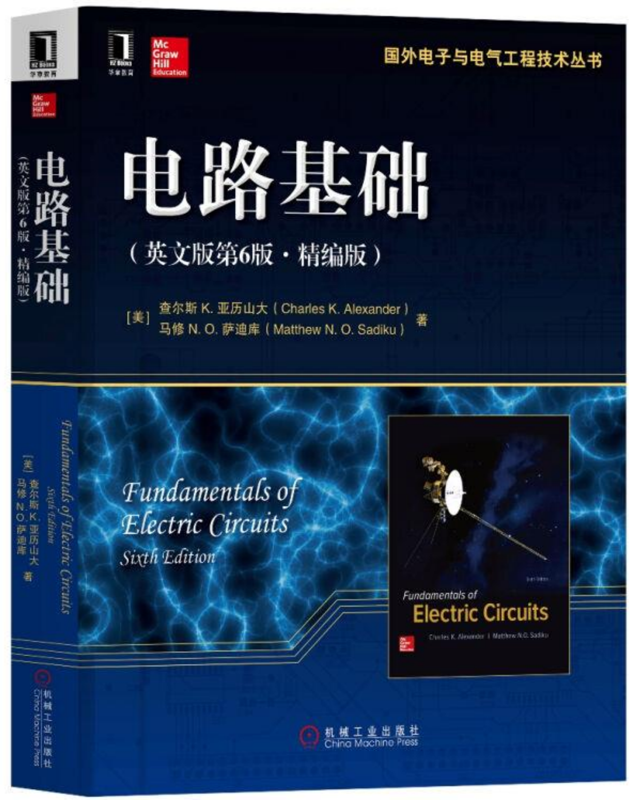
Fundamentals of Electric Circuits

School of Electronics and Information in NPU

2021.3

Textbook



Fundamentals of Electric Circuits

(Sixth Edition)

Charles K.Alexander

Matthew N.O.Sadiku

Recommended Reading

1. Hayt. Kemmerly,&Durbin, *Engineering Circuit Analysis,* McGraw-Hill, 8th, Edition
2. Alexander & Sadiku, *Fundamentals of Electric Circuits*, McGraw-Hill,2007
3. Nilsson & Ricedel, *Electric Circuits*, Pearson, 7th, 2005
4. Irawin, *Basic Engineering Circuit Analysis*, John Wiley & Sons, 7th,2002
5. G.Polya, *How to Solve It*, Princeton Press, the world-famous best-selling book teaches the reader how to develop winning strategies in the face seemingly impossible problem.

Some Requirements

- Attend the Lectures on Time

Student who miss the class for three times for no reason is not eligible for final exam

- Finish Assignment in Time

Student who does not submit homework three times is not eligible for final exam

Examination

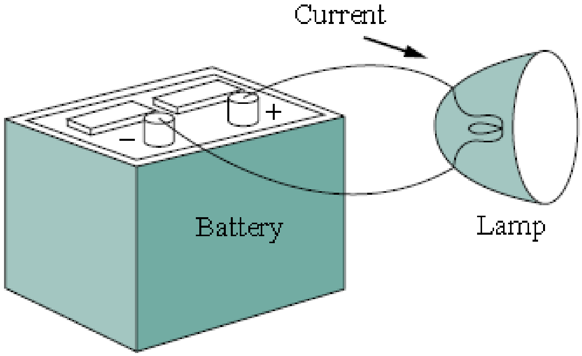
• Attendance/Assignment	10%
• Quiz and homework	20%
• Final examination	70%

chapter 0 Overview of the Course

1. What is a circuit?

- In electrical engineering, we are often interested in communicating or transferring energy from one point to another. To do this requires an interconnection of electrical devices.
- Such interconnection is referred to as an *electric circuit, and each component of the circuit is known as an element.*

An electric circuit is an interconnection of electrical elements.



Function:

(a) The transfer, distribute and transform of energy;

(b) The transfer, control and process of signals.

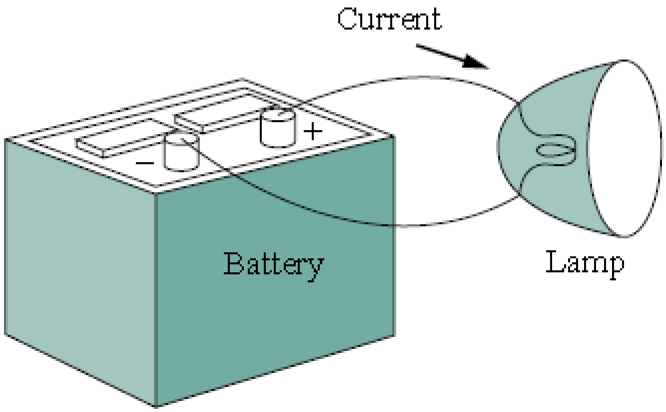


Figure 1.1 A simple electric circuit

It consists of three basic components: a battery, a lamp, and connecting wires.

Such a simple circuit can exist by itself; it has several applications, such as a torch light, a search light, and so forth.

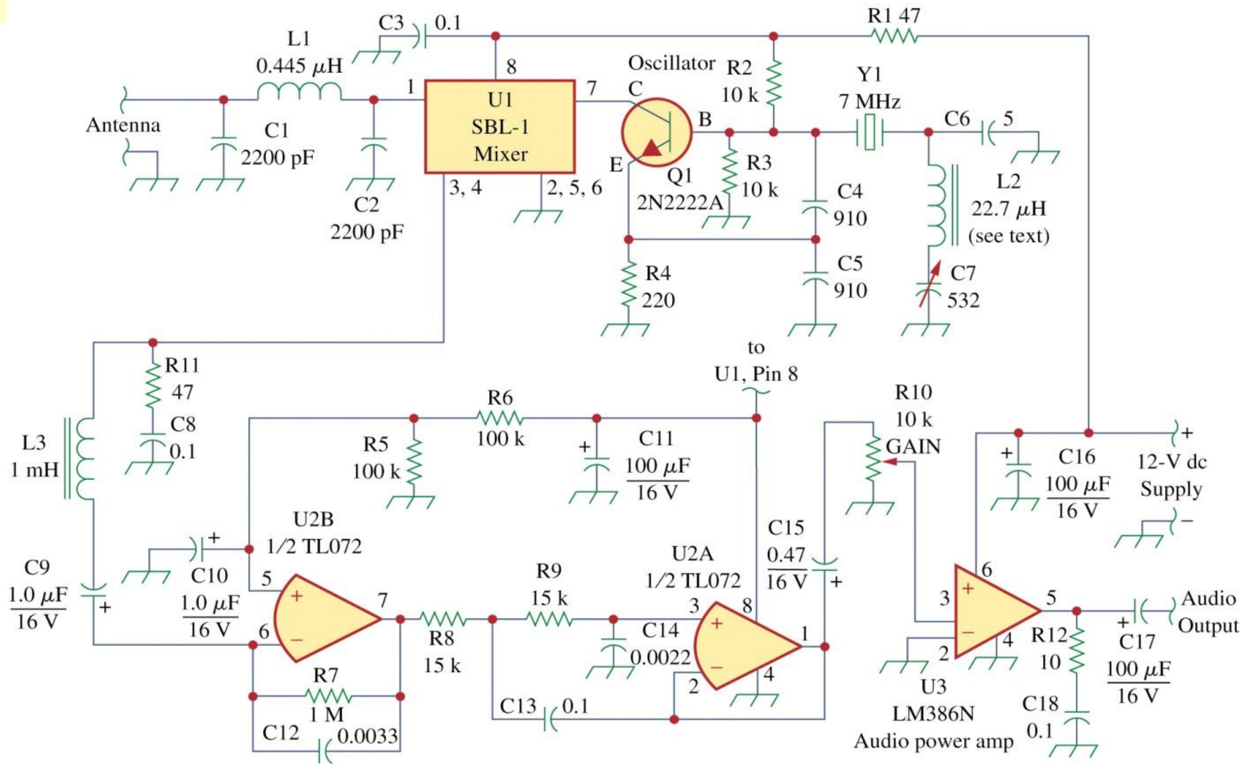


Figure 1.2 Electric circuit of a radio receiver-- a complicated real circuit

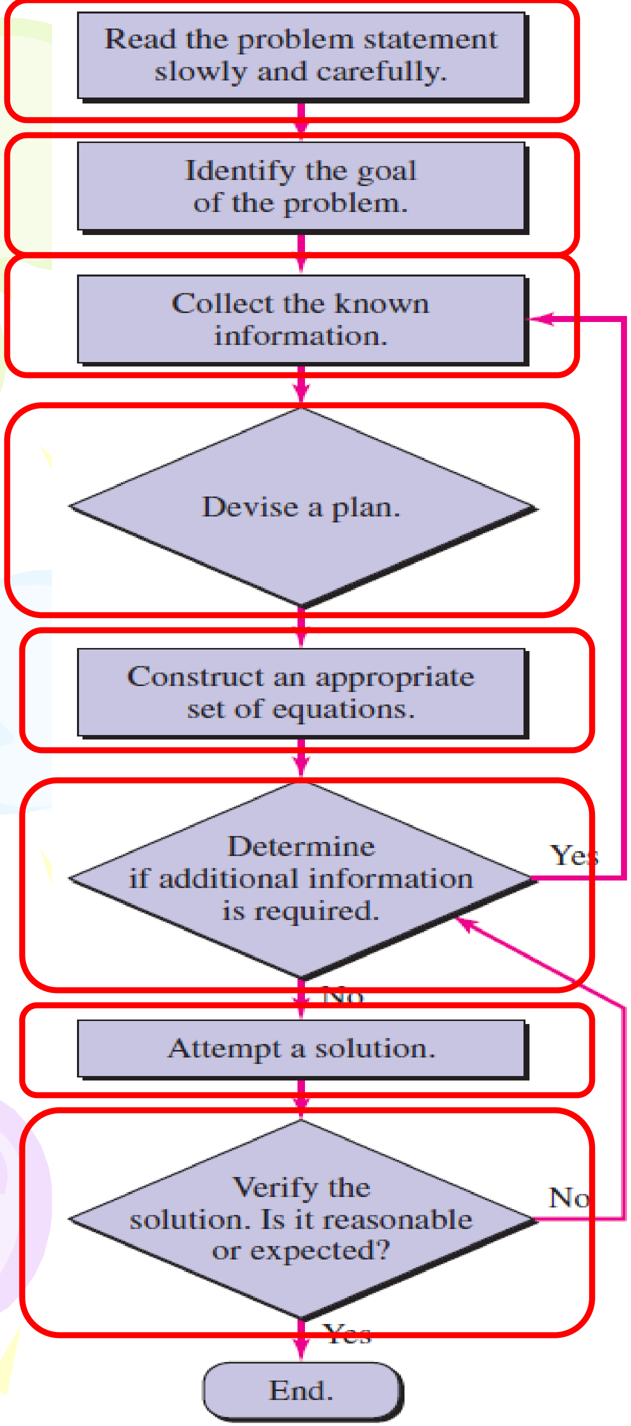
- Although it seems complicated, this circuit can be analyzed using the techniques we cover in this book.

2. What is circuit analysis?

Our major concern in this course is the how to study the behavior of the circuit: How does it respond to a given input? How do the interconnected elements and devices in the circuit interact?

We will define some basic concepts: charge, current, voltage, circuit elements, power, and energy.

Circuit analysis is the process of determining voltages across (or the currents through) the elements of the circuit.



For the circuit in Fig. 3.18, find the branch currents I_1, I_2, and I_3

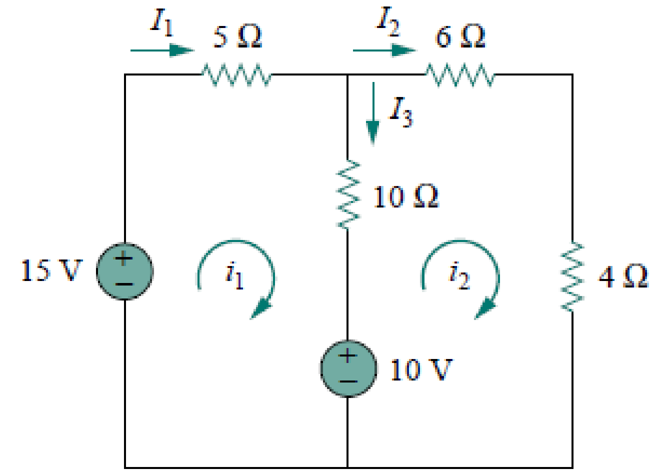


Figure 3.18 For Example 3.5.

Solution:

We first obtain the mesh currents using KVL.

For mesh 1,

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

For mesh 2,

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0$$

$$6i_2 - 3 - 2i_2 = 1 \quad \Longrightarrow \quad i_2 = 1 \text{ A}$$

From Eq. (3.5.2), $i_1 = 2i_2 - 1 = 2 - 1 = 1$ A. Thus,

$$I_1 = i_1 = 1 \text{ A}, \qquad I_2 = i_2 = 1 \text{ A}, \qquad I_3 = i_1 - i_2 = 0$$

3. Circuit analysis can be separated into two broad categories

–Linear circuit analysis

linear circuit: the equations(voltage–current relationship) characterized the circuits are linear algebraic or linear differential equations.

–Nonlinear circuit analysis

This course deals with linear circuit analysis.

4. Linear circuit analysis can be separated into four broad categories

–DC analysis: A direct current (dc) is a current that remains constant with time.

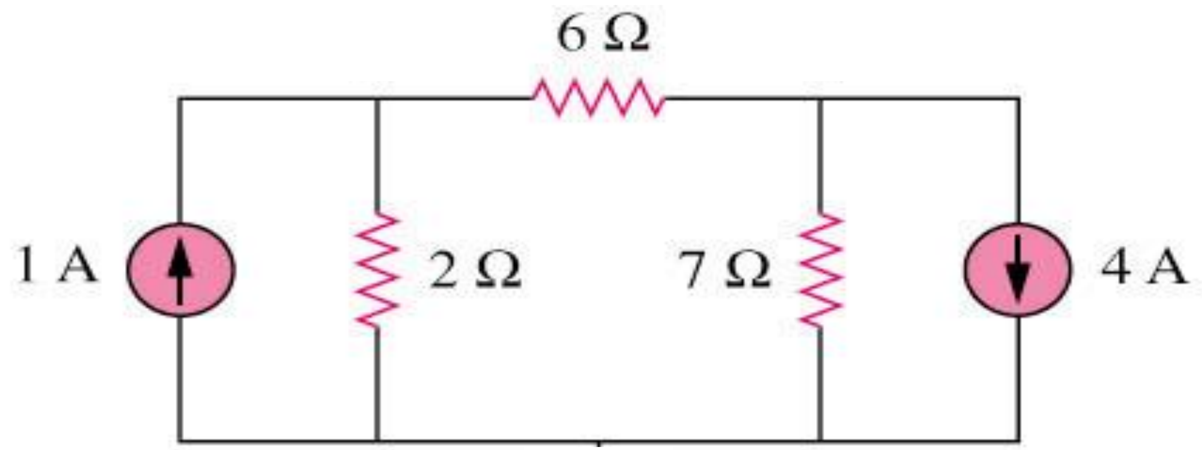
–Transient analysis: things often change quickly

–AC analysis: An alternating current (ac) is a current that varies sinusoidally with time.

–Frequency response analysis: the most general of the four categories, but typically assumes something is changing with frequency

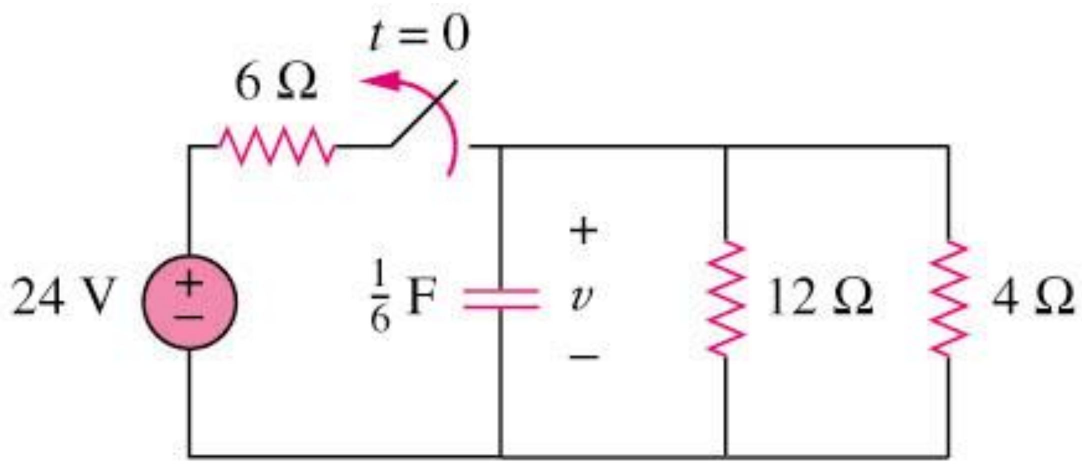
PART 1 : DC ANALYSIS

- Chapter 1 Basic Concepts
- Chapter 2 Basic Laws
- Chapter 3 Methods of Analysis
- Chapter 4 Circuit Theorems
- Chapter 5 Operational Amplifier



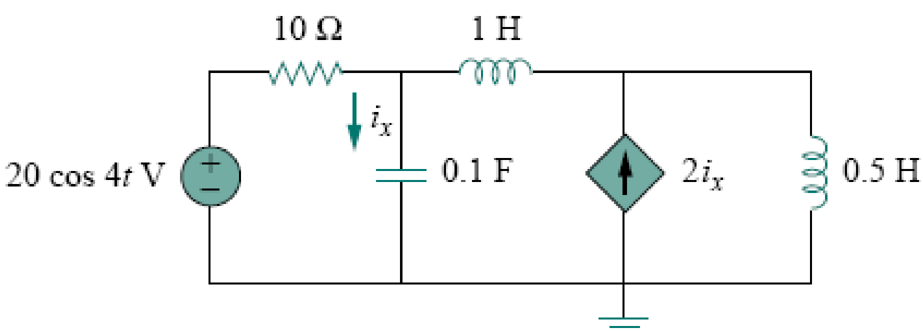
PART 2 : TRANSIENT ANALYSIS

- Chapter 6 Capacitors and Inductors
- Chapter 7 First-Order Circuits
- Chapter 8 Second-Order Circuits



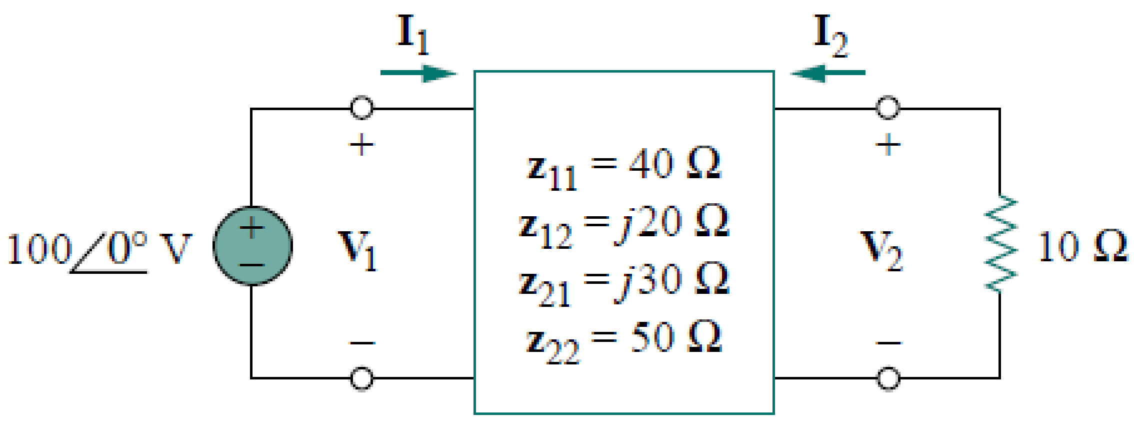
PART 3 : SINUSOIDAL CIRCUITS

- Chapter 9 Sinusoids and Phasors
- Chapter 10 Sinusoidal Steady-State Analysis
- Chapter 11 AC Power Analysis
- Chapter 12 Three-Phase Circuits
- Chapter 13 Magnetically Coupled Circuits
- Chapter 14 Frequency Response



PART 4 : ADVANCED CIRCUIT ANALYSIS

- Chapter 15 Introduction to the Laplace Transform
- Chapter 16 Applications of the Laplace Transform
- Chapter 17 The Fourier Series
- Chapter 18 Fourier Transform
- Chapter 19 Two-Port Networks



5. The simple fact of the matter is that no physical system is ever perfectly linear. Then why study linear analysis?

– A great many systems behave in a reasonably linear fashion over a limited range----allowing us to model them as linear systems if we keep the range limitations in mind.

– Linear problems are inherently more easily solved than their nonlinear counterparts.

Introductory Circuit Analysis

2021.3

Chapter 1 Basic Concept

Chapter 1 Basic Concept

1.1 Charge and Current

1.2 Voltage

1.3 Ass

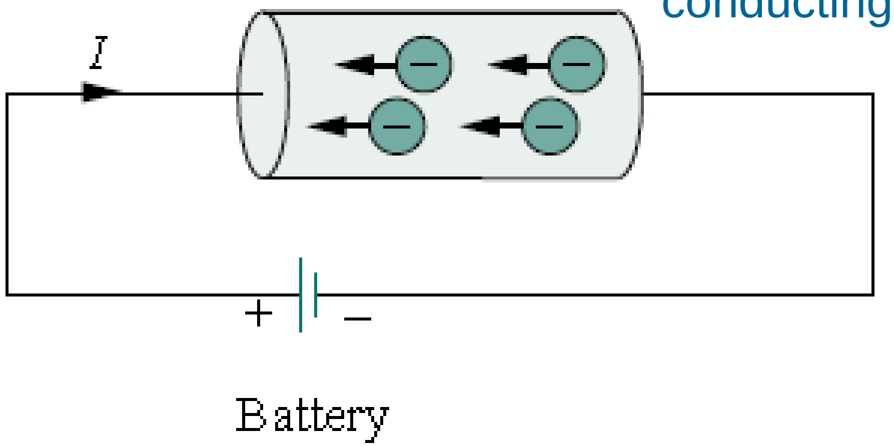
Circuit Elements

1.4 Summary and Review

1.1 Charge and Current

1.Charge

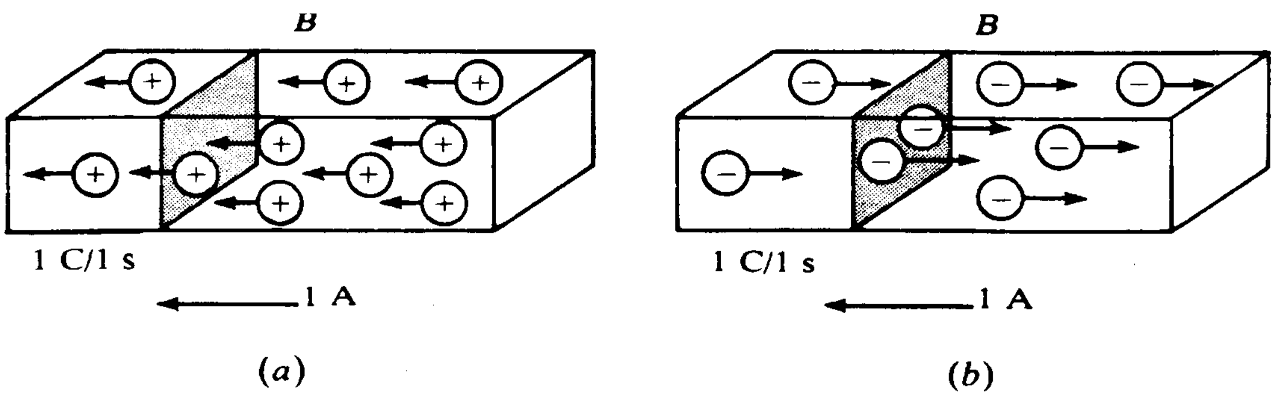
- We know from elementary physics that each atom consists of electrons, protons, and neutrons.
- Charge is an electrical property of the atomic particles of which matter consists, measured in coulombs (C).
- The charge e on one electron is negative and equal in magnitude to 1.602×10^{-19} C which is called as electronic charge.
- A unique feature of electric charge is the fact that it can be transferred from one place to another, where it can be converted to another form of energy.



- When a conducting wire (consisting of several atoms) is connected to a battery (a source of electromotive force), the charges are compelled to move;
- positive charges move in one direction while negative charges move in the opposite direction.
- This motion of charges creates electric current.

2. Current

- The motion of charges forms the electric current in a wire.
- The current has two factors: a numerical value ,a direction.



Positive ions　　　　Negative ions

It is conventional to take the current flow as the movement of positive charges, that is, opposite to the flow of negative charges

It is a measure of the rate at which charge is moving past a given reference point in a specified direction.

Electric current is the time rate of change of charge, measured in amperes (A).

- Electric current $$\boxed{i = \frac{dq}{dt}}$$ The differential of charge to time
- The unit of current is the ampere (A), and it can be derived as **1 A = 1C/s**.
- The charge transferred between time t_0 and t is obtained by integrating

$$\boxed{q = \int_{t_0}^{t} i \, dt}$$

EXAMPLE 1.1

The total charge entering a terminal is given by

$q = 5t \sin 4\pi t$ mC. Calculate the current at $t = 0.5$ s.

$$\boxed{i = \frac{dq}{dt}}$$

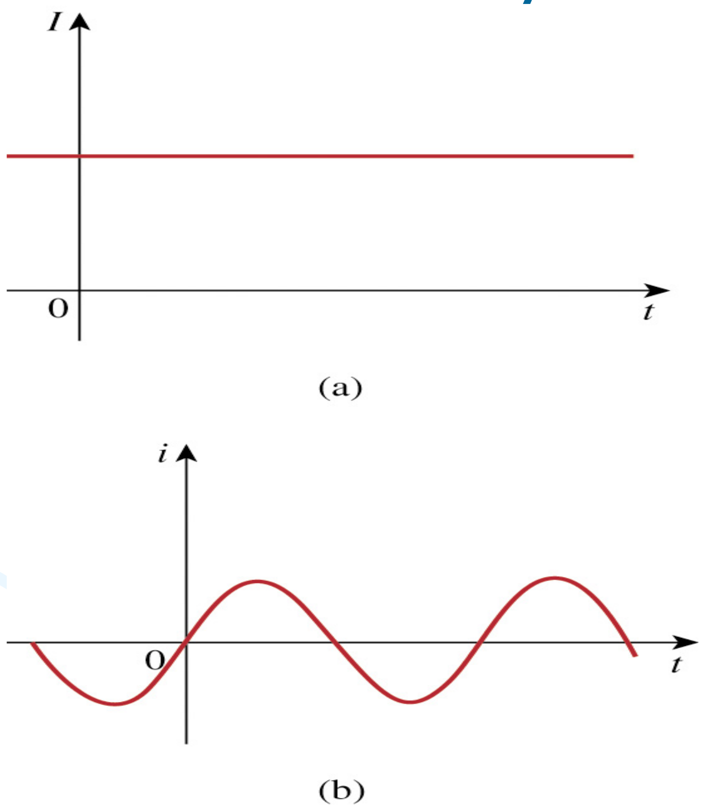
Solution:

$$i = \frac{dq}{dt} = \frac{d}{dt}(5t \sin 4\pi t) \text{ mC/s} = (5 \sin 4\pi t + 20\pi t \cos 4\pi t) \text{ mA}$$

At $t = 0.5$,

$$i = 5 \sin 2\pi + 10\pi \cos 2\pi = 0 + 10\pi = 31.42 \text{ mA}$$

- A direct current (dc) is a current that remains constant with time.
- An alternating current (ac) is a current that varies sinusoidally with time.



Constant current--I

time-varying current--i

Figure 1.2 (a) A direct current (dc) (b) An alternating current (ac)

Current direction

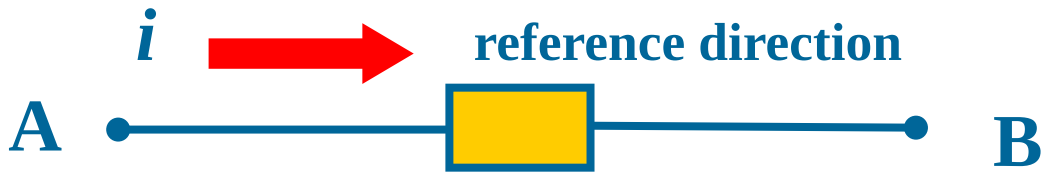
- The direction of current flow is conventionally taken as the direction of positive charge movement

Question:

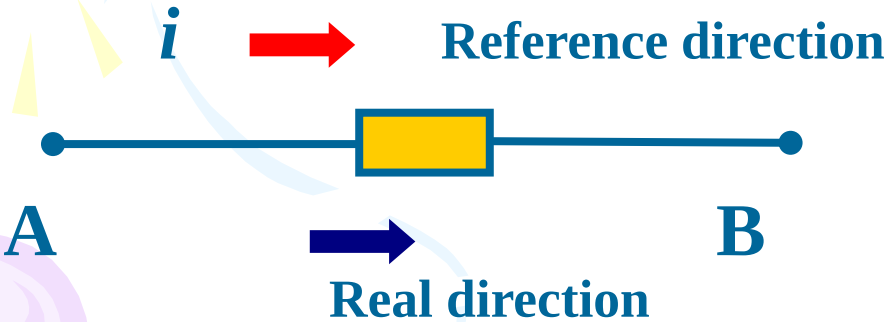
In complex circuit or direction of current flow change with time， how to estimate real direction of current?

Reference direction

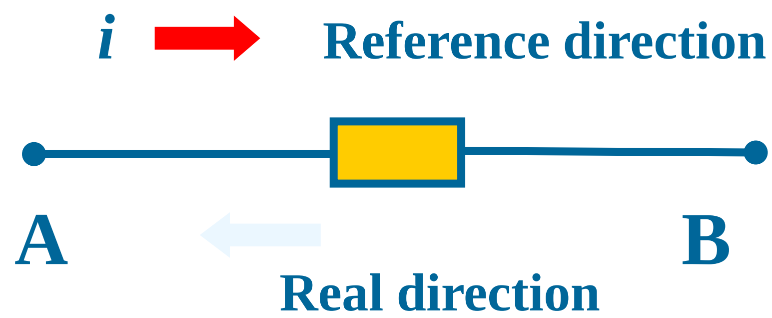
The direction of positive charge movement



Relation of reference direction and real direction:



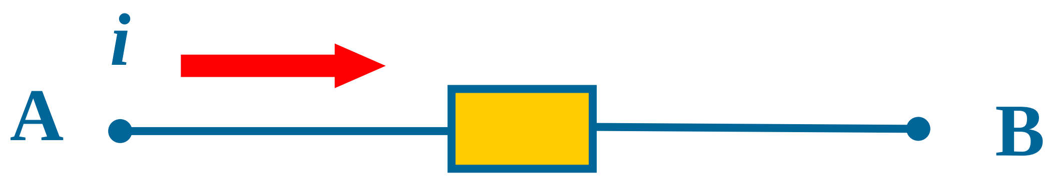
$i > 0$



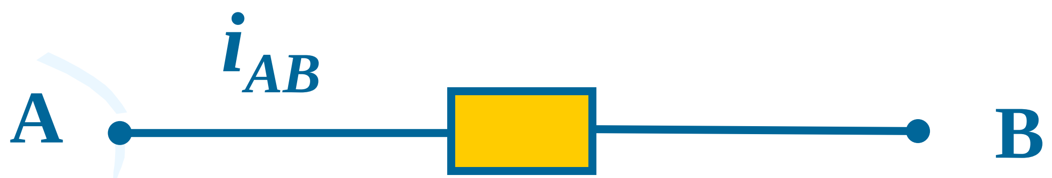
$i < 0$

Two express of reference direction：

- Arrowhead：Arrowhead point to current referent direction



- Subscript：such as i_{AB} ; referent direction: from A to B。



- We should pay close attention to that a numerical value and a direction are fundamental parts of the definition of the current!

For example, Fig. 1.3a is the proper definition, whereas Fig. 1.3b and c are meaningless representation of i(t) symbology.

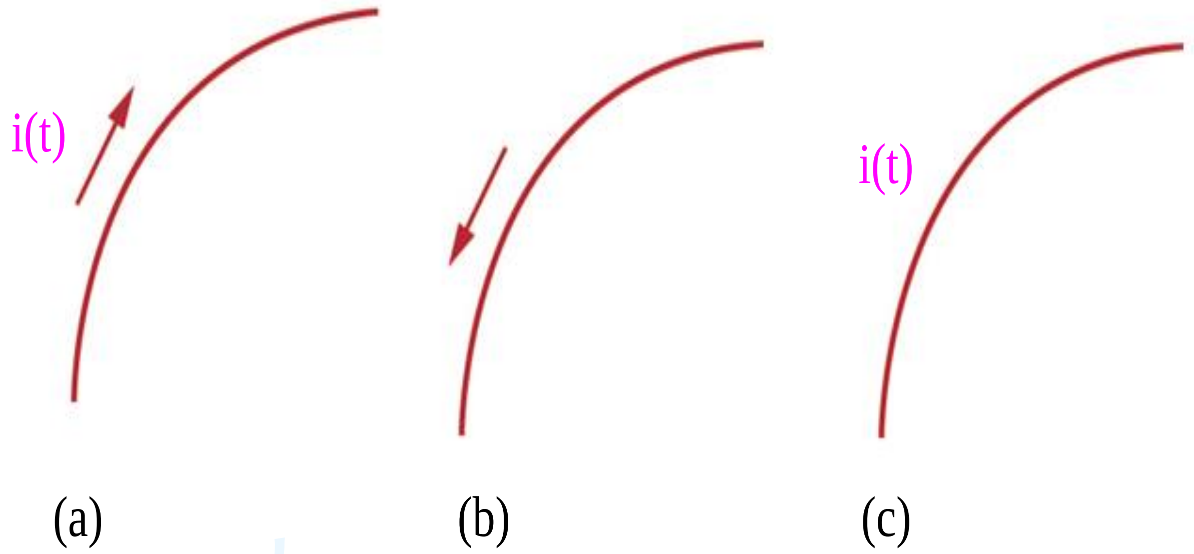


Figure 1.3 (a) the correct definition of i(t) (b, c) incomplete, improper and incorrect definitions of a current

Practice 1.1:

In the wire of Fig. 1.4 electrons moving from *left* to *right* to create a current of 1 mA. Determine I_1 and I_2.

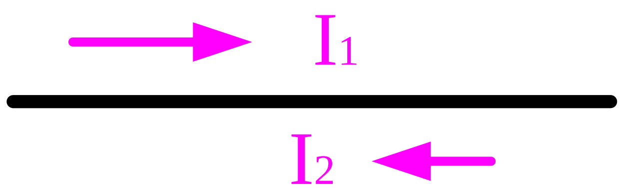


Figure 1.4

Ans: $I_1 = -1$ mA; $I_2 = +1$ mA.

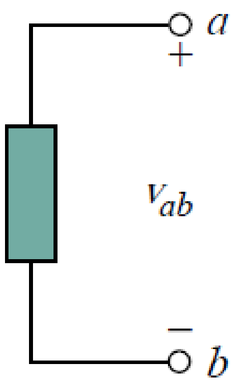
- The direction of current flow is conventionally taken as the direction of positive charge movement.

1.2 Voltage

- The voltage (or potential difference) across a terminal pair is the work (energy) required to move a unit charge through an element, measured in volts (V).
- Mathematically, $v_{ab} = \dfrac{dw}{dq}$ (volt)

> **Voltage** (or potential difference) is the energy required to move a unit charge through an element, measured in volts (V).

Reference Direction



- The plus (+) and minus (−) signs are used to define reference direction or voltage polarity.
- The v_{ab} can be interpreted in two ways:

(1) point *a* is at a potential of v_{ab} volts higher than point *b*;

(2) the potential at point *a* with respect to point *b* is v_{ab}.

$v_{ab} > 0$ means the potential of a is higher than potential of b.

$v_{ab} < 0$ means the potential of a is lower than potential of b.

- We should note that a voltage can exist between a pair of electrical terminals whether a current is flowing or not.

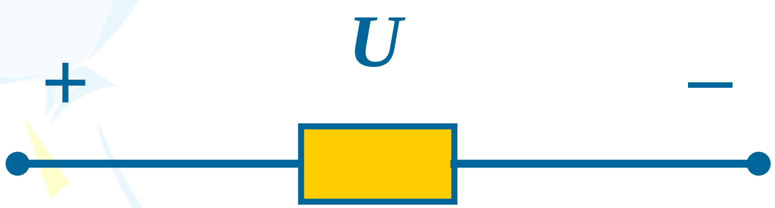
Note: The definition of any voltage must include plus-minus sign pair!

Question:

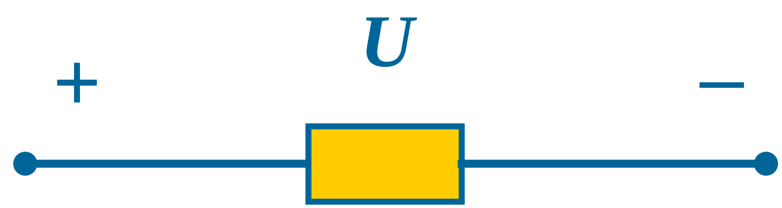
For complex circuit, how to make certain voltage direction?

Voltage drop reference direction

Reference Direction



Reference Direction



+ Real direction —

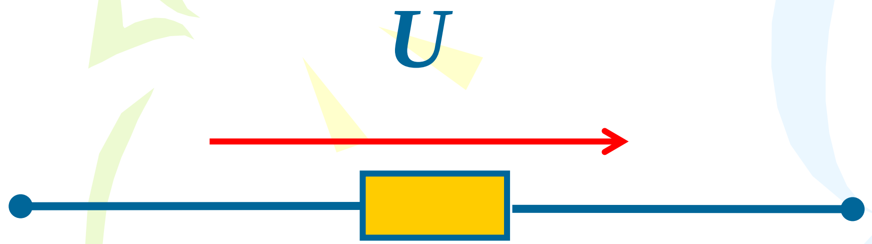
— Real direction +

$U > 0$

$U < 0$

Express for Voltage drop reference direction:

(1) Arrow :

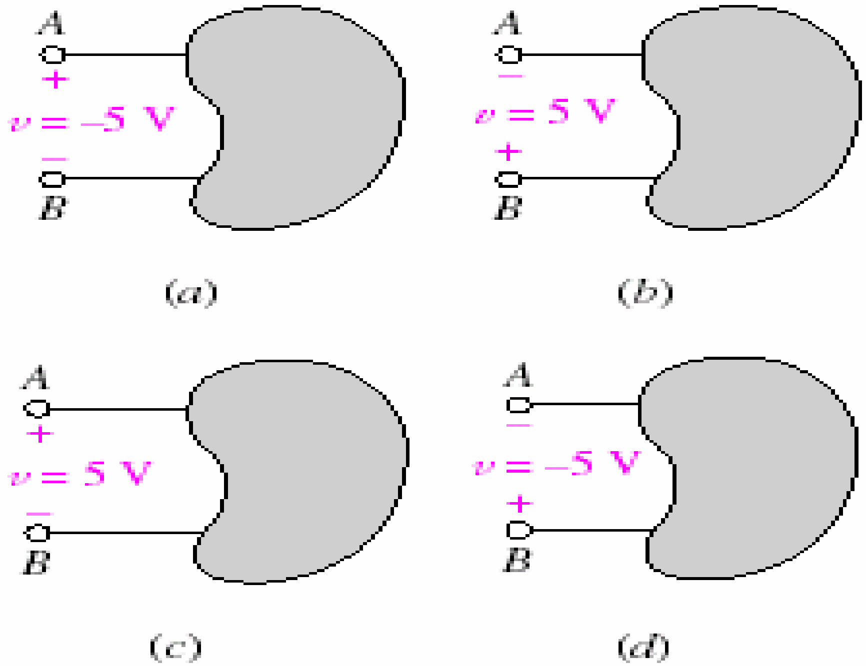


(2) Polarity of plus and minus:

(3) Subscript:

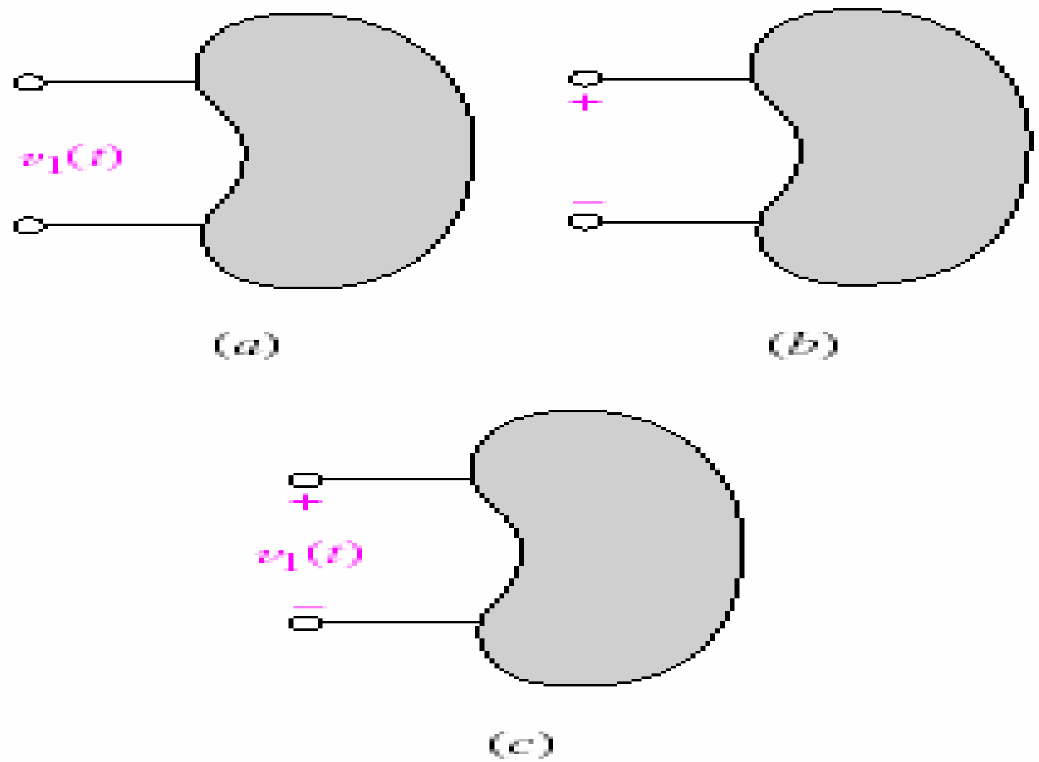
In Fig. 1.6a, for example, the placement of +sign in terminal A indicates that terminal A is *v* volts positive with respect to terminal B.

Figure 1.6



(*a*, *b*) Terminal *B* is 5 V positive with respect to terminal *A*; (*c*, *d*) terminal *A* is 5 V positive with respect to terminal *B*.

Figure 1.8



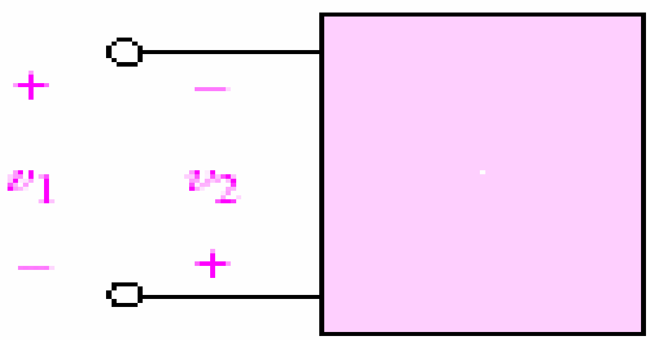
(a, b) These are inadequate definitions of a voltage. (c) A correct definition includes both a symbol for the variable and a plus-minus symbol pair.

Practice 1.2

For the element in Fig. 1.7, v_1 =17 V. Determine v_2.

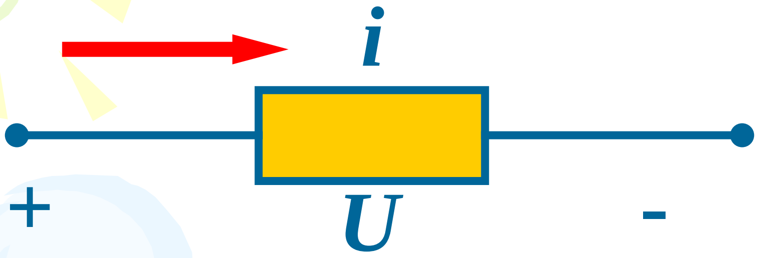
Ans: v_2 = - 17 V

Figure 1.7

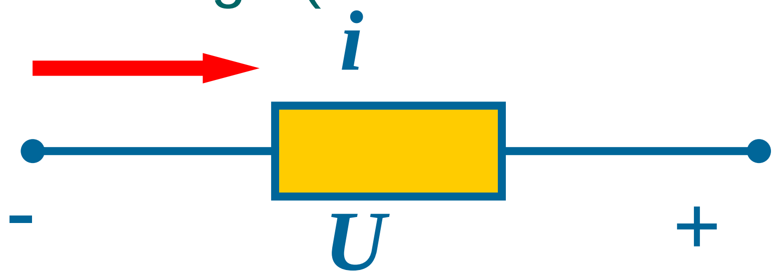


1.3 Passive sign convention

- Passive sign convention: current(reference direction) enters through the positive polarity of the voltage (reference direction).

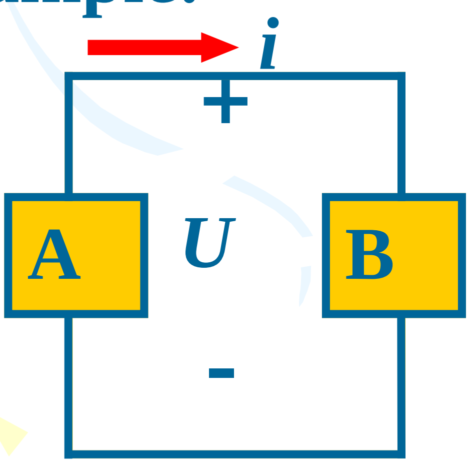


Passive sign convention
Associate Reference Direction



Active sign convention
Non-associate Reference Direction

Example:



Solution:

A Active sign convention

Non-associate Reference Direction

B Passive sign convention

Associate Reference Direction

1.4 Power and Energy

- Power is the time rate of expending or absorbing energy, measured in watts.

- We write this relationship as

$$p = \frac{dw}{dt}$$

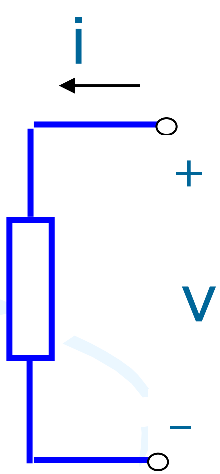
- Where p is power in watts (W), w is energy in joules (J), and t is time in second (s).

$$p = \frac{dw}{dt} = \frac{dw}{dq}\frac{dq}{dt} = ui$$

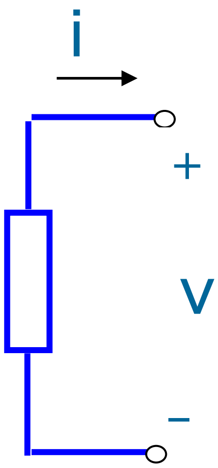
$$u = \frac{dw}{dq} \qquad i = \frac{dq}{dt}$$

- The power p is a time-varying quantity and is called the *instantaneous power*.

- By the passive sign convention, current enters through the positive polarity of the voltage. In this case, $p = +vi$ or $vi > 0$ implies that the element is absorbing power.
- However, if $p = -vi$ or $vi < 0$, the element is releasing or supplying power.

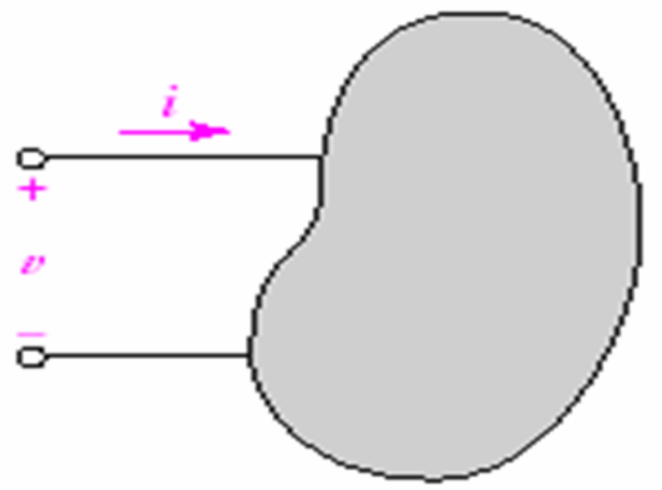


P = +vi
absorbing power



P = –vi
supplying power

Passive sign convention is satisfied when the current enters through the positive terminal of an element and $p = +vi$. If the current enters through the negative terminal, $p = -vi$.



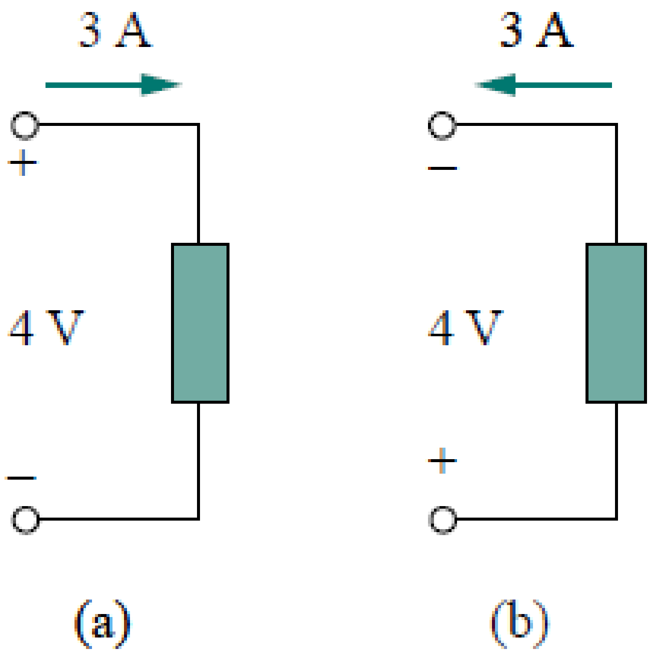


Figure 1.9 Two cases of an element with an absorbing power of 12 W:
(a) $p = 4 \times 3 = 12$ W,
(b) $p = 4 \times 3 = 12$ W.

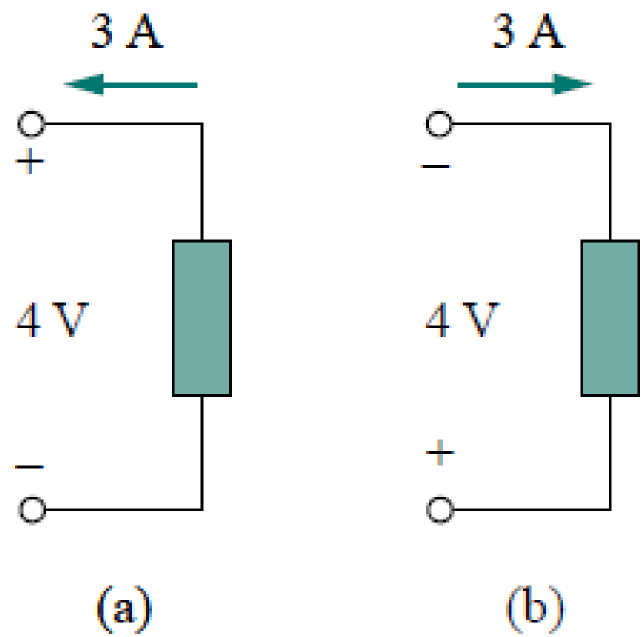


Figure 1.10 Two cases of an element with a supplying power of 12 W:
(a) $p = 4 \times (-3) = -12$ W,
(b) $p = 4 \times (-3) = -12$ W.

EXAMPLE 1.5

Find the power delivered to an element at $t = 3$ ms if the current entering its positive terminal is

$$i = 5\cos 60\pi t \text{ A}$$

and the voltage is: (a) $v = 3i$, (b) $v = 3\,di/dt$.

Solution:

(a) The voltage is $v = 3i = 15\cos 60\pi t$; hence, the power is

$$p = vi = 75\cos^2 60\pi t \text{ W}$$

EXAMPLE 1.5

Find the power delivered to an element at $t = 3$ ms if the current entering its positive terminal is

$$i = 5 \cos 60\pi t \text{ A}$$

and the voltage is: (a) $v = 3i$, (b) $v = 3\,di/dt$.

Solution:

(b) We find the voltage and the power as

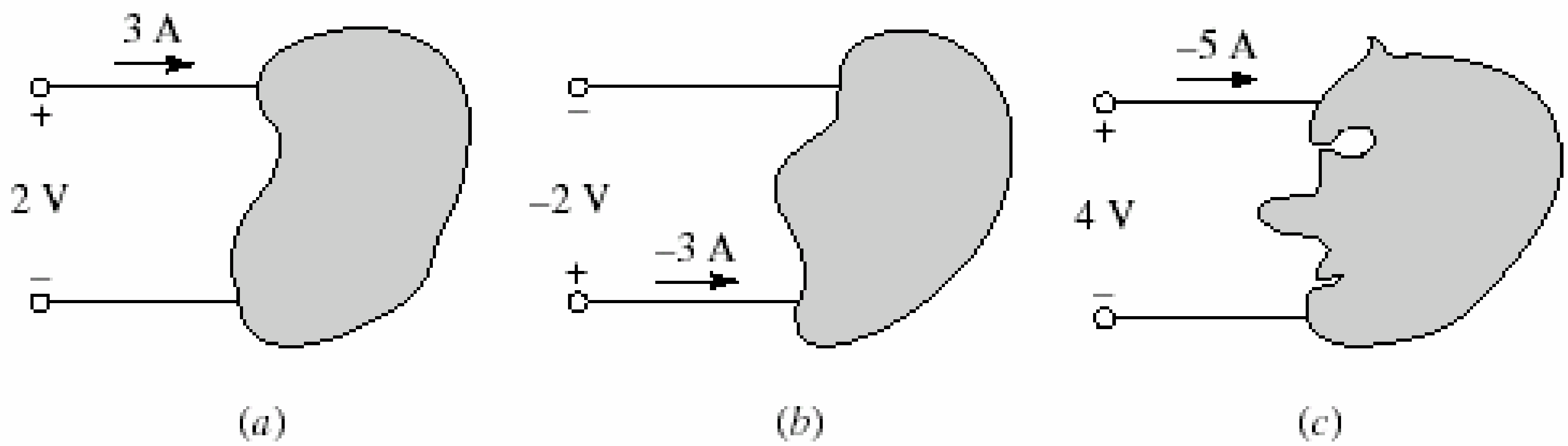
$$v = 3\frac{di}{dt} = 3(-60\pi)5 \sin 60\pi t = -900\pi \sin 60\pi t \text{ V}$$

$$p = vi = -4500\pi \sin 60\pi t \cos 60\pi t \text{ W}$$

Example

Compute the power absorbed by each part in Fig. 1.10

Figure 1.10



(a, b, c) Three examples of two-terminal elements.

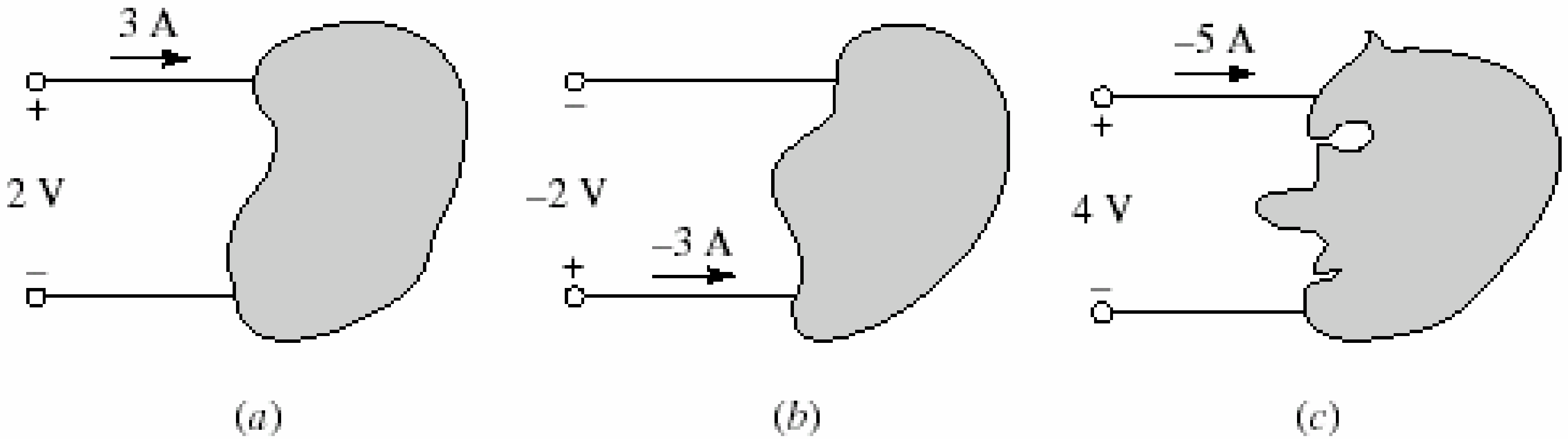
Solution

In Figure 1.10a, with +3 A flowing into the positive reference terminal, we compute

$$P = (2\text{ V})\,(3\text{ A}) = 6\text{ W}$$

of the power absorbed by the element.

Figure 1.10



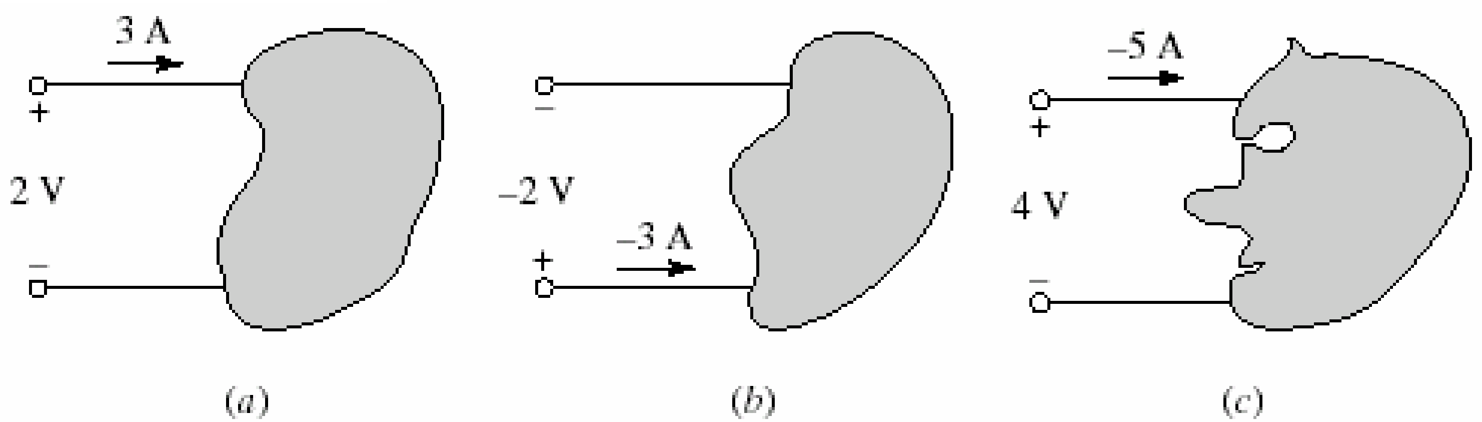
(a, b, c) Three examples of two-terminal elements.

Solution

Fig. 1.10b shows a slightly different picture. Now we have a current of -3 A flowing into the positive reference terminal. However, the voltage as defined is negative. This gives us an absorbed power

$$P = (-2\ V)\ (-3\ A) = 6\ W$$

Figure 1.10



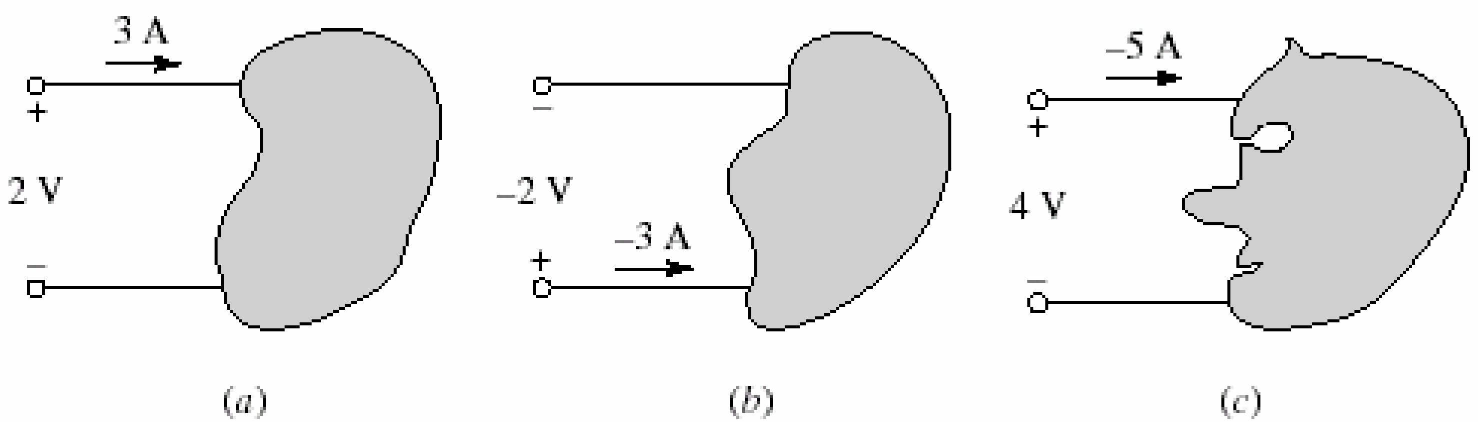
(a, b, c) Three examples of two-terminal elements.

Reference to Fig. 1.10c, we again apply the passive sign convention rulers and compute an absorbed power

$$P = (4\ V)\ (-5\ A) = -20\ W$$

Since we computed a negative absorbed power, this tells us that the element in Fig. 1.10c is actually supplying +20 W (i.e., it’s a source of energy).

Figure 1.10

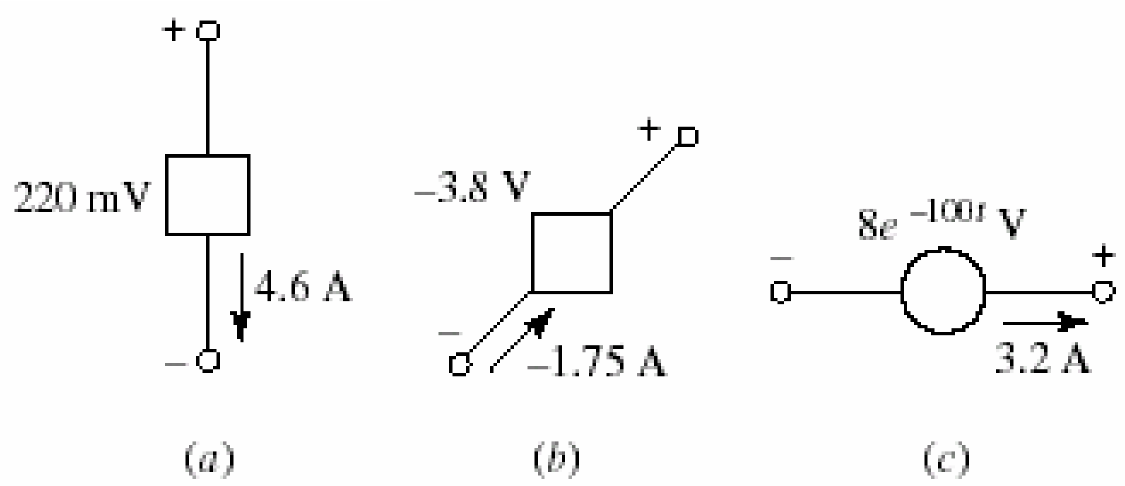


(a, b, c) Three examples of two-terminal elements.

Practice 1.3

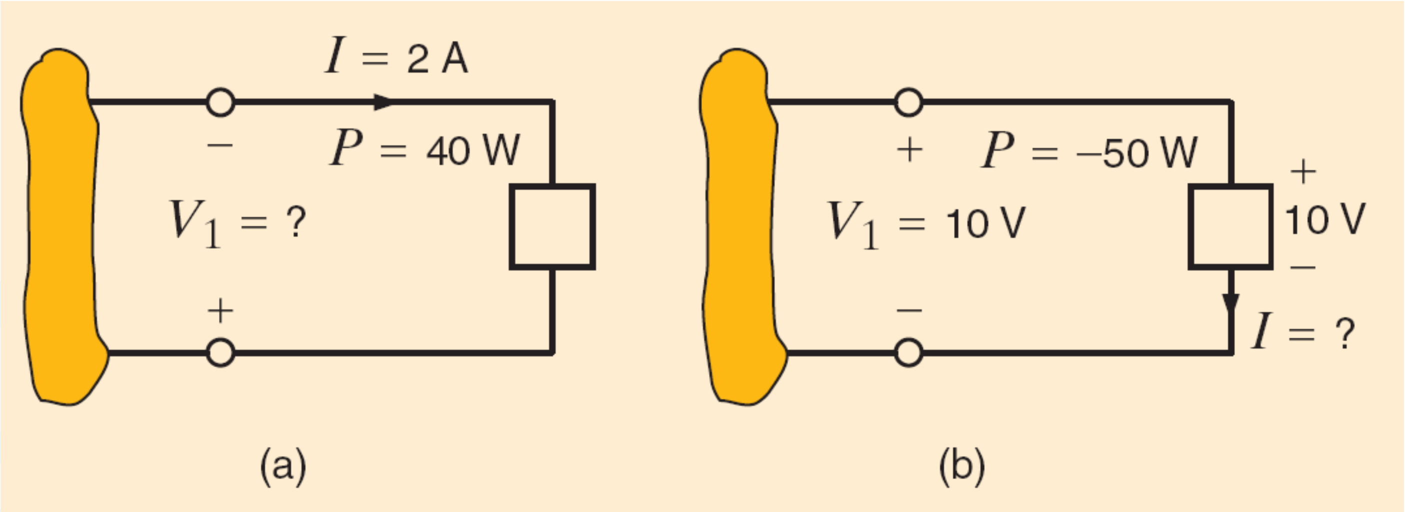
1. Find the power being absorbed by the circuit element in Fig. 1.11a.
2. Find the power being generated by the circuit element in Fig. 1.11b.
3. Find the power being delivered to the circuit element in Fig. 1.11c at t = 5 ms.

Figure 1.11



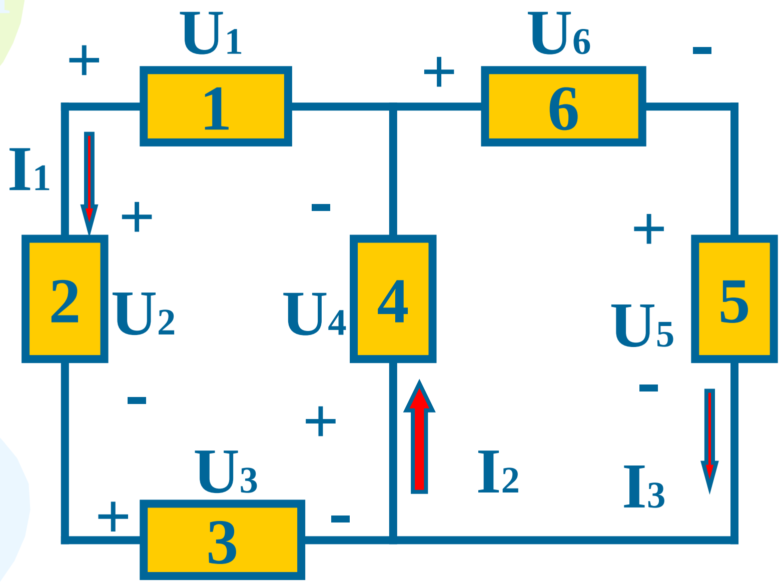
Answer : 1.012; 6.65w; -15.53

Practice : Determine the unknown variable in each circuit?



- Circuit (a): V_1 = -20V
- Circuit (b): I=-5A

Example:



Assume:
U_1=1V, U_2= -3V,
U_3=8V, U_4= -4V,
U_5=7V, U_6= -3V
I1=2A, I2=1A,
I3= -1A

Solution:

$$P_1 = -U_1 I_1 = -1\times 2 = -2W \text{(supplying power)}$$

$$P_2 = U_2 I_1 = (-3)\times 2 = -6W \text{(supplying power)}$$

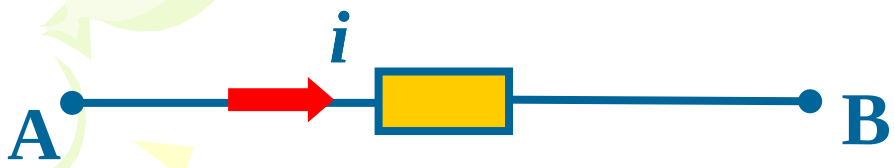
$$P_3 = U_3 I_1 = 8\times 2 = 16W \text{(absorbing power)}$$

$$P_4 = U_4 I_2 = (-4)\times 1 = -4W \text{(supplying power)}$$

$$P_5 = U_5 I_3 = 7\times(-1) = -7W \text{(supplying power)}$$

$$P_6 = U_6 I_3 = (-3)\times(-1) = 3W \text{(absoring power)}$$

Main content of last class



The current has two factors:

value, reference direction

If i>0, reference direction and real direction are the same.

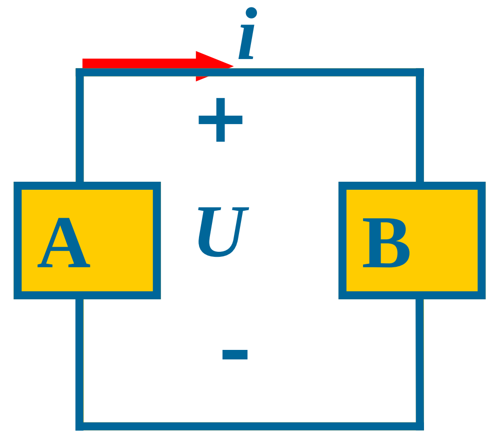
If i<0, reference direction and real direction are the opposite.

The voltage has two factors:

value, reference direction

If u>0, reference direction and real direction are the same.

If u<0, reference direction and real direction are the opposite.



For A: Active sign convention

Non-associate Reference Direction

For B: Passive sign convention

Associate Reference Direction

Absorbing power

For A: **P = -vi**

For B: **P = vi**

For an element:

Supplying power=Absorbing power

1.5 Sources

- An electric circuit is simply an interconnection of the elements.
- There are two types of elements found in electric circuits:
  - ✓Passive elements: An passive element is not capable of generating energy. Example passive element are resistors, capacitors, and inductors.
  - ✓Active elements: An active element is capable of generating energy. Type active element include generators, batteries, and operational amplifiers.
- For each element, its definition is based on its voltage –current relationship.

Notice: In this course, elements are ideal elements or elements model, not real elements.

- The most important active elements are voltage or current sources that generally deliver power to the circuit connected to them.
- There are two kinds of sources: independent sources and dependent sources.

An ideal independent source is an active element that provides a specified voltage or current that is completely independent of other circuit variables.

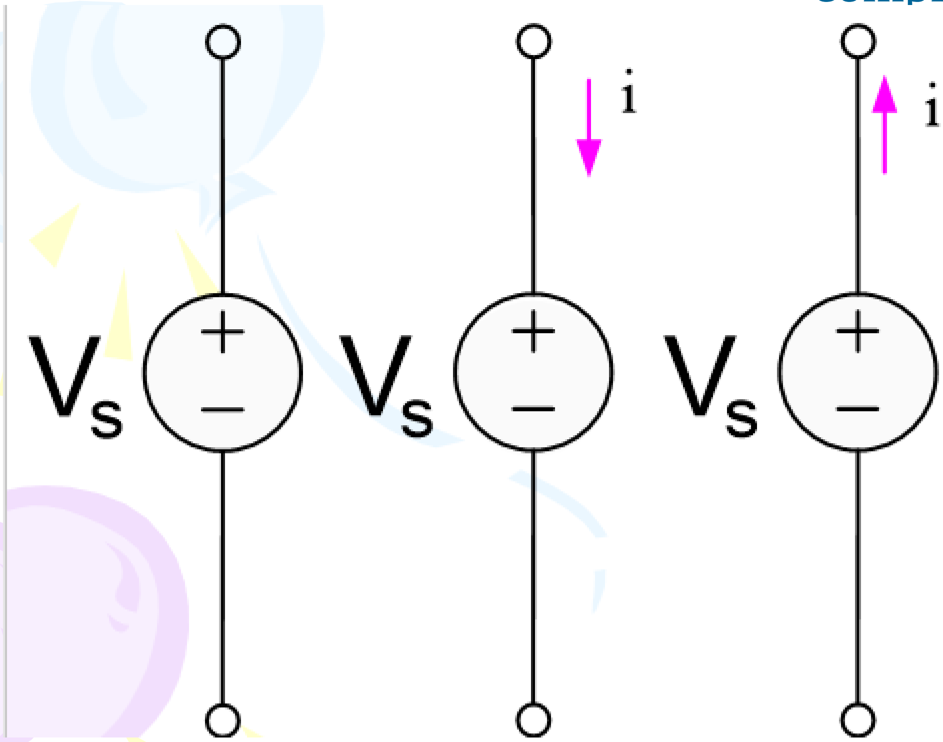
An ideal dependent (or controlled) source is an active element in which the source quantity is controlled by another voltage or current.

Independent Voltage Sources

The first element we will consider is the independent voltage source.

- Circuit symbol

- An independent voltage source is characterized by a terminal voltage which is completely independent of the current through



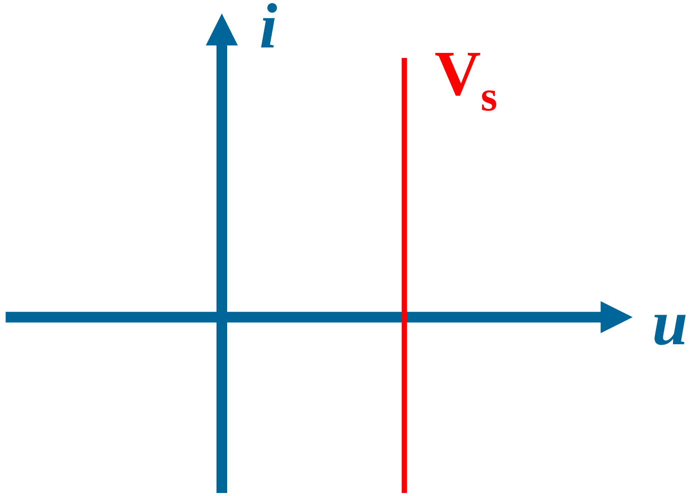
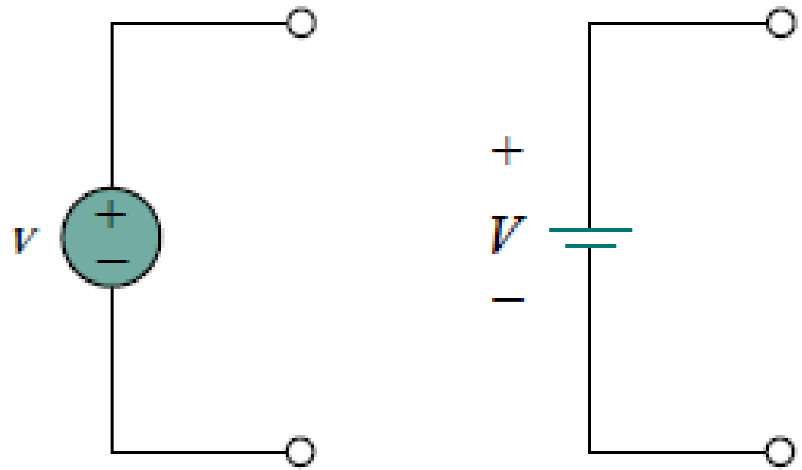


Figure 1.12 Circuit symbol of the independent voltage source



(a) (b)

Both symbols in Fig. (a) and (b) can be used to represent a dc voltage source,

but only the symbol in Fig. (a) can be used for a time-varying voltage source.

Independent Current Sources

Another ideal source which we will need is the independent current source.

- Circuit symbol is shown in Fig.13
- The current through the element is completely independent of the voltage across it.

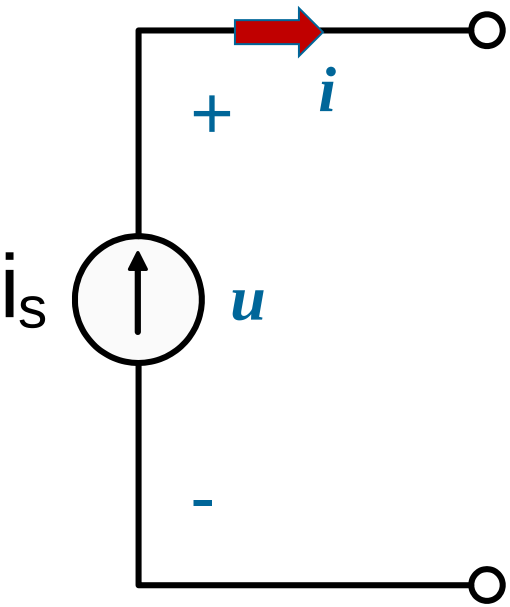
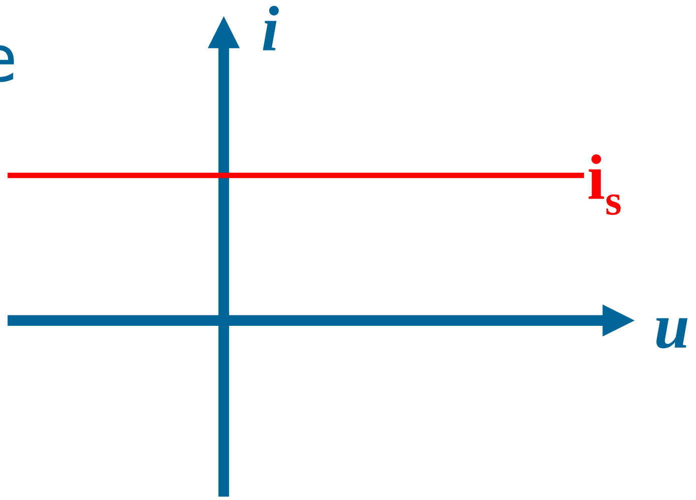


Figure1.13a Circuit symbol for the independent current source



Dependent (controlled) Sources

The source quantity of dependent sources is determined by a voltage or current existing at some other location in the system being analyzed.

There are four kind of Controlled Sources, that is,

- current-controlled current source, CCCS;
- voltage-controlled current source, VCCS;
- voltage-controlled voltage source, VCVS;
- current-controlled voltage source, CCVS .

Circuit Symbols is shown in Fig.1.14(a)

diamond-shaped symbols

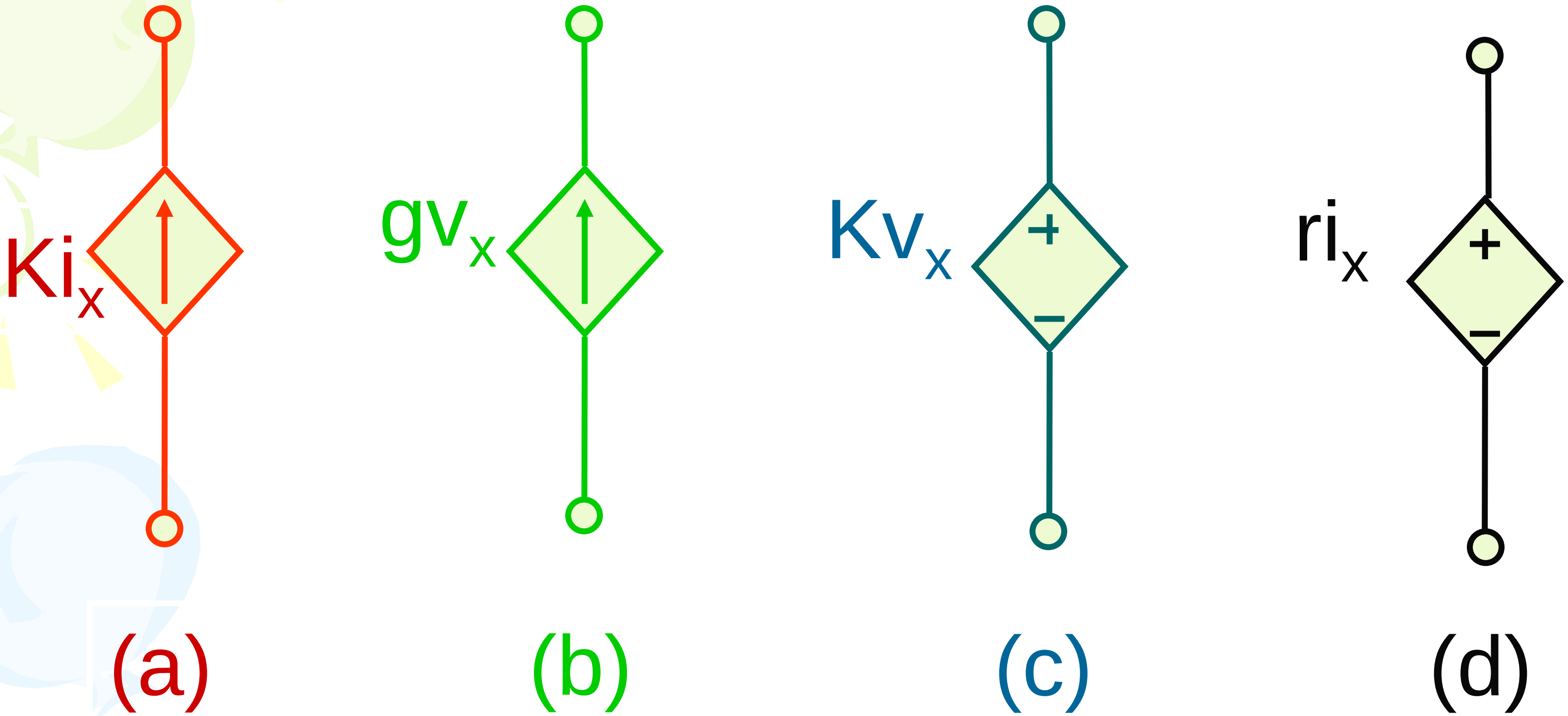


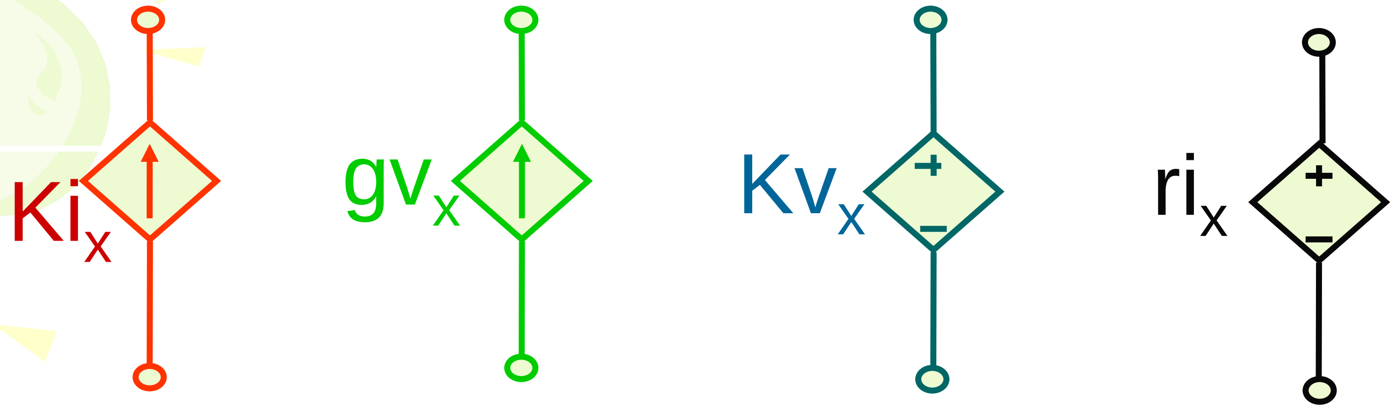
Figure 1.14(a) The four different types of dependent sources:

(a) current-controlled current source;

(b) voltage-controlled current source;

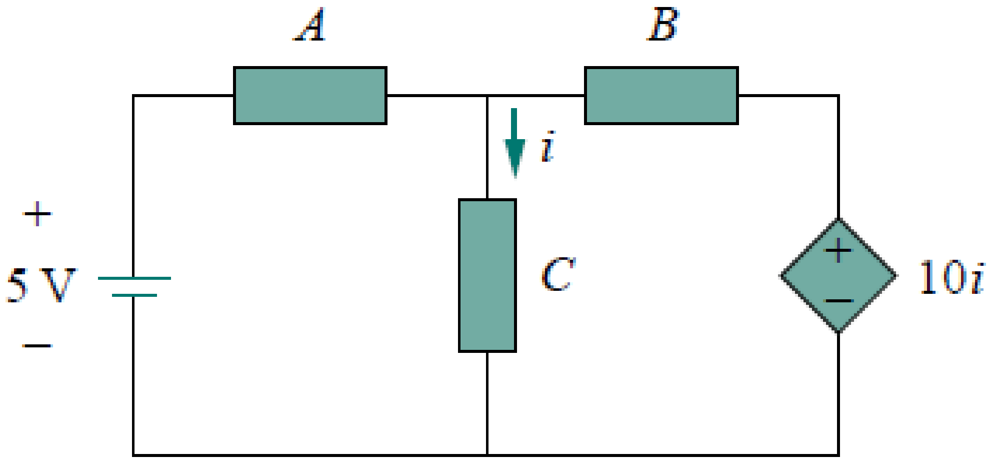
(c) voltage-controlled voltage source;

(d) current-controlled voltage source.



(a) (b) (c) (d)

- The key idea to keep in mind is that a voltage source comes with polarities (+ −) in its symbol, while a current source comes with an arrow, irrespective of what it depends on.
- Dependent sources are useful in modeling elements such as transistors, operational amplifiers and integrated circuits.



- The voltage **10i** of the voltage source depends on the current **i** through element C.

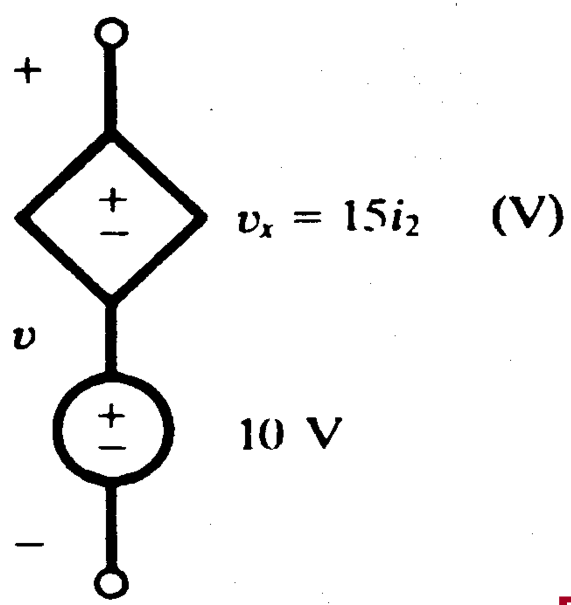
The most important elements in this book

- Passive elements
  - Resistors
  - Inductors
  - Capacitors
- Active elements
  - Independent source
    - Independent voltage source
    - Independent current source
  - Dependent source
    - CCCS
    - CCVS
    - VCVS
    - VCCS

- When dependent sources are given, their controlling current i_x and the controlling voltage v_x must be defined in the circuit.

Example 3

Obtain the voltage v in the branch shown in Figure 1.15 for $i_2 = 1A$.



$$v = 10 + v_x = 10 + 15(1) = 25 \text{ V}$$

Figure 1.15

Example 1.17

Calculate the power supplied or absorbed by each element in Fig. 1.15.

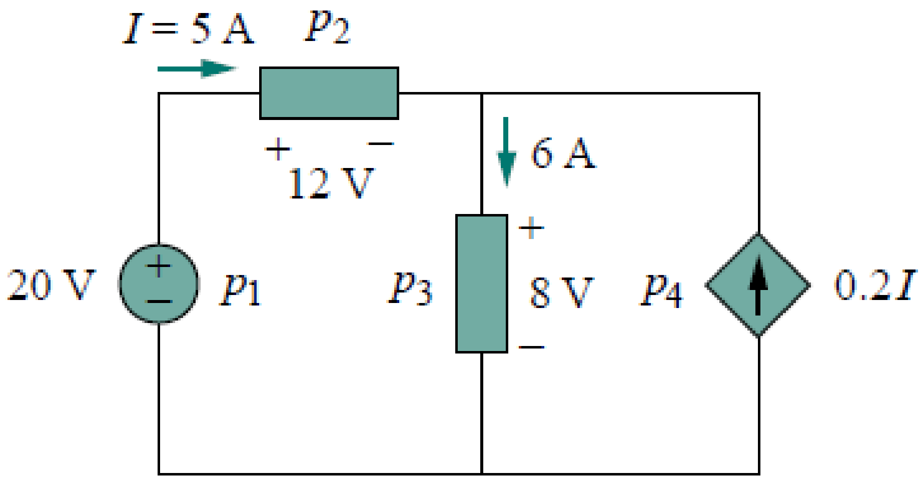


Figure 1.15 For Example 1.7.

Solution: $p_1 = 20(-5) = -100$ W Supplied power

$p_2 = 12(5) = 60$ W Absorbed power

$p_3 = 8(6) = 48$ W Absorbed power

$p_4 = 8(-0.2I) = 8(-0.2 \times 5) = -8$ W Supplied power

Power absorbed $= -$Power supplied

Practice

Find the power absorbed by each element in the circuit in Fig. 1.16.

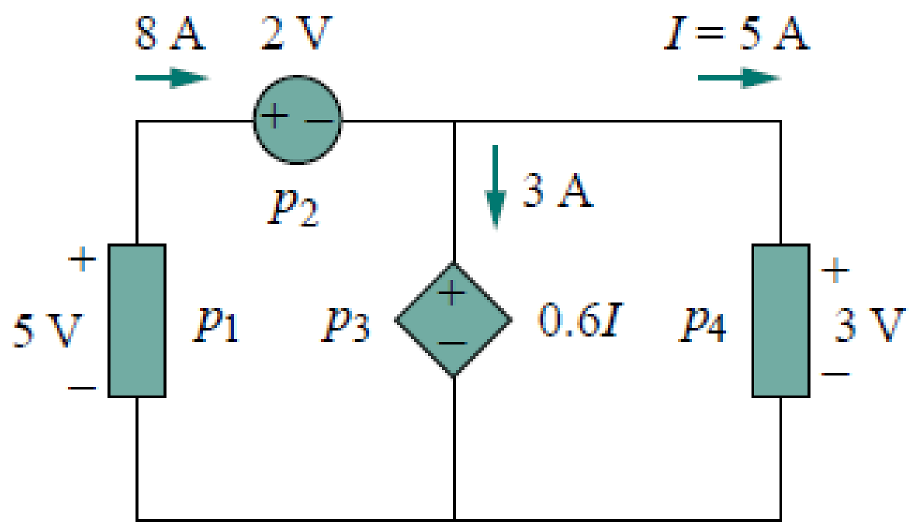


Figure 1.16 For Practice Prob. 1.7.

Answer: $p_1 = -40$ W, $p_2 = 16$ W, $p_3 = 9$ W, $p_4 = 15$ W.

