Set Operations

Section 2.2

Section Summary

- Set Operations
  - Union
  - Intersection
  - Complement
  - Difference
- Set Identities
- Proving Identities

Union

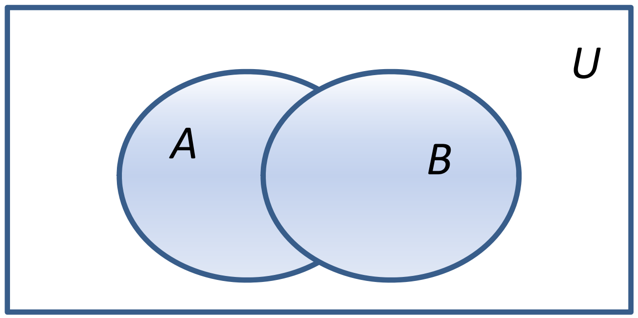
- **Definition**: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by $A \cup B$, is the set:

$$\{x \mid x \in A \lor x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

- **Definition**: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is

$$\{x | x \in A \land x \in B\}$$

- Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.
- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?

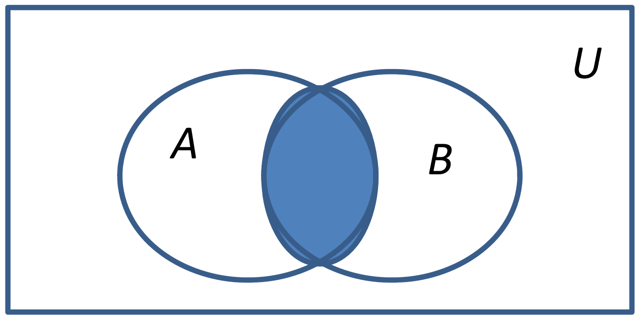
  Solution: $\{3\}$

- **Example:** What is?

  $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\emptyset$

Venn Diagram for $A \cap B$

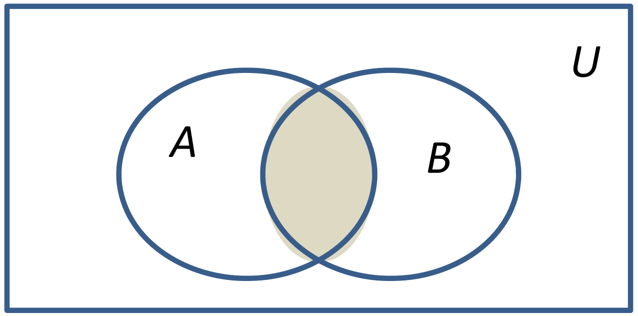


The Cardinality of the Union of Two Sets

- Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- **Example**: Let *A* be the math majors in your class and *B* be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.



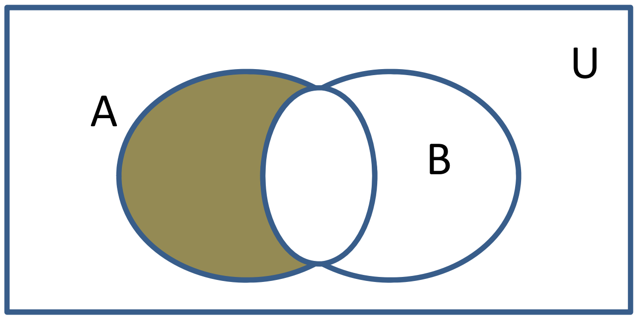
Venn Diagram for A, B, $A \cap B$, $A \cup B$

Difference

- **Definition**: Let *A* and *B* be sets. The *difference* of *A* and *B*, denoted by *A* – *B*, is the set containing the elements of *A* that are not in *B*. The difference of *A* and *B* is also called the complement of *B* with respect to *A*.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

