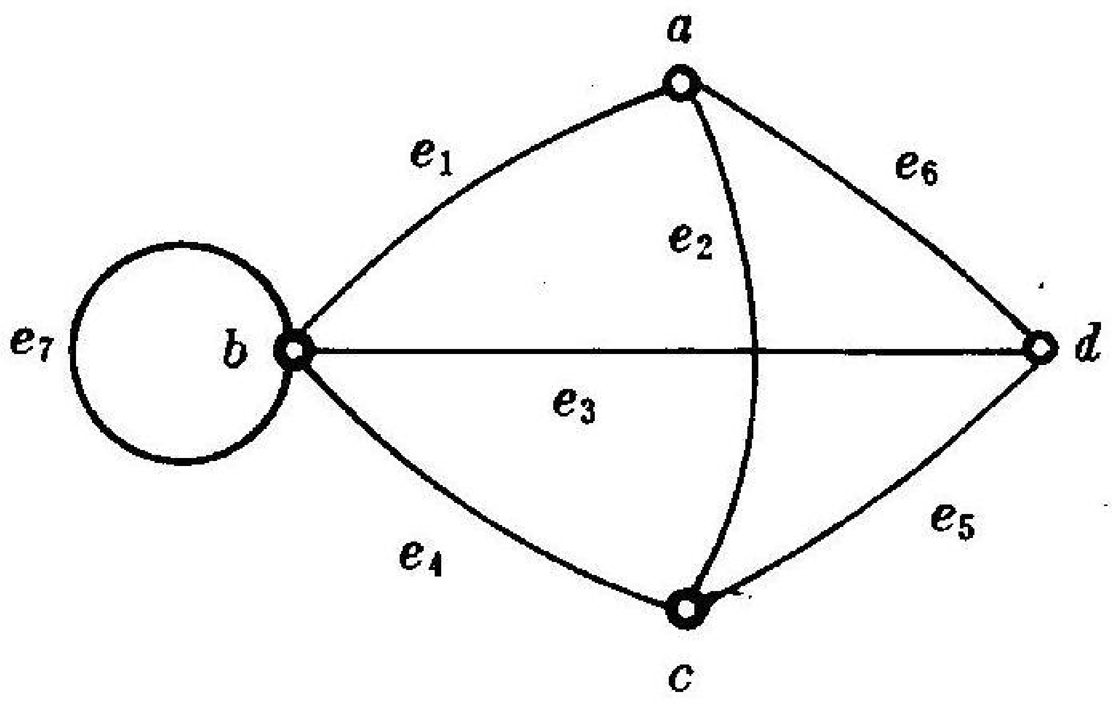


(d)

Definitions for undirected graph

- Definition1: Two vertices u and v in an undirected graph G are called adjacent in G if u and v are endpoints of an edge e of G. such an edge is called incident with the vertices u and v and e is said to connect u and v.
- Definition2: The set of all neighbors of a vertex v of G=(V,E) is called the neighborhood of v.
- Definition3: The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex, the degree of the vertex v is denoted by deg(v).



(a)

Handshaking theorem

- Let G=(V,E) be an undirected graph with m edges. Then $$\sum_{i=1}^{n} \deg(v_i) = 2m$$
- The sum of the degrees of all vertices is twice the number of the edges.
- An undirected graph has an even number of vertices of odd degree.

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

Definitions for directed graph

- Definition: when (u,v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called the initial vertex of (u,v) and v is the terminal or end vertex.
- In a graph with directed edges, the in-degree of a vertex v, denoted by $\deg^{-}(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $\deg^{+}(v)$, is the number of edges with v as their initial vertex.