Summary

- Current $$i_{(t)} = \frac{dq(t)}{dt}$$ Amp
- Charge $$q(t) = \int_{-\infty}^{t} i(x)dx$$ Coulombs
- Voltage $$v = \frac{dw}{dq}$$ Volts
- Power $$p = vi = \frac{dw}{dq}\frac{dq}{dt} = \frac{dw}{dt}$$ Watts

An element is the basic building block of a circuit

- Passive elements
  - Resistors
  - Inductors
  - Capacitors
- Active elements
  - Generators
  - Batteries

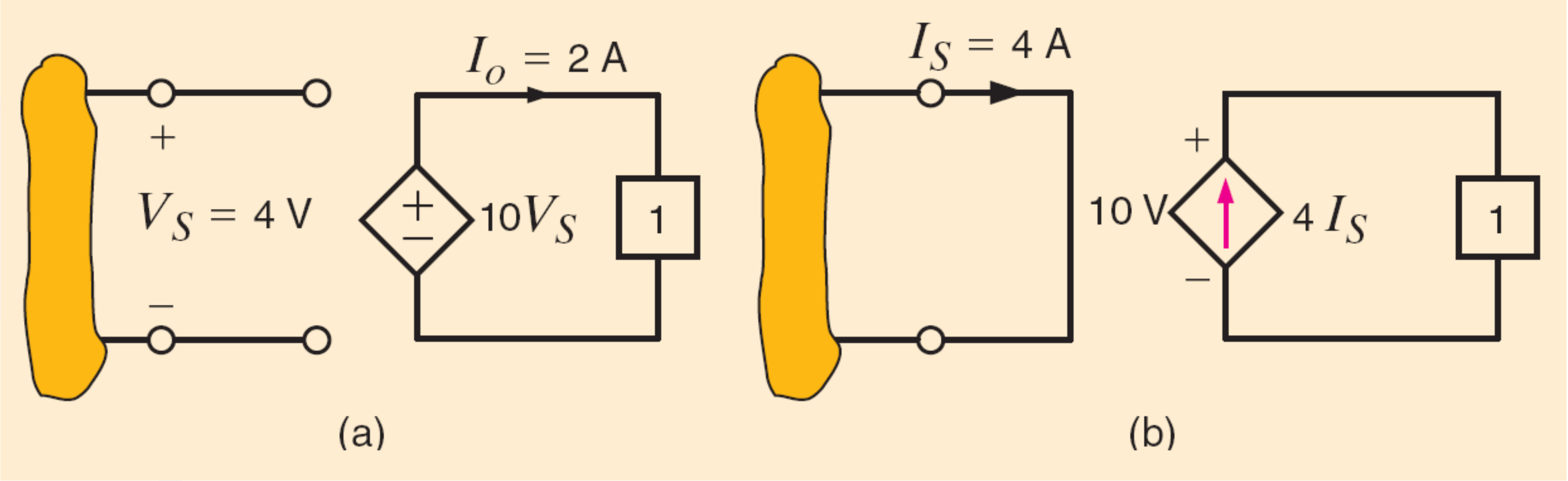
The most important active elements

- Voltage source
- Current source

- Independent source
- Dependent source
  - CCCS
  - CCVS
  - VCVS
  - VCCS

Practice

Determine the power supplied by the dependent sources



SUMMARY OF CHAPTER 1

- 1. An electric circuit consists of electrical elements connected together.
- 2. The International System of Units (SI) is the international measurement language, which enables engineers to communicate their results. From the six principal units, the units of other physical quantities can be derived.

SUMMARY

- 3. Current is the rate of charge flow.

$$i = \frac{dq}{dt}$$

- 4. Voltage is the energy required to move 1 C of charge through an element.

$$v = \frac{dw}{dq}$$

- 5. Power is the energy supplied or absorbed per unit time. It is also the product of voltage and current.

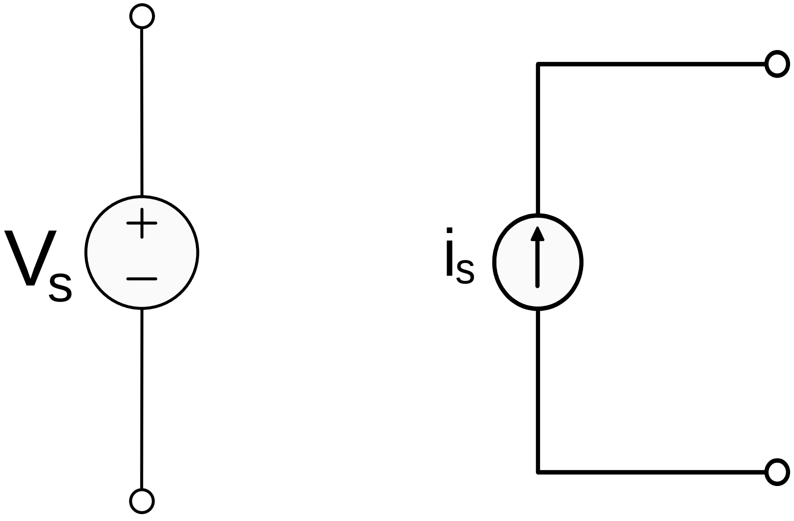
$$p = \frac{dw}{dt} = vi$$

SUMMARY

- 6. According to the passive sign convention, power assumes a positive sign when the current enters the positive polarity of the voltage across an element.

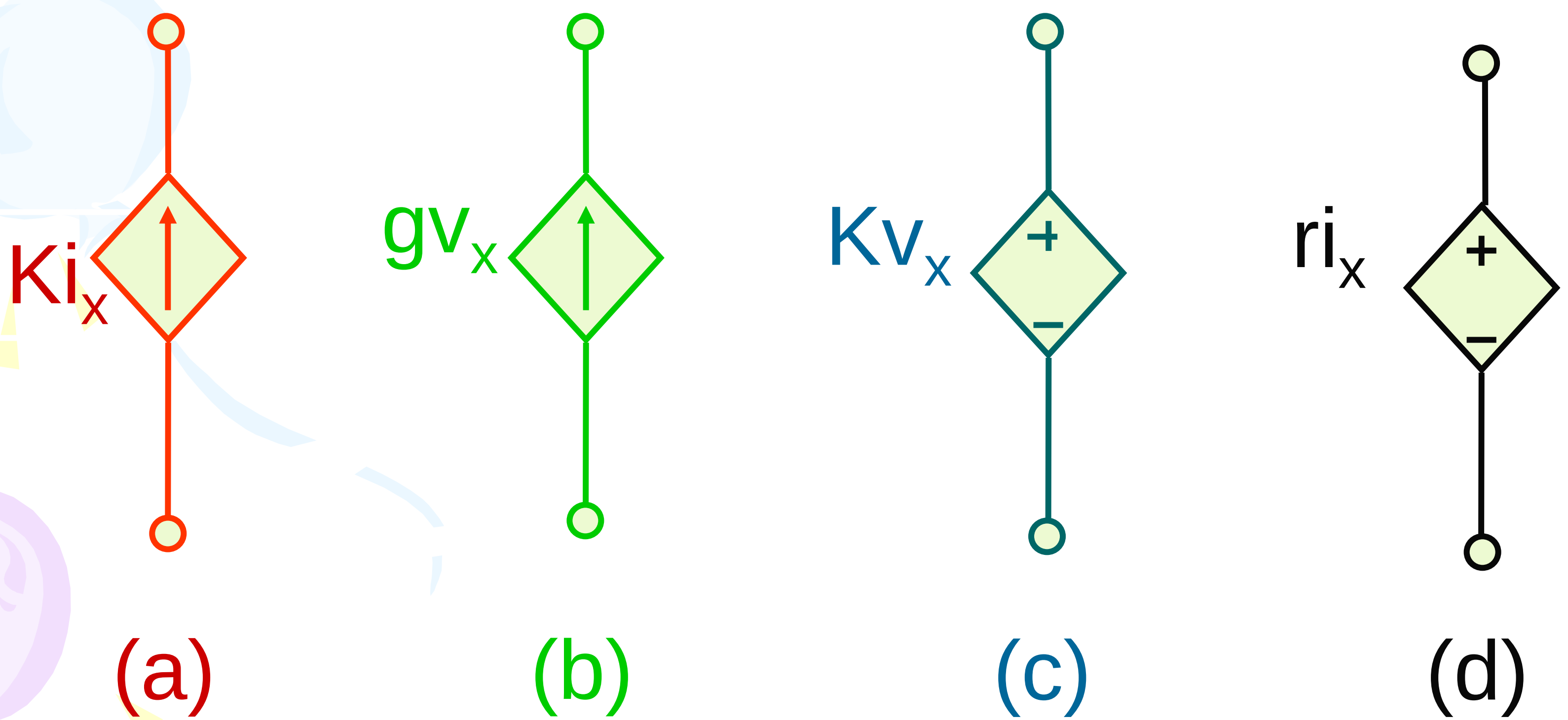
SUMMARY

- 7. An ideal voltage source produces a specific potential difference across its terminals regardless of what is connected to it.
- 8. An ideal current source produces a specific current through its terminals regardless of what is connected to it.



SUMMARY

- 9. Voltage and current sources can be dependent or independent. A dependent source is one whose value depends on some other circuit variable.



Practice

Find the power absorbed by each element in the circuit in Fig. 1.16.

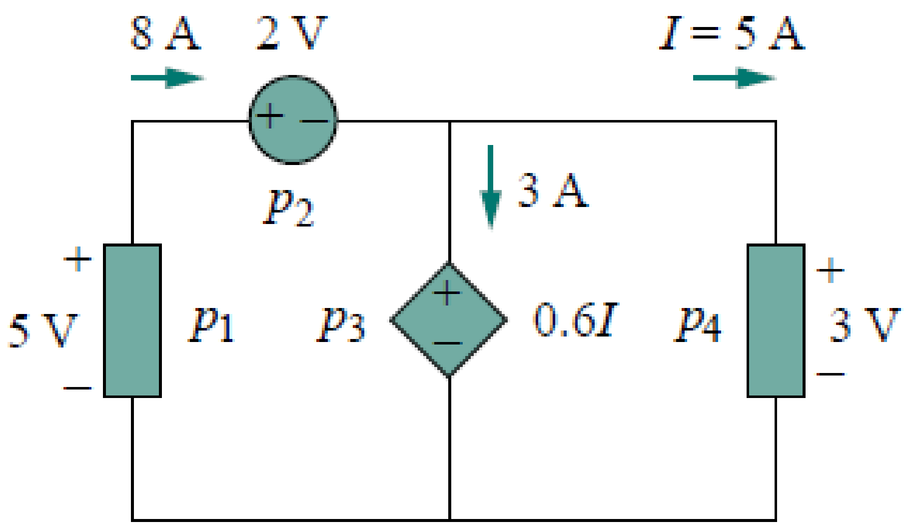
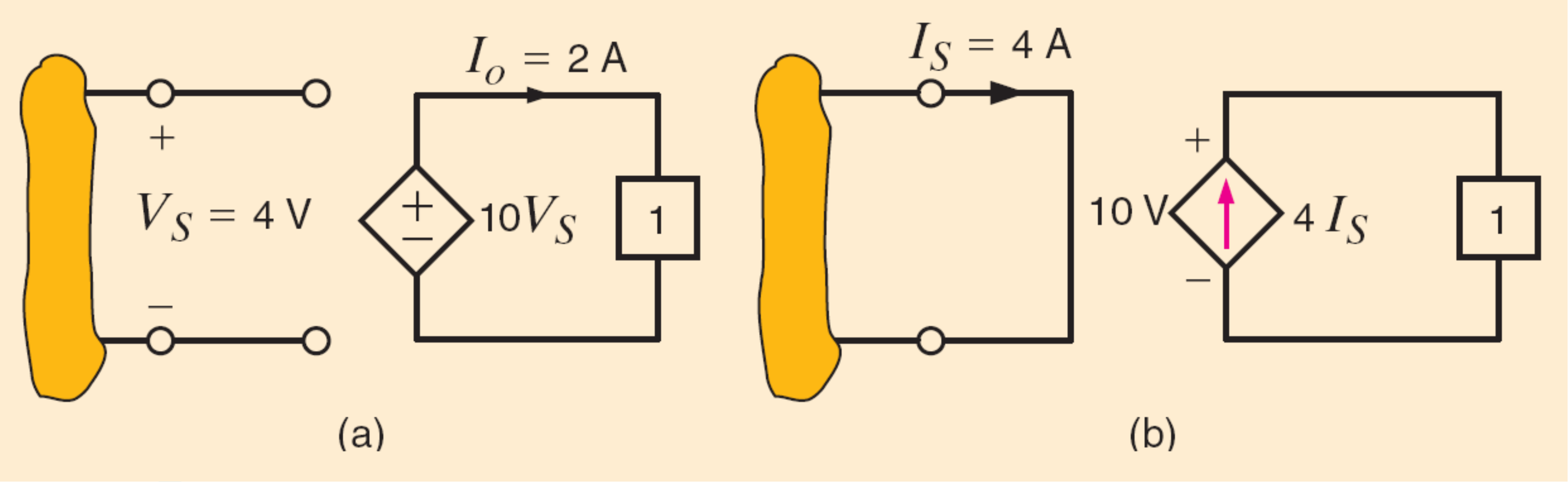


Figure 1.16 For Practice Prob. 1.7.

Answer: $p_1 = -40$ W, $p_2 = 16$ W, $p_3 = 9$ W, $p_4 = 15$ W.

Practice

Determine the power supplied by the dependent sources



Assignment (page 24)

Problems 1.2, 1.13, 1.15, 1.18, 1.20

1.2 Determine the current flowing through an element if the charge flow is given by

(a) $q(t) = (3t + 8)$ mC

(b) $q(t) = (8t^2 + 4t - 2)$ C

(c) $q(t) = (3e^{-t} - 5e^{-2t})$ nC

(d) $q(t) = 10 \sin 120\pi t$ pC

(e) $q(t) = 20e^{-4t} \cos 50t$ μC

1.13 The charge entering the positive terminal of an element is

$$q = 5 \sin 4\pi t \text{ mC}$$

while the voltage across the element (plus to minus) is

$$v = 3 \cos 4\pi t \text{ V}$$

(a) Find the power delivered to the element at $t = 0.3$ s.

(b) Calculate the energy delivered to the element between 0 and 0.6 s.

1.15 The current entering the positive terminal of a device is $i(t) = 6e^{-2t}$ mA and the voltage across the device is $v(t) = 10di/dt$ V.

(a) Find the charge delivered to the device between $t = 0$ and $t = 2$ s.

(b) Calculate the power absorbed.

(c) Determine the energy absorbed in 3 s.

1.18 Find the power absorbed by each of the elements in Fig. 1.29.

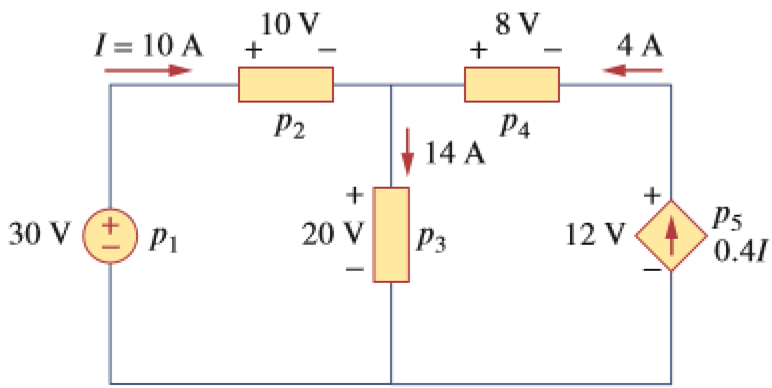


Figure 1.29

1.20 Find V_o and the power absorbed by each element in the circuit of Fig. 1.31.

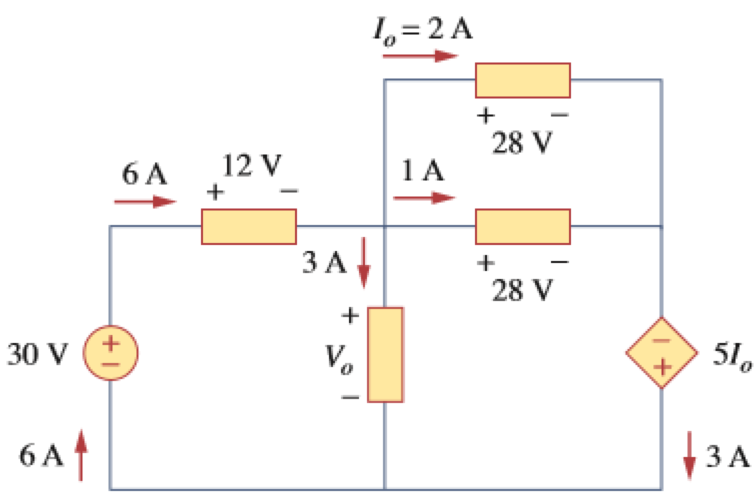


Figure 1.31

Chapter 2

Basic Laws

Chapter2 Basic Laws

2.1 Ohm's Law and Linear Resistor.

2.2 Nodes, Branches, and Loops.

2.3 Kirchhoff's Laws.

2.4 Equivalent Subcircuits.

2.5 Series Resistors and Voltage Division.

2.6 Parallel Resistors and Current Division.

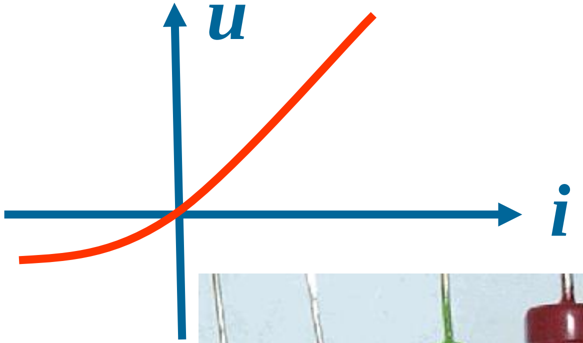
2.7 Wye-Delta Transformations.

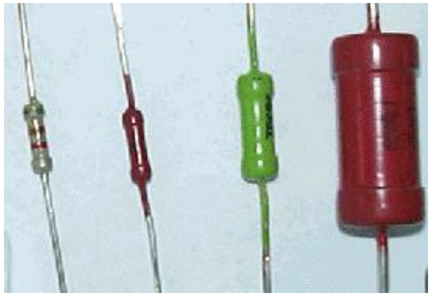
2.1 Ohm's Law and Linear Resistor

A element which dissipates energy but stores none is said to consist solely of resistance.

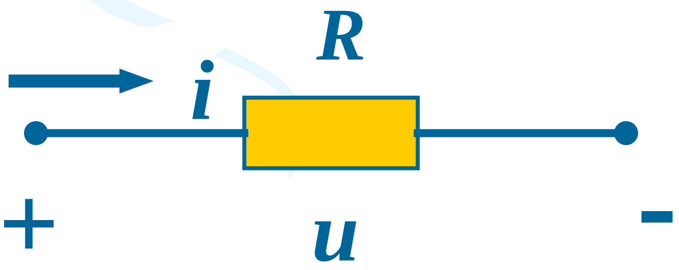
$$f\,(u, i) = 0$$

Volt-ampere relationship





- **Symbol for linear resistance**



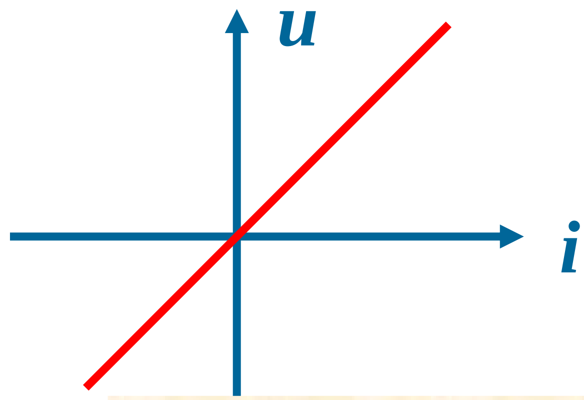
2.1 Ohm's Law and Linear Resistor

- $u \sim i$ relations

Ohm's Law

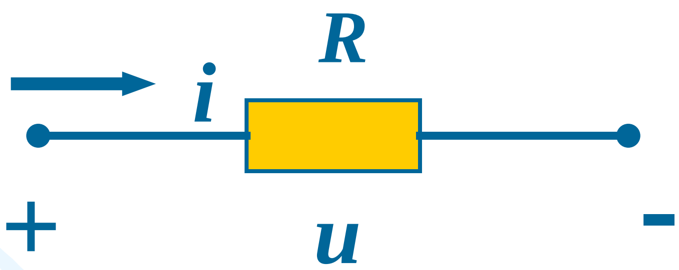
$$u = Ri \qquad R = u/i$$

$$i = u/R = Gu$$



Volt-ampere : A line through origin

u、i : Associate reference direction



- in S.I. units

R--Resistance： **Ω (Ohm)**

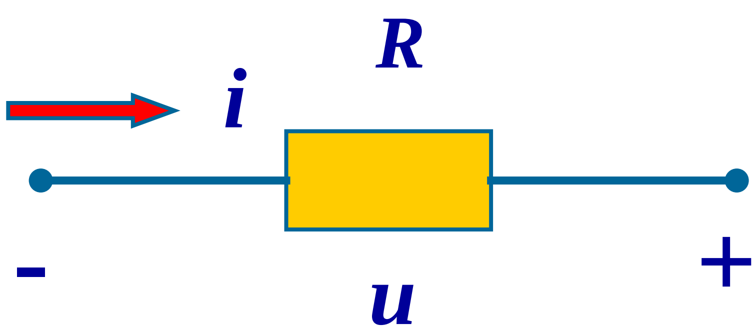
G –conductance: **S (Siemens)**

2.1 Ohm's Law and Linear Resistor

Notice:

Ohm's Law

- **(1) Be the same with linearity resistance**
- **(2) Imply linearity resistance is nonmemory**
- **(3) In non- Associate reference direction**

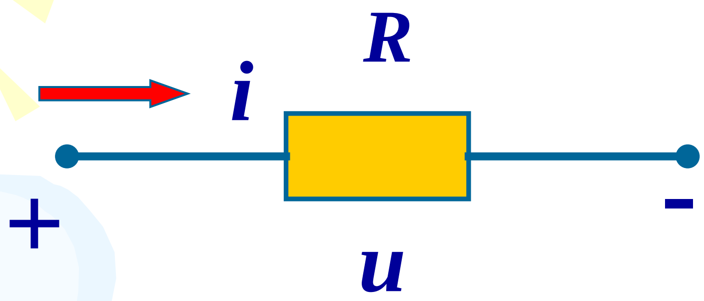


The expression for Ohm's Law is

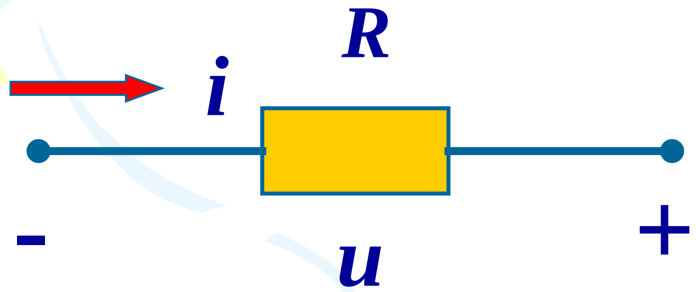
$$u = -R\,i \qquad i = -G\,u$$

2.1 Ohm's Law and Linear Resistor

- **Power of Resistance**



$$p = u\,i = i^2 R = u^2 / R$$



$$p = -u\,i = -(-R\,i)\,i = i^2 R$$

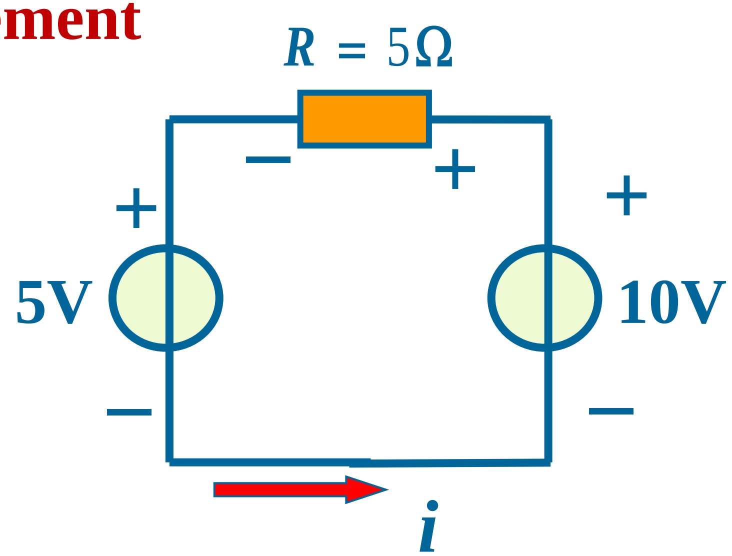
$$= -u(-u/R) = u^2/R$$

Resistor dissipate energy

2.1 Ohm’s Law and Linear Resistor

Example

Calculate the power of circuit element



Solution

$$u_R = (10-5) = 5V$$

$$i = {}^{u_R}\!/_{R} = {}^{5}\!/_{5} = 1A$$

$$P_{10V} = u_S i = 10\times 1 = 10W$$ **Deliver power**

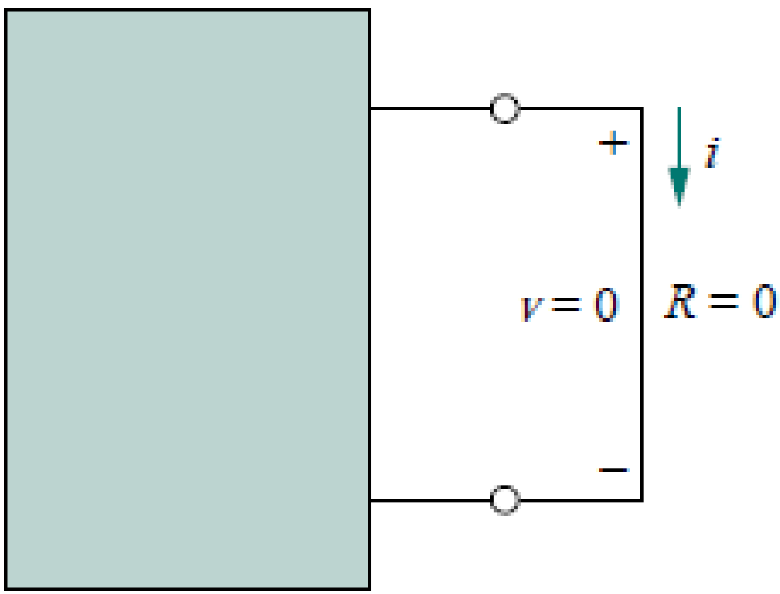
$$P_{5V} = u_S i = 5\times 1 = 5W$$ **Absorb power**

$$P_R = Ri^2 = 5\times 1 = 5W$$ **Absorb power**

satisfy： *P*（deliver） = *P*（absorb）

two extreme possible values of R

- An element with R = 0 is called a *short circuit*.



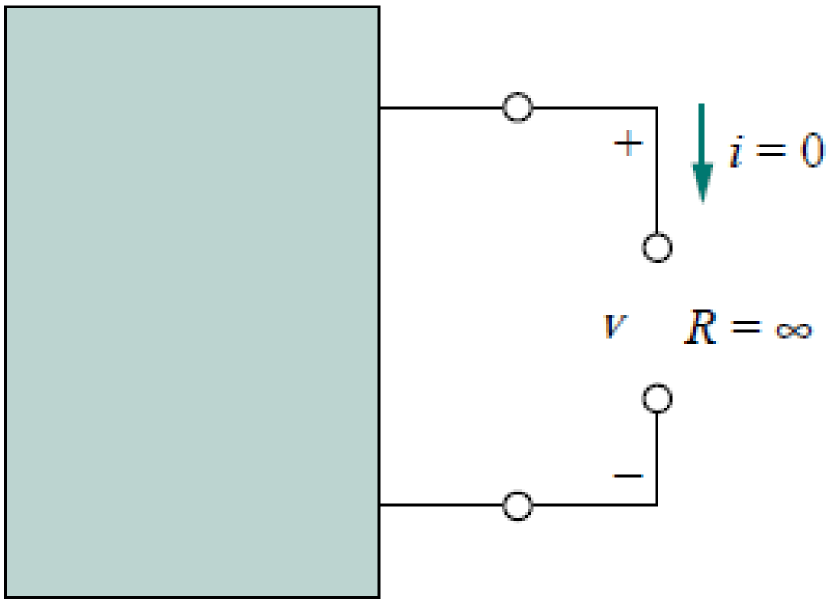
(a)

$$v = i\,R = 0$$

A short circuit is a circuit element with resistance approaching zero.

two extreme possible values of R

- An element with $R = \infty$ is known as an *open circuit*.

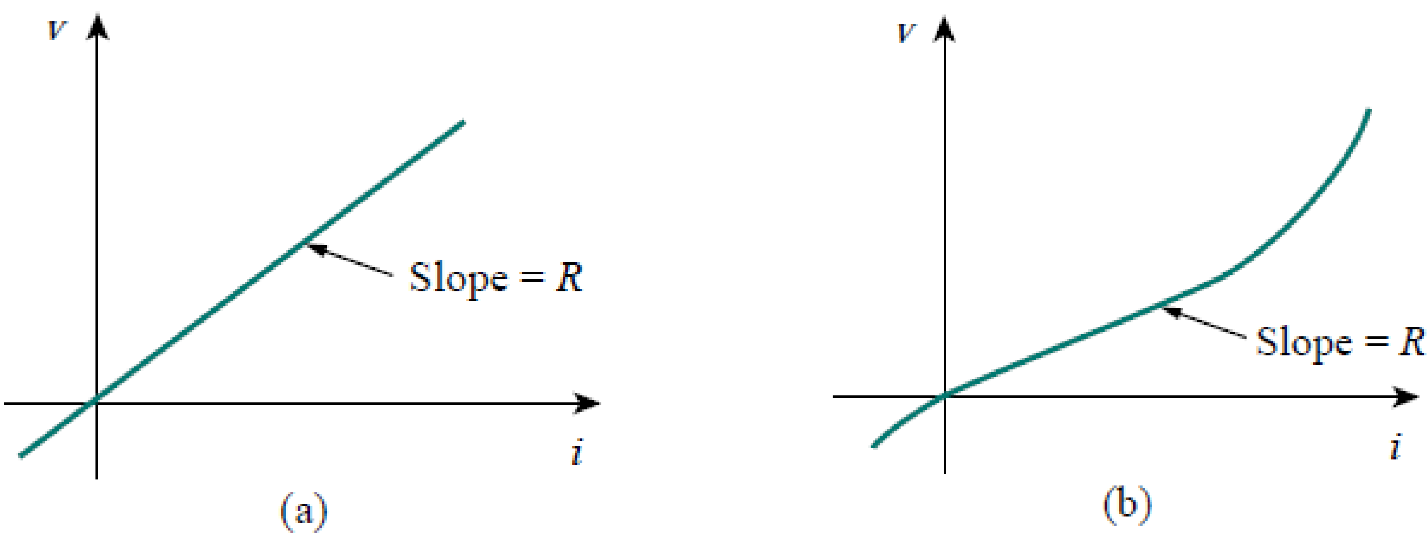


$$i = \lim_{R \to \infty} \frac{v}{R} = 0$$

(b)

An open circuit is a circuit element with resistance approaching infinity.

- It should be pointed out that not all resistors obey Ohm's law.
- A resistor that obeys Ohm's law is known as a linear resistor. It has a constant resistance.
- A nonlinear resistor does not obey Ohm's law. Its resistance varies with current.



(a) a linear resistor. (b) a nonlinear resistor.

conductance

- A useful quantity in circuit analysis is the reciprocal of resistance *R*, known as conductance and denoted by *G*:

$$\boxed{G = \frac{1}{R} = \frac{i}{v}}$$

- The conductance is a measure of how well an element will conduct electric current.

> **Conductance** is the ability of an element to conduct electric current; it is measured in mhos (℧) or siemens (S).

Power

- The power dissipated by a resistor can be expressed in terms of *R*. $p = vi = i^2 R = \dfrac{v^2}{R}$
- The power dissipated by a resistor may also be expressed in terms of *G*. $p = vi = v^2 G = \dfrac{i^2}{G}$

- 1. The power dissipated in a resistor is a nonlinear function of either current or voltage.
- 2. Since *R* and *G* are positive quantities, the power dissipated in a resistor is always positive. Thus, a resistor always absorbs power from the circuit. This confirms the idea that a resistor is a passive element, incapable of generating energy.

Example 2.1

An electric iron draws 2 A at 120 V. Find its resistance.

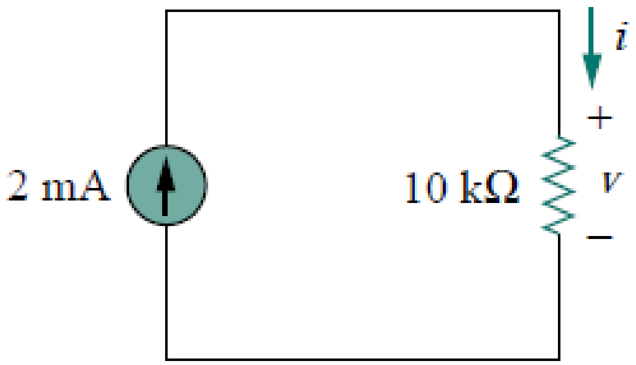
Solution:

From Ohm's law,

$$R = \frac{v}{i} = \frac{120}{2} = 60\ \Omega$$

PRACTICE 2.2

For the circuit shown in Fig. 2.9, calculate the voltage *v*, the conductance *G*, and the power *p*.



Answer: 20 V, 100 μS, 40 mW.

Figure 2.9 For Practice Prob. 2.2

2.2 Branches and Loops , Nodes

- Branch : A *branch* represents a single element such as a voltage source or a resistor.
- Loop: A *loop* is any closed path in a circuit.
- Node : A *node* is the point of connection between two or more branches.

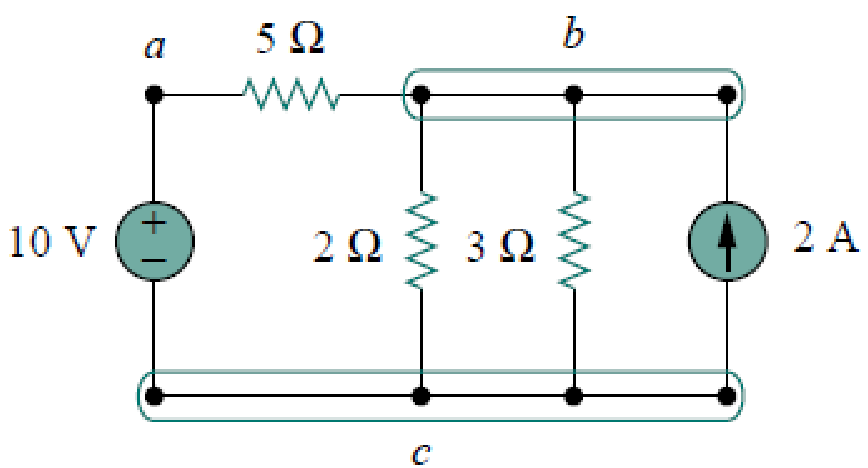


Figure 2.10 Nodes, branches, and loops.

- Node can be indicated in diagrams in two ways:
  - by a thin line enclosing the node, as with nodes ***b*** and ***c***;
  - by marking a typical point within the node, as with node ***a***.

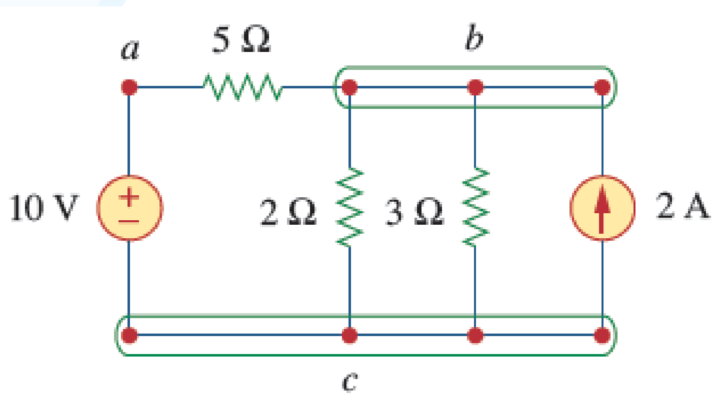


Figure 2.10
Nodes, branches, and loops.

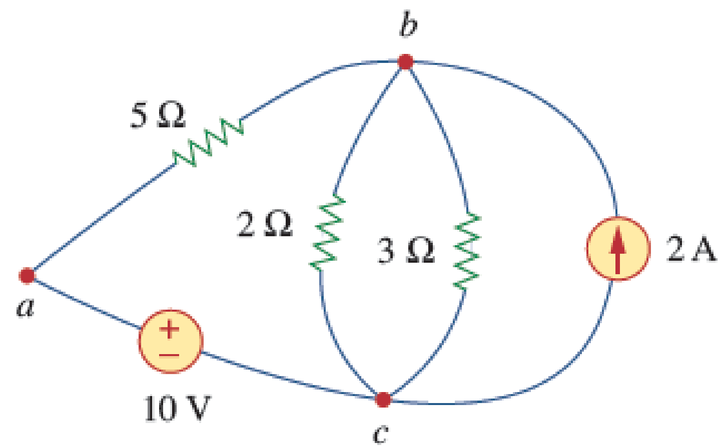


Figure 2.11
The three-node circuit of Fig. 2.10 is redrawn.

- Two or more elements are **in series** if they are cascaded or connected sequentially and consequently carry the same current.
- Two or more elements are **in parallel** if they are connected to the same two nodes and consequently have the same voltage across them.

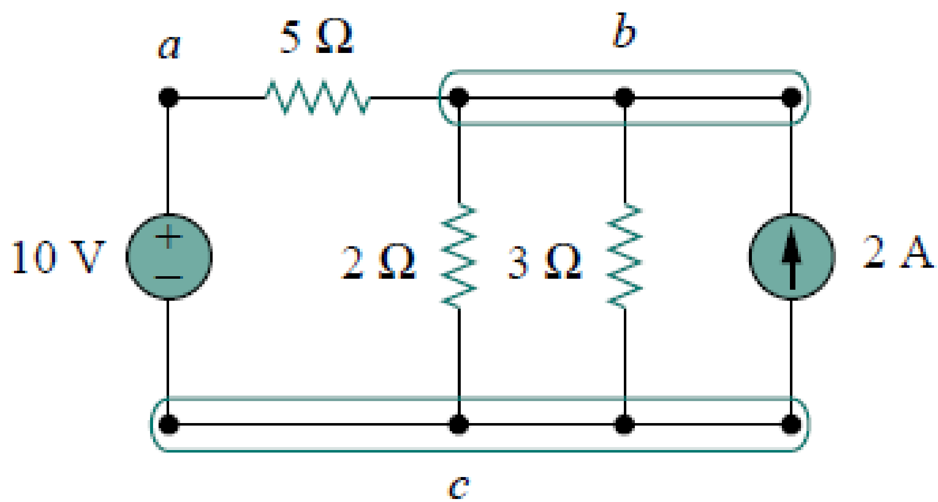


Figure 2.10 Nodes, branches, and loops.

2.2 Branches and Loops , Nodes(2)

Example 1

Determine the number of branches and nodes in the circuit shown in Fig. 2.2

Solution :

Since there are five elements in the circuit, the circuit has five branches. In addition, the circuit has three nodes, *a*, *b* and *c*

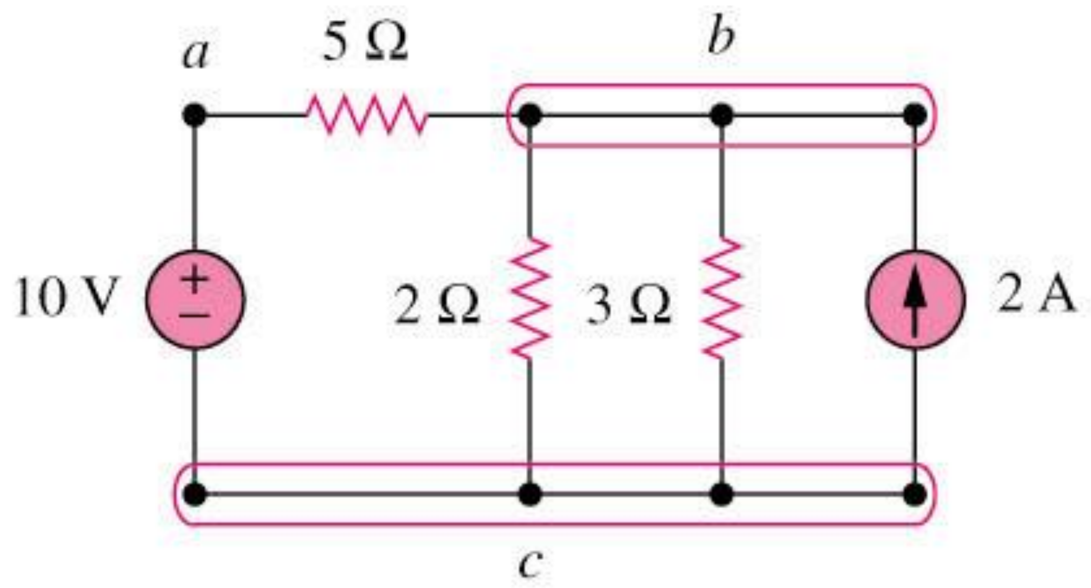


Fig. 2.2

Example 2

How many branches, nodes and loops are there?

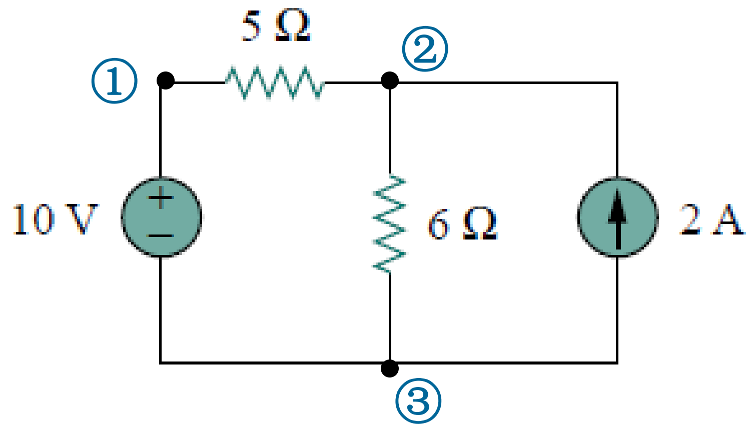


Figure 2.12 For Example 2.4.

Since there are four elements in the circuit, the circuit has four branches: 10 V, 5Ω , 6 Ω, and 2 A.

The circuit has three nodes.

The 5-Ω resistor is in series with the 10-V voltage source because the same current would flow in both.

The 6-Ω resistor is in parallel with the 2-A current source because both are connected to the same nodes 2 and 3.

Practice

How many branches and nodes does the circuit in Fig. 2.14 have? Identify the elements that are in series and in parallel.

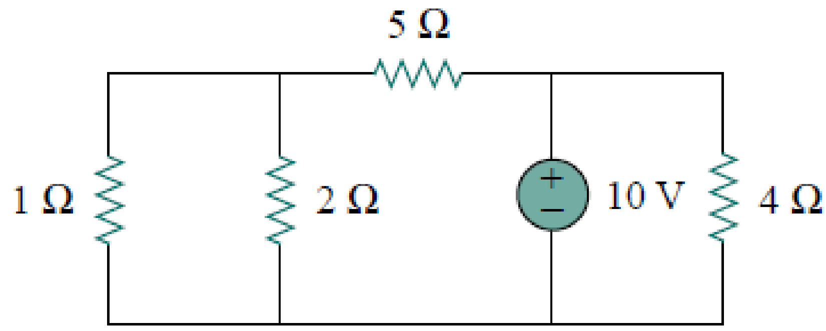


Figure 2.14 For Practice Prob. 2.4.

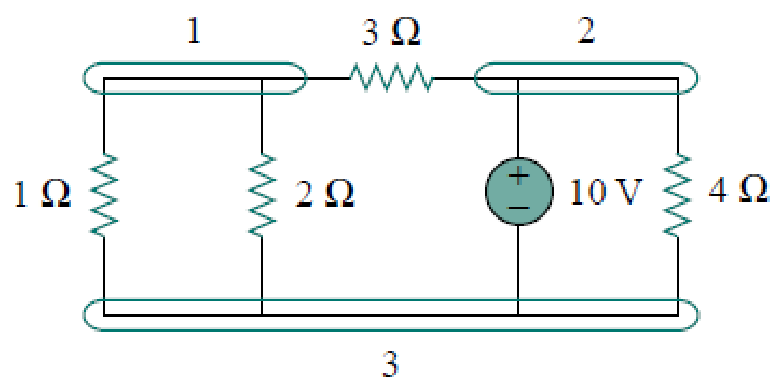
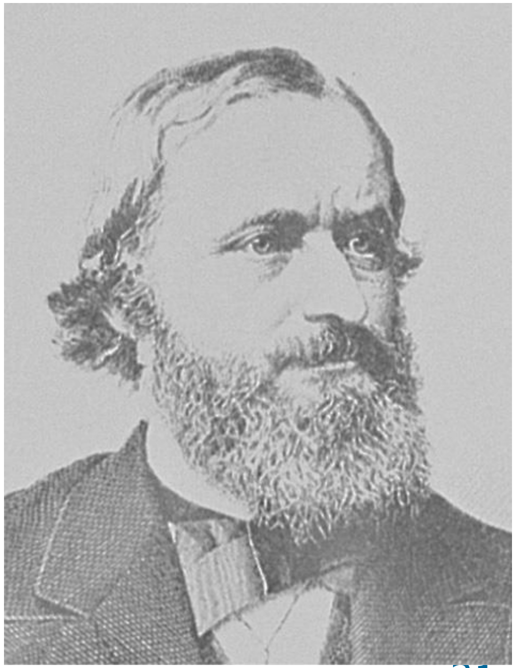


Figure 2.15 Answer for Practice Prob. 2.4.

2.3 Kirchhoff's Laws

- Kirchhoff's laws were first introduced in 1847 by the German physicist Gustav Robert Kirchhoff (1824–1887). These laws are formally known as
  - Kirchhoff's current law (KCL)
  - Kirchhoff's voltage law (KVL).



2.3 Kirchhoff's Laws (1)

- ***Kirchhoff's current law (KCL)*** states that the **algebraic sum** of currents entering a node (or a closed boundary) is zero.

Mathematically, KCL implies that

$$\sum_{n=1}^{N} i_n = 0$$

Where N is the number of branches connected to the node and i_n is the nth current entering the node .

Kirchhoff's Current Law

Kirchhoff's current law (KCL) states that the algebraic sum of currents entering a node (or a closed boundary) is zero.

$$i_1 - i_2 + i_3 + i_4 - i_5 = 0$$

$$i_1 + i_3 + i_4 = i_2 + i_5$$

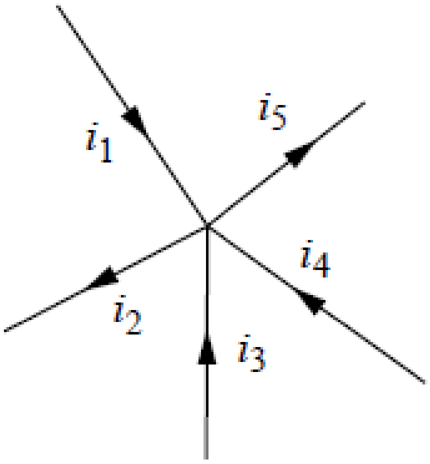


Figure 2.16 Currents at a node illustrating KCL.

The sum of the currents entering a node is equal to the sum of the currents leaving the node.

- Note that ***KCL*** also applies to a closed boundary. In two dimensions, a closed boundary is the same as a closed path.
- As Fig.2.4 show, the total current entering the closed surface is equal to the total current leaving the surface.

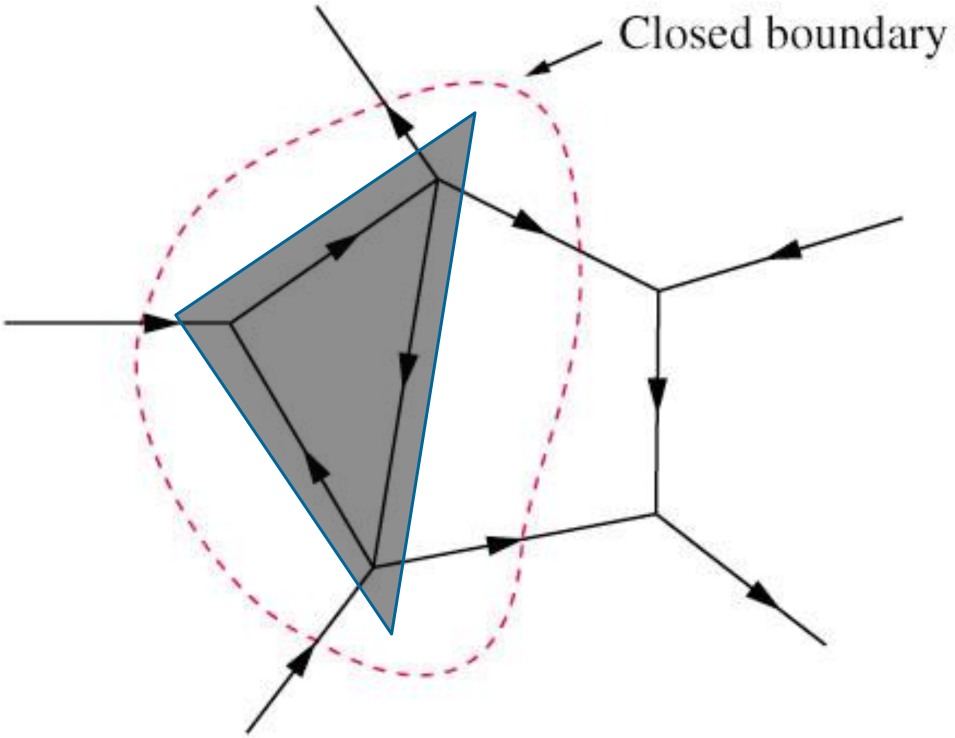
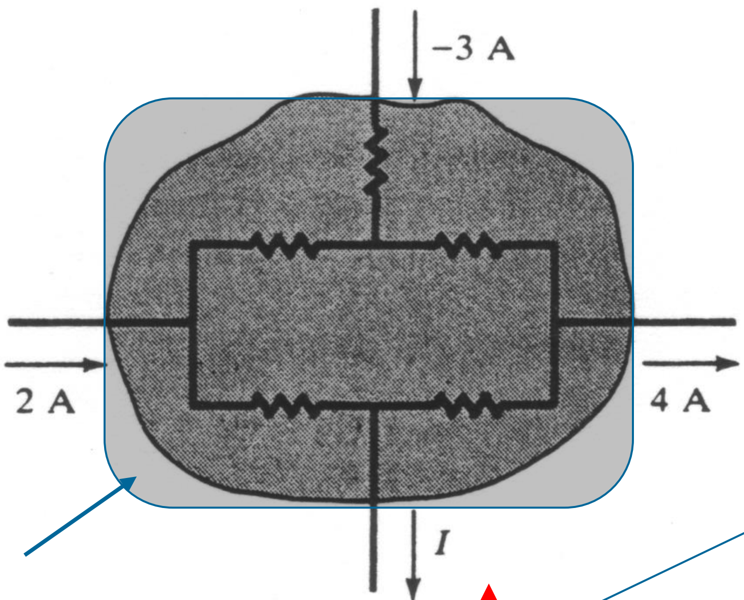


Fig.2.4 Applying KCL to closed boundary

Example 3

- Determine the current *I* for the circuit shown in the figure below.



We can consider the whole enclosed area as one "node".

I + 4-(-3)-2 = 0

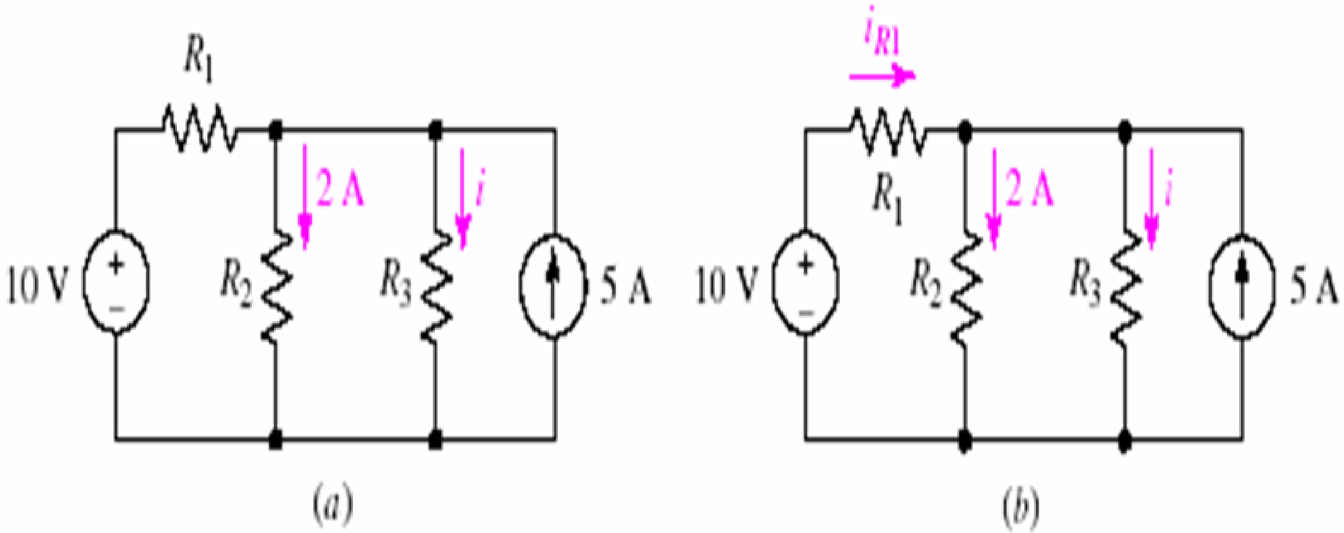
⇒I = -5A

This indicates that the actual current for *I* is flowing in the opposite direction.

Example 4

For the circuit in Fig. 2.5a, compute the current through resistor R_3 if it is known that the voltage source supplies a current of 3 A.

Fig. 2.5



$i_{R1} - 2 - i + 5 = 0$

$i_{R1} = 3$

So i=6A

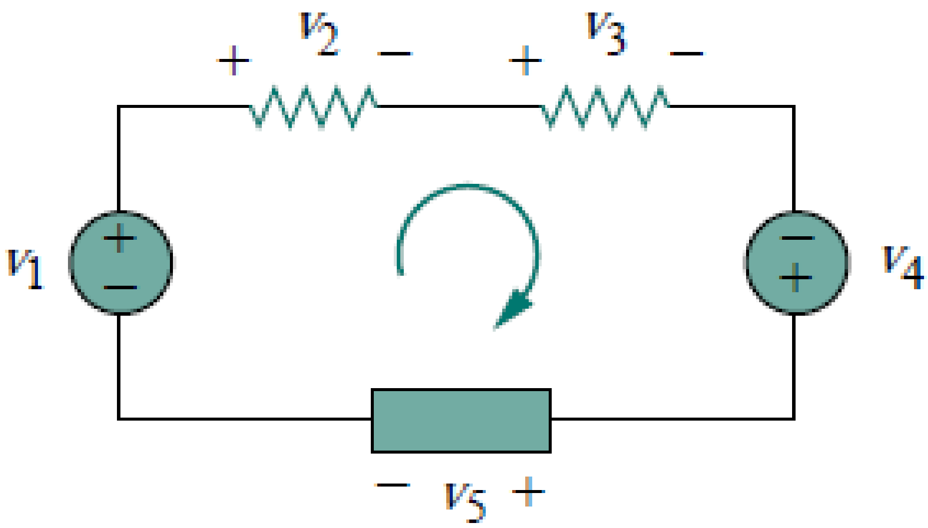
(*a*) Simple circuit for which the current through resistor R_3 is desired. (*b*) The current through resistor R_1 is labeled so that a KCL equation can be written. (*c*) The currents into the top node of R_3 are redrawn for clarity.

2.3 Kirchhoff's Laws (3)

- ***Kirchhoff's voltage law (KVL)*** states that the **algebraic sum** of all voltages around a closed path (or loop) is zero.

Expressed mathematically, KVL states that

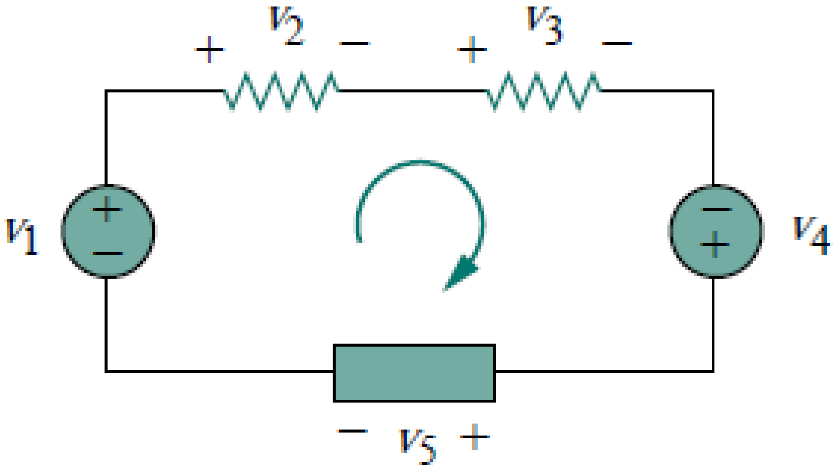
$$\sum_{m=1}^{M} v_n = 0$$



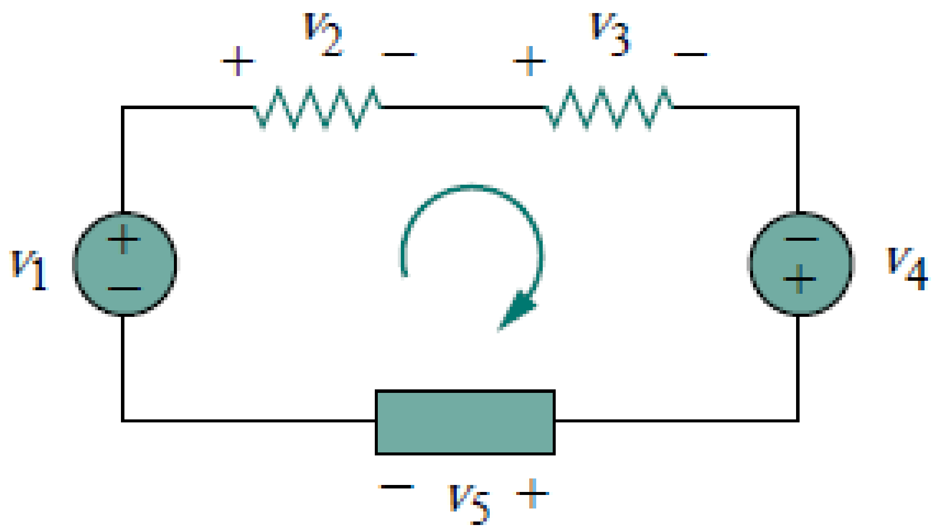
Kirchhoff's Voltage Law

- The sign on each voltage is the polarity of the terminal encountered first as we travel around the loop.
- We can start with any branch and go around the loop either clockwise or counterclockwise.

$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$



Alternative Form of KVL



$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$

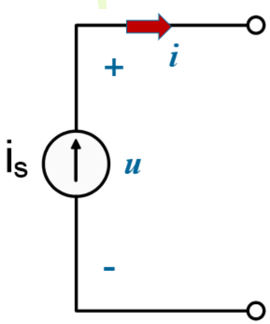
$$v_2 + v_3 + v_5 = v_1 + v_4$$

$$\boxed{\text{Sum of voltage drops} = \text{Sum of voltage rises}}$$

Main content of last class

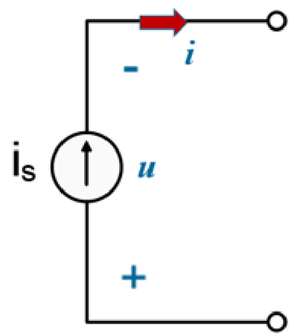
1. Two basis for Circuit analysis

(1) Voltage-current relationship-independent current sources



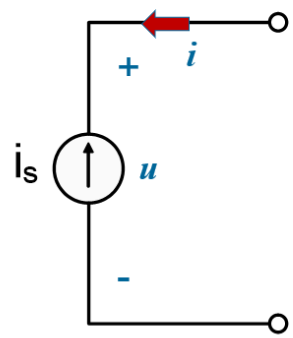
$i = i_s$

$p_{supply} = ui$



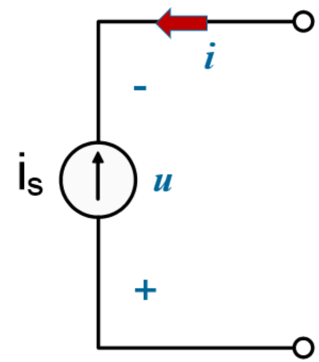
$i = i_s$

$p_{supply} = -ui$



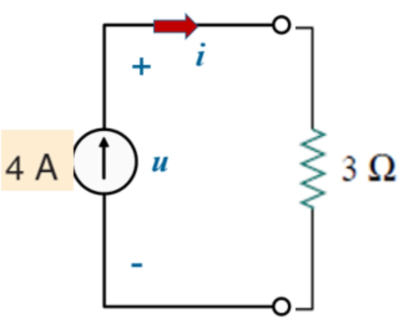
$i = -i_s$

$p_{supply} = -ui$



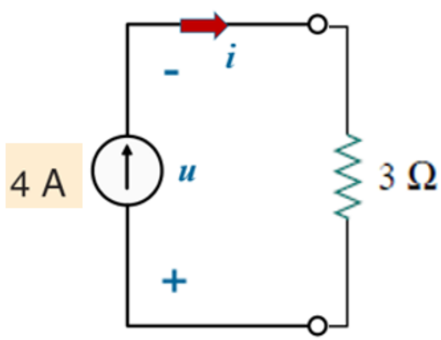
$i = -i_s$

$p_{supply} = ui$



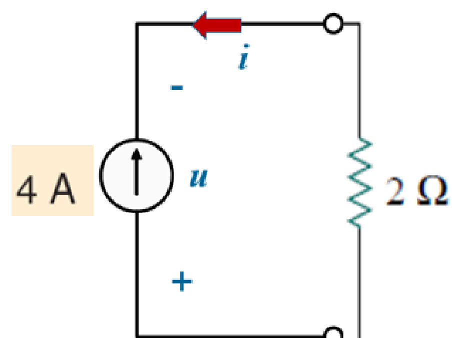
$i = 4A$

$u = 3 \times 4 = 12V$



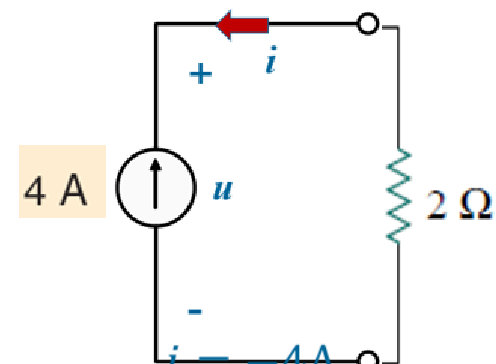
$i = 4A$

$u = -3 \times 4 = -12V$



$i = -4A$

$u = 2 \times (-4) = -8V$



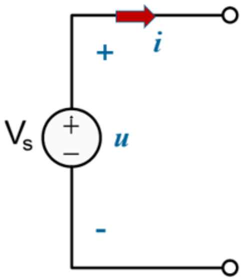
$i = -4A$

$u = -2 \times (-4) = 8V$

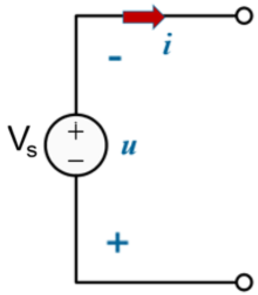
Main content of last class

1. Two basis for Circuit analysis

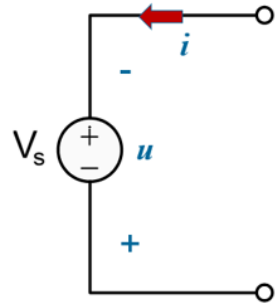
(1) Voltage-current relationship-independent voltage sources



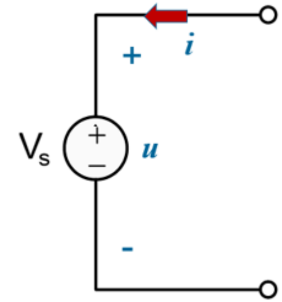
$$u = V_s$$
$$p_{supply} = ui$$



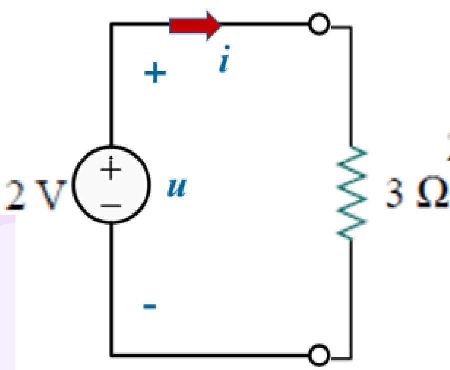
$$i = i_s$$
$$p_{supply} = -ui$$



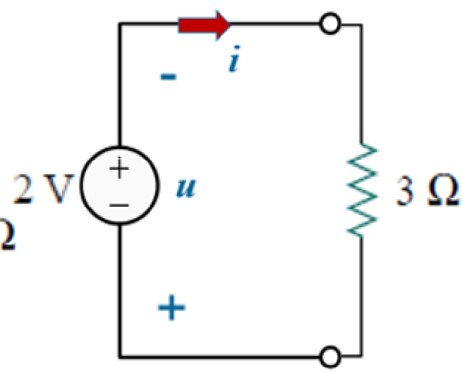
$$i = -i_s$$
$$p_{supply} = -ui$$



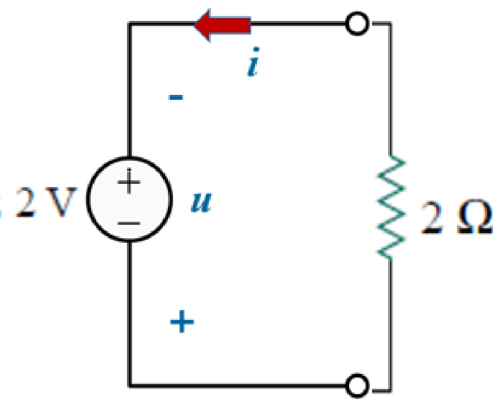
$$i = -i_s$$
$$p_{supply} = ui$$



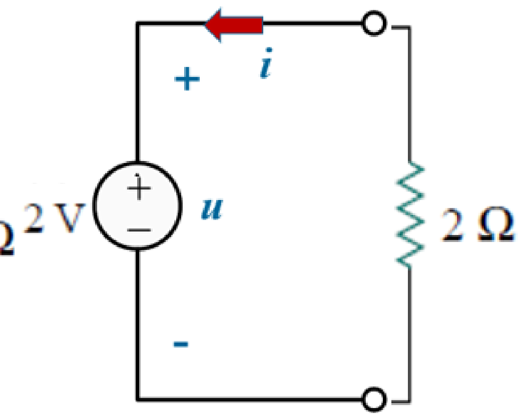
$$u = 2V$$
$$i = \frac{u}{3} = \frac{2}{3}A$$



$$u = -2V$$
$$i = -\frac{u}{3} = \frac{2}{3}A$$



$$u = -2V$$
$$i = \frac{u}{2} = -1A$$

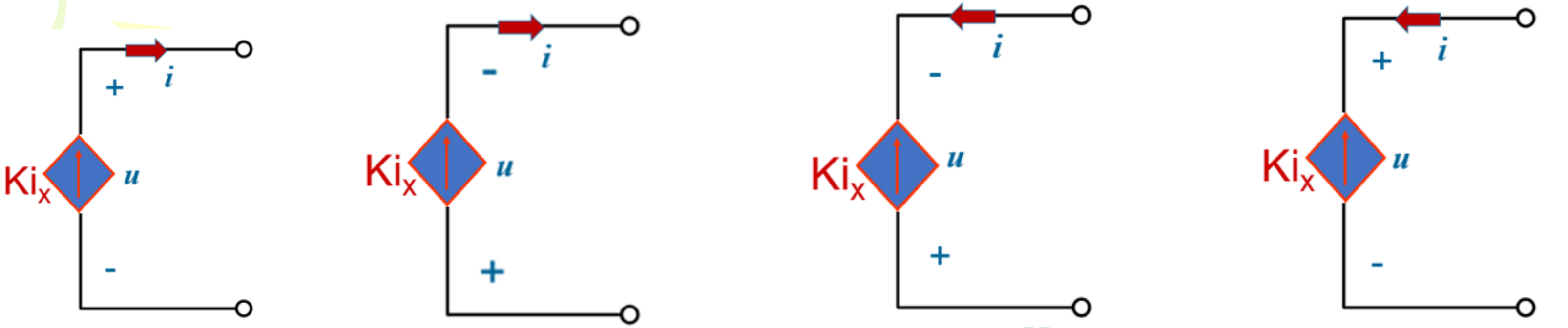


$$u = 2V$$
$$i = -\frac{u}{2} = -1A$$

Main content of last class

1. **Two basis for Circuit analysis**

(1) Voltage-current relationship -dependent current sources

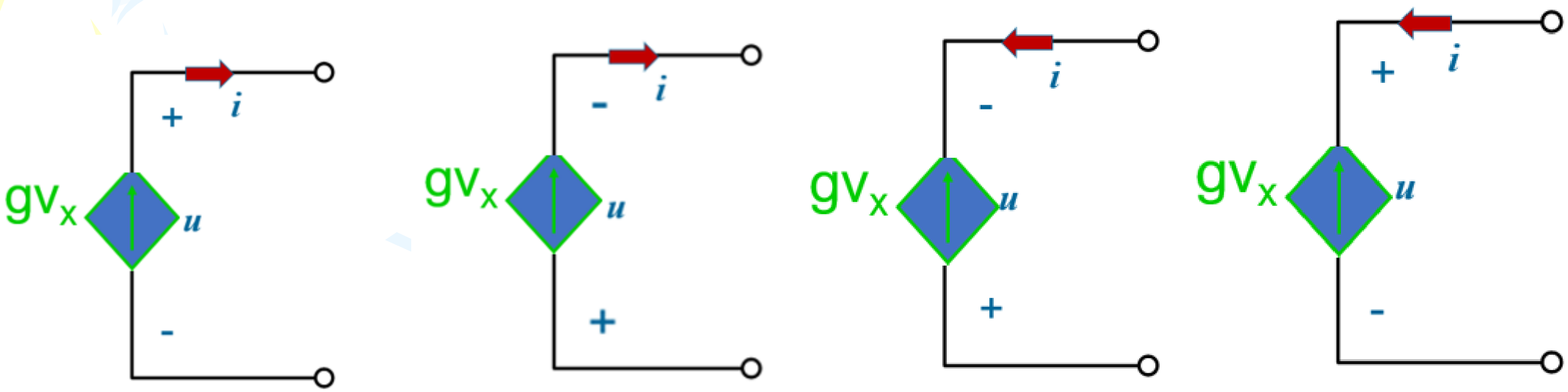


$i = Ki_x$ $\quad p_{supply} = ui$

$i = Ki_x$ $\quad p_{supply} = -ui$

$i = -Ki_x$ $\quad p_{supply} = -ui$

$i = -Ki_x$ $\quad p_{supply} = ui$



$i = gV_x$ $\quad p_{supply} = ui$

$i = gV_x$ $\quad p_{supply} = -ui$

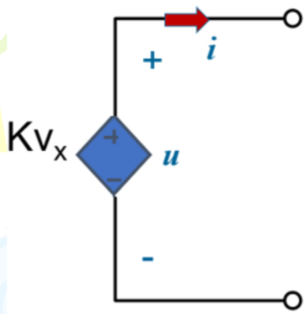
$i = -gV_x$ $\quad p_{supply} = -ui$

$i = -gV_x$ $\quad p_{supply} = ui$

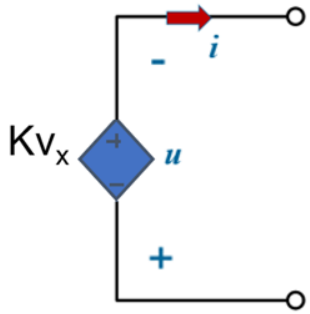
Main content of last class

1. Two basis for Circuit analysis

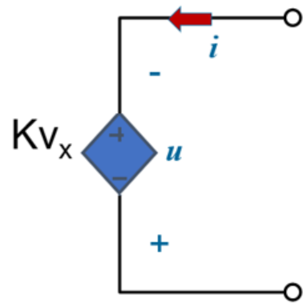
(1) Voltage-current relationship -dependent voltage sources



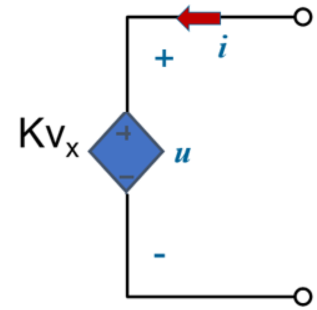
$$u = KV_x$$
$$p_{supply} = ui$$



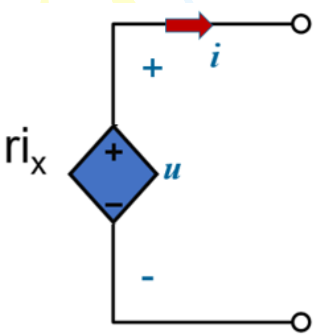
$$u = -KV_x$$
$$p_{supply} = -ui$$



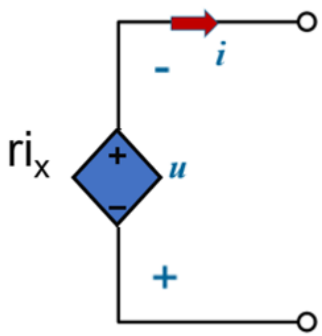
$$u = -KV_x$$
$$p_{supply} = -ui$$



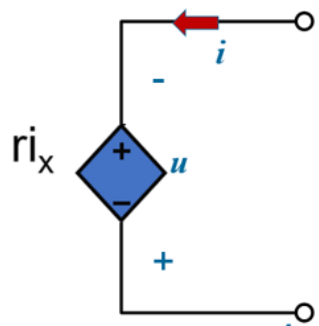
$$u = KV_x$$
$$p_{supply} = ui$$



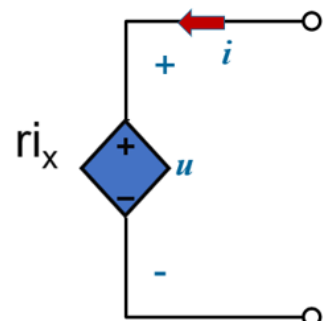
$$u = ri_x$$
$$p_{supply} = ui$$



$$u = -ri_x$$
$$p_{supply} = -ui$$



$$u = -ri_x$$
$$p_{supply} = -ui$$

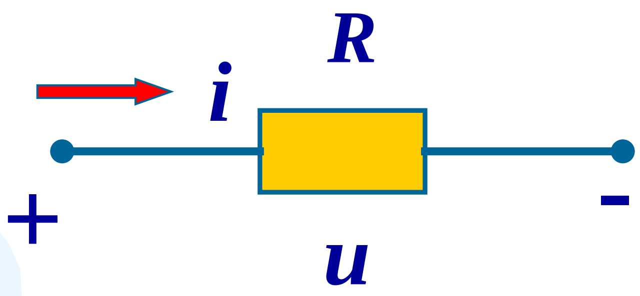


$$u = ri_x$$
$$p_{supply} = ui$$

Main content of last class

1. Two basis for Circuit analysis

(1) Voltage-current relationship -linear resistor

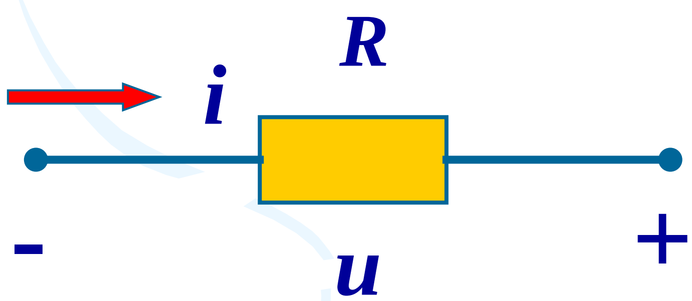


Passive sign convention

$$u = Ri$$

Ohm's Law

$$p_{absorbed} = u\,i = i^2R = u^2/R$$



Active sign convention

$$u = -Ri$$

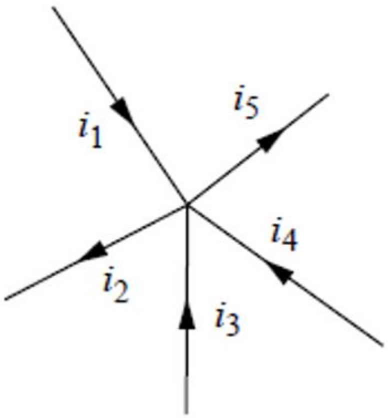
Ohm's Law

$$p_{absorbed} = -u\,i = i^2R = u^2/R$$

Main content of last class

1. Two basis for Circuit analysis

（2）Kirchhoff's Laws-Kirchhoff's current law (KCL)



$$-i_1 + i_2 - i_3 - i_4 + i_5 = 0$$

Kirchhoff's current law (KCL) states that the algebraic sum of currents entering a node (or a closed boundary) is zero.

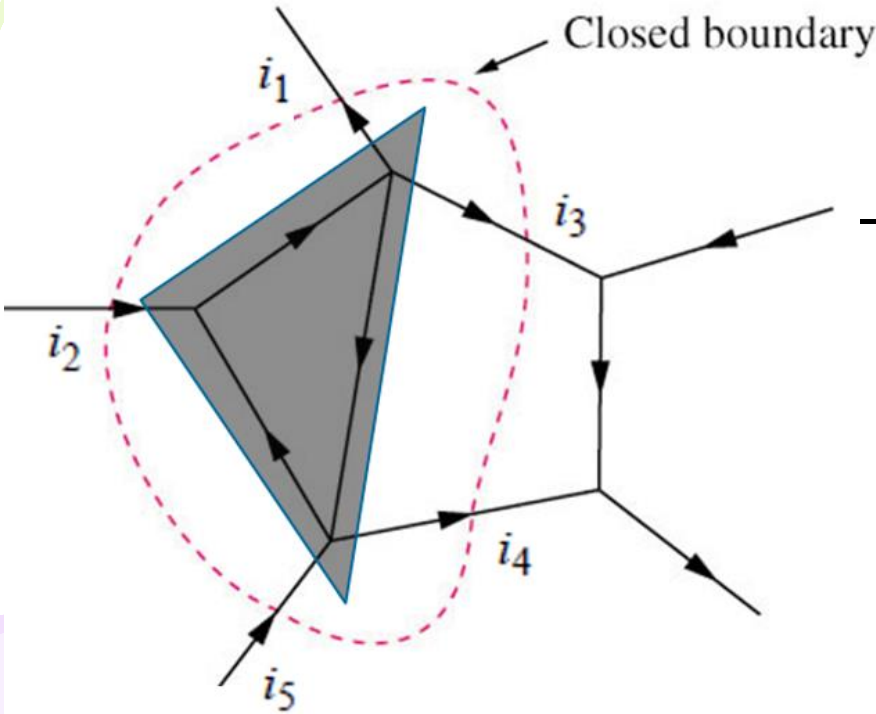
$$i_1 + i_3 + i_4 = i_2 + i_5$$

The sum of the currents entering a node is equal to the sum of the currents leaving the node.

Main content of last class

1. Two basis for Circuit analysis

（2）Kirchhoff's Laws-Kirchhoff's current law (KCL)



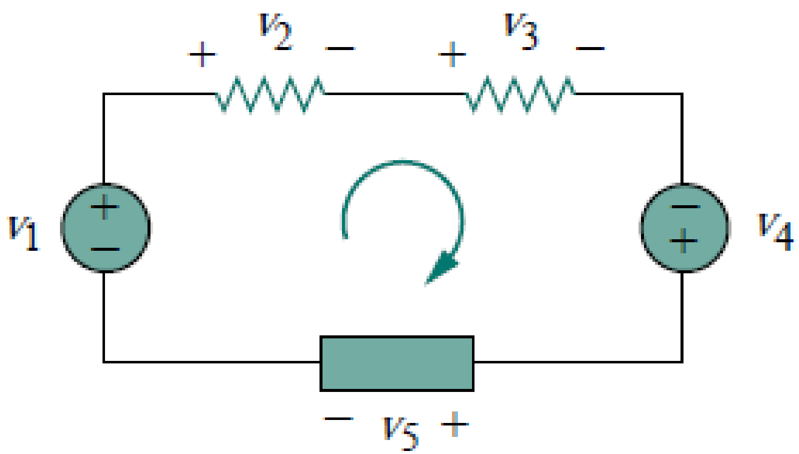
$$-i_1 + i_2 - i_3 - i_4 + i_5 = 0$$

$$i_1 + i_3 + i_4 = i_2 + i_5$$

Main content of last class

1. Two basis for Circuit analysis

（2）Kirchhoff's Laws-Kirchhoff's voltage law (KVL)



clockwise

$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$

Kirchhoff's voltage law (KVL) states that the algebraic sum of all voltages around a closed path (or loop) is zero.

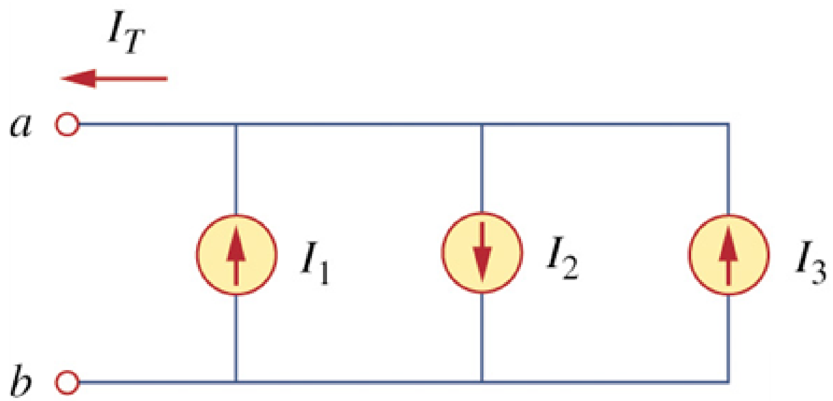
$$v_2 + v_3 + v_5 = v_1 + v_4$$

$$\boxed{\text{Sum of voltage drops} = \text{Sum of voltage rises}}$$

Main content of last class

2. Examples of circuit analysis

Example 6 Find the current I_T in Fig. 2.7(a)

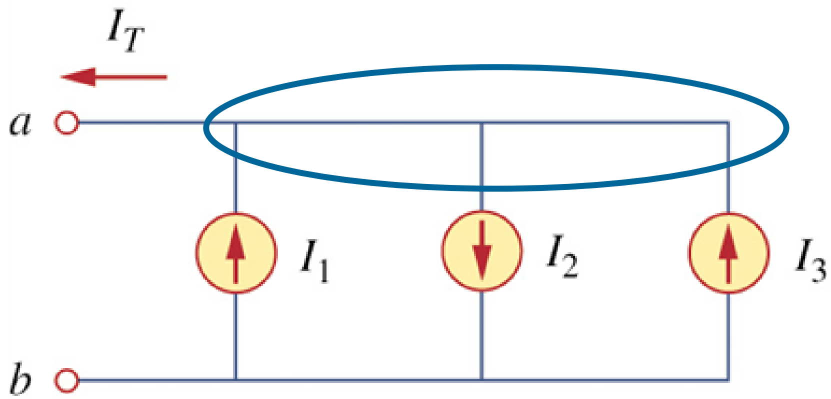


Solution :

Applying KCL to node *a* yields

$$I_T+I_2=I_1+I_3$$

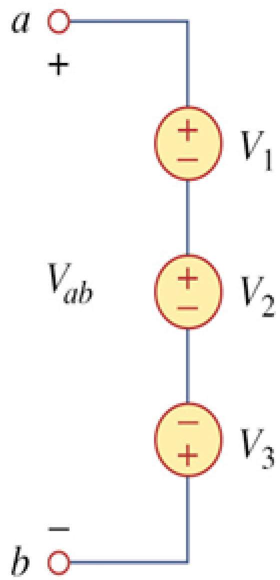
So $I_T=I_1-I_2+I_3$



Main content of last class

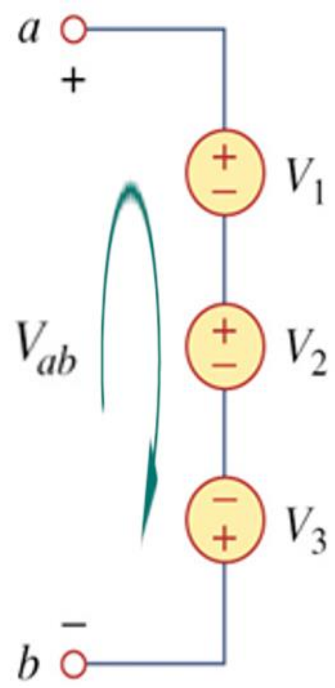
2. Examples of circuit analysis

Example 7 Determine the voltage v_{ab} in Fig. 2.8(a)



Solution

By applying KVL , we obtain



$$-V_{ab} + V_1 + V_2 - V_3 = 0$$

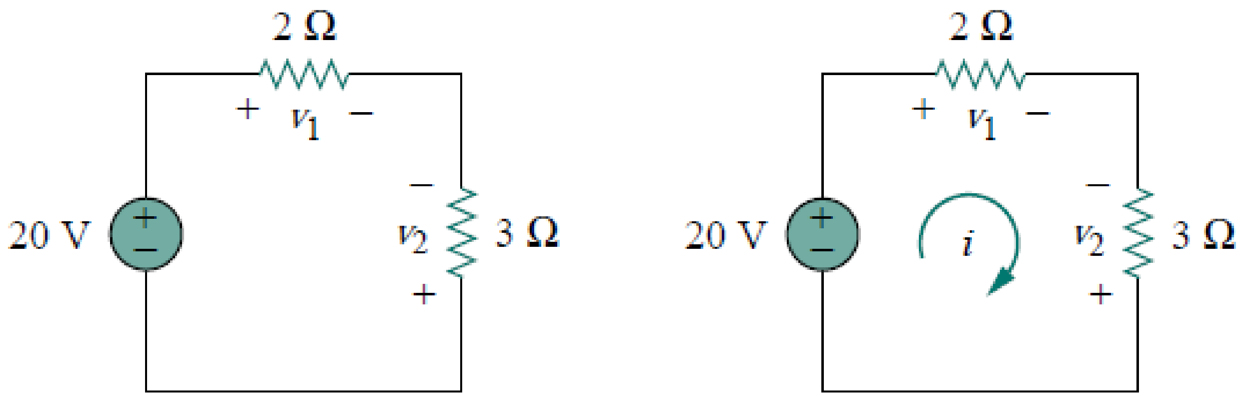
or

$$V_{ab} = V_1 + V_2 - V_3$$

Main content of last class

2. Examples of circuit analysis

Example 8 For the circuit in Fig. 2.21(a), find voltages v_1 and v_2.



(a) (b)

Solution:

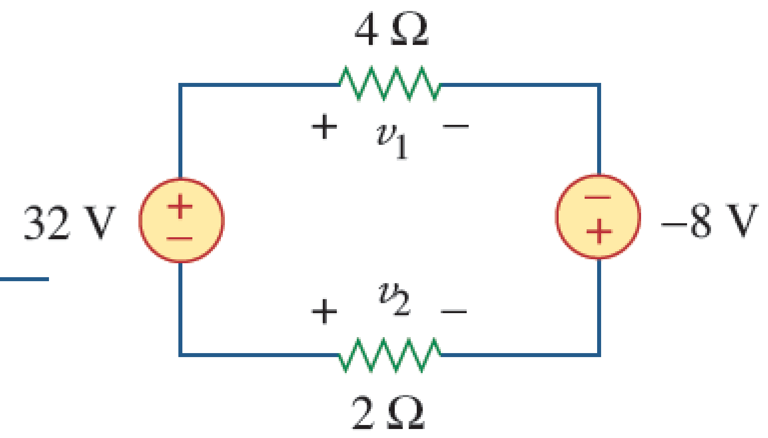
Applying Ohm's law : $v_1=2i$, $v_2=-3i$

Applying KVL : $v_1=20+v_2$

So $v_1= 8V$, $v_2= -12V$

Main content of last class

2. Examples of circuit analysis



Practice Problem 2.5

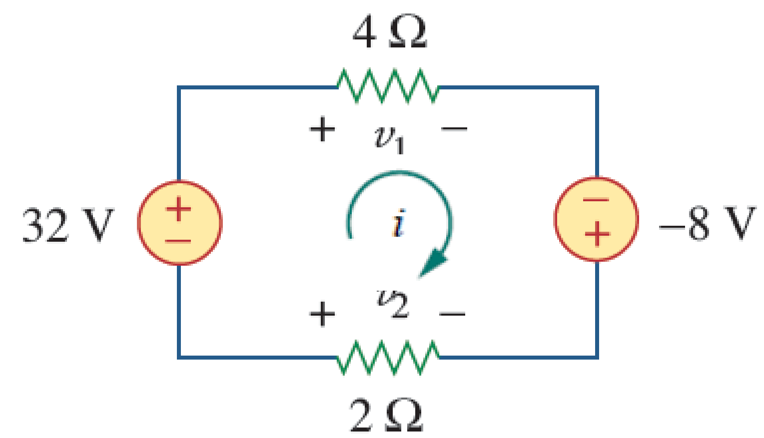
Find v_1 and v_2 in the circuit of Fig. 2.22.

Solution:

Applying Ohm's law: $v_1=4i$, $v_2=-2i$

Applying KVL : $v_1=-8+v_2+32$

So $v_1= 16V$, $v_2= -8V$



Answer: 16 V, −8 V.

Main content of last class

2. Examples of circuit analysis

Example 10

Determine current i_0 and voltage v_0 and In the circuit shown in Fig.2.25

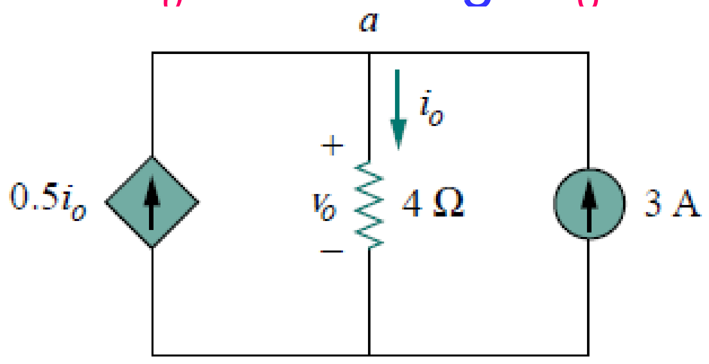
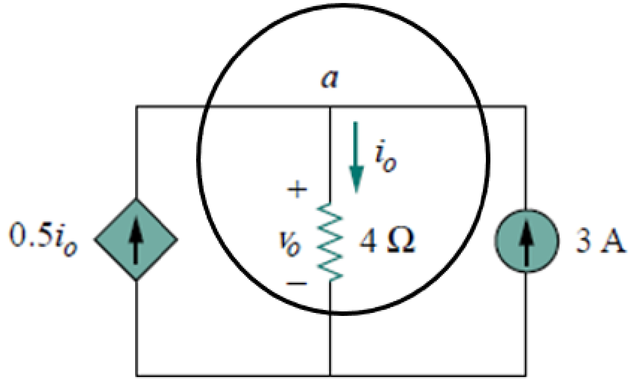


Figure 2.25 For Example 2.7.



