1.4 Predicates and Quantifiers

谓词与量词

Summary for propositioal logic(1.1-1.3)

- Using propositions and operators to denote long sentences(Translation)
- Constructing the truth table for all propositional formula by means of the truth table for logic operator(precedence of operator)
- Get truth table by hand with a small number of variables.
- Remember key laws and rules in your brain.(implication)
- Using laws to calculate and get the equivalence of compound proposition.

Example (using truth table)

Prove $P \rightarrow (Q \rightarrow R) = (P \land Q) \rightarrow R$ The last column is always omitted

P Q R	$Q \rightarrow R$	$P \land Q$	$P \rightarrow (Q \rightarrow R)$	$P \land Q \rightarrow R$	$(P \rightarrow (Q \rightarrow R)) \leftrightarrow (P \land Q \rightarrow R)$
0 0 0	1	0	1	1	1
0 0 1	1	0	1	1	1
0 1 0	0	0	1	1	1
0 1 1	1	0	1	1	1
1 0 0	1	0	1	1	1
1 0 1	1	0	1	1	1
1 1 0	0	1	0	0	1
1 1 1	1	1	1	1	1

Example(using laws to calculate)

- *Prove* $P \rightarrow (Q \rightarrow R) = (P \wedge Q) \rightarrow R$

Proof : $P \rightarrow (Q \rightarrow R)$

$$= \neg P \vee (\neg Q \vee R)$$

$$= (\neg P \vee \neg Q) \vee R$$

$$= \neg (P \wedge Q) \vee R$$

$$= (P \wedge Q) \rightarrow R$$

Limitations of the propositional logic

"All persons are mortal."

"Socrates is a person."

"Socrates is mortal."

$$(P \land Q) \rightarrow R$$

- **This conclusion is valid, but this compound proposition cannot show it is a tautology.**
- **Propositional logic cannot express enough the meaning of all statements.**
- **It is very necessary to introduce more powerful logic.**

Predicates

- **The statement has two parts:**

  predicate and subject

- **Predicates** represent properties or relations among objects.
- Uppercase letter is used to denote the predicate. Lowercase letter denotes the subject.

EX: P (x) : x is greater than 3

P(x) becomes a proposition when a value is assigned to x

- All propositions belong to predicates, but not all predicates are propositions.

Predicates

- Let Q(x, y) denote the statement "x = y + 3.“
- Q denotes …=…+ 3; it shows the relation between x and y.
- x and y are subjects.
- What are the truth values of the propositions Q(1, 2) and Q(3, 0)?

One way to make predicate become proposition is that x is assigned constant(particular value). Sometimes if predicate is unknown, it is also assigned.

Quantification

- Another way to make predicate become proposition.
- Focus on two types of quantified statements:

Universal

Example: ‘ all CS NWPU graduates have to pass discrete math”

– the statement is true for all graduates

Existential

Example: ‘Some CS NWPU students graduate with honor.’

– the statement is true for some people

Universal quantifier

Definition: The universal quantification of P(x) is the proposition: "*P(x) is true for all values of x in the domain of discourse*." The notation **$\forall$x P(x)** denotes the universal quantification of P(x), and is expressed as **for every x, P(x).**

Example:

- Let P(x) denote x > x - 1.
- the universe of discourse of x is all real numbers.
- What is the truth value of $\forall$x P(x)?
- **Answer:** Since every number x is greater than itself minus 1. Therefore, $\forall$**x P(x) is true.**

Existential quantifier

Definition: The **existential quantification** of P(x) is the proposition "*There exists an element in the domain (universe) of discourse such that P(x) is true*." The notation **∃x P(x)** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.

Example 1:

- Let T(x) denote x > 5 and x is from Real numbers.
- What is the truth value of ∃ x T(x)?
- **Answer:**
- Since 10 > 5 is true. Therefore, it is **true that ∃ x T(x).**

Quantifiers

◆ **The domain must always be specified when a quantifier is used**

◆ **The truth value of the proposition depends on the selection of the domain .**

Example:

Let P(x) be the statement"x=3".

- Domain is only 3
- Domain consists of all real numbers
- $\forall xP(x)$
- $\exists xP(x)$

Example

- What is the truth value of $\exists x P(x)$ where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 3? $\forall x P(x)$? $\exists x P(x)$

$\forall x P(x) = P(1) \wedge P(2) \wedge P(3) = T.$

Summary of quantified statements

- When $\forall$**x P(x)** and $\exists$**x P(x)** are true and false?
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Statement	When true?	When false?
$\forall x\ P(x)$	P(x) true for all x	There is an x where P(x) is false.
$\exists x\ P(x)$	There is some x for which P(x) is true.	P(x) is false for all x.

