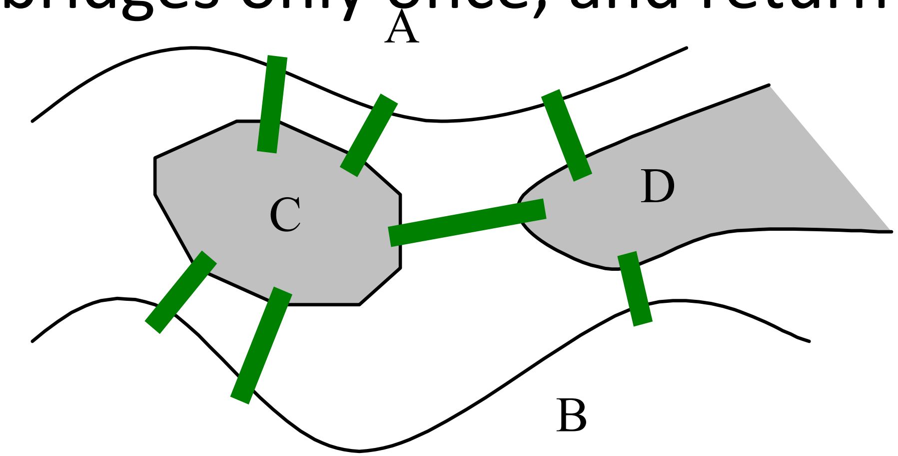
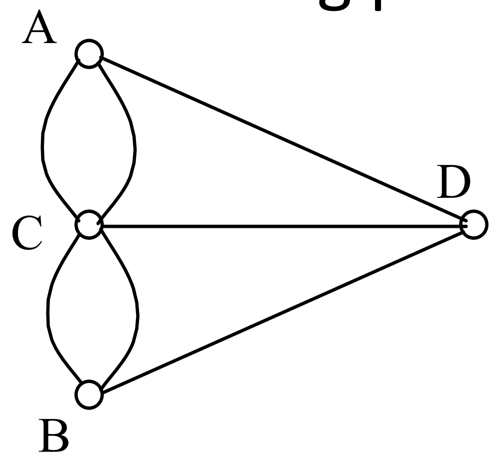
Euler Path

- Seven Bridges at Königsberg
- This town was divided into four sections by the branches of the river.
- Seven bridges connect these sections.
- The local people wondered whether it was possible to start from one location, travel across all the bridges only once, and return to the starting point.





- The Swiss mathematician Euler solved this problem using graph theory.
- This is the first use of graph theory.
- It is Euler that established the theory of graph.
- Euler path is named after him.

Euler Path and Euler Circuit

- Definition:
- An Euler path in G is a path which passes each edge in G exactly once.
- An Euler circuit in G is a circuit which passes each edge in G exactly once.
- An Euler graph is a graph which contains a Euler circuit.

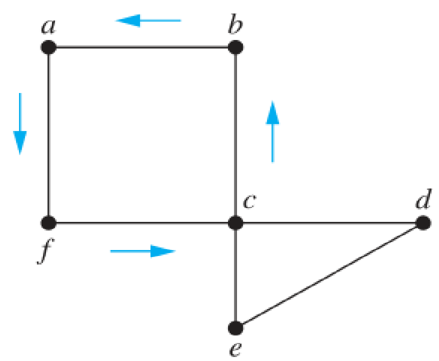
How to determine

Theorem:

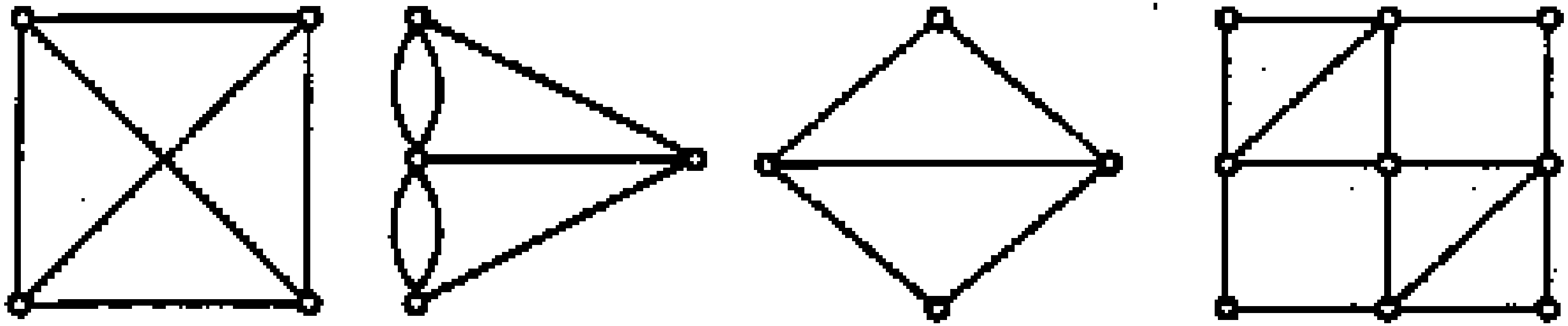
G is an Euler graph(has an Euler circuit) if and only if , for any vertex *v* in G, $d(v)$ is even.