Solution:

Applying KCL to node a, we obtain

$$3 + 0.5i_0 = i_0 \quad \Rightarrow \quad i_0 = 6\text{ A}$$

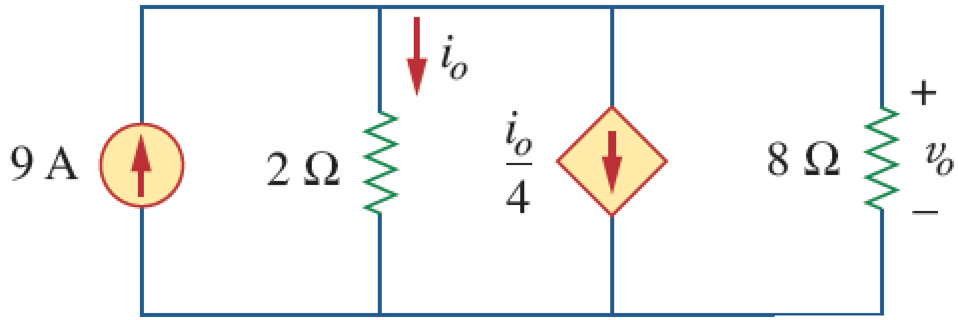
$$so \qquad v_0 = 4i_0 = 24\text{V}$$

Main content of last class

2. Examples of circuit analysis

Practice Problem 2.7

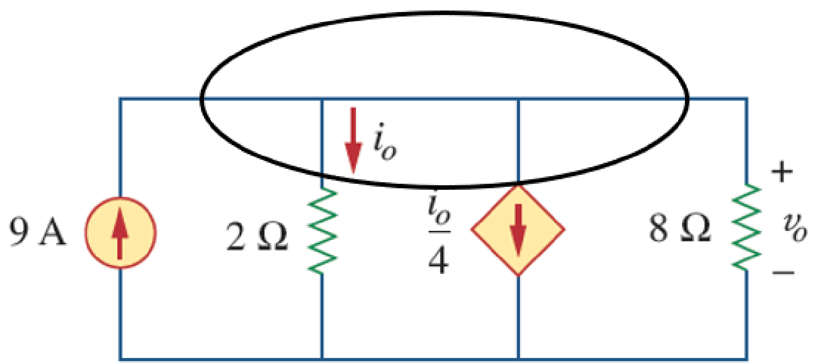
Find v_o and i_o in the circuit of Fig. 2.26.



Solution:

Applying KCL to node a, we obtain

$$\frac{v_0}{8} + \frac{i_0}{4} + i_0 = 9$$



Applying Ohm's law: $v_0 = 2\, i_0$

So $i_0 = 6$ A $v_0 = 2i_0 = 12$V

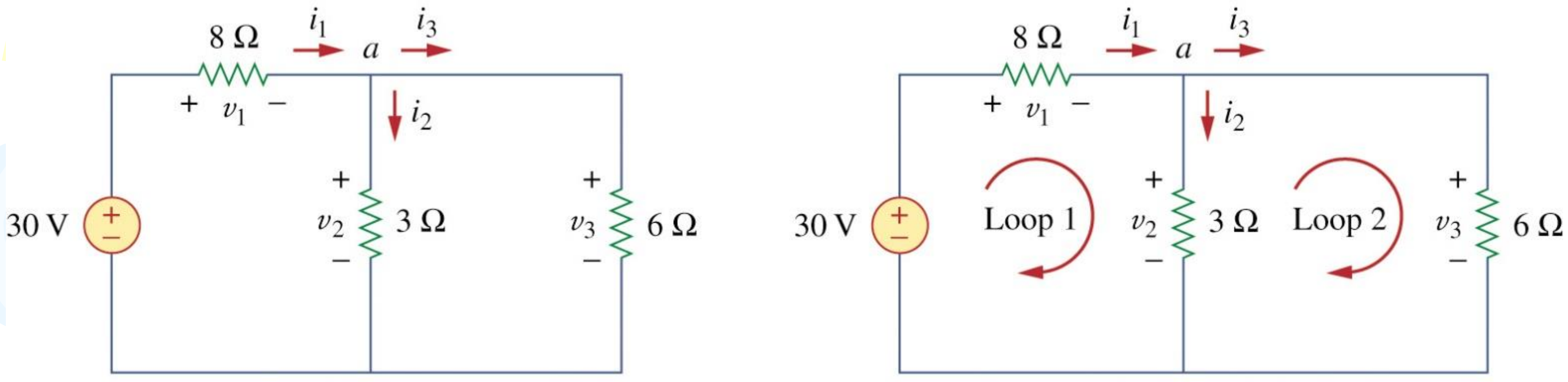
Answer: 12 V, 6 A.

Main content of last class

2. Examples of circuit analysis

Example 2.8

Find the currents and voltages in the circuit shown in Fig. 2.27(a).



(a) (b)

Solution

Apply Ohm's law: $v_1 = 8i_1$, $v_2 = 3i_2$, $v_3 = 6i_3$

Apply KCL: $\boldsymbol{i_1 - i_2 - i_3 = 0}$

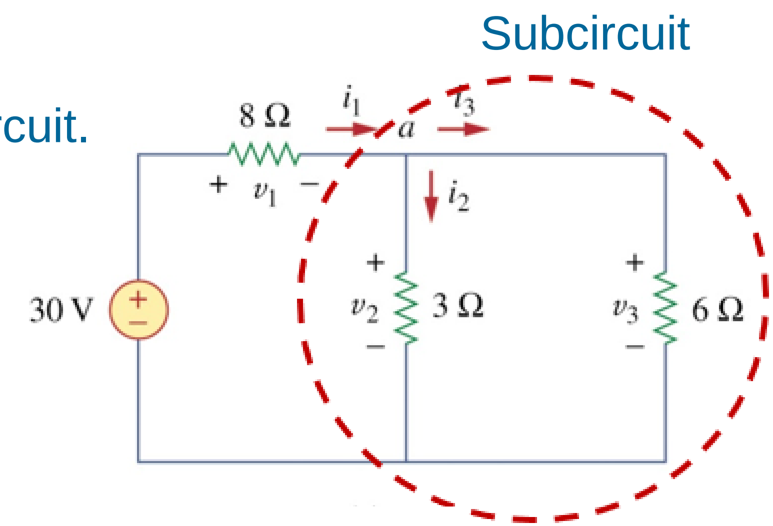
Applying KVL: $\boldsymbol{v_1 + v_2 = 30}$ $\boldsymbol{v_2 = v_3}$

So $i_1 = 3\text{A}$, $i_2 = 2\text{A}$, $i_3 = 1\text{A}$, $v_1 = 24\text{V}$, $v_2 = 6\text{V}$, $v_3 = 6\text{V}$

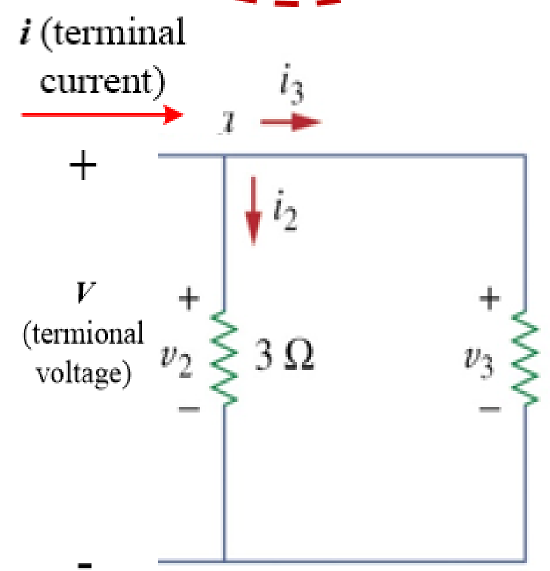
2.4 Equivalent Subcircuits

1. Some concepts

Subcircuit: A subcircuit is any part of a circuit.



Two-terminal subcircuit: A subcircuit with two accessible terminals, is called two-terminal subcircuit.

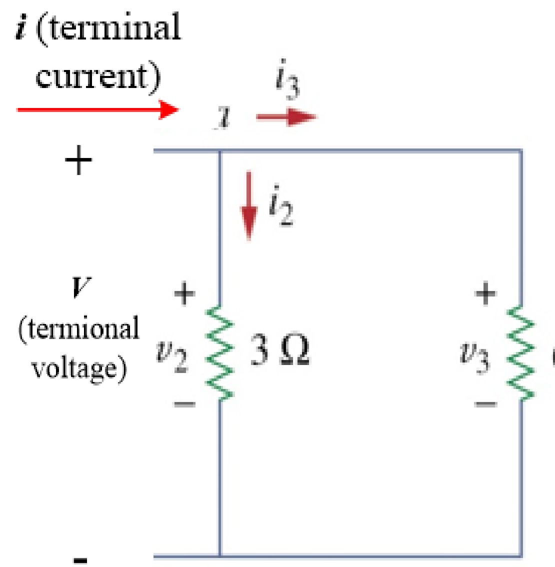


Terminal voltage and terminal current: The voltage across and current into these terminals are called the terminal voltage and terminal current of the two-terminal subcircuit

2.4 Equivalent Subcircuits

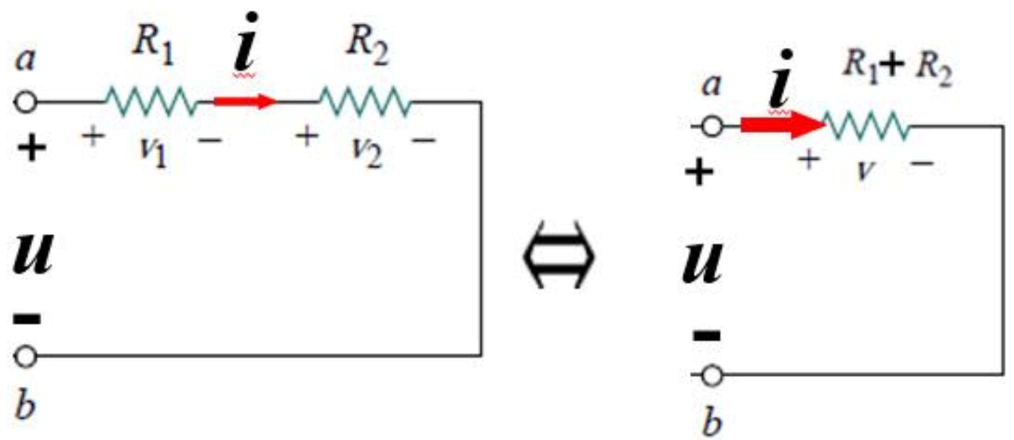
1. Some concepts

terminal law : the voltage-current relationship of a two-terminal subcircuit.



$$v = f(i) \ \text{ or } \ i = g(v)$$

Equivalent subcircuits: Two two-terminal subcircuit are said to be equivalent if they have same terminal law.



$u = R_1 i + R_2 i$

$u = (R_1 + R_2) i$

2.4 Equivalent Subcircuits

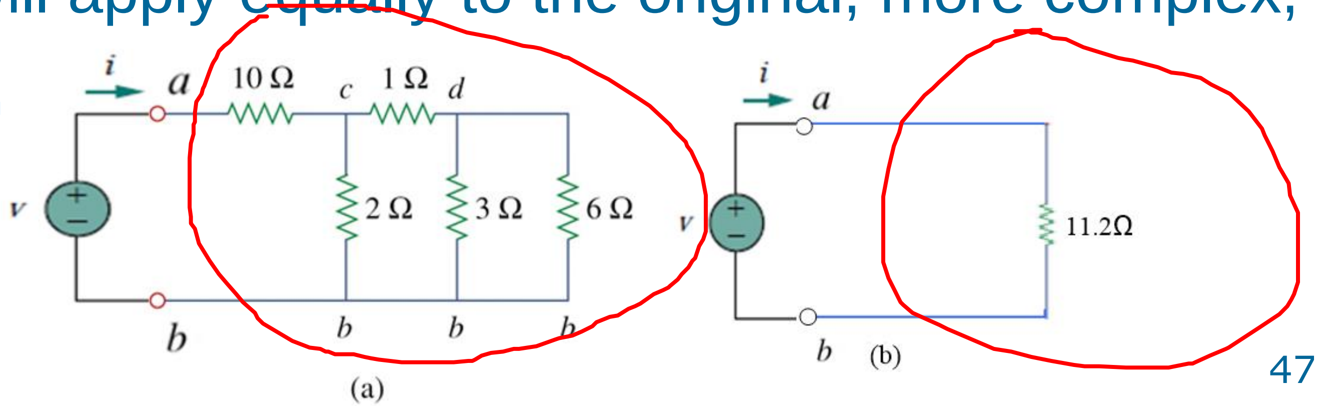
1. Some concepts

Why equivalent subcircuits:

A generally useful strategy in analyzing electric circuits is to *simplify wherever possible*.

-Replacing a part of a circuit with a simple ***equivalent subcircuit*** contains fewer elements, without altering any current or voltage outside that part (or region).

-The simpler circuit can then be analyzed, and the results will apply equally to the original, more complex, circuit.

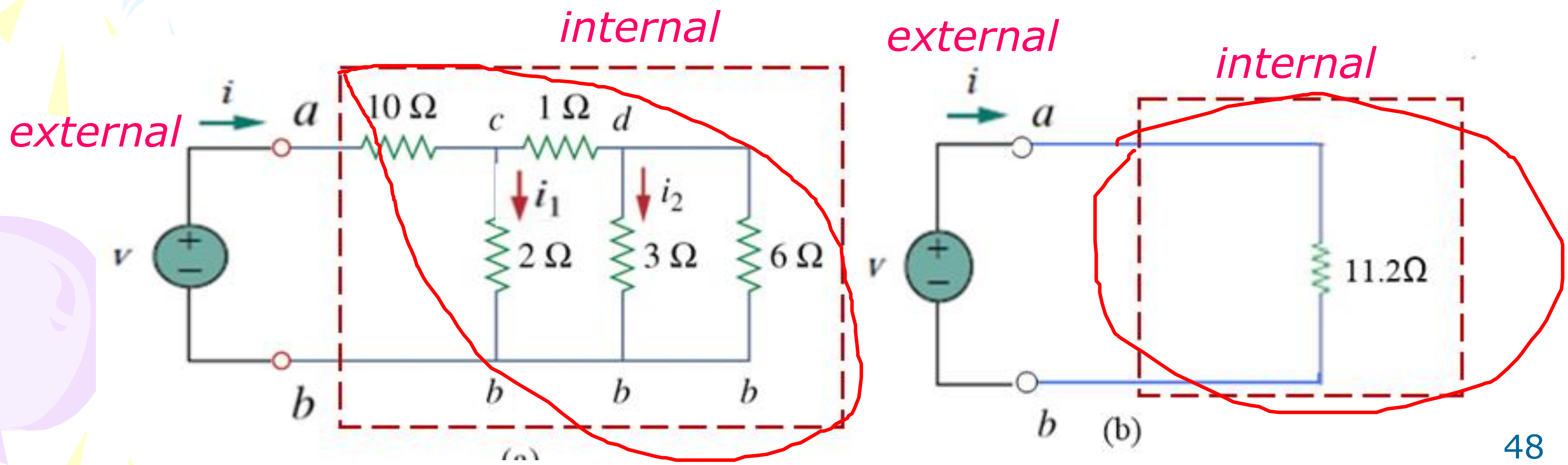


We can get same v,I with both circuit!

2.4 Equivalent Subcircuits

2. Attention to *external and internal part of* equivalent subcircuit

Attention: Only the currents and voltages *external* to the equivalent subcircuit will not be changed when one is exchanged for the other in any circuit; the currents and voltages *internal to the subcircuit* may be quite different.



2.5 Series Resistors and Voltage Division

- ***Series:*** Two or more elements are in series if they are cascaded or connected sequentially and consequently carry the same current.

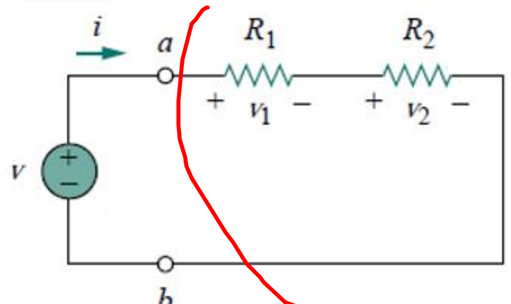


Figure 2.29 A single-loop circuit with two resistors in series.

Apply Ohm's law: $v_1 = i R_1, \qquad v_2 = i R_2$

Applying KVL: $-v + v_1 + v_2 = 0$

So $v = v_1 + v_2 = i(R_1 + R_2)$

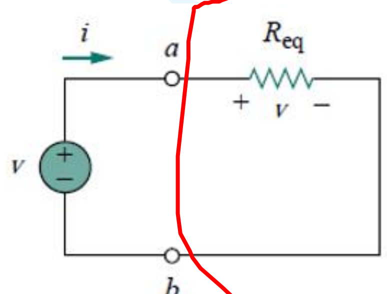


Figure 2.30 Equivalent circuit of the Fig. 2.29 circuit.

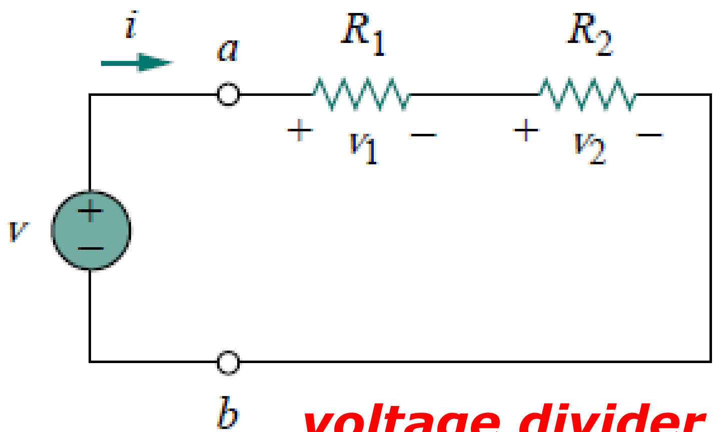
$$v = i R_{eq}$$

So $R_{eq} = R_1 + R_2$

The two circuits in Figs. 2.29 and 2.30 are equivalent when they exhibit the same voltage-current relationships at the terminals *a-b.*

2.5 Series Resistors and Voltage Division

principle of voltage division

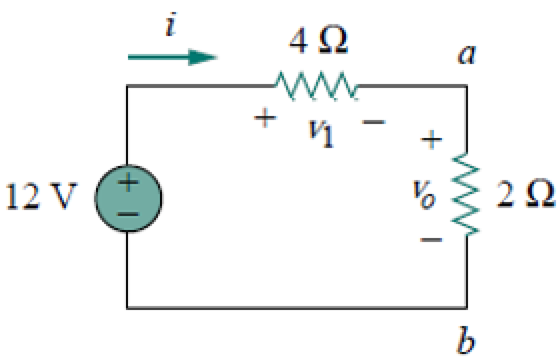


$$v_1 = iR_1, \qquad v_2 = iR_2$$

$$v = v_1 + v_2 = i(R_1 + R_2) \qquad i = \frac{v}{R_1 + R_2}$$

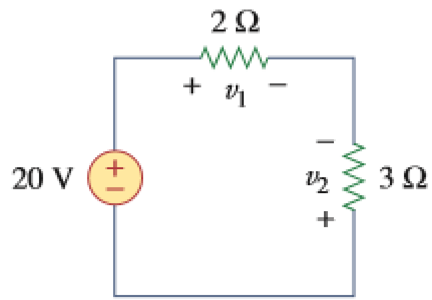
$$\boxed{v_1 = \frac{R_1}{R_1 + R_2} v, \qquad v_2 = \frac{R_2}{R_1 + R_2} v}$$

voltage divider



$$v_0 = \frac{2}{4+2} \times 12 = 4V$$

$$v_1 = \frac{4}{4+2} \times 12 = 8V$$



$$v_1 = \frac{2}{2+3} \times 20 = 8V$$

$$v_2 = -\frac{3}{2+3} \times 20 = \text{-}12V$$

2.5 Series Resistors and Voltage Division

- The equivalent resistance of any number of resistors connected in a series is the sum of the individual resistances.

$$R_{eq} = R_1 + R_2 + \cdots + R_N = \sum_{n=1}^{N} R_n$$

Resistors in series behave as a single resistor whose resistance is equal to the sum of the resistances of the individual resistors.

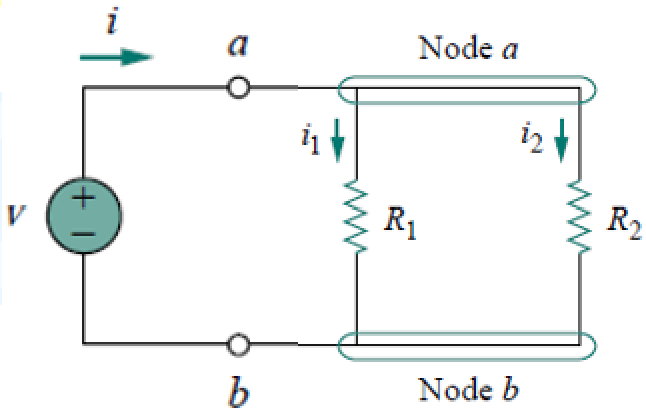
principle of voltage division

- In general, if a voltage divider has N resistors ($R_1, R_2, \ldots, R_N$) in series with the source voltage v, the *nth* resistor (R_n) will have a voltage drop of

$$v_n = \frac{R_n}{R_1 + R_2 + \cdots + R_N} v$$

2.6 Parallel Resistors and Current Division (1)

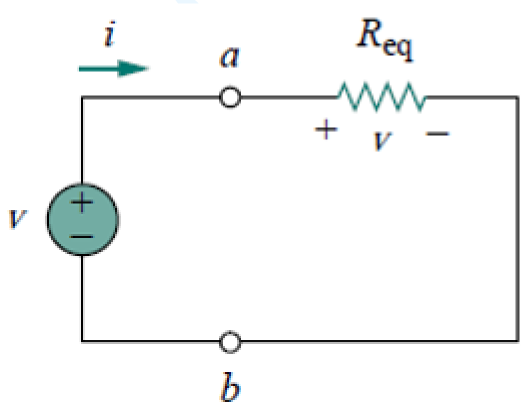
- Parallel: Two or more elements are in parallel if they are connected to the same two nodes and consequently have the <u>same voltage across</u> them.



$$i_1 = \frac{v}{R_1}, \qquad i_2 = \frac{v}{R_2}$$

$$i = i_1 + i_2$$

$$i = \frac{v}{R_1} + \frac{v}{R_2} = v\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

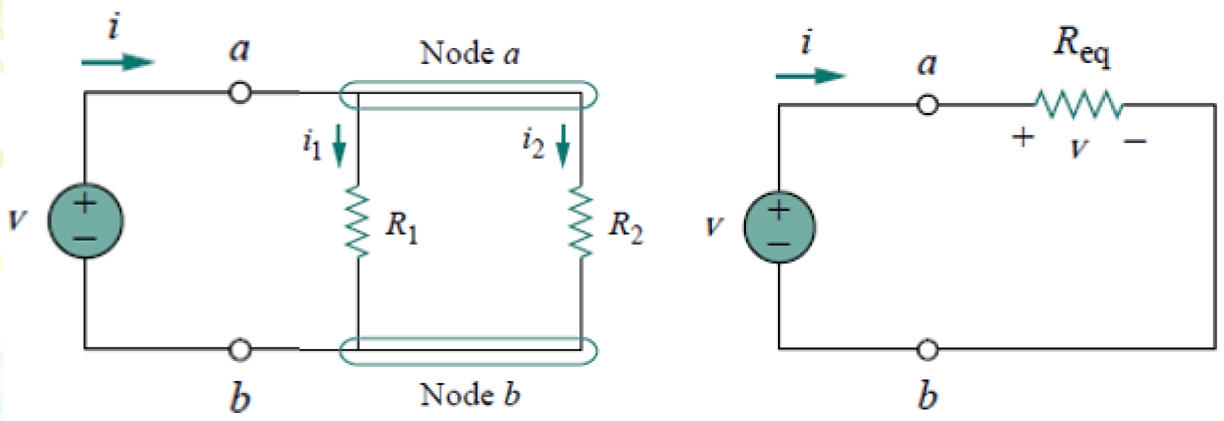


$$i = \frac{v}{R_{eq}} \qquad \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\boxed{R_{eq} = \frac{R_1 R_2}{R_1 + R_2}}$$

The equivalent resistance of two parallel resistors is equal to the product of their resistances divided by their sum.

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$



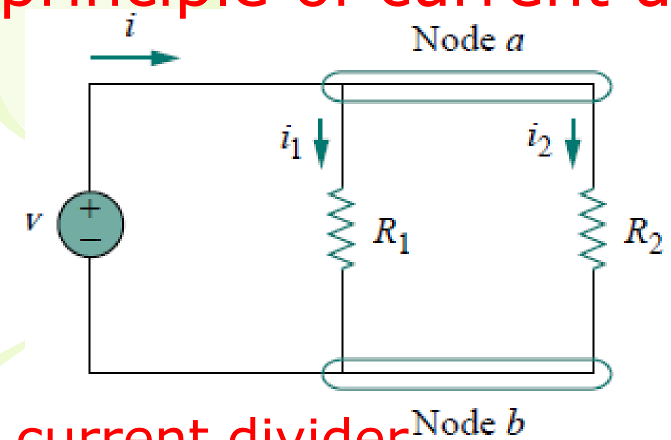
- The equivalent resistance of a circuit with N resistors in parallel is:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}$$

$$G_{eq} = G_1 + G_2 + G_3 + \cdots + G_N$$

The equivalent conductance of resistors connected in parallel is the sum of their individual conductances.

principle of current division



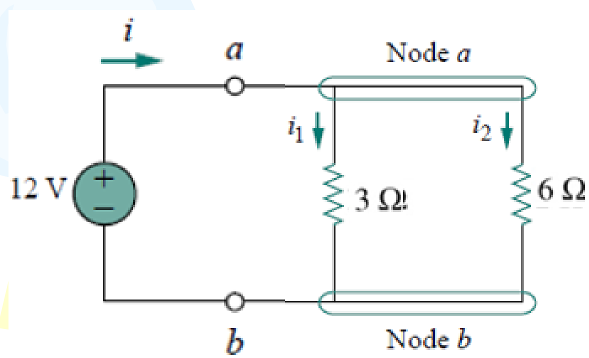
current divider

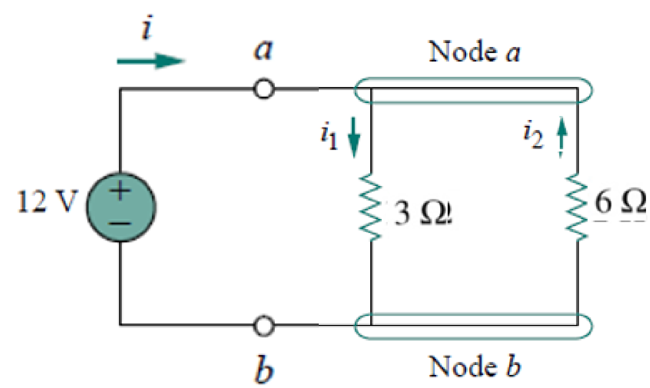
$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

$$v = i\,R_{eq} = \frac{i\,R_1 R_2}{R_1 + R_2}$$

$$i_1 = \frac{R_2\, i}{R_1 + R_2}, \qquad i_2 = \frac{R_1\, i}{R_1 + R_2}$$

Notice: the larger current flows through the smaller resistance.





$R_{eq} = \frac{3\times 6}{3+6} = 2\Omega$ $\quad i = \frac{12}{2} = 6\text{A}$

$i_1 = \frac{6}{3+6} \times 6 = 4\text{A}$ $\quad i_2 = \frac{3}{3+6} \times 6 = 2\text{A}$

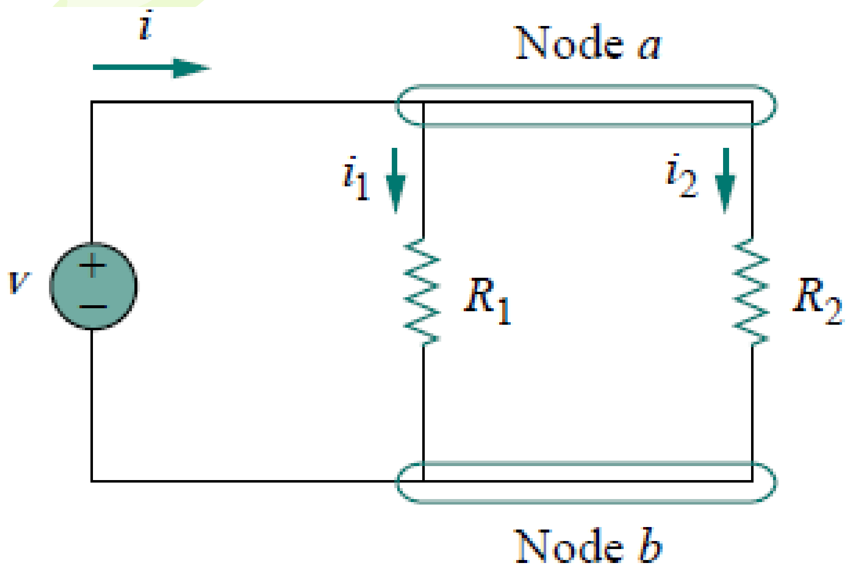
$R_{eq} = \frac{3\times 6}{3+6} = 2\Omega$ $\quad i = \frac{12}{2} = 6\text{A}$

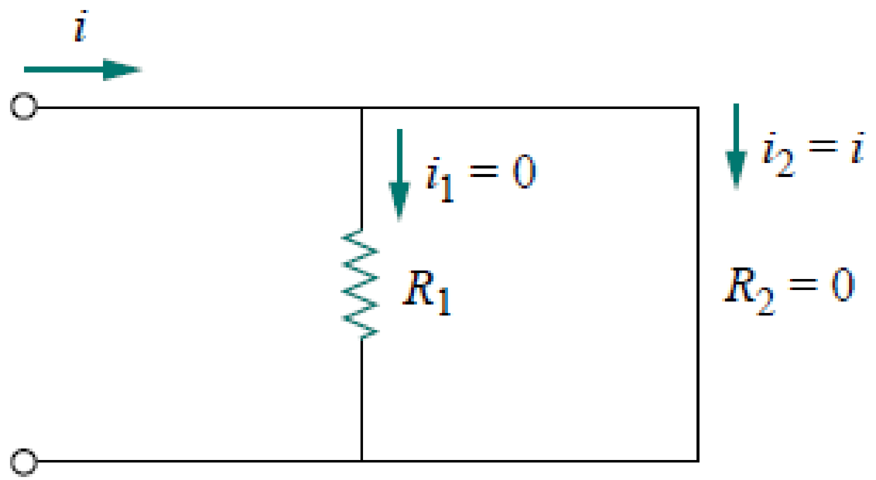
$i_1 = \frac{6}{3+6} \times 6 = 4\text{A}$ $\quad i_2 = -\frac{3}{3+6} \times 6 = -2\text{A}$

- The total current ***i*** is shared by the resistors in inverse proportion to their resistances. The current divider can be expressed as:

$$i_n = \frac{v}{R_n} = \frac{iR_{eq}}{R_n}$$

Parallel Resistors and Current Division ---- an extreme case 1





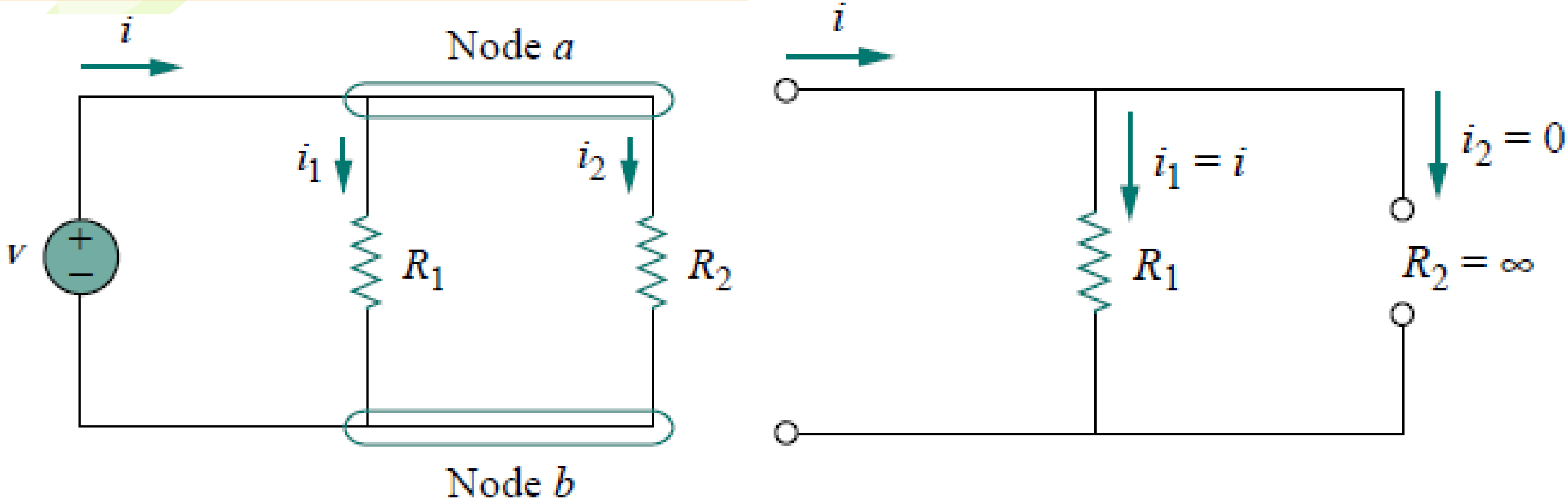
$$i_1 = \frac{R_2\, i}{R_1 + R_2}, \qquad i_2 = \frac{R_1\, i}{R_1 + R_2}$$

$$R_2 = 0$$

$$i_1 = 0, \quad i_2 = i$$

This means that the entire current ***i*** bypasses R_1 and flows through the short circuit $R_2 = 0$, the path of least resistance.

Parallel Resistors and Current Division ---- an extreme case 2



$$\boxed{i_1 = \frac{R_2\, i}{R_1 + R_2}, \qquad i_2 = \frac{R_1\, i}{R_1 + R_2}}$$

$$\boxed{R_{eq} = \frac{R_1 R_2}{R_1 + R_2}}$$

$$R_2 \rightarrow \infty$$

$$R_{eq} = R_1$$

The current still flows through the path of least resistance, R_1.

Example 2.9 Find R_{eq} for the circuit shown in Fig. 2.34

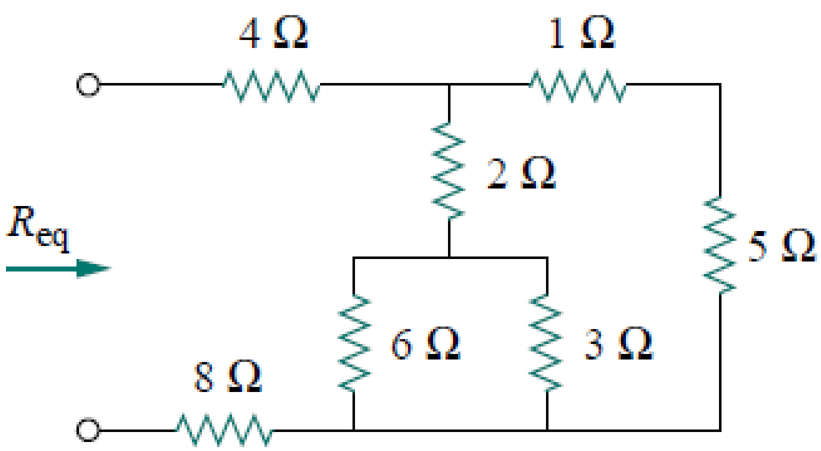
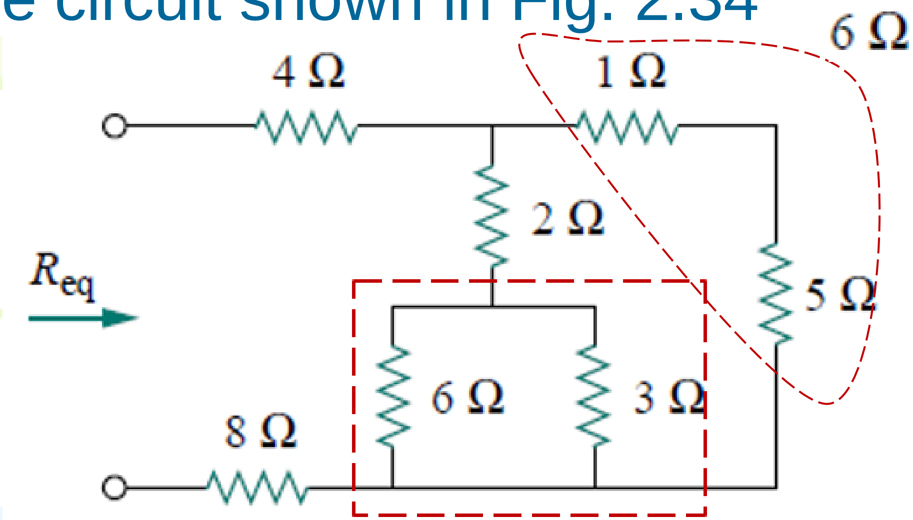
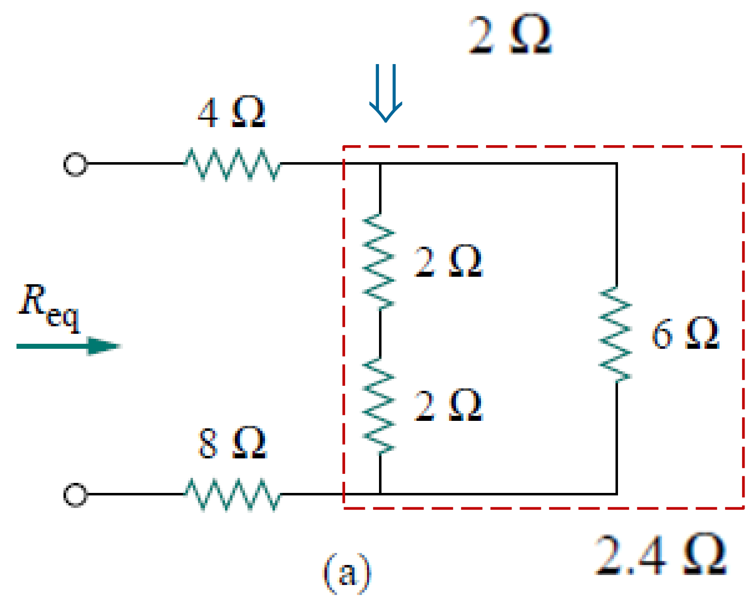
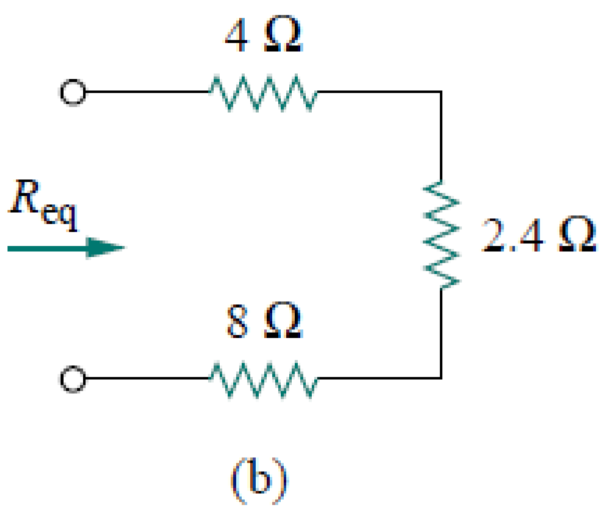


Figure 2.34 For Example 2.9.







R_{eq}=14.4Ω

Practice

By combining the resistors in Fig.2.36, find R_{eq}

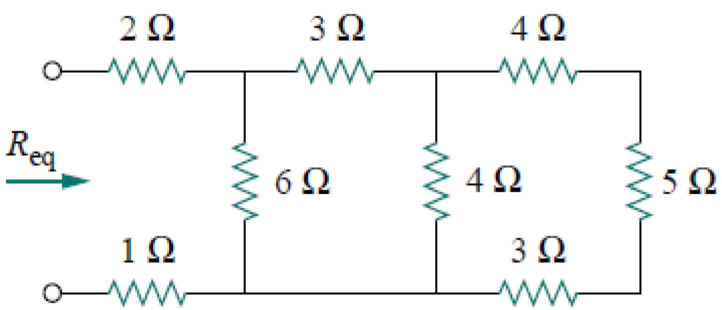
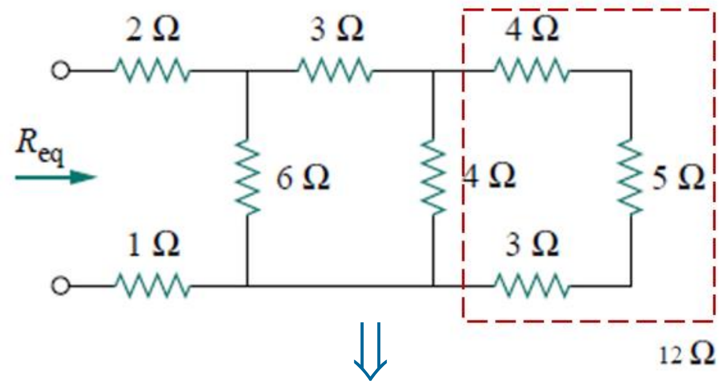
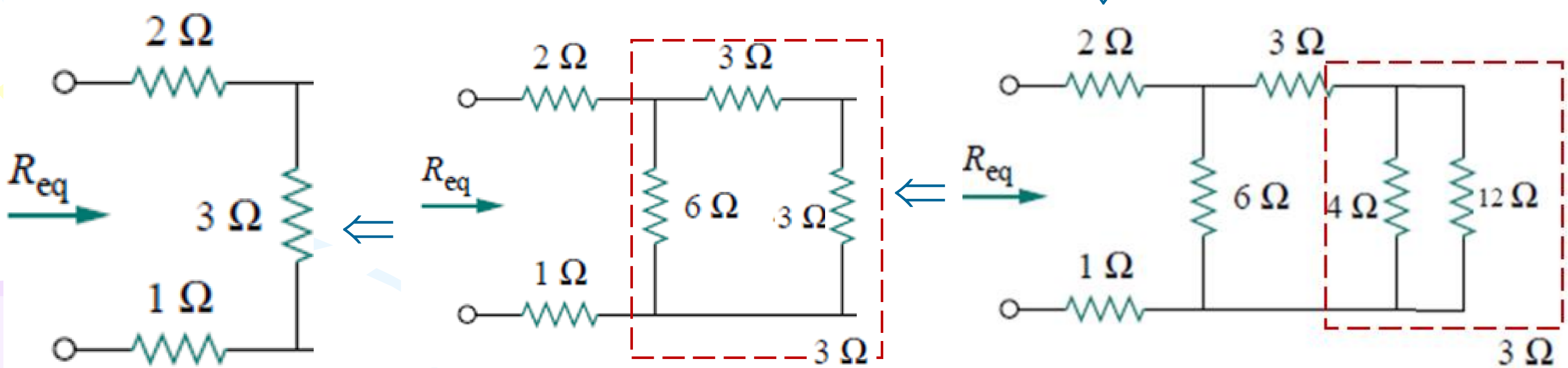


Figure 2.36 For Practice Prob. 2.9.





Ans: 6Ω

Example 2.10

Calculate the equivalent resistance R_{ab} in the circuit in Fig.2.37

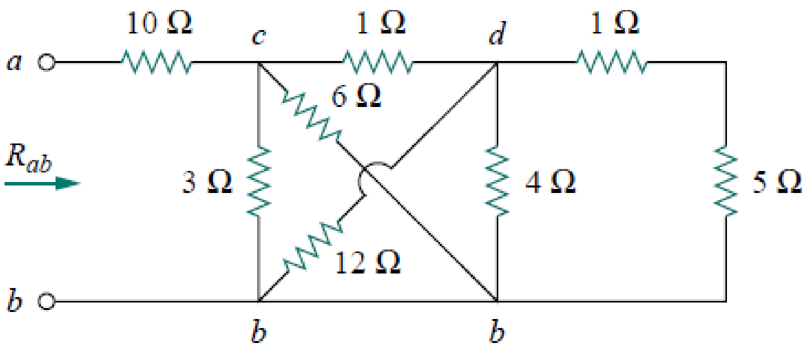
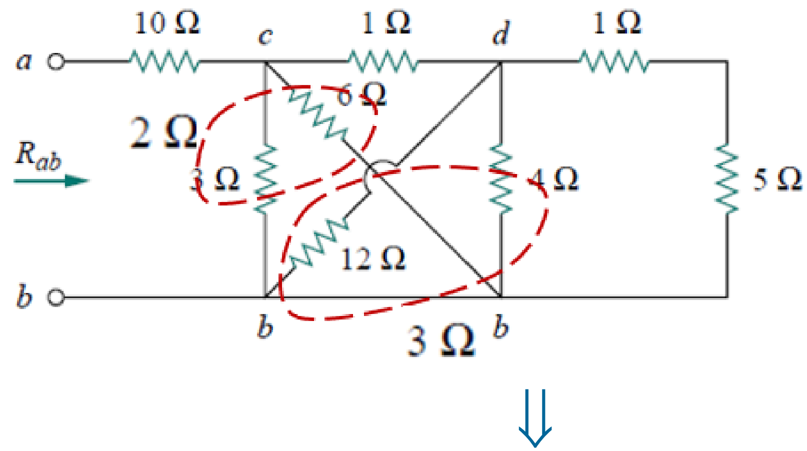
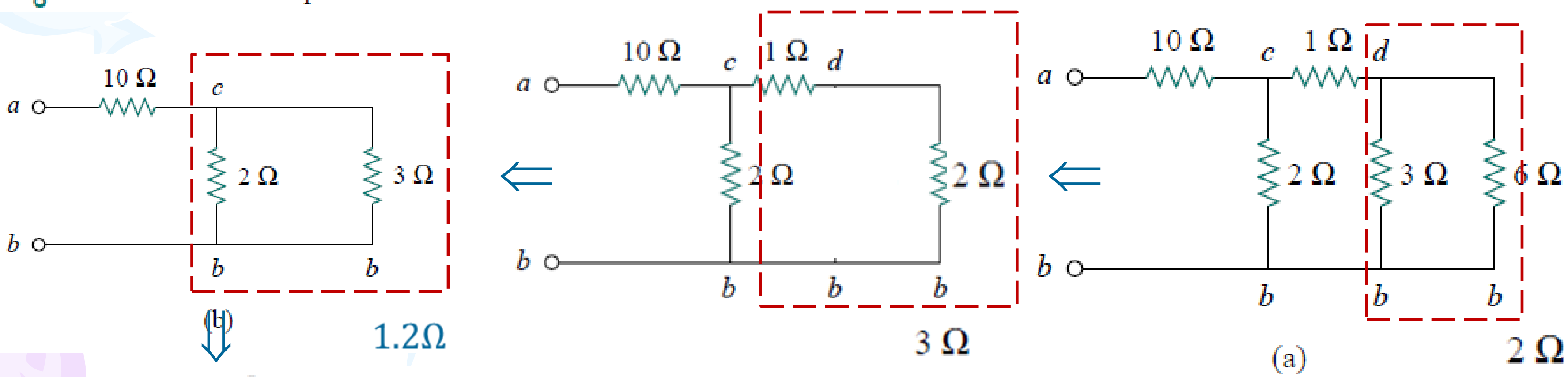
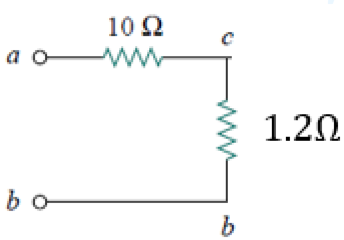


Figure 2.37 For Example 2.10.







R_{ab}=11.2 Ω

Practice Find R_{ab} for the circuit in Fig.2.39.

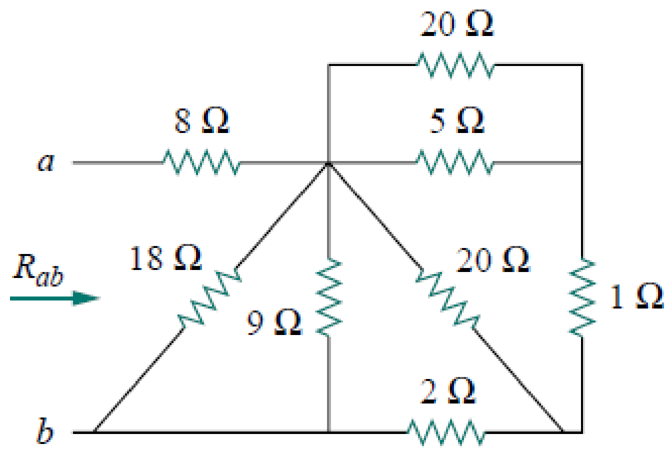
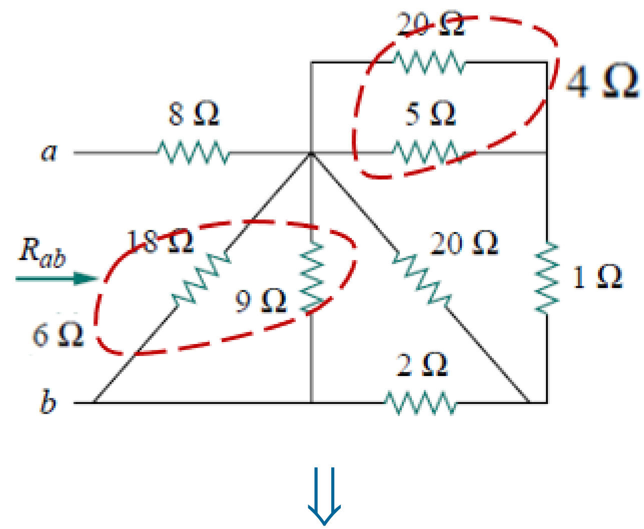
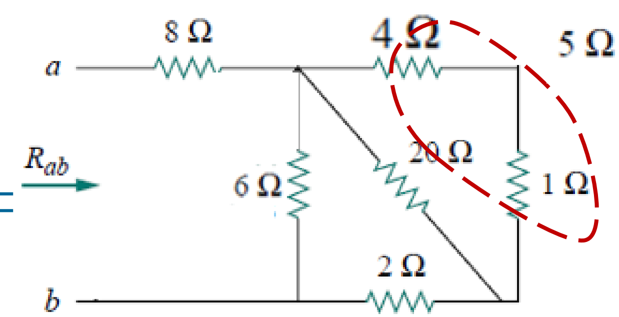
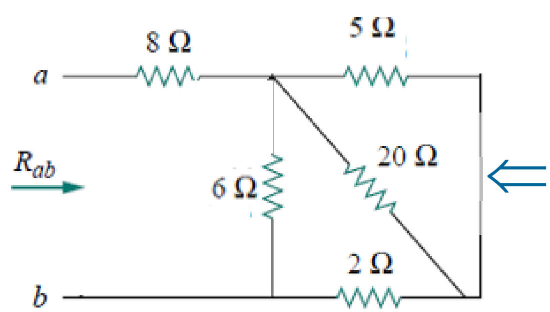
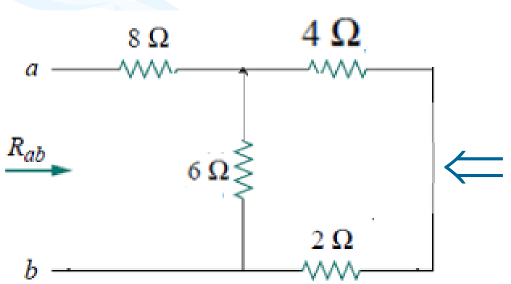


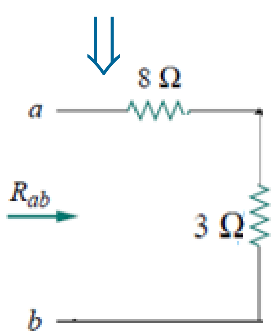
Figure 2.39 For Practice Prob. 2.10.





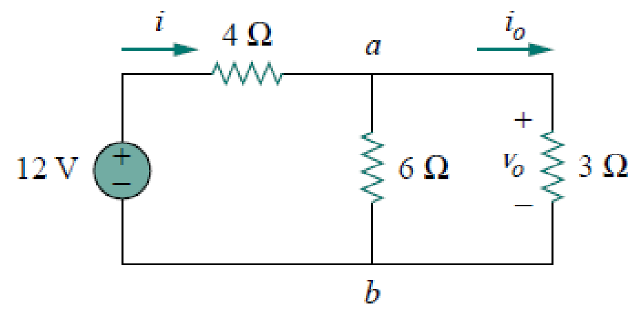






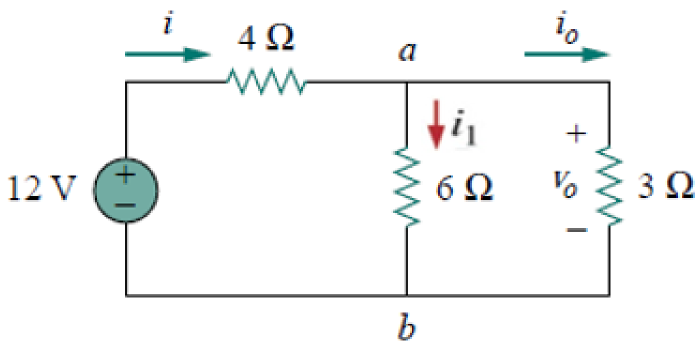
Example 2.12

Find i_0 and v_0 in the circuit shown in Fig.2.42. calculate the power dissipated in the 3-Ω resistor.



(a)

Solution1



Apply KCL: $i = i_0 + i_1$

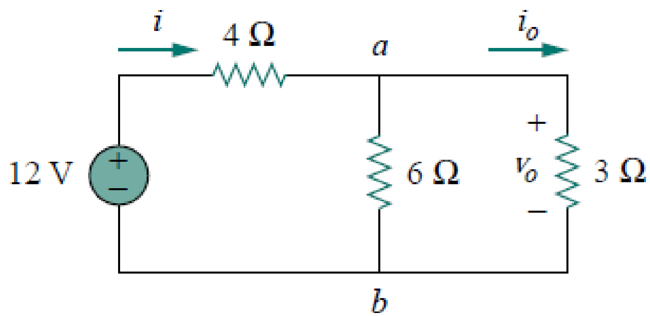
Apply KVL: $3i_0 = 6i_1$, 12=4i+ $3i_0$

So i = 2A, i_0 = 4/3A, i_0 = 2/3A,

v_0 = $3i_0$ = 4V, p_0 = $v_0\, i_0$ =5.333W

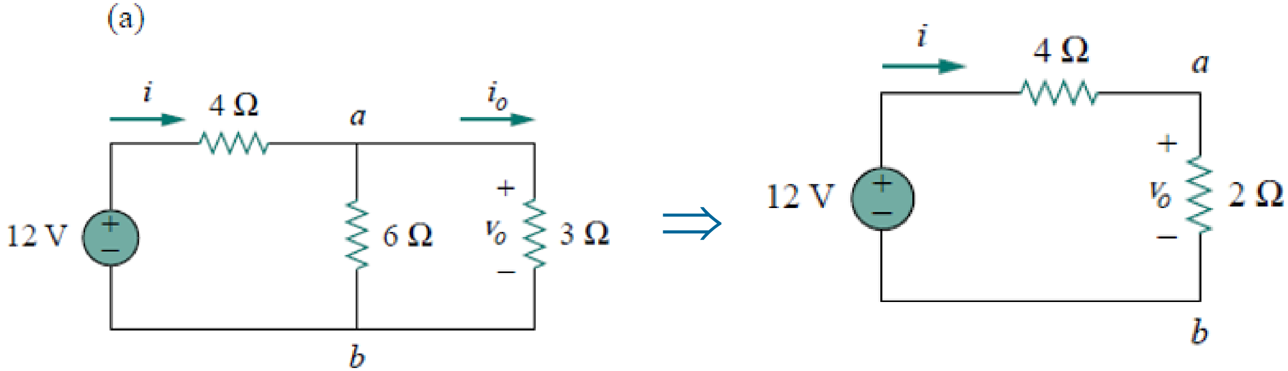
Example 2.12

Find i_0 and v_0 in the circuit shown in Fig.2.42. calculate the power dissipated in the 3-Ω resistor.



(a)

Solution2



$$i = \frac{12}{4+2} = 2\text{ A} \qquad v_o = \frac{2}{2+4}(12\text{ V}) = 4\text{ V}$$

$$v_o = 3i_o = 4 \quad \Longrightarrow \quad i_o = \frac{4}{3}\text{ A}$$

or

$$i_o = \frac{6}{6+3}i = \frac{2}{3}(2\text{ A}) = \frac{4}{3}\text{ A}$$

$$p_o = v_o i_o = 4\left(\frac{4}{3}\right) = 5.333\text{ W}$$

PRACTICE PROBLEM 2.12

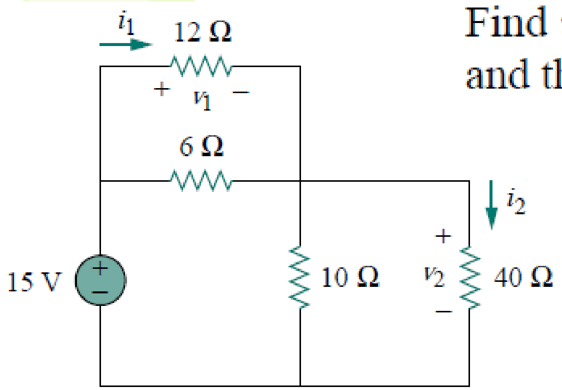
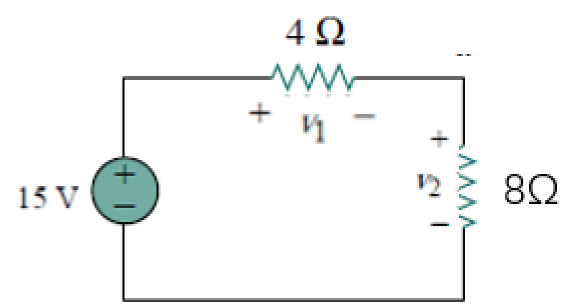


Figure 2.43 For Practice Prob. 2.12.

Find v_1 and v_2 in the circuit shown in Fig. 2.43. Also calculate i_1 and i_2 and the power dissipated in the 12-Ω and 40-Ω resistors.

Answer: $v_1 = 5$ V, $i_1 = 416.7$ mA, $p_1 = 2.083$ W, $v_2 = 10$ V, $i_2 = 250$ mA, $p_2 = 2.5$ W.



Solution2

Apply voltage division : $v_1 = \frac{4}{8+4} \times 15 = 5\ V$, $v_2 = \frac{8}{8+4} \times 15 = 10V$

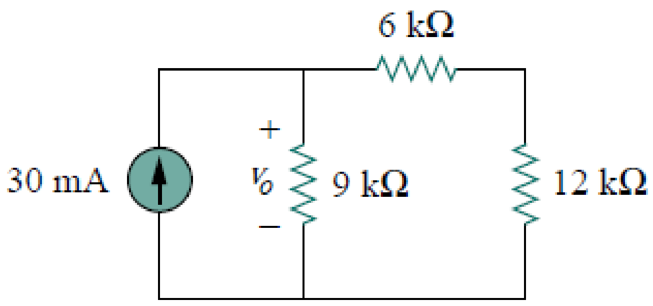
Apply Ohm's law : $i_1 = \frac{v_1}{12} = \frac{5}{12}A$, $i_2 = \frac{v_2}{40} = \frac{10}{40}A = 250mA$

So $p_{12\Omega} = v_1 i_1 = 2.083W$

$p_{40\Omega} = v_2 i_2 = 2.5W$

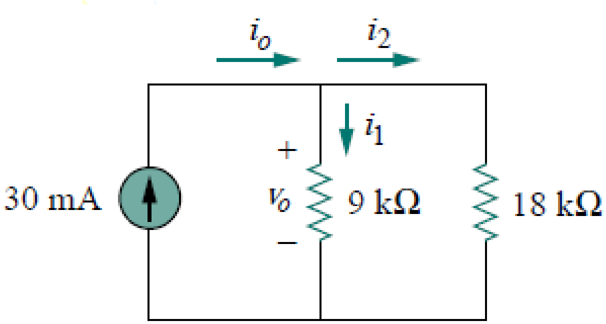
Example 2.13

For the circuit shown in Fig. 2.44(a), determine: (a) the voltage v_o, (b)the power supplied by the current source, (c) the power absorbed by each resistor.



(a)

Solution1



(b)

Apply KCL:$i_0 = i_1 + i_2$

Apply current division :

$i_1 = \frac{18}{9+18} \times 30 = 20mA$, $i_1 = \frac{9}{9+18} \times 30 = 10mA$

Apply Ohm's law: $v_0 = 9i_1 = 180V$

So $p_{30mA} = v_0 i_0 = 5.4W$ $p_{9K\Omega} = v_0 i_1 = 3.6W$ $p_{18K\Omega} = v_0 i_2 = 1.8W$

Practice

For the circuit shown in Fig.2.23.,find (a) v_1 and v_2, (b) the power dissipated in 3-kΩ and 20-kΩ resistors, and (c) the power supplied by the current source.

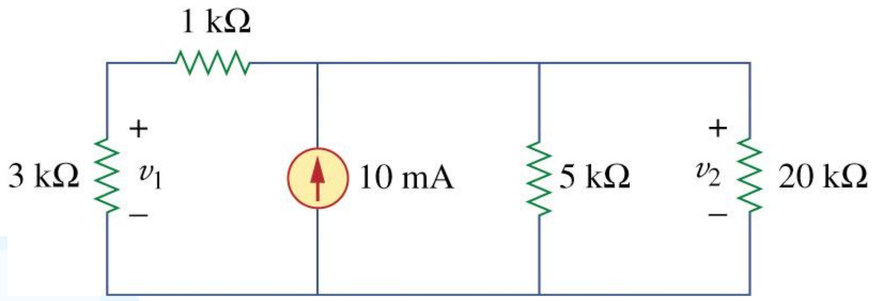
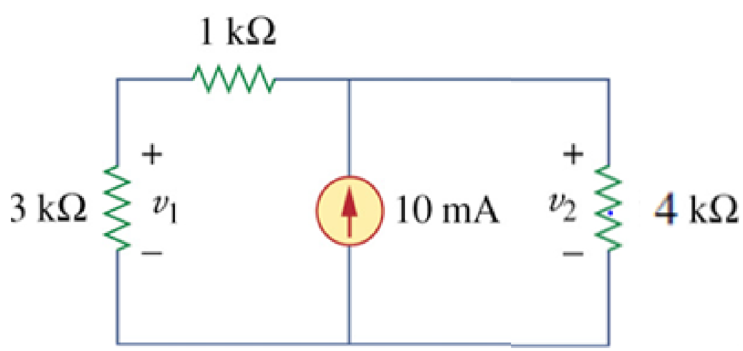


Fig. 2.23

Solution1



Apply voltage division and ohm's law:

$$v_1 = \frac{4}{4+4} \times 10 \times 3 = 15V,$$

$$v_2 = \frac{4}{4+4} \times 10 \times 4 = 20V,$$

$$p_{3k\Omega} = \frac{v_1^2}{3} = 75mW \qquad p_{20k\Omega} = \frac{v_2^2}{20} = 20mW$$

$$p_{10mA} = 10 \times v_2 = 200mW$$

Answer: (a) 15 V, 20 V, (b) 75 mW, 20 mW, (c) 200 mW.

2.7 Wye-Delta Transformations(1)

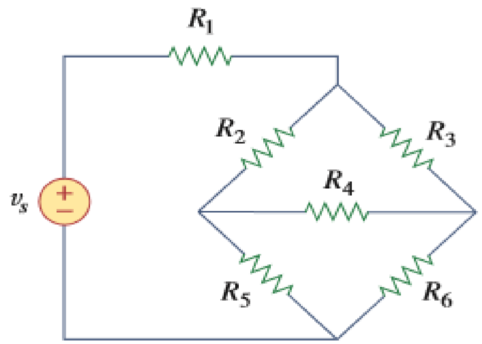


Figure 2.46
The bridge network.

There are some situations while the resistors are neither in parallel nor in series. How to simplify them? Many circuits of the type shown in Fig. 2.46 can be simplified by using three-terminal equivalent networks.

These are the wye(Y) or tee(T) network in Fig,2.47. and the delta(Δ) or pi(π) network in Fig,2.48.

Our main interest: how to identify them when they occur as part of a network; how to apply wye-delta transformation in the analysis of that network.

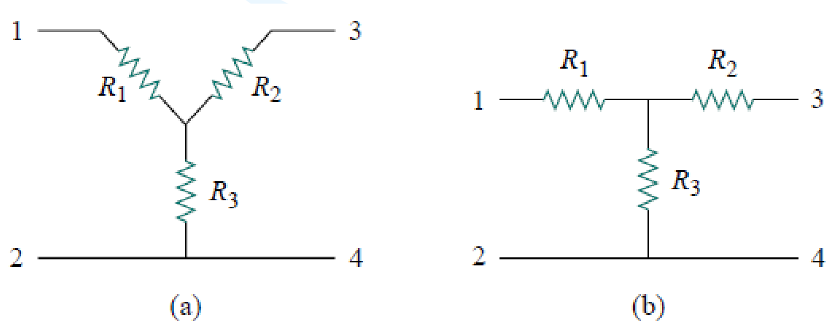


Figure 2.47 Two forms of the same network: (a) Y, (b) T.

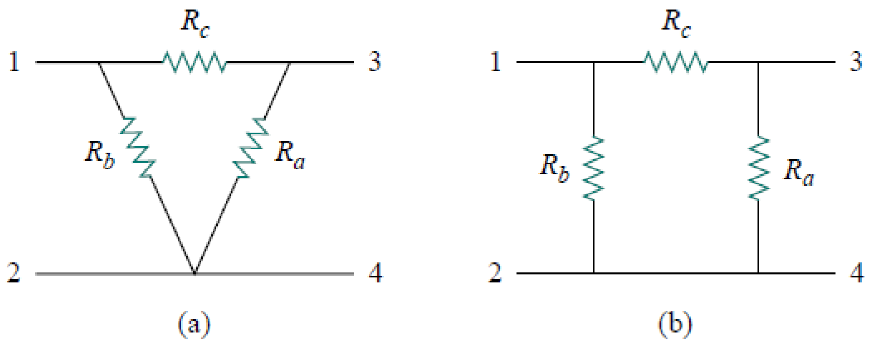
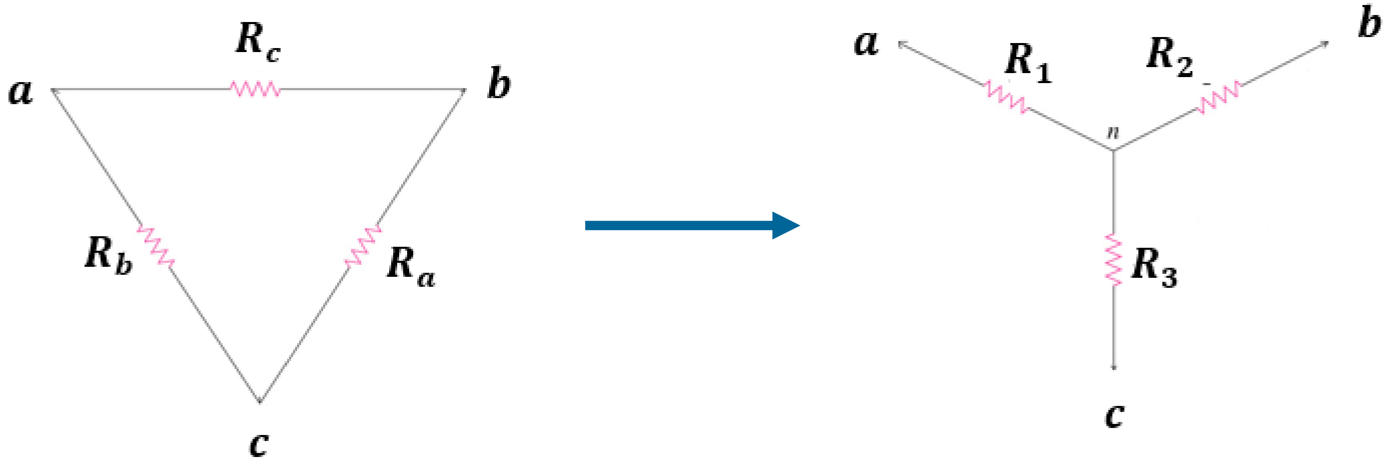


Figure 2.48 Two forms of the same network: (a) Δ, (b) Π.

Delta -> Wye

Given Δ construction, and R_a, R_b, R_c

Find Y construction, and R_1, R_2, R_3



$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} \qquad R_2 = \frac{R_c R_a}{R_a + R_b + R_c} \qquad R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

Each resistor in the Y network is the product of the resistors in the two adjacent Δbranches, divided by the sum of the threeΔresistors

Example 16

Find the Δ construction in the following circuit.

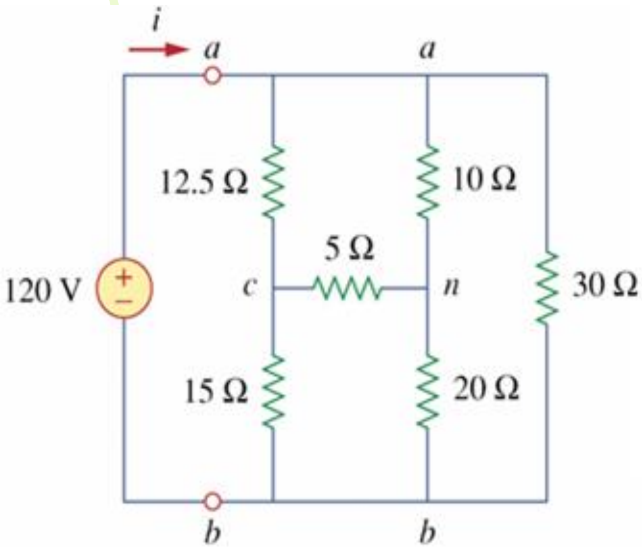
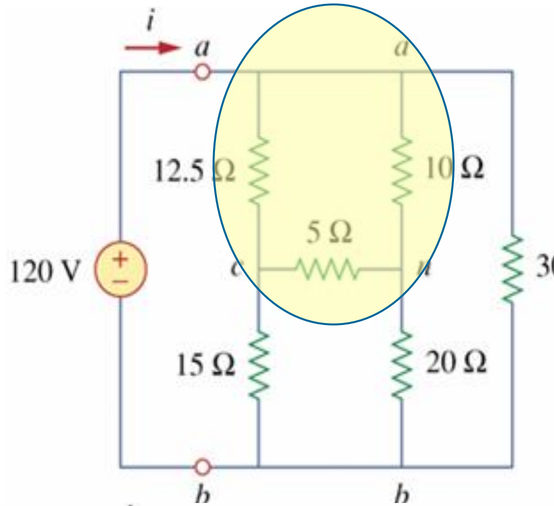
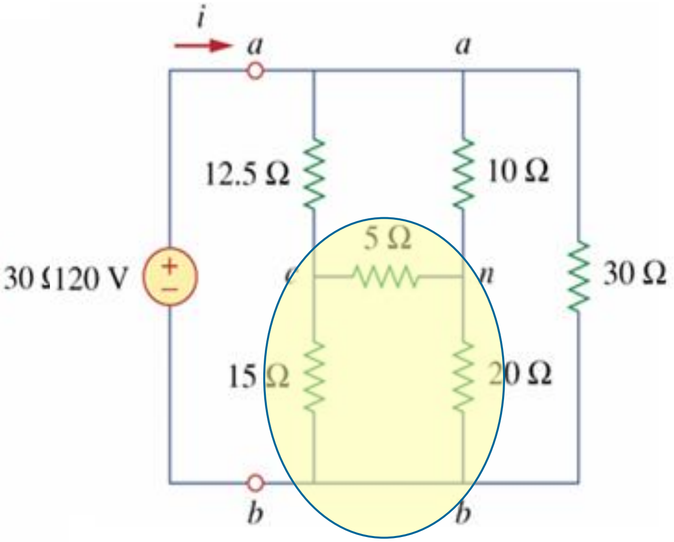
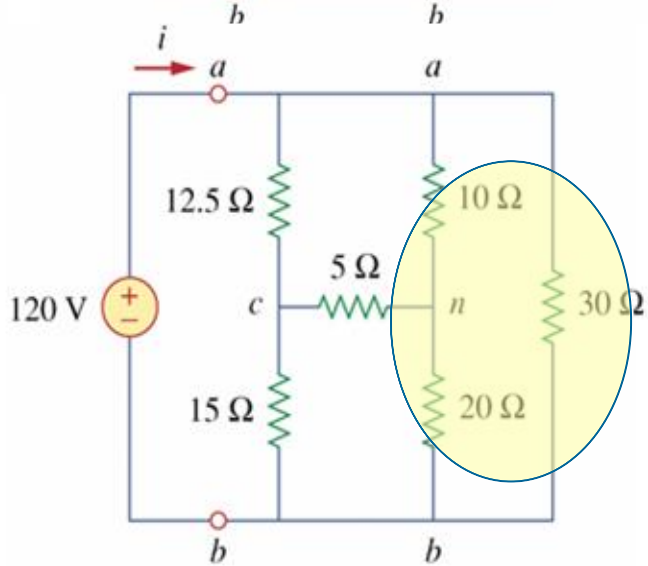


Fig.2.24(a)

Δ construction:

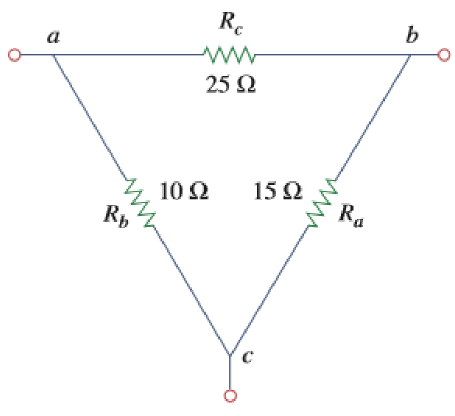






Example 2.14

Convert the Δ network in Fig. 2.50(a) to an equivalent Y network.

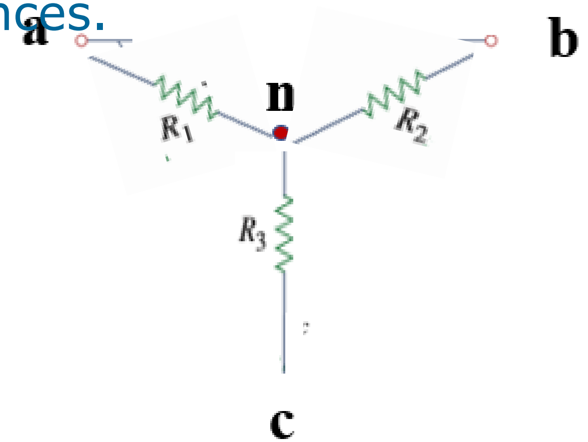


(a)

Figure 2.50

Delta -> Wye steps :

1. Keep node a,b,c unchanged.
2. Add node n.
3. Add 3 branches.
4. Compute resistances.



$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} = \frac{10 \times 25}{15 + 10 + 25} = \frac{250}{50} = 5\ \Omega$$

$$R_2 = \frac{R_c R_a}{R_a + R_b + R_c} = \frac{25 \times 15}{50} = 7.5\ \Omega$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c} = \frac{15 \times 10}{50} = 3\ \Omega$$

Example 2.15

Obtain the equivalent resistance R_{ab} for the circuit in Fig. 2.52 and use it to find current i.

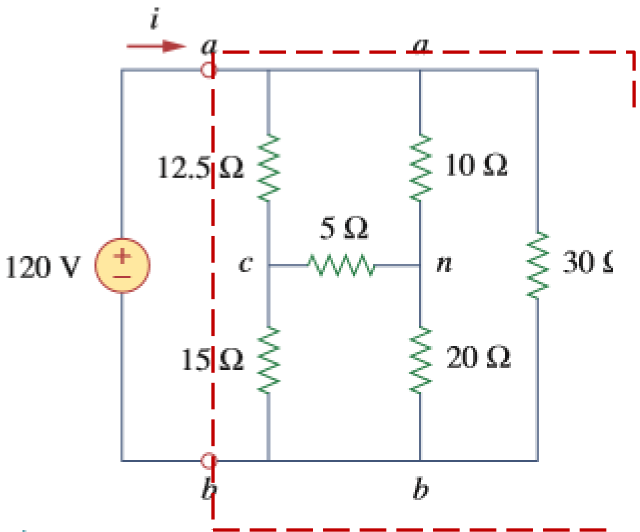
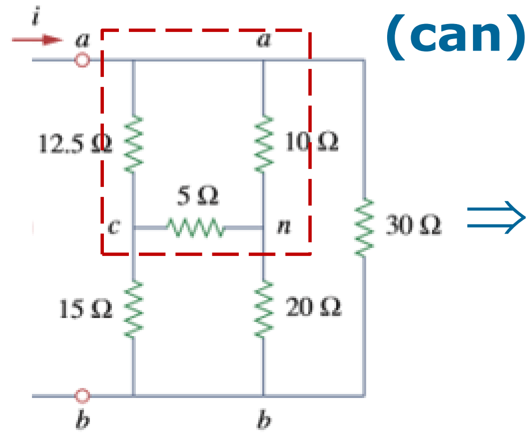
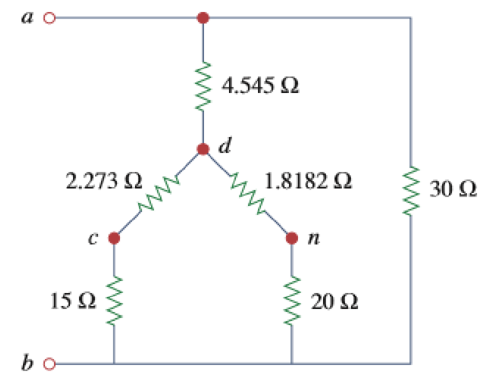


Figure 2.52

Solution:

Delta -> Wye (can)





$$R_{ad} = \frac{R_c R_n}{R_a + R_c + R_n} = \frac{10 \times 12.5}{5 + 10 + 12.5} = 4.545\ \Omega$$

$$R_{cd} = \frac{R_a R_n}{27.5} = \frac{5 \times 12.5}{27.5} = 2.273\ \Omega$$

$$R_{nd} = \frac{R_a R_c}{27.5} = \frac{5 \times 10}{27.5} = 1.8182\ \Omega$$

$$R_{db} = \frac{(2.273 + 15)(1.8182 + 20)}{2.273 + 15 + 1.8182 + 20} = \frac{376.9}{39.09} = 9.642\ \Omega$$

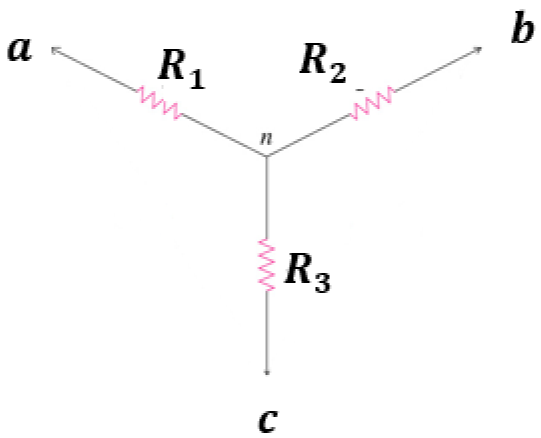
$$R_{ab} = \frac{(9.642 + 4.545)30}{9.642 + 4.545 + 30} = \frac{425.6}{44.19} = \mathbf{9.631\ \Omega}$$

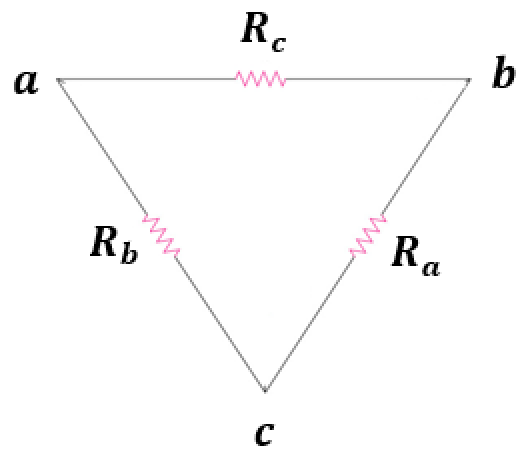
$$i = \frac{v_s}{R_{ab}} = \frac{120}{9.631} = \mathbf{12.46\ A}$$

Wye -> Delta

Given Y construction, and R_1, R_2, R_3

Find Δ construction, and R_a, R_b, R_c





$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

Each resistor in the Δnetwork is the sum of all possible products of Y resistors taken two at time, divided by the opposite Y resistor.

Example 16

Find the Y construction in the following circuit.

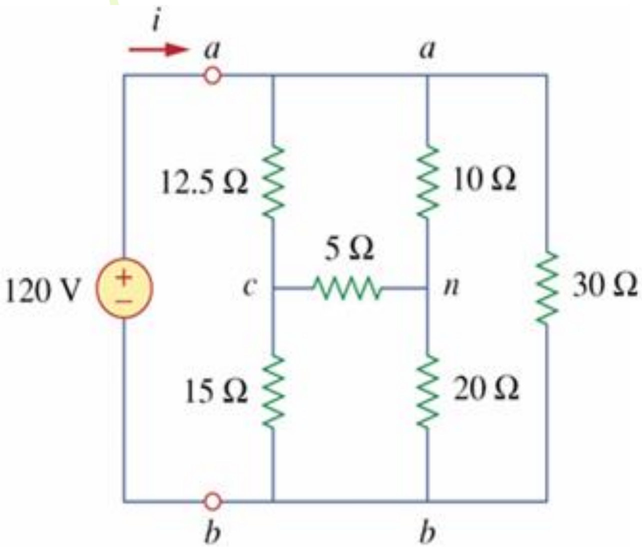
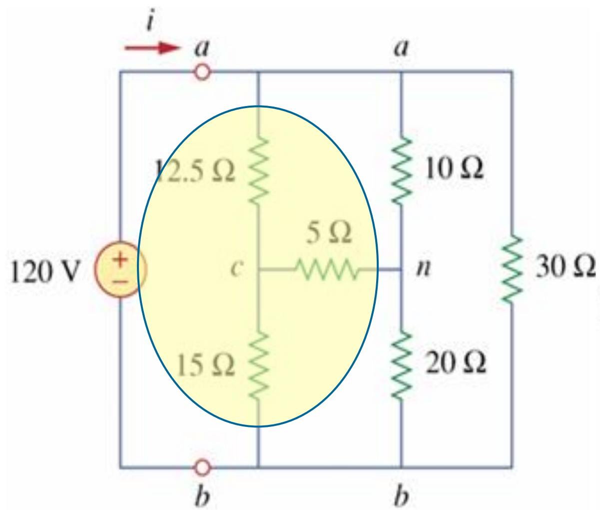
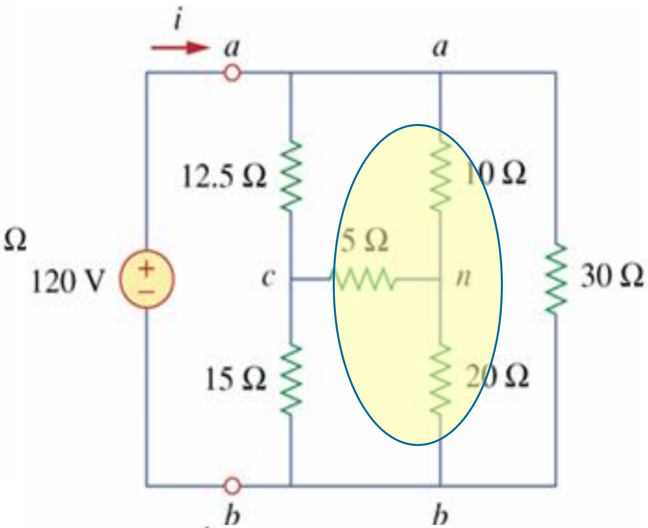
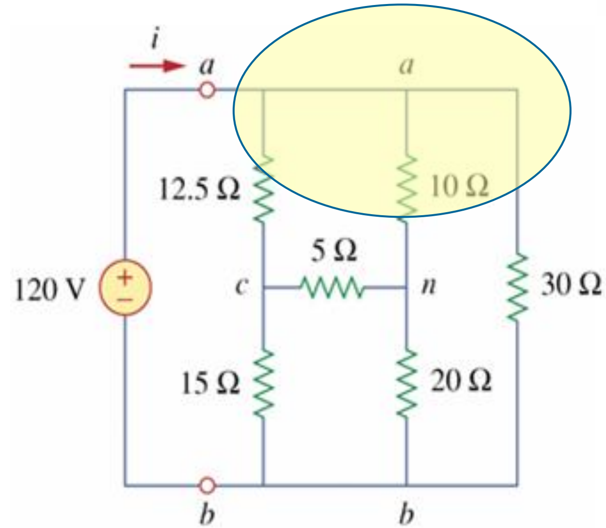


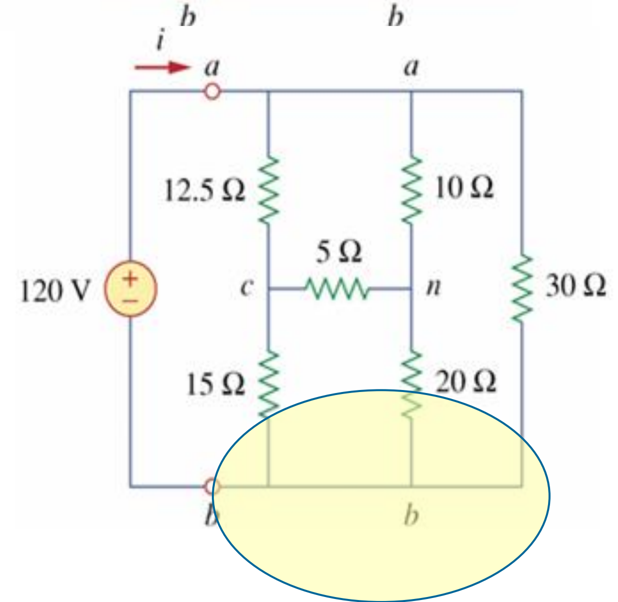
Fig.2.24(a)

Y construction:



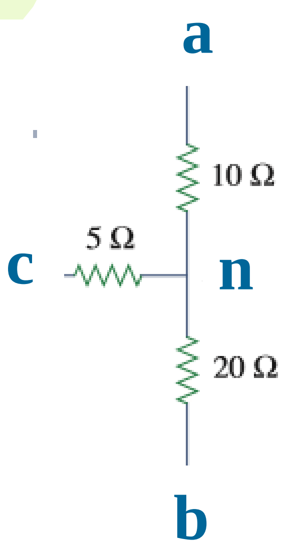






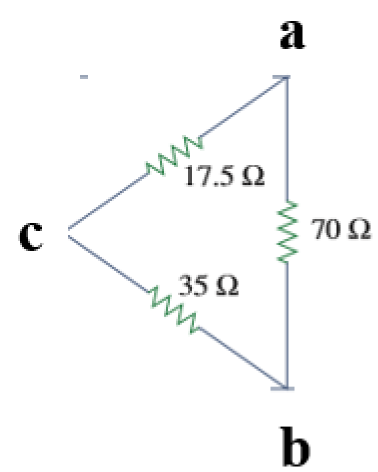
Example 2.14

Convert **Wye** network to an equivalent **Delta** network.



Wye -> Delta steps :

1. Delete node n.
2. delete branches of Wye network.
3. Add 3 branches.
4. Compute resistances.



$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} = \frac{10 \times 20 + 20 \times 5 + 5 \times 10}{10}$$

$$= \frac{350}{10} = 35\ \Omega$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} = \frac{350}{20} = 17.5\ \Omega$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} = \frac{350}{5} = 70\ \Omega$$

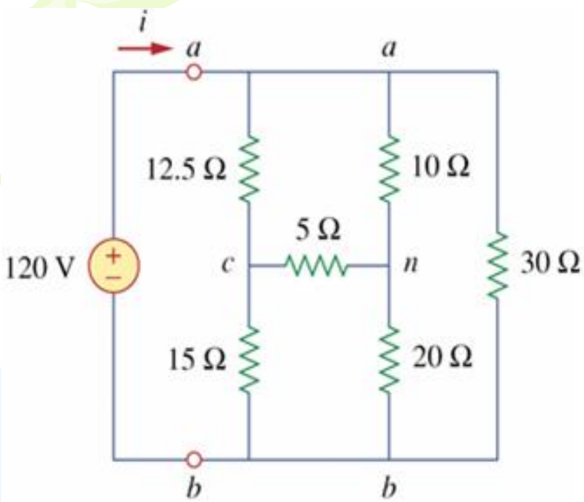
If $R_1=R_2=R_3=R_Y$ and $R_a=R_b=R_c=R_\Delta$, the Y and Δ networks are said to *be balanced*

Under these conditions, conversion formulas become

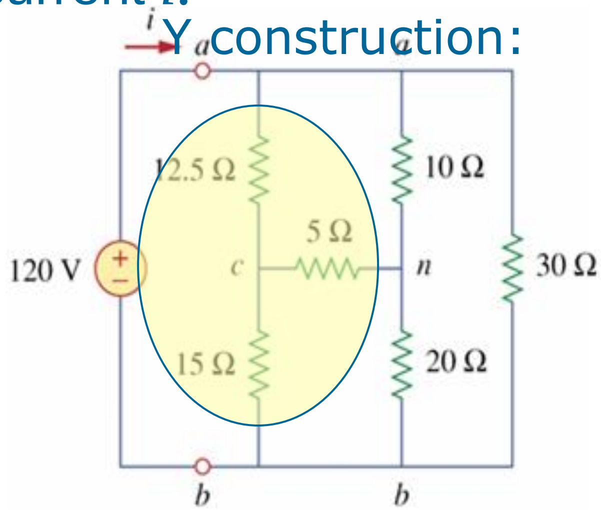
$$R_Y = \frac{R_\Delta}{3} \qquad or \qquad R_\Delta = 3R_Y$$

Example 16

Obtain the equivalent resistance R_{ab} for the circuit shown in Fig.2.24. and use it to calculate current i.



Y construction:



Example 16

Obtain the equivalent resistance R_{ab} for the circuit shown in Fig.2.24. and use it to calculate current i.

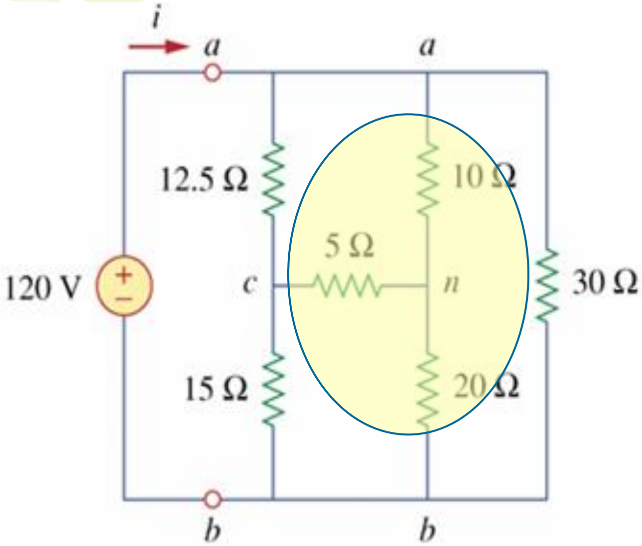


Fig.2.24(a)

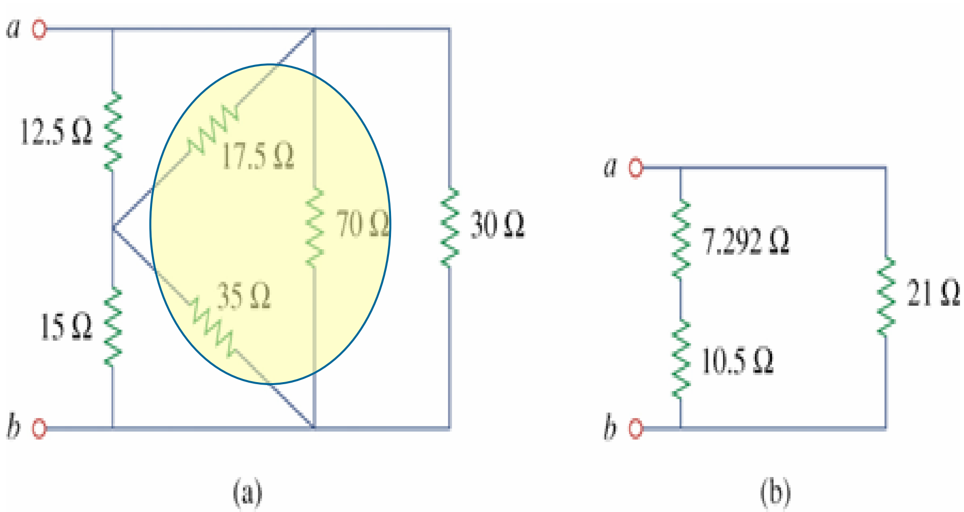


Fig.2.24(b)

Practice For the bridge network in Fig.2.25.,find R_{ab} and i.