- Suppose the elements in the universe of discourse are finite, such as x1, x2, ..., xN
- **$\forall$x P(x) is true whenever P(x1) $\wedge$ P(x2) $\wedge$ ... $\wedge$ P(xN) is true**
- **$\exists$x P(x) is true whenever P(x1) $\vee$ P(x2) $\vee$ ... $\vee$ P(xN) is true**.

Translating

- Express the statement " Every student in this class has studied calculus" using predicates and quantifiers.
- C(x) "x has studied calculus"
- If the domain consists of the students in the class, the answer is $\forall x\, C(x)$
- If the domain consists of all things in our world
- S(x): x is a student in this class
- the statement is expressed as follows: $\forall x\, (S(x) \rightarrow C(x))$
- For every thing, if x is a student in this class then x has studied calculus.

EXAMPLE

- Express the statements "Some student in this class has visited Mexico“
- If the domain consists of the students in the class, the answer is $\exists x\, C(x)$
- If the domain consists of all things in our world, the statement is expressed as follows: $\exists x\, (S(x) \wedge M(x))$
- There exists one thing x that x is a student in this class and x has visited Mexico”.

Example

- All lions are fierce.
- Some lions do not drink coffee.
- The domain consists of all things in the world.
- S(x)   " x is a lion"
- $\forall x\ (S(x) \rightarrow C(x))$
- $\exists x\ (S(x) \wedge M(x))$

Example

- There is no one who doesn’t make mistakes;
- M(x): x is a person   F(X): x makes mistakes
- $\neg \exists x(M(x) \land \neg F(X))$

NEGATIONS

- $\neg(\forall x\, P(x)) \Leftrightarrow \exists x \neg P(x)$
- $\neg(\exists x\, P(x)) \Leftrightarrow \forall x \neg P(x)$

They are logically equivalent no matter what the predicates is and what the domain is

Eg: All students have sent homework online. $\forall x\, P(x)$

- It is not the case that all students have sent homework online.
- Some students haven't sent homework online.

- It is not the case that some students have sent homework online
- None of students have sent homework online.

Binding Variables

$(\forall x\, P(x)) \wedge Q(x)$

- When a quantifier is used on x, we say that the occurrence of x is bound.

- The variable *x* is outside of the *scope* of the $\forall x$ quantifier, and is therefore free. Not a complete proposition!

- In order to distinguish bound x from free x, we can change bound variable.

  $(\forall y\, P(y)) \wedge Q(x)$

- $(\forall x\, P(x)) \wedge (\exists x\, Q(x))$ all are bound, no free variable.

Examples

(1) $\forall x P(x) \to Q(x)$

(2) $\exists x(P(x, y) \to Q(x, y)) \lor P(y, z)$

(3) $\forall x(F(x) \to G(x,y)) \to \exists y(H(x) \land L(x,y,z))$

(4) $\forall x(F(x,y) \to \exists y\ H(x,y))$

Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.
- Example: “Every real number has an inverse” is
- $\forall x\, \exists y(x + y = 0)$
- where the domains of x and y are the real numbers.

Order of Quantifiers

- Examples:
- Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\, \forall y P(x,y)$ and $\forall y\, \forall x P(x,y)$ have the same truth value.
- $\forall x\, \forall y P(x,y) \Leftrightarrow \forall y\, \forall x P(x,y)$
- $\exists y\, \exists x P(x,y) \Leftrightarrow \exists x\, \exists y P(x,y)$

Order of Quantifiers

- Let Q(x,y) be the statement "x + y = 0." Assume that U is the real numbers. Then $\forall x \exists y P(x,y)$ is true, but $\exists y \forall x P(x,y)$ is false.
- $\forall x \exists y P(x,y)$ and $\exists y \forall x P(x,y)$ are not logical equivalent.

"For all real numbers x there is a real number y such that x + y = 0"

the statement is true.

"There is a real number y such that for all real numbers x it is true that x + y = 0"

the statement is false.

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y

The order of the quantifiers is important unless all the quantifiers are universal quantifiers or all are existential quantifiers

Homework

- § 1.4 – 10, 12, 18, 21, 23, 29, 45,46