Partial Orderings

Section 9.6

Section Summary

- Partial Orderings
- Hasse Diagrams
- Special elements

Partial Orderings

Definition 1: A relation *R* on a set S is called a *partial ordering*（偏序）, or *partial order,* if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering *R* is called a *partially ordered set*, or *poset*(偏序集), and is denoted by (*S*, *R*). Members of *S* are called *elements* of the poset.

- Use ≼ to denote partial ordering, we read "less than or equals"
- $\langle x,y\rangle \in R \Leftrightarrow xRy \Leftrightarrow x \preccurlyeq y \Leftrightarrow x \le y$

Partial Orderings (*continued*)

Example 1: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a.
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

Partial Orderings (*continued*)

Example 2: Show that the divisibility relation (|) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a.
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

- $(\boldsymbol{Z}^+, |)$ is a poset.

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

$\{1\} \preccurlyeq \{1,2\}$

The properties all follow from the definition of set inclusion.

Comparability

Definition 2: The elements *a* and *b* of a poset $(S, \preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When *a* and *b* are elements of *S* so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then *a* and *b* are called *incomparable*.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

Definition 3: If $(S, \preccurlyeq)$ is a poset and every two elements of *S* are comparable, *S* is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain.*

Hasse Diagrams

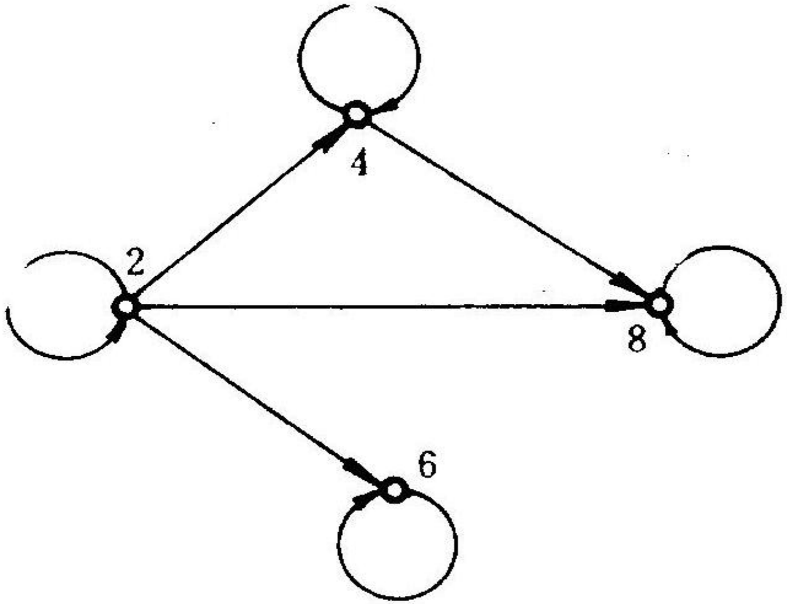
A *Hasse diagram*(哈斯图) is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

It is simpler that digraph. (simplified version)

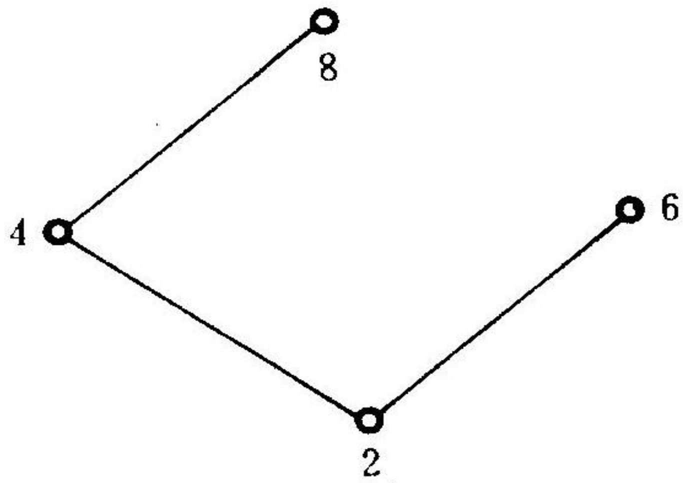
Procedure for Constructing a Hasse Diagram