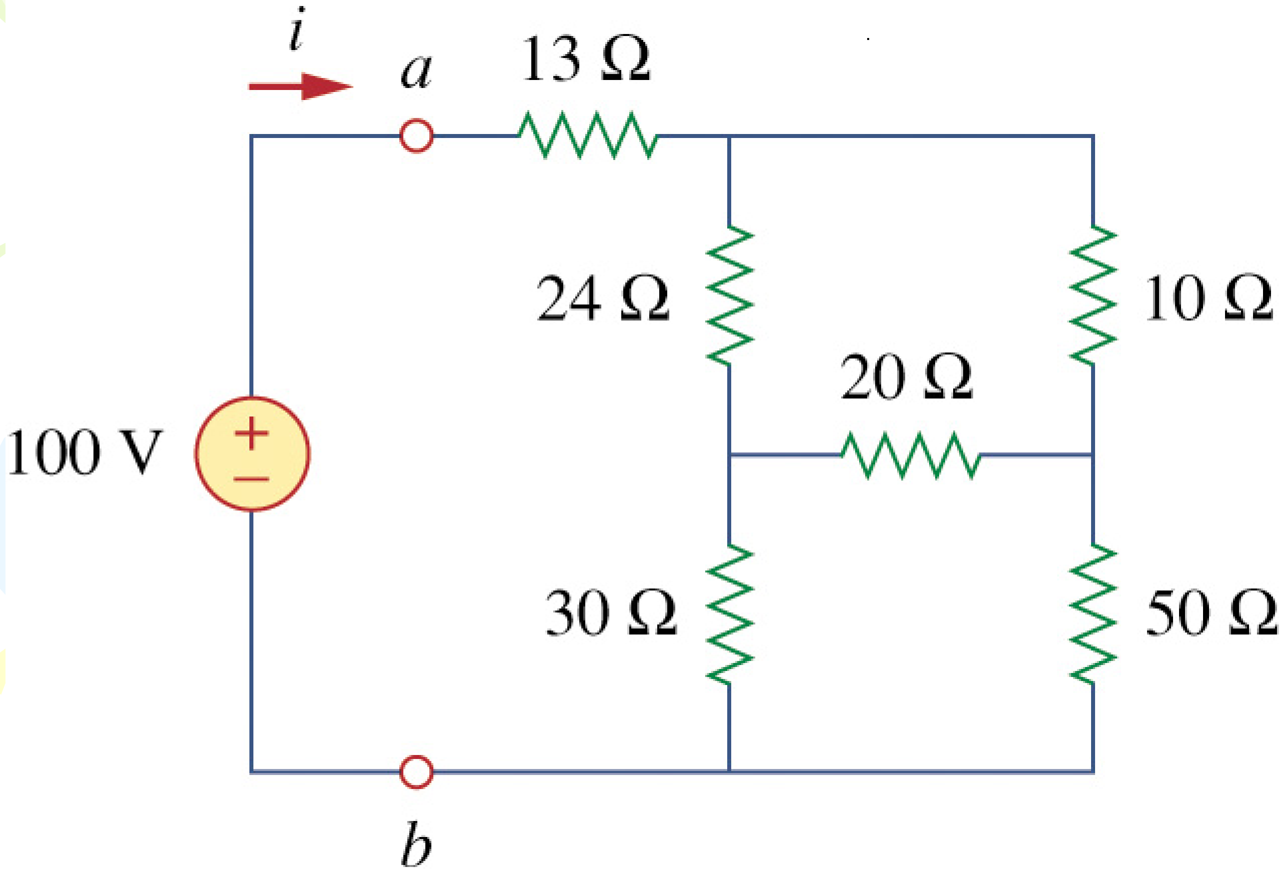


Fig. 2.25

Summary and Review

- For a linear resistor:

Ohm's law | The power dissipated

Passive sign condition: $v = i\,R$ $\quad p = v\,i = i^2R = v^2/R$

Active sign condition: $v = -i\,R$ $\quad p =- v\,i = i^2R = v^2/R$

- Kirchhoff's laws:

KCL:
$$\sum_{n=1}^{N} i_n = 0 \qquad \sum i_{in} = \sum i_{out}$$

KVL:
$$\sum_{m=1}^{M} v_n = 0 \qquad \sum v_{drops} = \sum v_{rises}$$

Summary and Review

- in series :All elements in a circuit that carry the same current are said to be connected in series.
- A series combination of N resistors can be replaced by a single resistor having the value

$$R_{eq} = R_1 + R_2 + \ldots + R_N.$$ Voltage division

- in parallel :Elements in a circuit having a common voltage across them are said to be connected in parallel.
- A parallel combination of N resistors can be replaced by a single resistor having the value

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots + \frac{1}{R_N}$$ Current division

Fundamentals of Electric Circuits 2021.3

Chapter 3
Methods of Analysis

Chapter3 Methods of Analysis

3.1 Motivation

3.2 Nodal analysis.

3.3 Nodal analysis with voltage sources.

3.4 Loop (Mesh)analysis.

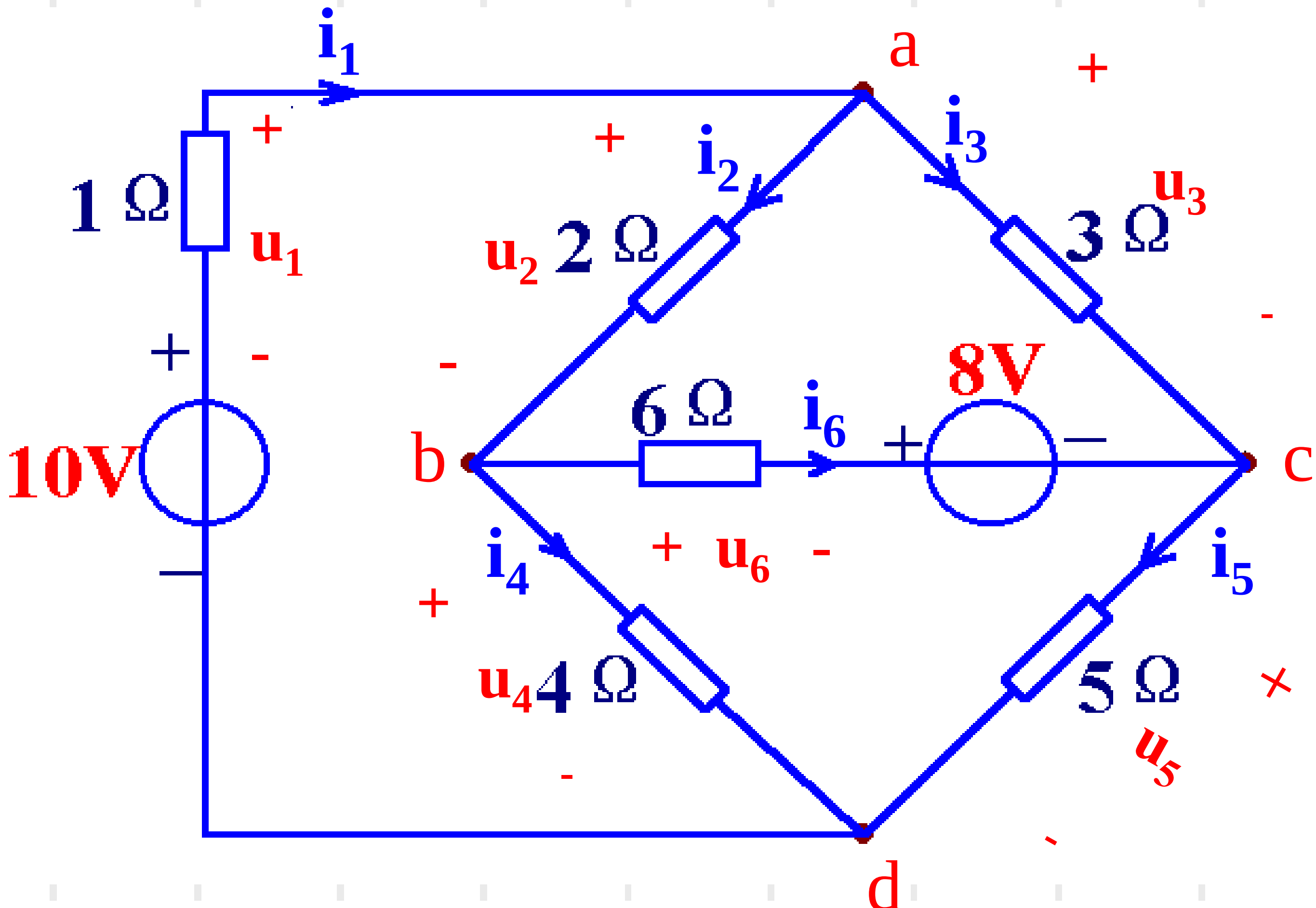
3.5 Loop (Mesh) analysis with current sources.

3.6 Nodal versus mesh analysis.

3.7 Summary

3.1 Motivation

If you are given the following circuit, how can we determine (1) the voltage across each resistor, (2) current through each resistor. (3) power generated by each current source, etc.



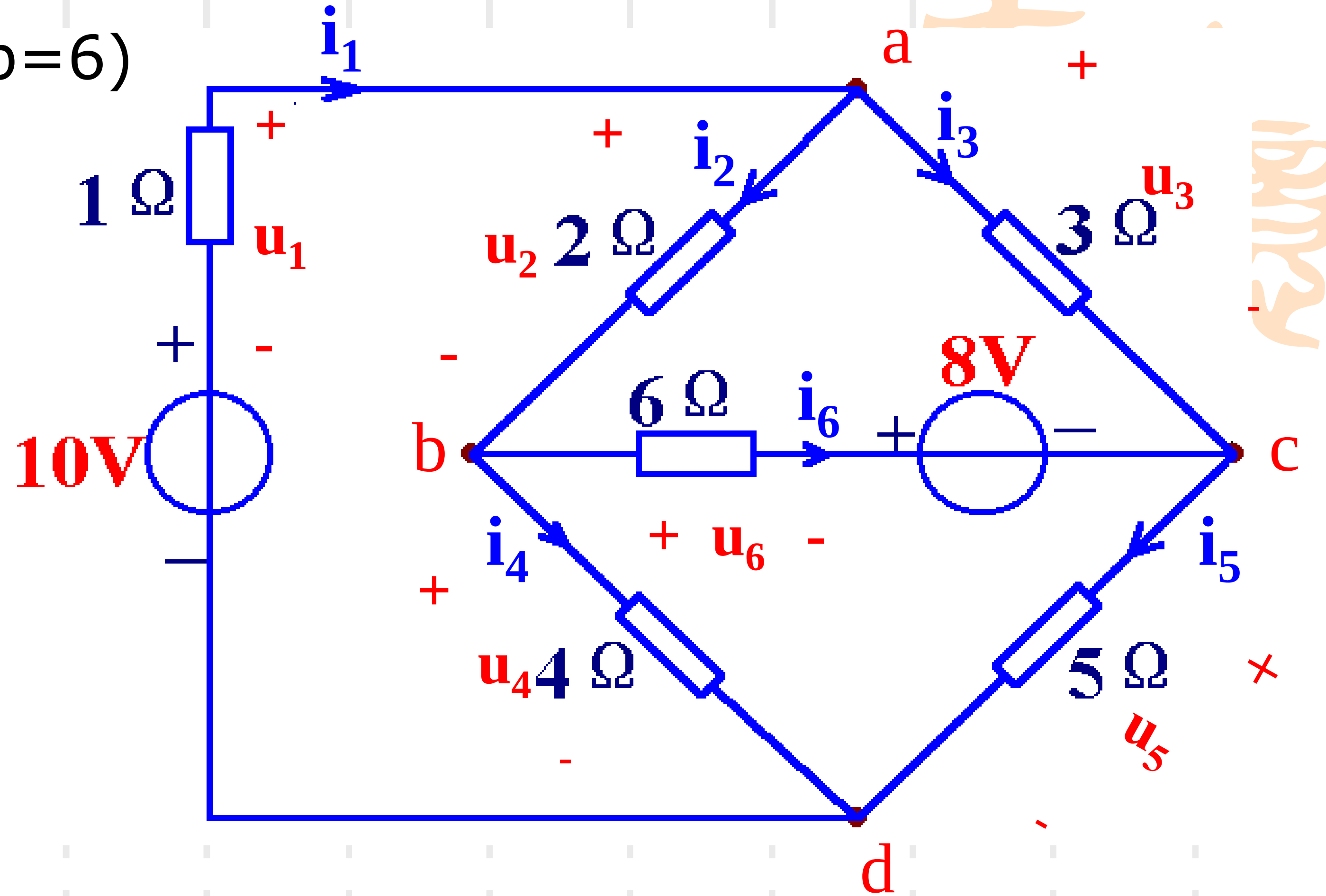
3.1 Motivation

Things we need to know in solving any resistive circuit with current and voltage sources only:

- Kirchhoff's Current Laws (KCL)
- Kirchhoff's Voltage Laws (KVL)
- Ohm's Law

How should we apply these laws to determine the answers?

Example(n=4,b=6)



KCL

a: $-i_1 + i_2 + i_3 = 0$

b: $-i_2 + i_4 + i_6 = 0$

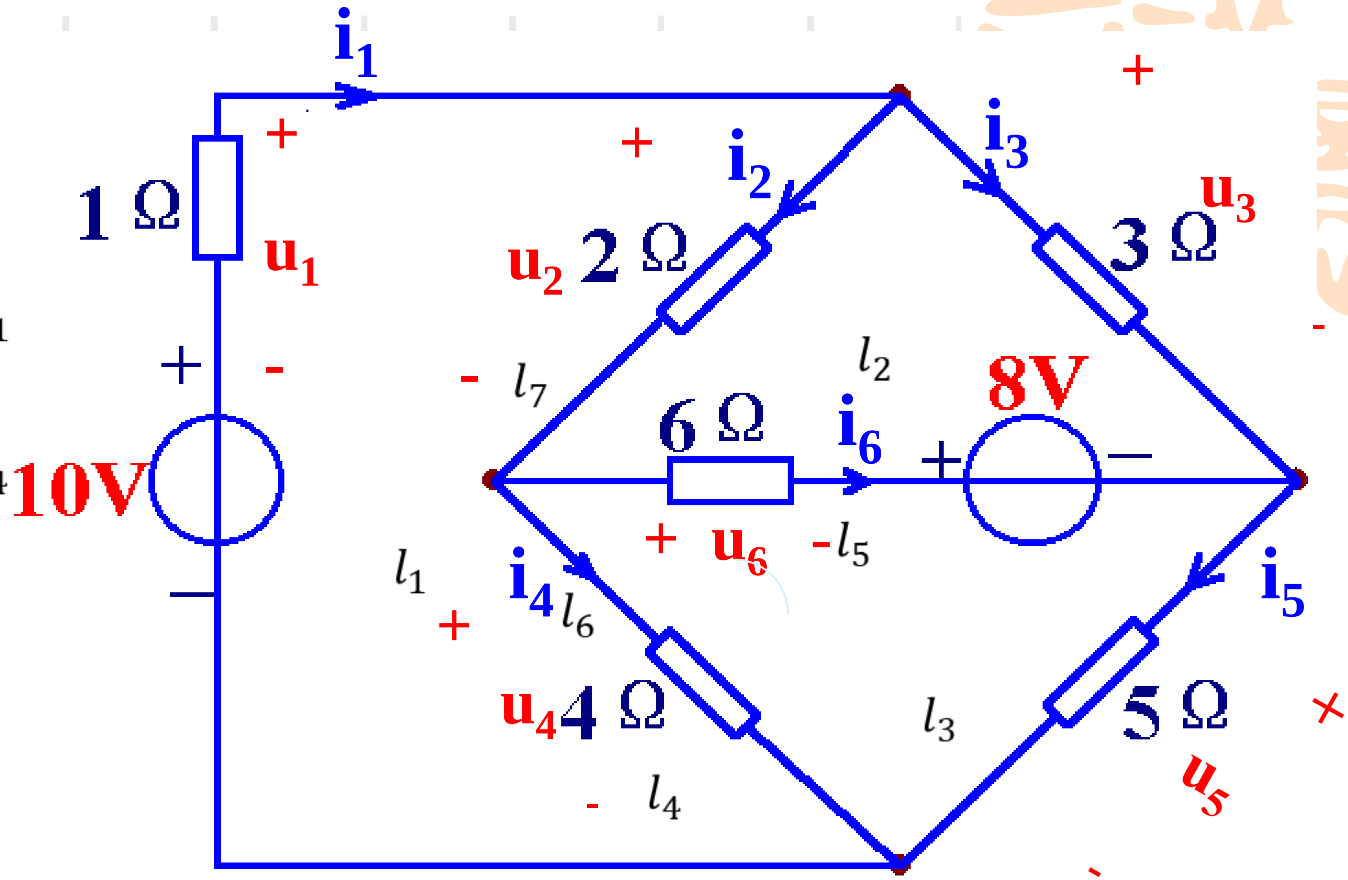
c: $-i_3 + i_5 - i_6 = 0$

d:

n-1 independent KCL equations

n-1 independent nodes, 1 reference node

Example(n=4,b=6)



KVL

l_1: $u_2 + u_4 = 10 + u_1$

l_2: $u_3 = 8 + u_6 + u_2$

l_3: $u_6 + 8 + +u_5 = u_4$

l_4:

l_5:

l_6:

l_7:

b-(n-1) independent KVL equations

Ohm's law: $u_1 = -i_1,\ u_2 = 2i_2,\ u_3 = 3i_3,\ u_4 = 4i_4,\ u_5 = 5i_5,\ u_6 = 6i_6$

Example(n=4,b=6)

KCL（n-1）

$$-i_1 + i_2 + i_3 = 0$$
$$-i_2 + i_4 + i_6 = 0$$
$$-i_3 + i_5 - i_6 = 0$$

KVL（b-(n-1））

$$l_1: \quad u_2 + u_4 = 10 + u_1$$
$$l_2: \quad u_3 = 8 + u_6 + u_2$$
$$l_3: \quad u_6 + 8 + +u_5 = u_4$$

Ohm's law:

$$u_1 = -i_1,\ u_2 = 2i_2,\ u_3 = 3i_3,$$
$$u_4 = 4i_4,\ u_5 = 5i_5,\ u_6 = 6i_6$$

KCL（n-1）

$$-i_1 + i_2 + i_3 = 0$$
$$-i_2 + i_4 + i_6 = 0$$
$$-i_3 + i_5 - i_6 = 0$$

KVL（b-(n-1））

$$l_1: \quad 2i_2 + 4i_4 = 10 - i_1$$
$$l_2: \quad 3i_3 = 8 + 6i_6 + 2i_2$$
$$l_3: \quad 6i_6 + 8 + 5i_5 = 4i_4$$

KCL（n-1）

$$u_1 + \frac{u_2}{2} + \frac{u_3}{3} = 0$$
$$-\frac{u_2}{2} + \frac{u_4}{4} + \frac{u_6}{6} = 0$$
$$-\frac{u_3}{3} + \frac{u_5}{5} - \frac{u_6}{6} = 0$$

KVL（b-(n-1））

$$l_1: \quad u_2 + u_4 = 10 + u_1$$
$$l_2: \quad u_3 = 8 + u_6 + u_2$$
$$l_3: \quad u_6 + 8 + +u_5 = u_4$$

b currents and b voltages

2b independent equations

b currents

b independent equations

b voltagess

b independent equations

Is it possible to use fewer variables and write fewer equations？

3.1 Motivation (2)

- In this chapter， we will study two powerful techniques for circuit analysis. These methods are based on the above laws(Kirchhoff's Laws, Ohm's Law).

  - Nodal analysis and Mesh analysis

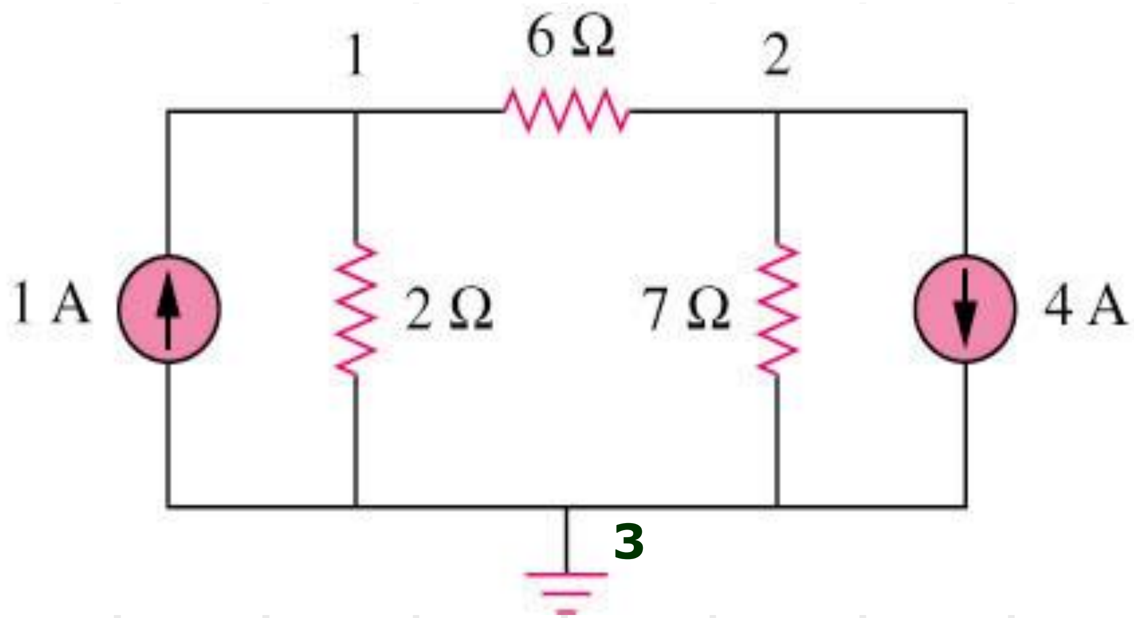
  - Please pay attention: Circuit variables, how to write equations

3.2 Nodal Analysis

Circuit variables: **node voltages**

Equations: **KCL for independent nodes, replace current with node voltages**

Example 1



2 independent nodes, 1 reference node

We can select node 3 as reference node.

3.2 Nodal Analysis (2)

Steps to determine the node voltages:

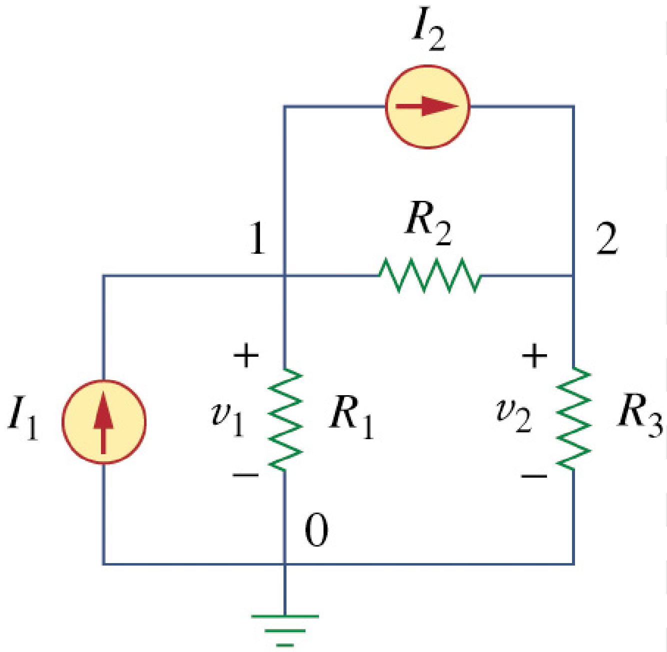
1. Select a node as the reference node.
2. Assign voltages $v_1, v_2, \ldots, v_{n-1}$ to the remaining n-1 nodes. The voltages are referenced with respect to the reference node.
3. Apply KCL to each of the n-1 non-reference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
4. Solve the resulting simultaneous equations to obtain the unknown node voltages.

1.Select node 0 as reference node, assigned voltages v_1 and v_2 ,respectively.

2.Applying KCL to node 1 and node 2.

Suppose each resister's voltage and current are passive sign convention.

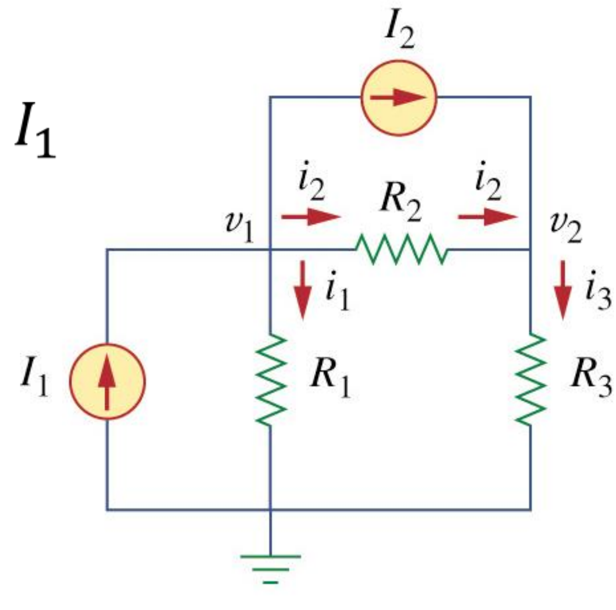
At Node 1 and node 2



(a)

$$\frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} + I_2 = I_1$$

$$\frac{v_1 - v_2}{R_2} + I_2 = \frac{v_2}{R_3}$$



(b)

3.2 Nodal Analysis (3)

Example 3 – Calculate the node voltages in the circuit shown in Fig. 3.3(a).

circuit independent current source only

At Node 1 $$5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2}$$

At Node 2 $$\frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$$

So

$$v_1 = \frac{40}{3} = 13.33 \text{ V}$$

$$v_2 = 20 \text{ V}$$

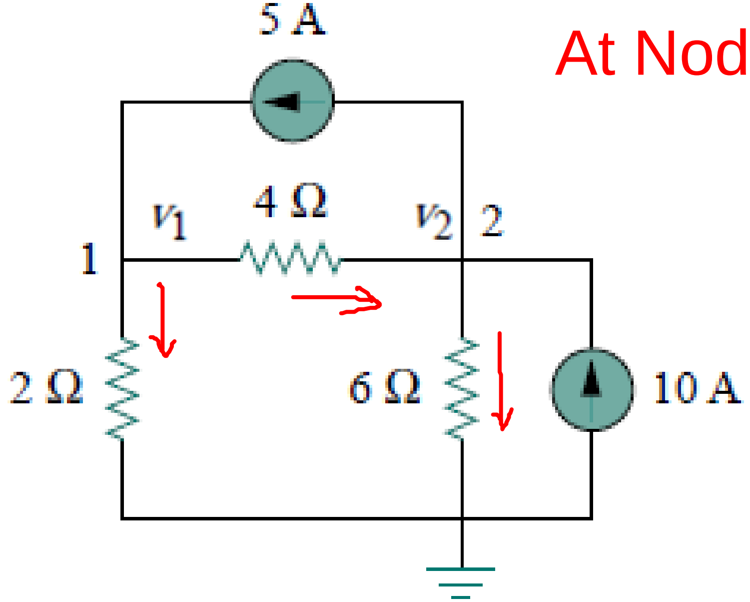


Fig. 3.3(a) original circuit.

PRACTICE PROBLEM 3.1

Obtain the node voltages in the circuit in Fig. 3.4.

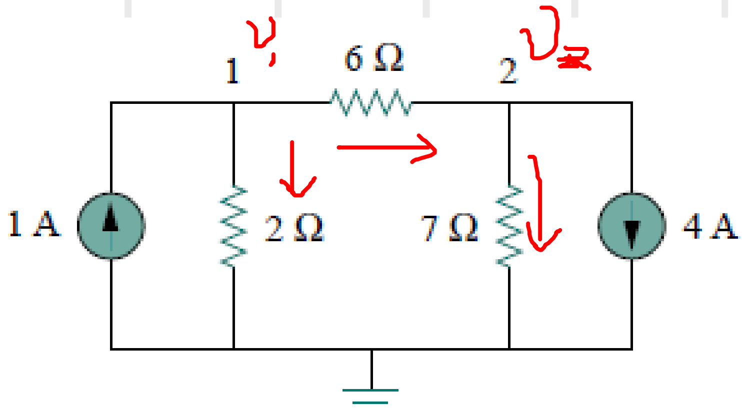
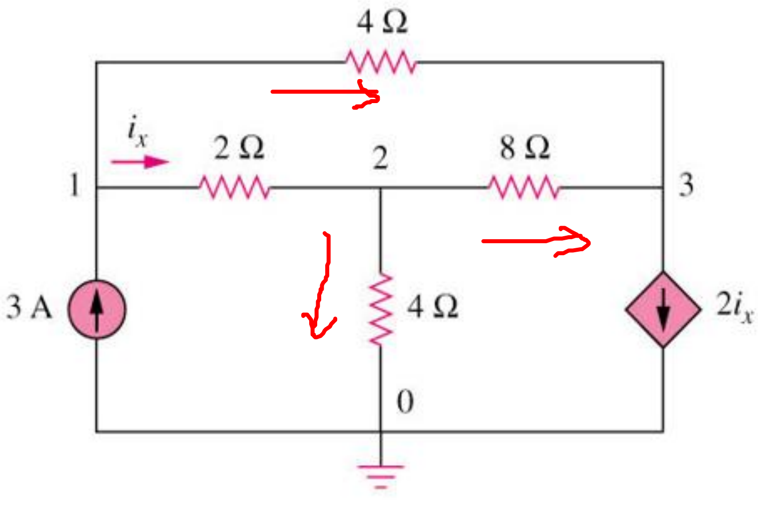


Figure 3.4 For Practice Prob. 3.1.

Answer: $v_1 = -2$ V, $v_2 = -14$ V.

3.2 Nodal Analysis (4)

<u>Example 3</u> – current with dependant current source



(a)

At node 1, $3 = \dfrac{v_1 - v_3}{4} + \dfrac{v_1 - v_2}{2}$

At node 2, $\dfrac{v_1 - v_2}{2} = \dfrac{v_2 - v_3}{8} + \dfrac{v_2 - 0}{4}$

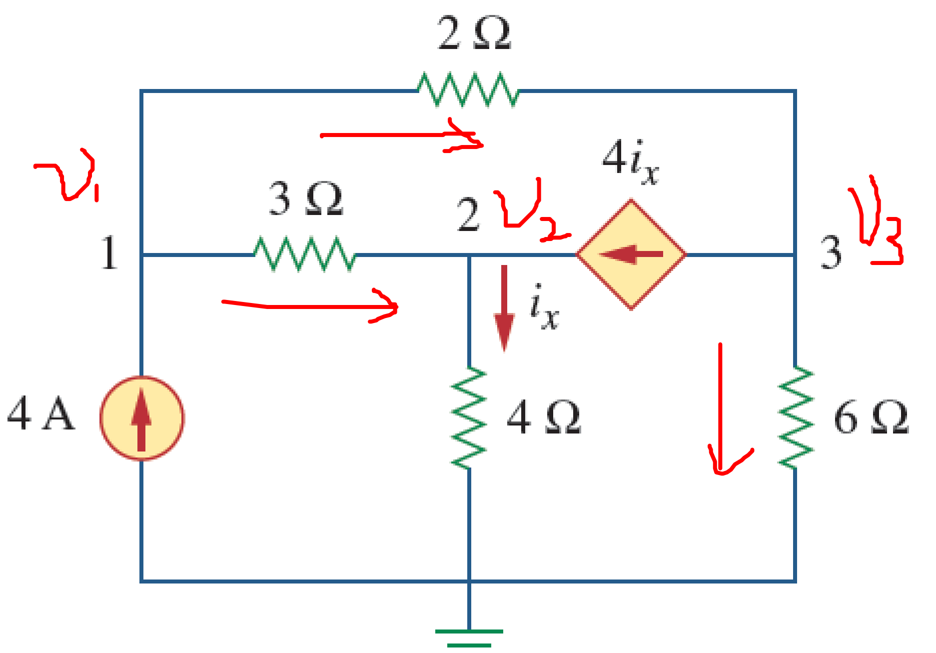
At node 3, $\dfrac{v_1 - v_3}{4} + \dfrac{v_2 - v_3}{8} = \dfrac{2(v_1 - v_2)}{2}$

$$i_x = \frac{v_1 - v_2}{2}$$

Write the equation that specifies the relationship of the dependent source to the controlling parameter.

Practice Problem 3.2

Find the voltages at the three nonreference nodes in the circuit of Fig. 3.6.

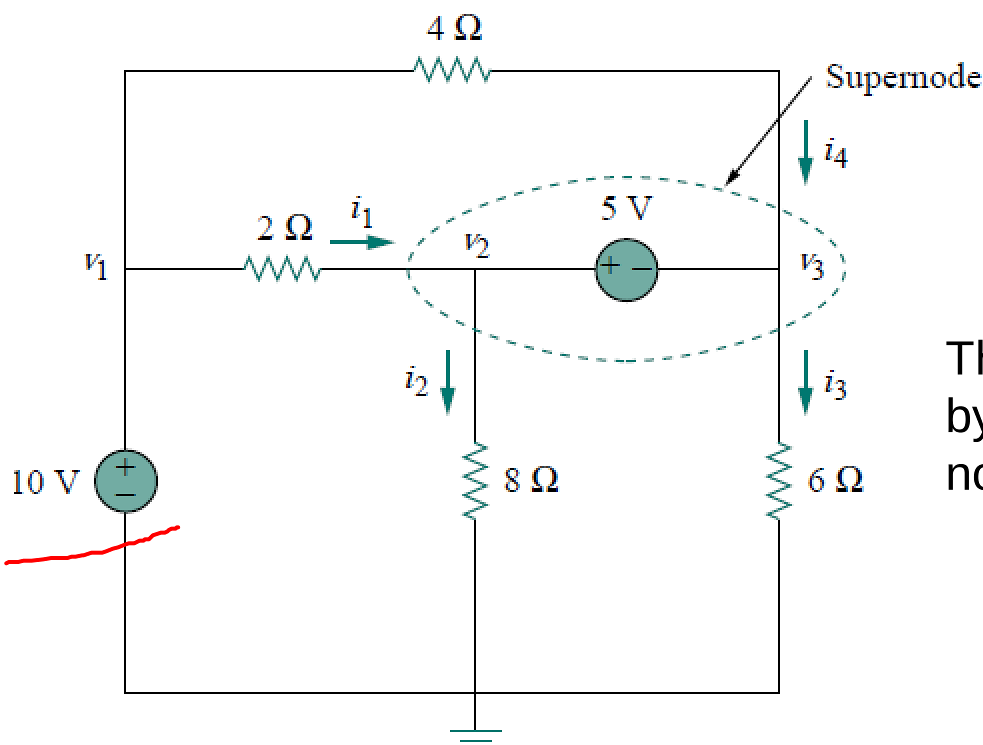


Answer: $v_1 = 32$ V, $v_2 = -25.6$ V, $v_3 = 62.4$ V.

3.3 Nodal Analysis with Voltage Source (1)

CASE I

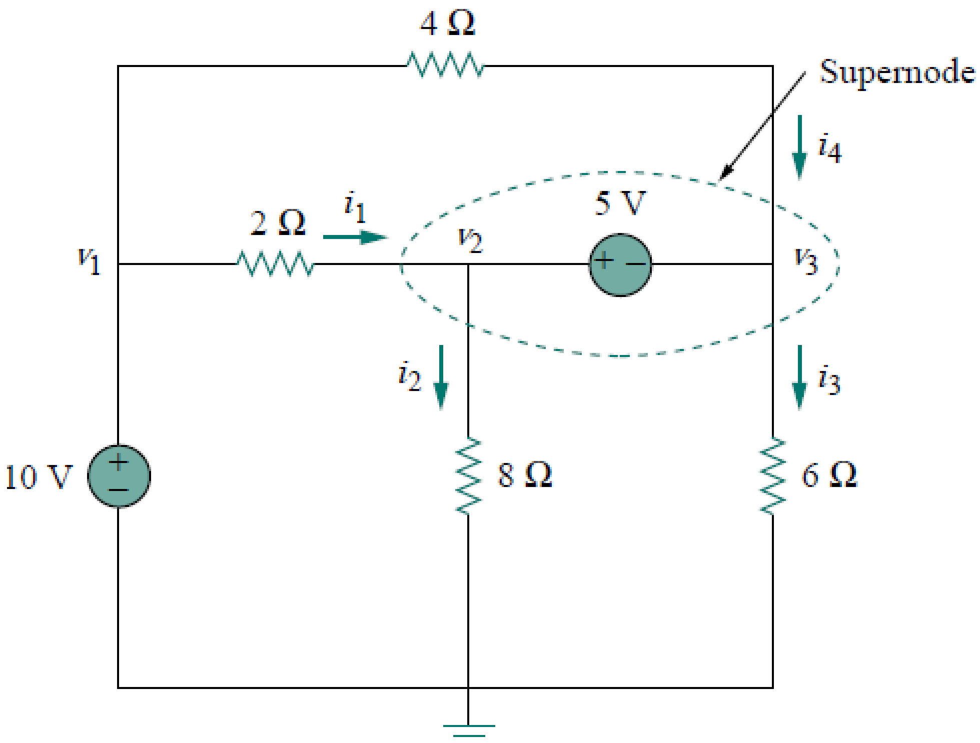
If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source.



$$v_1 = 10\text{ V}$$

Thus our analysis is somewhat simplified by this knowledge of the voltage at this node.

CASE 2 If the voltage source (dependent or independent) is connected between two nonreference nodes, the two nonreference nodes form a *generalized node* or *supernode*; we apply both KCL and KVL to determine the node voltages.



(1) Apply KCL to super-nodes;
(2) Write the equation that defines the voltage relationship between the two nonreference nodes as a result of the presences of the voltage source.

A supernode is formed by enclosing a (dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it.

3.3 Nodal Analysis with Voltage Source (2)

<u>Example 5</u> – For the circuit shown in Fig. 3.9, find the node voltages

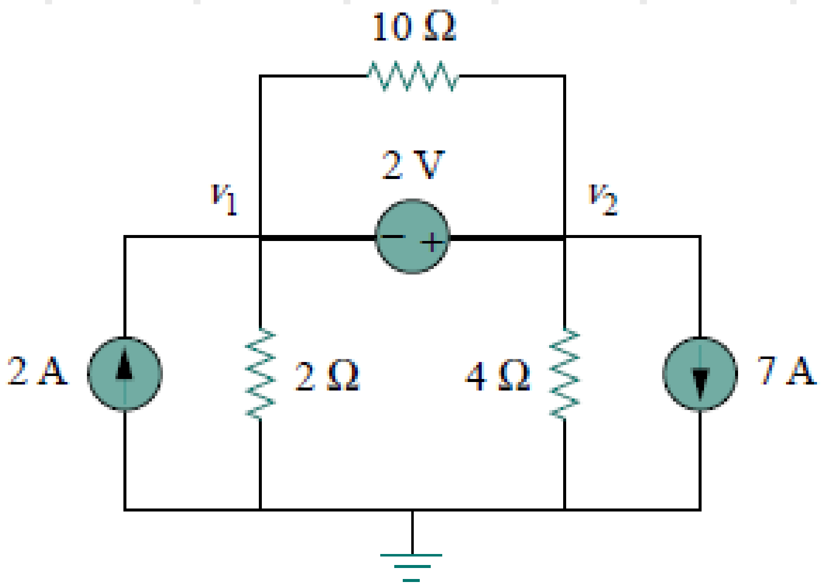
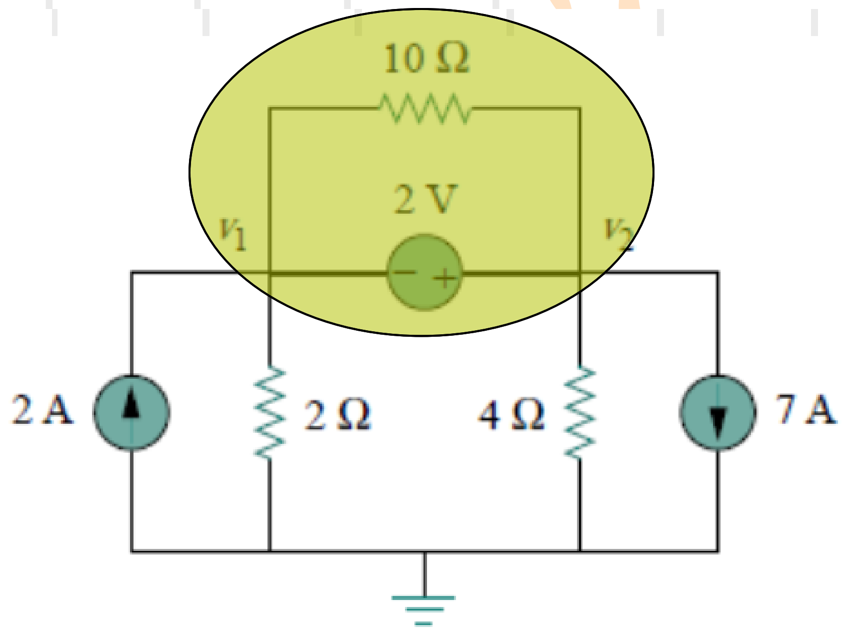


Figure 3.9 For Example 3.3.

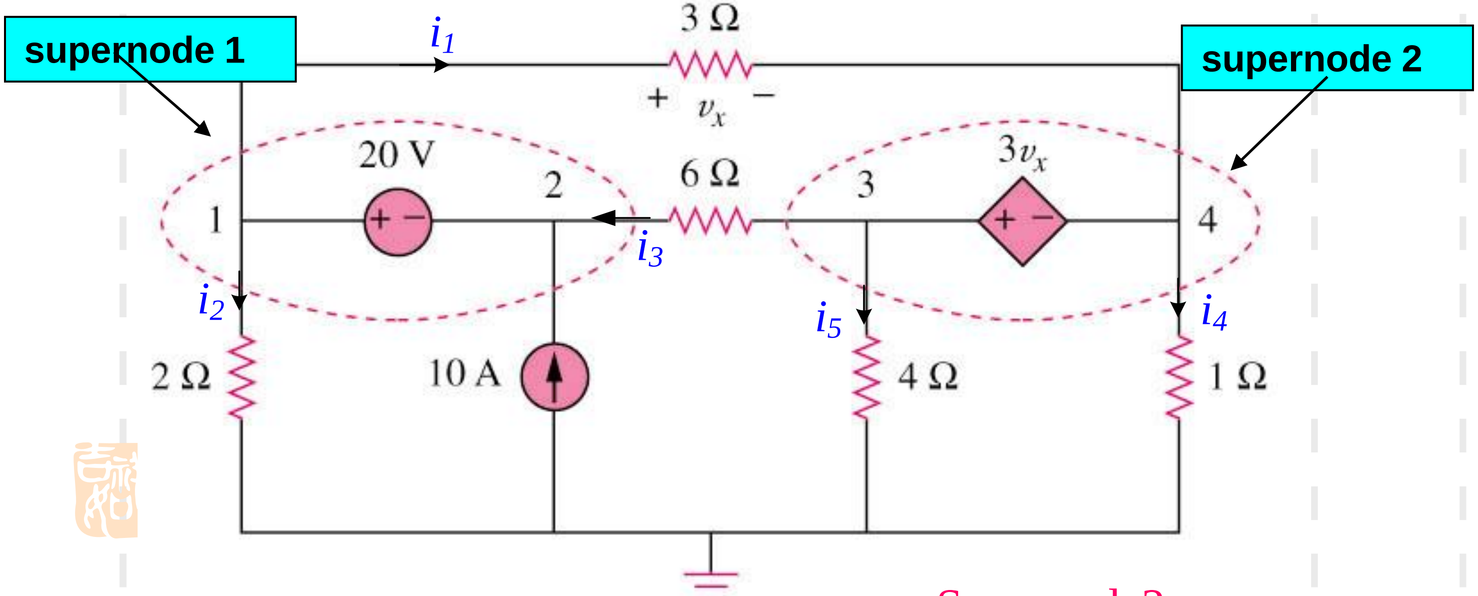


$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7$$

$v_2 - v_1 = 2$

3.3 Nodal Analysis with Voltage Source (4)

<u>Example 7</u> – circuit with one independent voltage source and one dependent voltage source



Supernode1,

$$\frac{v_3 - v_2}{6} + 10 = \frac{v_1 - v_4}{3} + \frac{v_1}{2}$$

$v_1 - v_2 = 20$

$v_1 - v_4 = v_x$

Supernode2,

$$\frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + \frac{v_4}{1} + \frac{v_3}{4}$$

$v_3 - v_4 = 3v_x$

PRACTICE PROBLEM 3.4

Find v_1, v_2, and v_3 in the circuit in Fig. 3.14 using nodal analysis.

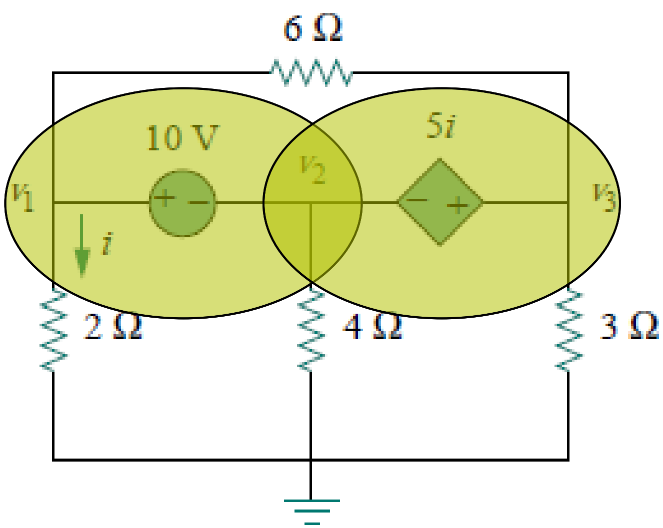
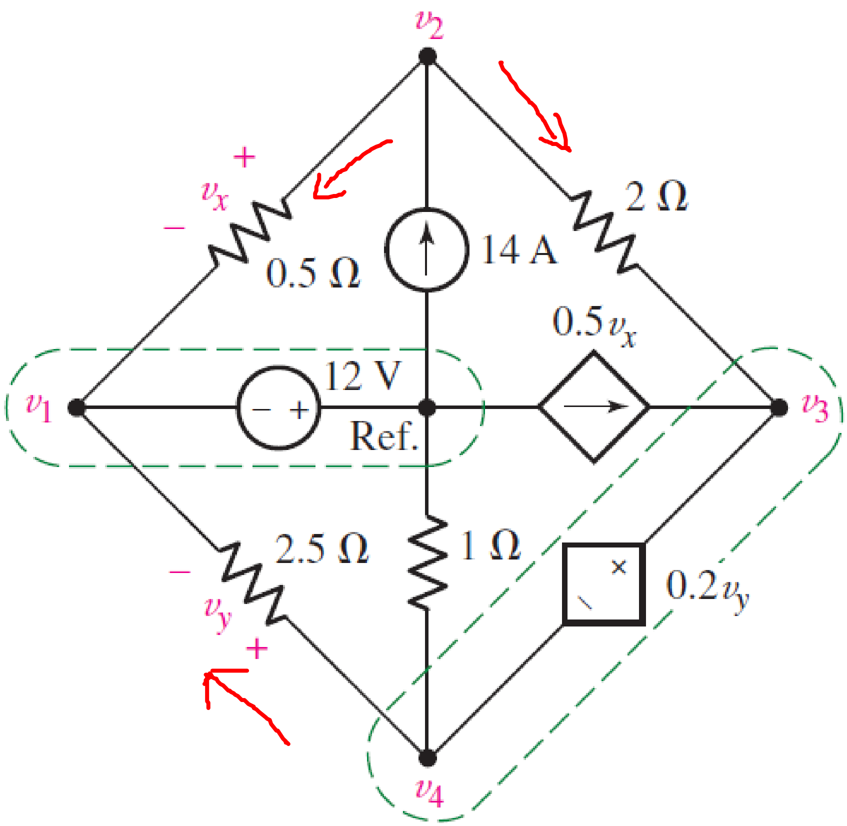


Figure 3.14 For Practice Prob. 3.4.

Answer: $v_1 = 3.043$ V, $v_2 = -6.956$ V, $v_3 = 0.6522$ V.

Determine the node-to-reference voltages in the circuit



$$v_1 = -12 \text{ V.}$$

At node 2,

$$\frac{v_2 - v_1}{0.5} + \frac{v_2 - v_3}{2} = 14$$

at the 3-4 supernode,

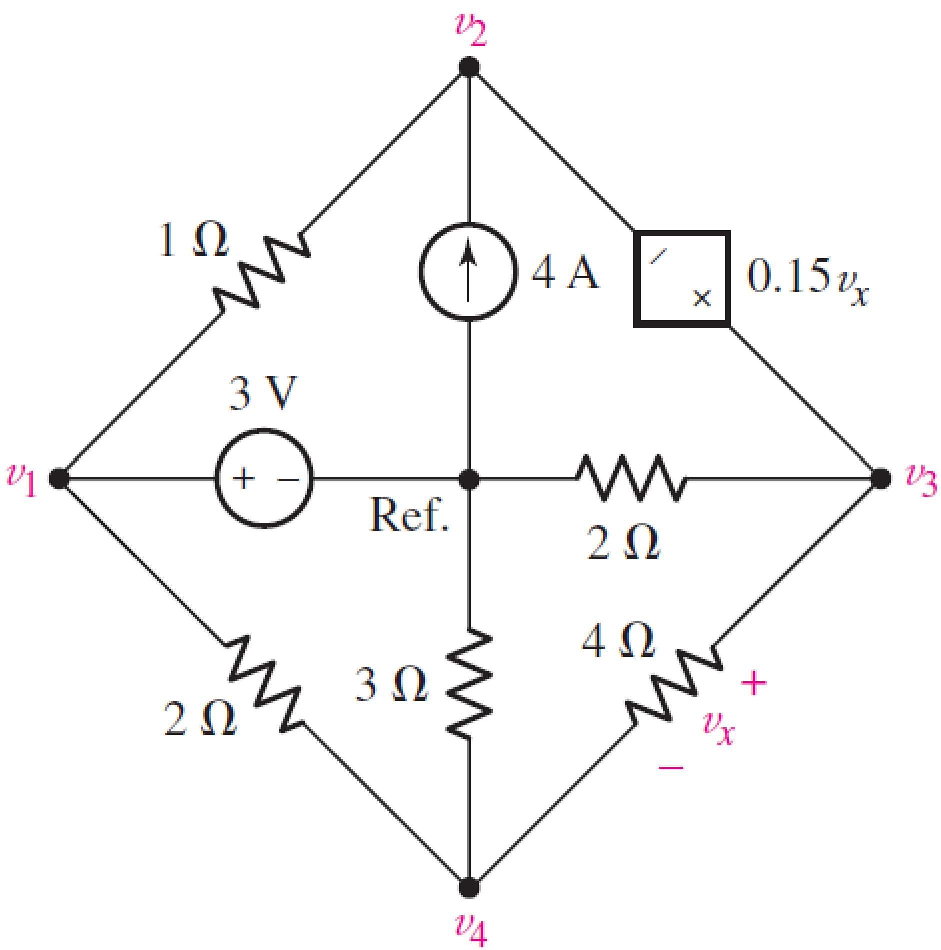
$$0.5v_x = \frac{v_3 - v_2}{2} + \frac{v_4}{1} + \frac{v_4 - v_1}{2.5}$$

$$v_3 - v_4 = 0.2v_y$$

$$0.2v_y = 0.2(v_4 - v_1)$$

$$0.5v_x = 0.5(v_2 - v_1)$$

PRACTICE Determine the nodal voltages in the circuit



Summary of last class

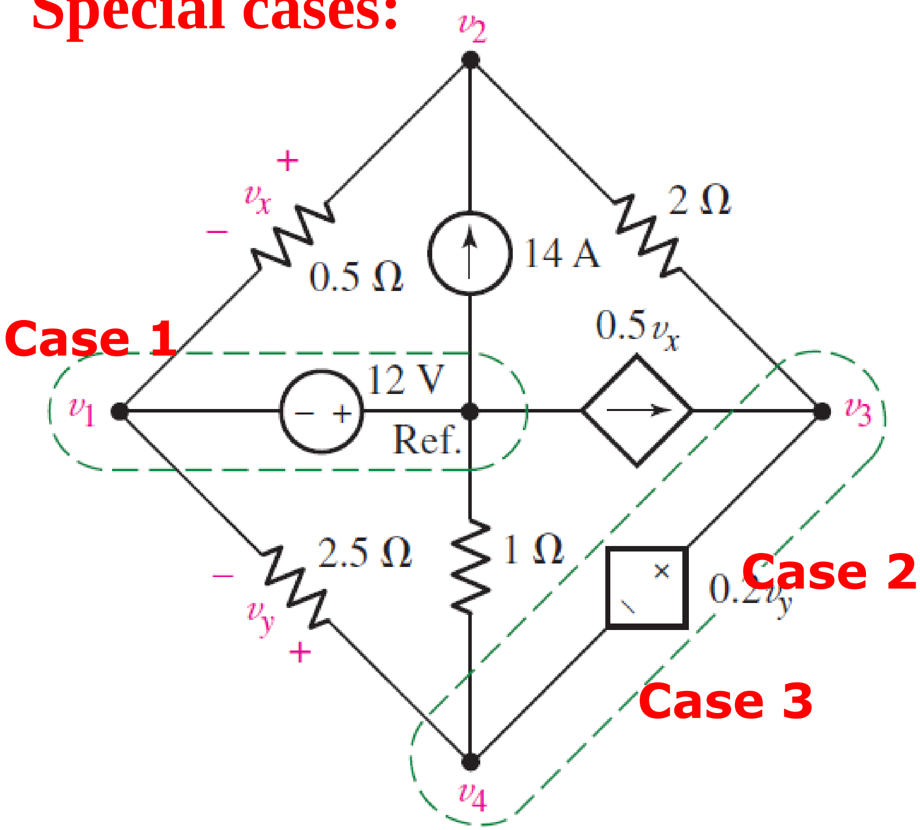
Nodal analysis method

Circuit variables: <u>node voltages</u>
Equations: <u>KCL for independent nodes, replace current with node voltages</u>

<u>Steps to determine the node voltages:</u>

1. **<u>Select</u> the reference node.**
2. **<u>Assign</u> node voltages $v_1, v_2, \ldots, v_{n-1}$ to the remaining n-1 nodes.**
3. **<u>Apply</u> KCL to each of the n-1 non-reference nodes.**
   Use Ohm's law to express the branch currents in terms of node voltages.
4. **<u>Solve</u> the equations to obtain the unknown node voltages.**
5. **Compute branch(element) currents and voltages.**

Special cases:



Case 1: **A voltage source is connected between the reference node and a nonreference node**

v_1 =-12V

Case 2: **A voltage source is connected between two nonreference nodes**

Select supernode

(1) Apply KCL to supernodes;

(2) $0.2v_y = v_3 - v_4$

Case 3: dependent sources

(1) Regard it as independent source;

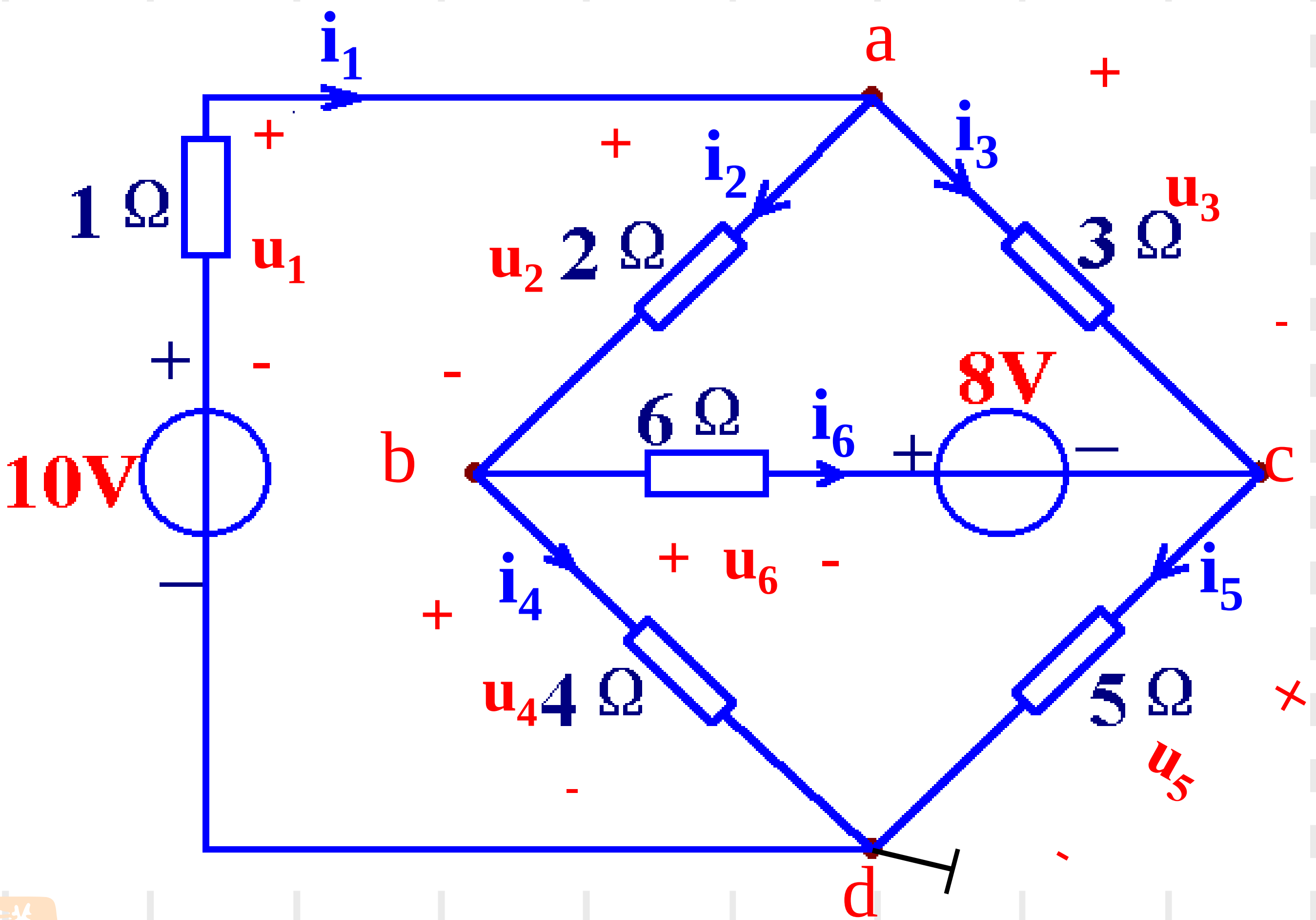
(2) Express the controlling variable with node voltages

$v_y = v_1 - v_4$, $v_x = v_2 - v_1$

branch(element) currents and voltages

$u_1 = u_a - 10$ $u_2 = u_a - u_b$ $u_3 = u_a - u_c$ $u_4 = u_b$ $u_5 = u_c$ $u_6 = u_b - u_c - 8$

$i_1 = u_1/1$ $i_2 = u_2/2$ $i_3 = u_3/3$ $i_4 = u_4/4$ $i_5 = u_5/5$ $i_6 = u_6/6$



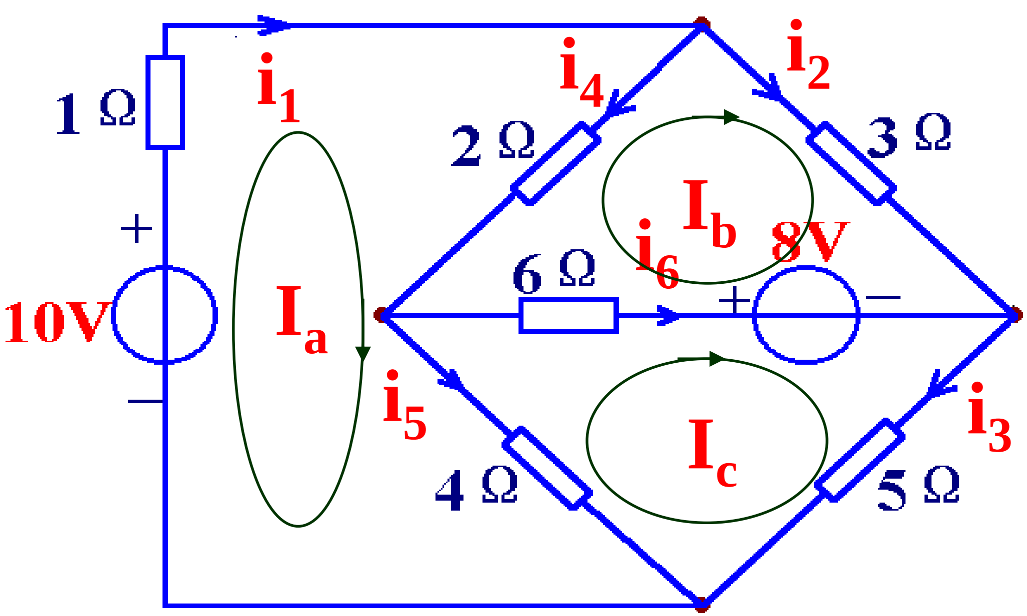
3.4 Mesh Analysis

1.Mesh

A *mesh* is a loop which does not contain any other loops within it.

2.Mesh current

We define a mesh current as a current that flows only around the perimeter of a mesh.



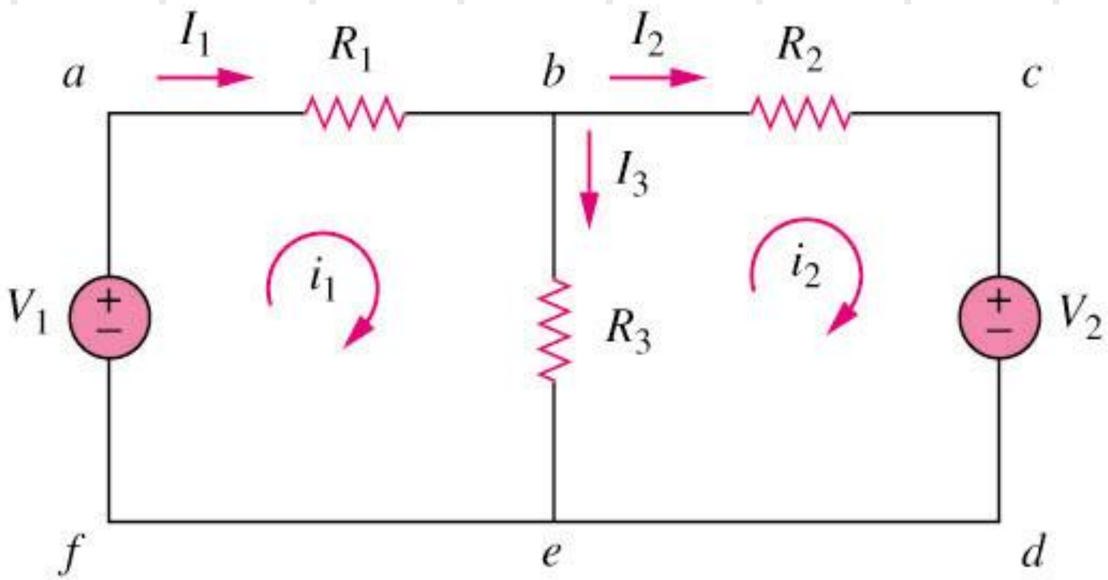
Circuit variables: Mesh current

Equations: KVL for meshes, replace voltage with Mesh current

We should notice that the mesh current is imaginative, not measurable directly.

3.4 Mesh Analysis

Example 8 – circuit with independent voltage sources



1.Assign mesh currents i_1, i_2, …, in to the *n* meshes.

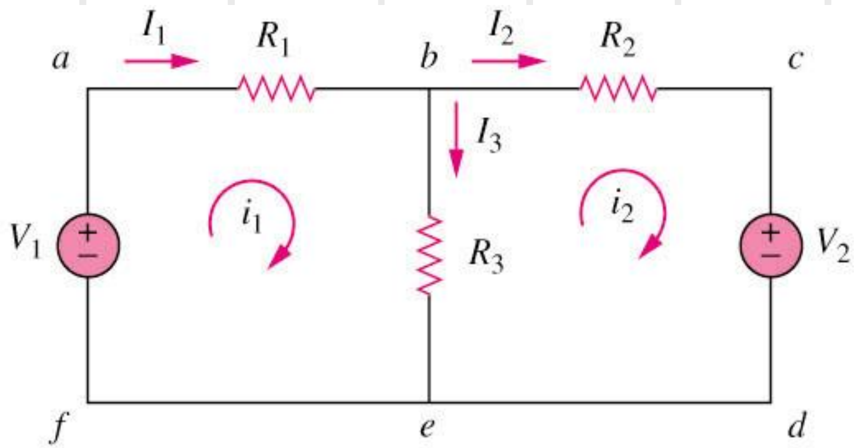
Note:

i_1 and i_2 are mesh current (imaginative, not measurable directly)

I_1, I_2 and I_3 are branch current (real, measurable directly)

$I_1 = i_1$; $I_2 = i_2$; $I_3 = i_1 - i_2$

2. Apply KVL to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.



Applying KVL to mesh 1,

$$-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0 \qquad \text{or} \qquad (R_1 + R_3)i_1 - R_3 i_2 = V_1$$

Applying KVL to mesh 2,

$$R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0 \qquad \text{or} \qquad -R_3 i_1 + (R_2 + R_3)i_2 = -V_2$$

3. Solve the resulting 2 simultaneous equations to get the mesh currents.

$$I_1 = i_1, \qquad I_2 = i_2, \qquad I_3 = i_1 - i_2$$

EXAMPLE 3.5 **For the circuit in Fig. 3.18, find the branch currents I_1 , I_2 , and I_3 using mesh analysis.**

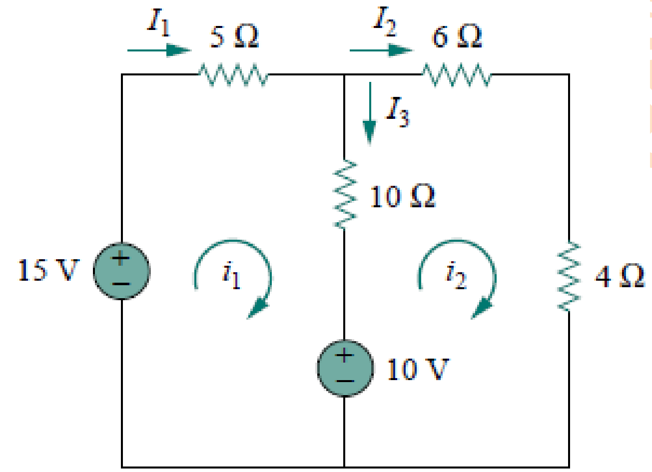


Figure 3.18 For Example 3.5.

Solution:

For mesh 1,

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

For mesh 2,

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0$$

$$i_1 = 1A, i_2 = 1A$$

So

$$I_1 = i_1 = 1A, I_2 = i_2 = 1A, I_3 = i_1\text{-}i_2 = 0A$$

PRACTICE PROBLEM 3.5

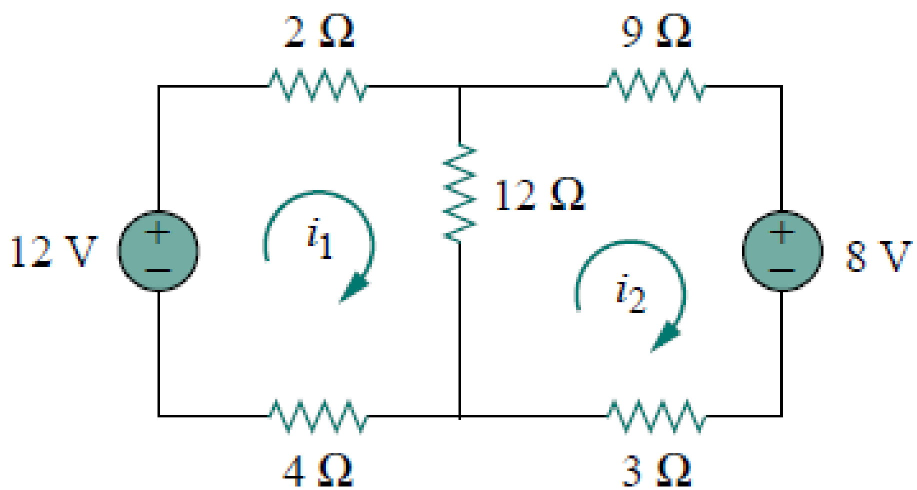


Figure 3.19 For Practice Prob. 3.5.

Calculate the mesh currents i_1 and i_2 in the circuit of Fig. 3.19.

Answer: $i_1 = \frac{2}{3}$ A, $i_2 = 0$ A.

3.4 Mesh Analysis

Dependent sources

Method: First, treat dependent sources as an independent source when writing the KVL equations. Then write the controlling equation for the dependent source.

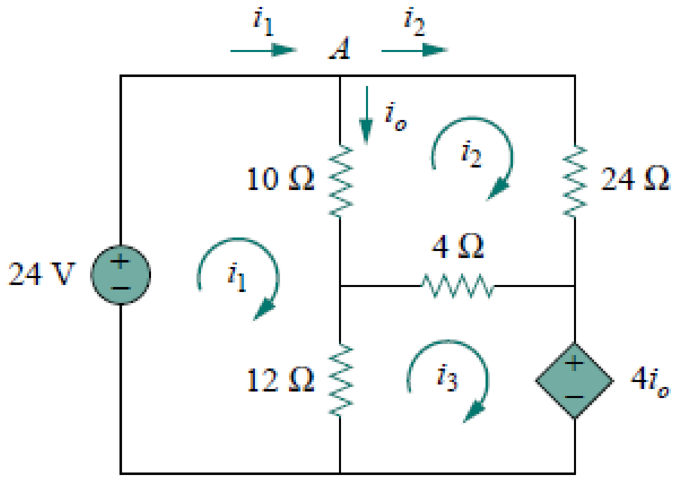


Figure 3.20 For Example 3.6.

EXAMPLE 3.6 **Use mesh analysis to find the current i_o in the circuit in Fig. 3.20.**

Solution:

Apply KVL to the three meshes in turn.

For mesh 1,

$$-24 + 10(i_1 - i_2) + 12(i_1 - i_3) = 0$$

For mesh 2,

$$24i_2 + 4(i_2 - i_3) + 10(i_2 - i_1) = 0$$

For mesh 3,

$$4i_o + 12(i_3 - i_1) + 4(i_3 - i_2) = 0$$

$$i_o = i_1 - i_2.$$

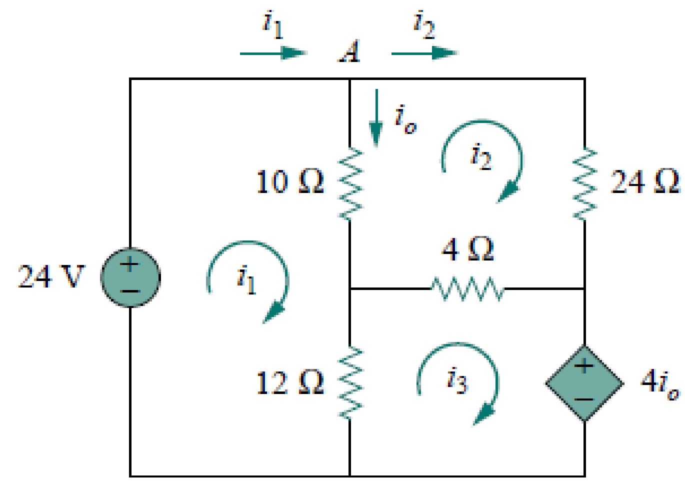


Figure 3.20 For Example 3.6.

PRACTICE PROBLEM 3.6

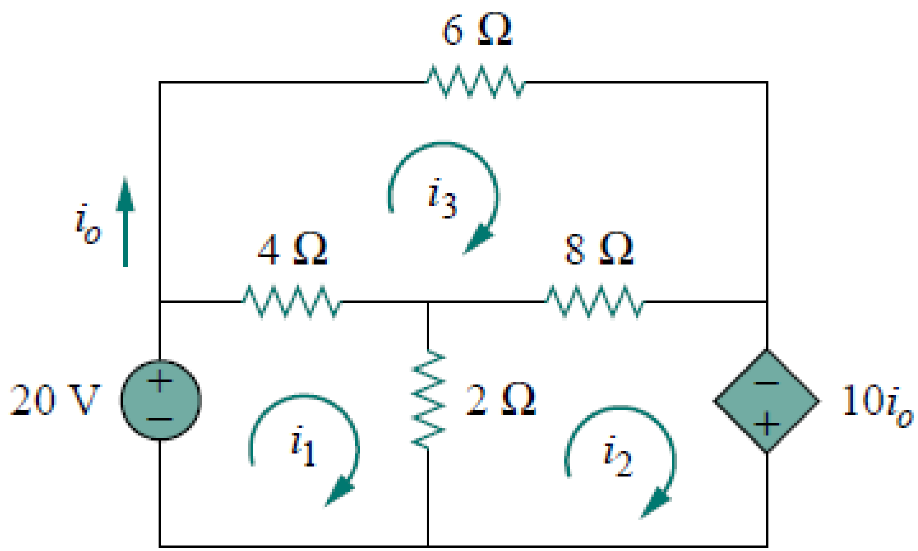


Figure 3.21 For Practice Prob. 3.6.

Using mesh analysis, find i_o in the circuit in Fig. 3.21.

Answer: -5 A.

3.5 Mesh Analysis with Current Source

Case 1. When a current source exists only in one mesh.

For the mesh containing the current source, its mesh current is known.

Set $i_2 = -5\,A$

For other meshes, write the mesh current equation using KVL.

$$4i_1 + 6(i_1 - i_2) = 10$$

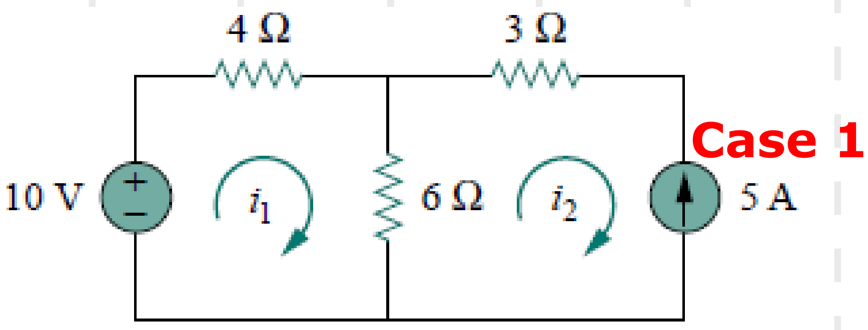
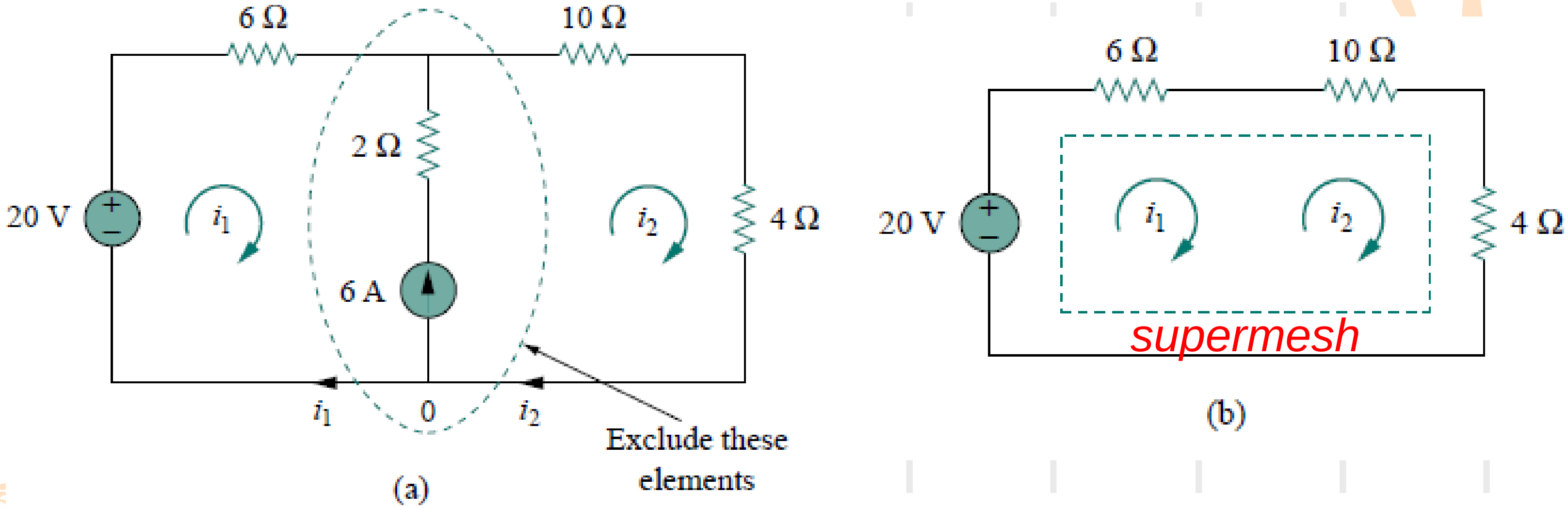


Figure 3.22 A circuit with a current source.

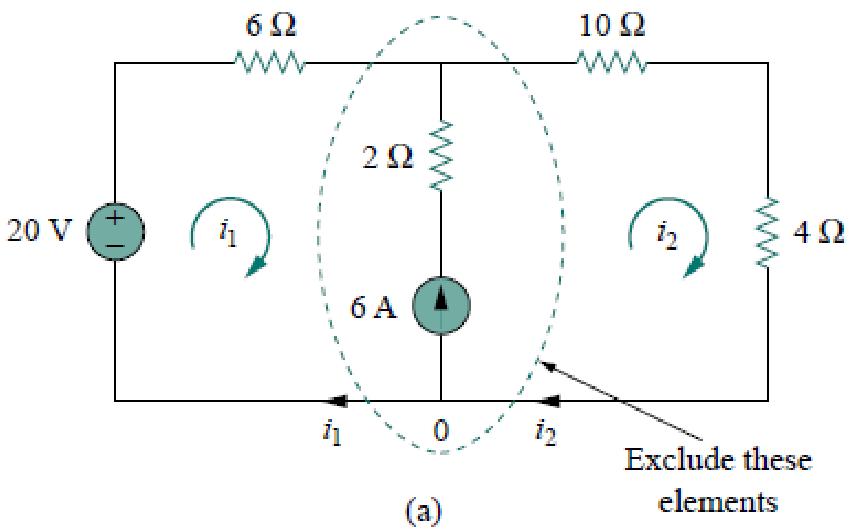
Case 2. A current source exists between two meshes .

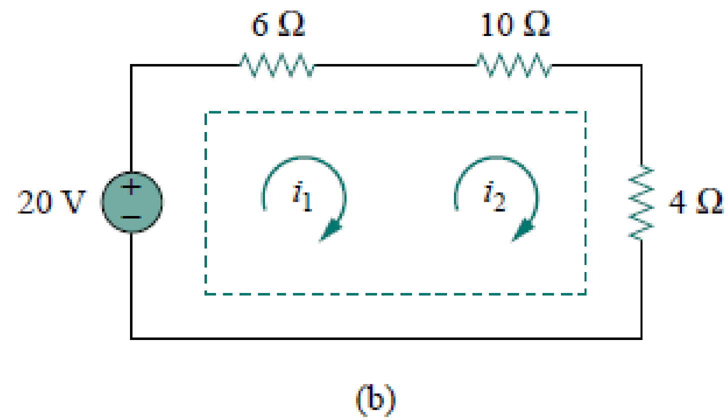


We create a *supermesh by excluding* the current source and any elements connected in series with it, as shown in Fig. 3.23(b).

A supermesh results when two meshes have a (dependent or independent) current source in common.

Case 2. A current source exists between two meshes .





Applying KVL to the supermesh

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$

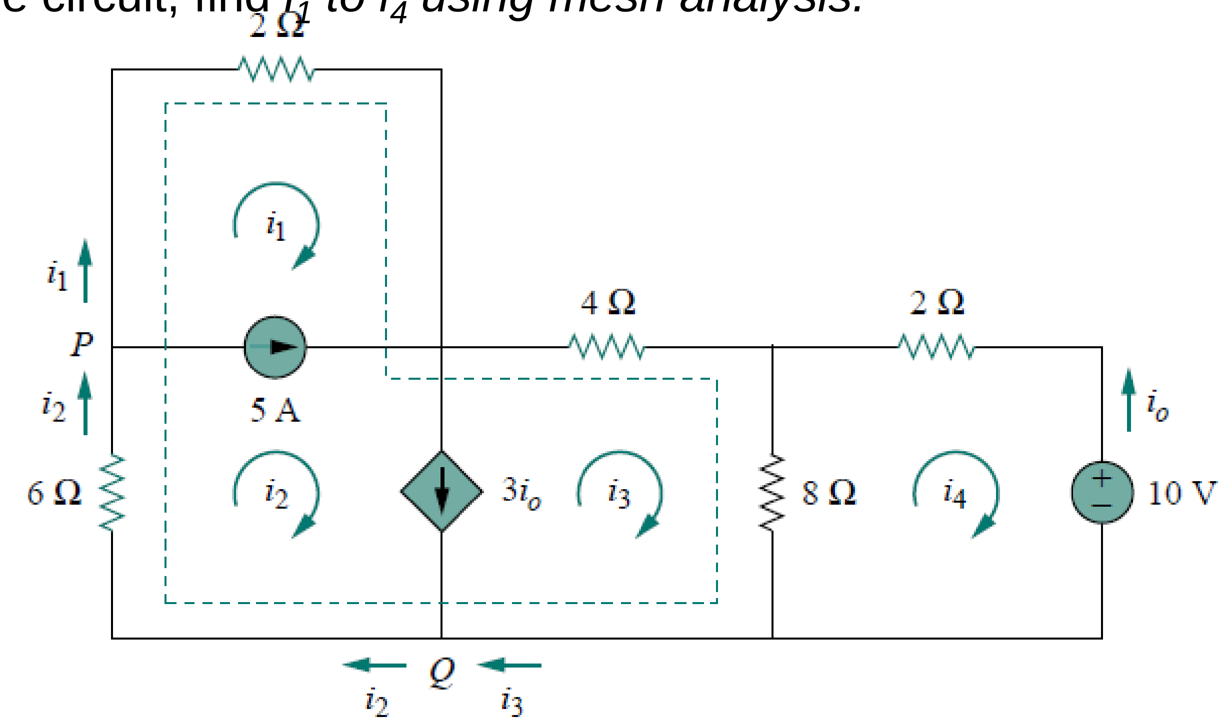
Apply KCL to a node in the branch where the two meshes intersect.

Applying KCL to node 0, $i_2 = i_1 + 6$

A super-mesh requires the application of both KVL and KCL!

If a circuit has two or more supermeshes that intersect, they should be combined to form a larger supermesh.

For the circuit, find i_1 *to* i_4 *using mesh analysis.*



Applying KVL to the larger supermesh, $2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0$

Applying KCL to node *P:* $i_2 = i_1 + 5$

Applying KCL to node *Q:* $i_2 = i_3 + 3i_o$

Applying KVL in mesh 4, $2i_4 + 8(i_4 - i_3) + 10 = 0$ $i_o = -i_4$

$i_1 = -7.5$ A, $i_2 = -2.5$ A, $i_3 = 3.93$ A, $i_4 = 2.143$ A

PRACTICE PROBLEM 3.7

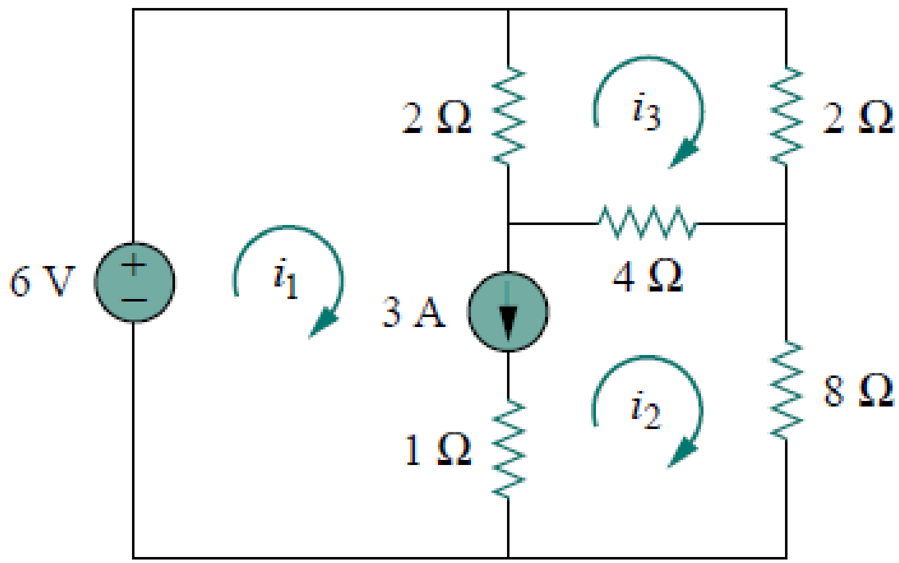


Figure 3.25 For Practice Prob. 3.7.

Use mesh analysis to determine *i_1*, *i_2*, and *i_3* in Fig. 3.25.

Answer: $i_1 = 3.474$ A, $i_2 = 0.4737$ A, $i_3 = 1.1052$ A.

3.6 Nodal versus Mesh Analysis (1)

To select the method that results in the smaller number of equations. For example:

1. Choose nodal analysis for circuit with fewer nodes than meshes.

   Choose mesh analysis for circuit with fewer meshes than nodes.

   *Networks that contain many series-connected elements, voltage sources, or supermeshes are more suitable for mesh analysis.

   *Networks with parallel-connected elements, current sources, or supernodes are more suitable for nodal analysis.

2. If node voltages are required, it may be expedient to apply nodal analysis. If branch or mesh currents are required, it may be better to use mesh analysis.

3.7 Summary and Review

- Nodal analysis:

If nodal analysis is the chosen approach, choose one of the nodes as a reference node. Then label the node voltage v_1, v_2, …, $v_{N\text{-}1}$, remembering that each is understood to be measured with respect to the reference node.

If the circuit contains only current sources, apply KCL at each nonreference node.

If the circuit contains voltages sources, form a supernode about each one, and then proceed to apply KVL at all nonreference nodes and supernodes.

- Mesh analysis:

Assign a clockwise mesh current in each mesh: i_1, i_2, …, i_M.

If the circuit contains only voltage sources, apply KVL around each mesh.

If the circuit contains current sources, create a supermesh for each one current source that is common to two meshes, and then apply KVL around each mesh and supermesh.

Assignments (page 114)

Problems

3.2

3.17

3.23

3.41

3.44

3.52

Fundamentals of Electric Circuits 2021.3

Chapter 4
Circuit Theorems

Chapter 4 Circuit Theorems

4.1 Motivation

4.2 Linearity Property

4.3 Superposition

4.4 Source Transformation

4.5 Substitution theorem

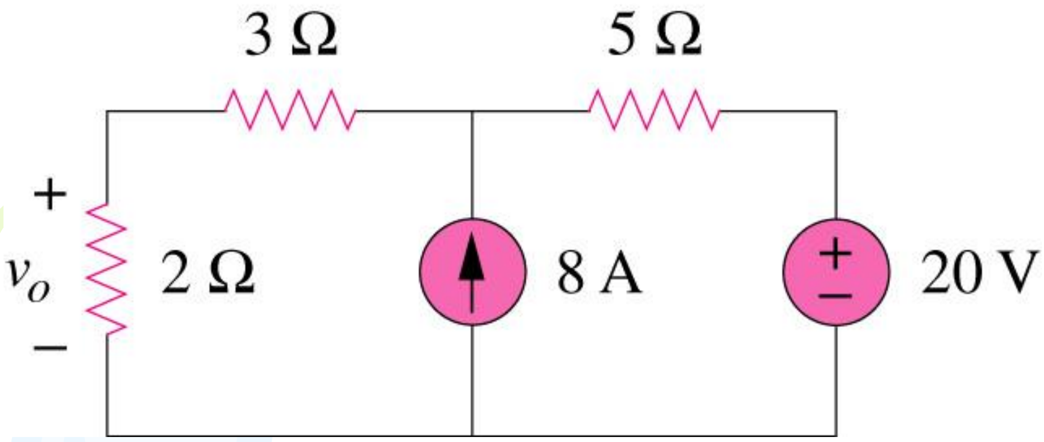
4.6 Simplification of a one-port network

4.7 Thevenin's Theorem

4.8 Norton's Theorem

4.9 Maximum Power Transfer

4.1 Motivation



If you are required to determine the voltage across 2Ω resistor, we can first replace the two resistors with only one resistor to simplify the circuit.

Are there any other method to simplify circuit analysis ?

What are they? And how?

Can you work it out by inspection?

4.1 Motivation

- Some theorems are developed to simplify circuit analysis. Such theorems include Thevenin's theorems and Norton's theorems.
- Since these theorems are applicable to linear circuits, we first discuss the concept of circuit linearity.
- In addition to circuit theorems, we discuss the concepts of superposition, source transformation, and maximum power transfer in this chapter.

4.2 Linearity Property

- **linear circuit**

Linear element: Linear relationship

A linear circuit: one circuit whose output is linearly related (or directly proportional) to its input.

Linearity is a combination of both the homogeneity (scaling) property and the additivity property.

4.2 Linearity Property

The homogeneity property: if the input (also called the *excitation*) is multiplied by a constant, then the output (also called the *response*) is multiplied by the same constant.

For a resistor, for example, Ohm's law relates the input i to the output v.

$$v = iR$$

If the current is increased by a constant k, then the voltage increases correspondingly by k, that is,

Homogeneity (scaling) property

$$v = i R \quad \longrightarrow \quad k v = k i R$$

4.2 Linearity Property

The additivity property: the response to a sum of inputs is the sum of the responses to each input applied separately.

Using the voltage-current relationship of a resistor, if

$$v_1 = i_1R \qquad\qquad v_2 = i_2R$$

then applying $(i_1 + i_2)$ gives

Additive property

$$\longrightarrow \; v = (i_1 + i_2)\,R = i_1R + i_2R = v_1 + v_2$$

A linear circuit is one whose output is linearly related (or directly proportional) to its input.

Consider the linear circuit shown in Fig. 4.1.

Suppose v_s = 10 V gives i = 2 A. According to the linearity principle, v_s = 20 V will give i = 4 A.

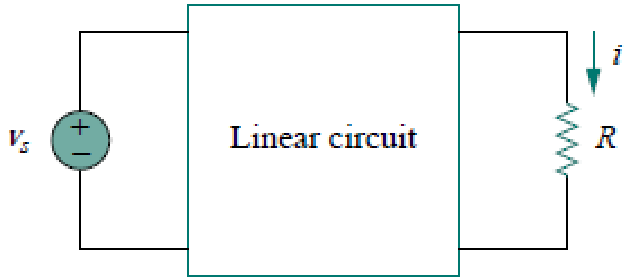
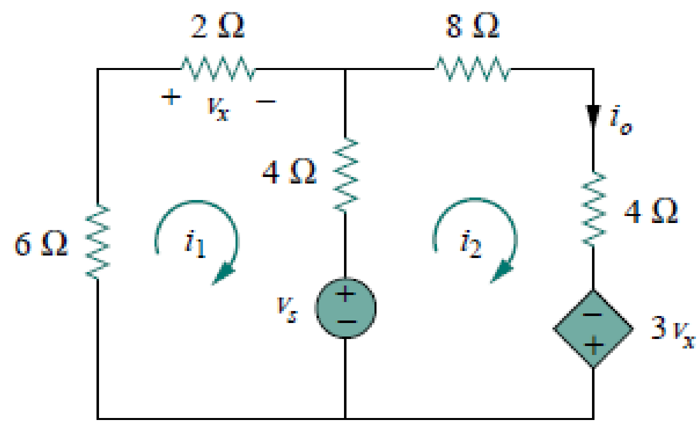


Figure 4.1 A linear circuit with input v_s and output i.

EXAMPLE 4.1

For the circuit in Fig. 4.2, find i_o when $v_s = 12$ V and $v_s = 24$ V.



Solution:

Applying KVL to the two loops, we obtain

$$12i_1 - 4i_2 + v_s = 0 \tag{4.1.1}$$

$$-4i_1 + 16i_2 - 3v_x - v_s = 0 \tag{4.1.2}$$

Controlling equation $v_x = 2i_1$.