$\Rightarrow$There are three types of vertices: starting, terminal, other vertices passed through. The circuit enters via an edge incident with these vertices and leaves via another edge. Then it contributes two to the degree of these vertices. For circuit, starting and terminal are the same vertex, so this vertex's degree is even.

- $\Leftarrow$ when the degrees of all vertices are even, now, we can find a Euler circuit.
- Start from any vertex, because its degree is even, so there are at least two edges incident with this vertex, we come out via any of these edges, and reach the second vertex, then come out via another edge, different from the previous edge, until we entered the starting vertex via another edge, different from the first edge. If we walk all edges, this is circuit, if some edges are left, continue to add edges left using the above ways.



- Therorem
- A connected multigraph has an Euler path if and only if it has exactly two vertices of odd degree.



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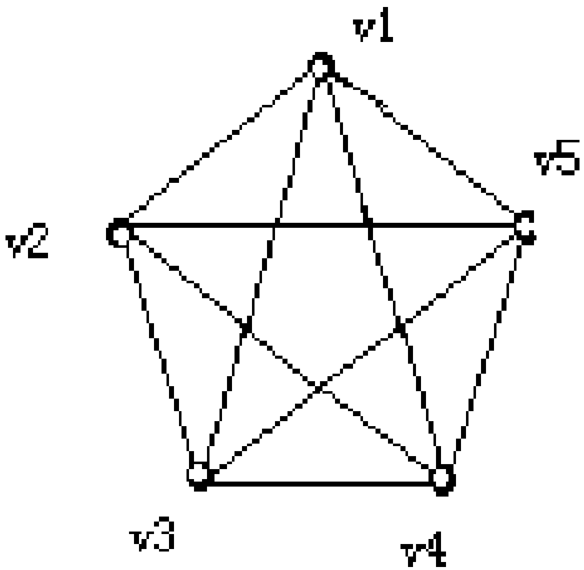
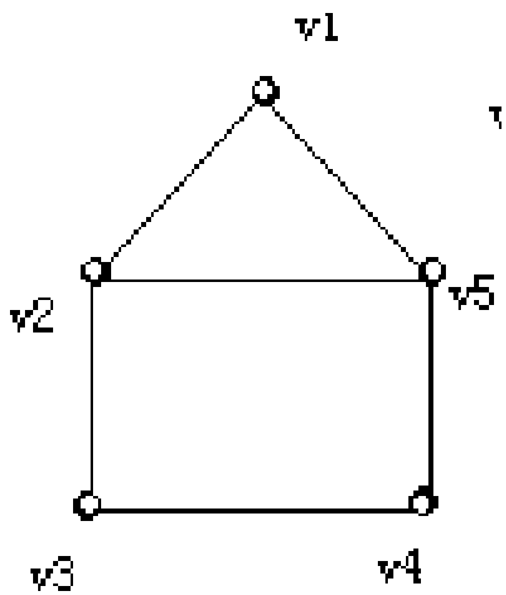




