Relations and Their Properties

Section 9.1

Section Summary

- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

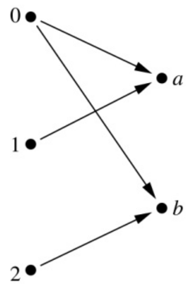
Binary Relations

Definition: A *binary relation(二元关系)* R from a set A to a set B is a subset $R \subseteq A \times B$.

Use aRb to denote (a, b)∈R

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a), (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A.

Binary Relation on a Set

Definition: A binary relation *R on a set A* is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b),(a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a,b) \mid a \text{ divides } b\}$ are $(1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_2 = \{(a,b) \mid a > b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_5 = \{(a,b) \mid a = b + 1\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

$$(1,1),\ (1,\ 2),\ (2,\ 1),\ (1,\ -1),\ \text{and } (2,\ 2)?$$

Solution: Checking the conditions that define each relation, we see that the pair $(1,1)$ is in R_1, R_3, R_4 , and R_6: $(1,2)$ is in R_1 and R_6: $(2,1)$ is in R_2, R_5, and R_6: $(1, -1)$ is in R_2, R_3, and R_6 : $(2,2)$ is in R_1, R_3, and R_4.

Example

$A=\{1, 2\}$, 则

$R= \{(1,1), (1,2), (2,1), (2,2)\}$

$I = \{(1,1), (2,2)\}$

Reflexive Relations

Definition: R is *reflexive(自反)* iff $(a,a) \in R$ for every element $a \in A$. R is reflexive if and only if

$$\forall x[x \in A \longrightarrow (x,x) \in R]$$

- Let $A=\{1,2,3\}$, $R \subseteq A \times A$
  - $\{(1,1),(1,3),(2,2),(2,1),(3,3)\}$
  - $\{(1,2),(2,3),(3,1)\}$
  - $\{(1,2),(2,2),(2,3),(3,1)\}$

Example

The following relations on the integers are reflexive:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

Symmetric Relations

Definition: R is *symmetric*(对称) iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. R is symmetric if and only if

$$\forall x \forall y\, [(x,y) \in R \longrightarrow (y,x) \in R]$$

Example: The following relations on the integers are symmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

The following are not symmetric:

$R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$),

$R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Antisymmetric Relations

Definition:A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric(反对称)*. Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y\, [(x,y) \in R \wedge (y,x) \in R \longrightarrow x = y]$$

- **Example**: The following relations on the integers are antisymmetric:

  $R_1 = \{(a,b) \mid a \leq b\}$,
  $R_2 = \{(a,b) \mid a > b\}$,
  $R_4 = \{(a,b) \mid a = b\}$,
  $R_5 = \{(a,b) \mid a = b + 1\}$.

  For any integer, if a $a \leq b$ and $b \leq a$, then $a = b$.

  The following relations are not antisymmetric:

  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
  (note that both $(1,-1)$ and $(-1,1)$ belong to R_3),
  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

$$\forall x \forall y (xRy \land yRx \to x=y)$$

$$\equiv \forall x \forall y (xRy \land x \neq y \to \neg yRx)$$

Proof: $\forall x \forall y (xRy \land yRx \to x=y)$

$$\equiv \forall x \forall y (\neg (xRy \land yRx) \lor x=y)$$

$$\equiv \forall x \forall y (\neg (xRy) \lor \neg (yRx) \lor x=y)$$

$$\equiv \forall x \forall y (\neg (xRy \land x \neq y) \lor \neg yRx)$$

$$\equiv \forall x \forall y (xRy \land x \neq y \to \neg yRx)$$

Antisymmetry is not the negative of symmetry

Examples

- Let $A=\{1,2,3\}$, $R\subseteq A\times A$
  - $\{(1,1),(1,2),(1,3),(2,1),(3,1),(3,3)\}$ (symmetric)
  - $\{(1,2),(2,3),(2,2),(3,1)\}$ (antisymmetric)
  - $\{(1,2),(2,3),(3,1)\}$ (antisymmetric)
  - $\{(1,2),(2,1),(2,2),(3,1)\}$
    (not symmetric and not antisymmetric)
  - $\{(1,1),(2,2)\}$ (symmetric and antisymmetric)
  - ϕ (symmetric and antisymmetric)

Transitive Relations

Definition: A relation R on a set A is called transitive(传递) if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z[(x,y) \in R \land (y,z) \in \mathrm{R} \longrightarrow (x,z) \in R\ ]$$

- **Example**: The following relations on the integers are transitive:

  $R_1 = \{(a,b) \mid a \leq b\}$,

  For every integer, $a \leq b$ and $b \leq c$, then $b \leq c$.

  $R_2 = \{(a,b) \mid a > b\}$,

  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

  $R_4 = \{(a,b) \mid a = b\}$.

  The following are not transitive:

  $R_5 = \{(a,b) \mid a = b + 1\}$ (note that both (3,2) and (4,3) belong to R_5, but not (3,3)),

  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both (2,1) and (1,2) belong to R_6, but not (2,2)).