- To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:
  - Remove the loops (a, a) present at every vertex due to the reflexive property.
  - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

- A={2, 4, 6, 8}, D={ ⟨2,2⟩ , ⟨4,4⟩ , ⟨6,6⟩ , ⟨8,8⟩ , ⟨2,4⟩ , ⟨2,6⟩ , ⟨2,8⟩ , ⟨4,8⟩ }



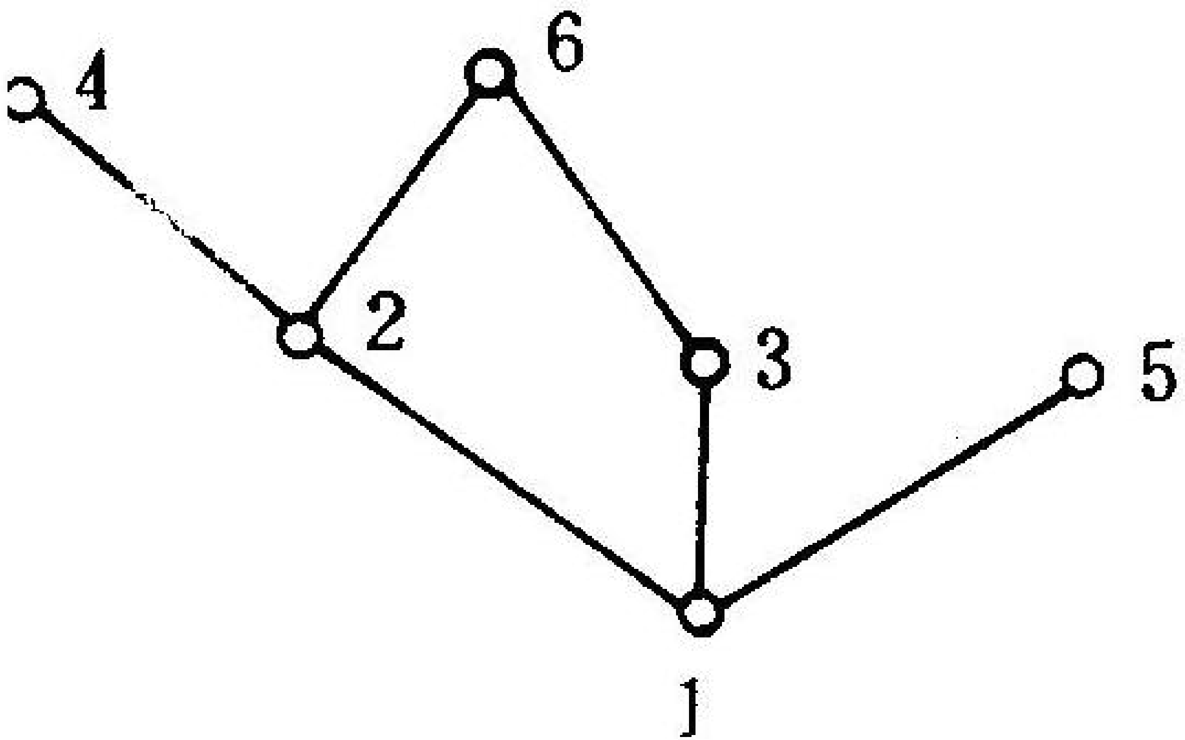
(a)



(b)

