

1.4 Predicates and Quantifiers

谓词与量词

Summary for propositioal logic(1.1-1.3) Mathematics

- Using propositions and operators to denote long sentences(Translation)
- Constructing the truth table for all propositional formula by means of the truth table for logic operator(precedence of operator)
- Get truth table by hand with a small number of variables.
- Remember key laws and rules in your brain.(implication)
- Using laws to calculate and get the equivalence of compound proposition.

Example (using truth table)

Discrete Mathematics

Prove $P \rightarrow (Q \rightarrow R) = (P \land Q) \rightarrow R$ The last column is always omitted

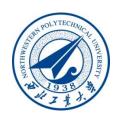
P Q R	$Q \rightarrow R$	P∧Q	$P \rightarrow (Q \rightarrow R)$	$P \land Q \rightarrow R$	$(P \rightarrow (Q \rightarrow R))$
					\leftrightarrow $(P \land Q \rightarrow R)$
0 0 0	1	0	1	1	1
0 0 1	1	0	1	1	1
0 1 0	0	0	1	1	1
0 1 1	1	0	1	1	1
1 0 0	1	0	1	1	1
1 0 1	1	0	1	1	1
1 1 0	0	1	0	0	1
1 1 1	1	1	1	1	1



Example(using laws to calculate)

• Prove
$$P \rightarrow (Q \rightarrow R) = (P \land Q) \rightarrow R$$

Proof: $P \rightarrow (Q \rightarrow R)$
 $= \neg P \lor (\neg Q \lor R)$
 $= (\neg P \lor \neg Q) \lor R$
 $= \neg (P \land Q) \lor R$
 $= (P \land Q) \rightarrow R$



Limitations of the propositional logic

"All persons are mortal."

"Socrates is a person."

"Socrates is mortal."

$$(P \land Q) \rightarrow R$$

- This conclusion is valid, but this compound proposition cannot show it is a tautology.
- Propositional logic cannot express enough the meaning of all statements.
- It is very necessary to introduce more powerful logic.



Predicates

- The statement has two parts:
 - predicate and subject
- **Predicates** represent properties or relations among objects.
- Uppercase letter is used to denote the predicate.
 Lowercase letter denotes the subject.

EX: P(x): x is greater than 3
P(x) becomes a proposition when a value is assigned to x

 All propositions belong to predicates, but not all predicates are propositions.



Predicates

- Let Q(x, y) denote the statement "x = y + 3."
- Q denotes ...=...+ 3; it shows the relation between x and y.
- x and y are subjects.
- What are the truth values of the propositions Q(1, 2) and Q(3, 0)?

One way to make predicate become proposition is that x is assigned constant(particular value). Sometimes if predicate is unknown, it is also assigned.



Quantification

- Another way to make predicate become proposition.
- Focus on two types of quantified statements:

Universal

Example: 'all CS NWPU graduates have to pass discrete math"

the statement is true for all graduates

Existential

Example: 'Some CS NWPU students graduate with honor.'

– the statement is true for some people



Universal quantifier

Definition: The universal quantification of P(x) is the proposition: "P(x) is true for all values of x in the domain of discourse." The notation $\forall x P(x)$ denotes the universal quantification of P(x), and is expressed as **for every x**, P(x).

Example:

- Let P(x) denote x > x 1.
- the universe of discourse of x is all real numbers.
- What is the truth value of $\forall x P(x)$?
- Answer: Since every number x is greater than itself minus 1. Therefore, $\forall x P(x)$ is true.



Existential quantifier

Definition: The **existential quantification** of P(x) is the proposition "There exists an element in the domain (universe) of discourse such that P(x) is true." The notation $\exists x P(x)$ denotes the existential quantification of P(x), and is expressed as **there is** an x such that P(x) is true.

Example 1:

- Let T(x) denote x > 5 and x is from Real numbers.
- What is the truth value of $\exists x T(x)$?
- Answer:
- Since 10 > 5 is true. Therefore, it is **true that** $\exists x T(x)$.



Quantifiers

- ◆ The domain must always be specified when a quantifier is used
- ◆ The truth value of the proposition depends on the selection of the domain .

Example:

Let P(x) be the statement"x=3".

- Domain is only 3
- Domain consists of all real numbers
- $\forall x P(x)$
- ∃xP(x)



Example

• What is the truth value of $\exists x P(x)$ where P(x) is the statement " $x^2 > 10$ " and the universe of discourse consists of the positive integers not exceeding $\exists x P(x)$ $\exists x P(x)$

 $\forall x P(x)=P(1) \land P(2) \land P(3)=T.$



Summary of quantified statements

• When $\forall x P(x)$ and $\exists x P(x)$ are true and false?