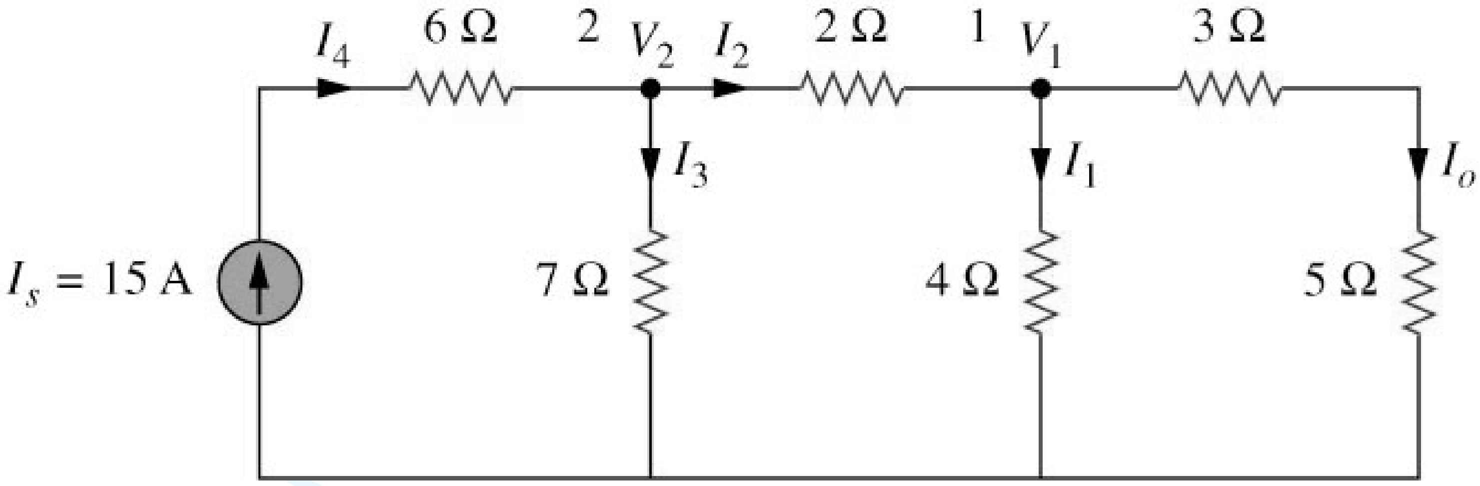
When $v_s = 12$ V, $\quad i_o = i_2 = \dfrac{12}{76}$ A

When $v_s = 24$ V, $\quad i_o = i_2 = \dfrac{24}{76}$ A

4.2 Linearity Property (2)

Example 4.2

Use linearity to find the actual value of I_o in the circuit shown below.



By assume $I_o = 1$ A,

Answer $I_o = 3\text{A}$

PRACTICE PROBLEM 4.1

For the circuit in Fig. 4.3, find v_o when $i_s = 15$ and $i_s = 30$ A.

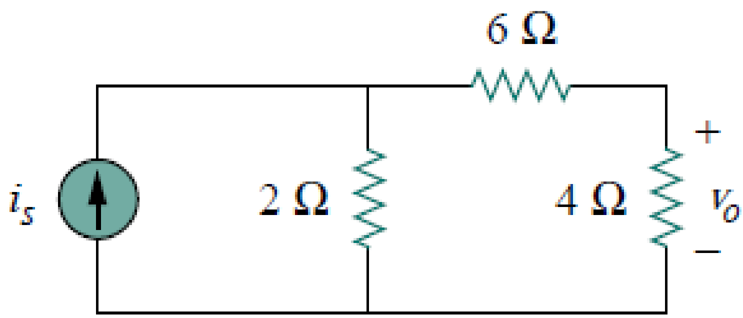


Figure 4.3 For Practice Prob. 4.1.

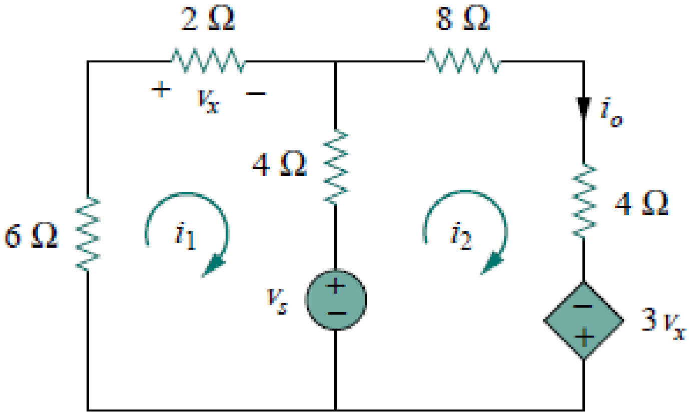
Answer: 10 V, 20 V.

Summary of last class

A linear circuit: one circuit whose output is **linearly related** (or directly proportional) to its input.

Linearity is a combination of both the **homogeneity (scaling) property and the additivity property.**

The homogeneity property: if the input (also called the *excitation*) is multiplied by a constant, then the output (also called the *response*) is multiplied by the same constant.



When $v_s = 12$ V, $\quad i_o = i_2 = \dfrac{12}{76}$ A

When $v_s = 24$ V, $\quad i_o = i_2 = \dfrac{24}{76}$ A

The additivity property: the response to a sum of inputs is the sum of the responses to each input applied separately.

4.3 Superposition Theorem

If a circuit has two or more independent sources:

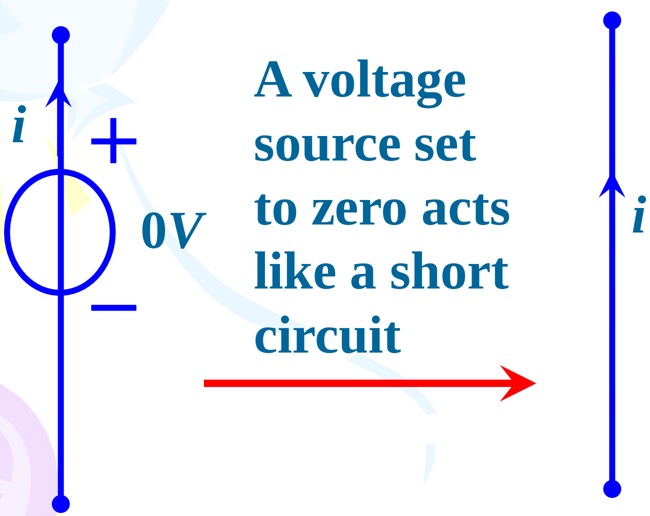
Method 1: Nodal or mesh analysis method.

Method 2: Superposition.

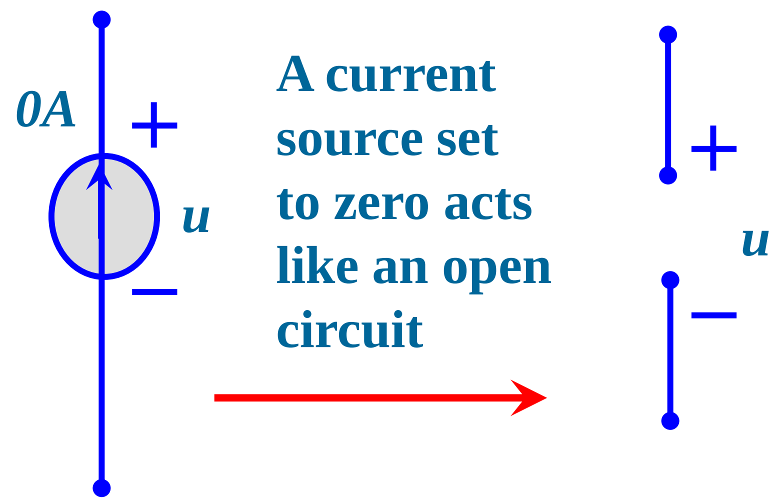
Superposition: If a linear circuit with more than one independent source, its response is the algebraic sum of the contribution of each independent source acting alone.

4.3 Superposition Theorem

1. One independent source acting alone, means all other independent sources are turned off. This implies that we replace
   - every voltage source by 0 V (or a short circuit),
   - every current source by 0 A (or an open circuit).



"turned off" or "zeroed out",



"turned off" or "zeroed out",

2. Dependent sources are left.

4.3 Superposition Theorem (3)

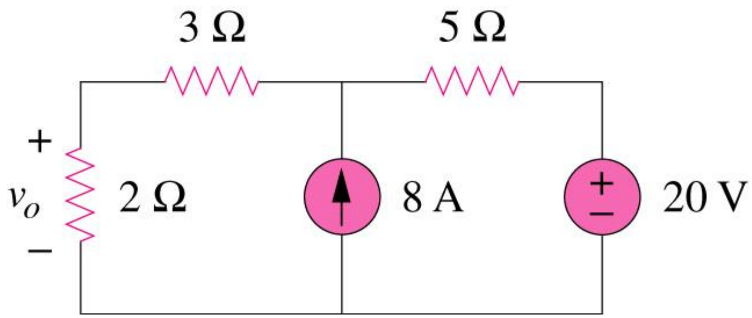
Steps to apply superposition principle

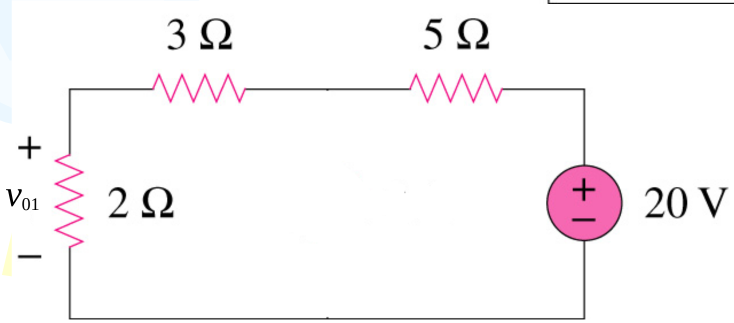
1. Let one independent source acting alone, and find the output (voltage or current) due to that active source using nodal or mesh analysis.
2. ***Repeat step 1*** for each of the other independent sources.
3. ***Find*** the total contribution by adding ***algebraically*** all the contributions.

Pay attention to the reference direction.

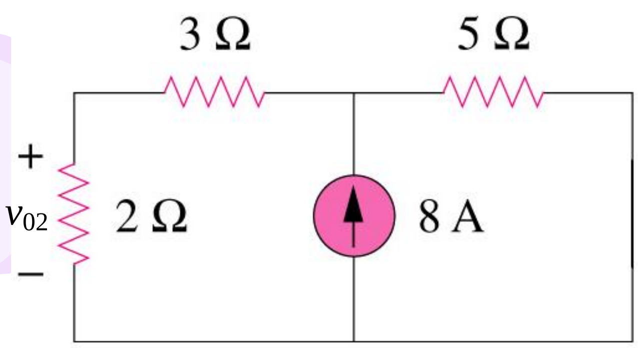
4.3 Superposition Theorem

We consider the effects of 8A and 20V one by one, then add the two effects together for final v_o.





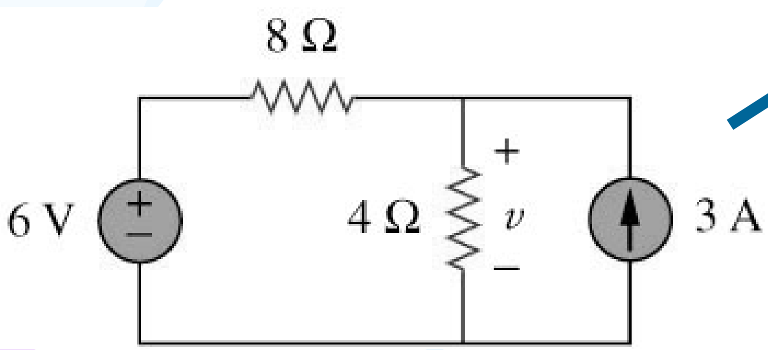
(a)



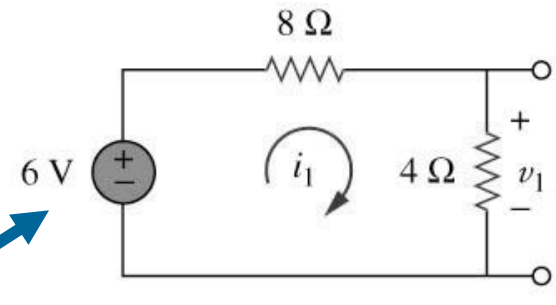
(b)

4.3 Superposition Theorem (5)

Example 2 Use the superposition theorem to find *v* in the circuit shown below.

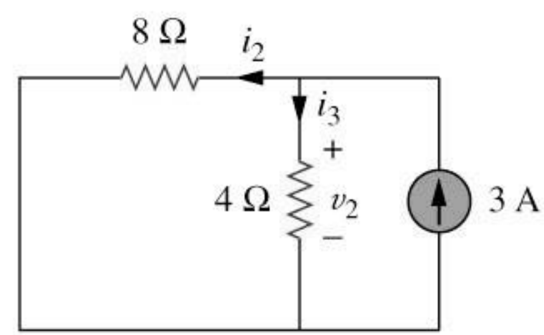


3A is discarded by open-circuit



(a)

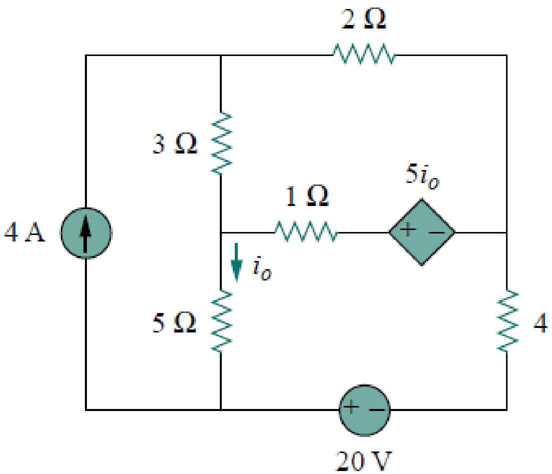
6V is discarded by short-circuit

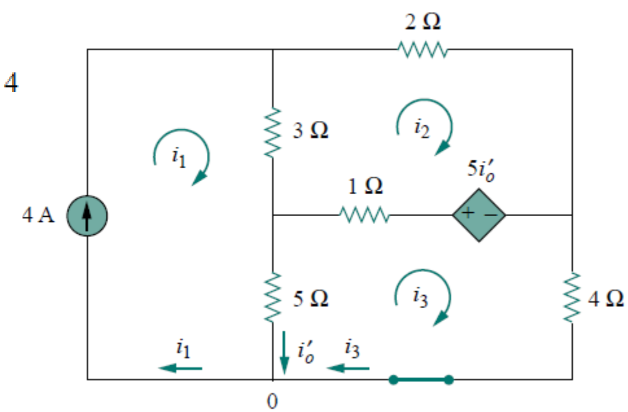


(b)

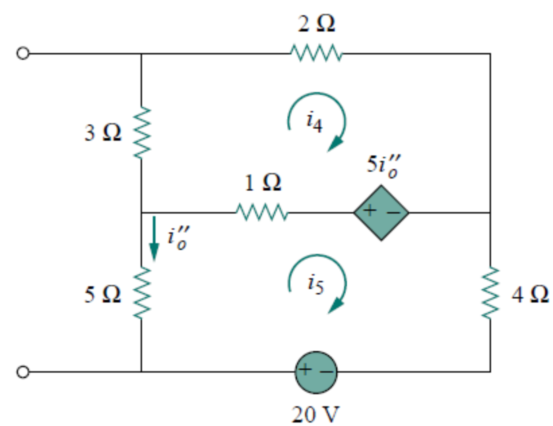
Answer v = 10V

Example 4 Use superposition to find i_0 in the circuit below.





(a)

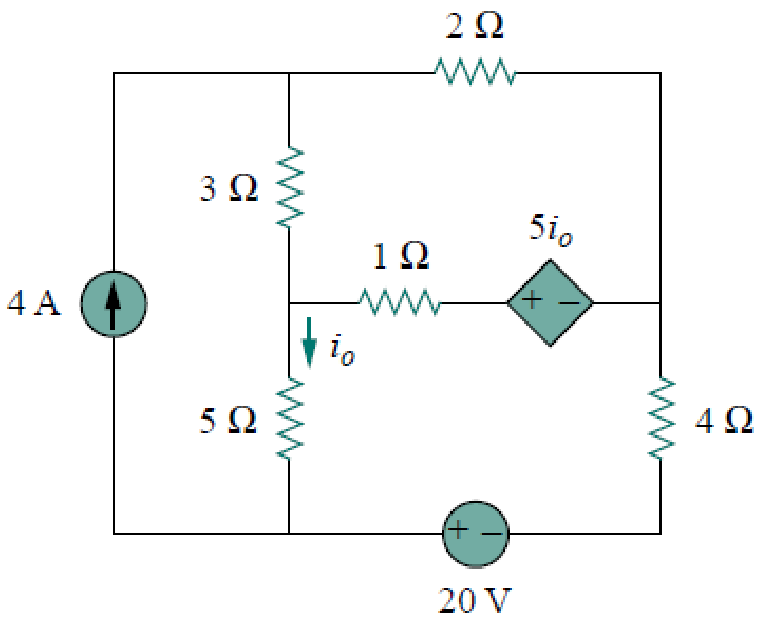


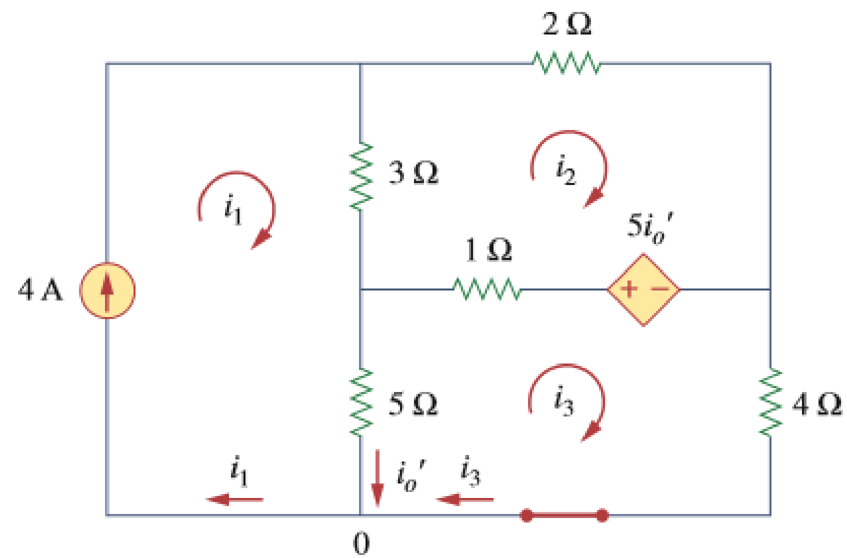
(b)

The circuit involves a dependent source, which must be left intact. We let.

$$i_o = i_o' + i_o'' \tag{4.4.1}$$

where i_o' and i_o'' are due to the 4-A current source and 20-V voltage source





To obtain i_o', we turn off the 20-V source so that we have the circuit in Fig. 4.10(a). We apply mesh analysis in order to obtain i_o'.

For loop 1,

$$i_1 = 4 \text{ A} \tag{4.4.2}$$

For loop 2,

$$-3i_1 + 6i_2 - 1i_3 - 5i_o' = 0 \tag{4.4.3}$$

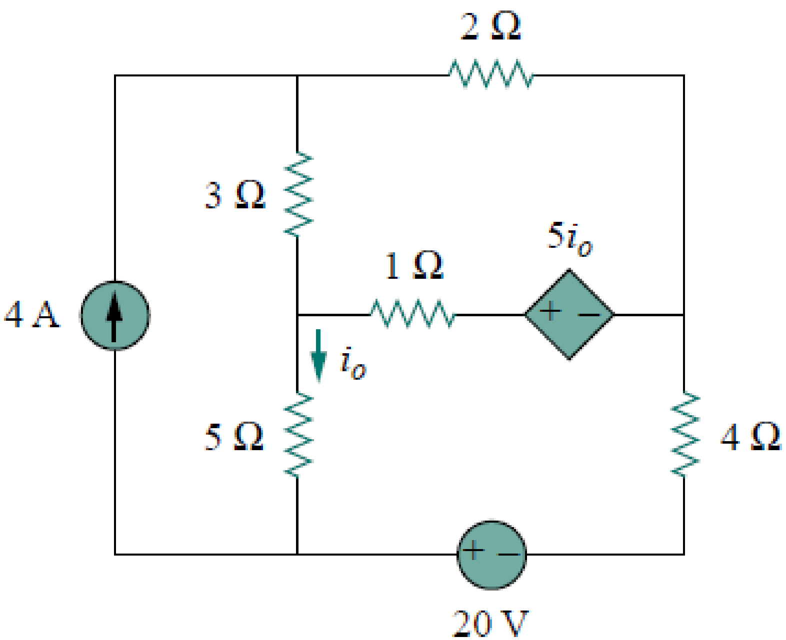
$$i_o' = \frac{52}{17} \text{ A}$$

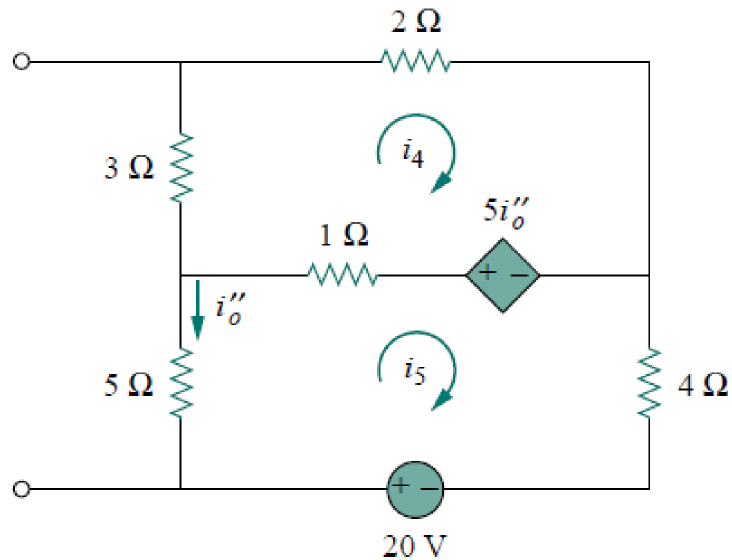
For loop 3,

$$-5i_1 - 1i_2 + 10i_3 + 5i_o' = 0 \tag{4.4.4}$$

But at node 0,

$$i_3 = i_1 - i_o' = 4 - i_o' \tag{4.4.5}$$





(b)

To obtain i_o'', we turn off the 4-A current source so that we have the circuit in Fig. 4.10(b). We apply mesh analysis in order to obtain i_o''.

For loop 4, KVL gives $6i_4 - i_5 - 5i_o'' = 0$ (4.4.9)

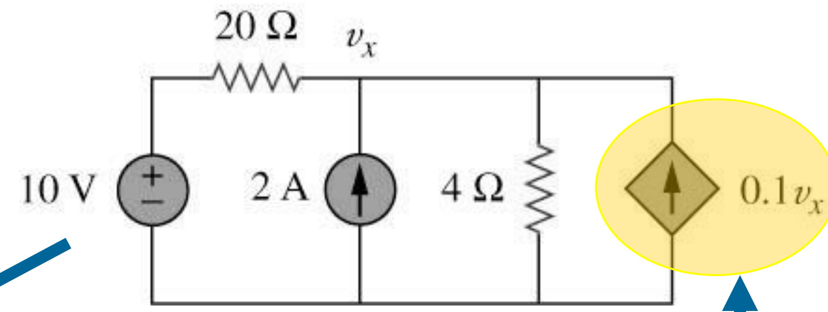
For loop 5, KVL gives $-i_4 + 10i_5 - 20 + 5i_o'' = 0$ (4.4.10)

But $i_5 = -i_o''$.

$$i_o'' = -\frac{60}{17}\text{ A} \qquad i_o' = \frac{52}{17}\text{ A} \qquad i_o = -\frac{8}{17} = -0.4706\text{ A}$$

Example 6

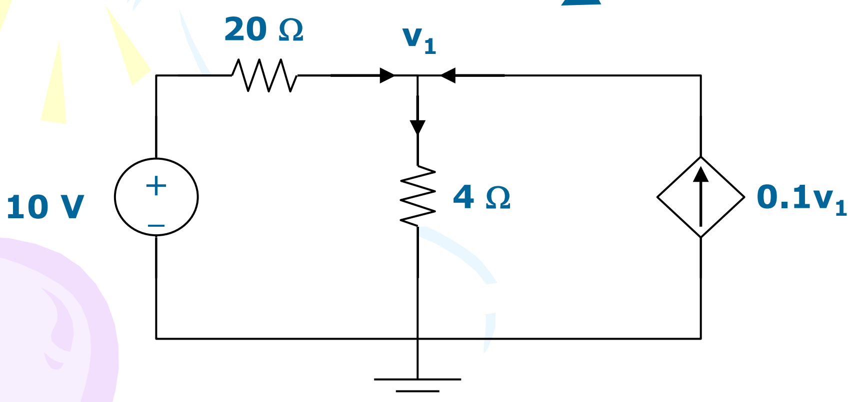
Use superposition to find v_x in the circuit below.



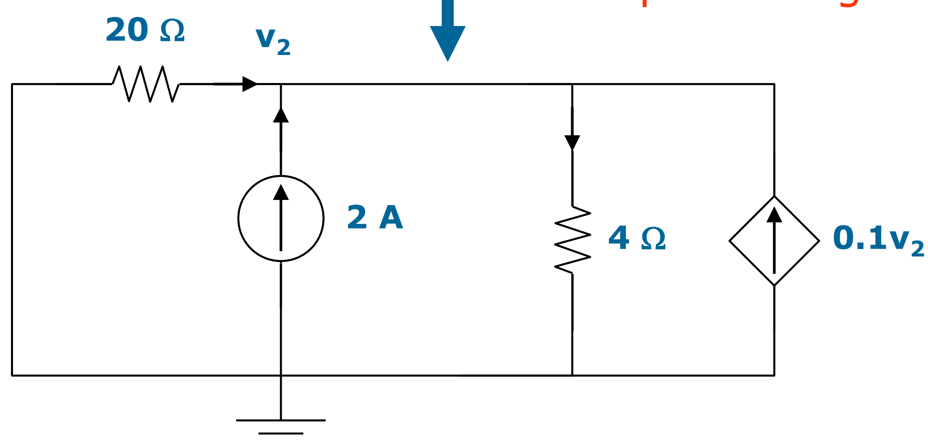
2A is discarded by open-circuit

10V is discarded by short-circuit

Dependant source keep unchanged



(a)



(b)

Answer Vx = 12.5V

For the circuit in Fig. 4.12, use the superposition theorem to find i.

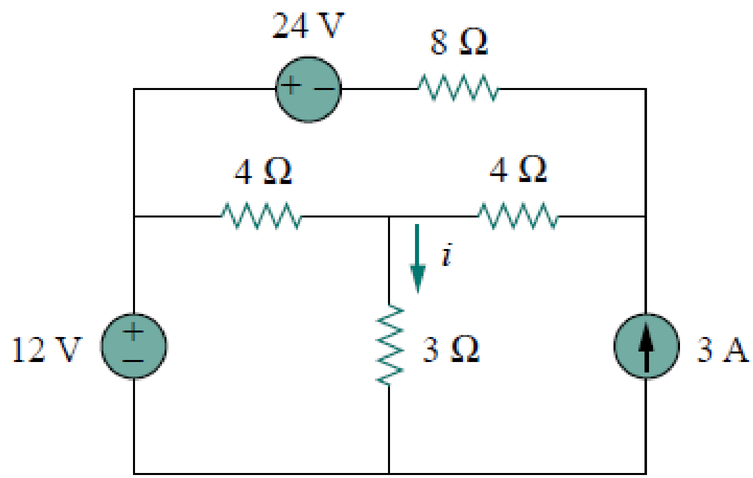
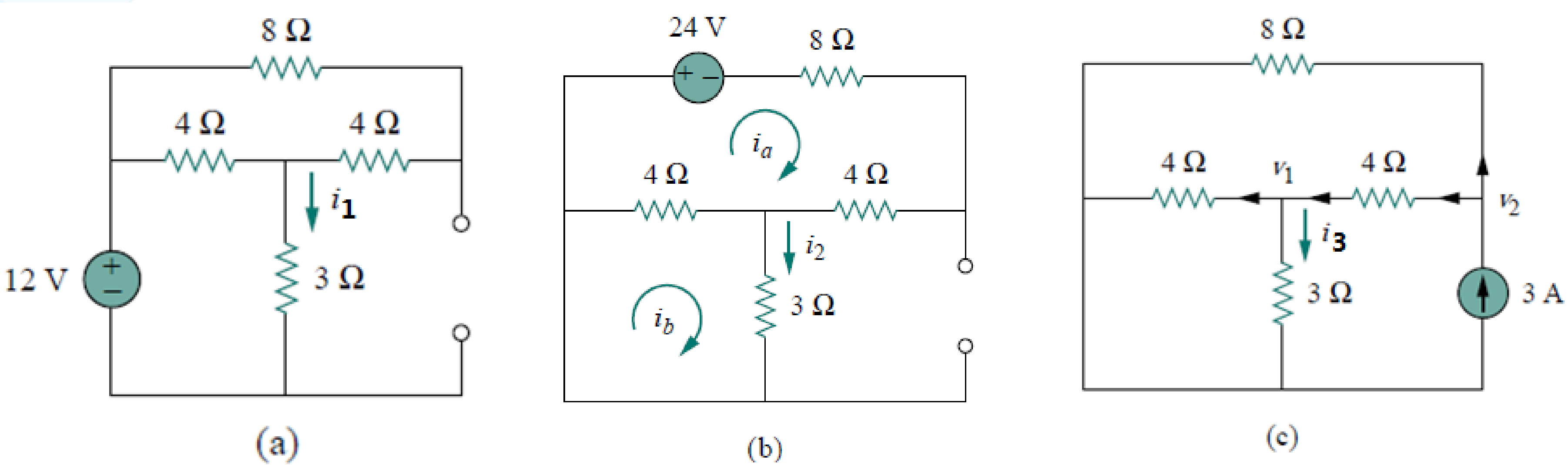
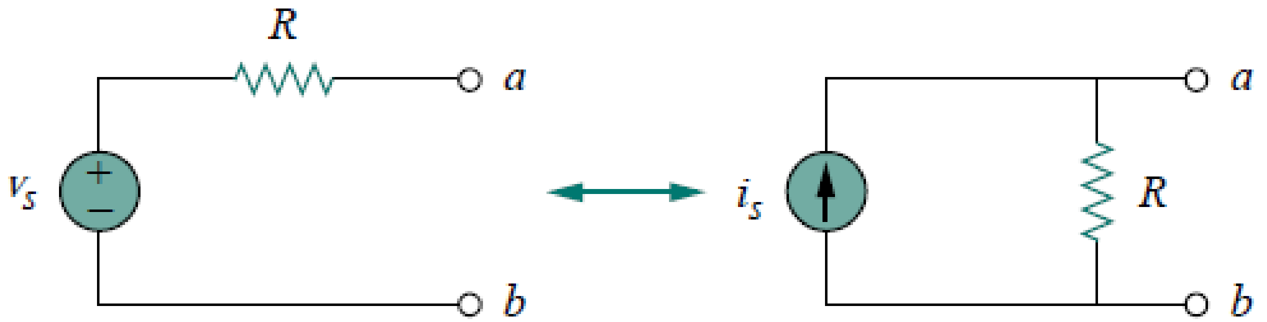


Figure 4.12 For Example 4.5.

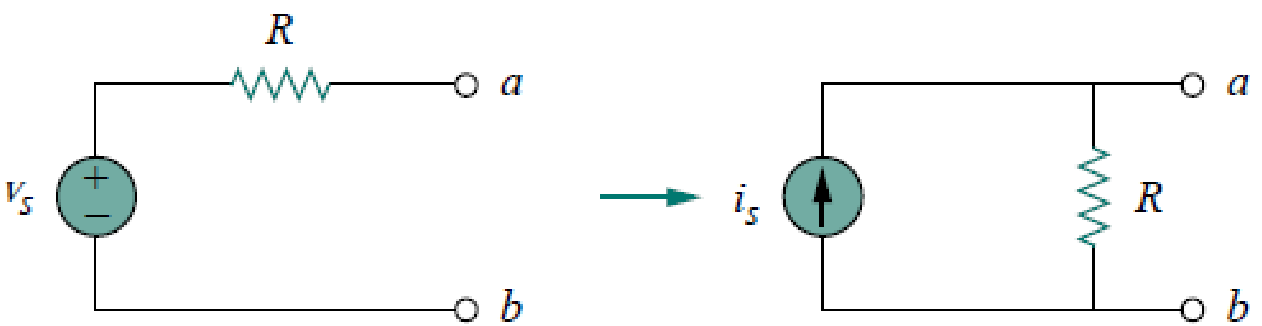


4.4 Source Transformation

- Source transformation is another tool for simplifying circuits.
- It is the process of replacing ***a voltage source v_s in series with a resistor R*** **by a** ***current source i_s in parallel with a resistor*** *R*, or vice versa.



4.4 Source Transformation (1)



(a) (b)

For (a): (b)

Steps to replace (a) by (b):

1.Structure:

before : **a *voltage source v_S in series with a resistor R***

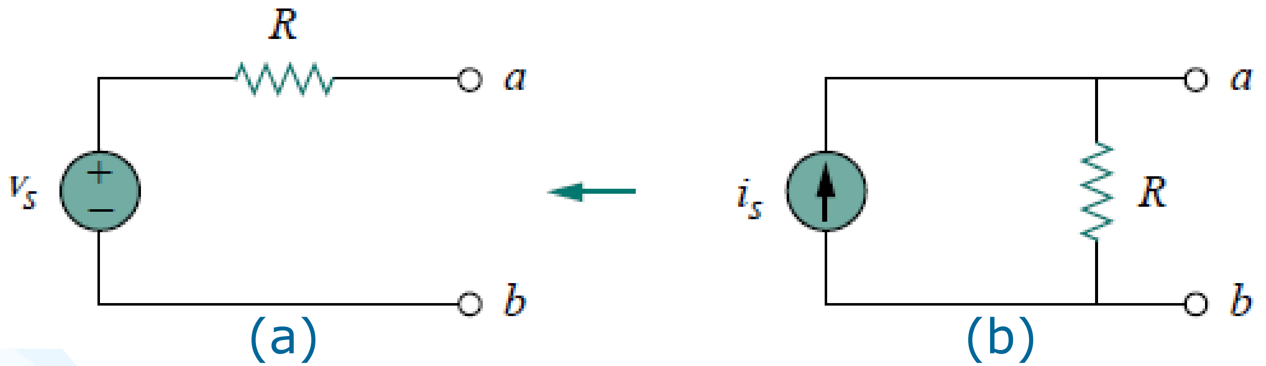
Now: **a *current source i_S in parallel with a resistor R***

2. Value: R the same, $i_s = \dfrac{v_s}{R}$

3. Source direction:

the '-' to '+' direction of voltage source= current source direction.

4.4 Source Transformation (1)



Steps to replace (b) by (a):

1.Structure:

before : **a *current source i_s in parallel with a resistor* *R***

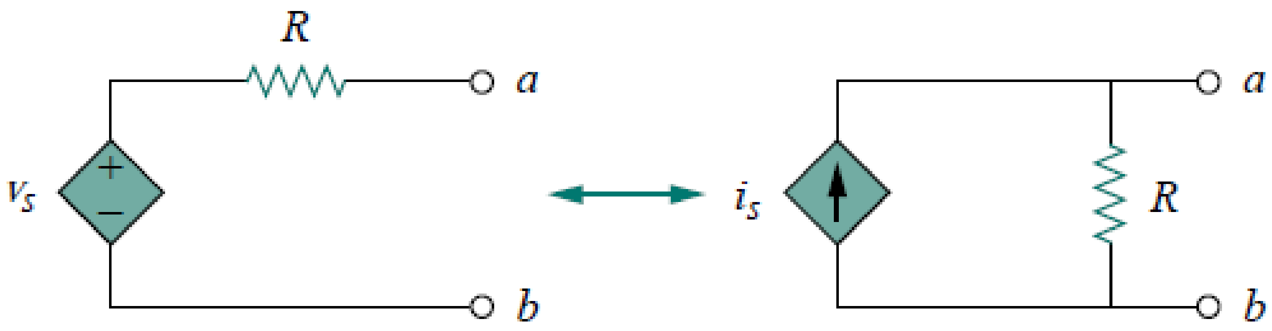
Now: **a *voltage source v_s in series with a resistor R***

2. Value: R the same, $v_s = i_s R$

3. Source direction:

current source direction= the '-' to '+' direction of voltage source.

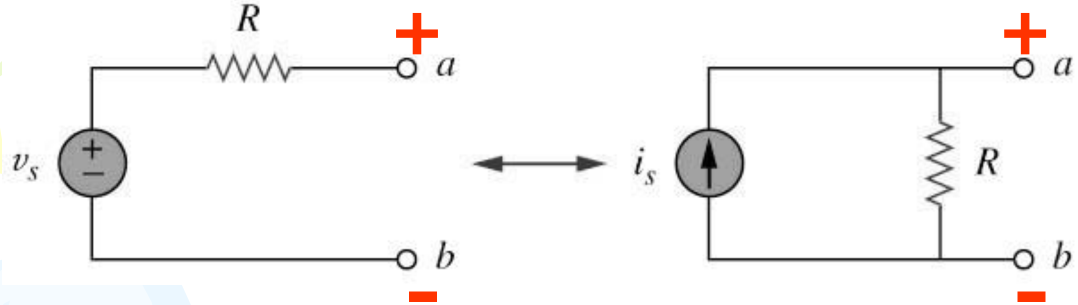
4.4 Source Transformation (1)



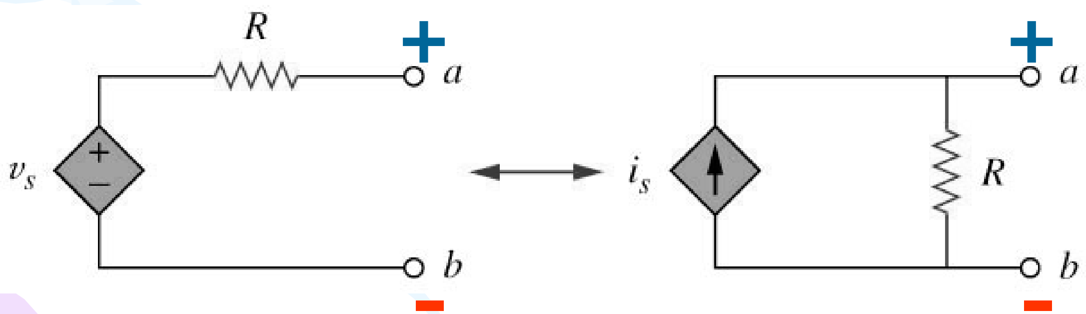
$$v_s = i_s R$$

$$i_s = \frac{v_s}{R}$$

4.4 Source Transformation (2)



(a) Independent source transform

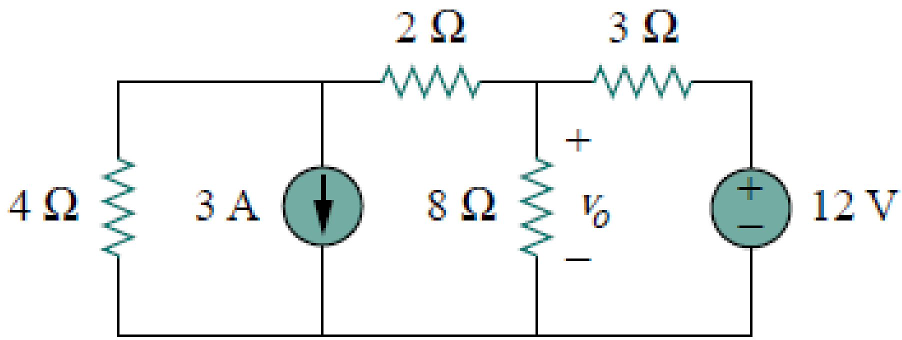


(b) Dependent source transform

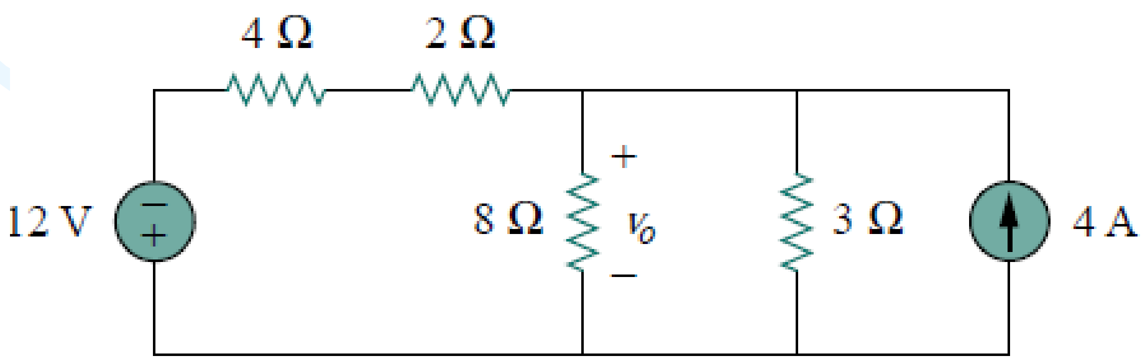
- The arrow of the current source is directed toward the positive terminal of the voltage source.
- The source transformation is not possible when R = 0 for voltage source and R = ∞ for current source.

Example 4.6

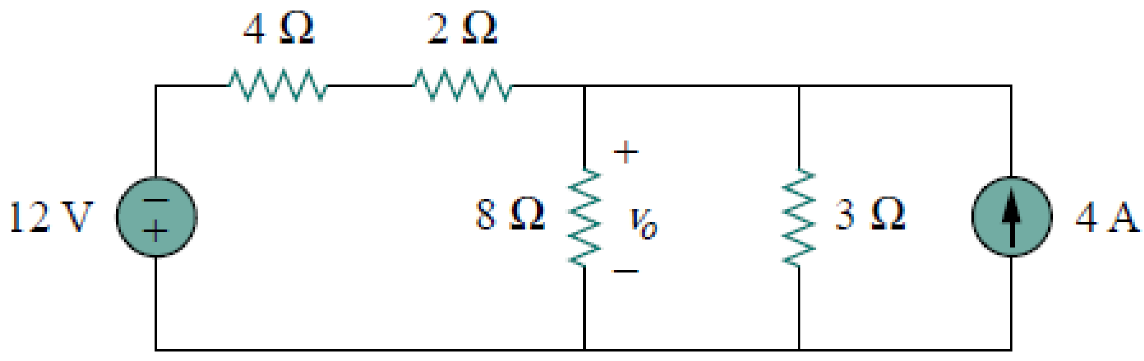
Find v_o in the circuit shown below using source transformation.



We first transform the current and voltage sources to obtain the circuit in Fig. 4.18(a).

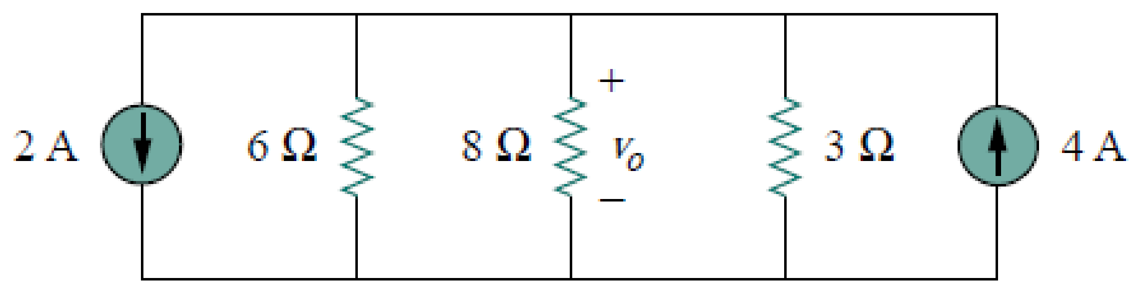


(a)

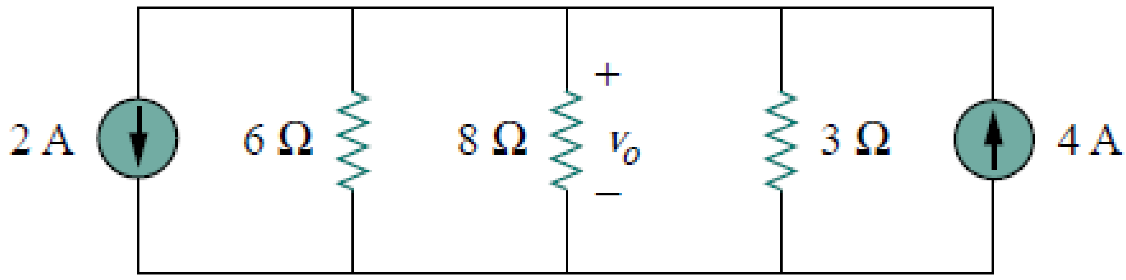


(a)

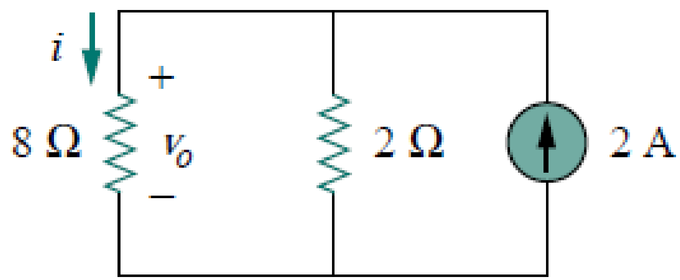
Combining the 4-Ω and 2- Ω resistors in series and transforming the 12-V voltage source gives us Fig. 4.18(b).



(b)



(b)



(c)

We use current division in Fig. 4.18(c) to get

$$i = \frac{2}{2+8}(2) = 0.4$$

$$v_o = 8i = 8(0.4) = 3.2 \text{ V}$$

EXAMPLE 4.7

Find v_x in Fig. 4.20 using source transformation.

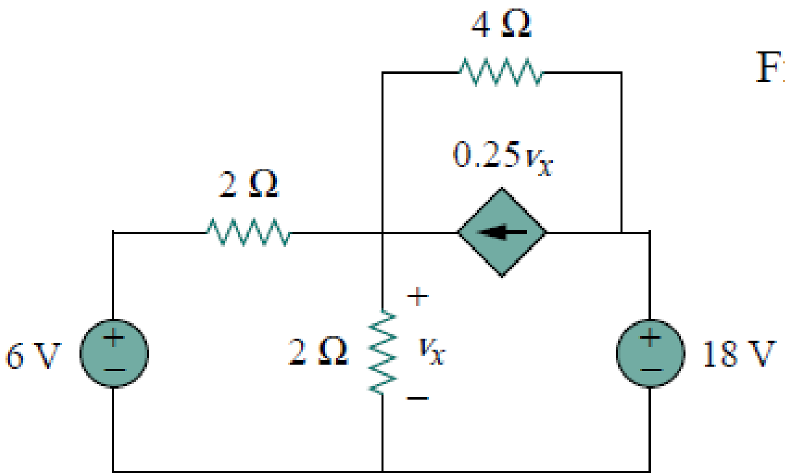
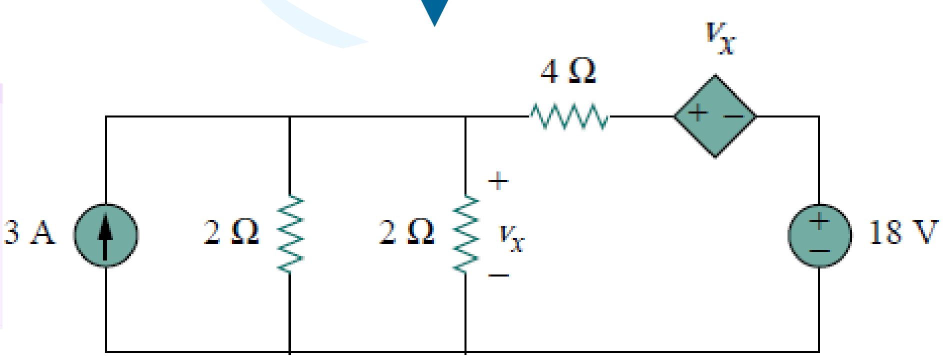
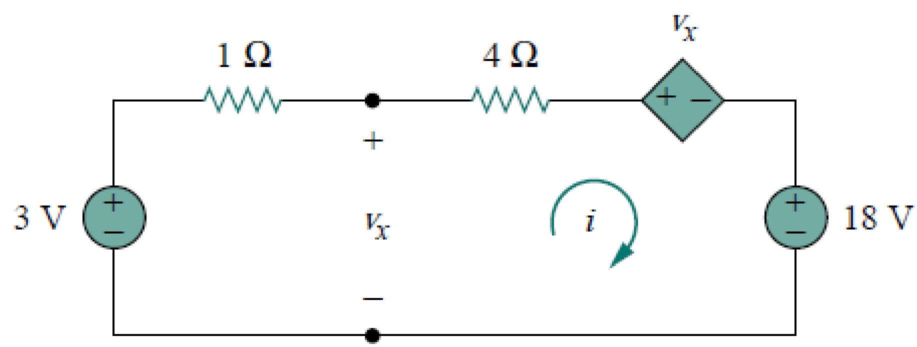


Figure 4.20 For Example 4.7.





$$-3 + 5i + v_x + 18 = 0$$

$$-3 + 1i + v_x = 0$$

$$i = -4.5\ \text{A}$$

PRACTICE PROBLEM 4.7

Use source transformation to find i_x in the circuit shown in Fig. 4.22.

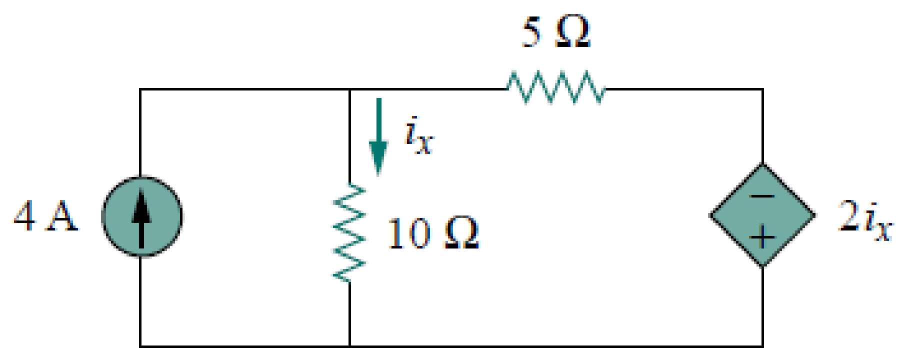


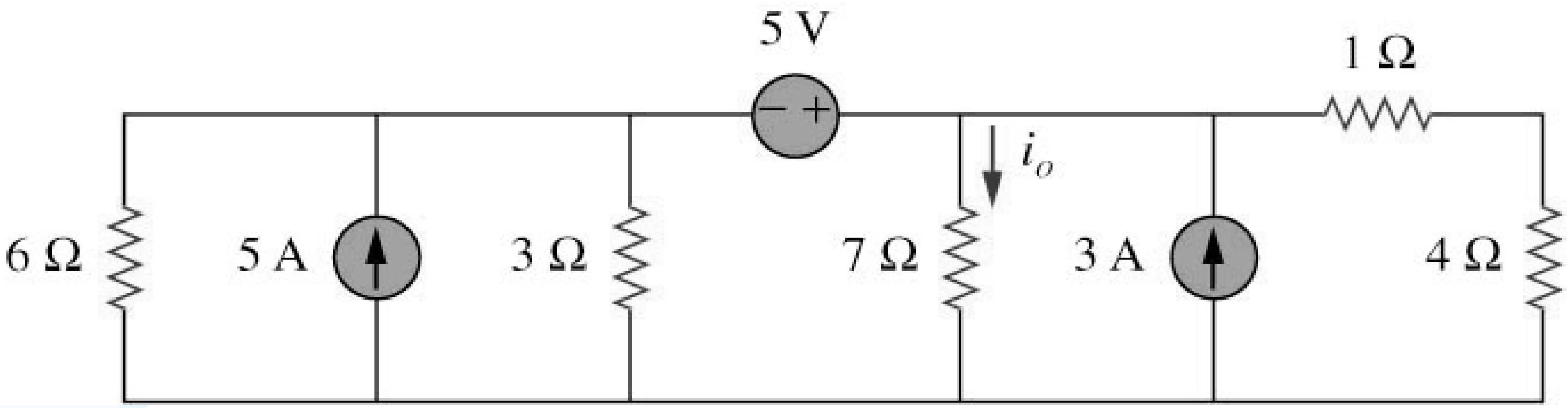
Figure 4.22 For Practice Prob. 4.7.

Answer: 1.176 A.

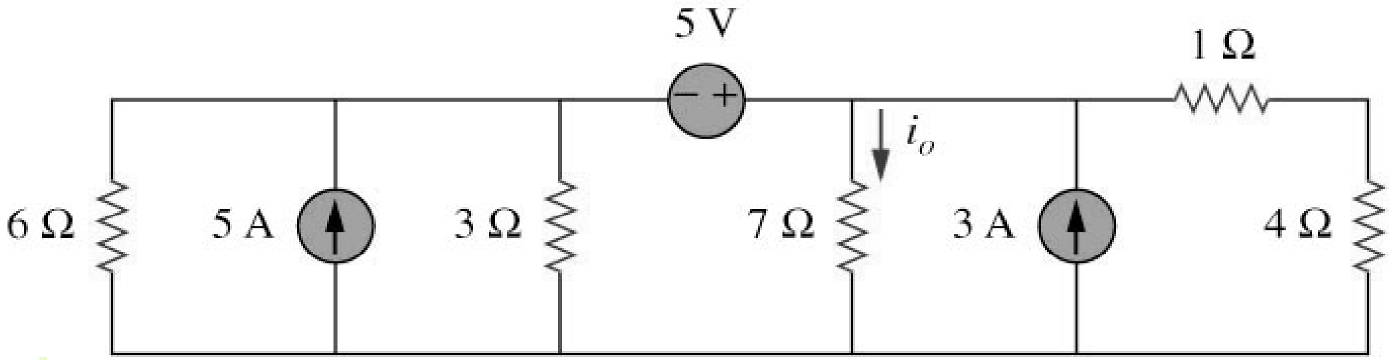
Practice 4

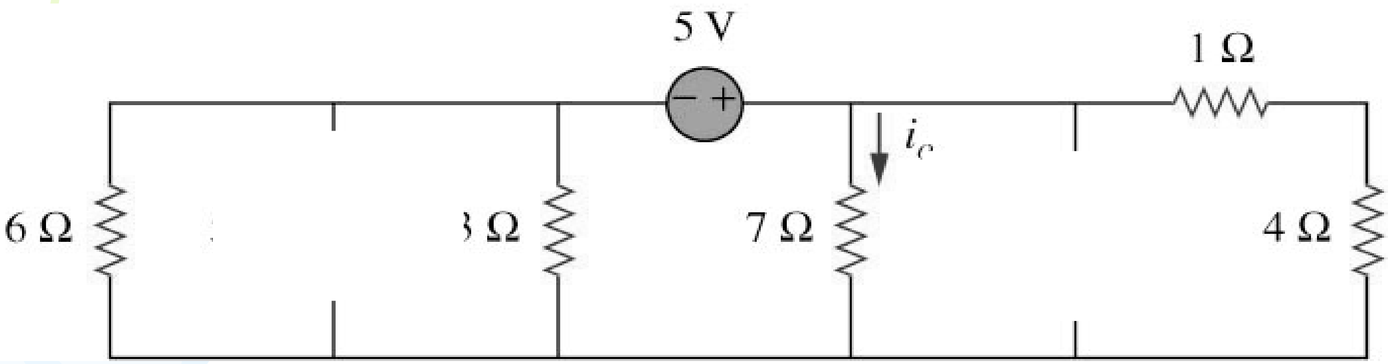
i_0 = 1.78A

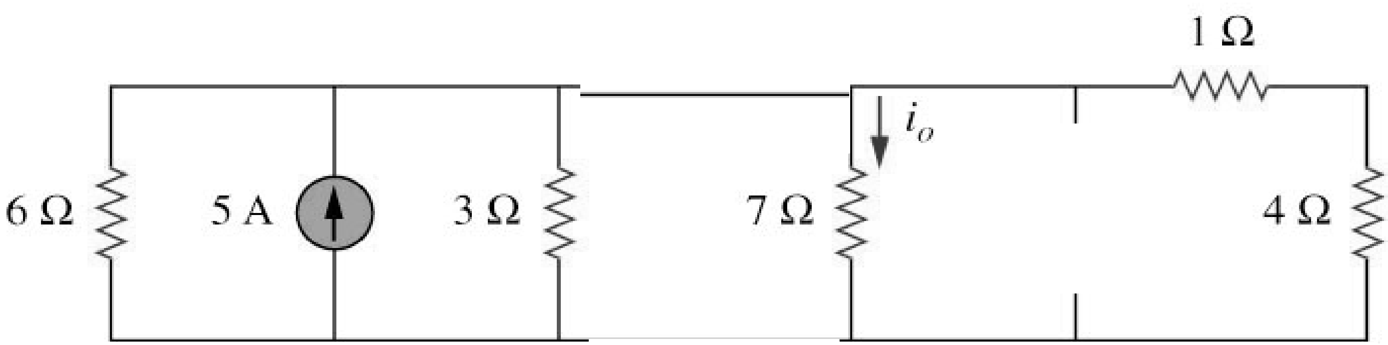
Find i_o in the circuit shown below using superposition theorem and source transformation.

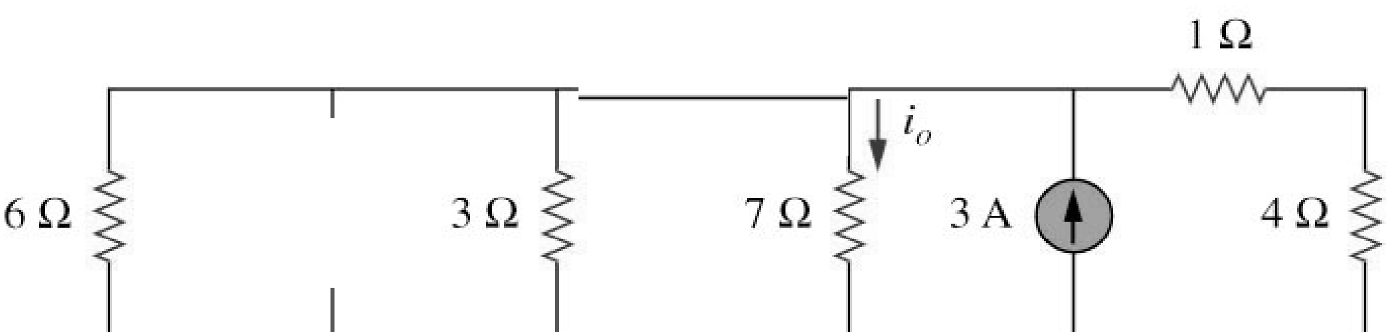


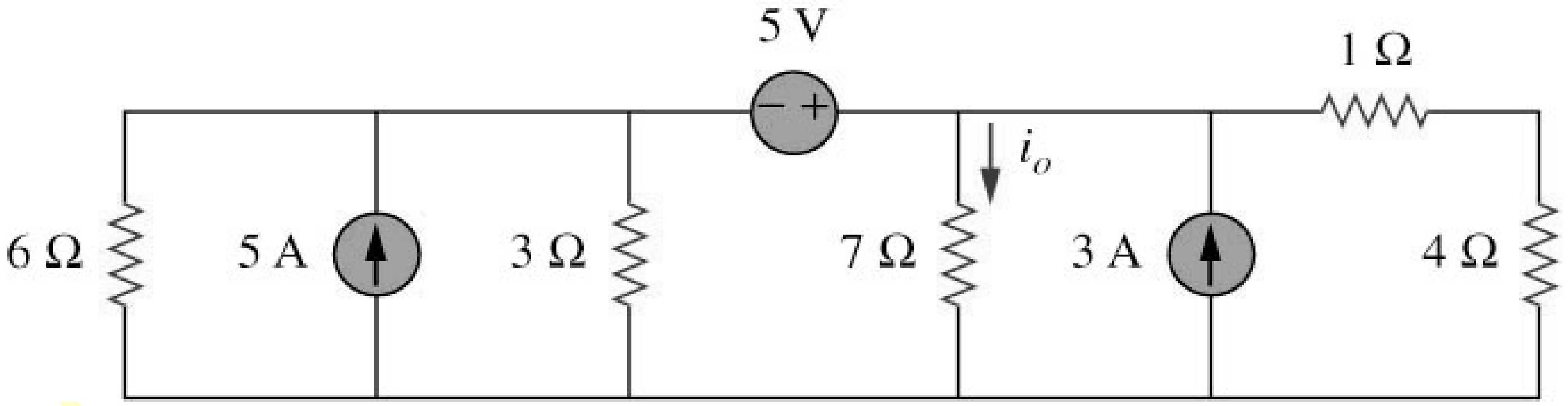
Method 1: superposition theorem


5 V
1 Ω
i_o
6 Ω
5 A
3 Ω
7 Ω
3 A
4 Ω


5 V
1 Ω
i_o
6 Ω
3 Ω
7 Ω
4 Ω

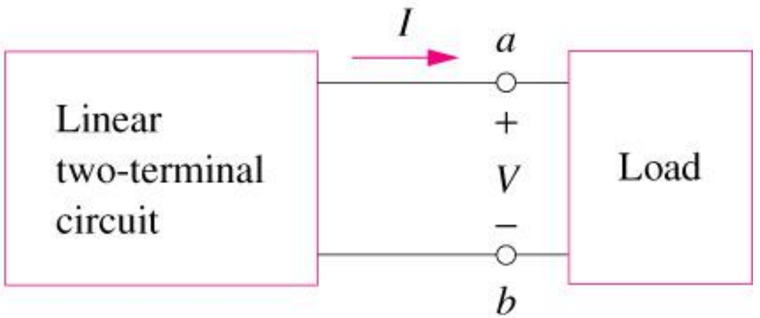

1 Ω
i_o
6 Ω
5 A
3 Ω
7 Ω
4 Ω


1 Ω
i_o
6 Ω
3 Ω
7 Ω
3 A
4 Ω

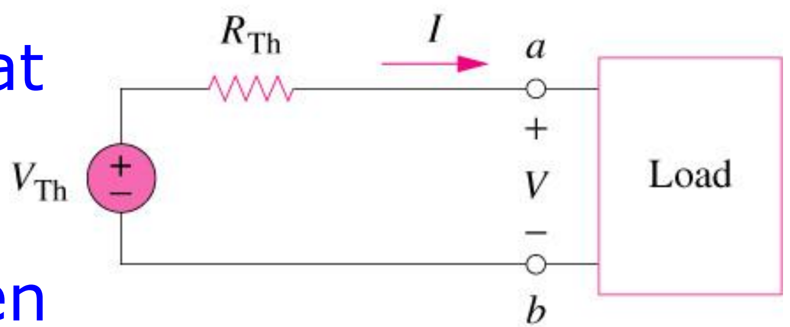

5 V
1 Ω
i_o
6 Ω
5 A
3 Ω
7 Ω
3 A
4 Ω

4.6 Thevenin's Theorem (1)

It states that a linear two-terminal circuit (Fig. a) can be replaced by an equivalent circuit (Fig. b) consisting of **a voltage source V_{TH} in series with a resistor R_{TH},** where



(a)

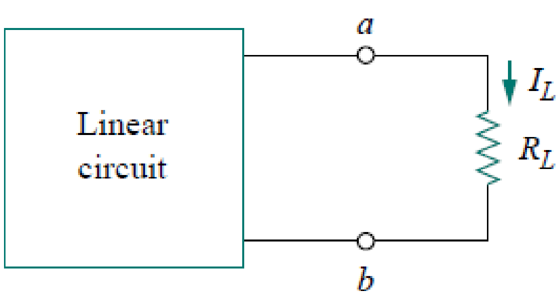


(b)

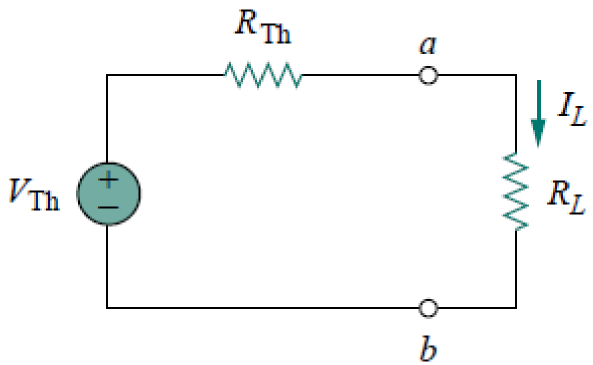
- V_{TH} is the open-circuit voltage at the terminals.
- R_{TH} is the input or equivalent resistance at the terminals when the independent sources are turned off.

4.6 Thevenin's Theorem (1)

Thevenin's theorem is very important in circuit analysis. It helps simplify a circuit. A large circuit may be replaced by a single independent voltage source and a single resistor. This replacement technique is a powerful tool in circuit design.



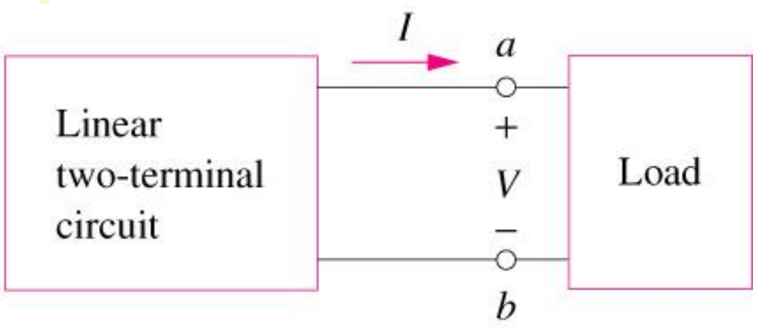
(a)



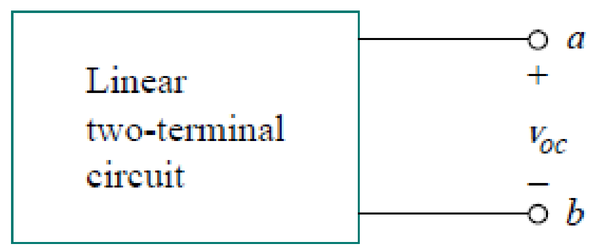
4.6 Thevenin's Theorem (1)

Finding V_{Th} and R_{Th}

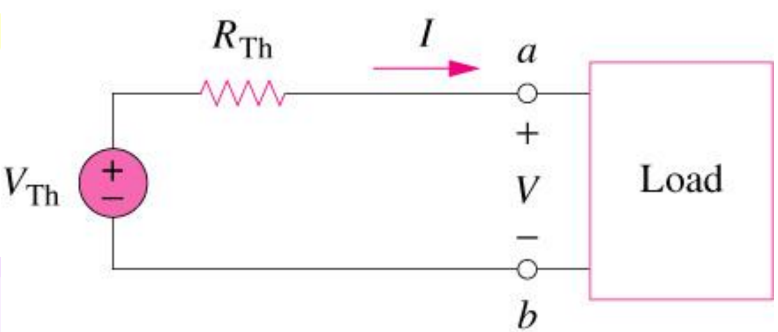
V_{Th} is the open-circuit voltage across the terminals



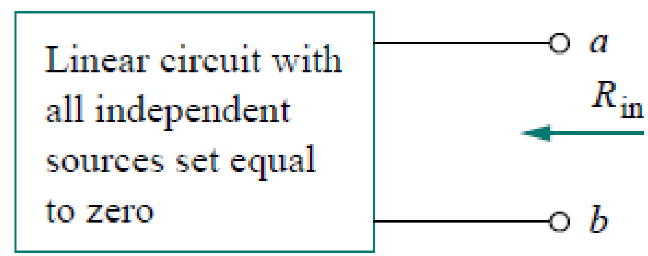
(a)



$$V_{Th} = v_{oc}$$

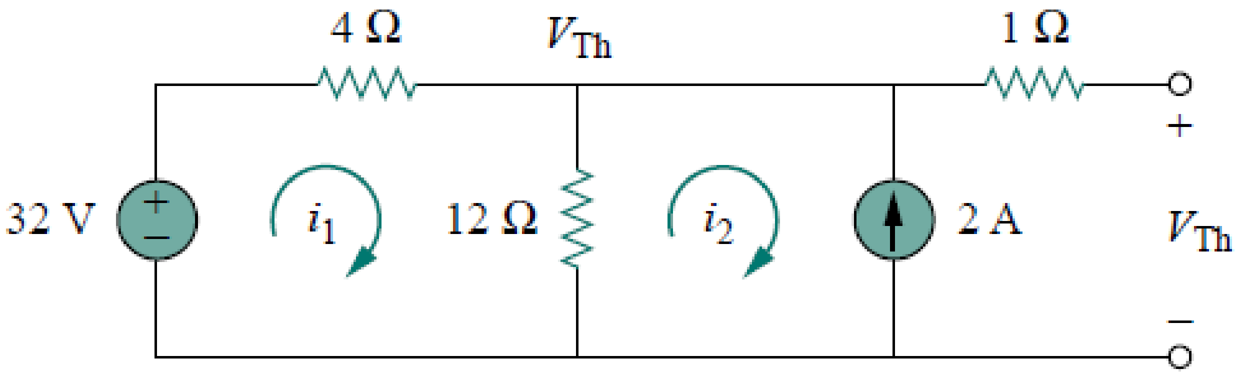


(b)



$$R_{Th} = R_{in}$$

R_{Th} is the input resistance at the terminals when the independent sources are turned off.



$$-32 + 4i_1 + 12(i_1 - i_2) = 0,$$

$$i_2 = -2\ \text{A}$$

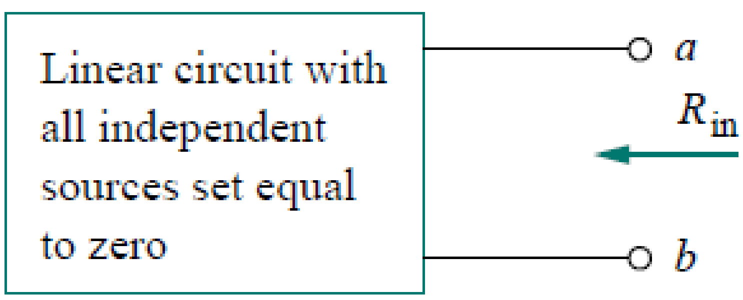
Solving for i_1, we get $i_1 = 0.5$ A. Thus,

$$V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30\ \text{V}$$

4.6 Thevenin's Theorem (1)

To apply this idea in finding the Thevenin resistance R_{Th}, we need to consider two cases.

CASE I If the network has no dependent sources, we turn off all independent sources. R_{Th} is the input resistance of the network looking between terminals a and b, as shown in Fig. 4.24(b).

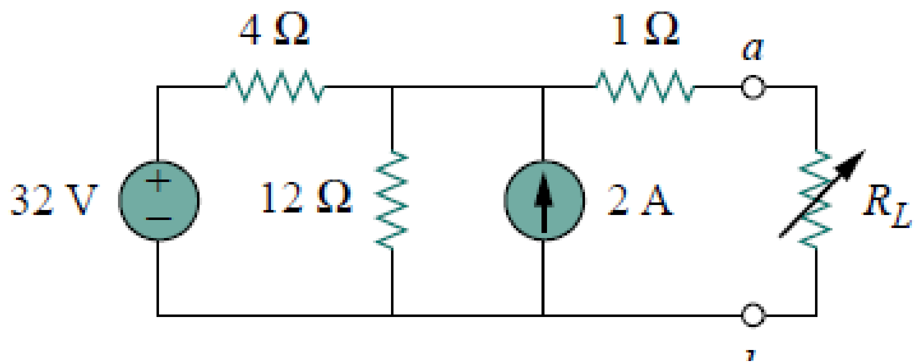


$R_{Th} = R_{in}$

(b)

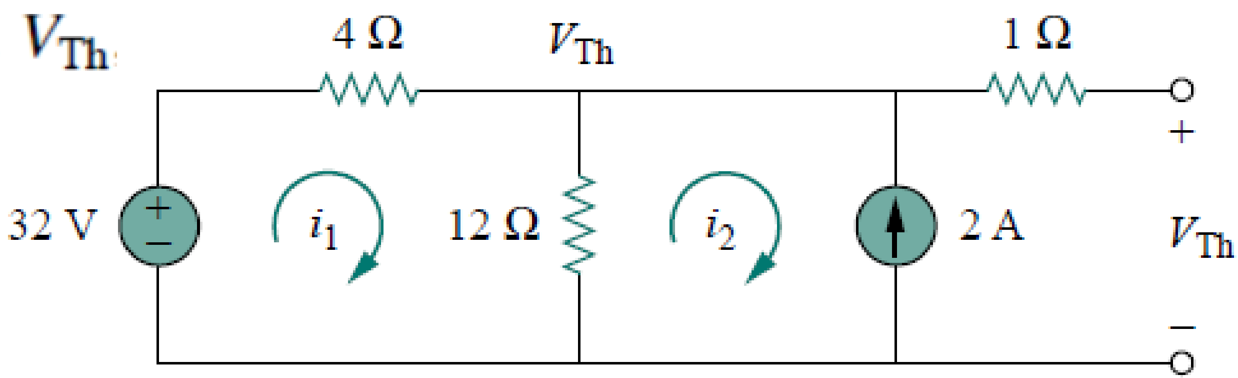
EXAMPLE 4.8

Find the Thevenin equivalent circuit of the circuit shown in Fig. 4.27, to the left of the terminals a-b. Then find the current through $R_L = 6, 16,$ and $36\ \Omega$.



Solution:

find V_{Th}:



$i_2 = -2$ A

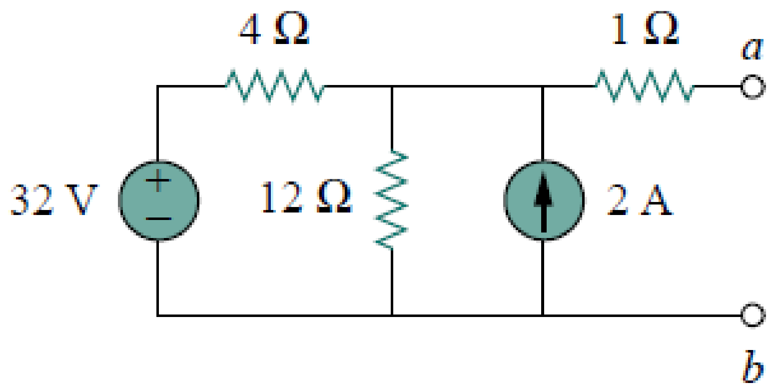
$$-32 + 4i_1 + 12(i_1 - i_2) = 0,$$

Solving for i_1, we get $i_1 = 0.5$ A. Thus,

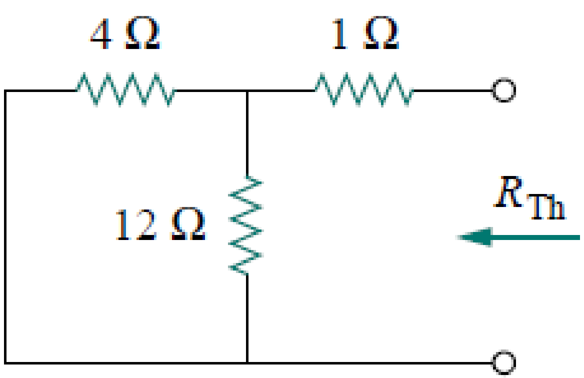
$$V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30\text{ V}$$

EXAMPLE 4.8

Find the Thevenin equivalent circuit of the circuit shown in Fig. 4.27, to the left of the terminals a-b. Then find the current through $R_L = 6, 16,$ and $36\ \Omega$.

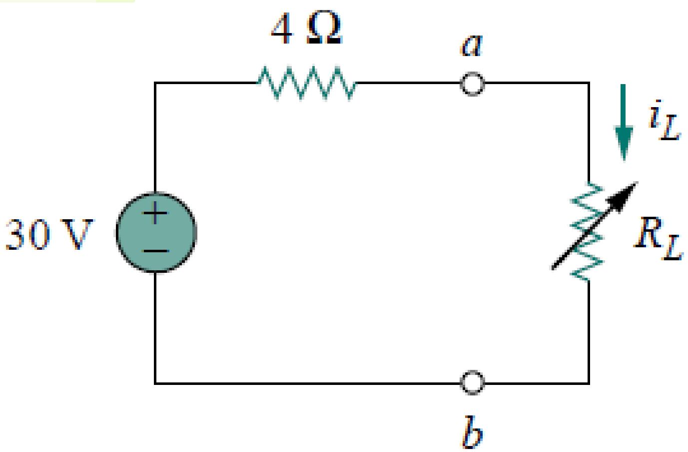


Find R_{TH}:



$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$

The Thevenin equivalent circuit is



$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L}$$

When $R_L = 6$,

$$I_L = \frac{30}{10} = 3 \text{ A}$$

When $R_L = 16$,

$$I_L = \frac{30}{20} = 1.5 \text{ A}$$

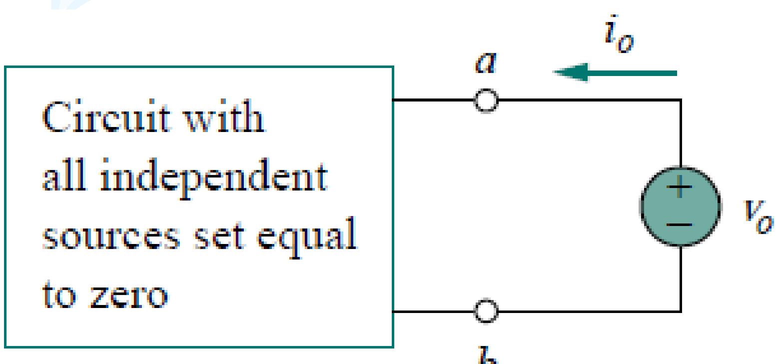
When $R_L = 36$,

$$I_L = \frac{30}{40} = 0.75 \text{ A}$$

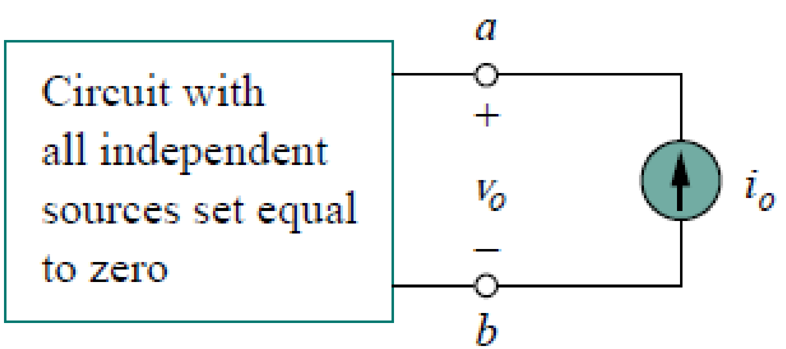
4.6 Thevenin's Theorem (1)

To apply this idea in finding the Thevenin resistance R_{Th}, we need to consider two cases.

CASE 2 If the network has dependent sources, we turn off all independent sources. As with superposition, dependent sources are not to be turned off because they are controlled by circuit variables. We apply a voltage source v_o at terminals a and b and determine the resulting current i_o. Then $R_{Th} = v_o/i_o$, as shown in Fig. 4.25(a). Alternatively, we



$$R_{Th} = \frac{v_o}{i_o}$$



$$R_{Th} = \frac{v_o}{i_o}$$

(b)

EXAMPLE 4.9

Find the Thevenin equivalent of the circuit in Fig. 4.31.

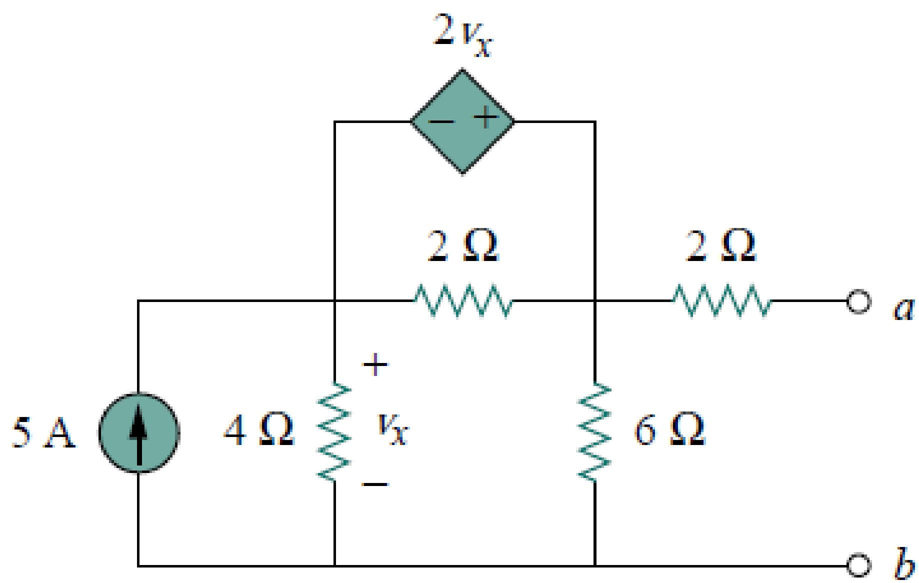
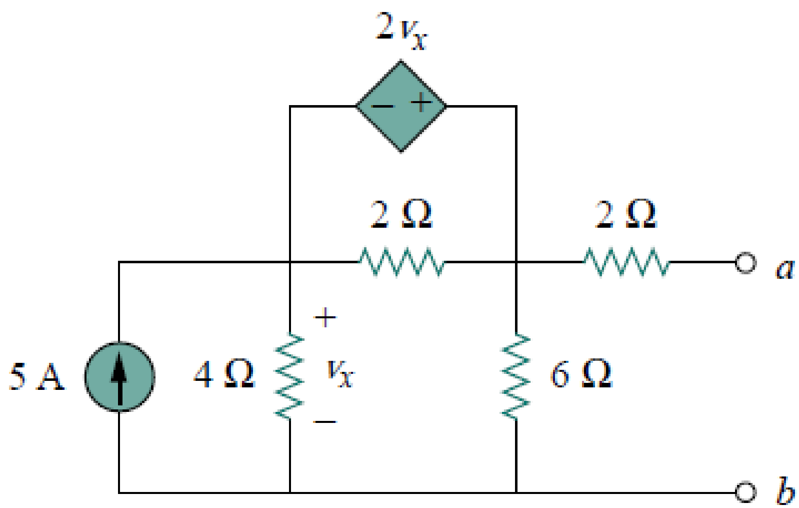


Figure 4.31 For Example 4.9.



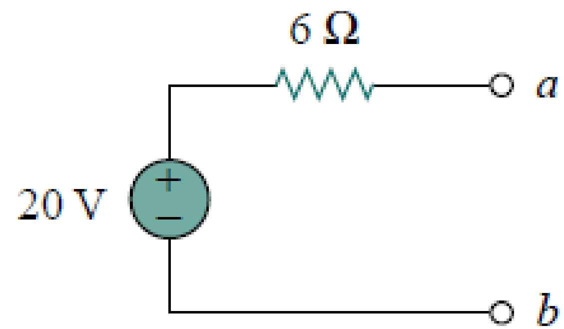
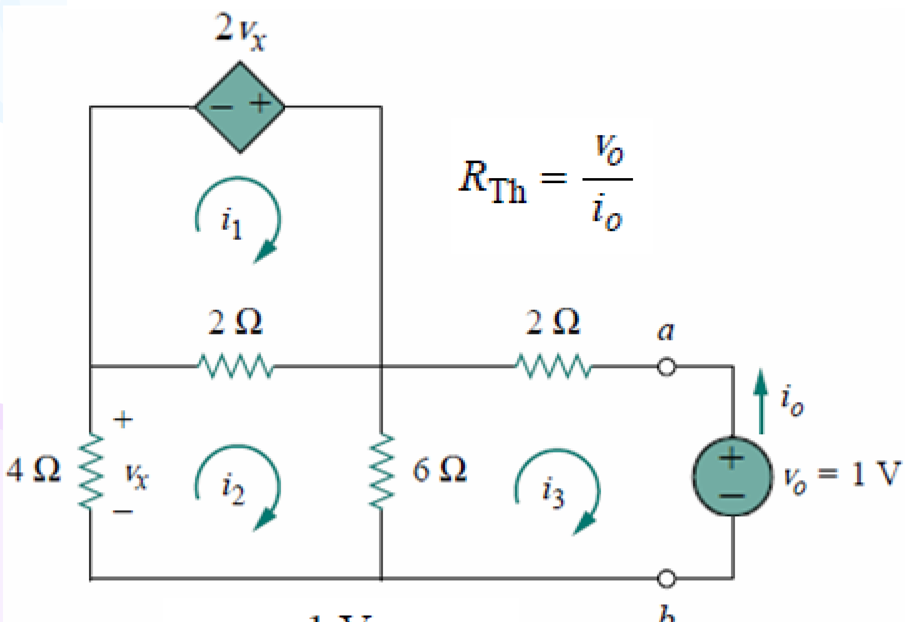
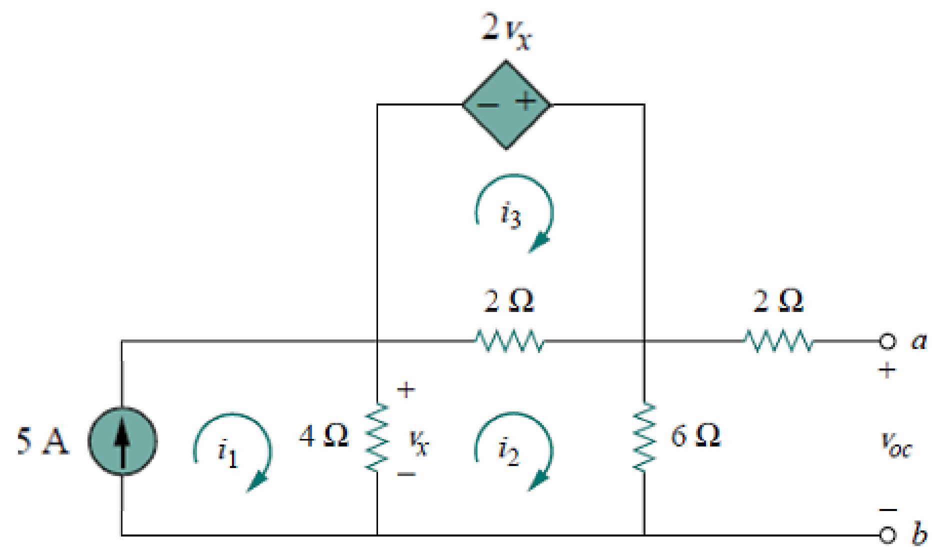


Figure 4.31 For Example 4.9.



$$R_{\mathrm{Th}} = \frac{v_o}{i_o}$$

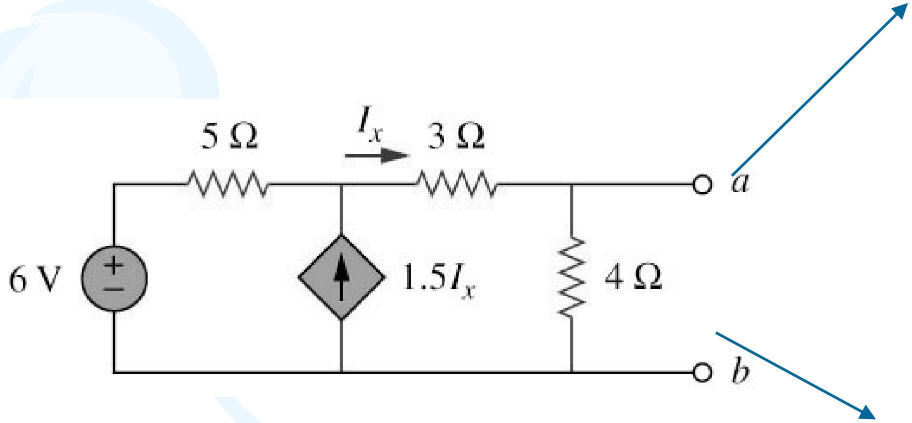
$$R_{\mathrm{Th}} = \frac{1\ \mathrm{V}}{i_o} = 6\ \Omega$$



$$V_{\mathrm{Th}} = v_{oc} = 6i_2 = 20\ \mathrm{V}$$

Example 6

Find the Thevenin equivalent circuit of the circuit shown below to the left of the terminals.



V_{TH} **= 5.33V,** R_{TH} **= 3Ω**

4.44 For the circuit in Fig. 4.111, obtain the Thevenin equivalent as seen from terminals:

(a) *a-b* (b) *b-c*

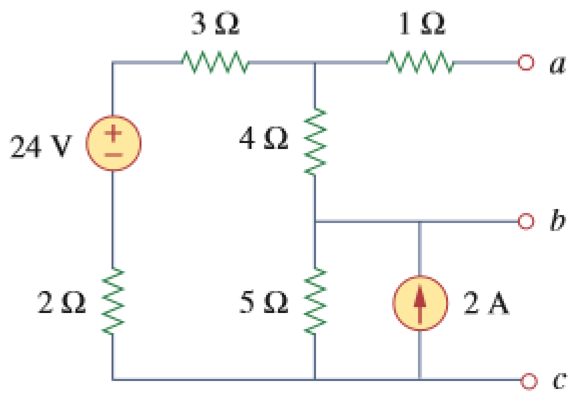
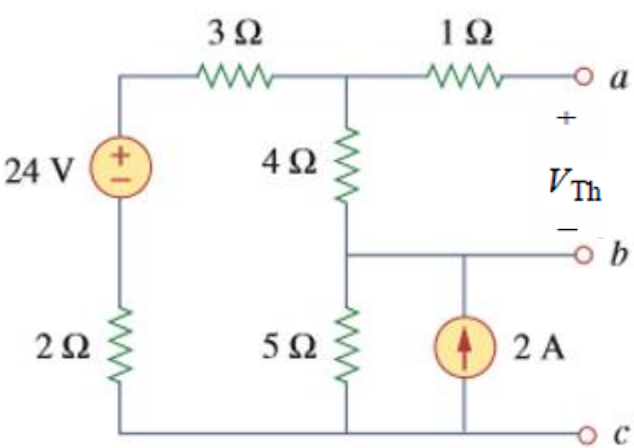
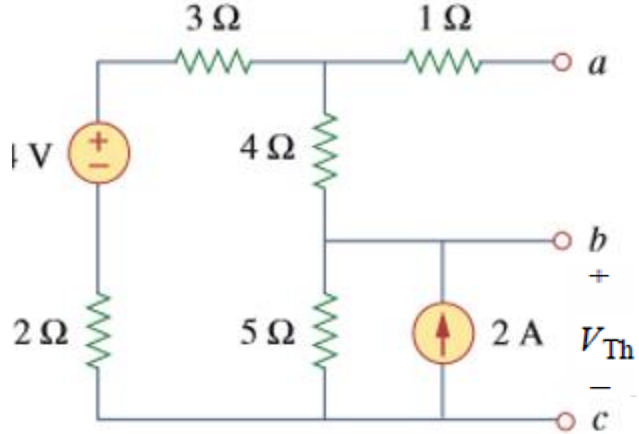


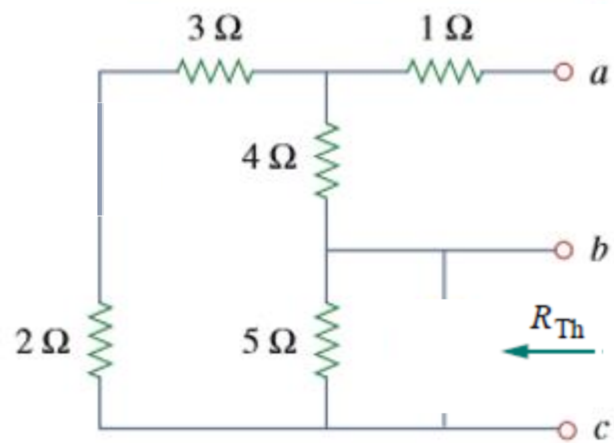
Figure 4.111

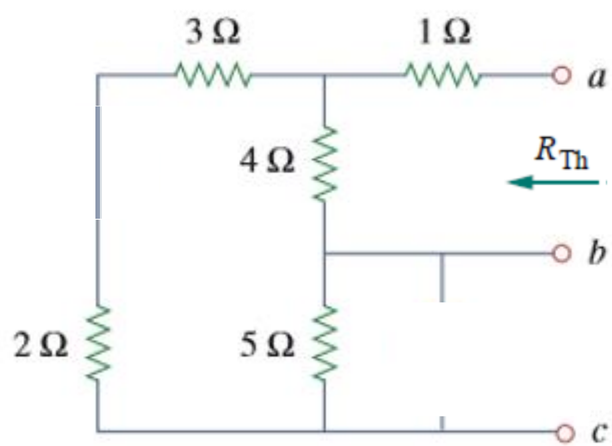
（a） from terminal a-b:



（b） from terminal b-c:







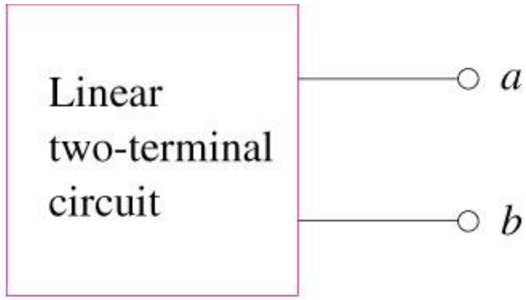
4.8 Norton's Theorem

It states that a linear two-terminal circuit can be replaced by an equivalent circuit of a current source I_N in parallel with a resistor R_N,

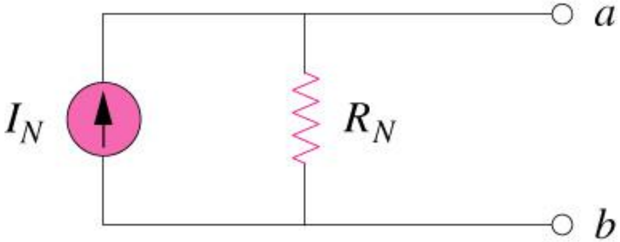
Where

- I_N is the short circuit current through the terminals.