Handshaking theorem2

Theorem

Let G=(V,E) be a graph with directed edges, then

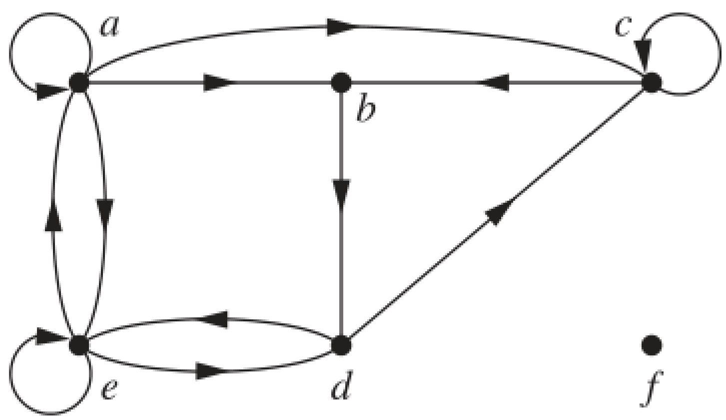
$\sum_{v\in V} deg^{-}(v) = \sum_{v\in V} deg^{+}(v) = |E|$

The sum of the in degrees is same as the sum of the out degrees.

Example

- Find the in-degree and out-degree of each vertex in the graph G with directed edge

The in-degrees in G are deg−(a) = 2, deg−(b) = 2, deg−(c) = 3, deg−(d) = 2,deg−(e) = 3, and deg−(f) = 0. The out-degrees are deg+(a) = 4, deg+(b) = 1, deg+(c) = 2,deg+(d) = 2, deg+(e) = 3, and deg+(f) = 0.

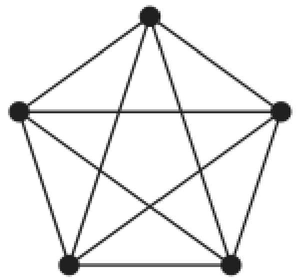


G

Special simple graphs

- Complete graph

A complete graph on n vertices, denoted by Kn, is a simple graph that contains exactly one edge between each pair of distinct vertices.



K_5

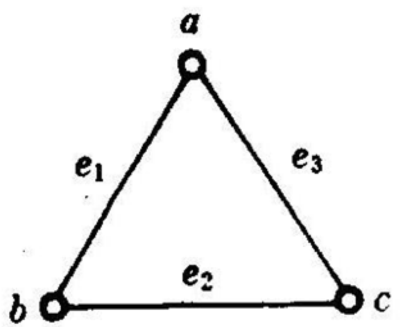
- Cycles
- Wheels

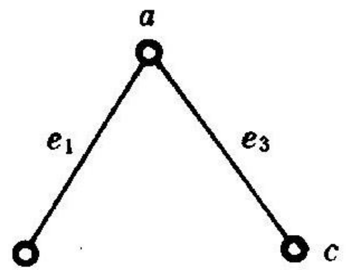
subgraph

- **Assume there are two graphs G=(V,E) and G'=(V',E'). if V'⊆V, and E'⊆E, G' is called subgraph of G.**
- Removing edge

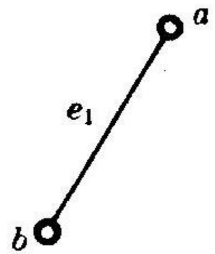
  G-e=(V, E-{e})
- Removing vertices from a graph

  G-v=(V-v, E') , where E' is the set of edges of G not incident to v.





G_1 删去边 e_2



G_1 删去结点 c

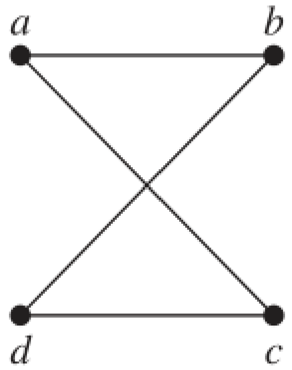
Representing Graphs

- Adjacency lists (table)
- Adjacency matrices
- Incidence matrices

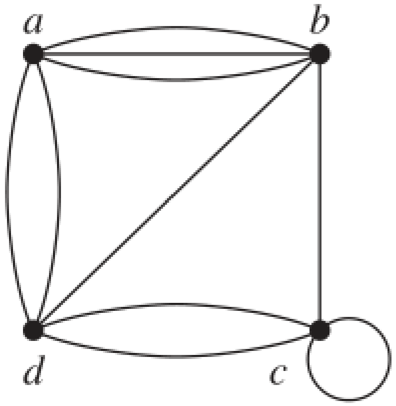
Adjacency matrices

- zero-one matrices
- List all vertices in any order. $v_1, v_2, \ldots, v_n$
- $a_{ij} = \begin{cases} 1 & if\ \{v_i, v_j\} is\ an\ edge\ of\ G \\ 0 & otherwise \end{cases}$
- It can be used to represent graphs with loops and multiple edges.