Eg. $\{1,2,3\} - \{3,4,5\} = \{1,2\}$



Venn Diagram for $A - B$

Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set U - A

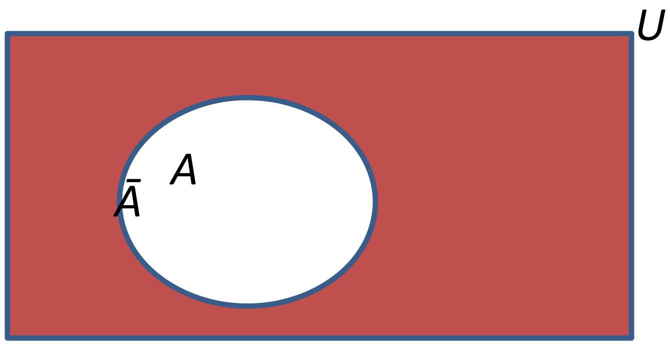
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement



Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

6. $B - A$

   Solution: $\{6,7,8\}$

Proving Set Identities

- Different ways to prove set identities:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Use set builder notation and propositional logic.
  3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complement law

$$\overline{(\overline{A})} = A$$

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Set Identities

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

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Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Proof of Second De Morgan Law

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

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Proof of Second De Morgan Law

These steps show that:

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

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Proof of Second De Morgan Law

These steps show that:

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: Second De Morgan Law

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x \mid x \notin A \cap B\} & \text{by defn. of complement} \\
 & = & \{x \mid \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
 & = & \{x \mid \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
 & = & \{x \mid \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
 & & & \text{for Prop Logic} \\
 & = & \{x \mid x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
 & = & \{x \mid x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
 & = & \{x \mid x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
 & = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Example

- Prove

$$\overline{A \cup (B \cap C)} = \bar{A} \cap (\bar{C} \cup \bar{B})$$

Symmetric Difference (*optional*)

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$A \oplus B = \{x \mid (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}$$

$$A \oplus B = (A-B) \cup (B-A) = (A \cup B) - (A \cap B)$$

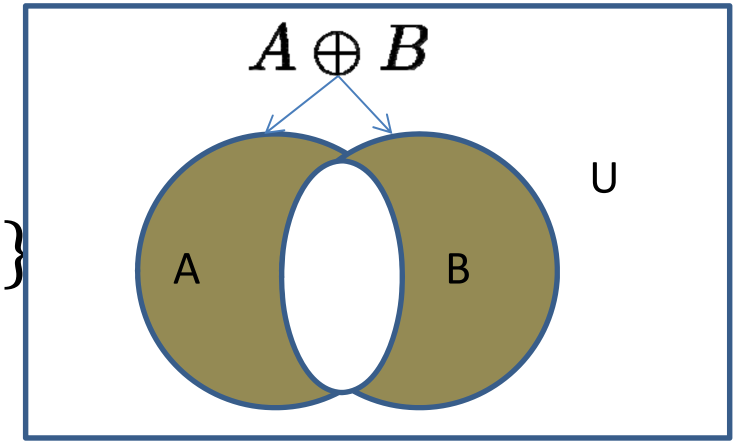
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is:

– **Solution**: $\{1,2,3,6,7,8\}$



Venn Diagram

Generalized Unions and Intersections

- Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

The union of a collection of sets

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

The intersection of a collection of sets

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

- For $i = 1,2,\ldots,$ let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n$$

Computer Representation of sets

- Bit strings are often used to represent information.
- They also can be used to represent set.
- Assume that the universal set U is finite.
- Specify an arbitrary ordering the elements of U. Represent a subset A with the bit string, where the ith bit in this strings is 1 if ai belongs to A and is 0 if ai does not belong to A.

Example

- U={1,2,3,4,5,6,7,8,9,10}
- This bit string 11 1110 0000 represents the subset {1,2,3,4,5}
- 10 1010 1010 {1,3,5,7,9}
- The union of these sets are
- 10 1010 1010 $\vee$ 11 1110 0000= 11 1110 1010
- this strings represent {1,2,3,4,5,7,9}

Homework

- P144 2.2
- 4, 19, 26, 27, 32, 35, 56