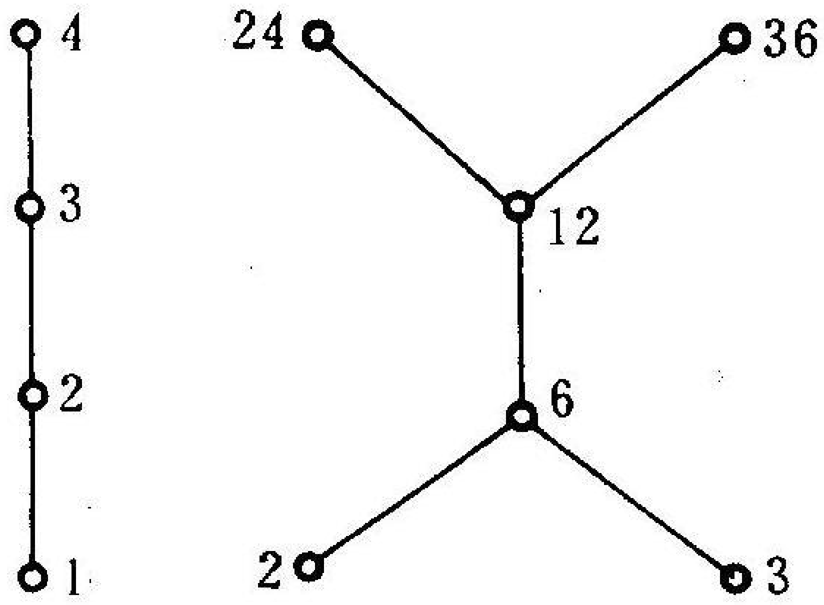
Example

- ({1,2,3,4,5,6}, |)



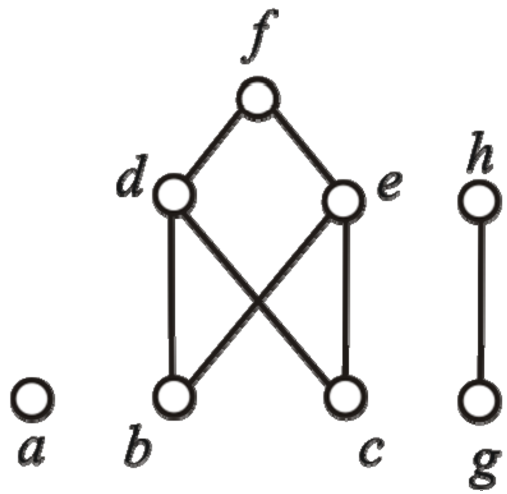
Examples

- (a)$P=\{1,2,3,4\}$, $\langle P,\leq\rangle$
- (b)$A=\{2,3,6,12,24,36\}$, $\langle A,|\rangle$



Example

Try to get the original set and partial ordering by means of the following graph.



Solution: $A=\{a, b, c, d, e, f, g, h\}$

$R=\{\langle b,d\rangle,\langle b,e\rangle,\langle b,f\rangle,\langle c,d\rangle,\langle c,e\rangle,\langle c,f\rangle,\langle d,f\rangle,\langle e,f\rangle,\langle g,h\rangle\} \cup I_A$

Maximal and Minimal Elements

- Let $<B,\preccurlyeq>$ be a poset,
- y is maximal element in B
  $\Leftrightarrow \forall x(x\in B \land y\preccurlyeq x \rightarrow x=y)$

There is no x such that $y \prec x$

- y is minimal element in B
  $\Leftrightarrow \forall x(x\in B \land x\preccurlyeq y \rightarrow x=y)$