Examples

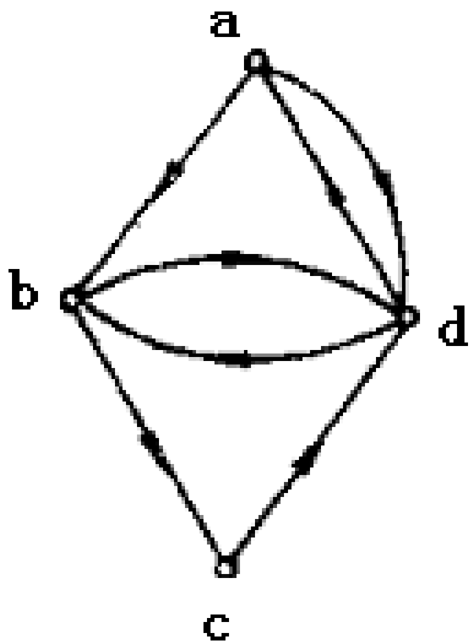


$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

Examples

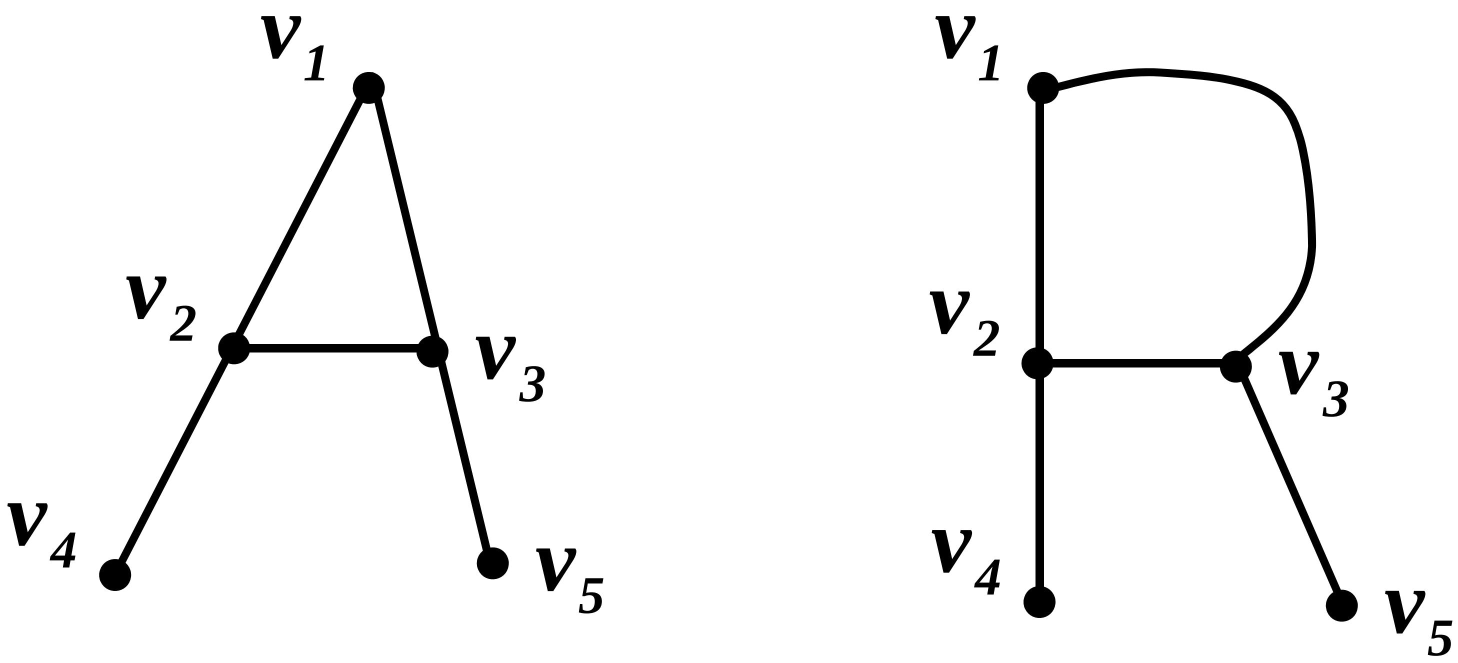


$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

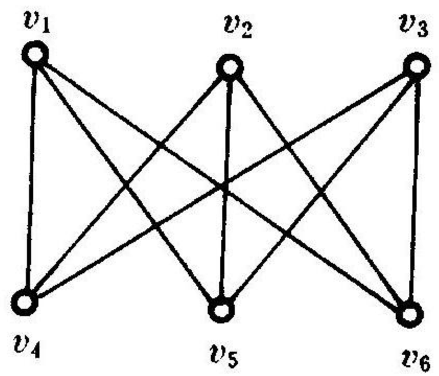
Isomorphism of graphs

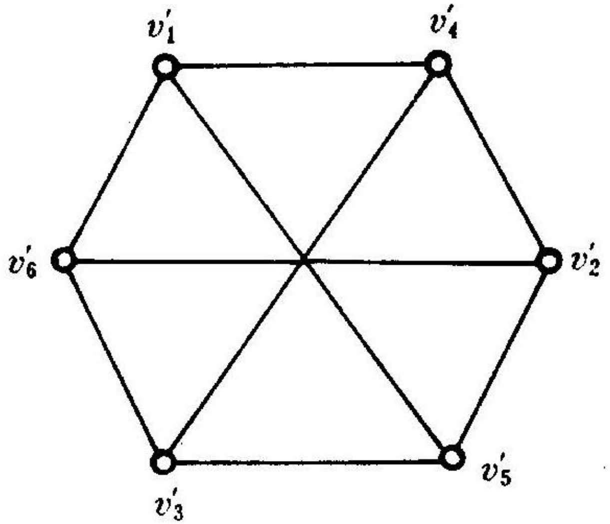
- Definition
- the simple graph $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there exists a bijective function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2, for all a and b in V1. Such a function f is called an isomorphism.

