Section 5 independence of events

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一、Mutually independent events

(一) The independence of the two events

By **conditional probability**,

$$P(A|B)=\frac{P(AB)}{P(B)}$$

generally，$P(A|B)\neq P(A)$

yielding that： the event that *B happens influences* *P(A)*.

However， in some cases：

$$P(A|B)=P(A)$$

1.e.g. Given one box containing 5 balls（3 green, 2 red）, take out one ball from the box twice with replacement.

A="the ball taken out is green in the first time."

B="the ball taken out is green in the second time."

Then, $P(B|A)=\dfrac{3}{5}=P(B)$

Which shows that the happening of event A does not affect P(B).

$$\text{Given } P(A)>0, \quad \text{then}$$

$$P(B|A)=P(B) \Leftrightarrow \boxed{P(AB)=P(A)P(B)}$$

2. Def. 1.9 **Given A, B are two events, if**

$$P(AB) = P(A)\,P(B)$$

then A and B are called mutually independent.

Note. 1° *if* $P(A) > 0$, then

$$P(B|A) = P(B) \Leftrightarrow P(AB) = P(A)P(B)$$

2° ***A*** **and** ***B are mutually independent*****. This shows that the happening of** ***A does not affect P(B)*****.**

2 º Relationship mutually independent and mutually exclusive.

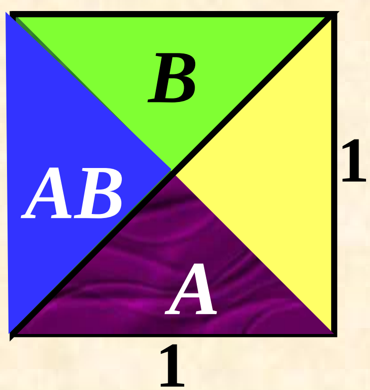
They are different concepts.

Mutually independent $P(AB) = P(A)P(B)$

Mutually exclusive $AB = \varnothing$

No relationship

e.g.,



if $P(A) = \frac{1}{2}, P(B) = \frac{1}{2},$

then $P(AB) = P(A)P(B).$

mutually independent $\not\longrightarrow$ Mutually exclusive.

3. Properties 1.5

(1) Certain event Ω (or impossible event $\varnothing$) and A are mutually independent.

Proof: $\because$ $\Omega A=A$, $P(\Omega)=1$

$\therefore$ $P(\Omega A) = P(A)=1\cdot P(A)= P(\Omega)\, P(A)$

Thus, Ω and A are independent.

$\because$ $\varnothing A=\varnothing$, $P(\varnothing)=0$

$\therefore$ $P(\varnothing A) = P(\varnothing)=0= P(\varnothing)\, P(A)$

Therefore, $\varnothing$ and A are independent.

(2) If A and B are mutually independent, then the following events are independent.

① $A \text{ and } \overline{B}$;

② $\overline{A} \text{ and } B$;

③ $\overline{A} \text{ and } \overline{B}$.

Proof: ① $\because A = A\Omega = A(B + \overline{B}) = AB + A\overline{B}$

$\therefore \quad P(A) = P(AB) + P(A\overline{B})$

$P(A\overline{B}) = P(A) - P(AB)$

$\because$ *A* and *B* *are mutually independent,*

$$\therefore \quad P(A\overline{B}) = P(A) - P(AB)$$

$$= P(A) - P(A)P(B)$$

$$= P(A)[1 - P(B)]$$

$$= P(A)P(\overline{B})$$

③ $\because \overline{A}\,\overline{B} = \overline{A \cup B}$

$$\therefore \quad P(\overline{A}\,\overline{B}) = P(\overline{A \cup B})$$

$$= 1 - P(A \cup B)$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(AB)]$$

$$= 1 - [P(A) + P(B) - P(A)P(B)]$$

$$= [1 - P(A)] - P(B)[1 - P(A)]$$

$$= [1 - P(A)] \cdot [1 - P(B)]$$

$$= P(\overline{A})P(\overline{B}).$$

e.g.1

Two soldiers (1,2) fire at one enemy at the same time. Given the probability of the enemy is hit by 1 and 2 are 0.6 and 0.5 respectively, what is the probability of the event that the enemy is hit?

Solution: A={ the enemy is hit by 1}

B={ the enemy is hit by 2 }

C={the enemy is hit }

$$Then,\quad C = A \cup B.$$

Since, $P(A) = 0.6, \quad P(B) = 0.5$

Because 1，2 fire at the same time, *A* and *B* are mutually independent.

$$\therefore \quad P(C) = P(A \cup B)$$

$$= P(A) + P(B) - P(AB)$$

$$= P(A) + P(B) - P(A)P(B)$$

$$= 0.5 + 0.6 - 0.5 \times 0.6$$

$$= 0.8$$

(二) generalization of independence

1. Three events A,B,C are called pairwise independent, if

$$\begin{cases} P(AB) = P(A)P(B), \\ P(BC) = P(B)P(C), \\ P(AC) = P(A)P(C). \end{cases}$$

2. Def. Three events A, B, C are called mutually independent if

$$\begin{cases} P(AB) = P(A)P(B), \\ P(BC) = P(B)P(C), \\ P(AC) = P(A)P(C), \\ P(ABC) = P(A)P(B)P(C). \end{cases}$$

3. Def.

A_1, A_2 , … , A_n are pairwise independent, if for any $1 \le i < j \le n$,

$$P(A_iA_j) = P(A_i)P(A_j)$$

$$\boxed{\begin{aligned} &C_n^2 + C_n^3 + \cdots + C_n^n \\ &= (1+1)^n - C_n^0 - C_n^1 \\ &= 2^n - 1 - n. \end{aligned}}$$

