Rules for Quantified Statements

- How to deal with Quantified Statements for valid argument?
- Truth table and equivalent calculation don't work well for quantified statements. **P(x)**
- The rules of inference for quantified statements will be introduced to determine valid argument.

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

C is a particular member of the domain

Example:

Our domain consists of all dogs and Fido is a dog.

“All dogs are cute.”

“Therefore, Fido is cute.”

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

C must be an arbitrary, not a specific, member of the domain

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

We have no knowledge of what c is, only that it exists.

Example:

"There is someone who got an A in the course."
"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."
"Therefore, someone got an A in the class."

Inference Rules for Quantifiers

- $$\frac{\forall x\, P(x)}{\therefore P(o)}$$ **Universal instantiation**
- $$\frac{P(g)}{\therefore \forall x\, P(x)}$$ **Universal generalization**
- $$\frac{\exists x\, P(x)}{\therefore P(c)}$$ **Existential instantiation**
- $$\frac{P(o)}{\therefore \exists x\, P(x)}$$ **Existential generalization**

Returning to the Socrates Example

$\forall x(M(x) \rightarrow D(x))$

$M(a)$

$\therefore \quad D(a)$

M(x): x is a person
D(x): x is mortal
a: Socrates
Domain: all things in the world

Valid Argument

①	$\forall x\,(M(x) \rightarrow D(x))$	P
②	$M(a) \rightarrow D(a)$	① UI
③	$M(a)$	P
④	$D(a)$	②③ MP

Using Rules of Inference

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Solution: Let $C(x)$ denote "x is in this class," $B(x)$ denote "x has read the book," and $P(x)$ denote "x passed the first exam."

First we translate the premises and conclusion into symbolic form.

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

Continued on next slide →

Example

Valid Argument:

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	EG from (8)

Summary

- Generally, if we determine the argument with quantifiers is valid, the first step, we use UI or EI rules to remove quantifiers, then, we use rules of inference or basic logic equivalences to proof, the last step, if there exists quantifiers in conclusion, here, we need to use UG or EG rules to add it.
- If there are existential and universal quantifiers in premises, EI rule is usually first used, then UI rule.

Example

Identify the error or errors in this argument that supposedly shows that if $\exists x P(x) \land \exists x Q(x)$ is true then $\exists x(P(x) \land Q(x))$ is true.

1.	$\exists x P(x) \lor \exists x Q(x)$	Premise
2.	$\exists x P(x)$	Simplification from (1)
3.	$P(c)$	Existential instantiation from (2)
4.	$\exists x Q(x)$	Simplification from (1)
5.	$Q(c)$	Existential instantiation from (4)
6.	$P(c) \land Q(c)$	Conjunction from (3) and (5)
7.	$\exists x(P(x) \land Q(x))$	Existential generalization

$$\exists x(P(x) \land Q(x)) \Rightarrow \exists x P(x) \land \exists x Q(x)$$

Example

前提： $\forall x(F(x) \vee G(x))$，$\neg\exists x\, G(x)$. 结论： $\exists x\, F(x)$.

证明：① $\neg\exists x\, G(x)$　　P

② $\forall x \neg G(x)$　　① 替换规则

③ $\neg G(a)$　　② US

④ $\forall x(F(x) \vee G(x))$　　P

⑤ $F(a) \vee G(a)$　　④ US

⑥ $F(a)$　　③⑤ 析取三段论

⑦ $\exists x\, F(x)$　　⑥ EG

homework

45. Determine whether $\forall x(P(x) \to Q(x))$ and $\forall xP(x) \to \forall xQ(x)$ are logically equivalent. Justify your answer.

46. Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x\ P(x) \leftrightarrow \forall xQ(x)$ are logically equivalent. Justify your answer.

- **Domain: natural number**
- **P(x): x is an even number ; Q(x): x is divided by 3**

$\forall x Q(x)$ is 0 $\forall x\ P(x)$ is 0

So $\forall x\ P(x) \to \forall x\ Q(x)$ is 1 , $\forall x Q(x) \leftrightarrow x\ P(x)$ is 1

But $\forall x\ (P(x) \to Q(x))$ is 0 , $\forall x\ (P(x) \leftrightarrow Q(x))$ is 0

Therefore, $\forall x\ P(x) \to \forall x\ Q(x) \neq \forall x\ (P(x) \to Q(x))$

$\forall x Q(x) \leftrightarrow x\ P(x) \neq \forall x\ (P(x) \leftrightarrow Q(x))$

homework

- 1.6: P83 18, 24, 27, 29, 31