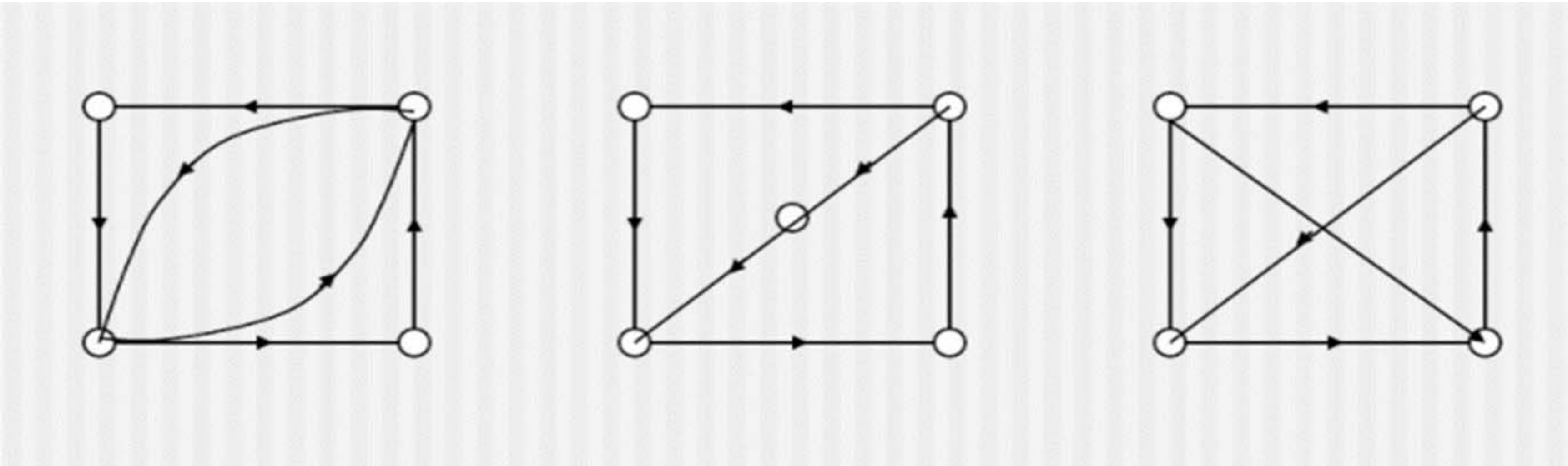
Directed graph

Theorem:

For connected digraph G, G has an Euler circuit if and only if , for any vertex *v* in G, *out-degrees of v* is the same as its in-degrees.

G has an Euler path if and only if , out-degrees of one vertex minus its in-degrees is one, in-degrees of another vertex minus its out-degrees is one, for any other vertices *out-degrees* is the same as its in-degrees.

