Generalized Permutations and Combinations

Section 6.5

Section Summary

- Permutations with Repetition(可重复)
- Combinations with Repetition
- Permutations with Indistinguishable Objects
- Distributing Objects into Boxes

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed. Hence, by the product rule there are n^r r-permutations with repetition. ◀

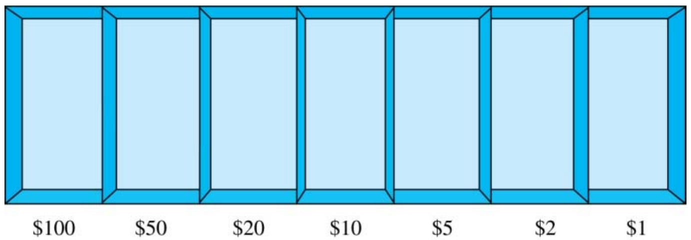
Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

Combinations with Repetition

Example: How many ways are there to select five bills from a box containing the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100? Assume that the order in which the bills are chosen does not matter, there are at least five bills of each type.

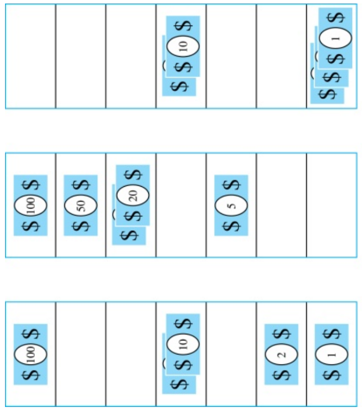
Solution:



continued →

Combinations with Repetition

- Some possible ways of selecting the five bills:



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- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

Combinations with Repetition

Theorem 2: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Proof: Each r-combination of a set with n elements with repetition allowed can be represented by a list of $n - 1$ bars and r stars. The bars mark the n cells containing a star.

The number of such lists is $C(n + r - 1, r)$, because each list is a choice of the r positions to place the stars, from the total of $n + r - 1$ positions to place the stars and the bars. This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars.

◀

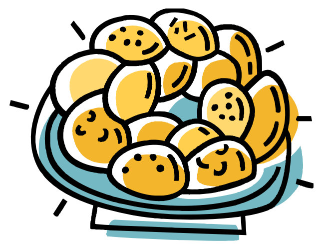
Combinations with Repetition

Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.



Combinations with Repetition

Example: How many solutions does the equation

$x_1 + x_2 + x_3 = 11$

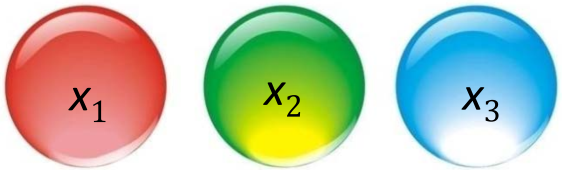
have, where x_1 , x_2 and x_3 are nonnegative integers?

Solution: Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three.

By Theorem 2 it follows that there are

$$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$$

solutions.



Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where the variables are integers

with $x1 \geq 1$, $x2 \geq 2$, and $x3 \geq 3$.

Solution: $C(3 + 5 - 1, 5) = C(7, 5) = = 21$

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

The reasoning can be generalized to the following theorem. →

Permutations with Indistinguishable Objects

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Distributing Objects into Boxes

- Many counting problems can be solved by counting the ways objects can be placed in boxes.
  - The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
  - The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

Distributing Objects into Boxes

- *Distinguishable objects* and *distinguishable boxes*.
  - There are $n!/(n_1!n_2!\cdots n_k!)$ ways to distribute n distinguishable objects into k distinguishable boxes.
- *Indistinguishable objects* and *distinguishable boxes*.
  - There are $C(n+r-1, n-1)$ ways to place r indistinguishable objects into n distinguishable boxes.
  - Proof based on one-to-one correspondence between n-combinations from a set with k-elements when repetition is allowed and the ways to place n indistinguishable objects into k distinguishable boxes.

Distributing Objects into Boxes

- *Distinguishable objects* and *indistinguishable boxes*.
  - There is no simple closed formula for the number of ways to distribute n distinguishable objects into j indistinguishable boxes.
- *Indistinguishable objects* and *indistinguishable boxes*.
  - No simple closed formula exists for this number.

Homework

- § 6.5 13, 14, 16, 32, 38