Isomorphic

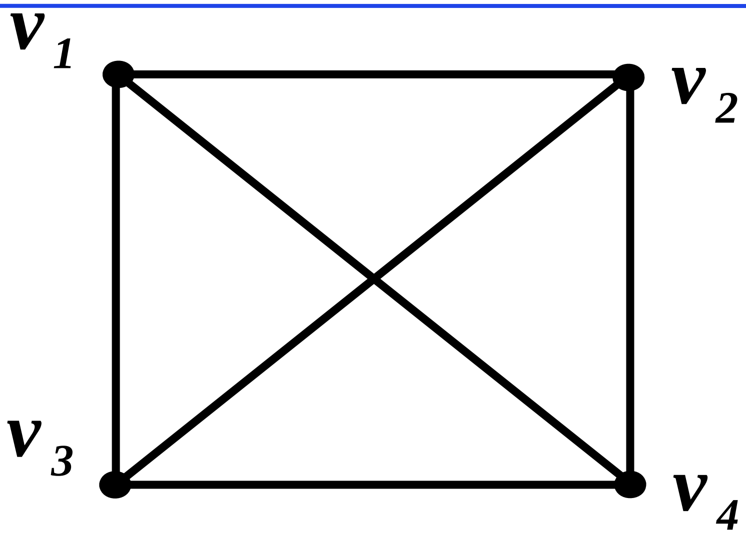


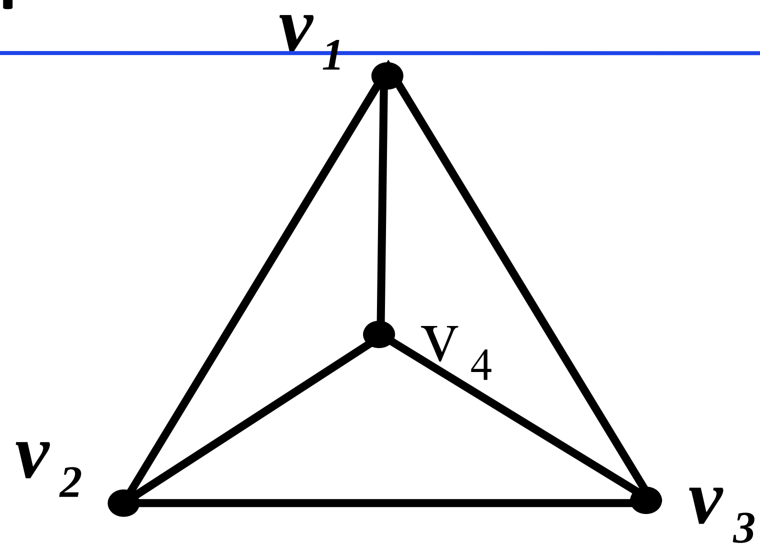
Isomorphic

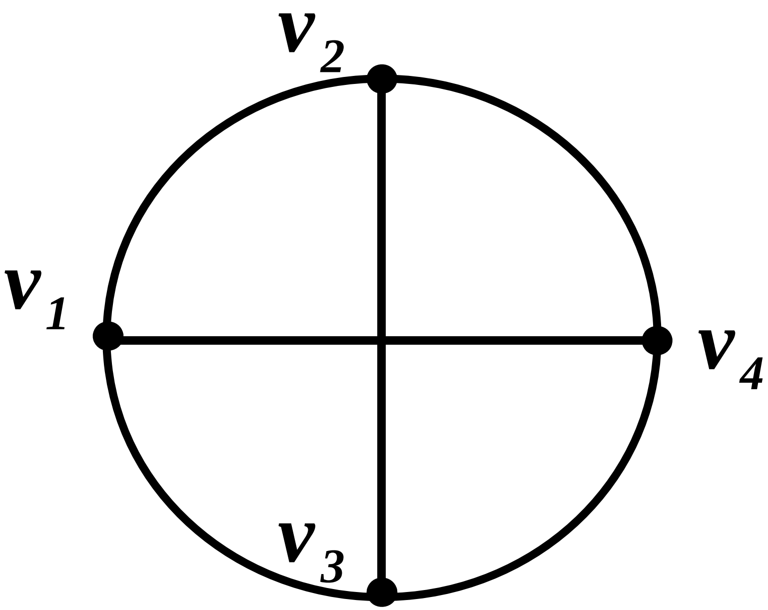


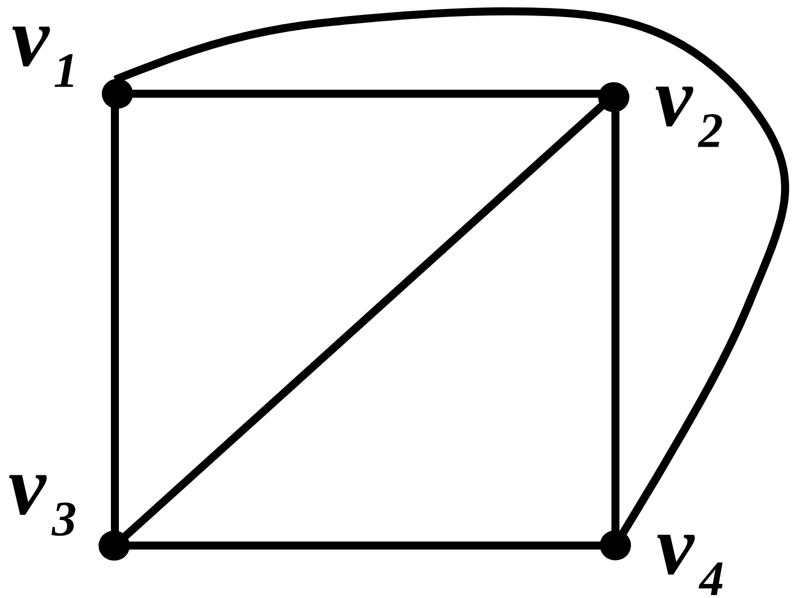


Isomorphic





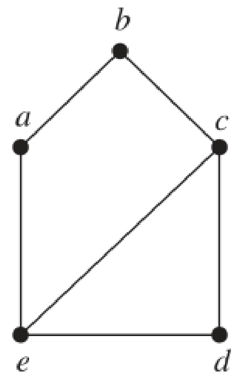




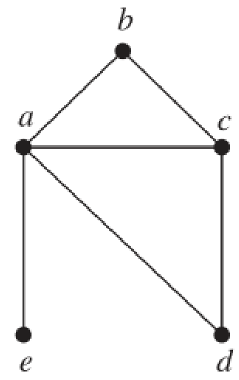
How to determine

- It is very difficult to determine
- We have design algorithms to determine
- Nauty software can determine whether two graphs with as many as 100 vertices are isomorphic in less than a second on a pc.