Statement	When true?	When false?
∀x P(x)	P(x) true for all x	There is an x where P(x) is false.
∃x P(x)	There is some x for which P(x) is true.	P(x) is false for all x.

- Suppose the elements in the universe of discourse are finite, such as x1, x2, ..., xN
- $\forall x P(x)$ is true whenever $P(x1) \land P(x2) \land ... \land P(xN)$ is true
- $\exists x P(x)$ is true whenever $P(x1) \lor P(x2) \lor ... \lor P(xN)$ is true.



Translating

- Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.
- C(x) "x has studied calculus"
- If the domain consists of the students in the class, the answer is $\forall x \ C(x)$
- If the domain consists of all things in our world
- S(x): x is a student in this class
- the statement is expressed as follows: $\forall x \ (S(x) \to C(x))$
- For every thing, if x is a student in this class then x has studied calculus.



EXAMPLE

- Express the statements "Some student in this class has visited Mexico"
- If the domain consists of the students in the class, the answer is $\exists x \ C(x)$
- If the domain consists of all things in our world, the statement is expressed as follows: $\exists x \ (S(x) \land M(x))$
- There exists one thing x that x is a student in this class and x has visited Mexico".



Example

- All lions are fierce.
- Some lions do not drink coffee.
- The domain consists of all things in the world.
- S(x) "x is a lion"
- $\forall x (S(x) \rightarrow C(x))$
- $\exists x (S(x) \land M(x))$



Example

- There is no one who doesn't make mistakes;
- M(x): x is a person F(X): x makes mistakes
- $_{7}\exists x(M(x) \land_{7}F(X))$



NEGATIONS

- $\neg (\forall x P(x)) \iff \exists x \neg P(x)$
- $\neg (\exists x P(x)) \iff \forall x \neg P(x)$

They are logically equivalent no matter what the predicates is and what the domain is

Eg: All students have sent homework online. $\forall x P(x)$

- It is not the case that all students have sent homework online.
- Some students haven't sent homework online.
- It is not the case that some students have sent homework online
- None of students have sent homework online.



Binding Variables

$(\forall x P(x)) \land Q(x)$

- When a quantifier is used on x, we say that the occurrence of x is bound.
- The variable x is outside of the scope of the $\forall x$ quantifier, and is therefore free. Not a complete proposition!
- In order to distinguish bound x from free x, we can change bound variable.

$$(\forall y P(y)) \land Q(x)$$

• $(\forall x P(x)) \land (\exists x Q(x))$ all are bound, no free variable.



Examples

- $(1) \forall x P(x) \rightarrow Q(x)$
- (2) $\exists x(P(x, y) \rightarrow Q(x, y)) \lor P(y, z)$
- (3) $\forall x(F(x) \rightarrow G(x,y)) \rightarrow \exists y(H(x) \land L(x,y,z))$
- (4) $\forall x(F(x,y) \rightarrow \exists y H(x,y))$



Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.
- Example: "Every real number has an inverse" is
- $\forall x \exists y(x + y = 0)$
- where the domains of x and y are the real numbers.



Order of Quantifiers

- Examples:
- Let P(x,y) be the statement "x + y = y + x." Assume that U is the real numbers. Then $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ have the same truth value.
- $\forall x \forall y P(x,y) \Leftrightarrow \forall y \forall x P(x,y)$
- $\exists y \exists x P(x,y) \Leftrightarrow \exists x \exists y P(x,y)$



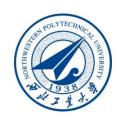
Order of Quantifiers

- Let Q(x,y) be the statement "x + y = 0." Assume that U is the real numbers. Then $\forall x$ $\exists y P(x,y)$ is true, but $\exists y \forall x P(x,y)$ is false.
- ∀x ∃yP(x,y) and ∃y∀xP(x,y) are not logical equivalent.

"For all real numbers x there is a real number y such that x + y = 0"

the statement is true.

"There is a real number y such that for all real numbers x it is true that x + y = 0" the statement is false.



Quantifications of Two Variables Mathematics

Statement	When True?	When False
orall x orall y P(x,y) $orall y orall x P(x,y)$	P(x,y) is true for every pair x,y .	There is a pair x , y for which $P(x,y)$ is false.
$orall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y .
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y .	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y P(x,y) \ \exists y \exists x P(x,y)$	There is a pair x , y for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y

The order of the quantifiers is important unless all the quantifiers are universal quantifiers or all are existential quantifiers



Homework

• § 1.4 – 10, 12, 18, 21, 23, 29, 45,46