Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

- Set is very important for us. There are so many objects in our world. How do we classify them?

  How can we construct relation among them?
- Everyone knows set, even can do some set operations;
- Today, we restudy this, we will use logic to understand its some definitions and operations again.

Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences and Summations
  - Types of Sequences
  - Summation Formulae
- Set Cardinality
  - Countable Sets
- Matrices
  - Matrix Arithmetic

Sets

Section 2.1

Section Summary

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Subsets and Set Equality
- Cardinality of Sets
- Power sets
- Tuples
- Cartesian Product

Sets

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

- Elements are listed between braces
- $S = \{a,b,c,d\}$
- Order is not important

$$S = \{a,b,c,d\} = \{b,c,a,d\}$$

- Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$$

- Ellipsis (…) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a,b,c,d, \ldots\ldots,z\}$$

Roster Method

- Set of all vowels in the English alphabet:

$$V = \{a,e,i,o,u\}$$

- Set of all odd positive integers less than 10:

$$O = \{1,3,5,7,9\}$$

- Set of all positive integers less than 100:

$$S = \{1,2,3,\ldots\ldots,99\}$$

- Set of all integers less than 0:

$$S = \{\ldots, -3,-2,-1\}$$

Some Important Sets

$\mathbf{N}$ = *natural numbers* = $\{0,1,2,3\ldots.\}$

$\mathbf{Z}$ = *integers* = $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

$\mathbf{Z^+}$ = *positive integers* = $\{1,2,3,\ldots..\}$

$\mathbf{R}$ = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

$\mathbf{C}$ = set of *complex numbers*.

$\mathbf{Q}$ = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

$$S = \{x \mid x \text{ is a positive integer less than } 100\}$$

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

$$O = \{x \in \mathbf{Z^+} \mid x \text{ is odd and } x < 10\}$$

- A predicate may be used:

$$S = \{x \mid \mathrm{P}(x)\}$$

- Example: $S = \{x \mid \mathrm{x} \text{ is a person}\}$
- Positive rational numbers:

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$$

Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$

$$[a,b) = \{x \mid a \le x < b\}$$

$$(a,b] = \{x \mid a < x \le b\}$$

$$(a,b) = \{x \mid a < x < b\}$$

closed interval [a,b]

open interval (a,b)

Universal Set and Empty Set

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized ∅, but {} also used.

Some things to remember

- Sets can be elements of sets.

  $\{\{1,2,3\}, a, \{b, c\}\}$

  $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{\emptyset\}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B.)$
- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.
- $\emptyset \subseteq S$
- $S \subseteq S$

1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

★ $\mathbf{A} \not\subseteq \mathbf{B} \Leftrightarrow \exists x(x \in A \wedge x \notin B)$

$\mathbf{A} \not\subseteq \mathbf{B} \Leftrightarrow \neg(A \subseteq B)$

$\Leftrightarrow \neg\forall x(x \in A \rightarrow x \in B)$

$\Leftrightarrow \exists x \neg(x \in A \rightarrow x \in B)$

$\Leftrightarrow \exists x \neg(\neg x \in A \vee x \in B)$

$\Leftrightarrow \exists x(x \in A \wedge \neg x \in B)$

Showing a Set is a Subset of Another Set or not

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Another look at Equality of Sets

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \to x \in B) \wedge (x \in B \to x \in A)]$$

- This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

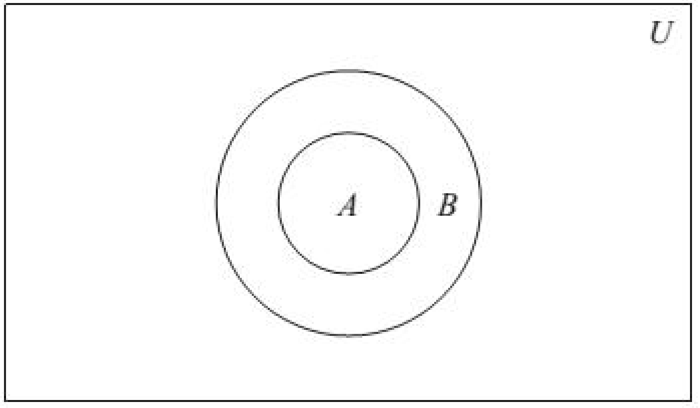
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \to x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



$$\star\ A \neq B \Leftrightarrow \neg(A = B)$$
$$\Leftrightarrow \neg(\forall x(x \in A \rightarrow x \in B) \wedge \forall x(x \in B \rightarrow x \in A))$$
$$\Leftrightarrow \exists x \neg(x \in A \rightarrow x \in B) \vee \exists x \neg(x \in B \rightarrow x \in A))$$
$$\Leftrightarrow \exists x(x \in A \wedge x \notin B) \vee \exists x(x \in B \wedge x \notin A))$$

$$\star\ \forall x(x \in A \rightarrow x \in B)$$
$$\Leftrightarrow \forall x(\neg x \in A \vee x \in B)$$
$$\Leftrightarrow \forall x \neg(x \in A \wedge \neg x \in B)$$
$$\Leftrightarrow \neg \exists x(x \in A \wedge x \notin B)$$

Set Cardinality

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set *A,* denoted by $|A|$, is the number of (distinct) elements of *A*.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set *A*, denoted P(*A*), is called the *power set* of *A*.

Example: If $A = \{a,b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

- If a set has *n* elements, then the cardinality of the power set is 2^n.

Exercise

11. Determine whether each of these statements is true or false.

a) $0 \in \emptyset$

b) $\emptyset \in \{0\}$

c) $\{0\} \subset \emptyset$

d) $\emptyset \subset \{0\}$

e) $\{0\} \in \{0\}$

f) $\{0\} \subset \{0\}$

g) $\{\emptyset\} \subseteq \{\emptyset\}$

Example

What are the *power sets* of the following sets?

$$\{\Phi\}$$

$$\{\Phi, \{\Phi\}\}$$

$$\{a, \{\Phi, a\}\}$$

$$\{\{a, b\}, \{a, a, b\}, \{b, a, b\}\}$$

Tuples

- The *ordered n-tuple* $(a_1, a_2, \ldots\ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product(笛卡尔乘积)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

- **Definition**: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

René Descartes (1596-1650)

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_{i} for i = 1, … n.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times$ C where A = {0,1}, B = {1,2} and C = {0,1,2}

Solution: $A \times B \times$ C **=** {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)}

Truth Sets of Quantifiers

- Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D | P(x)\}$$

- **Example**: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "$|x| = 1$" is the set $\{-1,1\}$

Homework

- 2.1 P132
- 7, 9, 12, 13 , 22, 23, 29, 34