- Let $A=\{1,2,3\}$, $R\subseteq A\times A$
  - $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3)\}$ is transitive
  - $\{(1,2),(2,3),(3,1)\}$ is not transitive.
  - $\{(1,3)\}$? ϕ ? (transitive)

Combining Relations

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations: $R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$

  $R_1 \cap R_2 = \{(1,1)\}$ $\quad$ $R_1 - R_2 = \{(2,2),(3,3)\}$

  $R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

Composition

Definition: Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

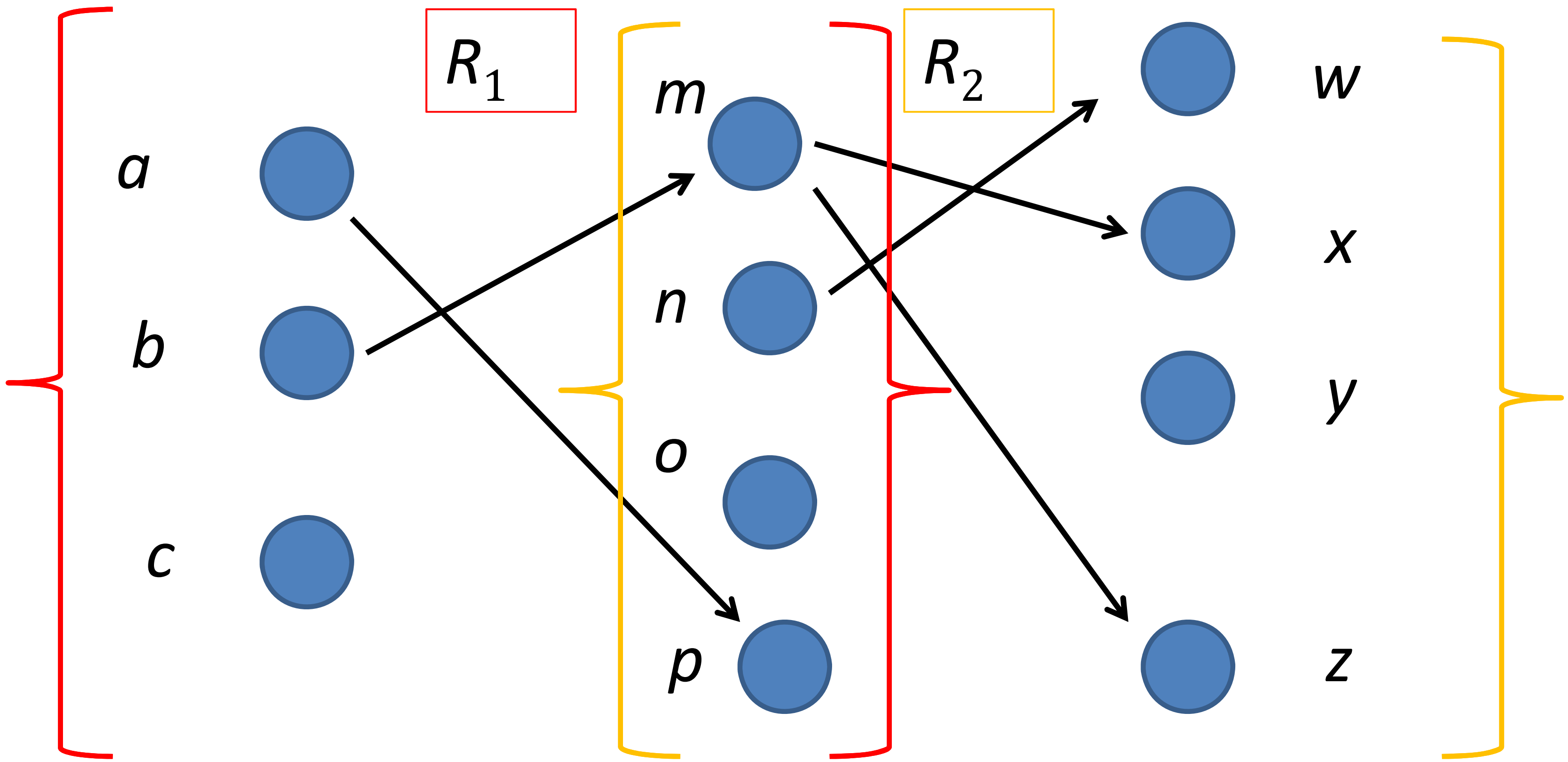
Then the *composition*(合成) (or *composite*) of R_1 and R_2, is a relation from A to C where

- if (a,b) is a member of R_1 and (b,c) is a member of R_2, then (a,c) is a member of $R_2 \circ R_1$.

$R_2R_1 = R_2 \circ R_1$

$= \{(a,c) \mid \exists b \, [b \in B \wedge (a,b) \in R_1 \wedge (b,c) \in R_2] \}$

Representing the Composition of a Relation



$$R_2 \circ R_1 = \{(b,x),(b,z)\}$$

- Let $A=\{a,b,c,d\}$, R_1, R_2 are relations on A:

  $R_1 = \{(a,a),(a,b),(b,d)\}$

  $R_2 = \{(a,d),(b,c),(b,d),(c,b)\}$

  then:

  $R_2 \circ R_1 = \{(a,d),(a,c)\}$

  $R_1 \circ R_2 = \{(c,d)\}$

  $R_1 \circ R_1 = \{(a,a),(a,b),(a,d)\}$

  $R_2 \circ R_2 = \{(b,d), (c,c), (c,d)\}$

Powers of a Relation

Definition: Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

Theorem 1: The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1,2,3 \ldots.$

(*see the text for a proof via mathematical induction*)

Homework

- 9.1 P608
- 3, 4, 5, 6(a,c,e), 10, 26, 32, 33,47