- R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

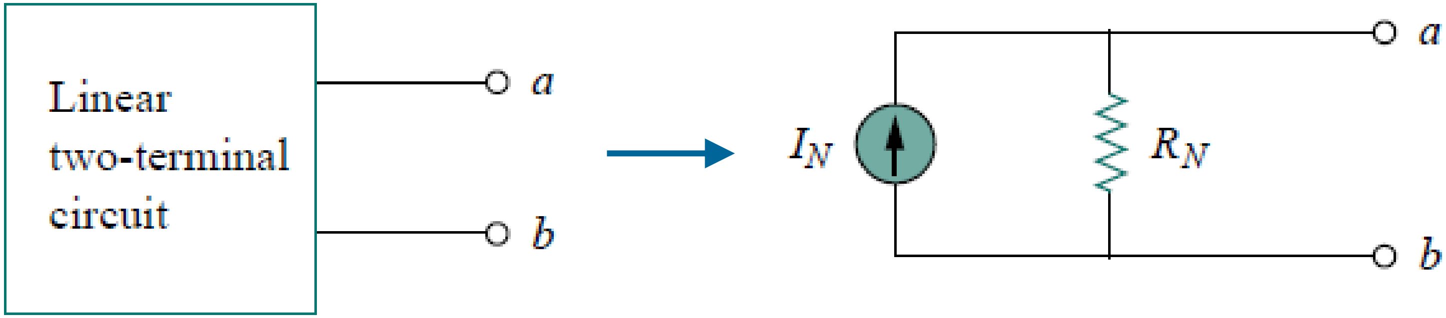


(a)



(b)

4.8 Norton's Theorem



To find the Norton current I_N, we determine the short-circuit current flowing from terminal a to b in both circuits in Fig. 4.37. It is evident that the short-circuit current in Fig. 4.37(b) is I_N. This must be the same short-circuit current from terminal a to b in Fig. 4.37(a), since the two circuits are equivalent. Thus,

$$I_N = i_{sc}$$

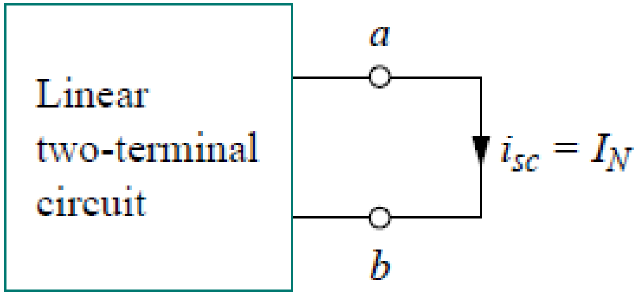
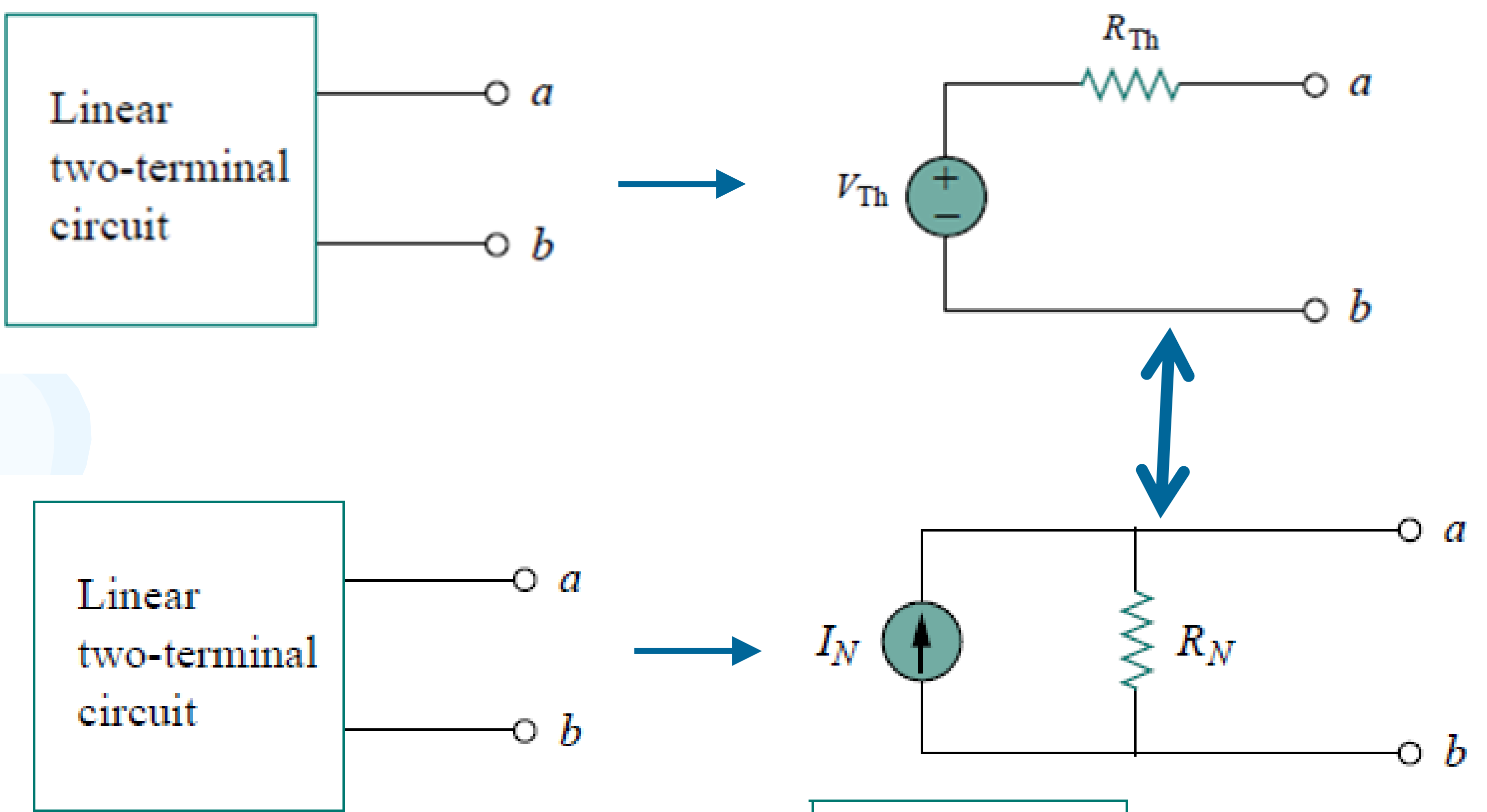


Figure 4.38 Finding Norton current I_N.

4.8 Norton's Theorem

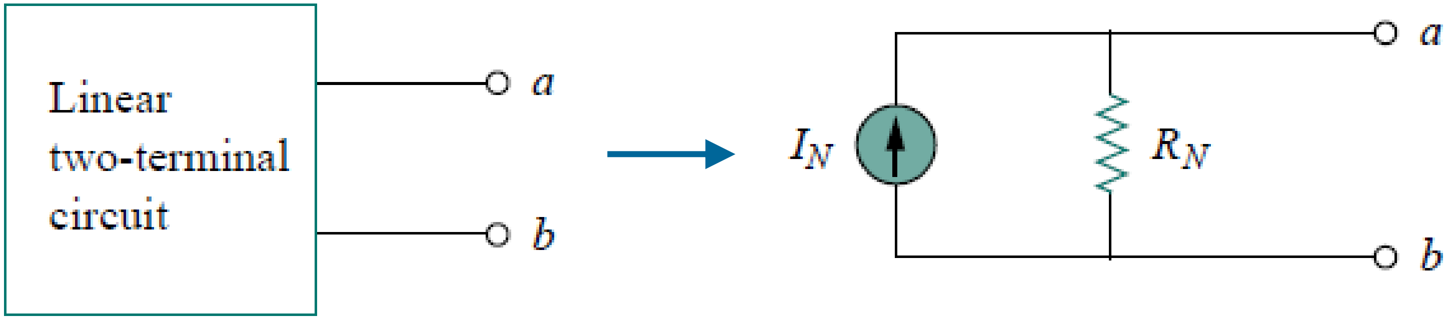


$$\boxed{R_N = R_{Th}}$$

$$\boxed{I_N = \frac{V_{Th}}{R_{Th}}}$$

The Thevenin's and Norton equivalent circuits are related by a source transformation.

4.8 Norton's Theorem



Observe the close relationship between Norton's and Thevenin's theorems: $R_N = R_{Th}$ as in Eq. (4.9), and

$$\boxed{I_N = \frac{V_{Th}}{R_{Th}}}$$

This is essentially source transformation. For this reason, source transformation is often called Thevenin-Norton transformation.

4.8 Norton's Theorem

Since V_{Th}, I_N, and R_{Th} are related according to Eq. (4.11), to determine the Thevenin or Norton equivalent circuit requires that we find:

$$I_N = \frac{V_{\text{Th}}}{R_{\text{Th}}}$$

- The open-circuit voltage v_{oc} across terminals a and b.
- The short-circuit current i_{sc} at terminals a and b.
- The equivalent or input resistance R_{in} at terminals a and b when all independent sources are turned off.

We can calculate any two of the three using the method that takes the least effort and use them to get the third using Ohm's law. Example 4.11 will illustrate this. Also, since

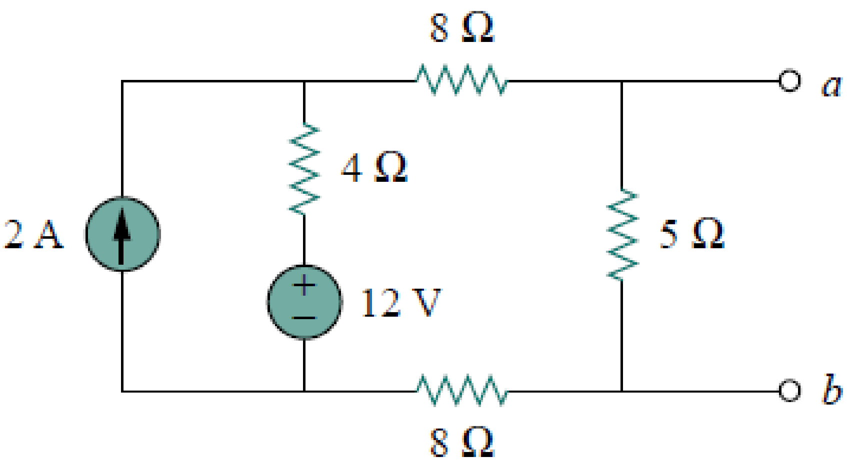
$$V_{\text{Th}} = v_{oc}$$

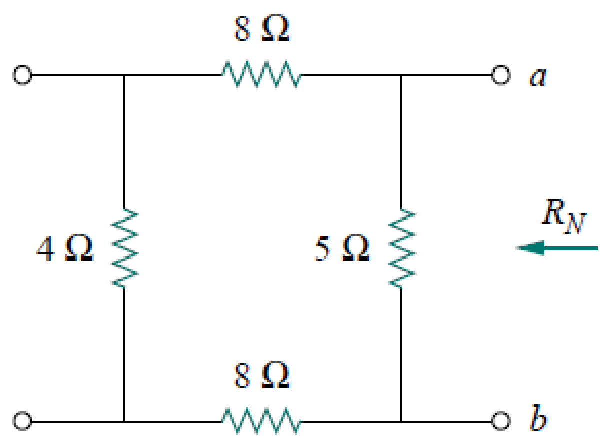
$$I_N = i_{sc}$$

$$R_{\text{Th}} = \frac{v_{oc}}{i_{sc}} = R_N$$

Find the Norton equivalent circuit of the circuit in Fig. 4.39.

EXAMPLE 4.11





(a)

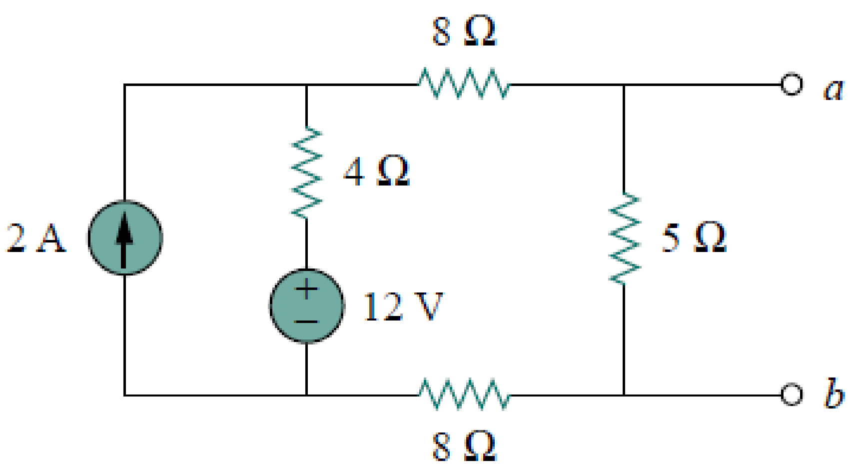
Figure 4.39 For Example 4.11.

We find R_N in the same way we find R_{Th} in the Thevenin equivalent circuit. Set the independent sources equal to zero. This leads to the circuit in Fig. 4.40(a), from which we find R_N. Thus,

$$R_N = 5 \parallel (8 + 4 + 8) = 5 \parallel 20 = \frac{20 \times 5}{25} = 4\ \Omega$$

Find the Norton equivalent circuit of the circuit in Fig. 4.39.

EXAMPLE 4.11



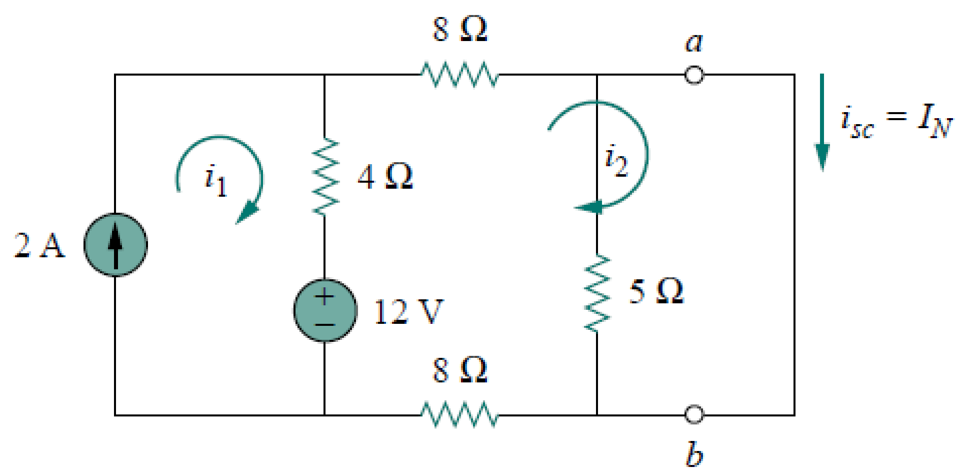


Figure 4.39 For Example 4.11.

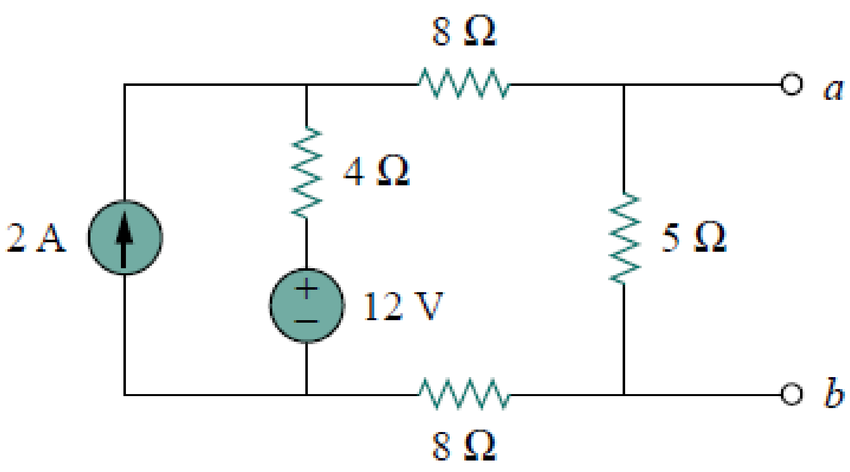
find I_N $\quad i_1 = 2\text{ A}, \qquad 20i_2 - 4i_1 - 12 = 0$

From these equations, we obtain

$$i_2 = 1\text{ A} = i_{sc} = I_N$$

Find the Norton equivalent circuit of the circuit in Fig. 4.39.

EXAMPLE 4.11



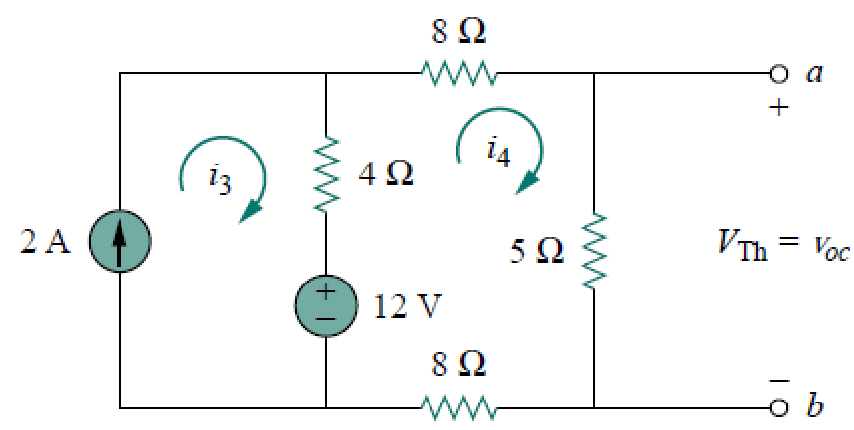


Figure 4.39 For Example 4.11.

Alternatively, we may determine I_N from V_{Th}/R_{Th}. We obtain V_{Th} as the open-circuit voltage across terminals a and b in Fig. 4.40(c). Using mesh analysis, we obtain

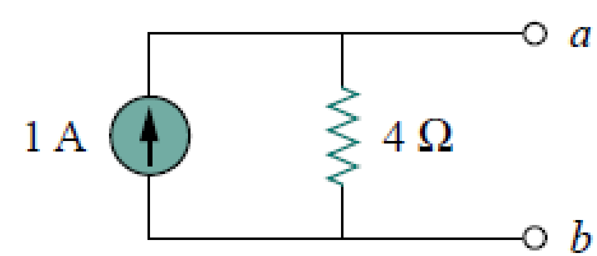
$$i_3 = 2\text{ A}$$

$$25i_4 - 4i_3 - 12 = 0$$

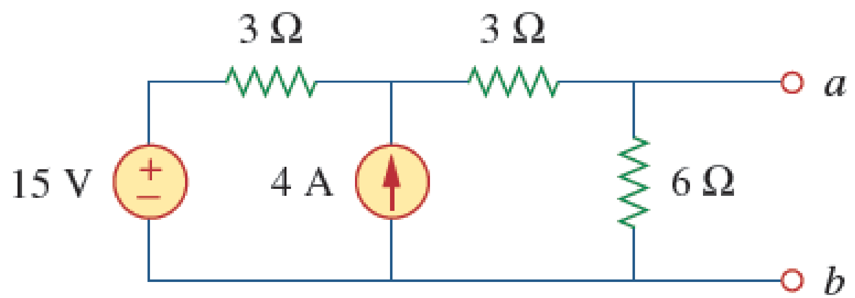
$$i_4 = 0.8\text{ A}$$

$$v_{oc} = V_{Th} = 5i_4 = 4\text{ V}$$

$$I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1\text{ A}$$

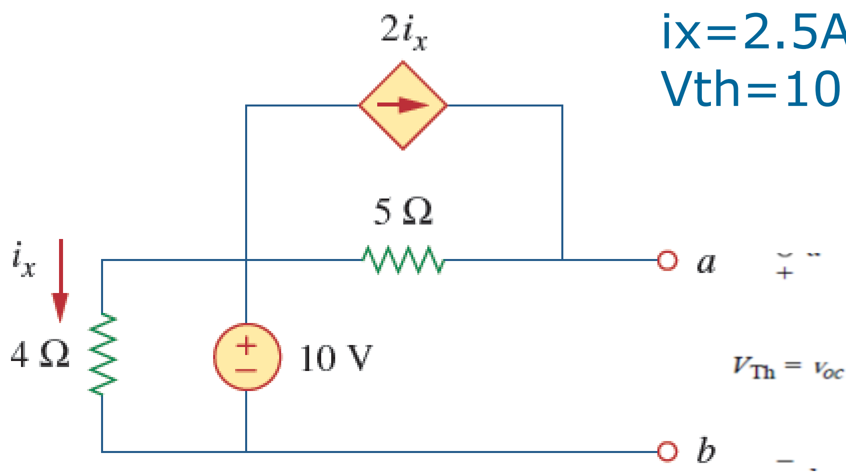


Find the Norton equivalent circuit for the circuit in Fig. 4.42, at terminals *a-b*.

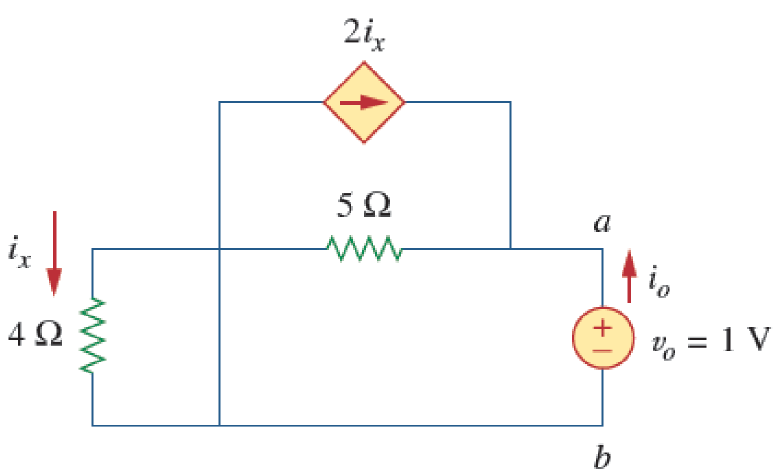


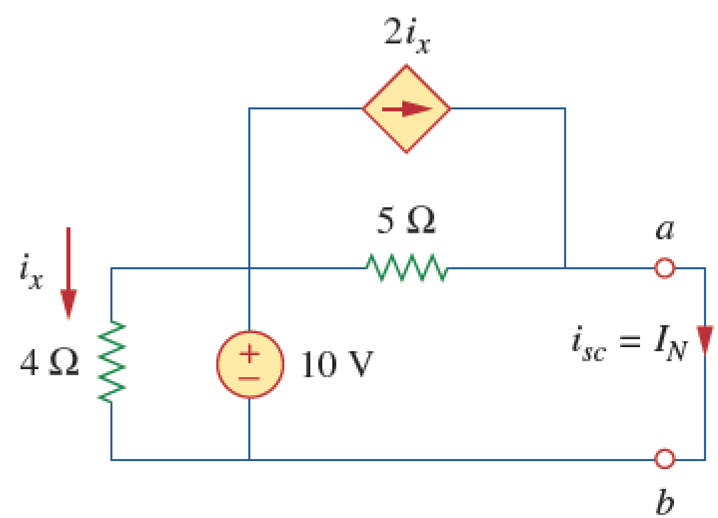
Answer: $R_N = 3\ \Omega$, $I_N = 4.5$ A.

Find the Norton equivalent circuit for the circuit in Fig. 4.43, at terminals *a-b*.



ix=2.5A
Vth=10+25=35V



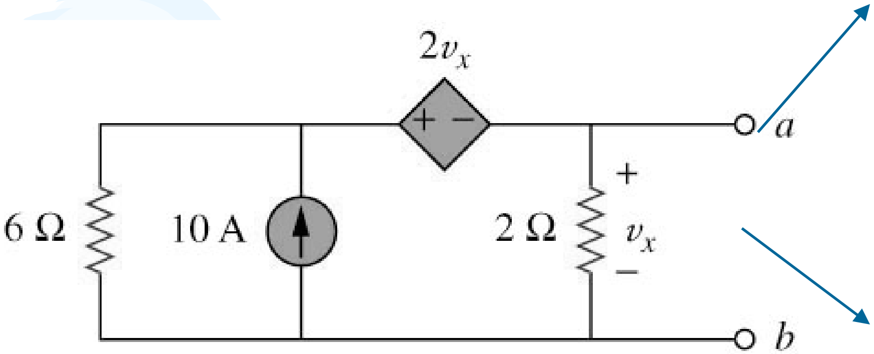


$$R_N = \frac{v_o}{i_o} = \frac{1}{0.2} = 5\ \Omega$$

$$i_{sc} = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7\ \text{A}$$

Example 7

Find the Norton equivalent circuit of the circuit shown below.



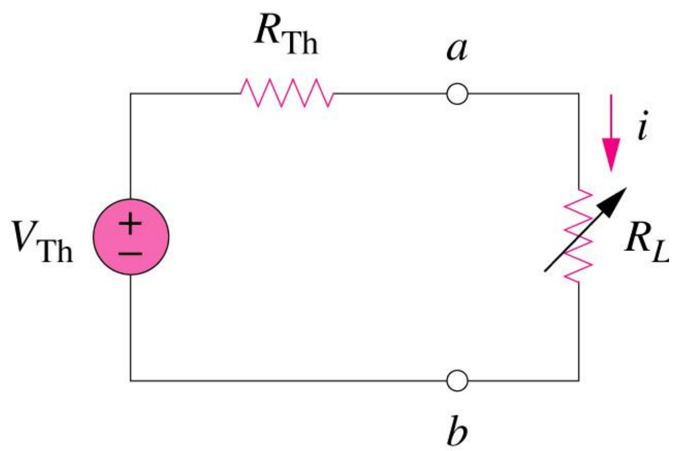
$R_N = 1\Omega$, $I_N = 10A$.

4.9 Maximum Power Transfer

The Thevenin equivalent is useful in finding the maximum power a linear circuit can deliver to a load. We assume that we can adjust the load resistance R_L.

4.9 Maximum Power Transfer

If the entire circuit is replaced by its Thevenin equivalent except for the load, the power delivered to the load is:



$$P = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$

$$\frac{dp}{dR_L} = V_{\mathrm{Th}}^2 \left[\frac{(R_{\mathrm{Th}} + R_L)^2 - 2R_L(R_{\mathrm{Th}} + R_L)}{(R_{\mathrm{Th}} + R_L)^4} \right]$$

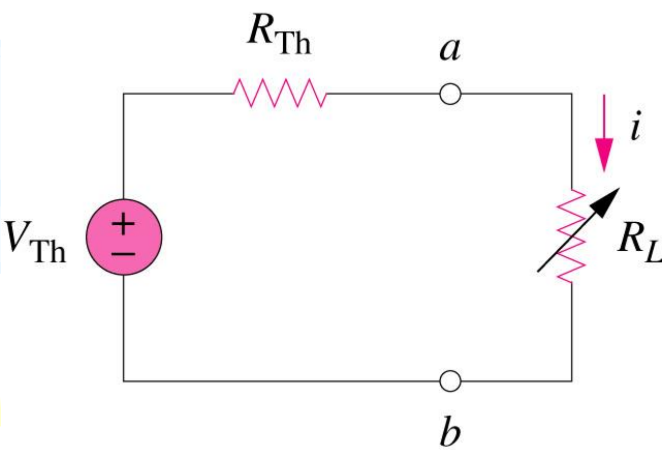
$$= V_{\mathrm{Th}}^2 \left[\frac{(R_{\mathrm{Th}} + R_L - 2R_L)}{(R_{\mathrm{Th}} + R_L)^3} \right] = 0$$

$$R_L = R_{TH} \quad \Rightarrow \quad P_{\max} = \frac{V_{Th}^{\;2}}{4R_L}$$

To prove the maximum power transfer theorem, we differentiate p in Eq. (4.21) with respect to R_L and set the result equal to zero. We obtain

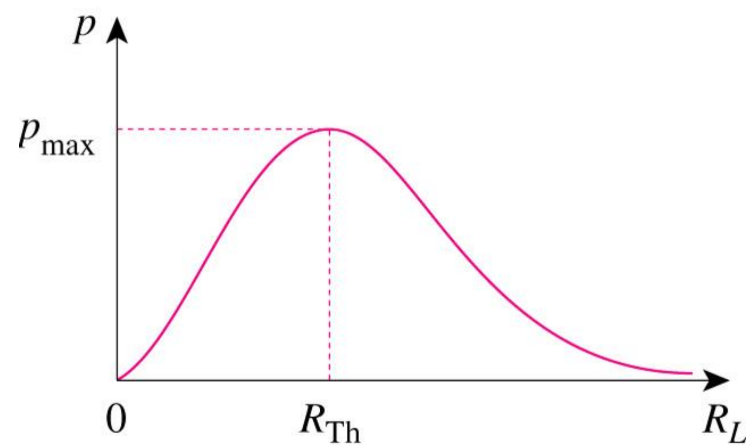
4.9 Maximum Power Transfer

Maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$).



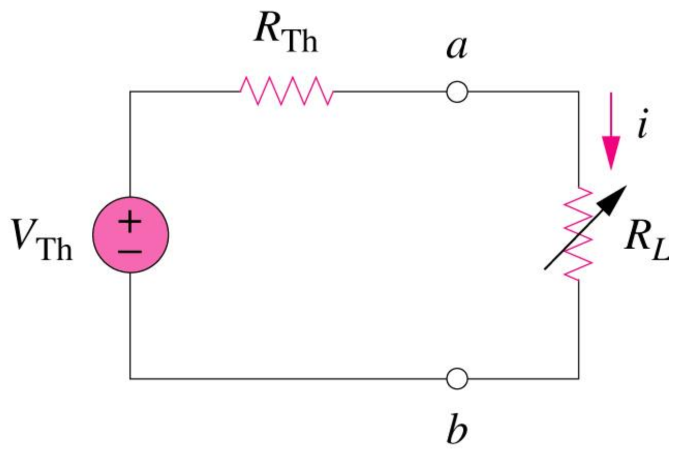
For maximum power dissipated in R_L, P_{max}, for a given R_{TH}, and V_{TH},

$$R_L = R_{TH} \quad \Rightarrow \quad P_{max} = \frac{V_{Th}^{2}}{4R_L}$$



The power transfer profile with different R_L

4.9 Maximum Power Transfer



$$R_L = R_{Th}$$

$$p_{max} = \frac{V_{Th}^2}{4R_{Th}}$$

EXAMPLE 4.13

Find the value of R_L for maximum power transfer in the circuit of Fig. 4.50. Find the maximum power.

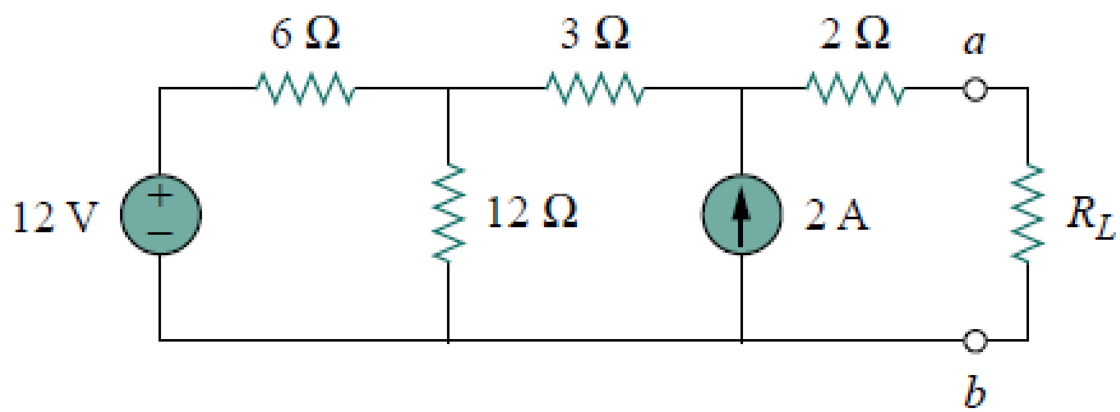
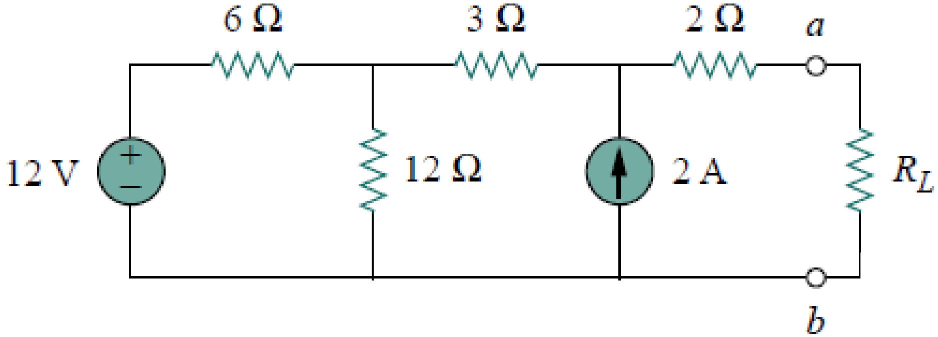
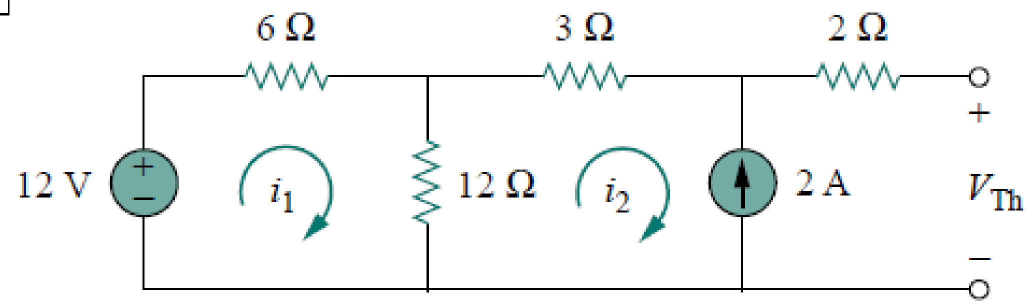


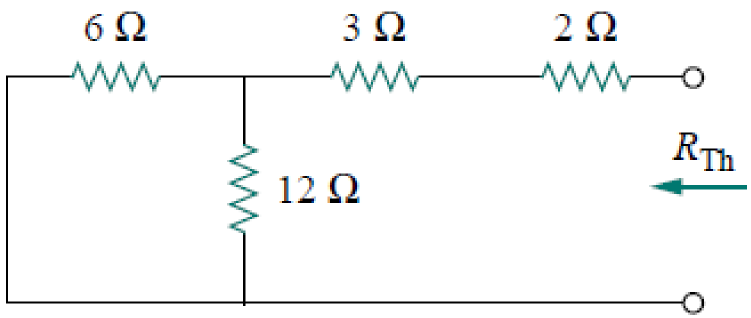
Figure 4.50 For Example 4.13.

Solution:

We need to find the Thevenin resistance R_{Th} and the Thevenin voltage V_{Th} across the terminals *a*-*b*. To get R_{Th}, we use the circuit in Fig. 4.51(a) and obtain







(a)

$$R_{Th} = 2 + 3 + 6 \parallel 12 = 5 + \frac{6 \times 12}{18} = 9\ \Omega$$

$$-12 + 18i_1 - 12i_2 = 0,$$

$$i_2 = -2\ \text{A}$$

$$V_{Th} = 22\ \text{V}$$

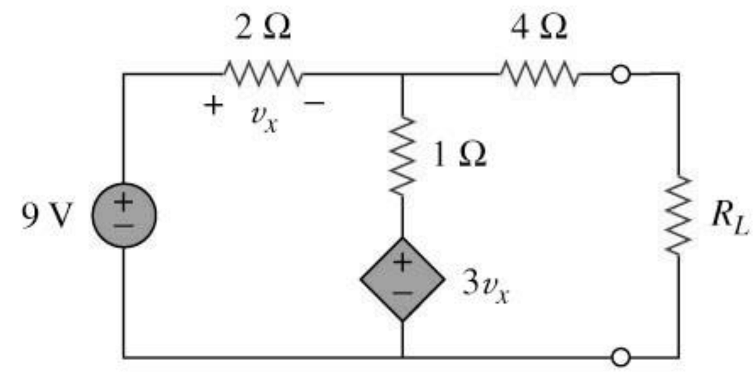
For maximum power transfer,

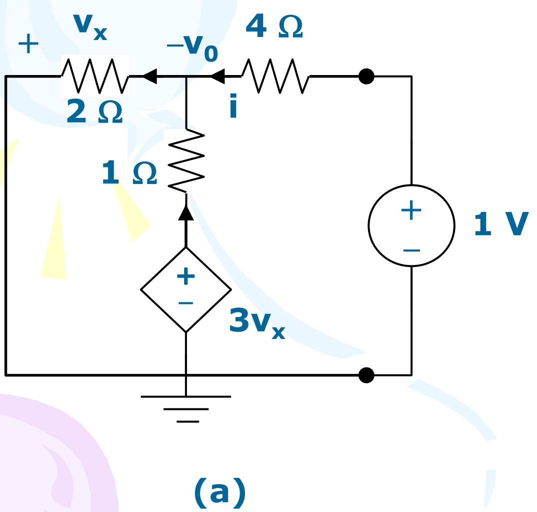
$$R_L = R_{Th} = 9\ \Omega$$

$$p_{max} = \frac{V_{Th}^2}{4R_L} = \frac{22^2}{4 \times 9} = 13.44\ \text{W}$$

Example 8

Determine the value of R_L that will draw the maximum power from the rest of the circuit shown below. Calculate the maximum power.





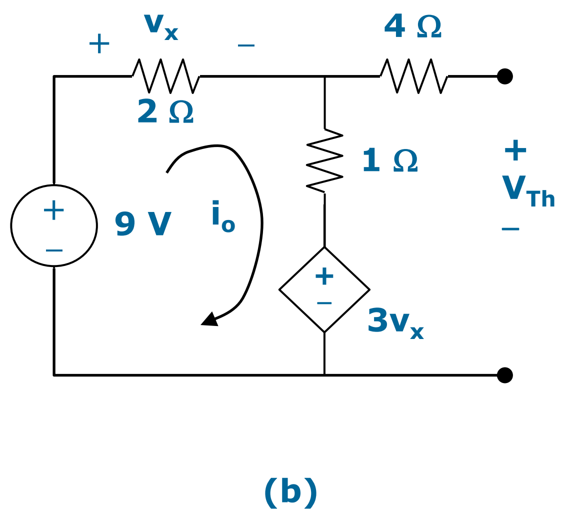


Fig. a

=> To determine R_{TH}

Fig. b

=> To determine V_{TH}

$R_L = 4.22\Omega$, $P_m = 2.901\text{W}$

Summary and Review

- The principle of superposition states that the response in a linear circuit can be obtained by adding the individual responses caused by the separate independent sources acting alone.

- Superposition is most often used when it is necessary to determine the individual contribution of each source to a particular response.

- Source transformations allow us to convert voltage source in series with a resistor into a practical current source in parallel with a resistor , and vice versa.

- Repeated source transformations can greatly simplify analysis of a circuit by providing the means to combine resistors and source.

- The Thévenin equivalent of a network is a resistor in series with an independent voltage source. The Norton equivalent is the same resistor in parallel with an independent current source.
- There are several ways to obtain the Thévenin equivalent resistance, depending on whether or not dependent sources are present in the network.
- Maximum power transfer occurs when the load resistor matches the Thévenin equivalent resistance of the network to which it is connected.

Assignment (page 162)

Problems 4.11, 4.15, 4.23, 4.31, 4.41,

4.47, 4.57, 4.66, 4.71,

Fundamentals of Electric Circuit

2021.4

Chapter 5

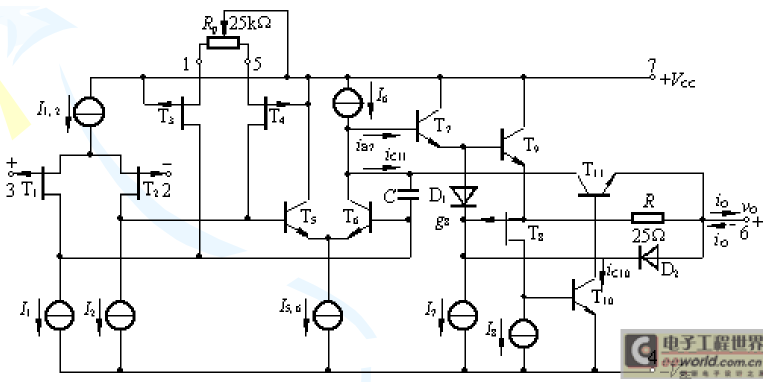
The Operational Amplifier

Chapter 5 The Operational Amplifier

5.1 Introduction

● In this chapter, we will study an active circuit element: **the operational amplifier**, or **op amp** for short.

The op amp is an electronic unit that behaves like a voltage-controlled voltage source.



Some types of operational amplifiers

5.1 Introduction

Chapter objectives

- **Master two basic rules that must be applied when analyzing ideal op amp circuits.**

- **Be able to analyze simple circuits containing ideal op amp, and recognize the following op amplifier circuits:**

  - ***inverting* and *noninverting amplifiers,***
  - ***adder, subtracter,***
  - ***integrator, differentiator, voltage follower.***

5.2 The Operational Amplifier

● An operational amplifier is designed to perform some mathematical operations with external components, such as resistors and capacitors.

An op amp is an active circuit element designed to perform mathematical operations of addition, subtraction, multiplication, division, differentiation, and integration.

In the circuit analysis, we are only concerned with the voltage and current relationships existing between the input and output terminals.

5.2 The Operational Amplifier

1. The inverting input, pin 2.
2. The noninverting input, pin 3.
3. The output, pin 6.
4. The positive power supply V +, pin 7.
5. The negative power supply V −, pin 4.

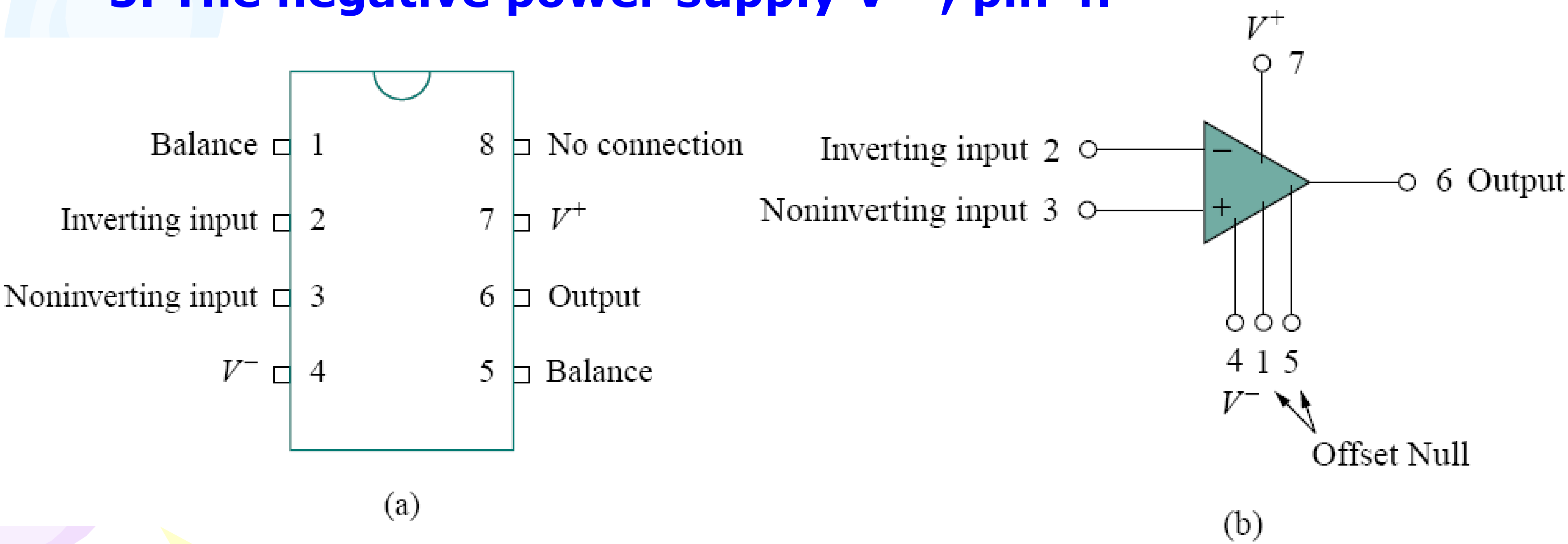


Figure 5.2 A typical op amp: (a) pin configuration, (b) circuit symbol.

5.2 The Operational Amplifier

As an active element, the op amp must be powered by a voltage supply as typically shown in Fig. 5.3.

By KCL,

$$i_o = i_1 + i_2 + i_+ + i_-$$

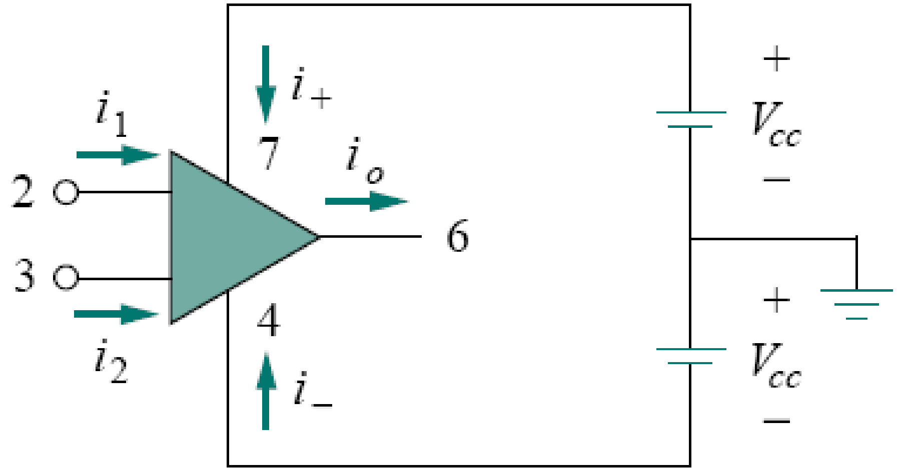


Figure 5.3 Powering the op amp.

5.2 The Operational Amplifier

The output section consists of a voltage-controlled source in series with the output resistance R_o.

The input section consists of a input resistance R_i

The differential input voltage v_d is given by $$v_d = v_2 - v_1$$

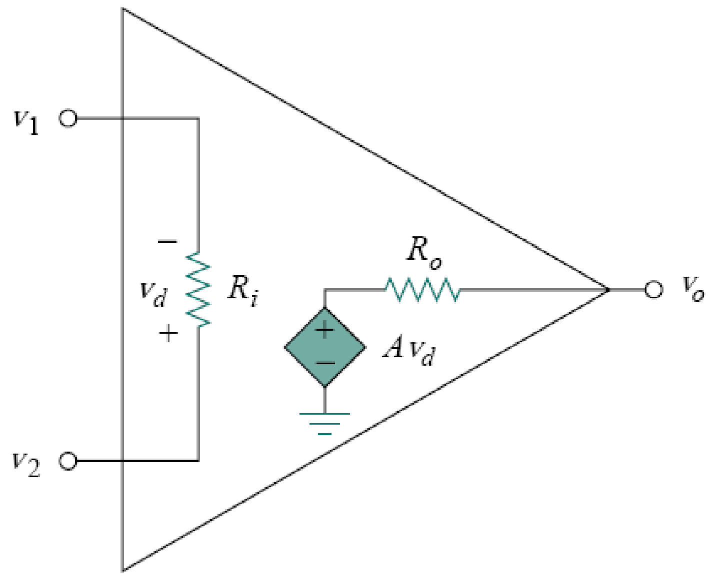


Figure 5.4 The equivalent circuit of the non-ideal op amp.

Thus, the output v_o is given by

$$\boxed{v_o = Av_d = A(v_2 - v_1)}$$

A is called the open-loop voltage gain because it is the gain of the op amp without any external feedback from output to input.

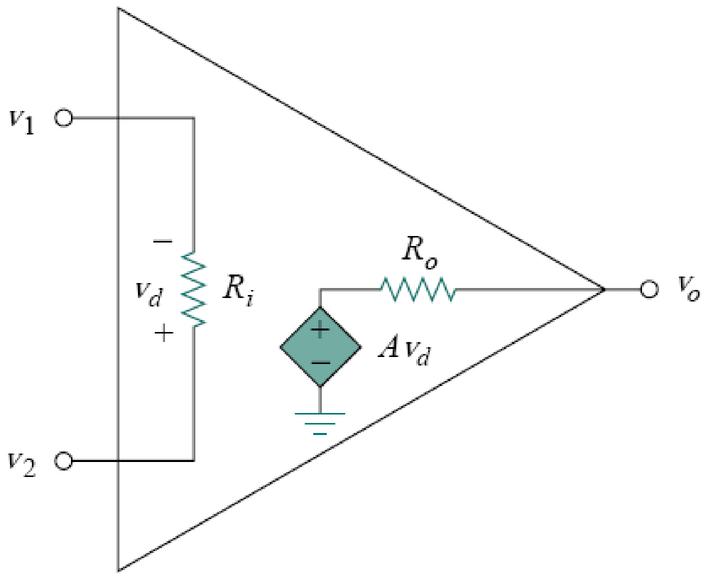


Figure 5.4 The equivalent circuit of the non-ideal op amp.

5.3. The ideal op amp

◆*Ideal conditions*

1. **Infinite open-loop gain, $A \approx \infty$**
2. **Infinite input resistance, $R_i \approx \infty$**
3. **Zero output resistance, $R_o \approx 0$**

An ideal op amp is an amplifier with infinite open-loop gain, infinite input resistance, and zero output resistance.

Although assuming an ideal op amp provides only an approximate analysis, most modern amplifiers have such large gains and input impedances that the approximate analysis is a good one. Unless stated otherwise, we will assume from now on that every op amp is ideal.

5.3. The ideal op amp

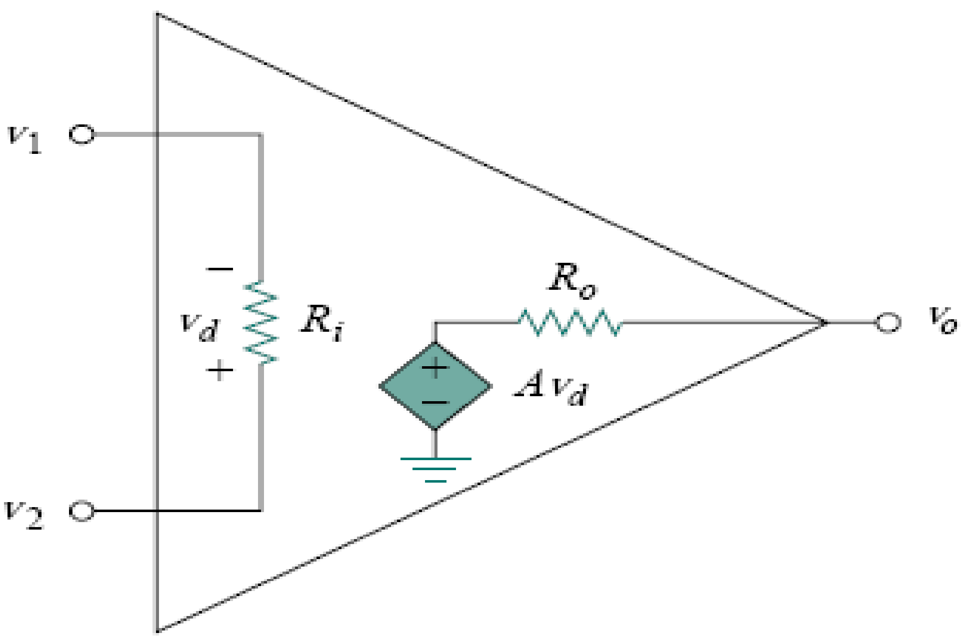
◆ ***Ideal conditions***

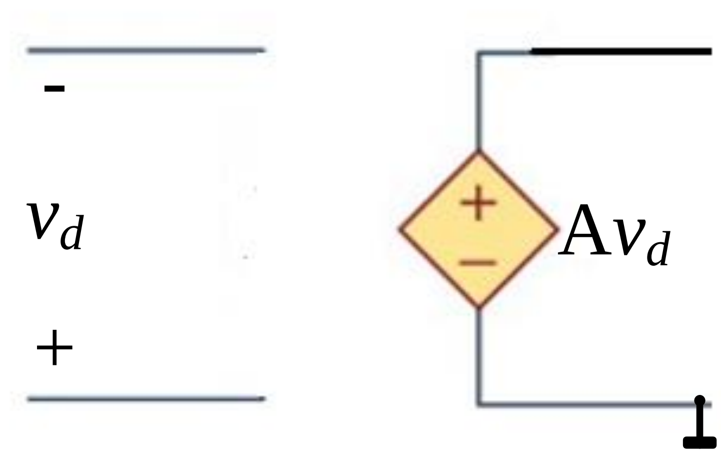
1. Infinite open-loop gain, $A \approx \infty$

2. Infinite input resistance, $R_i \approx \infty$

3. Zero output resistance, $R_o \approx 0$

◆ ***Ideal op amp model***





Equivalent circuit for ideal op amp model

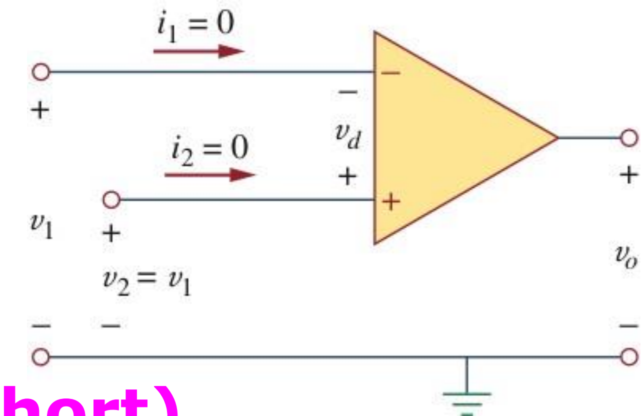
Two important characteristics of the ideal op amp are:

1. The currents into both input terminals are zero:

$$i_1 = 0, \qquad i_2 = 0$$

(virtual open)

2. The voltage across the input terminals is negligibly small; i.e.,



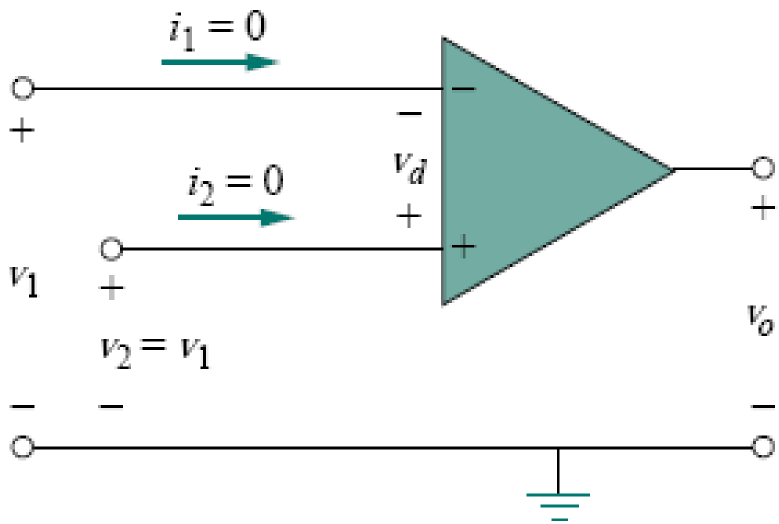
$$v_d = v_2 - v_1 \approx 0$$

or

$$v_2 = v_1$$

(virtual short)

The virtual short and virtual open are extremely important and should be regarded as the key handles to analyzing op amp circuits

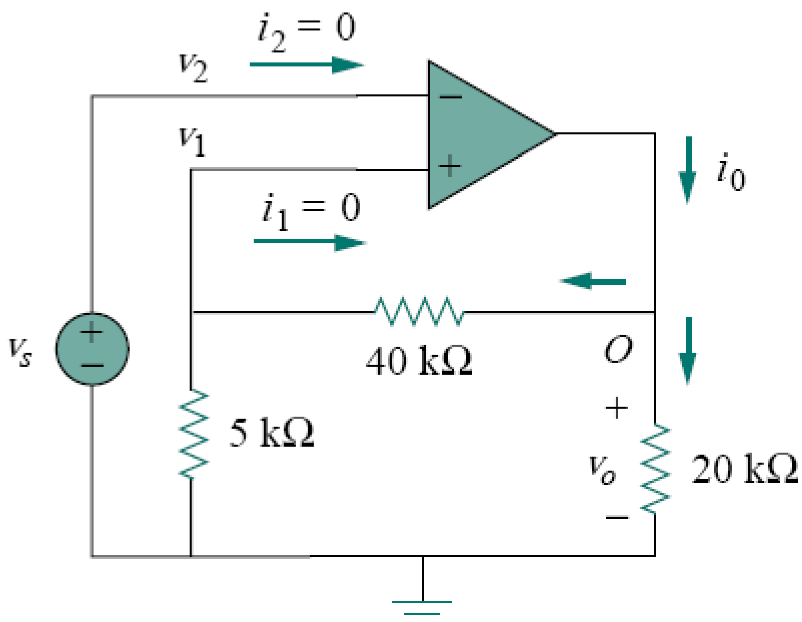


1. The currents into both input terminals are zero: $\boxed{i_1 = 0, \qquad i_2 = 0}$

2. The voltage across the input terminals is negligibly small; $\boxed{v_1 = v_2}$

EXAMPLE 5.2

If the ideal op amp is used in the circuit, calculate the closed-loop gain v_o/v_s , Find i_o when v_s = 1 V.



$$\boxed{i_1 = 0, \qquad i_2 = 0}$$

$$\boxed{v_1 = v_2}$$

$$v_1 = \frac{5}{5+40} v_o = \frac{v_o}{9}$$

$$v_2 = v_s$$

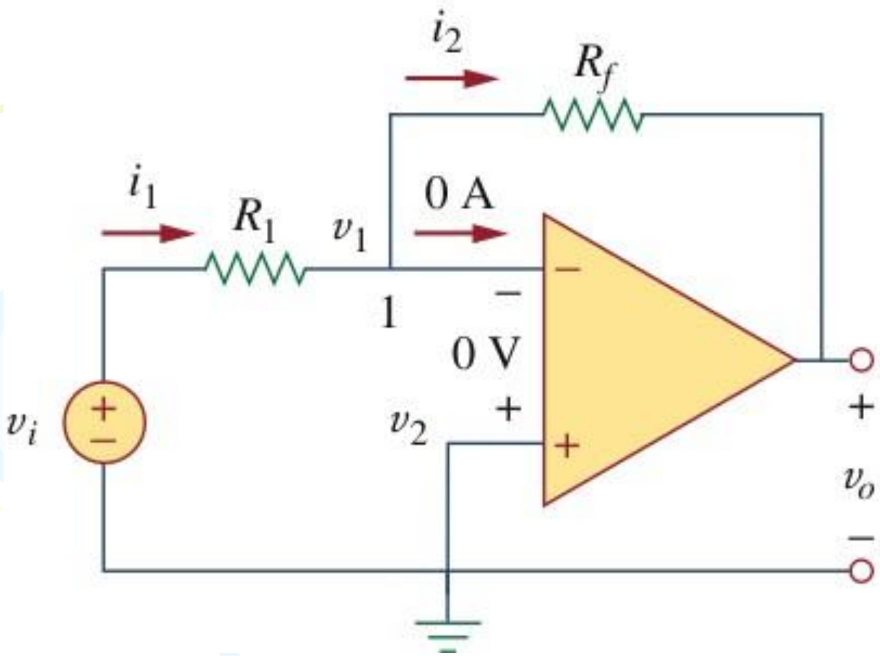
$$v_s = \frac{v_o}{9} \qquad \Longrightarrow \qquad \frac{v_o}{v_s} = 9$$

At node O, $\quad i_o = \dfrac{v_o}{40+5} + \dfrac{v_o}{20}$ mA

$$v_o = 9 \text{ V} \qquad i_o = 0.2 + 0.45 = 0.65 \text{ mA}$$

5.4 Inverting Amplifier

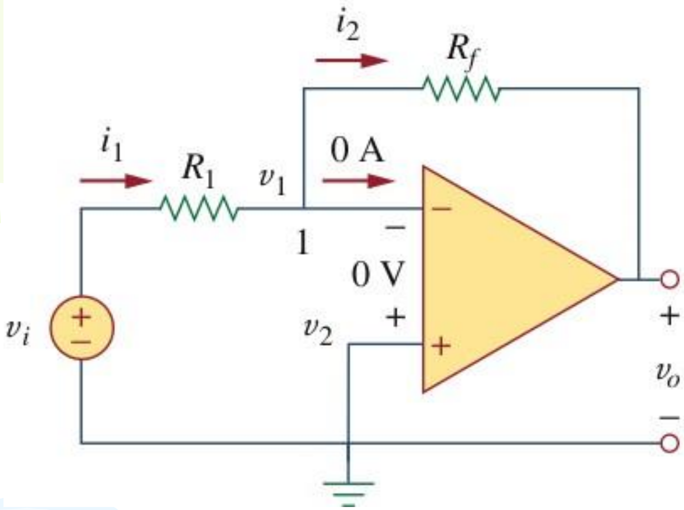
The inverting amplifier is shown in the below Figure.



***Notice:* both the input signal and the feedback are applied at the *inverting terminal* of the op amp**

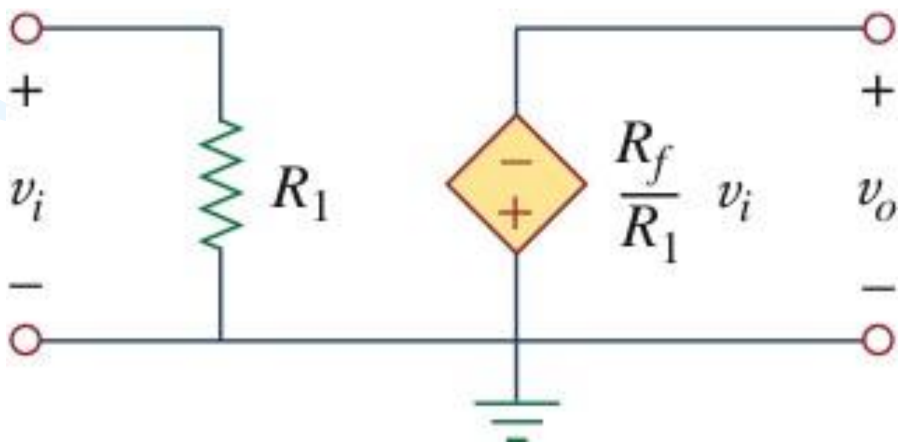
$$i_1 = i_2 \Rightarrow \frac{v_i - v_1}{R_1} = \frac{v_1 - v_o}{R_f}$$ *and* $$v_1 = v_2 \qquad v_1 = v_2 = 0$$

Hence, we obtain $$v_o = -\frac{R_f}{R_1} v_i$$



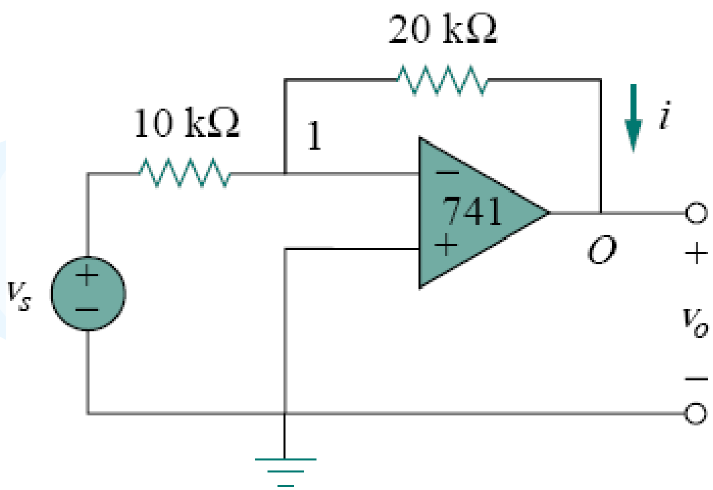
$$v_o = -\frac{R_f}{R_1} v_i$$

An inverting amplifier reverses the polarity of the input signal while amplifying it.

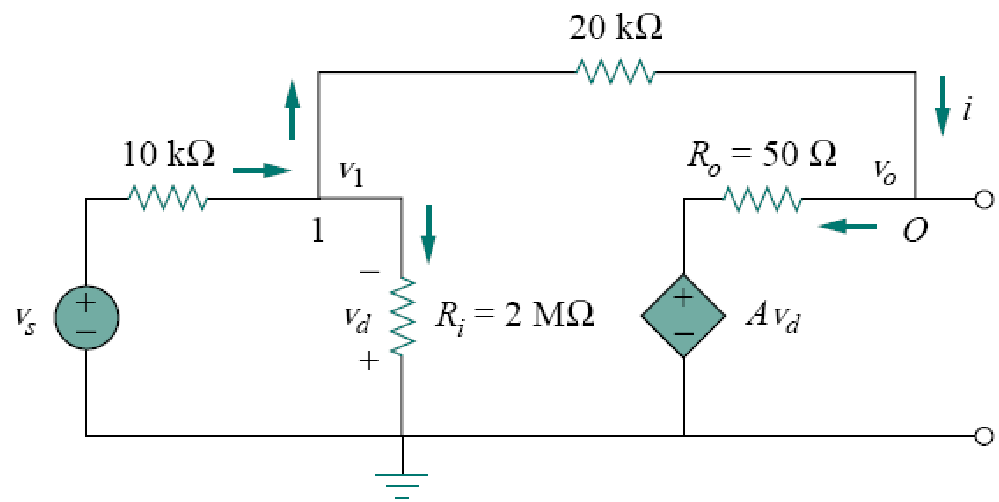


EXAMPLE 5.1

A 741 op amp has an open-loop voltage gain of 2×10^5, input resistance of 2 MΩ, and output resistance of 50 Ω. The op amp is used in the circuit of Fig. 5.6(a). Find the closed-loop gain v_o/v_s. Determine current i when $v_s = 2$ V.



(a)



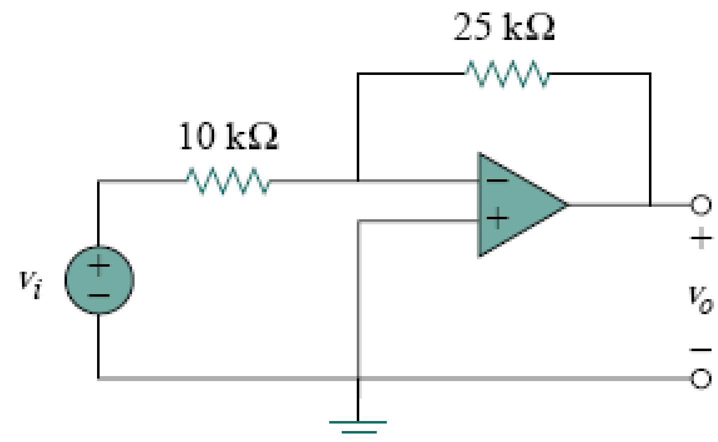
(b)

$$\frac{v_s - v_1}{10 \times 10^3} = \frac{v_1}{2000 \times 10^3} + \frac{v_1 - v_o}{20 \times 10^3} \qquad \frac{v_1 - v_o}{20 \times 10^3} = \frac{v_o - Av_d}{50}$$

$$\frac{v_o}{v_s} = -1.9999699$$

EXAMPLE 5.3

Refer to the op amp in Fig. 5.12. If $v_i = 0.5$ V, calculate: (a) the output voltage v_o, and (b) the current in the 10 kΩ resistor.



$$v_o = -\frac{R_f}{R_1} v_i$$

(a)
$$\frac{v_o}{v_i} = -\frac{R_f}{R_1} = -\frac{25}{10} = -2.5$$

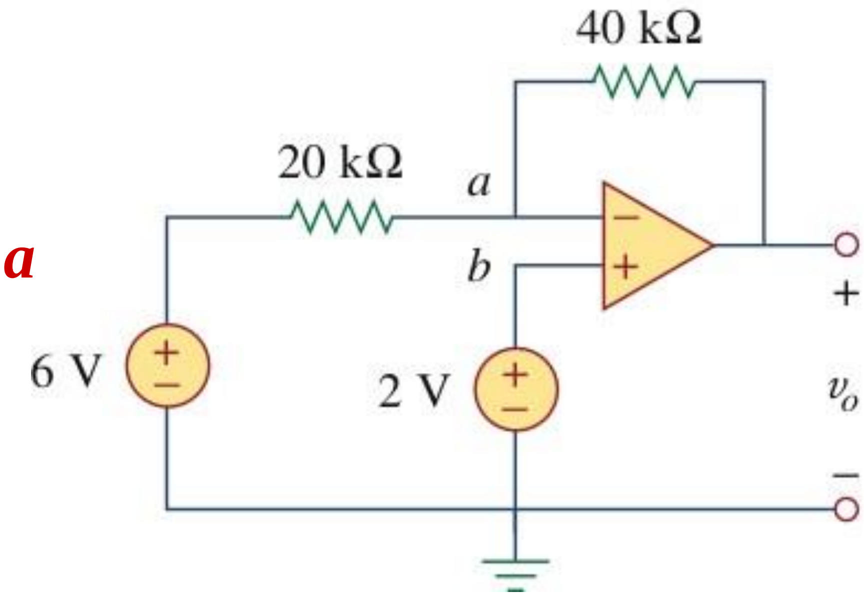
$$v_o = -2.5 v_i = -2.5(0.5) = -1.25 \text{ V}$$

(b) The current through the 10-kΩ resistor is

$$i = \frac{v_i - 0}{R_1} = \frac{0.5 - 0}{10 \times 10^3} = 50\ \mu\text{A}$$

Example 5.4

Determine the v_o in the op amp circuit shown in Fig.5.14



Solution:

Applying KCL at node *a*

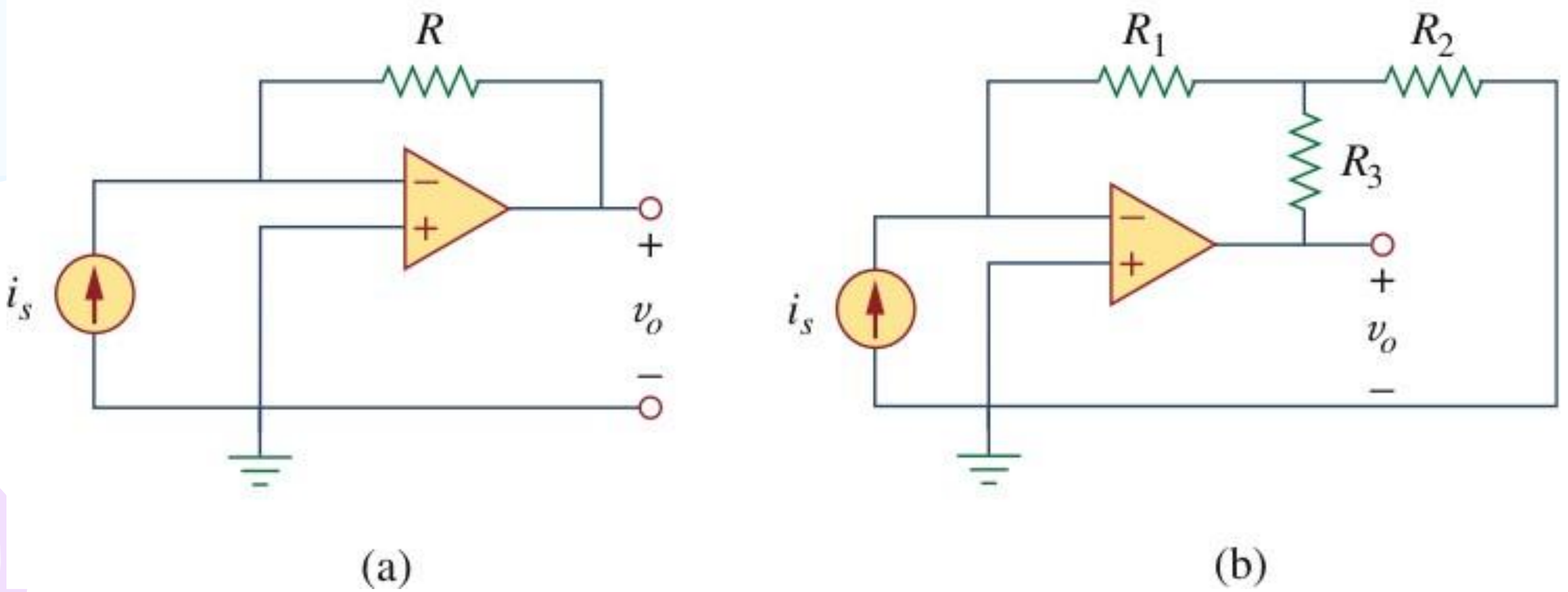
$$\frac{v_a - v_o}{40} = \frac{6 - v_a}{20}$$

$$v_a = v_b = 2\ \text{V}$$

$$v_o = 6 - 12 = -6\ \text{V}$$

Practice Problem

Two kinds of current-to-voltage converters (also known as *transresistance amplifier*) are shown in Fig. 5.2



$$\frac{v_o}{i_s} = -R$$

Fig. 5.2

$$\frac{v_o}{i_s} = -R_1\left(1 + \frac{R_3}{R_1} + \frac{R_3}{R_2}\right)$$

5.5 Noninverting Amplifier

applying KCL at the inverting terminal gives

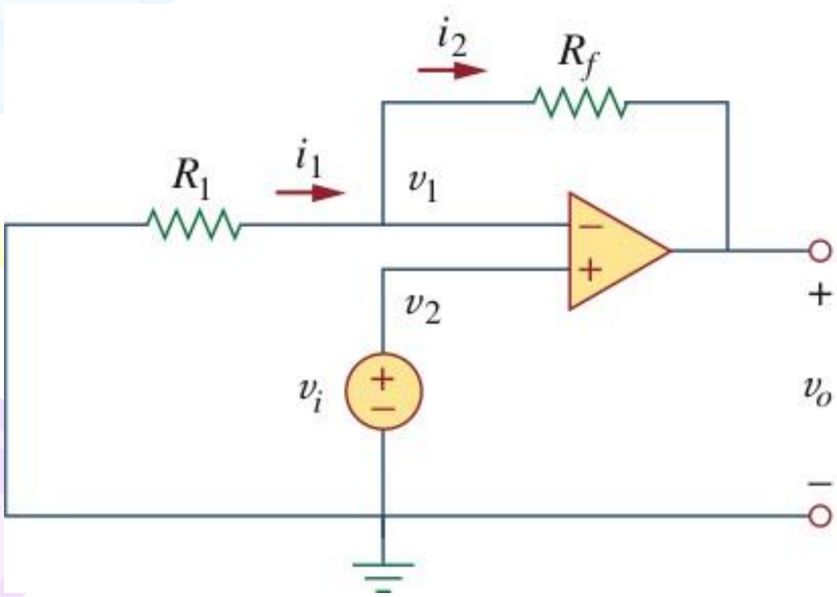


Fig.5.16

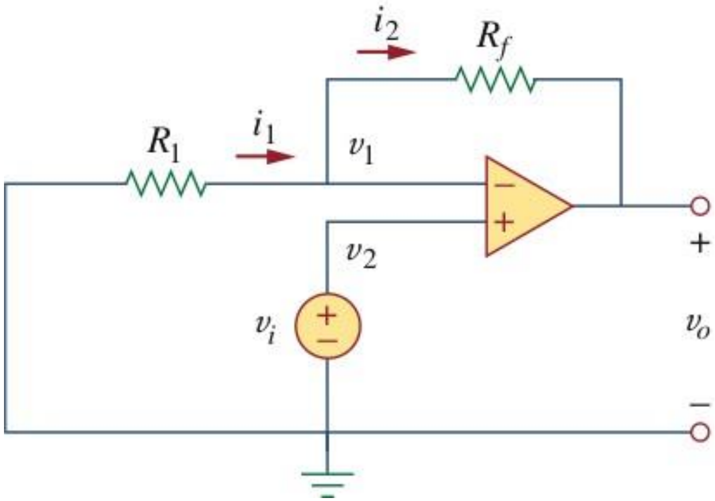
$$i_1 = i_2 \Rightarrow \frac{0 - v_1}{R_1} = \frac{v_1 - v_o}{R_f}$$

$$v_1 = v_2 = v_i$$

Hence, we obtain

$$v_o = (1 + \frac{R_f}{R_1}) v_i$$

$$v_o = (1 + \frac{R_f}{R_1})v_i$$



The voltage gain is

$$A_v = 1 + \frac{R_f}{R_1}$$

A noninverting amplifier is an op amp circuit designed to provide a positive voltage gain

Voltage Follower

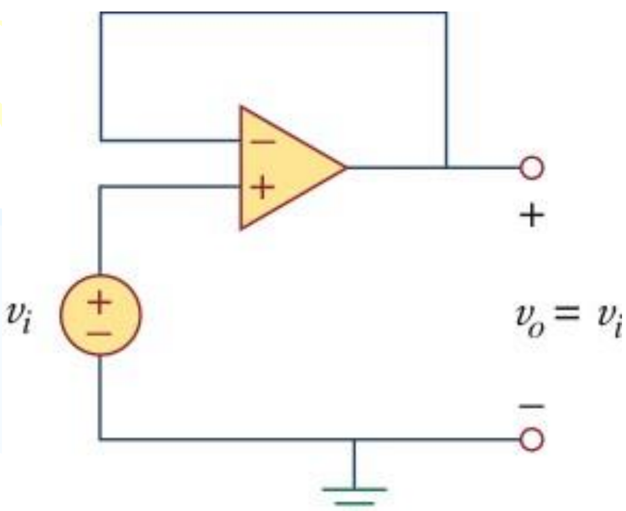
If R_f=0 or R_1=∞, the noninverting amplifier becomes into ***unity gain amplifier,*** which is called a ***voltage follower*** because the output follows the input.

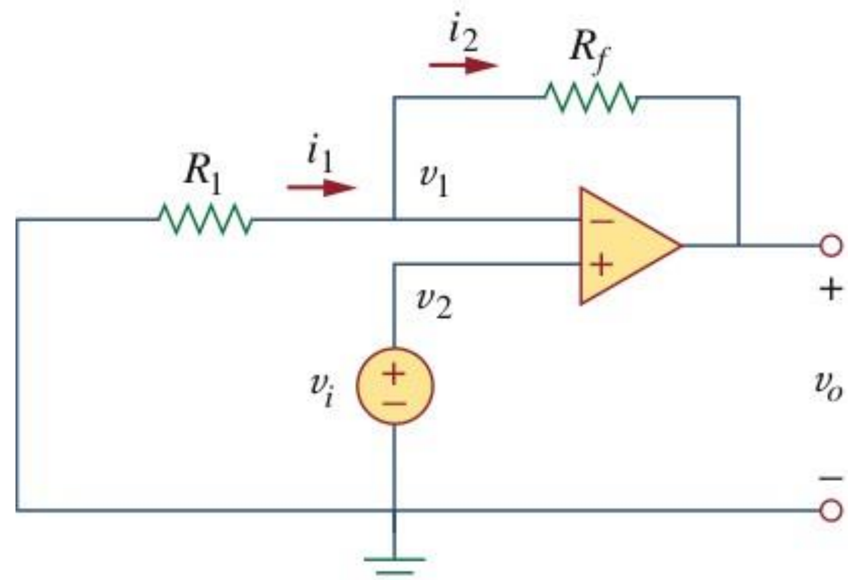
$$\boxed{v_o = v_i}$$

Thus, for a voltage follower

$R_f = 0$, $R_1 = \infty$

$$v_o = v_i$$





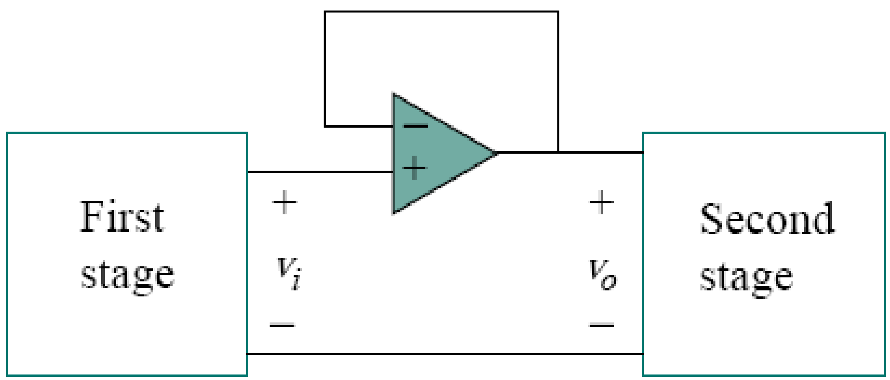
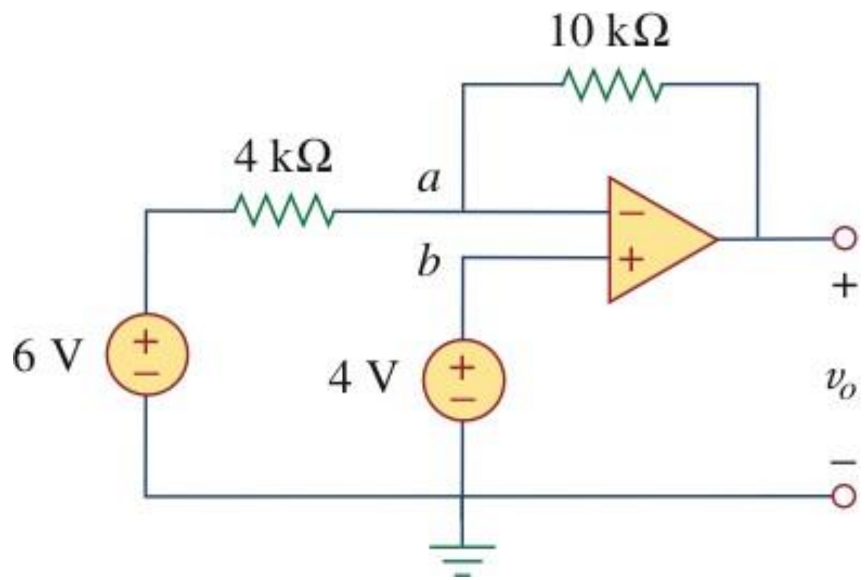


Fig.5.18 A voltage follower used to isolate two cascaded stages of a circuit

The voltage follower minimizes interaction between the two stages and eliminates inter-stage loading.

Example 5.5

For the right op amp circuit , calculate the output voltage v_o.



- **Method 1** Applying KCL at node *a*

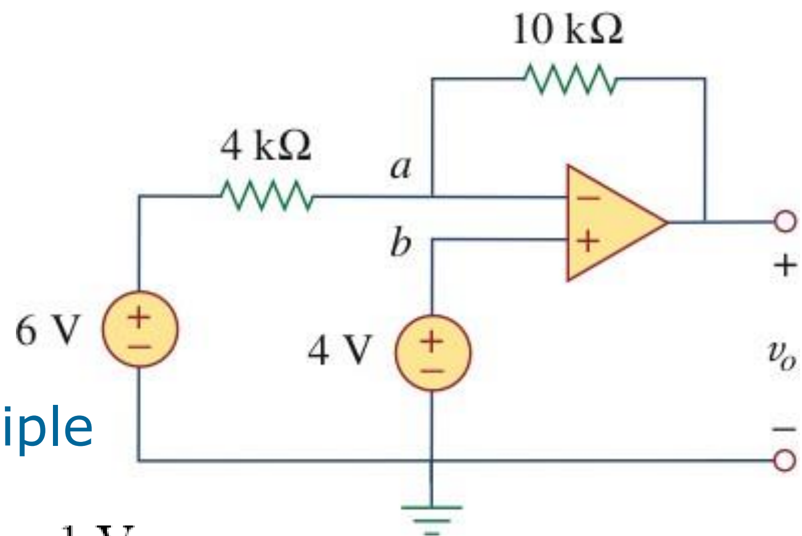
$$\frac{6 - v_a}{4} = \frac{v_a - v_o}{10}$$

$$v_a = v_b = 4,$$

$$v_o = -1 \text{ V}.$$

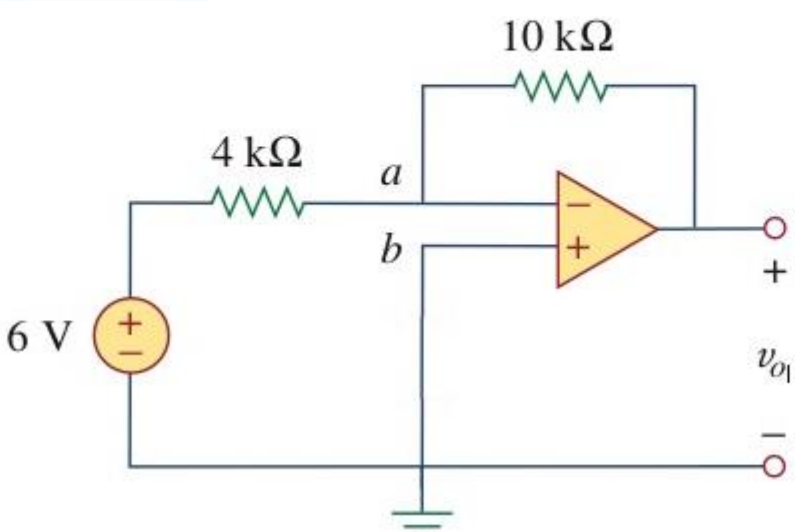
Example 5.5

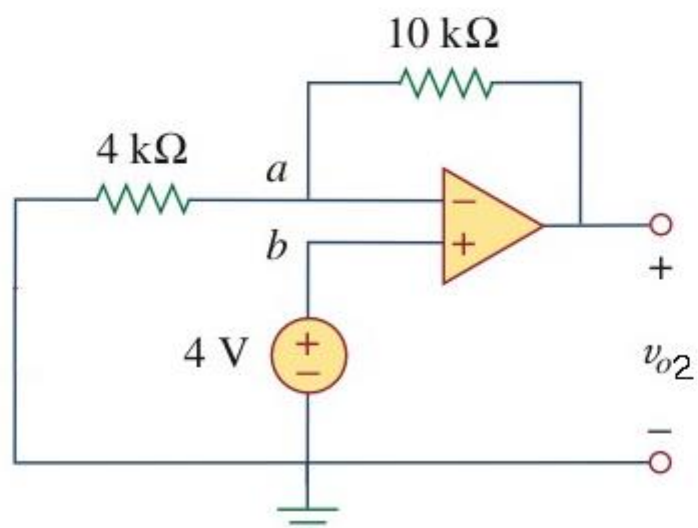
For the right op amp circuit , calculate the output voltage v_o.



- **Method 2** Using superposition principle

$$v_o = v_{o1} + v_{o2} = -15 + 14 = -1\text{ V}$$





$$v_{o1} = -\frac{10}{4}(6) = -15\text{ V}$$

$$v_{o2} = \left(1 + \frac{10}{4}\right)4 = 14\text{ V}$$

Practice Problem 5.5

Calculate v_o in the Fig.5.20

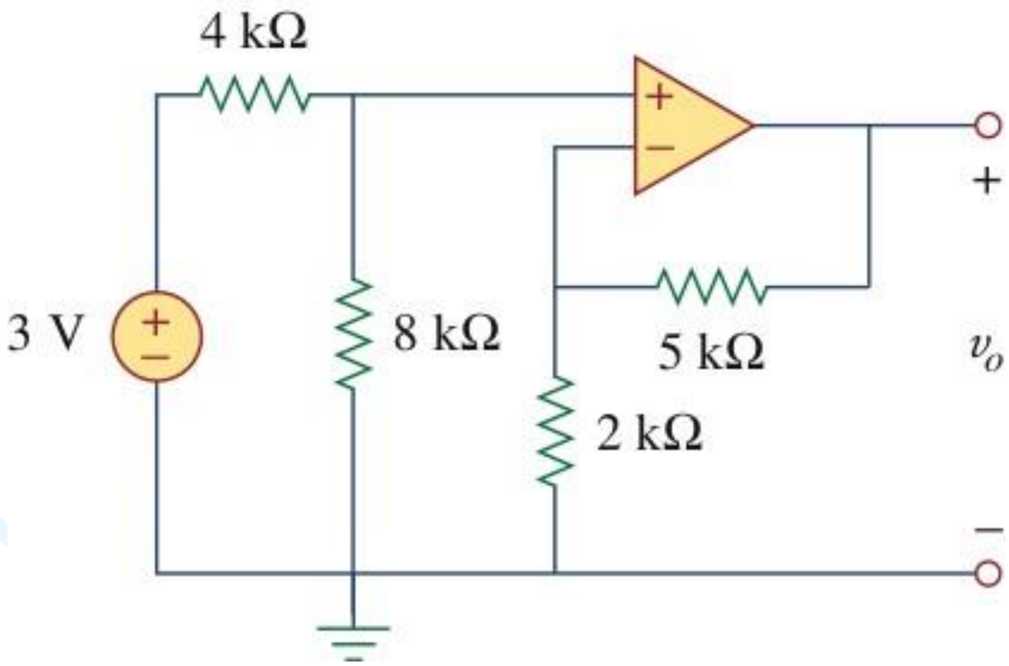
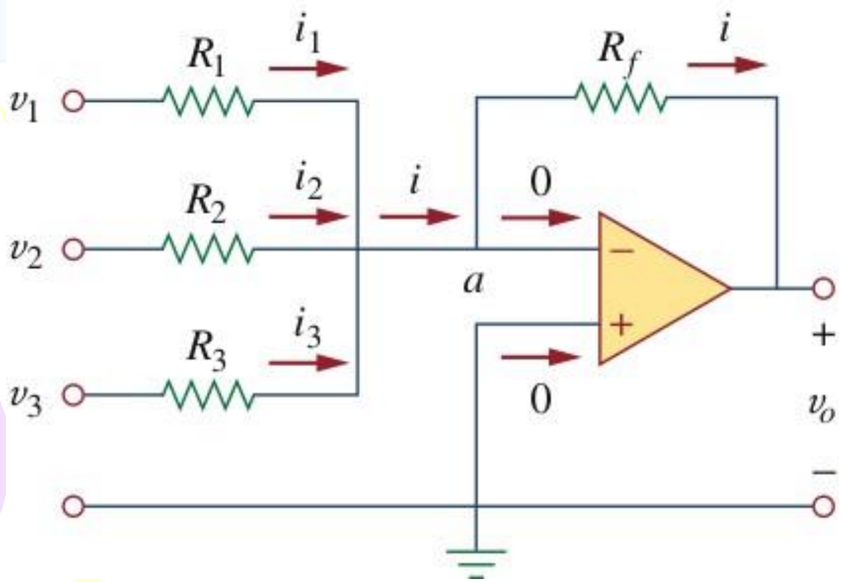


Fig.5.20

Answer: 7V

5.4 The Summing-amplifier (Summer, Adder)

The summing amplifier, shown in Fig.5.21, is a variation of the inverting amplifier. It takes advantage of the fact the inverting configuration can handle many inputs at the same time.