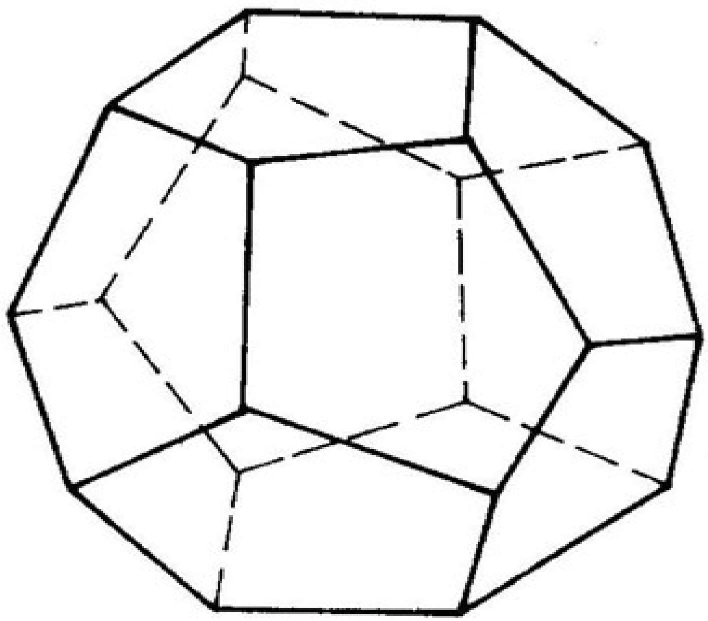
Examples

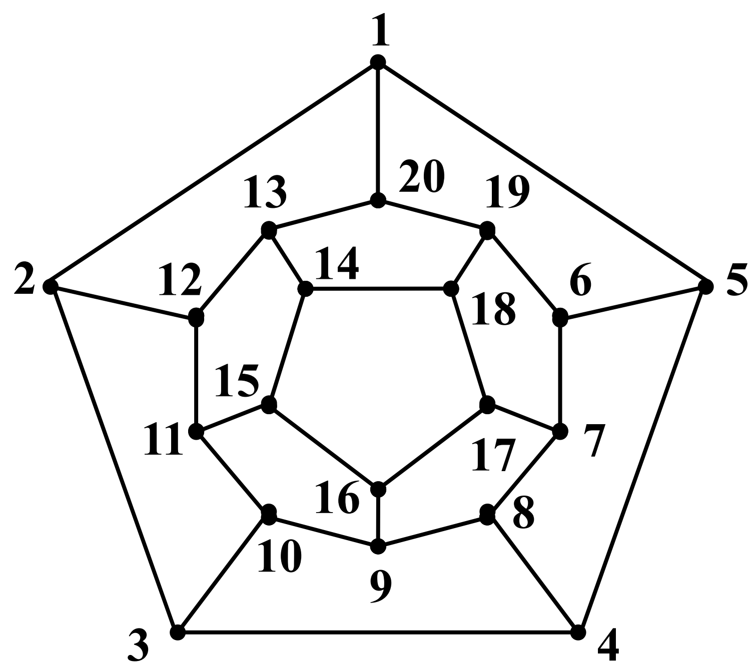


All Around the World

- In 1859, an Irish mathematician, Hamilton, introduced the game called “All around the world”

The player is required to start from a city, visit every city exactly once, along the edges, and return to the starting city finally.

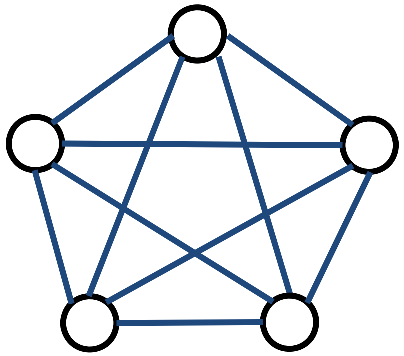




Hamilton Path and Circuit

- Definition:
- In a graph G, a circuit is called a ***Hamilton circuit*** if and only if it contains all the vertices in G exactly once. If G contains a Hamilton circuit, G itself is called a ***Hamilton graph***.
- A Hamilton path is a simple path which contains all vertices exactly once.
- If we pass an edge more than once, it cause us to pass vertex incident with this edge more than once.

There exists edges between any two vertices



The more edges the graph have, the more likely it is that this graph is ***Hamilton***

A Sufficient Condition for Hamilton Graph

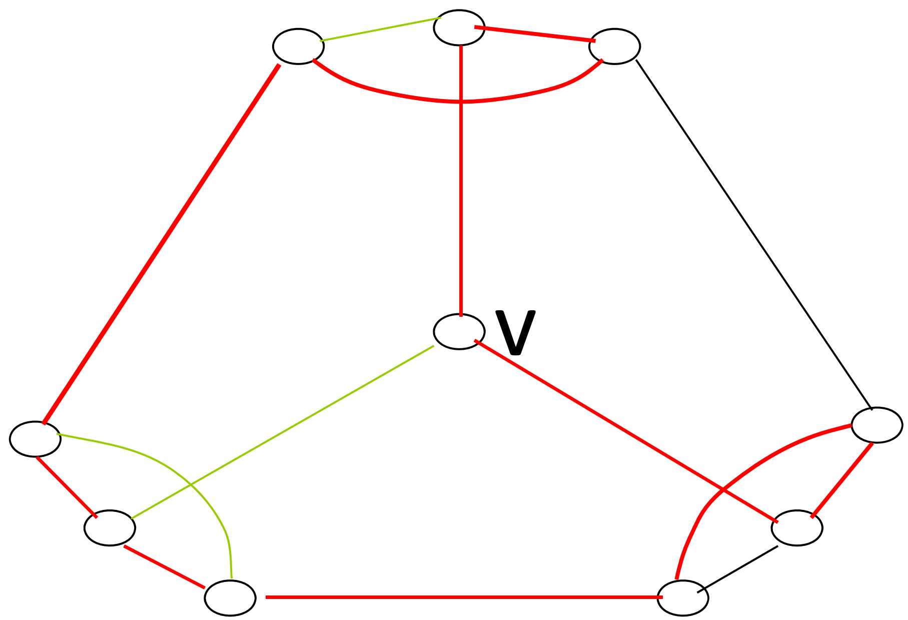
- G is a simple graph with n vertices with not less than 3. If for any vertices *u*,*v* not adjacent in G:

  $\boldsymbol{d(u)+d(v)\geq n}$ (n is $|V_G|$)

  Then G is a Hamilton graph.

- G is a simple graph with n vertices with not less than 3. **$d(v) \geq n/2$,** then G is a Hamilton graph.

- G is Hamilton graph , but d(v) = 3 < 10/2



Homework

- 10.5 P739
- 1-2, 13-14, 18-19, 30-32