Def.1.11

A_1, A_2 , … , A_n are called mutually independent, if for any $k(1 \le k \le n)$, and $1 \le i_1 < i_2 < \cdots < i_k \le n$

$$P(A_{i_1}A_{i_2}\cdots A_{i_k}) = P(A_{i_1})P(A_{i_2})\cdots P(A_{i_k}).$$

Note. $A_1, A_2, \cdots, A_n$ **are mutually independent**

$\underset{\not\longleftarrow}{\longrightarrow}$ $A_1, A_2, \cdots, A_n$ **are pairwise independent.**

Conclusions

1. $if\ \ A_1, A_2, \cdots, A_n\ (n \geq 2)$ **are mutually independent,**

then, any events are pairwise independent.

2. $if\ \ A_1, A_2, \cdots, A_n (n \geq 2),$ **are mutually independent,**

then $\overline{A_i}, \overline{A_j}, \ldots, \overline{A_k}, \overline{A_l} \ldots, \overline{A_m} (\mathrm{i} \neq \mathrm{j} \neq \mathrm{k})$ **are mutually**

independent.

Application:

Given $A_1, A_2, \cdots, A_n$ are mutually independent , then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - P(\overline{A_1 \cup A_2 \cup \cdots \cup A_n})$$

$$= 1 - P(\overline{A}_1 \overline{A}_2 \cdots \overline{A}_n)$$

$$= 1 - P(\overline{A}_1)P(\overline{A}_2) \cdots P(\overline{A}_n)$$

$\overline{A}_1, \overline{A}_2, \cdots, \overline{A}_n$ **are mutually independent**

i.e. given the events independent, the probability of the event that at least one event happens is equal to 1 minus the product of the $P(\overline{A}_i)$ **.**

Given $A_1, A_2, \cdots, A_n$ **are mutually independent, and**

$p_1, \cdots, p_n$,

Then **at least one of** $A_1, A_2, \cdots, A_n$ **happens = B**

$$\mathbf{P(B)} = P(A_1 \cup \ldots \cup A_n) = 1 - (1 - p_1) \ldots (1 - p_n).$$

Similarly,

at least one of $A_1, A_2, \cdots, A_n$ **does not happen=C**

$$\mathbf{P(C)} = P(\overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n) = 1 - P(A_1)P(A_2)\cdots P(A_n)$$

$$= 1 - p_1 \ldots p_n$$

二、independent experiment

1. def.1.12

Given E_i **(**i=1,2,…) are a series of experiments, the sample space of E_i **is** Ω_I, **and** A_k **is from** E_k, $A_k \subset \Omega_k$ **, if P(**A_k**)** ***does not depend on the outcome of*** E_i **(**$i \neq k$**), then{**E_i**} are independent.**

2. n multiple Bernoulli trials

Given the trials are done *n* times and have the following properties:

1) The outcome(two) of every trail is $A, \bar{A}$,

$and\ P(A) = p, P(\bar{A}) = 1 - p$ **(where *p is constant*)**

2) All the trails are independent.

Then the n times trails are called n multiple Bernoulli trials.

e. g. 1 Toss a balanced coin ten times, observe the situation of head and tail. Are the trails Bernoulli trials?

e. g. 2 toss a die n times, observe the no. on the top face if is “ 1 ” . Are the trails Bernoulli trials?

Generally, for n multiple Bernoulli trials, it holds：

3. The binomial probability formula

Theorem

for n multiple Bernoulli trials， if P(*A*)= *p* (0<*p*<1), then，

***A happens k* times = B ：**

$$\text{P(B)=}\quad P_n(k) = C_n^k p^k (1-p)^{n-k} = C_n^k p^k q^{n-k}$$

$$(k = 0,1,2,\cdots,n;\ q = 1-p)$$

$$\textit{and} \qquad \sum_{k=0}^{n} P_n(k) = 1.$$

- **Geometric distribution**

For the n multiple Bernoulli trials, denote the time(moment) of A happening for the first time in the trail by X, then X=1,2,…n.

i.e., X=2 means that A does not happen in the first trail, and happens in the second trail

X=k means that in the initial k-1 trails, A does not happen, while in the k trail, A happens. Denoted by B_k .

Then

Geometric distribution

$$B_k = \bar{A}_1 \bar{A}_2 \cdots \bar{A}_{k-1} A_k$$

$$P(B_k) = P(\bar{A}_1) \cdots P(\bar{A}_{k-1}) P(A_k) = (1-p)^{k-1} p$$

e.g.6

One man has n keys. He takes out one key to open the door. Given every key is chose equally. What is the probability for him to open the door in the kth time?

Solution: B_k **= open the door in the kth time, then**

$$P(B_k) = (1 - \frac{1}{n})^{k-1}\frac{1}{n} \qquad k = 1, 2, \cdots$$

三、conclusion

1. A, B are pairwise independent $\Leftrightarrow P(AB)=P(A)\,P(B)$

A, B and C are mutually independent

$$\Leftrightarrow \begin{cases} P(AB)=P(A)P(B), \\ P(BC)=P(B)P(C), \\ P(AC)=P(A)P(C), \\ P(ABC)=P(A)P(B)P(C). \end{cases}$$

2. A and B are mutually independent

$$\Leftrightarrow \overline{A} \text{ and } B,\ A \text{ and } \overline{B},\ \overline{A} \text{ and } \overline{B}.$$

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