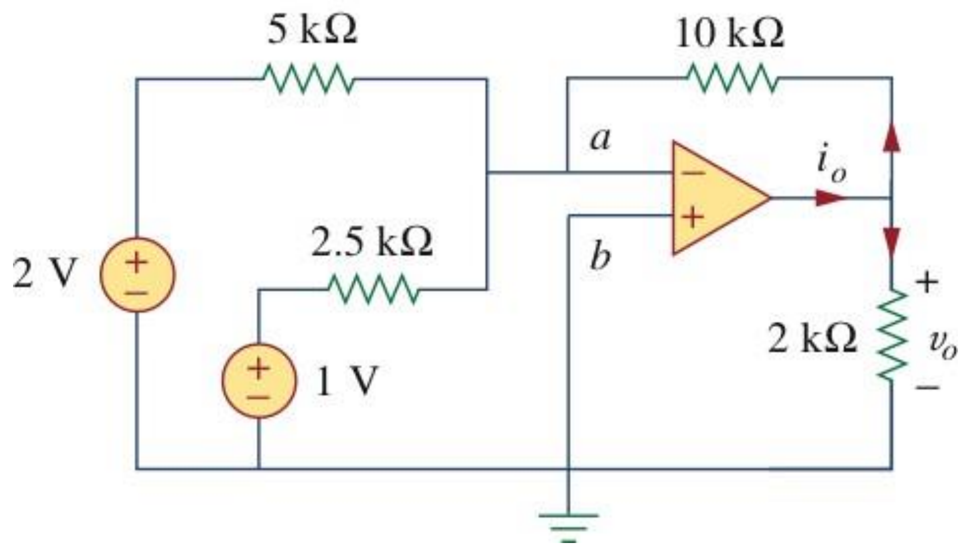


$$v_o = -\left(\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2 + \frac{R_f}{R_3}v_3\right)$$

Fig.5.21 The summing amplifier

Example 5.6 Calculate v_o and i_o in the circuit.

$$v_o = -\left(\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2 + \frac{R_f}{R_3}v_3\right)$$



$$v_0 = -\left(\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2\right)$$

This is a summer with two inputs.

$$v_o = -\left[\frac{10}{5}(2) + \frac{10}{2.5}(1)\right] = -(4+4) = -8\text{ V}$$

The current i_o is the sum of the currents through the 10-kΩ and 2-kΩ

$$i_o = \frac{v_o - 0}{10} + \frac{v_o - 0}{2}\text{ mA} = -0.8 - 0.4 = -1.2\text{ mA}$$

Practice Problem Find v_o and i_o in the circuit in Fig.5.10.

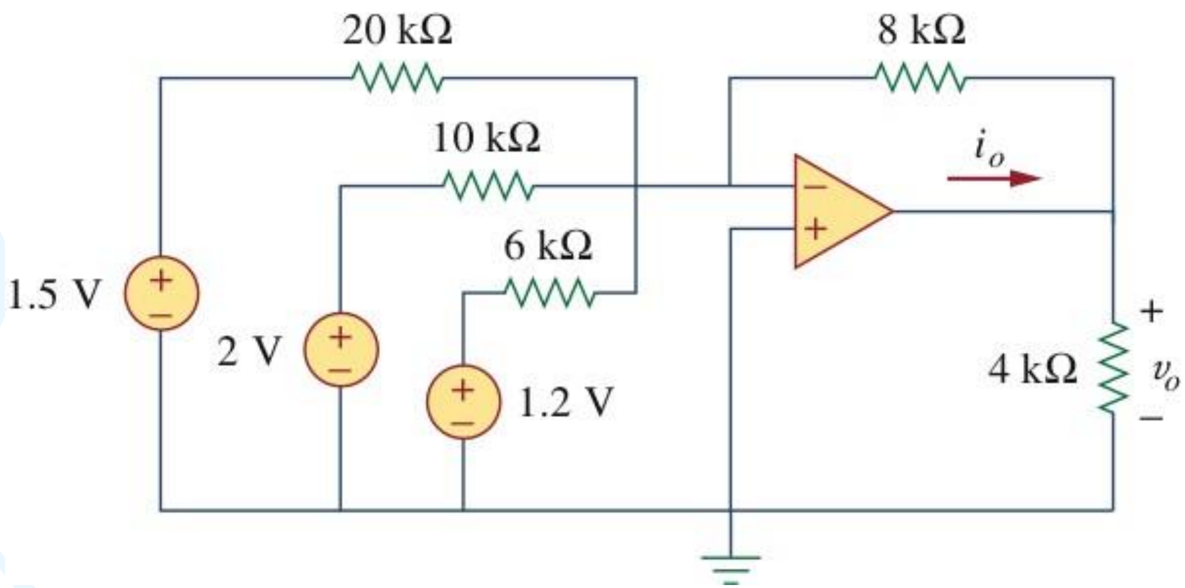
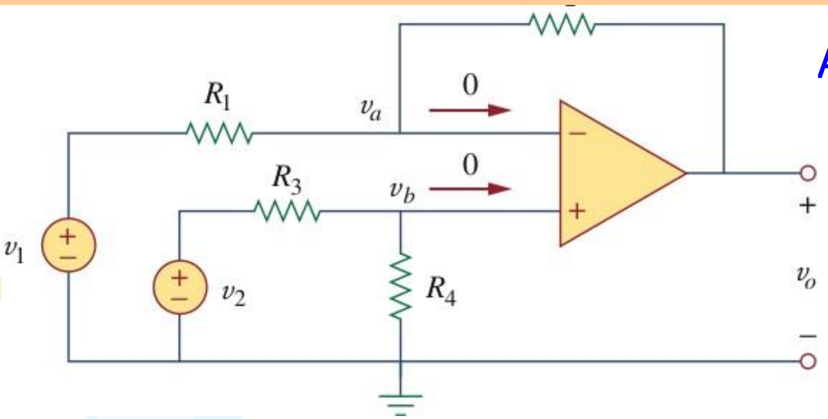


Fig.5.10

Answer: -3.8V, -1.425mA

5.5 Difference Amplifier



Applying KCL to node ***a***, $$\frac{v_1 - v_a}{R_1} = \frac{v_a - v_o}{R_2}$$

Applying KCL to node ***b*** $$\frac{v_2 - v_b}{R_3} = \frac{v_b}{R_4}$$

$$v_a = v_b$$

$$v_o = \left(\frac{R_2}{R_1} + 1\right)\frac{R_4}{R_3 + R_4} v_2 - \frac{R_2}{R_1} v_1$$

A difference amplifier is a device that amplified the difference between two inputs but rejects any signals common to the two inputs.

When $\frac{R_1}{R_2} = \frac{R_3}{R_4}$ $\quad v_o = \frac{R_2}{R_1}(v_2 - v_1)$

Subtractor:

Furthermore, if $R_1 = R_2 \text{ and } R_3 = R_4$ $\quad v_o = v_2 - v_1$

Example 5.7

An instrumentation amplifier shows in Fig.5.26 is an amplifier of low-level signal used in process control or measurement applications.

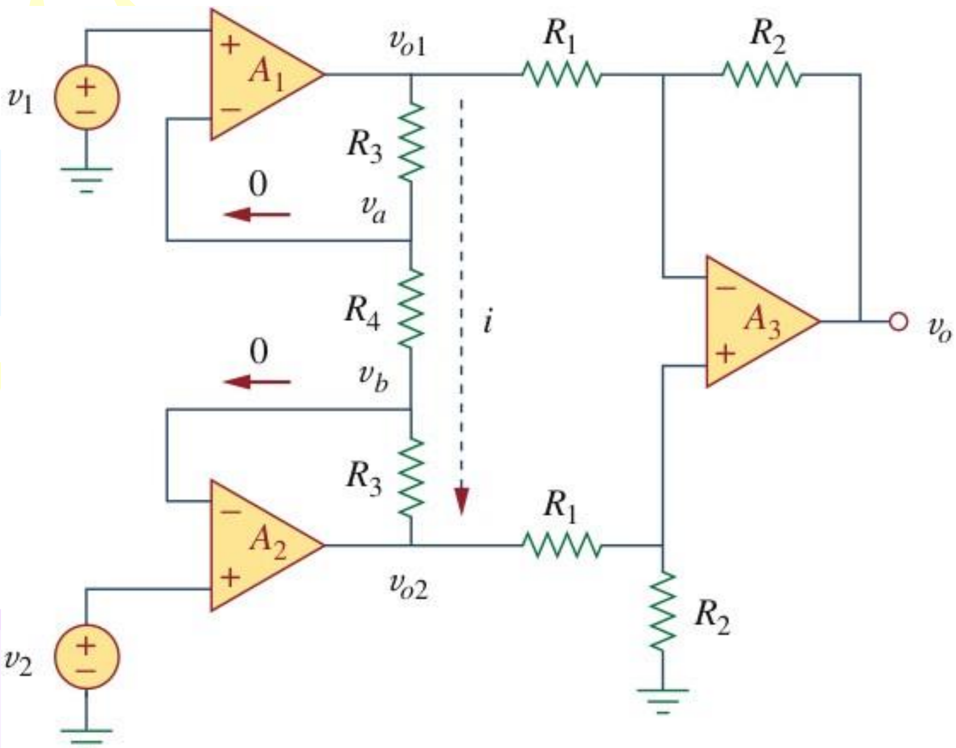


Fig. 5.26

$$v_o = \frac{R_2}{R_1}(v_{o2} - v_{o1})$$

$$v_{o1} - v_{o2} = i(R_3 + R_4 + R_3)$$
$$= i(2R_3 + R_4)$$

$$i = \frac{v_a - v_b}{R_4} \qquad v_a = v_1, \quad v_b = v_2.$$

$$i = \frac{v_1 - v_2}{R_4}$$

$$v_o = \frac{R_2}{R_1}(1 + \frac{2R_3}{R_4})(v_2 - v_1)$$

5.6 Cascaded Op Amp Circuits

- ***A cascade connection is a head-to- tail arrangement of two or more op amp circuits such that the output of one is the input of the next.***

$$A = A_1 A_2 A_3$$

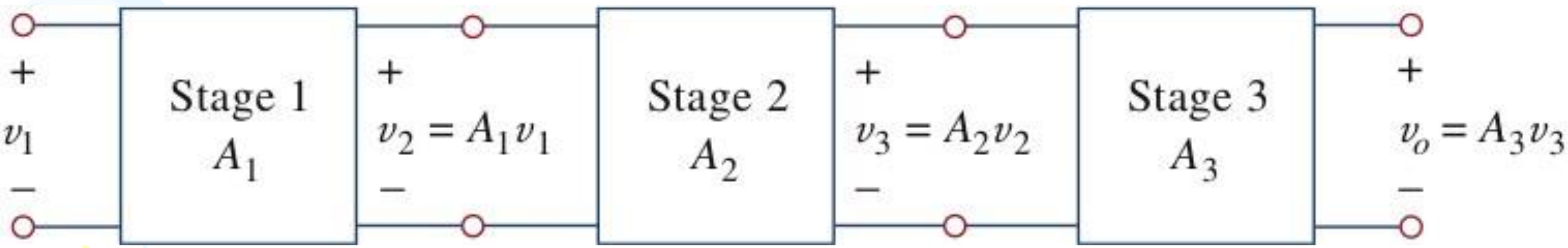
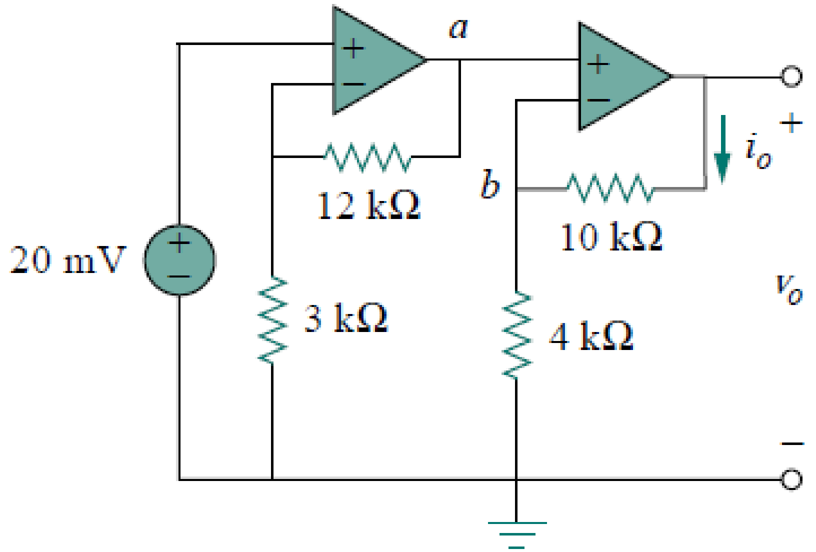


Fig. 5.13 A three-stage cascaded connection

EXAMPLE 5.9

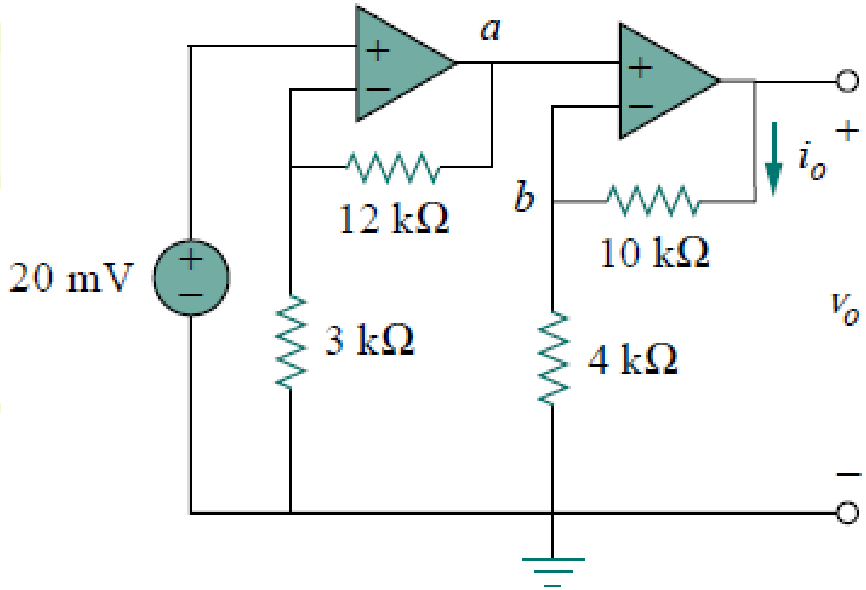
Find v_o and i_o in the circuit in Fig. 5.29.



Solution:

This circuit consists of two noninverting amplifiers cascaded. At the output of the first op amp,

$$v_a = \left(1 + \frac{12}{3}\right)(20) = 100\text{ mV}$$



$$v_a = \left(1 + \frac{12}{3}\right)(20) = 100 \text{ mV}$$

At the output of the second op amp,

$$v_o = \left(1 + \frac{10}{4}\right) v_a = (1 + 2.5)100 = 350 \text{ mV}$$

The required current i_o is the current through the 10-kΩ resistor.

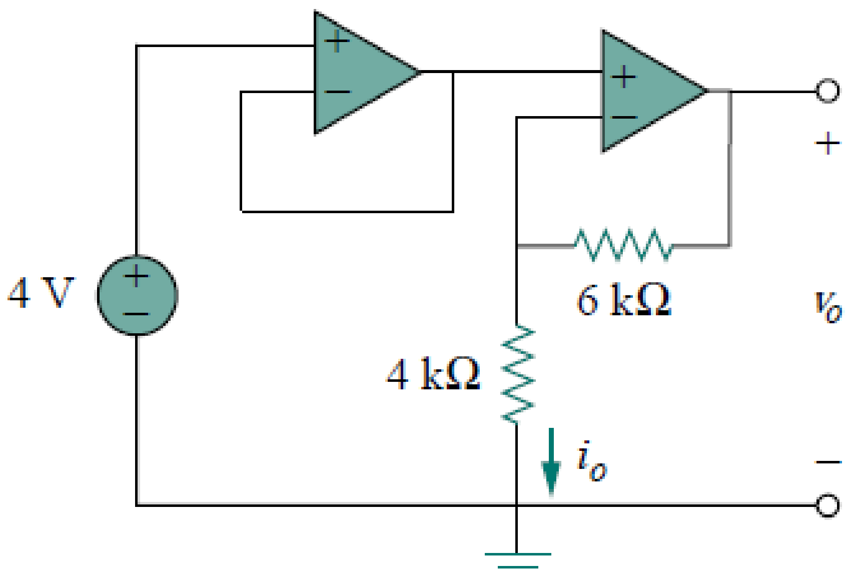
$$i_o = \frac{v_o - v_b}{10} \text{ mA}$$

But $v_b = v_a = 100$ mV. Hence,

$$i_o = \frac{(350 - 100) \times 10^{-3}}{10 \times 10^3} = 25\ \mu\text{A}$$

PRACTICE PROBLEM 5.9

Determine v_o and i_o in the op amp circuit in Fig. 5.30.



Answer: 10 V, 1 mA.

Example 5

If v_1=1V and v_2=2V find v_o in the amp circuit of Fig.5.14

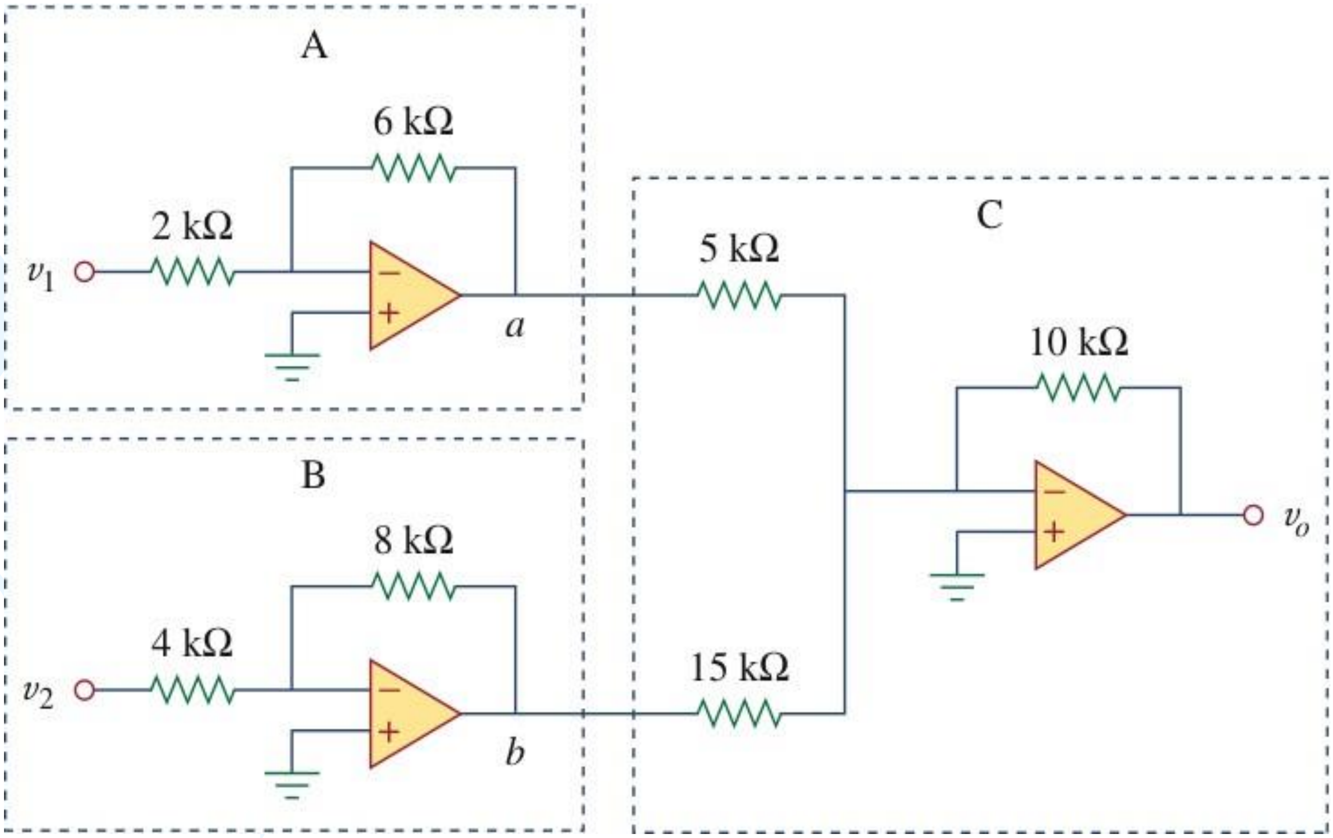
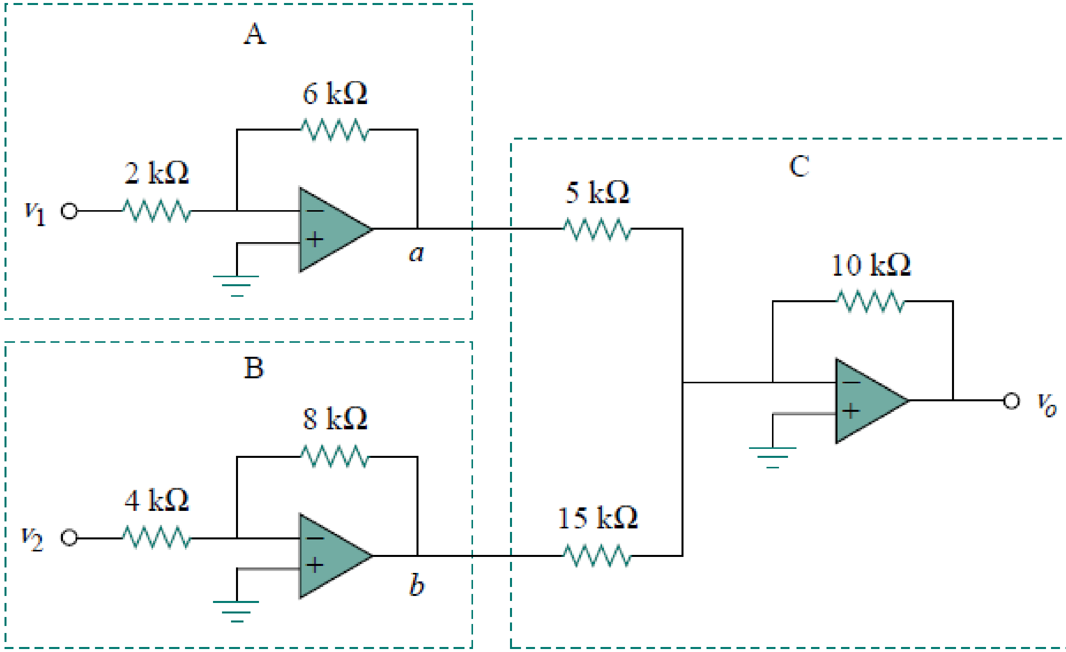


Fig.5.14



Solution:

The circuit consists of two inverters A and B and a summer C as shown in Fig. 5.31. We first find the outputs of the inverters.

$$v_a = -\frac{6}{2}(v_1) = -3(1) = -3 \text{ V}, \qquad v_b = -\frac{8}{4}(v_2) = -2(2) = -4 \text{ V}$$

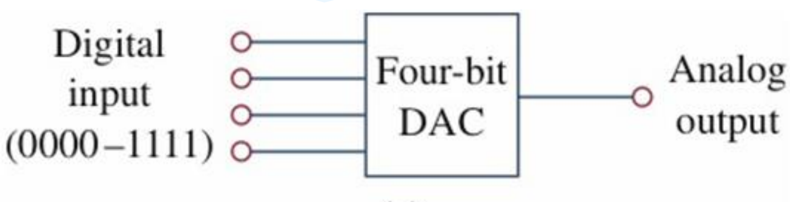
These become the inputs to the summer so that the output is obtained as

$$v_o = -\left(\frac{10}{5}v_a + \frac{10}{15}v_b\right) = -\left[2(-3) + \frac{2}{3}(-4)\right] = 8.333 \text{ V}$$

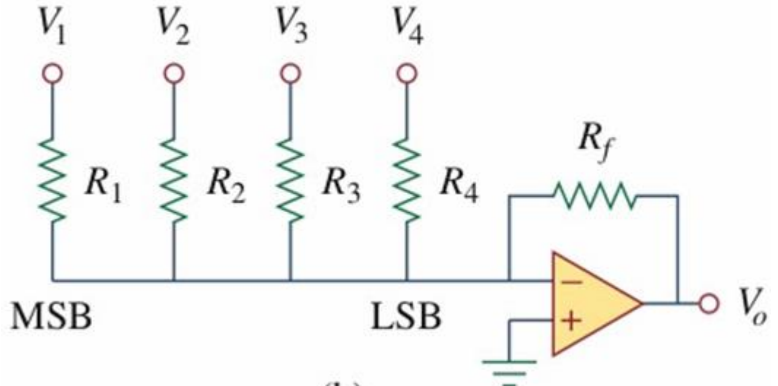
5.7 Applications

Digital-to-Analog Converter

- The Digital-to-Analog Converter (DAC) transforms digital signals into analog form.
- A typical example of a four-bit DAC is illustrated in Fig.5.15(a).
- The four-bit DAC can be realized in the ***binary weighted ladder***, shown in Fig.5.15(b).



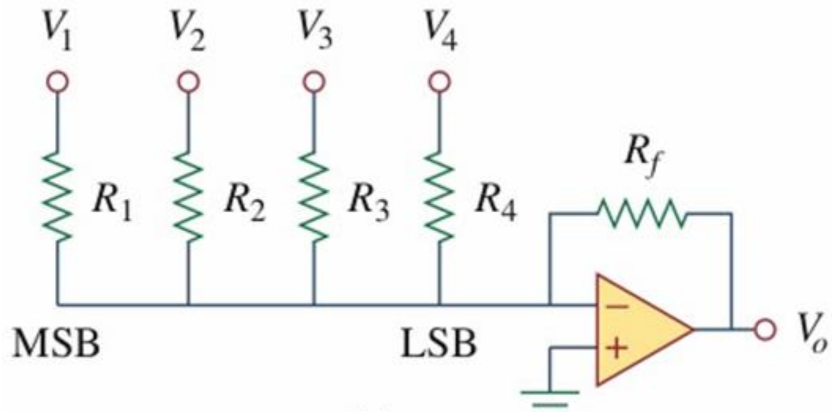
(a)



(b)

Digital-to-Analog Converter

- The bits are weights according to the magnitude of their place value, by descending value of R_f/R_n so that each lesser bit has half the weight of the next higher.



(b)

The output voltage

$$v_o = -R_f\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} + \frac{V_4}{R_4}\right)$$

The output voltage

$$v_o = -R_f \left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} + \frac{V_4}{R_4}\right)$$

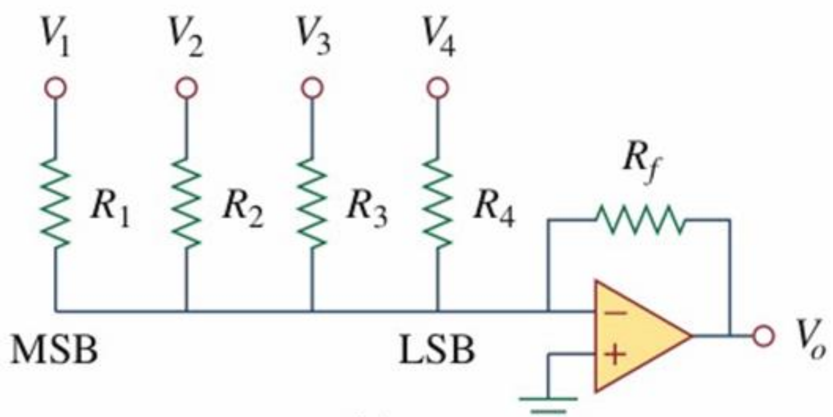
- Input voltage V_1 is called the most significant bit (MSB), and the input V_4 is the least significant (LSB)
- Each of the four binary inputs V_1 ,..., V_4 can assume only two voltage levels: 0 or 1 V.
- By using the proper input and feedback resistor values, the DAC provides a single output that is proportional to the inputs.

Example 6

In the op amp circuit of Fig.5.15(b), let R_f=10kΩ, R_1=10kΩ, R_2=20kΩ, R_3=40kΩ,and R_4=80kΩ. Find the analog output for binary [0000], [0001], [0010], …, [1111].

$$v_o = -R_f\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} + \frac{V_4}{R_4}\right)$$

$$-V_o = V_1 + 0.5V_2 + 0.25V_3 + 0.125V_4$$

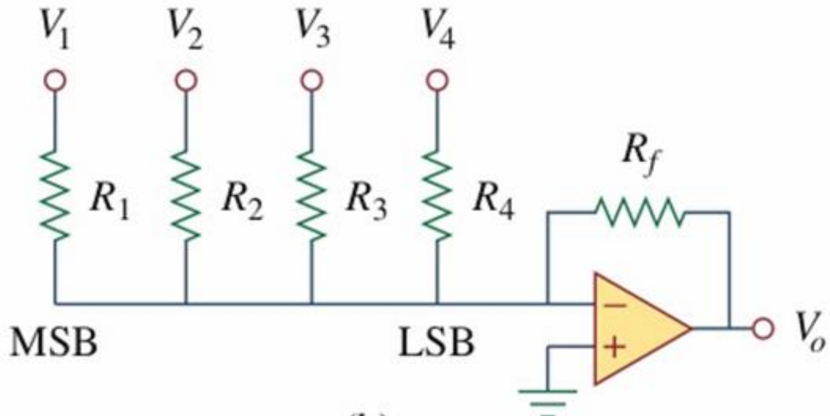


(b)

Using this equation, a digital input $[V_1V_2V_3V_4] = [0000]$ produces an analog output of $-V_o = 0$ V; $[V_1V_2V_3V_4] = [0001]$ gives $-V_o = 0.125$ V.

Example 6

$$v_o = -R_f\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} + \frac{V_4}{R_4}\right)$$



(b)

$$-V_o = V_1 + 0.5V_2 + 0.25V_3 + 0.125V_4$$

Using this equation, a digital input $[V_1V_2V_3V_4] = [0000]$ produces an analog output of $-V_o = 0$ V; $[V_1V_2V_3V_4] = [0001]$ gives $-V_o = 0.125$ V.

Similarly,

$$\begin{aligned}
[V_1\,V_2\,V_3\,V_4] &= [0010] &\Rightarrow\quad -V_o &= 0.25\text{ V}\\
[V_1\,V_2\,V_3\,V_4] &= [0011] &\Rightarrow\quad -V_o &= 0.25 + 0.125 = 0.375\text{ V}\\
[V_1\,V_2\,V_3\,V_4] &= [0100] &\Rightarrow\quad -V_o &= 0.5\text{ V}\\
&\ \vdots\\
[V_1\,V_2\,V_3\,V_4] &= [1111] &\Rightarrow\quad -V_o &= 1 + 0.5 + 0.25 + 0.125\\
&&&= 1.875\text{ V}
\end{aligned}$$

TABLE 5.2

Input and output values of the four-bit DAC.

Binary input $[V_1V_2V_3V_4]$	Decimal value	Output $-V_o$
0000	0	0
0001	1	0.125
0010	2	0.25
0011	3	0.375
0100	4	0.5
0101	5	0.625
0110	6	0.75
0111	7	0.875
1000	8	1.0
1001	9	1.125
1010	10	1.25
1011	11	1.375
1100	12	1.5
1101	13	1.625
1110	14	1.75
1111	15	1.875

Table 5.2 summarizes the result of the digital-to-analog conversion. Note that we have assumed that each bit has a value of 0.125 V. Thus, in this system, we cannot represent a voltage between 1.000 and 1.125, for example. This lack of resolution is a major limitation of digital-to-analog conversions. For greater accuracy, a word representation with a greater number of bits is required. Even then a digital representation of an analog voltage is never exact. In spite of this inexact representation, digital representation has been used to accomplish remarkable things such as audio CDs and digital photography.

Summary

- There are two fundamental rules that must be applied when analyzing ideal op amp circuits:

  1. No current ever flows into either input terminal.

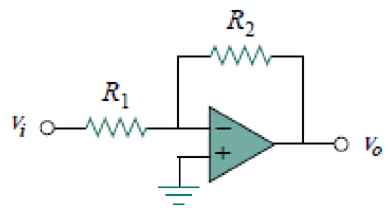
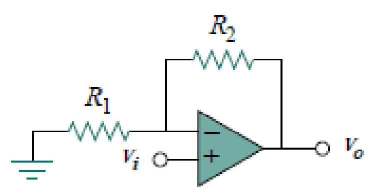
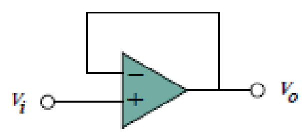
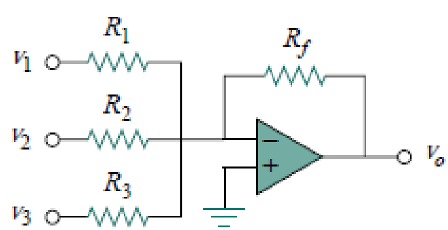
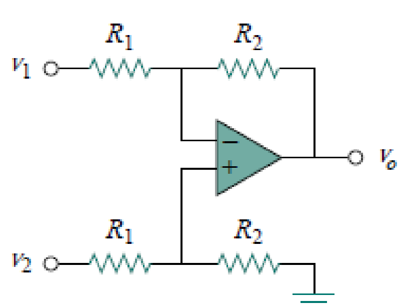
  2. No voltage ever exists between the input terminals.

- Op amp circuits are usually analyzed for an output voltage in terms of some input quantity or quantities.

- Nodal analysis is typically the best choice in analyzing op amp circuits, and it is usually better to begin at the input, and work toward the output.

- The output current of op amp cannot be assumed; it must be found after the output voltage has been determined independently.

TABLE 5.3 Summary of basic op amp circuits.

Op amp circuit	Name/output-input relationship
	Inverting amplifier $v_o = -\frac{R_2}{R_1} v_i$
	Noninverting amplifier $v_o = \left(1 + \frac{R_2}{R_1}\right) v_i$
	Voltage follower $v_o = v_i$
	Summer $v_o = -\left(\frac{R_f}{R_1} v_1 + \frac{R_f}{R_2} v_2 + \frac{R_f}{R_3} v_3\right)$
	Difference amplifier $v_o = \frac{R_2}{R_1}(v_2 - v_1)$

Fundamentals of Electric Circuit 2021.4

Chapter 6
Capacitors and Inductors

Chapter 6 Capacitors and Inductors

In this chapter, we are particular interested in the following objectives

- Finding the voltage-current relationship of ideal capacitors;
- Finding the voltage-current relationship of ideal inductors;
- Calculating the energy stored in inductors and capacitors;

- Methods for reducing series/parallel combinations of inductors;
- Methods for reducing series/parallel combinations of capacitor;
- Predicting the behavior of op amp circuits with capacitors.

Chapter 6 Capacitors and Inductors

6.1 Capacitors

6.2 Series and Parallel Capacitors

6.3 Inductors

6.4 Series and Parallel Inductors

6.5 Applications

6.1 Introduction

- Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called *storage* elements.
- The circuit analysis techniques covered in Chapters 3 and 4 are equally applicable to circuits with capacitors and inductors.

6.2 Capacitors

- A capacitor is a passive element designed to **store energy** in its **electric field.**
- The capacitor is said to store the electric charge.
- The amount of charge stored, represented by *q*, is directly proportional to the applied voltage *v*

$$q = Cv$$

- **Capacitance** *C* is the ratio of q to v, measured in farads (F).
- Typically, capacitors have values in the picofarad (pF) to microfarad (μF) range.

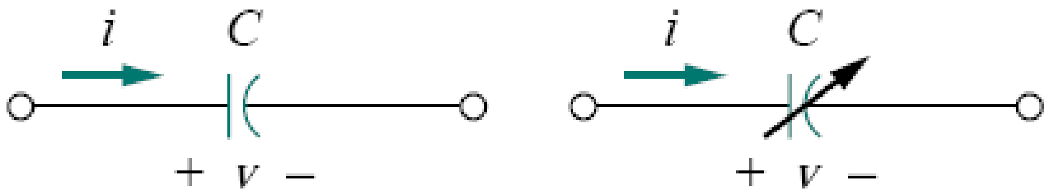


Figure 6.3 Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.

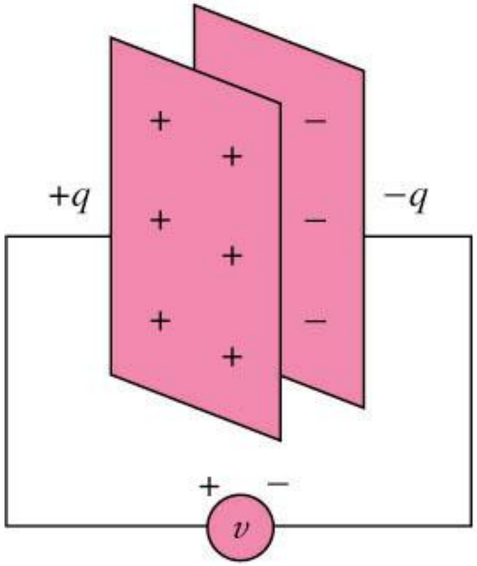
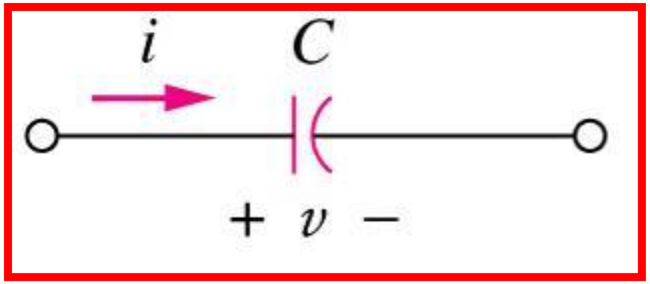


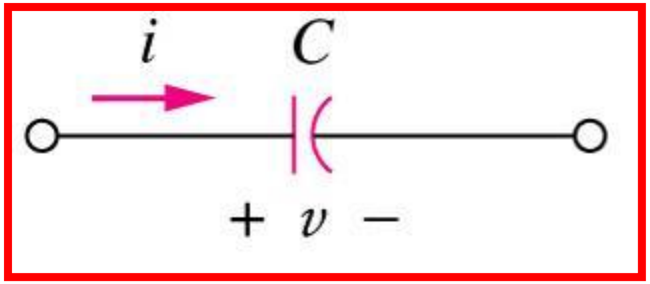
Figure 5.2 A capacitor with applied voltage *v*.

- According to the passive sign convention, current is considered to flow into the positive terminal of the capacitor when the capacitor is being charged, and out of the positive terminal when the capacitor is discharging.

- If i is flowing into the +v terminal of C
  - Charging => i is +v
  - Discharging => i is –v



$$q = C\,v$$



- The current-voltage relationship of capacitor according to above convention is

$$\boxed{i = C\,\frac{d\,v}{d\,t}}$$

$$v = \frac{1}{C}\int_{-\infty}^{t} i\;dt$$

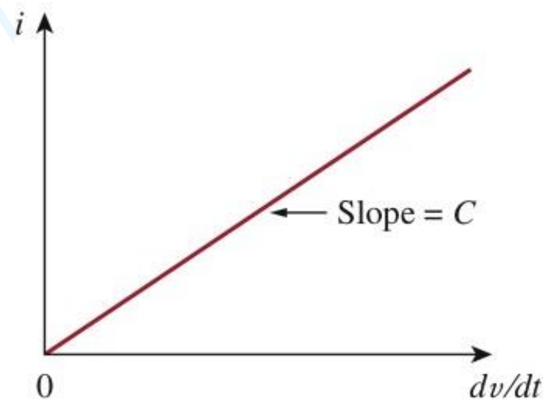
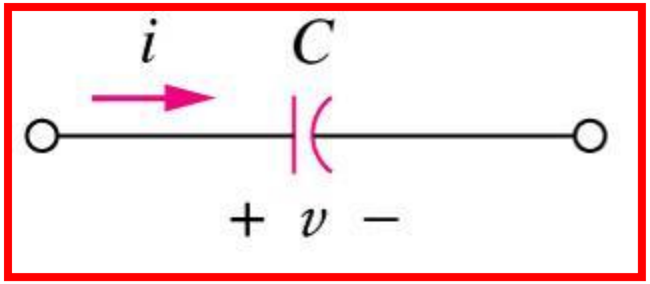


Fig.6.6 Current-Voltage relationship of a capacitor

$$\boxed{v = \frac{1}{C}\int_{t_0}^{t} i\,d\,t + v(t_0)}$$

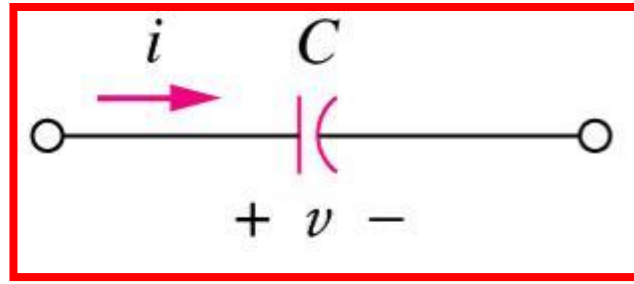
$$\boxed{v = \frac{1}{C}\int_{t_0}^{t} i \, dt + v(t_0)}$$



- Above Equation shows that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has memory—a property that is often exploited.

- The instantaneous power delivered to the capacitor is

$$p = vi = Cv\frac{dv}{dt}$$



The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p\,dt = C\int_{-\infty}^{t} v\frac{dv}{dt}dt = C\int_{-\infty}^{t} v\,dv = \frac{1}{2}Cv^2\Big|_{t=-\infty}^{t}$$

- The energy, ***w***, stored in the capacitor is

$$\boxed{w = \frac{1}{2}C\,v^2}$$

- The energy stored in the electric field that exists between the plates of the capacitor.

important properties of a capacitor:

$$\boxed{i = C\frac{dv}{dt}}$$

- 1. A capacitor is an **open circuit** to dc (dv/dt = 0).
- 2. The voltage on the capacitor must be continuous.
  - The voltage on a capacitor **cannot change abruptly**.

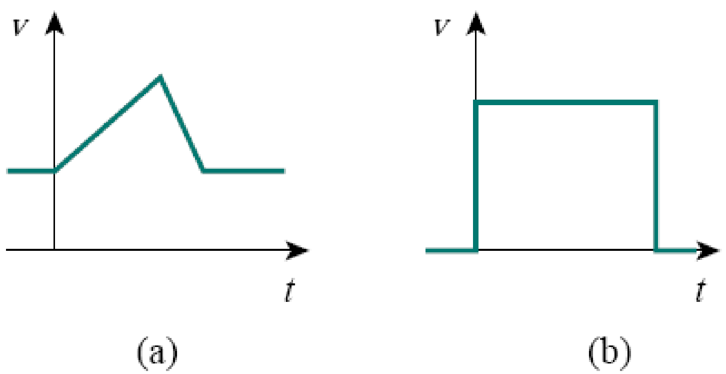
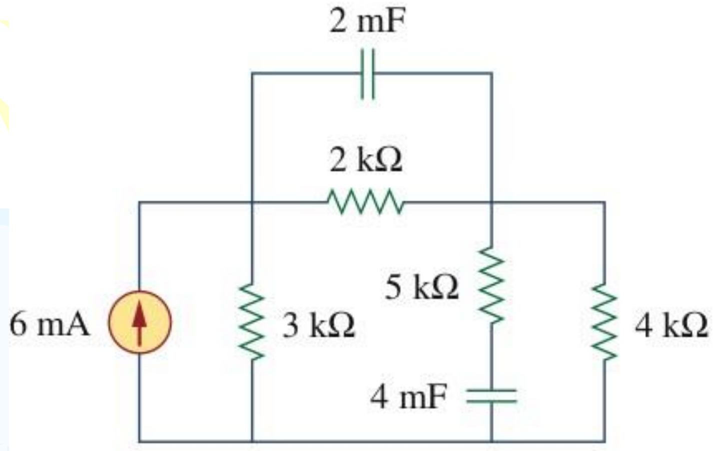


Figure 6.7 Voltage across a capacitor: (a) allowed, (b) not allowable; an abrupt change is not possible.

EXAMPLE 6.5

Find the energy stored in each capacitor in Fig.6.12(a) under dc conditions



(a)

Solution:

Fig.6.12

Under dc conditions, we replace each capacitor with an open circuit,

$$i = \frac{3}{3+2+4}(6\text{ mA}) = 2\text{ mA} \qquad v_1 = 2000i = 4\text{ V} \qquad v_2 = 4000i = 8\text{ V}$$

$$w_1 = \frac{1}{2}C_1 v_1^2 = \frac{1}{2}(2\times 10^{-3})(4)^2 = 16\text{ mJ} \quad w_2 = \frac{1}{2}C_2 v_2^2 = \frac{1}{2}(4\times 10^{-3})(8)^2 = 128\text{ mJ}$$

PRACTICE PROBLEM 6.5

Under dc conditions, find the energy stored in the capacitors in Fig. 6.13.

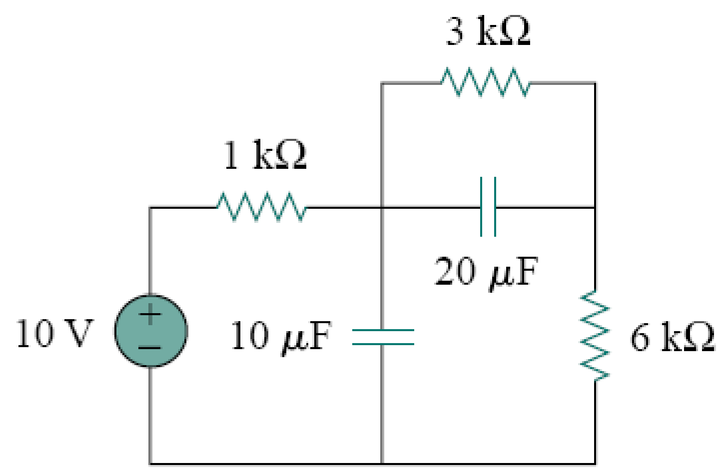
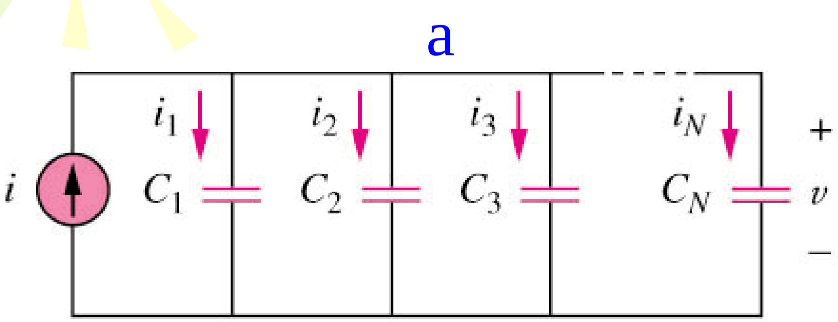


Figure 6.13 For Practice Prob. 6.5.

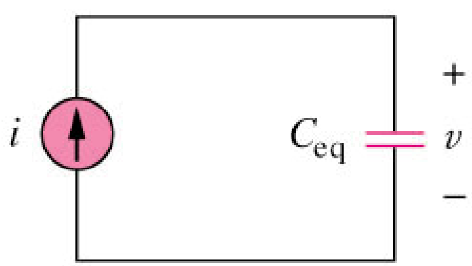
Answer: 405 μJ, 90 μJ.

6.2 Series and Parallel Capacitors

- The equivalent capacitance of *N* **parallel-connected** capacitors is the sum of the individual capacitances.



(a)



(b)

$$C_{eq} = C_1 + C_2 + \ldots + C_N$$

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

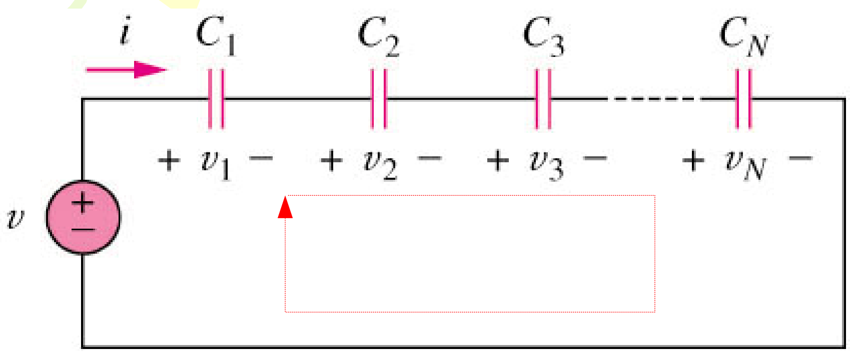
$$i_k = C_k \, dv/dt$$

$$i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \cdots + C_N \frac{dv}{dt}$$

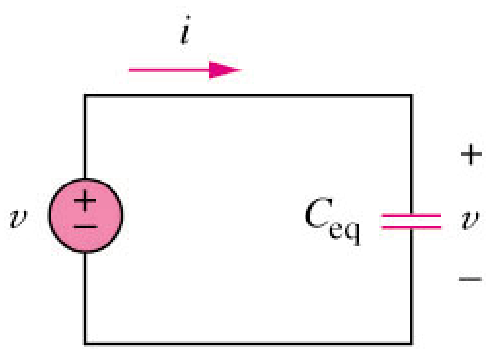
$$= \left(\sum_{k=1}^{N} C_k \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt}$$

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N}$$

- The equivalent capacitance of N **series-connected** capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.



(a)



(b)

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}$$

$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

But $v_k = \frac{1}{C_k}\int_{t_0}^{t} i(t)\,dt + v_k(t_0)$.

$$\begin{aligned} v &= \frac{1}{C_1}\int_{t_0}^{t} i(t)\,dt + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(t)\,dt + v_2(t_0) \\ &\qquad + \cdots + \frac{1}{C_N}\int_{t_0}^{t} i(t)\,dt + v_N(t_0) \\ &= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(t)\,dt + v_1(t_0) + v_2(t_0) \\ &\qquad + \cdots + v_N(t_0) \\ &= \frac{1}{C_{\mathrm{eq}}}\int_{t_0}^{t} i(t)\,dt + v(t_0) \end{aligned}$$

- We observe that capacitors in parallel combine in the same manner as resistors in series.
- Note that capacitors in series combine in the same manner as resistors in parallel.
- For N = 2 (i.e., two capacitors in series),

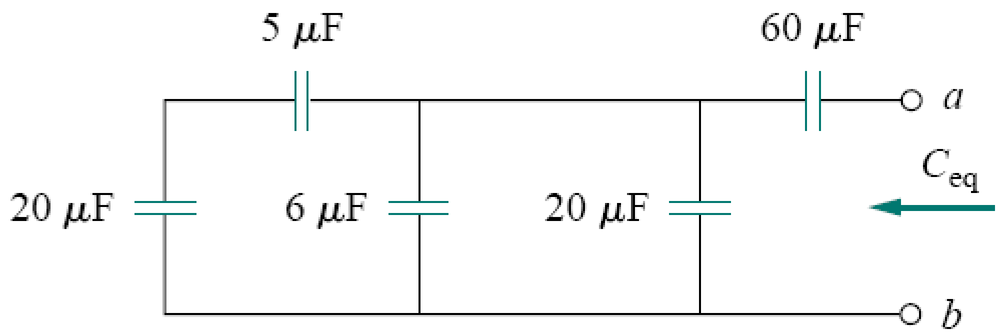
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$\boxed{C_{eq} = \frac{C_1 C_2}{C_1 + C_2}}$$

EXAMPLE 6.6

Find the equivalent capacitance seen between terminals *a* and *b* of the circuit in Fig. 6.16.



Solution:

The 20-μF and 5-μF capacitors are in series; their equivalent capacitance is

$$\frac{20 \times 5}{20 + 5} = 4\ \mu\text{F}$$

This 4-μF capacitor is in parallel with the 6-μF and 20-μF capacitors; their combined capacitance is

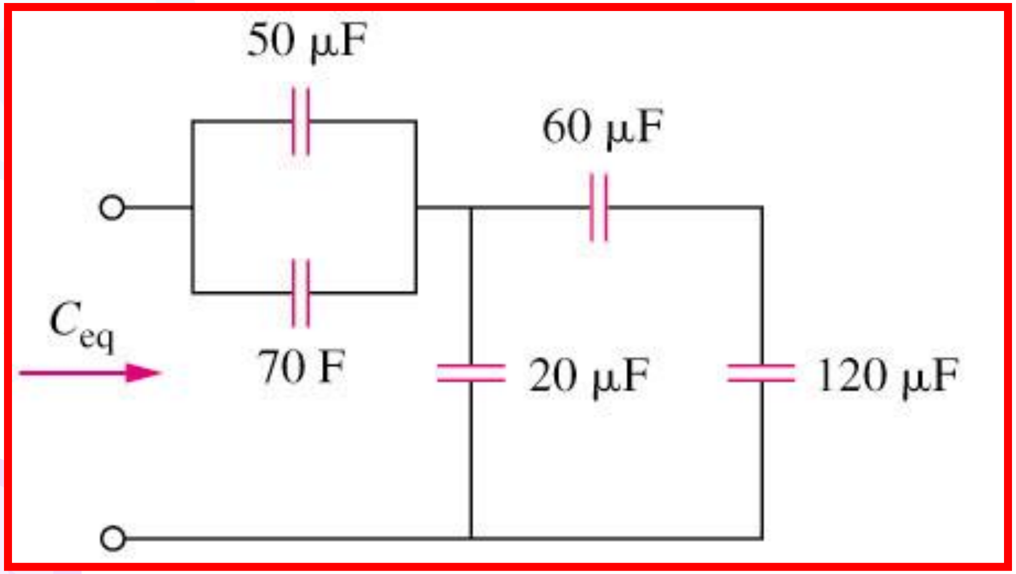
$$4 + 6 + 20 = 30\ \mu\text{F}$$

This 30-μF capacitor is in series with the 60-μF capacitor. Hence, the equivalent capacitance for the entire circuit is

$$C_{eq} = \frac{30 \times 60}{30 + 60} = 20\ \mu\text{F}$$

PRACTICE PROBLEM 6.6

Find the equivalent capacitance seen at the terminals of the circuit in the circuit shown below:

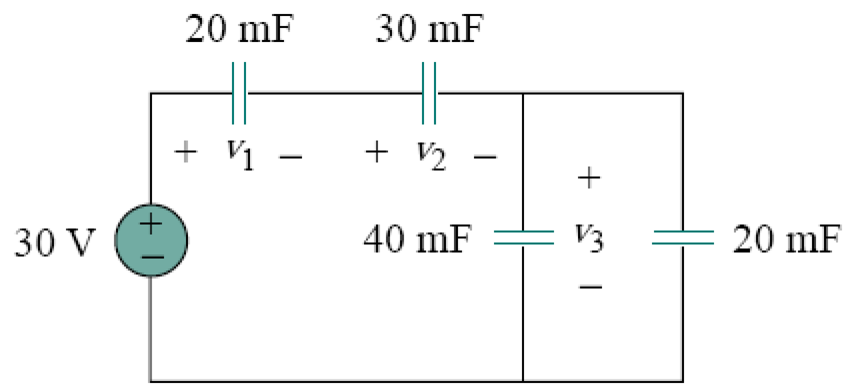


Answer:

C_{eq} = 40μF

EXAMPLE 6.7

For the circuit in Fig. 6.18, find the voltage across each capacitor.



Solution:

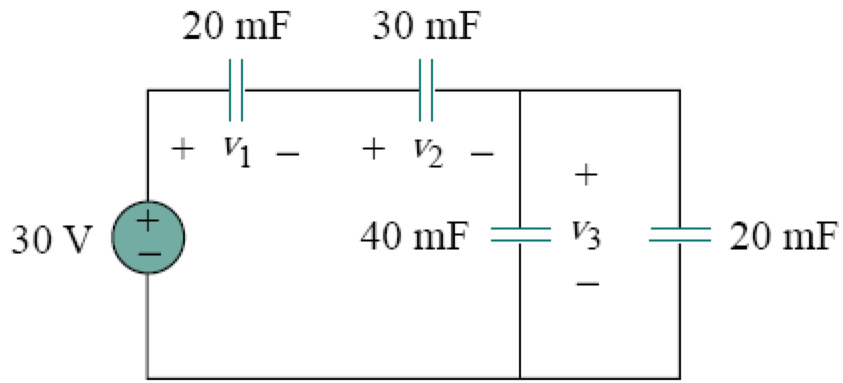
We first find the equivalent capacitance C_{eq}, shown in Fig. 6.19. The two parallel capacitors in Fig. 6.18 can be combined to get $40 + 20 = 60$ mF.

This 60-mF capacitor is in series with the 20-mF and 30-mF capacitors. Thus,

$$C_{eq} = \frac{1}{\frac{1}{60} + \frac{1}{30} + \frac{1}{20}} \text{ mF} = 10 \text{ mF}$$

The total charge is

$$q = C_{eq} v = 10 \times 10^{-3} \times 30 = 0.3 \text{ C}$$



$$q = C_{\text{eq}} v = 10 \times 10^{-3} \times 30 = 0.3 \text{ C}$$

This is the charge on the 20-mF and 30-mF capacitors, because they are in series with the 30-V source. (A crude way to see this is to imagine that charge acts like current, since $i = dq/dt$.) Therefore,

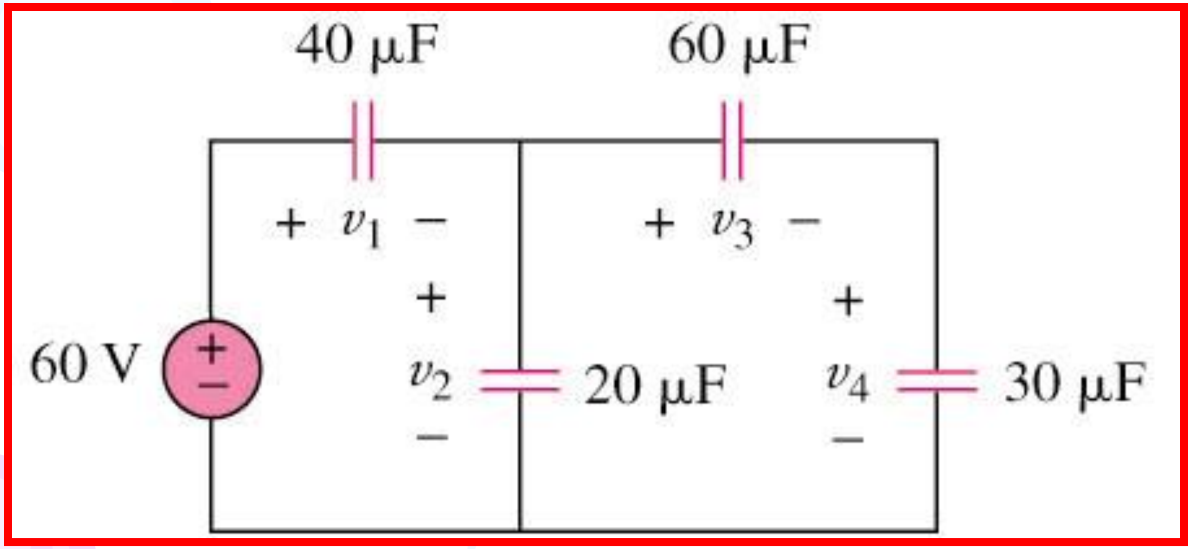
$$v_1 = \frac{q}{C_1} = \frac{0.3}{20 \times 10^{-3}} = 15 \text{ V}$$

$$v_2 = \frac{q}{C_2} = \frac{0.3}{30 \times 10^{-3}} = 10 \text{ V}$$

$$v_3 = 30 - v_1 - v_2 = 5 \text{ V}$$

Example 4

Find the voltage across each of the capacitors in the circuit shown below:



Answer:

v_1 = 30V

v_2 = 30V

v_3 = 10V

v_4 = 20V

6.3 Inductors

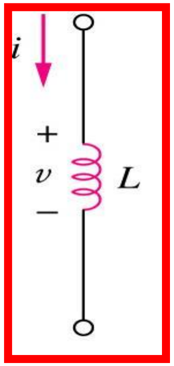
- An inductor is a passive element designed to store energy in its magnetic field.
- An inductor consists of a coil of conducting wire.
- If current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current. Using the passive sign convention,

$$v = L\frac{di}{dt}$$

- where L is the constant of proportionality called the *inductance* of the inductor. The unit of inductors is Henry (H), mH (10^{-3}) and μH (10^{-6}).

- The current-voltage relationship of an inductor:

$$v = L\frac{di}{dt} \qquad i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$$



- The inductor is designed to store energy in its magnetic field.
- The power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt}\right)i$$

$$w = \frac{1}{2}Li^2$$

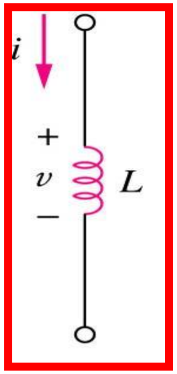
The energy stored is

$$w = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t}\left(L\frac{di}{dt}\right)i\,dt$$

$$= L\int_{-\infty}^{t} i\,di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty)$$

- 1.An inductor acts like a short circuit to dc ($di/dt = 0$) and its current cannot change abruptly.

$$v = L\frac{di}{dt}$$



- 2. ***The current of an inductor cannot change instantaneously.***
- 3.Like the ideal capacitor, the ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time.
  - The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.

EXAMPLE 6.8

The current through a 0.1-H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

Solution:

Since $v = L\,di/dt$ and $L = 0.1$ H,

$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1 - 5t)\text{ V}$$

The energy stored is

$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t}\text{ J}$$

EXAMPLE 6.9

Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

$$v = L\frac{di}{dt}$$

Also find the energy stored within $0 < t < 5$ s.

Solution:

Since $i = \dfrac{1}{L}\displaystyle\int_{t_0}^{t} v(t)\,dt + i(t_0)$ and $L = 5$ H,

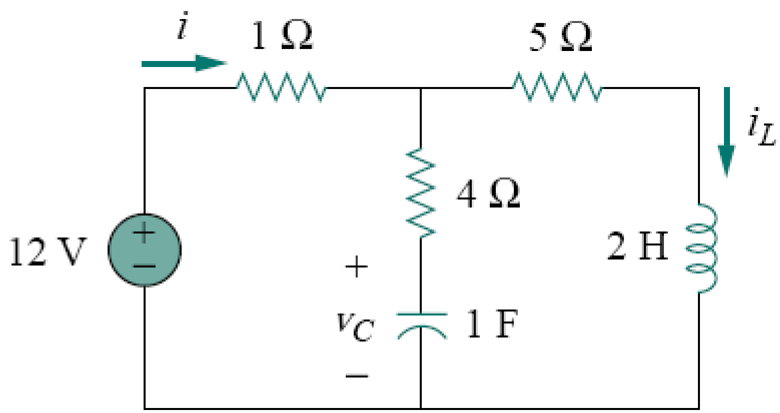
$$i = \frac{1}{5}\int_0^t 30t^2\,dt + 0 = 6 \times \frac{t^3}{3} = 2t^3 \text{ A}$$

The power $p = vi = 60t^5$, and the energy stored is then

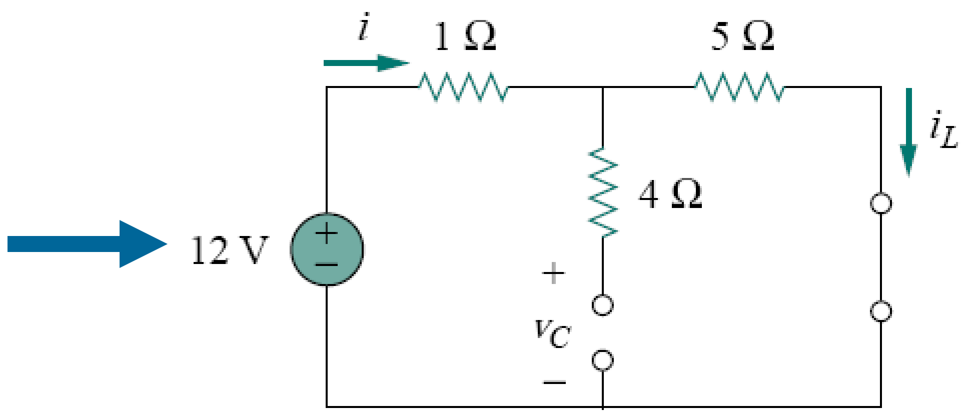
$$w = \int p\,dt = \int_0^5 60t^5\,dt = 60\frac{t^6}{6}\bigg|_0^5 = 156.25 \text{ kJ}$$

EXAMPLE 6.10

Consider the circuit in Fig. 6.27(a). Under dc conditions, find: (a) i, v_C, and i_L, (b) the energy stored in the capacitor and inductor.



(a)



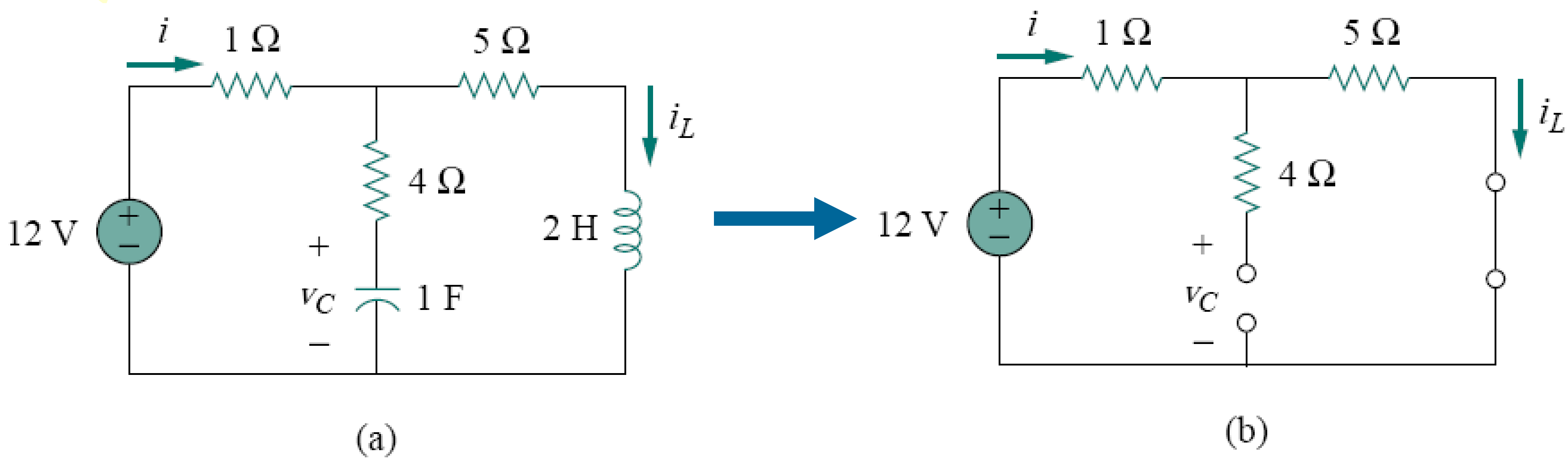
(b)

(a) Under dc conditions, we replace the capacitor with an open circuit and the inductor with a short circuit, as in Fig. 6.27(b). It is evident from Fig. 6.27(b) that

$$i = i_L = \frac{12}{1+5} = 2 \text{ A} \qquad v_C = 5i = 10 \text{ V}$$

EXAMPLE 6.10

Consider the circuit in Fig. 6.27(a). Under dc conditions, find: (a) i, v_C, and i_L, (b) the energy stored in the capacitor and inductor.



(b) The energy in the capacitor is

$$w_C = \frac{1}{2} C v_C^2 = \frac{1}{2}(1)(10^2) = 50 \text{ J}$$

and that in the inductor is

$$w_L = \frac{1}{2} L i_L^2 = \frac{1}{2}(2)(2^2) = 4 \text{ J}$$

PRACTICE PROBLEM 6.10

Determine v_C, i_L, and the energy stored in the capacitor and inductor in the circuit of Fig. 6.28 under dc conditions.

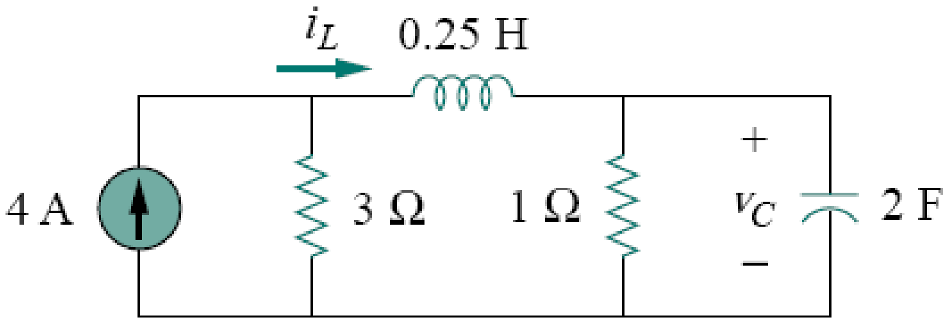


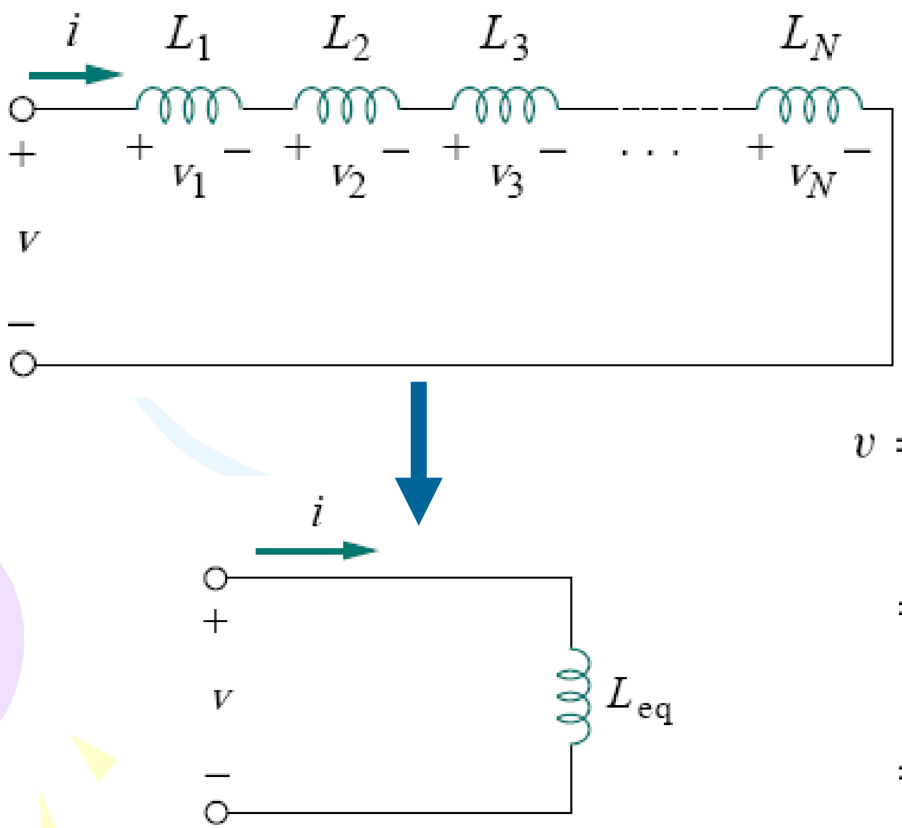
Figure 6.28 For Practice Prob. 6.10.

Answer: 3 V, 3 A, 9 J, 1.125 J.

6.4 Series and Parallel Inductors

- The equivalent inductance of **series-connected** inductors is the sum of the individual inductances.

$$L_{eq} = L_1 + L_2 + \ldots + L_N$$



$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

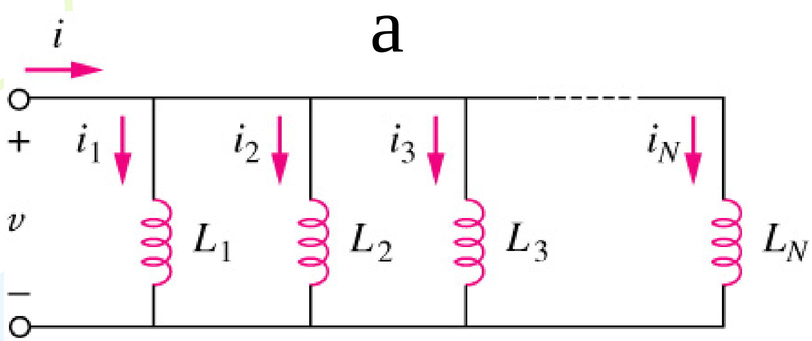
Substituting $v_k = L_k\, di/dt$ results in

$$v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} + \cdots + L_N \frac{di}{dt}$$

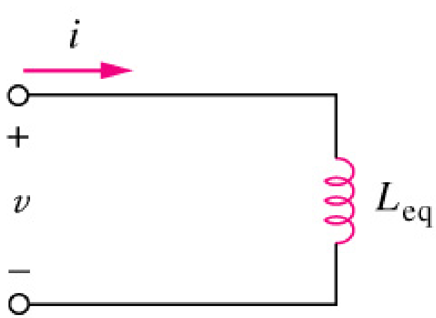
$$= (L_1 + L_2 + L_3 + \cdots + L_N) \frac{di}{dt}$$

$$= \left(\sum_{k=1}^{N} L_k \right) \frac{di}{dt} = L_{eq} \frac{di}{dt}$$

- The equivalent inductance of **parallel** inductors is the reciprocal of the sum of the reciprocals of the individual inductances.



(a)



(b)

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

But $i_k = \dfrac{1}{L_k}\displaystyle\int_{t_0}^{t} v\, dt + i_k(t_0)$; hence,

$$i = \frac{1}{L_1}\int_{t_0}^{t} v\, dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\, dt + i_2(t_0) + \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\, dt + i_N(t_0)$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\, dt + i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

$$= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v\, dt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{eq}}\int_{t_0}^{t} v\, dt + i(t_0)$$

$$\boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}}$$

EXAMPLE 6.11

Find the equivalent inductance of the circuit shown in Fig. 6.31.

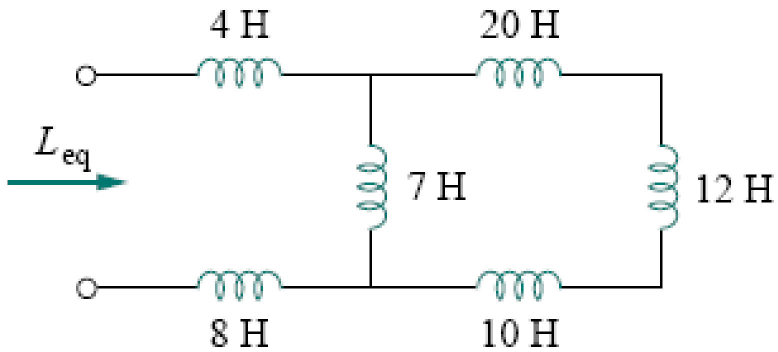


Figure 6.31 For Example 6.11.

Solution:

The 10-H, 12-H, and 20-H inductors are in series; thus, combining them gives a 42-H inductance. This 42-H inductor is in parallel with the 7-H inductor so that they are combined, to give

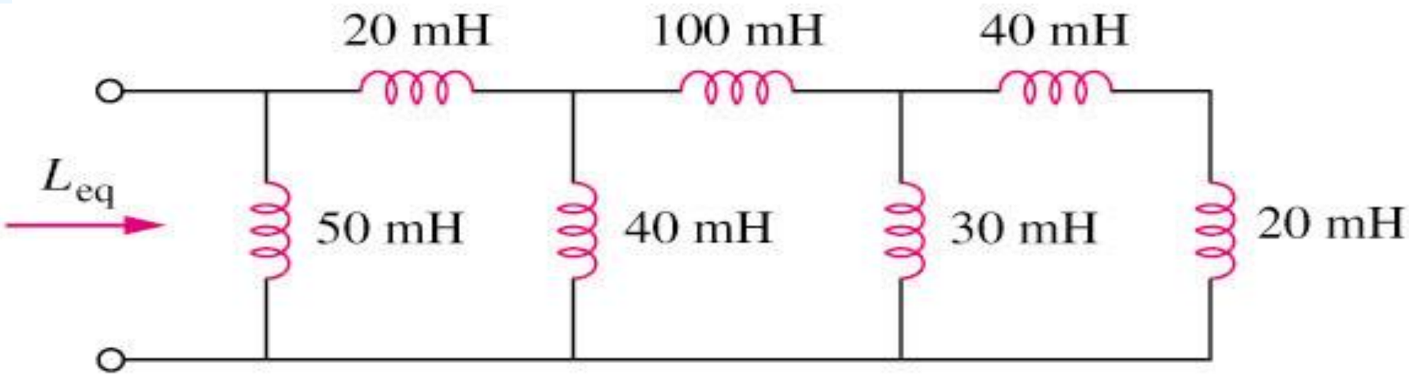
$$\frac{7 \times 42}{7 + 42} = 6 \text{ H}$$

This 6-H inductor is in series with the 4-H and 8-H inductors. Hence,

$$L_{eq} = 4 + 6 + 8 = 18 \text{ H}$$

Example 6.12

Calculate the equivalent inductance for the inductive ladder network in the circuit shown below:



Answer:

L_{eq} = 25mH

Current and voltage relationship for R, L, C

TABLE 6.1 Important characteristics of the basic elements.[†]

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} i\,dt + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

[†] *Passive sign convention is assumed.*

EXAMPLE 6.12

For the circuit in Fig. 6.33, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find: (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.

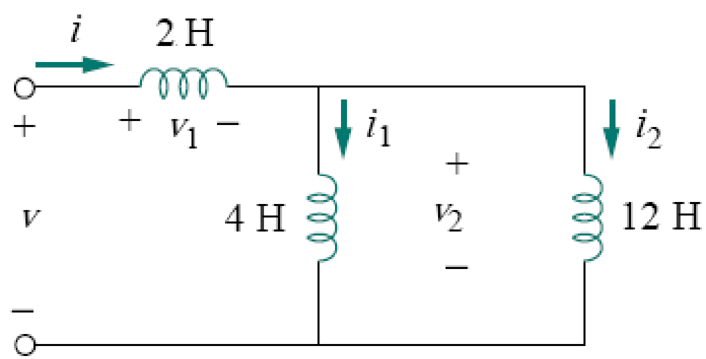


Figure 6.33 For Example 6.12.

Solution:

(a) From $i(t) = 4(2 - e^{-10t})$ mA, $i(0) = 4(2 - 1) = 4$ mA. Since $i = i_1 + i_2$,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

EXAMPLE 6.12

For the circuit in Fig. 6.33, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find: (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.

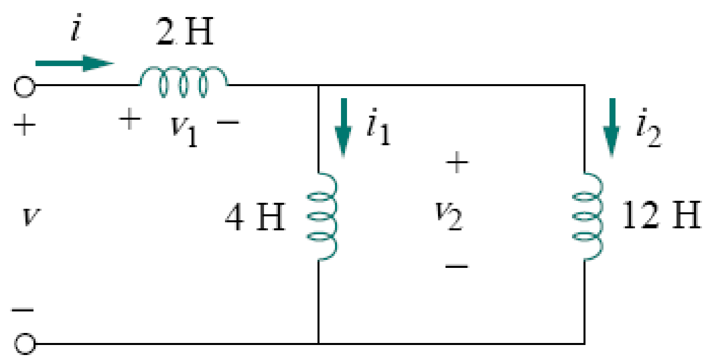


Figure 6.33 For Example 6.12.

(b) The equivalent inductance is

$$L_{eq} = 2 + 4 \parallel 12 = 2 + 3 = 5 \text{ H}$$

$$v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV} = 200e^{-10t} \text{ mV}$$

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV} = 80e^{-10t} \text{ mV}$$

Since $v = v_1 + v_2$,

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

EXAMPLE 6.12

For the circuit in Fig. 6.33, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find: (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.

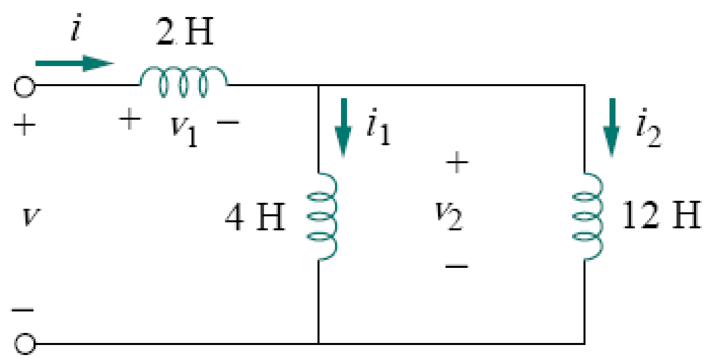


Figure 6.33 For Example 6.12.

(c) The current i_1 is obtained as

$$i_1(t) = \frac{1}{4}\int_0^t v_2\,dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\,dt + 5\text{ mA}$$

$$= -3e^{-10t}\Big|_0^t + 5\text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t}\text{ mA}$$

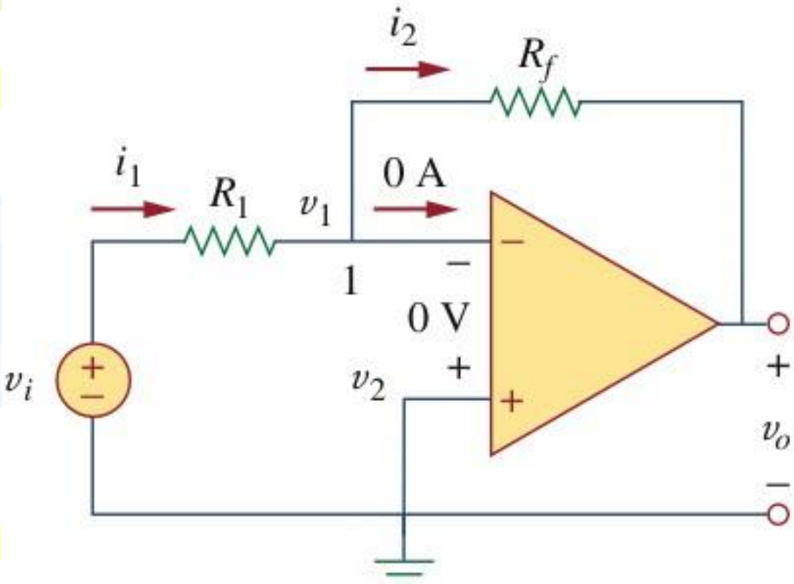
$$i_2(t) = \frac{1}{12}\int_0^t v_2\,dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\,dt - 1\text{ mA}$$

$$= -e^{-10t}\Big|_0^t - 1\text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t}\text{ mA}$$

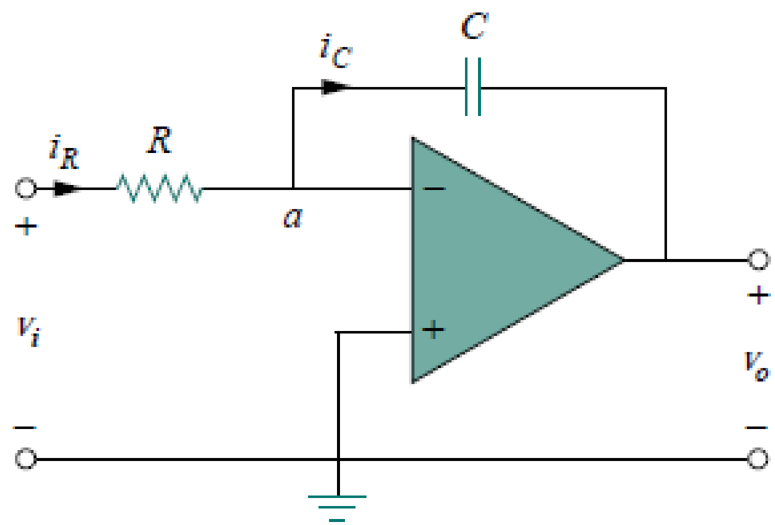
6.5 Applications

Capacitors and inductors possess the following three special properties that make them very useful in electric circuits:

- The capacity to store energy makes them useful as temporary voltage or current sources. Thus, they can be used for generating a large amount of current or voltage for a short period of time.
- Capacitors and inductors are frequency sensitive. This property makes them useful for frequency discrimination.

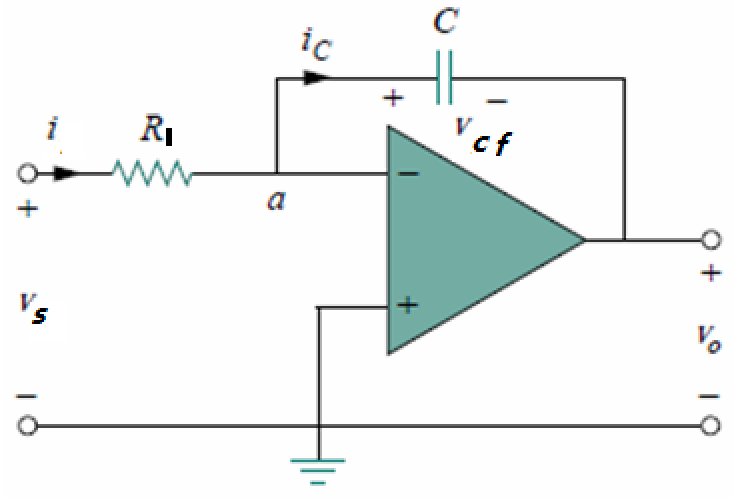


i_2
R_f
i_1
R_1
v_1
0 A
1
0 V
v_2
v_i
v_o



C
i_C
i_R
R
a
v_i
v_o

Performing nodal analysis at the inverting input,



$$i = \frac{v_s}{R_1} \qquad i_c = C\frac{dv_{C_f}}{dt} = -C\frac{dv_o}{dt}$$

Thus
$$\frac{v_s}{R_1} = -C\frac{dv_o}{dt}$$

Integrating both sides gives

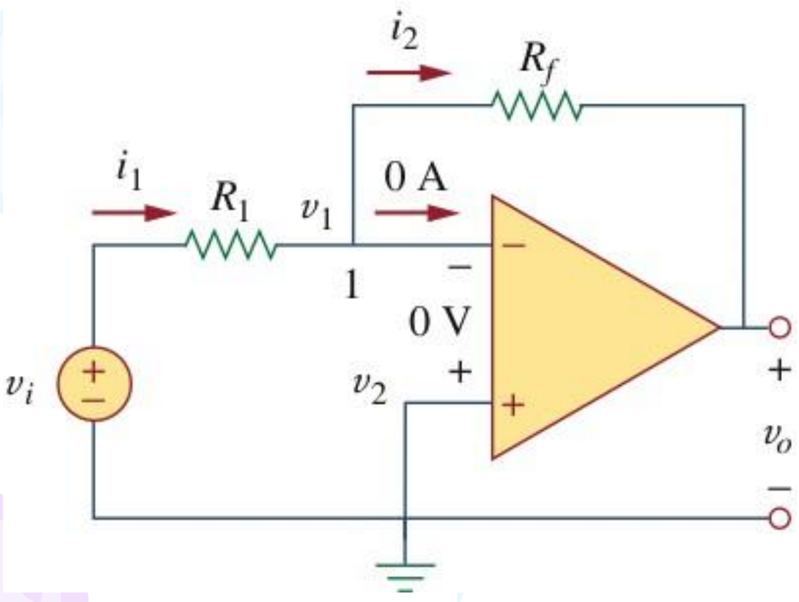
$$v_o(t) - v_o(0) = -\frac{1}{R_1 C_f}\int_0^t v_s \, dt$$

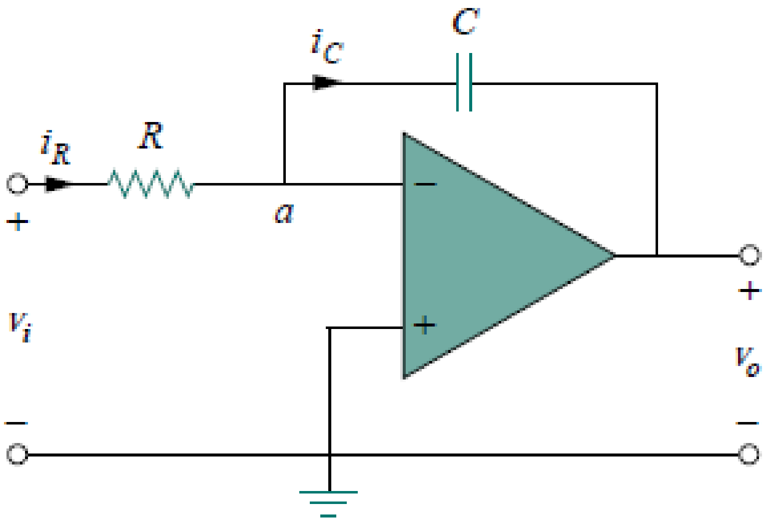
Assuming

$$v_o(0) = 0 \qquad v_o(t) = -\frac{1}{R_1 C_f}\int_0^t v_s \, dt$$

Integrator

An integrator is an op amp circuit whose output is proportional to the integral of the input signal.





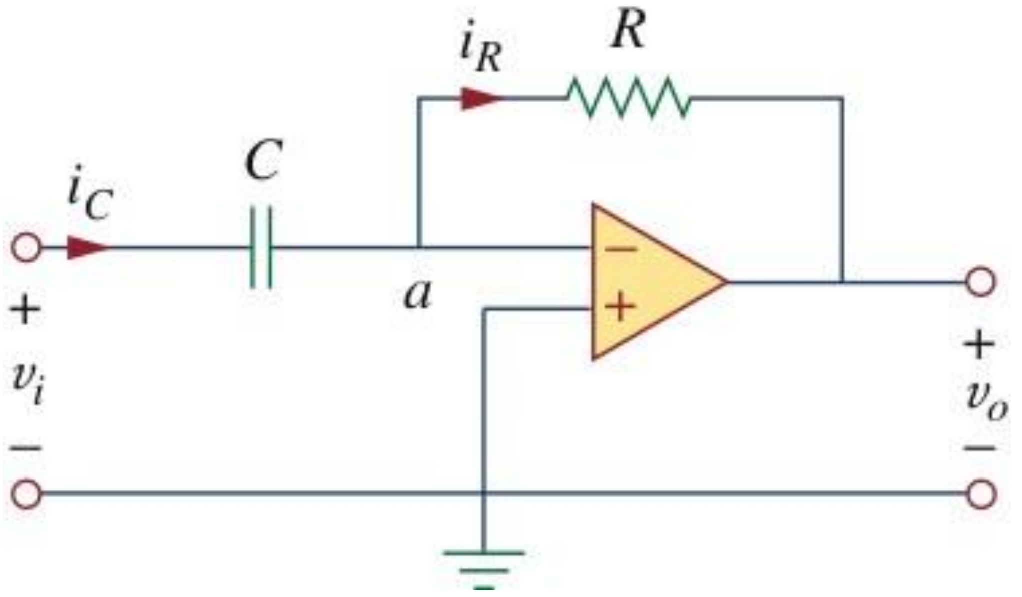
Differentiator

***A differentiator** is an op amp circuit whose output is proportional to the rate of change of the input signal.*

$$i_C = i_R$$

$$i_R = -\frac{v_0}{R}$$

$$i_C = C\frac{dv_i}{dt}$$



We can obtain from these equations

$$v_o = -RC\frac{dv_i}{dt}$$

Summary and Review

- The current through a capacitor is given by *$i = C\, dv/dt$*
- The voltage across a capacitor is related to its current by

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t')dt' + v(t_0)$$

- A capacitor is an *open circuit* to dc currents.

- The voltage across an inductor is given by $v = L\, di/dt$.
- The current through an inductor is related to its current by

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v\,dt' + i(t_0)$$

- An inductor is a *short circuit* to dc currents.

- Series and parallel combinations of inductors can be combined using the same equations as for resistors.

- Series and parallel combinations of capacitors work the opposite way as they do for resistor.

- A capacitor as the feedback element in an inverting op amp leads to an output voltage proportional to the integral of the input voltage. Swapping the input resistor and the feedback capacitor leads to an output voltage proportional to the derivative of the input voltage.

- Since capacitors and inductors are linear elements, KVL, KCL, superposition, Thévenin’s and Norton’s theorems, and nodal and mesh analysis apply to their circuits as well.