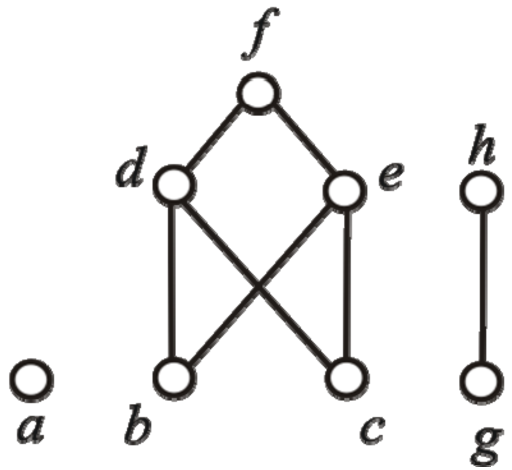
There is no x such that $x \prec y$

Greatest and least Elements

- Let $<B, \preccurlyeq>$ be a poset,
- y is the greatest element in B
  $\Leftrightarrow \forall x(\ x \in B \rightarrow x \preccurlyeq y\ )$
- y is the least element in B
  $\Leftrightarrow \forall x(\ x \in B \rightarrow y \preccurlyeq x\ )$
- The greatest and least elements are unique if they exist.

Example

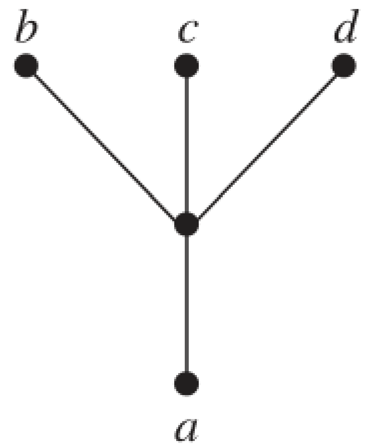
Try to spot some unusual elements, such as maximal element , minimal element, the greatest and least elements .



maximal element : a, f, h
minimal element: a, b, c, g

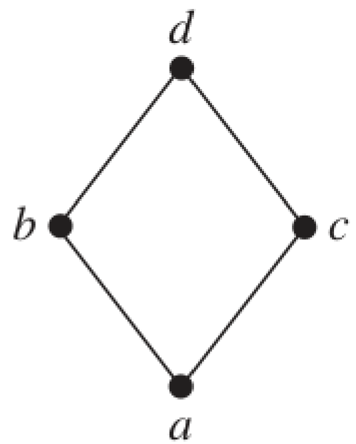
The greatest and least elements: none

Example



maximal element : b, c, d
minimal element: a

The greatest element: no
The least element: no



maximal element : d
minimal element: a

The greatest element: d
The least element: a

Upper and lower bounds

Let $<A,\preccurlyeq>$ be a poset, $B \subseteq A$, $y \in A$

y is upper bound in B

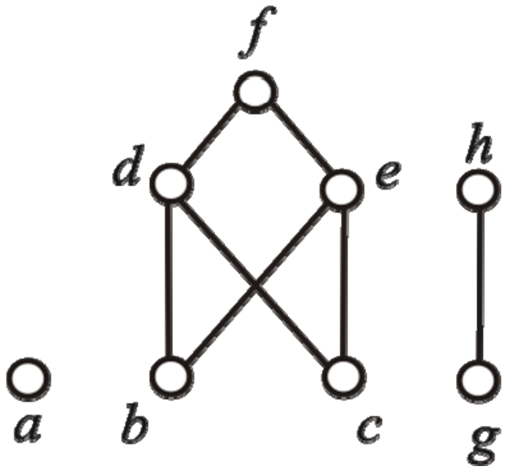
$$\Leftrightarrow \forall x(\, x \in B \rightarrow x \preccurlyeq y \,)$$

y is lower bound in B

$$\Leftrightarrow \forall x(\, x \in B \rightarrow y \preccurlyeq x \,)$$

Example

Find the lower and upper bounds of the subset B＝{ b,c,d } in the poset with the Hasse diagram as follows



the lower bounds: no
the upper bounds: d, f

Glb and lub

- The element x is called the least upper bound of the subset B if x is an upper bound that is less than every other upper bound of A.
- The element x is called the greatest lower bound of the subset B if x is a lower bound that is greater than every other upper bound of A.

Homework

- 9.6 P662

1(a)(c)(e) 9 10 15 22 25 32 33