1.5 Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.
- Example: “Every real number has an inverse” is
- $\forall x \, \exists y(x + y = 0)$
- where the domains of x and y are the real numbers.

Order of Quantifiers

- Examples:
- Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.
- $\forall x\ \forall y P(x,y) \Leftrightarrow \forall y\ \forall x P(x,y)$
- $\exists y\ \exists x P(x,y) \Leftrightarrow \exists x\ \exists y P(x,y)$

Order of Quantifiers

- Let Q(x,y) be the statement "x + y = 0." Assume that U is the real numbers. Then $\forall x \exists y P(x,y)$ is true, but $\exists y \forall x P(x,y)$ is false.
- $\forall x \exists y P(x,y)$ and $\exists y \forall x P(x,y)$ are not logical equivalent.

"For all real numbers x there is a real number y such that x + y = 0"

the statement is true.

"There is a real number y such that for all real numbers x it is true that x + y = 0"

the statement is false.

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

The order of the quantifiers is important unless all the quantifiers are universal quantifiers or all are existential quantifiers

Translating Mathematical Statements into Predicate Logic

- Example : Translate "The sum of two positive integers is always positive" into a logical expression.
- Solution:
  - Rewrite the statement to make the implied quantifiers and domains explicit:
    - "For every two integers, if these integers are both positive, then the sum of these integers is positive."
  - Introduce the variables x and y, and specify the domain, to obtain:
    - "For all positive integers x and y, x + y is positive."
  - The result is:
    - $\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$
    - where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

- Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."
- Solution:
  - Let P(w,f) be "w has taken f " and Q(f,a) be "f is a flight on a ."
  - The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
  - Then the statement can be expressed as:
  - $\exists w\, \forall a\, \exists f\ (P(w,f) \wedge Q(f,a))$

Example

- Express the statement "Everyone has exactly one best friend" as a logical expression.

$$\boxed{\forall x \exists y \forall z (B(x, y) \wedge ((z \neq y) \rightarrow \neg B(x,z))).}$$

- Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression.

$$\boxed{\forall x\, ((F(x) \wedge P(x)) \rightarrow \exists y\, M(x, y)).}$$

Ex： Everyone love her baby

- Let
  - $P(x)$： x refer to people
  - $C(x)$： x refer to baby
  - $I(x，y)$： x is baby of y
  - $L(x，y)$： x loves y
- Proposition:
  - $\forall x \forall y(P(y) \wedge C(x) \wedge I(x, y) \rightarrow L(y, x))$

Translating Nested Quantifiers into English

- Translate the statement
- $\forall x\,(C(x) \vee \exists y(C(y) \wedge F(x, y)))$
- into English, where C(x) is "x has a computer," F(x,y) is "x and y are friends," and the universe of discourse for both x and y is the set of all students in NWPU.

The statement says that for every student x in NWPU x has a computer or there is a student y such that y has a computer and x and y are friends. In other words, every student in NWPU has a computer or has a friend who has a computer.

Example

- Translate the statement
- $\exists x \forall y \forall z(((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z)))$
- into English, where F(a,b) means a and b are friends and the universe of discourse for x, y, and z is the set of all students in your school.

This statement says that there is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends. In other words, there is a student none of whose friends are also friends with each other.

1.6 Rules of Inference

- In mathematic proof, the following form often occurs.

If the premises $p_1, p_2, \ldots, p_n$ are true , then the conclusion q holds.

$(p_1 \land p_2 \land \ldots \land p_n) \to q$ is a tautology.

- Proofs are valid argument that establishing the truth of statement.
- An argument is a sequence of statements ends with *conclusion.*
- The argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false.
- How to deduce new statement from known statements? get valid conclusion?
- Our basic tools are rules of inference

P	Q	$P \to Q$
0	0	1
0	1	1
1	0	0
1	1	1

Revisiting the Socrates Example

- We have the two premises:
  - “All men are mortal.”
  - “Socrates is a man.”
- And the conclusion:
  - “Socrates is mortal.”
- How do we get the conclusion from the premises?

The Argument

- $\forall x(M(x) \rightarrow D(x)) \wedge M(a) \rightarrow D(a)$ (**Tautology**)
- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\begin{array}{ll} & \forall x(M(x) \rightarrow D(x)) \\ & M(a) \\ \hline \therefore & D(a) \end{array}$$

We will see shortly that this is a valid argument.

Rules of Inference
Simplification

$$\frac{p \land q}{\therefore q}$$

Corresponding Tautology:

$$(p \land q) \rightarrow p$$

Example:

Let *p* be "I will study discrete math."
Let *q* be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$$

Corresponding Tautology:

$((p) \land (q)) \rightarrow (p \land q)$

Example:

Let p be "I will study discrete math."

Let q be "I will study English literature."

"I will study discrete math."

"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$$p \rightarrow (p \vee q)$$

Example:

Let p be "I will study discrete math."

Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$(\neg p \wedge (p \vee q)) \rightarrow q$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math or I will study English literature."

"I will not study discrete math."

"Therefore , I will study English literature."

Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

Example:

Let p be "It is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$$(\neg q \wedge (p \to q)) \to \neg p$$

Example:
Let p be "it is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:

$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example:

Let *p* be "it snows."

Let *q* be "I will study discrete math."

Let *r* be "I will get an A."

"If it snows, then I will study discrete math."

"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Resolution

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology:

$$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$$

Example:
Let *p* be "I will study discrete math."
Let *r* be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will English literature."

Rules of Inference

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

- simple argument forms that will be used to construct more complex argument forms.

Other rules

- Note that logic equivalences are also can be used as rules of inference.
- Table 6 in p29
- $P \to Q \equiv \neg P \vee Q$

  $P \to Q \equiv \neg Q \to \neg P$

Example

- Show the argument

$(P \to Q) \land (R \to \neg Q) \land R \to \neg P$ is valid.

- Premises: $P \to Q$ $R \to \neg Q$ R
- Conclusion: $\neg P$

Method 1: Truth table

P	Q	R	P→Q	R→┐Q	(P→Q)∧(R→┐Q)∧R→┐P
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	1	1
0	1	1	1	0	1
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	1	1
1	1	1	1	0	1

Method 2: Proposition Calculation

$(P\to Q)\wedge(R\to\neg Q)\wedge R\to\neg P$

$\Leftrightarrow\neg((\neg P\vee Q)\wedge(\neg R\vee\neg Q)\wedge R)\vee\neg P$

$\Leftrightarrow\neg(\neg P\vee Q)\vee\neg(\neg R\vee\neg Q)\vee\neg R\vee\neg P$

$\Leftrightarrow(P\wedge\neg Q)\vee(R\wedge Q)\vee\neg R\vee\neg P$

$\Leftrightarrow(P\wedge\neg Q)\vee\neg P\vee(R\wedge Q)\vee\neg R$

$\Leftrightarrow(P\vee\neg P)\wedge(\neg Q\vee\neg P)\vee(R\vee\neg R)\wedge(Q\vee\neg R)$

$\Leftrightarrow(\neg Q\vee\neg P)\vee(Q\vee\neg R)$

$\Leftrightarrow T$

So $(P\to Q)\wedge(R\to\neg Q)\wedge R\Rightarrow\neg P$

Method 3: Reasoning method

- Premises: $P \to Q$, $R \to \neg Q$, R
- Conclusion: $\neg P$

Step	Statement	Reason
1	R	P
2	$R \to \neg Q$	P
3	$\neg Q$	Modus ponens using (1)(2)
4	$P \to Q$	P
5	$\neg P$	Modus tollens using (3)(4)

Example

- With these hypotheses:

  "It is not sunny this afternoon and it is colder than yesterday."

  "We will go swimming only if it is sunny."

  "If we do not go swimming, then we will take a canoe trip."

  "If we take a canoe trip, then we will be home by sunset."

- Using the inference rules, construct a valid argument for the conclusion:

  "We will be home by sunset."

Solution:

1. Choose propositional variables:

   p : "It is sunny this afternoon." *r* : "We will go swimming." *t :* "We will be home by sunset."

   q : "It is colder than yesterday." *s :* "We will take a canoe trip."

2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow s,\ s \rightarrow t$$
$$\text{Conclusion: } t$$

Build Arguments

3. Construct the Valid Argument

Hypotheses: $\neg p \wedge q$, $r \to p$, $\neg r \to s$, $s \to t$
Conclusion: t

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

Example

- Show that the hypotheses
  - “If you send me an e-mail message, then I will finish writing the program,”
  - “If you do not send me an e-mail message, then I will go to sleep early,”
  - “If I go to sleep early, then I will wake up feeling refreshed”
- lead to the conclusion
  - “If I do not finish writing the program, then I will wake up feeling refreshed.”

Proof Example

- p: send e-mail; q:finish writing program;r: sleep early; s:wake up refreshed
- premises: (1) p->q (2) ¬p->r (3) r->s
- conclusion: ¬q->s

Step	Reason
1. $p \rightarrow q$	Premise #1.
2. $\neg q \rightarrow \neg p$	Contrapositive of 1.
3. $\neg p \rightarrow r$	Premise #2.
4. $\neg q \rightarrow r$	Hypothetical syllogism using 2,3.
5. $r \rightarrow s$	Premise #3.
6. $\neg q \rightarrow s$	Hypothetical syllogism using 4,5.

Example

- Who is murder?
  - (1） A or B is murder
  - (2） If A is the murder, then murder occurs after midnight.
  - (3） If B say the truth, then the light is on after midnight.
  - (4） If B lied, the murder occurs before midnight.
  - (5） The light is off after midnight.

Ex(Cont)

- Proposition：
  - A： A is the murder.
  - B： B is the murder.
  - C： murder occurs after midnight.
  - D： B tell the truth
  - E： light is off after midnight

- compound proposition
  - （1） $A \vee B$
  - （2） $A \to C$
  - （3） $D \to \neg E$
  - （4） $\neg D \to \neg C$
  - （5） E

Who is the murder?

homework

- 1.5: P69 10, 28, 33

- 1.6: P82 4, 5, 6, 12