- Use degrees, the number of edges, subgraph adjacency matrix, and so on.

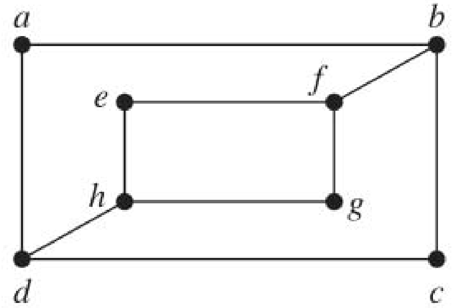
Not isomorphic



G



H



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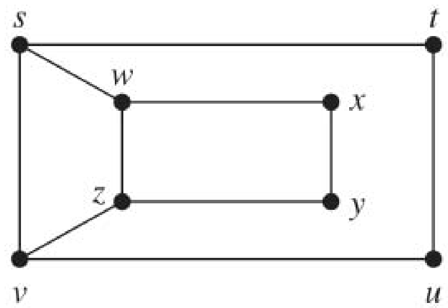
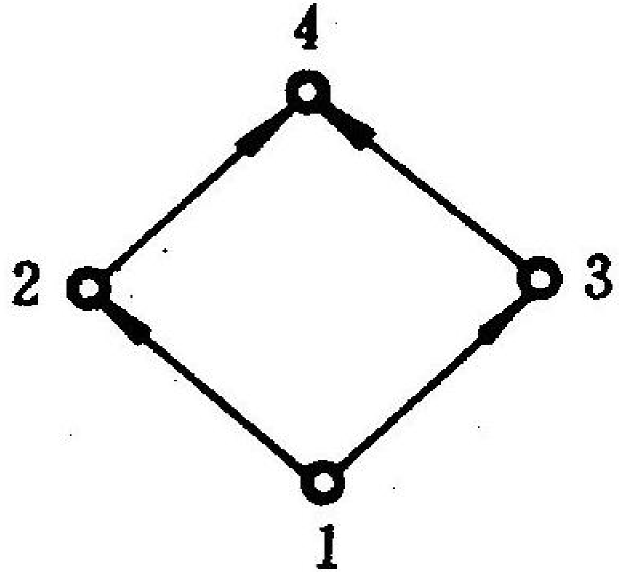
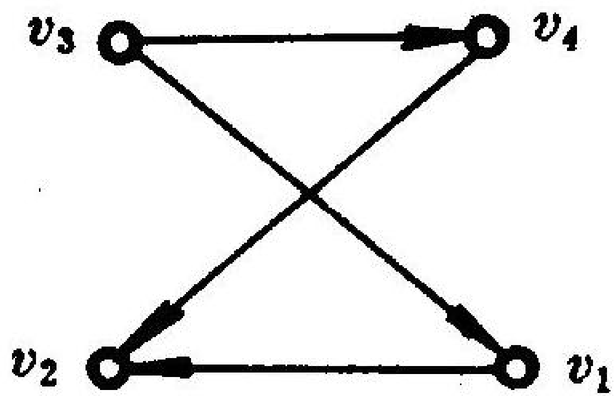


H

isomorphic



(a)



(b)

Homework

- 10.1 4,7
- 10.2 2, 8
- 10.3 5,7, 11, 15, 17, 20, 39, 40, 41