● Current and voltage relationship for R, L, C

TABLE 6.1 Important characteristics of the basic elements.†

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \dfrac{1}{C}\displaystyle\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\dfrac{di}{dt}$
i-v:	$i = v/R$	$i = C\dfrac{dv}{dt}$	$i = \dfrac{1}{L}\displaystyle\int_{t_0}^{t} i\,dt + i(t_0)$
p or w:	$p = i^2R = \dfrac{v^2}{R}$	$w = \dfrac{1}{2}Cv^2$	$w = \dfrac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \dfrac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \dfrac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \dfrac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

†*Passive sign convention is assumed.*

6.29 Determine C_{eq} for each circuit in Fig. 6.61.

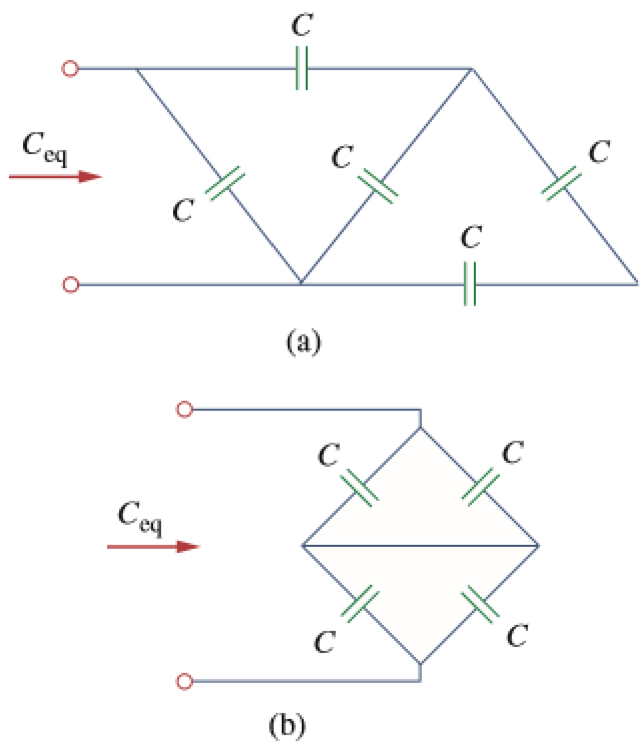


Figure 6.61
For Prob. 6.29.

6.31 If $v(0) = 0$, find $v(t)$, $i_1(t)$, and $i_2(t)$ in the circuit of Fig. 6.63.

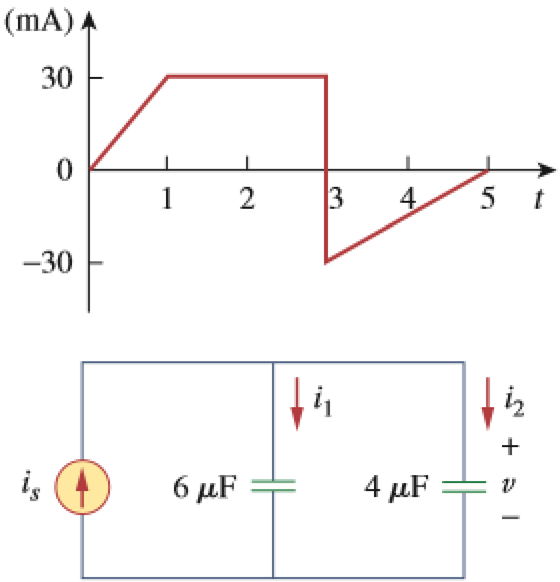


Figure 6.63
For Prob. 6.31.

6.32 In the circuit of Fig. 6.64, let $i_s = 50e^{-2t}$ mA and $v_1(0) = 50$ V, $v_2(0) = 20$ V. Determine: (a) $v_1(t)$ and $v_2(t)$, (b) the energy in each capacitor at $t = 0.5$ s.

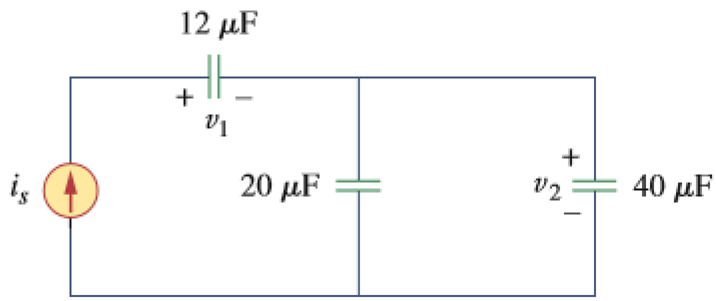


Figure 6.64
For Prob. 6.32.

6.48 Under steady-state dc conditions, find i and v in the circuit in Fig. 6.71.

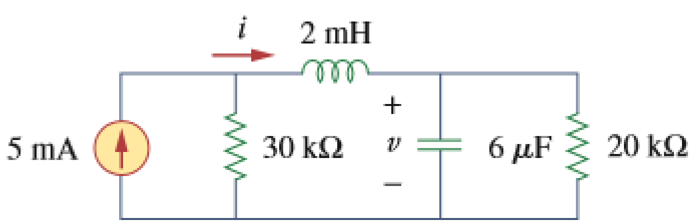


Figure 6.71
For Prob. 6.48.

6.40 The current through a 5-mH inductor is shown in Fig. 6.66. Determine the voltage across the inductor at $t = 1$, 3, and 5 ms.

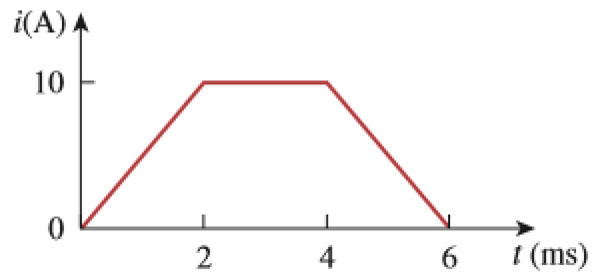


Figure 6.66
For Prob. 6.40.

6.53 Find L_{eq} at the terminals of the circuit in Fig. 6.75.

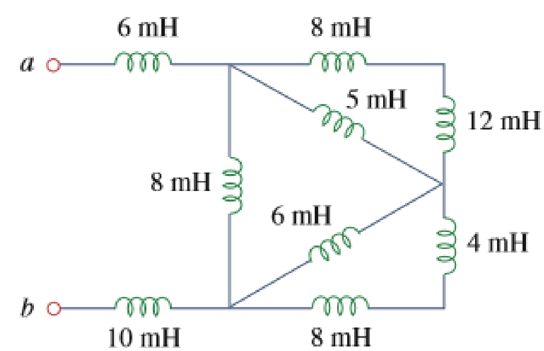


Figure 6.75
For Prob. 6.53.

6.73 Show that the circuit in Fig. 6.90 is a noninverting integrator.

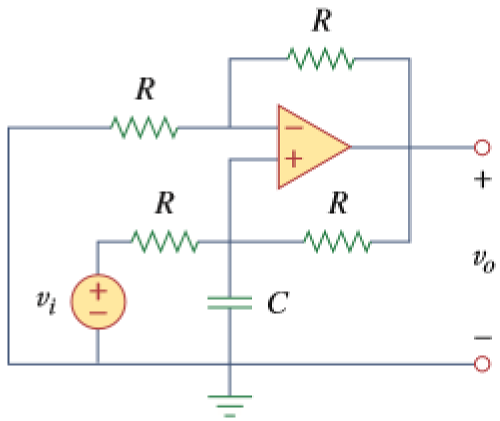


Figure 6.90
For Prob. 6.73.

Fundamentals of Electric Circuit 2021.4

Chapter 7 First-Order Circuits

Chapter 7 First-Order Circuits

7.0 Introduction

7.1 The Source-Free RC Circuit

7.2 The Source-Free RL Circuit

7.3 Unit-Step Function

7.4 The Step-Response of a RC Circuit

7.5 The Step-Response of a RL Circuit

7.0 introduction

- In this chapter, we will focus on circuits that consist only of sources, resistors, and either (but not both) inductors or capacitors.
- We call them as ***RC (resistor-capacitor) circuits*** and ***RL (resistor-inductor) circuits.***
- Method: Kirchhoff's laws.
- Difference: algebraic equations(Resistive corcuit)

  differential equations(RC,RL circuit).

  first order differential equations

  first-order circuits.

7.0 introduction

- A ***first-order circuit*** is characterized by a first-order differential equation.

Any circuit with

- ***a single energy storage element（C or L）***
- ***an arbitrary number of sources***
- ***an arbitrary number of resistors***

is a circuit of order 1.

- Two ways to excite the circuits:

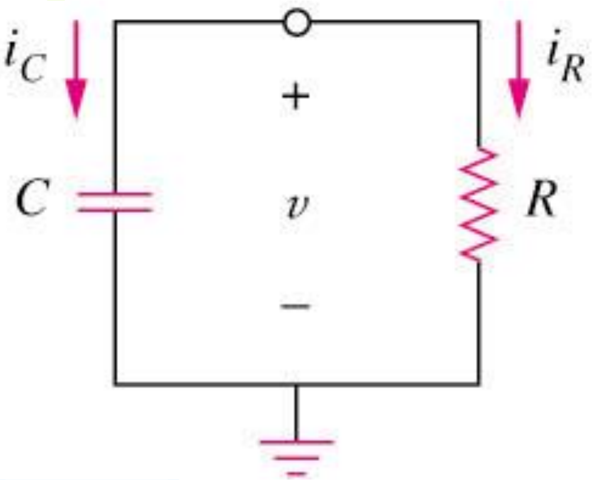
Initial conditions: energy stored in the capacitive or inductive element. (Source-free)

Independent sources

7.1 The Source-Free RC Circuit

$$v(0) = V_0 = A$$

$$w(0) = \frac{1}{2}CV_0^2$$



By KCL

$$i_R + i_C = 0 \rightarrow \frac{v}{R} + C\frac{dv}{dt} = 0$$

Ohms law

Capacitor law

Rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

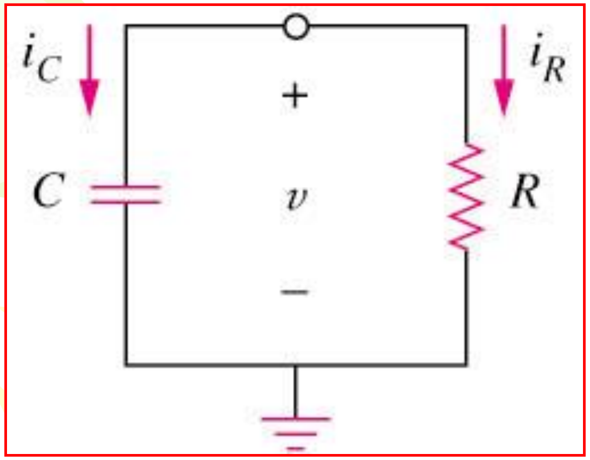
first-order differential equation.

Integrating both sides gives

$$\ln v = -\frac{1}{RC}t + \ln A$$

Take powers of *e* produces

$$v(t) = Ae^{-t/RC}\,\mathrm{V}$$



$$v(t) = Ae^{-t/RC}V$$

$$\boxed{v(t) = V_0\, e^{-t/\tau}}$$ where $$\boxed{\tau = RC}$$

$$\boxed{v(0) = V_0 = A}$$

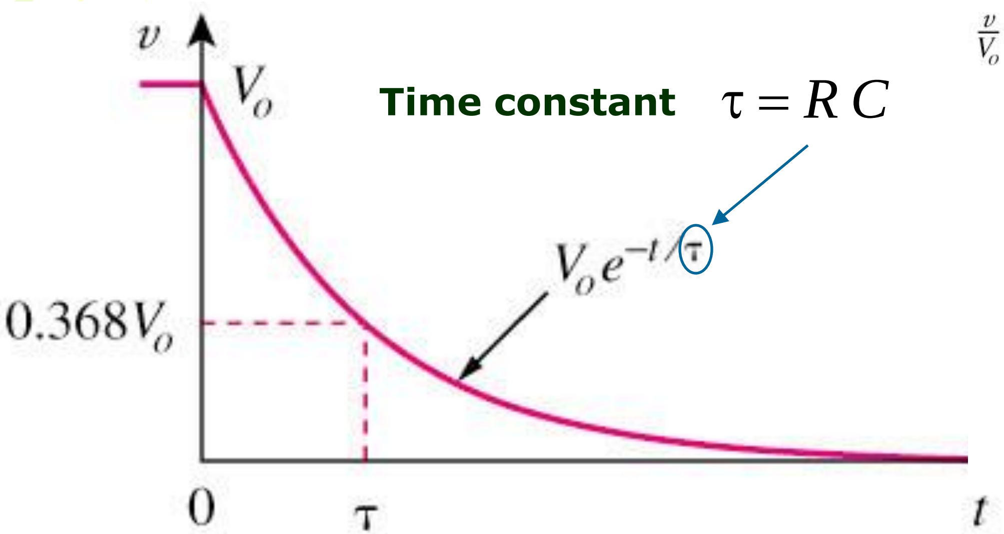
The response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the **natural response** of the circuit.

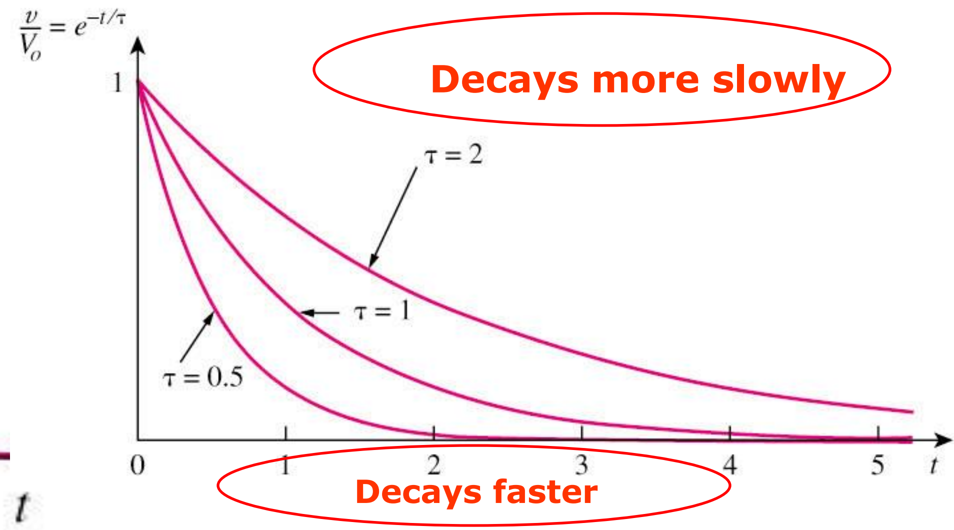
The key to working with a source-free RC circuit is finding:

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ = RC.

- The **natural response** of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with <u>no external sources of excitation</u>.

$$v(t) = V_0\, e^{-t/\tau} \quad \text{where} \quad \tau = RC$$





- The ***time constant*** τ of a circuit is the time required for the response to decay by a factor of <u>1/*e* or 36.8%</u> of its initial value.
- v decays faster for small τ and slower for large τ.

TABLE 7.1 Values of $v(t)/V_0 = e^{-t/\tau}$.

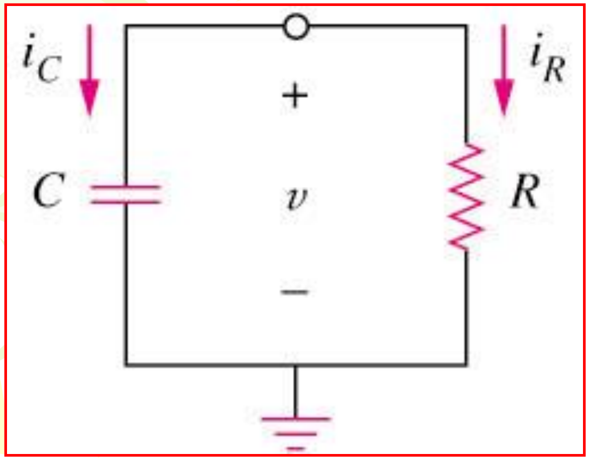
t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

$$v(t) = Ae^{-t/RC}\,\text{V}$$

$$v(t) = V_0\, e^{-t/\tau}$$ where $$\tau = RC$$

It is customary to assume that the capacitor is fully discharged (or charged) after **five** time constants.

In other words, it takes **5τ** for the circuit to reach its final state or steady state when no changes take place with time.



$$v(t) = Ae^{-t/RC}V$$

$$v(t) = V_0\, e^{-t/\tau}$$ where $$\tau = R\,C$$

The time constant is the same regardless of what the output is defined to be.

How to compute R:the Thevenin equivalent resistance at the terminals of the capacitor(take out the capacitor C and find R = R_{Th} at its terminals).

equivalent capacitor: combined them to form a single.

EXAMPLE 7.1

In Fig. 7.5, let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.

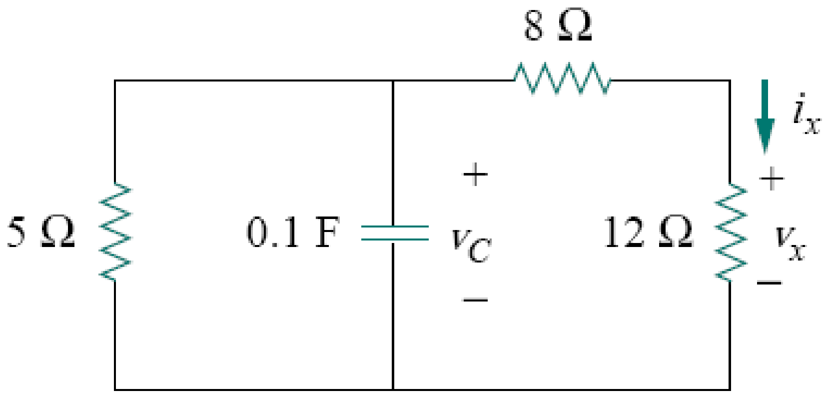


Figure 7.5 For Example 7.1.

We find the equivalent resistance or the Thevenin resistance at the capacitor terminals.

$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega \qquad \tau = R_{eq}C = 4(0.1) = 0.4 \text{ s}$$

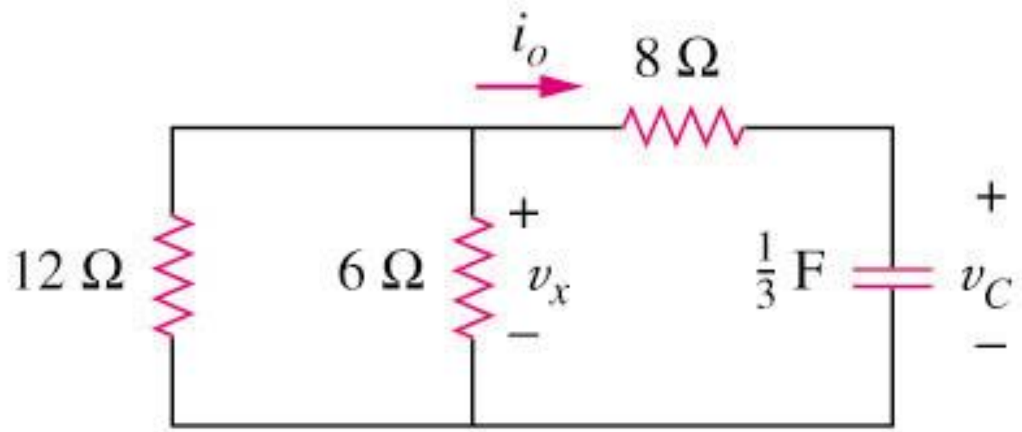
$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4} \text{ V} \qquad v_C = v = 15e^{-2.5t} \text{ V}$$

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t} \text{ V} \qquad i_x = \frac{v_x}{12} = 0.75e^{-2.5t} \text{ A}$$

Practice Problem 7.1

Refer to the circuit below, determine v_C, v_x, and i_o for $t \geq 0$.

Assume that $v_C(0) = 30$ V.

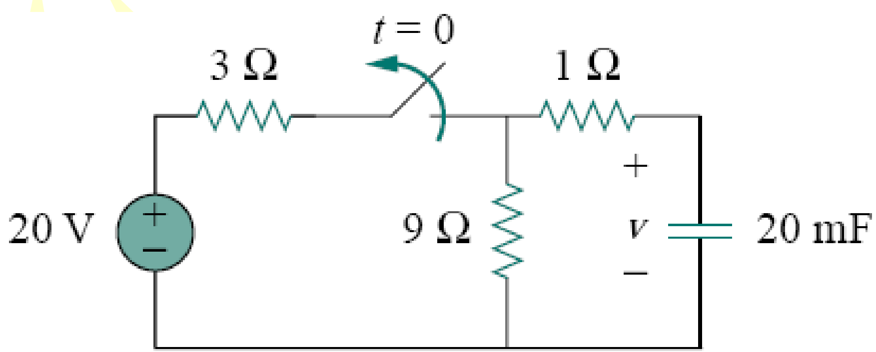


- Answer:

  $v_C = 30e^{-0.25t}$ V ; $v_x = 10e^{-0.25t}$; $i_o = -2.5e^{-0.25t}$ A

EXAMPLE 7.2

The switch in the circuit in Fig. 7.8 has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.



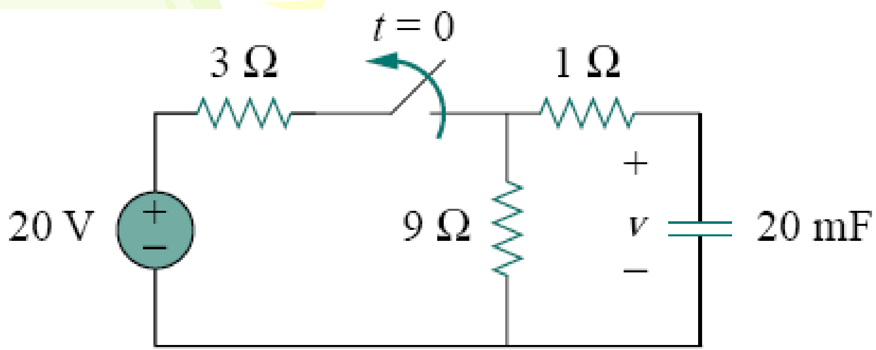
For $t < 0$, the switch is closed; the capacitor is an open circuit to dc

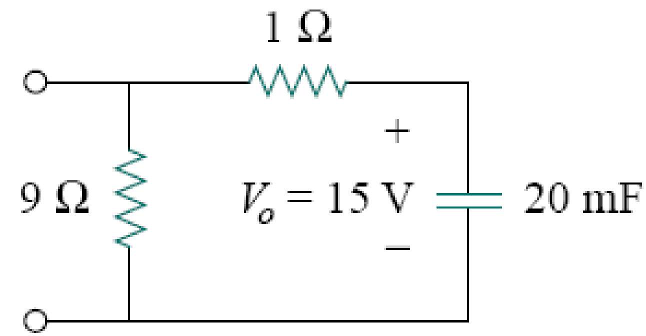
$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

$$v_C(0) = V_0 = 15 \text{ V}$$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25 \text{ J}$$





$$R_{eq} = 1 + 9 = 10\ \Omega$$

The time constant is $\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2$ s

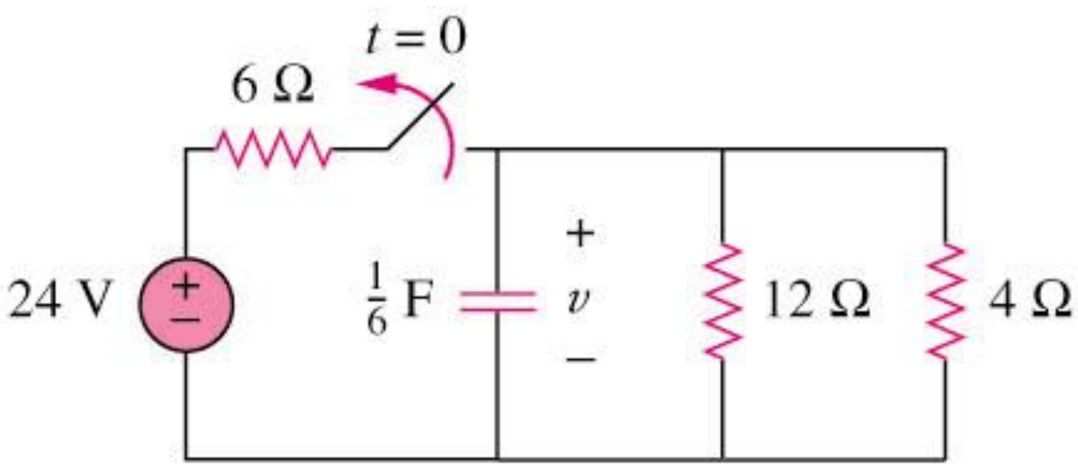
Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2}\ \text{V}$$

$$v(t) = 15e^{-5t}\ \text{V}$$

Example 2

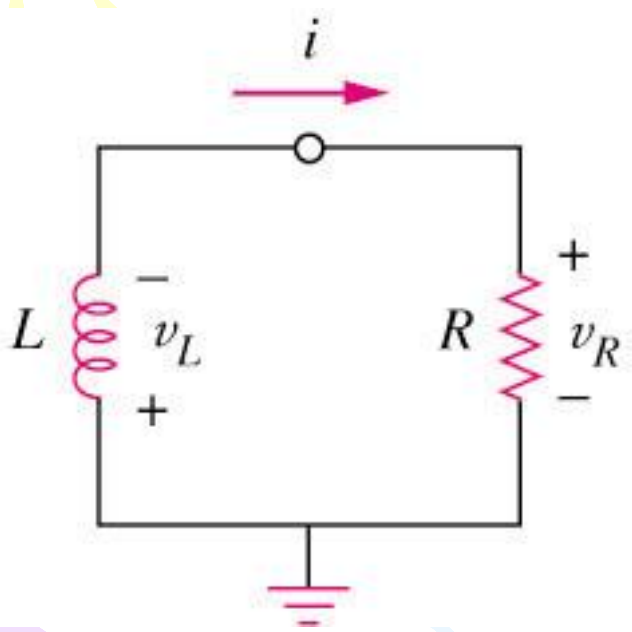
The switch in circuit below is opened at $t = 0$, find $v(t)$ for $t \geq 0$.



- Answer: $V(t) = 8e^{-2t}$ V

7.2 The Source-Free RL Circuit

- A **first-order RL circuit** consists of a inductor L (or its equivalent) and a resistor (or its equivalent)



By KVL $v_L + v_R = 0$

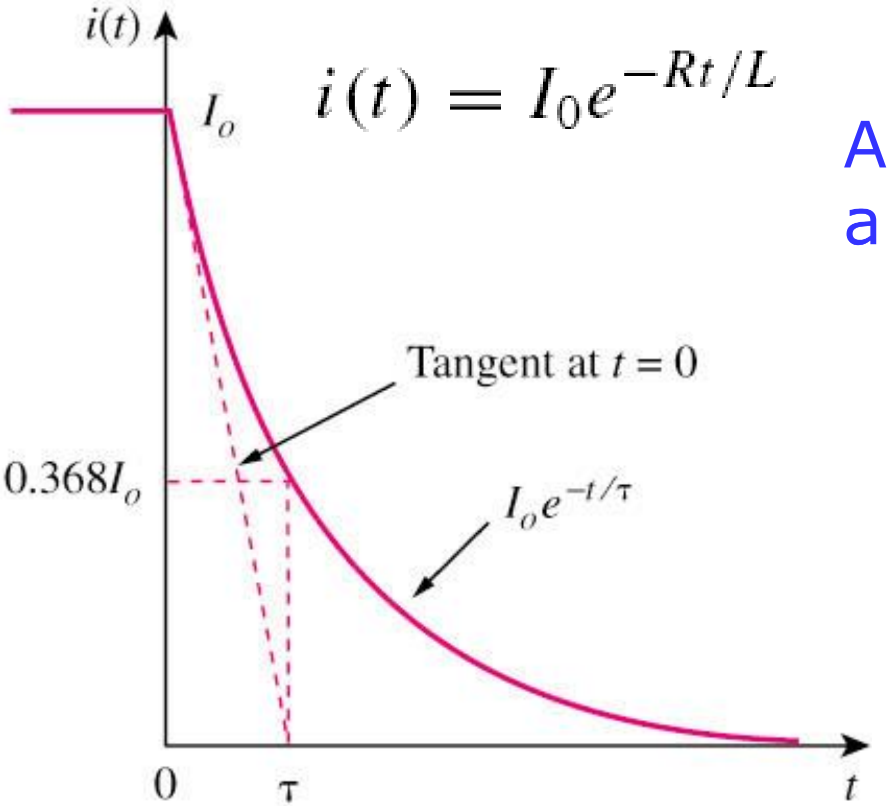
$$\rightarrow \quad L\frac{di}{dt} + iR = 0$$

Inductors law

Ohms law

$$\rightarrow \quad \frac{di}{i} = -\frac{R}{L}dt \quad \rightarrow \quad i(t) = I_0\, e^{-Rt/L}$$

the natural response of the *RL* circuit is an exponential decay of the initial current.



A general form representing a RL

$$i(t) = I_0\, e^{-t/\tau}$$

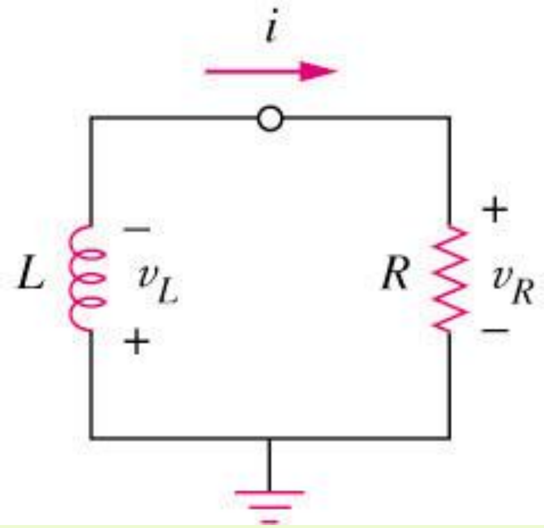
where $\tau = \dfrac{L}{R}$

- The **time constant** τ of a circuit is the time required for the response to decay by a factor of 1/*e* or 36.8% of its initial value.
- ***i(t)*** decays faster for small τ and slower for large τ.
- The general form is very similar to a RC source-free circuit.

Comparison between a RL and RC circuit

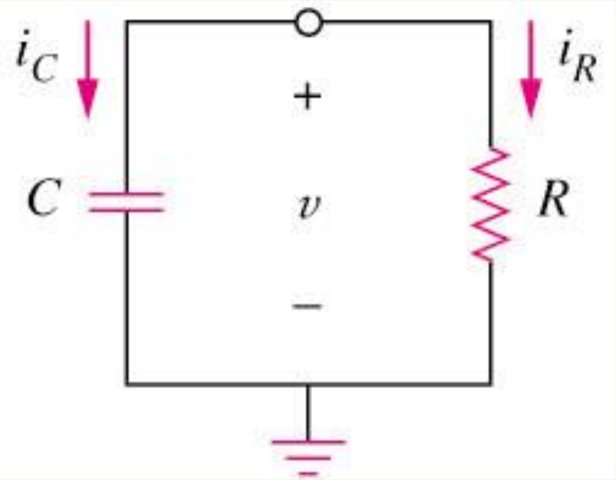
A RL source-free circuit

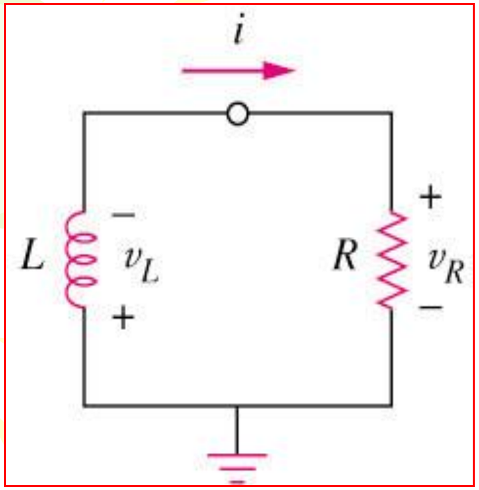
$$i(t) = I_0\, e^{-t/\tau} \quad \textbf{where} \quad \tau = \frac{L}{R}$$



A RC source-free circuit

$$v(t) = V_0\, e^{-t/\tau} \quad \textbf{where} \quad \tau = RC$$

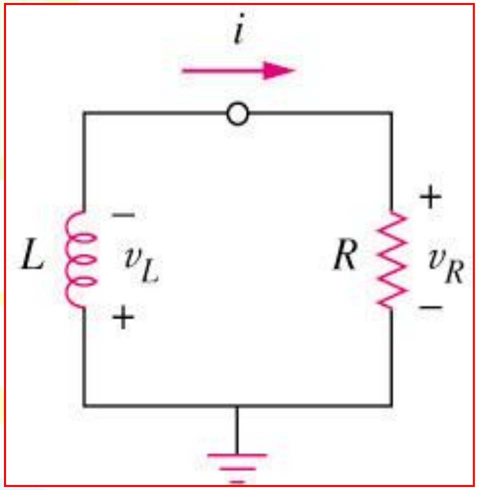




$$i(t) = I_0 \, e^{-t/\tau}$$ where $$\tau = \frac{L}{R}$$

The key to working with a source-free RL circuit is finding:

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant $\tau = L/R$.

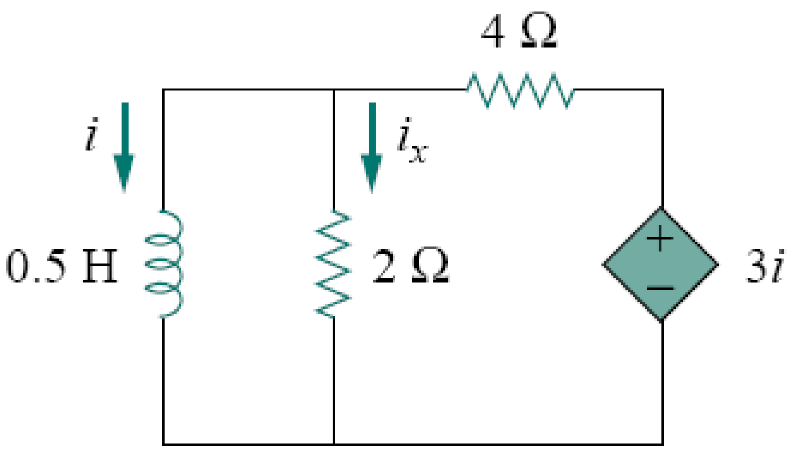


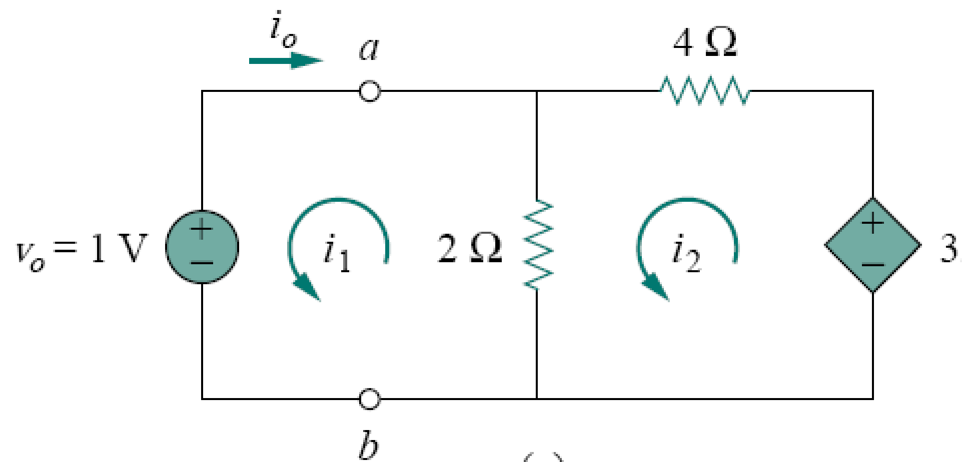
$$i(t) = I_0\, e^{-t/\tau}$$ where $$\tau = \frac{L}{R}$$

- When a circuit has a single inductor and several resistors and dependent sources, the Thevenin equivalent can be found at the terminals of the inductor to form a simple RL circuit.
- Also, one can use Thevenin’s theorem when several inductors can be combined to form a single equivalent inductor.

EXAMPLE 7.3

Assuming that $i(0) = 10$ A, calculate $i(t)$ and $i_x(t)$ in the circuit





(a)

Compute R_{eq} ,

Mesh analysis: $2(i_1 - i_2) + 1 = 0 \qquad 6i_2 - 2i_1 - 3i_1 = 0$

$$i_1 = -3 \text{ A}, \qquad i_o = -i_1 = 3 \text{ A}$$

$$R_{eq} = R_{Th} = \frac{v_o}{i_o} = \frac{1}{3}\ \Omega$$

The time constant is $\tau = \dfrac{L}{R_{eq}} = \dfrac{\frac{1}{2}}{\frac{1}{3}} = \dfrac{3}{2}$ s

So $i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t}$ A, $\quad t > 0$

Practice Problem 7.3

Find i and v_x in the circuit of Fig. 7.15. Let $i(0) = 12$ A.

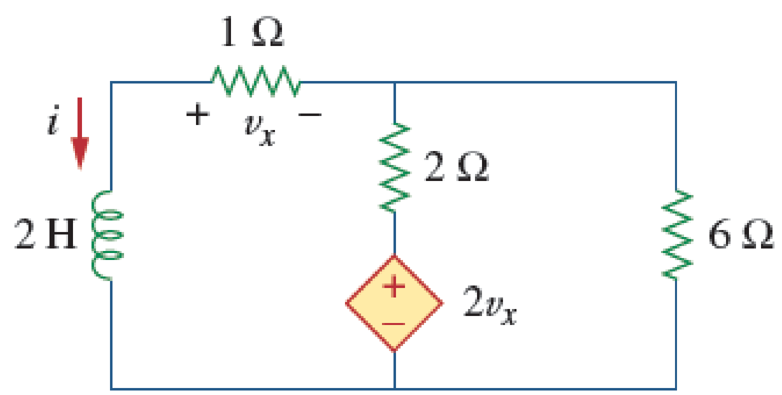
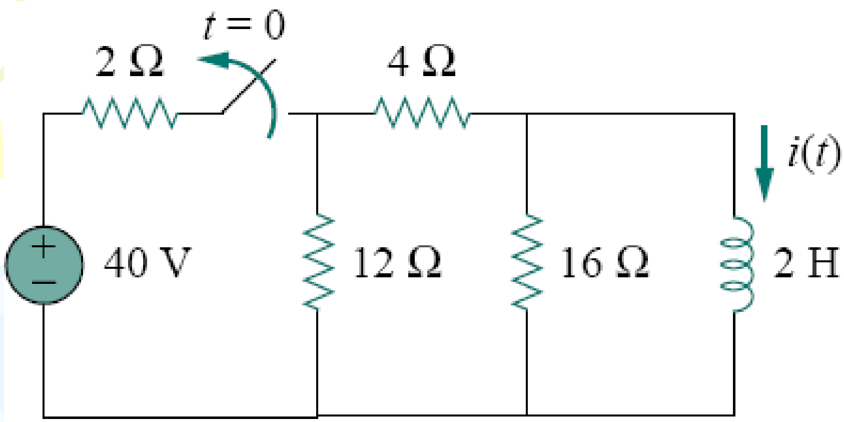


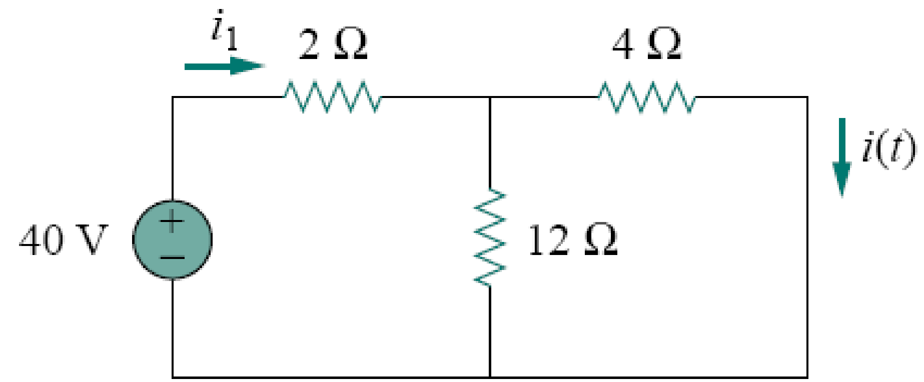
Figure 7.15
For Practice Prob. 7.3.

Answer: $12e^{-2t}$ A, $-12e^{-2t}$ V, $t > 0$.

EXAMPLE 7.4

The switch in the circuit of Fig. 7.16 has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.





(a)

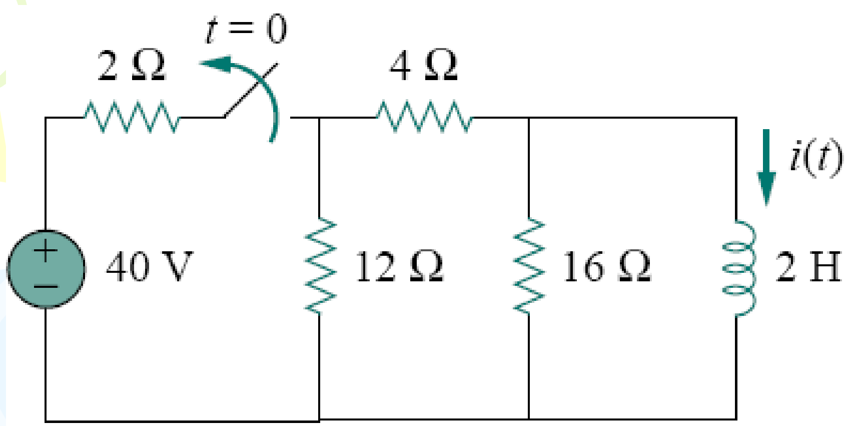
When $t < 0$, the switch is closed, and the inductor acts as a short circuit

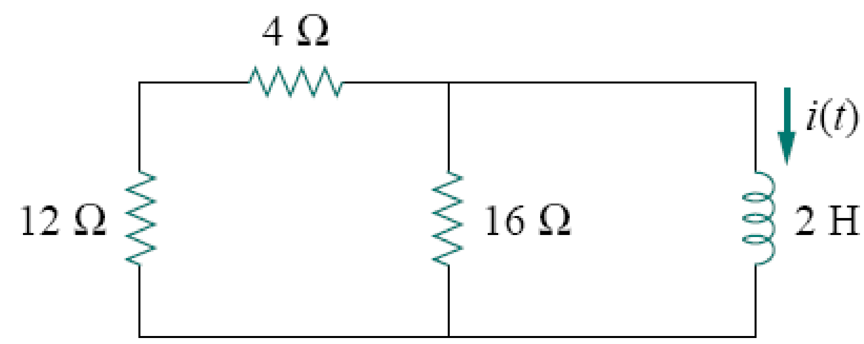
$$\frac{4 \times 12}{4 + 12} = 3\ \Omega \qquad i_1 = \frac{40}{2 + 3} = 8\ \text{A} \qquad i(t) = \frac{12}{12 + 4} i_1 = 6\ \text{A}, \qquad t < 0$$

$$i(0) = i(0^-) = 6\ \text{A}$$

EXAMPLE 7.4

The switch in the circuit of Fig. 7.16 has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.





(b)

When $t > 0$, the switch is open and the voltage source is disconnected. We now have the RL circuit in Fig. 7.17(b). Combining the resistors, we have

$$R_{\text{eq}} = (12 + 4) \parallel 16 = 8\ \Omega$$

The time constant is $$\tau = \frac{L}{R_{\text{eq}}} = \frac{2}{8} = \frac{1}{4}\ \text{s}$$ $$i(0) = i(0^-) = 6\ \text{A}$$

$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t}\ \text{A}$$

Practice Problem 7.4

For the circuit in Fig. 7.18, find $i(t)$ for $t > 0$.

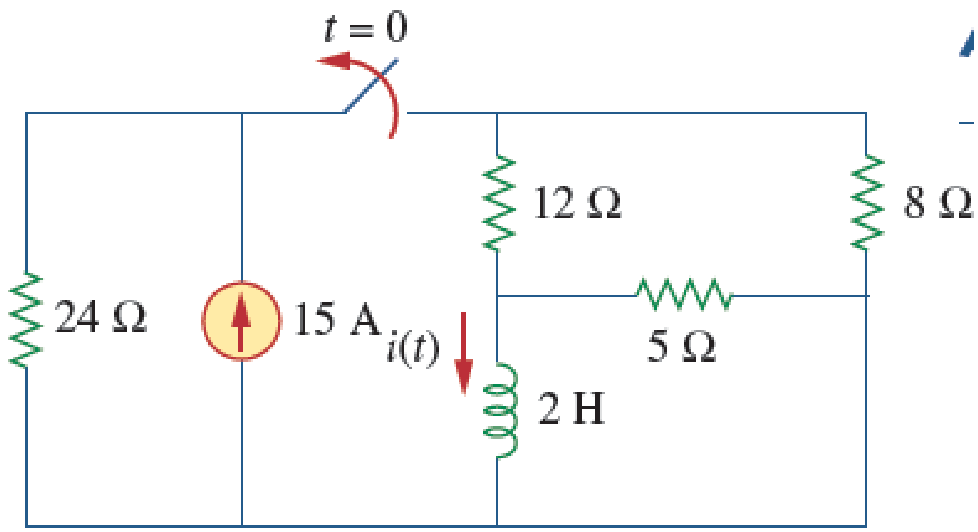
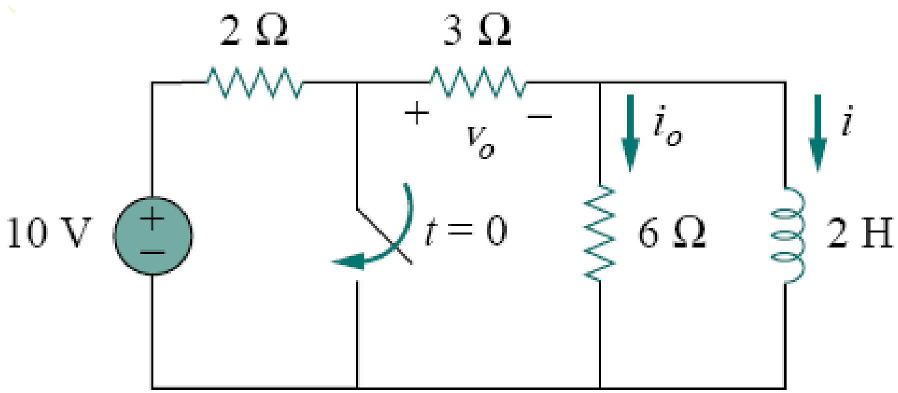


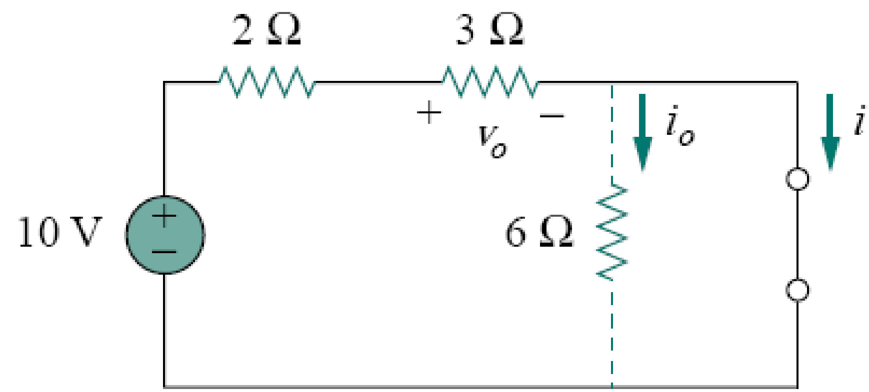
Figure 7.18
For Practice Prob. 7.4.

Answer: $5e^{-2t}$ A, $t > 0$.

EXAMPLE 7.5

In the circuit shown in Fig. 7.19, find i_o, v_o, and i for all time, assuming that the switch was open for a long time.





(a)

Solution:

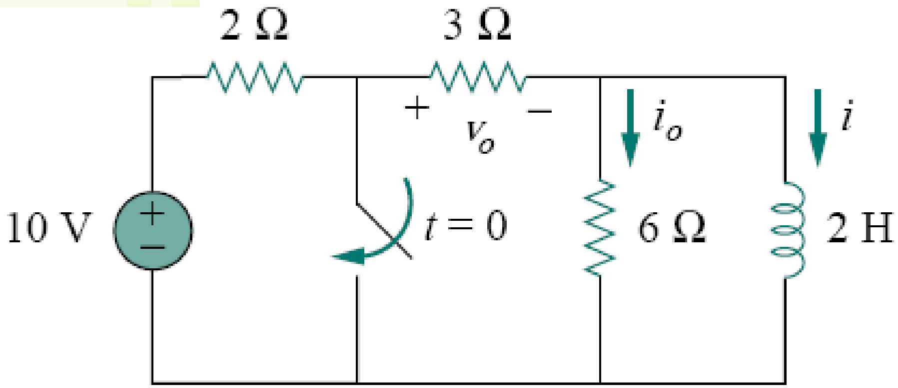
It is better to first find the inductor current i and then obtain other quantities from it.

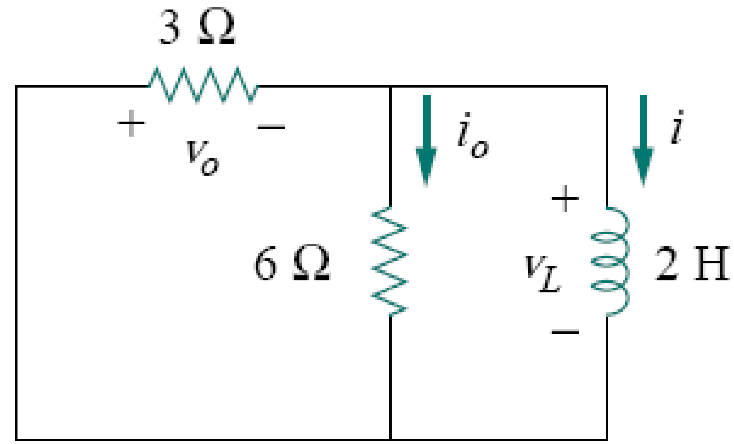
For $t < 0$, the switch is open. Since the inductor acts like a short circuit to dc, the 6-Ω resistor is short-circuited, so that we have the circuit shown in Fig. 7.20(a). Hence, $i_o = 0$, and

$$i(t) = \frac{10}{2+3} = 2 \text{ A}, \qquad t < 0 \qquad v_o(t) = 3i(t) = 6 \text{ V}, \qquad t < 0$$

Thus, $i(0) = 2$.

EXAMPLE 7.5





(b)

For $t > 0$, the switch is closed, so that the voltage source is short-circuited. We now have a source-free RL circuit as shown in Fig. 7.20(b). At the inductor terminals, $R_{\text{Th}} = 3 \parallel 6 = 2\ \Omega$

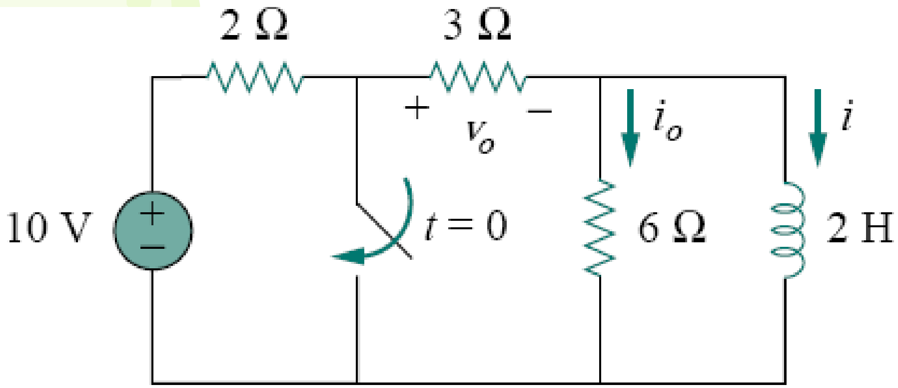
so that the time constant is $\tau = \dfrac{L}{R_{\text{Th}}} = 1\ \text{s}$

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t}\ \text{A}, \qquad t > 0$$

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\ \text{V}, \qquad t > 0$$

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}\ \text{A}, \qquad t > 0$$

EXAMPLE 7.5



Thus, for all time,

$$i_o(t) = \begin{cases} 0 \text{ A}, & t < 0 \\ -\frac{2}{3}e^{-t} \text{ A}, & t > 0 \end{cases},$$

$$v_o(t) = \begin{cases} 6 \text{ V}, & t < 0 \\ 4e^{-t} \text{ V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2 \text{ A}, & t < 0 \\ 2e^{-t} \text{ A}, & t \geq 0 \end{cases}$$

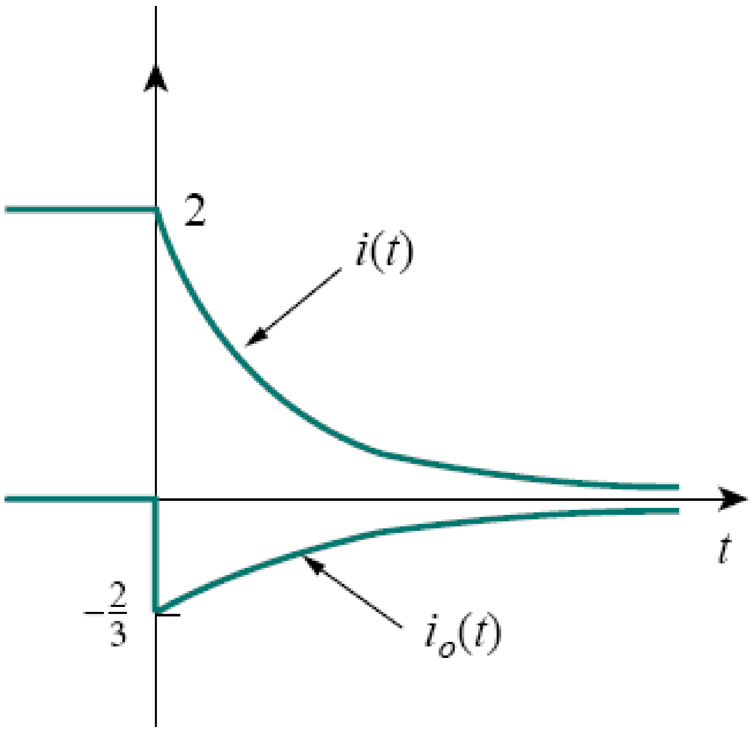


Figure 7.21 A plot of i and i_0.

7.4 Singularity functions

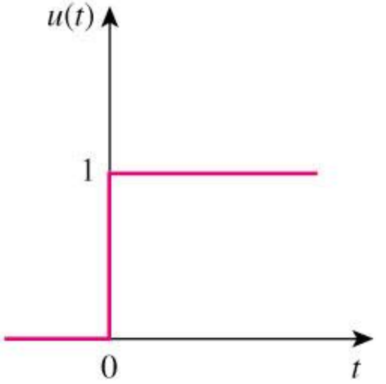
Singularity functions are functions that either are discontinuous or have discontinuous derivatives.

- The three most widely used singularity functions in circuit analysis are
- the unit step,
- the unit impulse,
- the unit ramp functions.

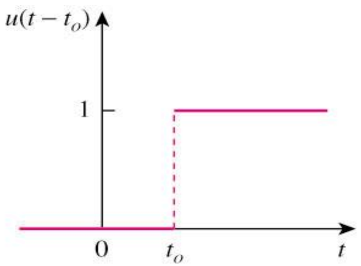
7.4 Singularity functions

- The ***unit step function u(t)*** is 0 for negative values of t and 1 for positive values of t.

$$u(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$$

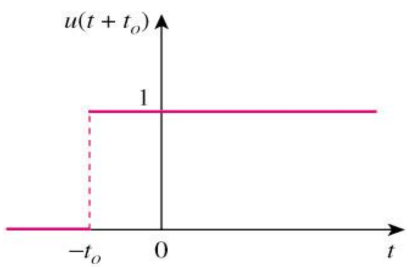


$$u(t-t_o)=\begin{cases}0, & t<t_o\\ 1, & t>t_o\end{cases}$$



(a)

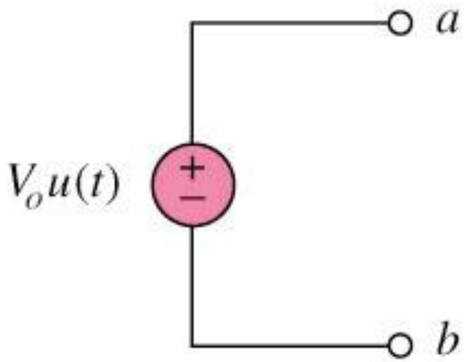
$$u(t+t_o)=\begin{cases}0, & t<-t_o\\ 1, & t>-t_o\end{cases}$$



(b)

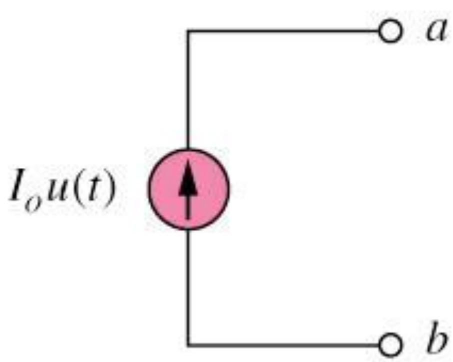
Represent an abrupt change for:

Voltage source.



(a)

Current source



(a)

switching functions

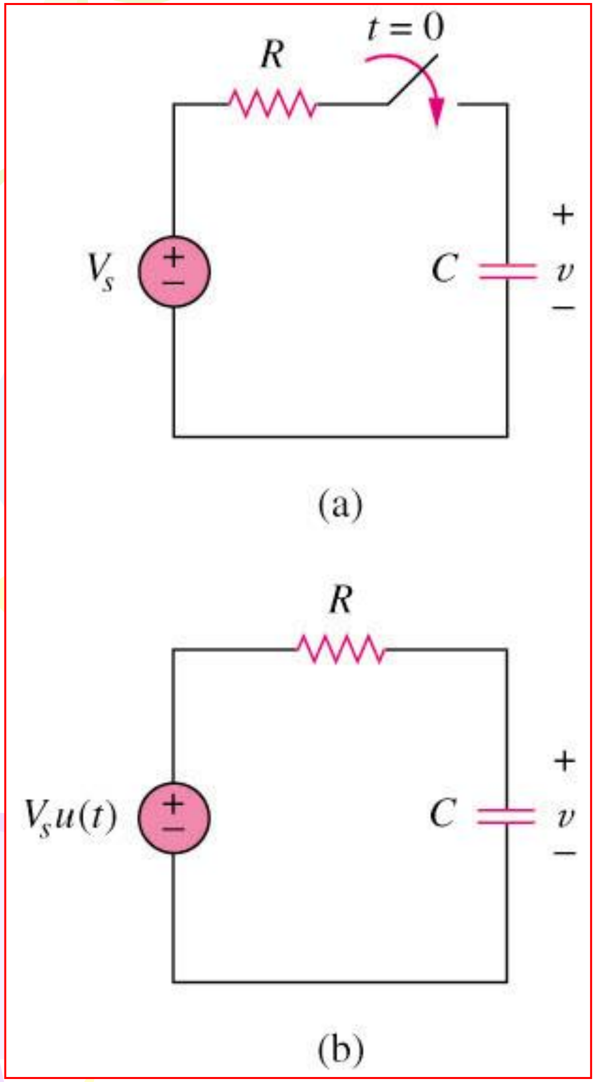
7.4 The Step-Response of an RC Circuit

- When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a **step response**.

- The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

- The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

7.4 The Step-Response of an RC Circuit



- **Initial condition**:

$$v(0-) = v(0+) = V_0$$

Applying KCL,

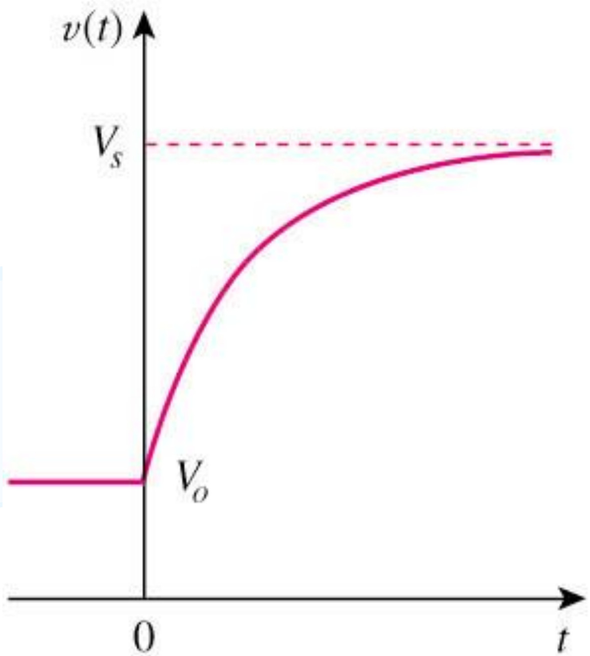
$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$$

- Where u(t) is the <u>unit-step function</u>

- Integrating both sides and considering the initial conditions, the solution of the equation is:



$$v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau} & t > 0 \end{cases}$$

Final value at t -> ∞

Initial value at t = 0

Source-free Response

$$\boxed{v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}}$$

Complete Response = Natural response + Forced Response

(stored energy) (independent source)

= $V_0e^{-t/\tau}$ + $V_s(1-e^{-t/\tau})$

Three steps to find out the step response of an RC circuit:

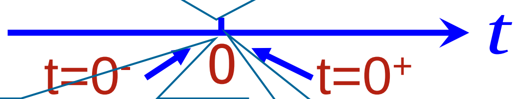
1. The initial capacitor voltage v(0).
2. The final capacitor voltage v(∞) — DC voltage across C.
3. The time constant τ.

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$.

Note: The above method is a **short-cut method**. You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

the instant when the circuit changes



the instant just prior to circuit changes
Initial state
Only $i_L(0^-)$ or $u_C(0^-)$

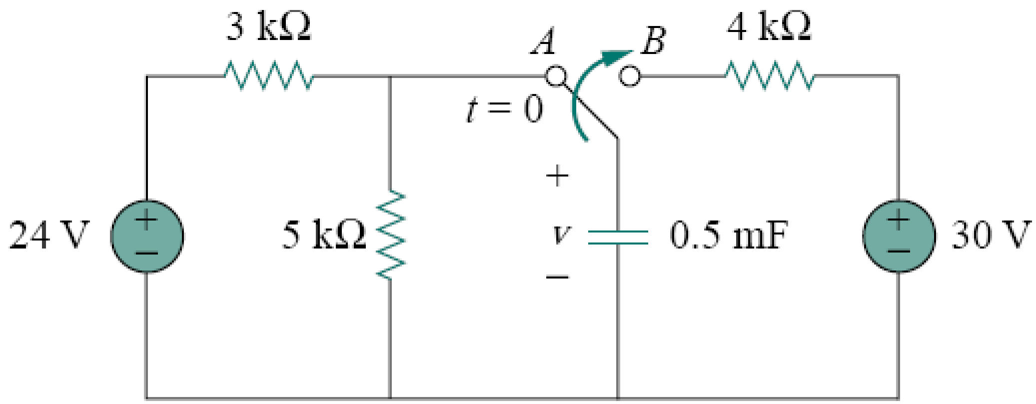
the instant immediately following to circuit changes

Initial value or
Initial condition
$y(0^+)$ y denote any current or voltage

EXAMPLE 7.10

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

The switch in Fig. 7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.



Solution:

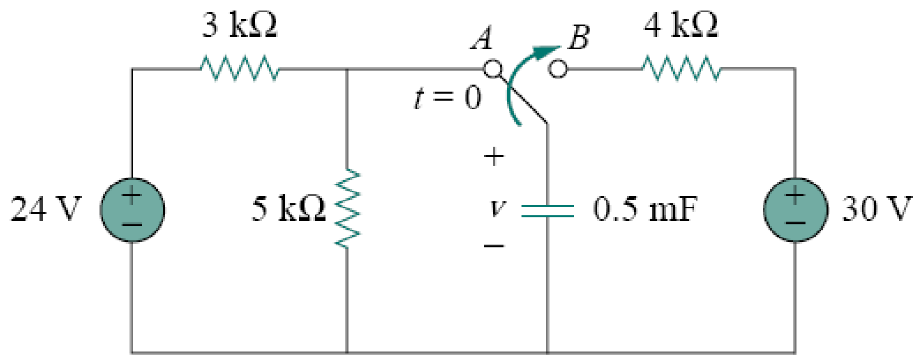
For $t < 0$, the switch is at position A. Since v is the same as the voltage across the 5-kΩ resistor, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as

$$v(0^-) = \frac{5}{5+3}(24) = 15\text{ V} \qquad v(0) = v(0^-) = v(0^+) = 15\text{ V}$$

EXAMPLE 7.10

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

The switch in Fig. 7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.



For $t > 0$, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{\text{Th}} = 4$ kΩ, and the time constant is

$$\tau = R_{\text{Th}}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$
$$= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t}) \text{ V}$$

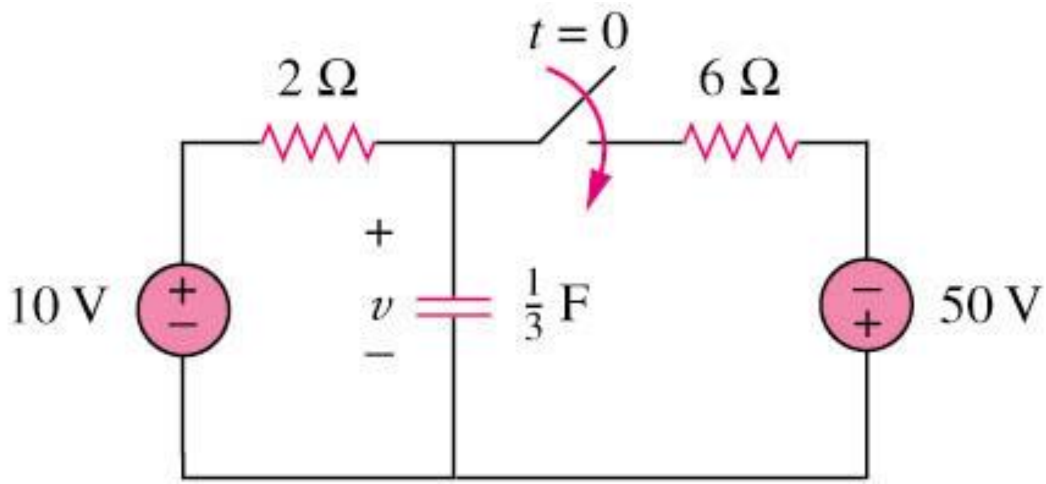
At $t = 1$, $\quad v(1) = 30 - 15e^{-0.5} = 20.902$ V

At $t = 4$, $\quad v(4) = 30 - 15e^{-2} = 27.97$ V

Example 5

Find v(t) for t > 0 in the circuit in below. Assume the switch has been open for a long time and is closed at t = 0.

Calculate v(t) at t = 0.5.

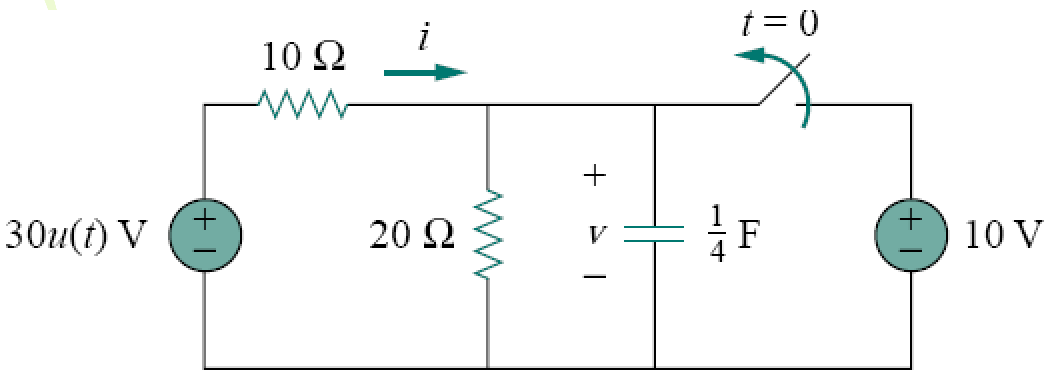


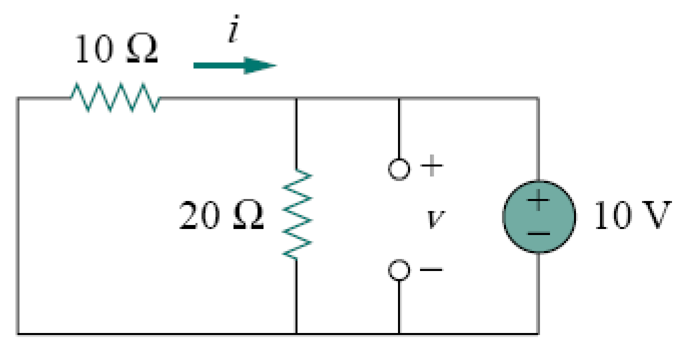
Answer:

$$v(t) = 15e^{-2t} - 5$$ *and v(0.5) = 0.5182V*

EXAMPLE 7.11

In Fig. 7.45, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.





Solution:

The resistor current i can be discontinuous at $t = 0$, while the capacitor voltage v cannot. Hence, it is always better to find v and then obtain i from v.

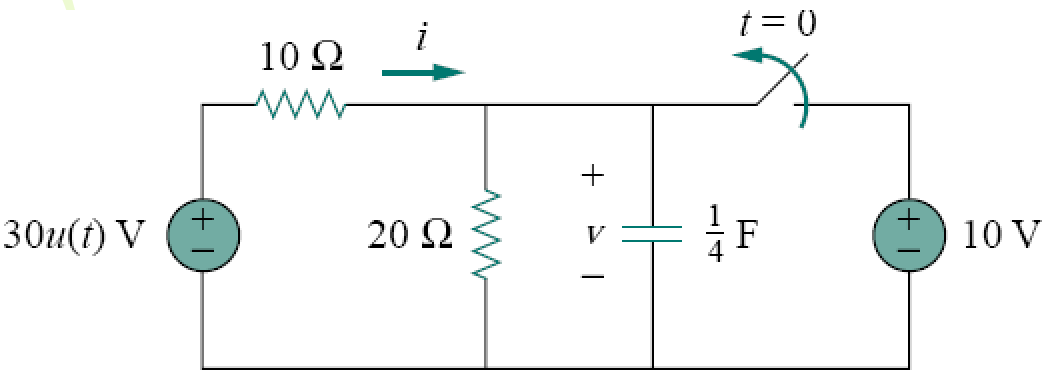
By definition of the unit step function, $30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$

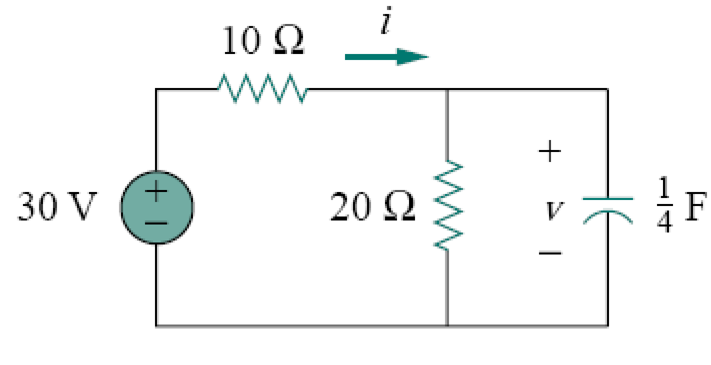
For $t < 0$, the switch is closed and $30u(t) = 0$

$$v = 10 \text{ V}, \qquad i = -\frac{v}{10} = -1 \text{ A} \qquad v(0) = v(0^-) = 10 \text{ V}$$

EXAMPLE 7.11

In Fig. 7.45, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.





(b)

For $t > 0$, the switch is opened and the 10-V voltage source is disconnected from the circuit. The $30u(t)$ voltage source is now operative, so the circuit becomes that shown in Fig. 7.46(b). After a long time, the circuit reaches steady state and the capacitor acts like an open circuit again. We obtain $v(\infty)$ by using voltage division, writing

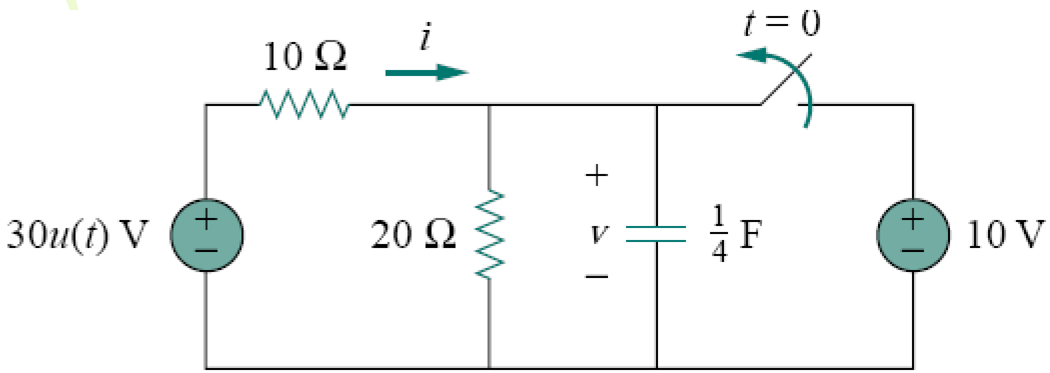
$$v(\infty) = \frac{20}{20+10}(30) = 20 \text{ V}$$

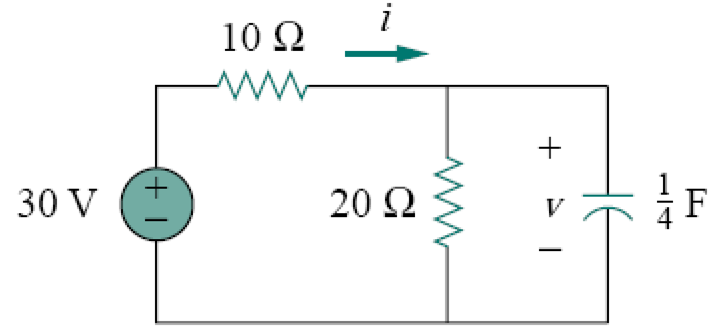
The Thevenin resistance at the capacitor terminals is

$$R_{\text{Th}} = 10 \parallel 20 = \frac{10 \times 20}{30} = \frac{20}{3}\ \Omega \qquad \tau = R_{\text{Th}}C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3} \text{ s}$$

EXAMPLE 7.11

In Fig. 7.45, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.





(b)

$$v(0) = v(0^-) = 10 \text{ V}$$

$$v(\infty) = \frac{20}{20+10}(30) = 20 \text{ V}$$

$$\tau = R_{\text{Th}}C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3} \text{ s}$$

For $t < 0$, $i = -\dfrac{v}{10} = -1 \text{ A}$

$$v = \begin{cases} 10 \text{ V}, & t < 0 \\ (20 - 10e^{-0.6t}) \text{ V}, & t \geq 0 \end{cases}$$

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t}) \text{ V}$$

$$i = \frac{v}{20} + C\frac{dv}{dt}$$

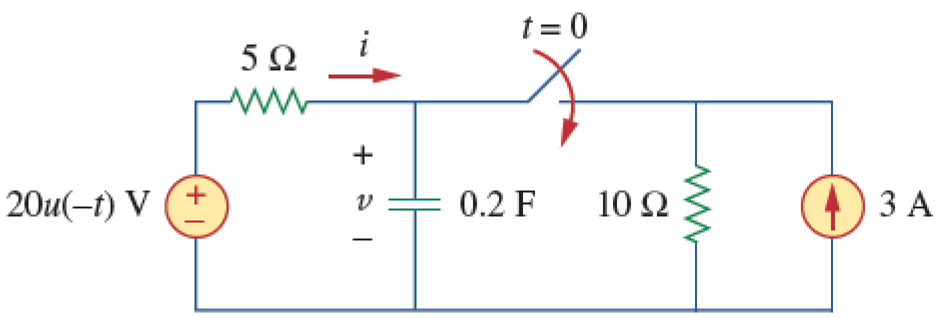
$$= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t}$$

$$= (1 + e^{-0.6t}) \text{ A}$$

$$i = \begin{cases} -1 \text{ A}, & t < 0 \\ (1 + e^{-0.6t}) \text{ A}, & t > 0 \end{cases}$$

Practice Problem 7.11

The switch in Fig. 7.47 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time.



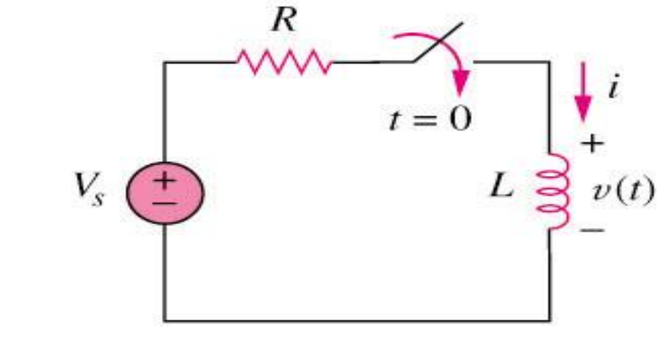
Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.

Answer: $i(t) = \begin{cases} 0, & t < 0 \\ -2(1 + e^{-1.5t}) \text{ A}, & t > 0, \end{cases}$

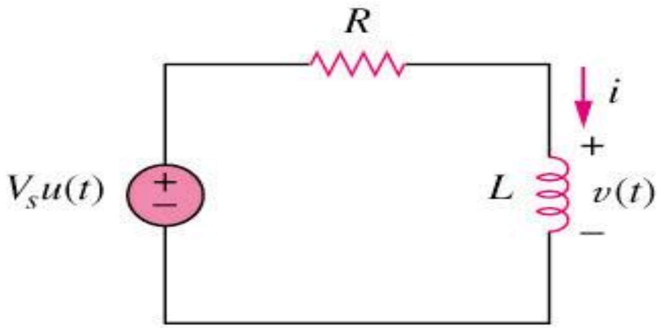
$$v = \begin{cases} 20 \text{ V}, & t < 0 \\ 10(1 + e^{-1.5t}) \text{ V}, & t > 0 \end{cases}$$

7.5 The Step-Response of a RL Circuit

- The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



(a)



(b)

- **Initial current**

  $i(0^-) = i(0^+) = I_o$

- **Final inductor current**

  $i(\infty) = V_s/R$

  Time constant $\tau = L/R$

$$i(t) = \frac{V_s}{R} + (I_o - \frac{V_s}{R})e^{-\frac{t}{\tau}}u(t)$$

$$i(t) = i(\infty) + [i(0+) - i(\infty)]\, e^{-t/\tau}$$

Three steps to find out the step response of an RL circuit:

1. The initial inductor current i(0) at t = 0+.
2. The final inductor current i(∞).
3. The time constant τ.

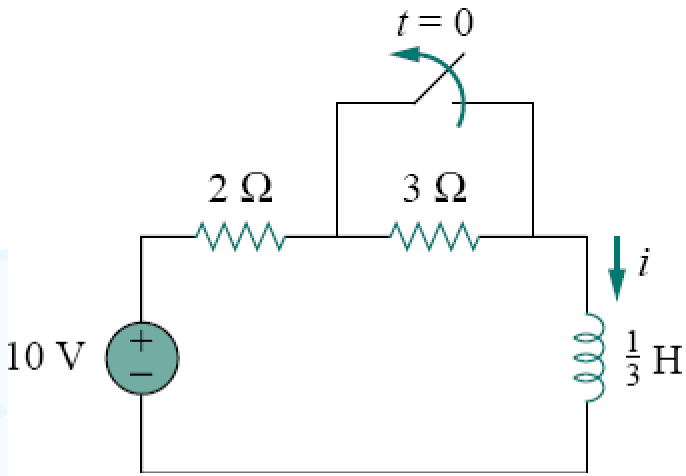
$$i(t) = i(\infty) + [i(0+) - i(\infty)]\, e^{-t/\tau}$$

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$.

Note: The above method is a ***short-cut method.***
You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

EXAMPLE 7.12

Find $i(t)$ in the circuit in Fig. 7.51 for $t > 0$. Assume that the switch has been closed for a long time.



$$i(t) = i(\infty) + [i(0+) - i(\infty)]\,e^{-t/\tau}$$

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 2 + (5-2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \qquad t > 0 \end{aligned}$$

Solution:

When $t < 0$, the 3-Ω resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t = 0^-$ (i.e., just before $t = 0$) is

$$i(0^-) = \frac{10}{2} = 5 \text{ A} \qquad i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$

When $t > 0$, the switch is open. The 2-Ω and 3-Ω resistors are in series, so that

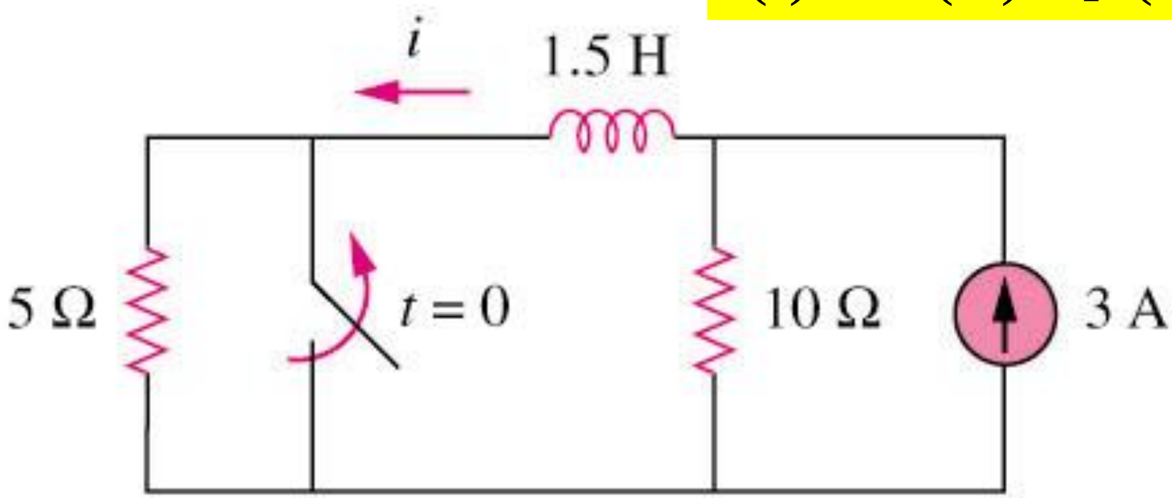
$$i(\infty) = \frac{10}{2+3} = 2 \text{ A} \qquad R_{\text{Th}} = 2 + 3 = 5\ \Omega \qquad \tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}$$

Example 6

The switch in the circuit shown below has been closed for a long time. It opens at t = 0.

Find i(t) for t > 0.

$$i(t) = i(\infty) + [i(0+) - i(\infty)]\, e^{-t/\tau}$$

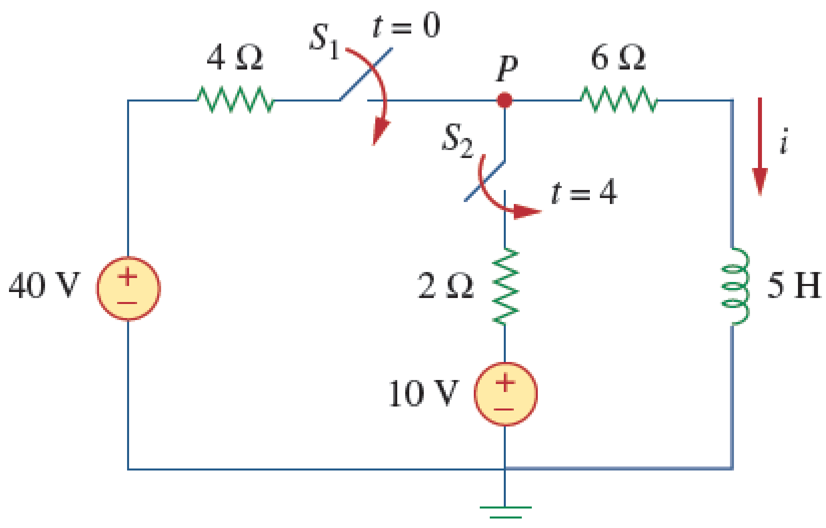


Answer:

$$i(t) = 2 + e^{-10t}$$

Example 7.13

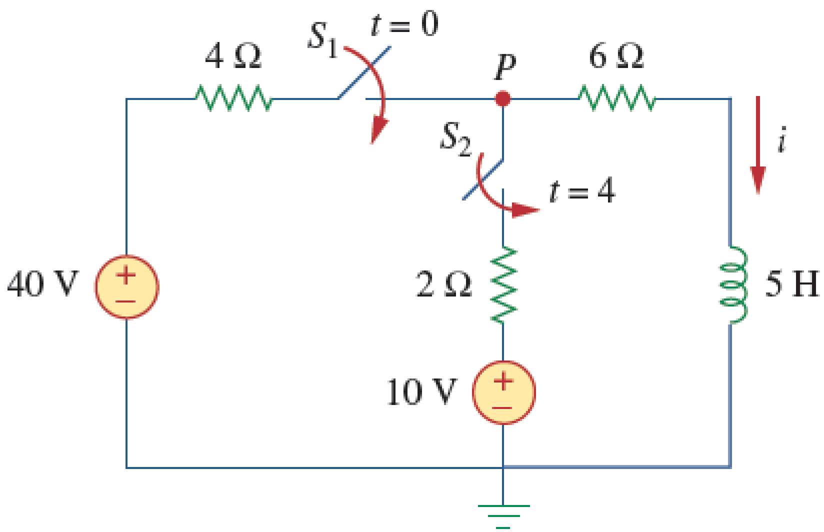
At $t = 0$, switch 1 in Fig. 7.53 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2$ s and $t = 5$ s.



Solution:

We need to consider the three time intervals $t \le 0$, $0 \le t \le 4$, and $t \ge 4$ separately. For $t < 0$, switches S_1 and S_2 are open so that $i = 0$. Since the inductor current cannot change instantly,

$$i(0^-) = i(0) = i(0^+) = 0$$



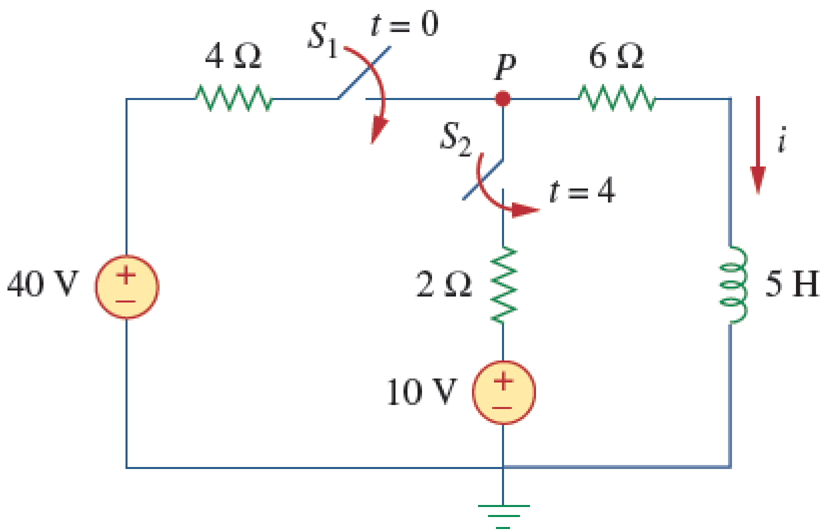
$$i(0^-) = i(0) = i(0^+) = 0$$

For $0 \leq t \leq 4$, S_1 is closed so that the 4-Ω and 6-Ω resistors are in series. (Remember, at this time, S_2 is still open.) Hence, assuming for now that S_1 is closed forever,

$$i(\infty) = \frac{40}{4 + 6} = 4 \text{ A}, \qquad R_{\text{Th}} = 4 + 6 = 10\ \Omega$$

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{5}{10} = \frac{1}{2} \text{ s}$$

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 4 + (0 - 4)e^{-2t} = 4(1 - e^{-2t}) \text{ A}, \qquad 0 \leq t \leq 4 \end{aligned}$$



$$i(0^-) = i(0) = i(0^+) = 0$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$= 4 + (0 - 4)e^{-2t} = 4(1 - e^{-2t})\text{ A}, \qquad 0 \le t \le 4$$

$$i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau}, \qquad t \ge 4$$

$$i(t) = 2.727 + (4 - 2.727)e^{-(t-4)/\tau}, \qquad \tau = \frac{15}{22}$$

$$= 2.727 + 1.273e^{-1.4667(t-4)}, \qquad t \ge 4$$

For $t \ge 4$, S_2 is closed; the 10-V voltage source is connected, and the circuit changes. This sudden change does not affect the inductor current because the current cannot change abruptly. Thus, the initial current is

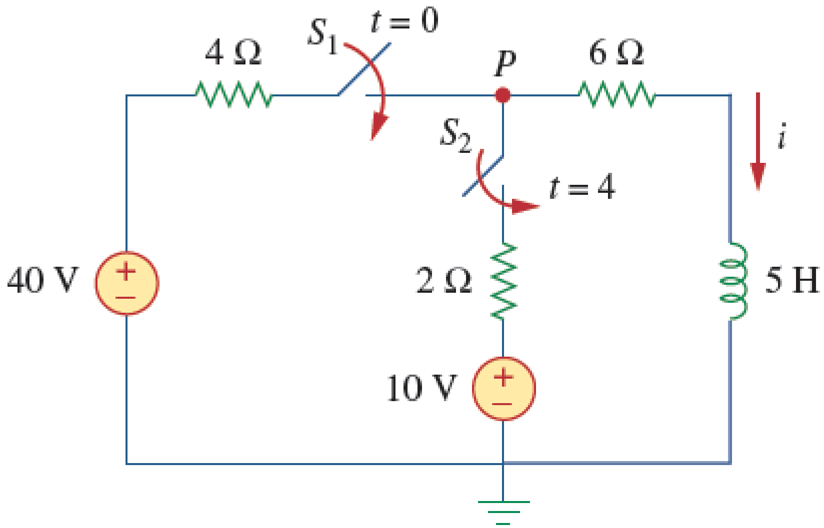
$$i(4) = i(4^-) = 4(1 - e^{-8}) \simeq 4\text{ A}$$

To find $i(\infty)$, let v be the voltage at node P in Fig. 7.53. Using KCL,

$$\frac{40 - v}{4} + \frac{10 - v}{2} = \frac{v}{6} \quad \Rightarrow \quad v = \frac{180}{11}\text{ V} \qquad i(\infty) = \frac{v}{6} = \frac{30}{11} = 2.727\text{ A}$$

The Thevenin resistance at the inductor terminals is

$$R_{Th} = 4 \parallel 2 + 6 = \frac{4 \times 2}{6} + 6 = \frac{22}{3}\,\Omega \qquad \tau = \frac{L}{R_{Th}} = \frac{5}{\frac{22}{3}} = \frac{15}{22}\text{ s}$$



$$i(0^-) = i(0) = i(0^+) = 0$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$
$$= 4 + (0 - 4)e^{-2t} = 4(1 - e^{-2t})\text{ A}, \qquad 0 \le t \le 4$$

$$i(t) = 2.727 + (4 - 2.727)e^{-(t-4)/\tau}, \qquad \tau = \frac{15}{22}$$
$$= 2.727 + 1.273e^{-1.4667(t-4)}, \qquad t \ge 4$$

Putting all this together,

$$i(t) = \begin{cases} 0, & t \le 0 \\ 4(1 - e^{-2t}), & 0 \le t \le 4 \\ 2.727 + 1.273e^{-1.4667(t-4)}, & t \ge 4 \end{cases}$$

At $t = 2$,

$$i(2) = 4(1 - e^{-4}) = 3.93\text{ A}$$

At $t = 5$,

$$i(5) = 2.727 + 1.273e^{-1.4667} = 3.02\text{ A}$$

Summary and Review

- The response of a circuit having sources suddenly switched in or out of a circuit containing capacitors and inductors will always be composed of two parts: **a natural response** and **a forced response.**

- A circuit reduced to a single equivalent inductance L and a single equivalent resistance R will have a natural response given by $i(t) = I_0 e^{-t/\tau}$, where $\tau = L/R$ is the circuit time constant.

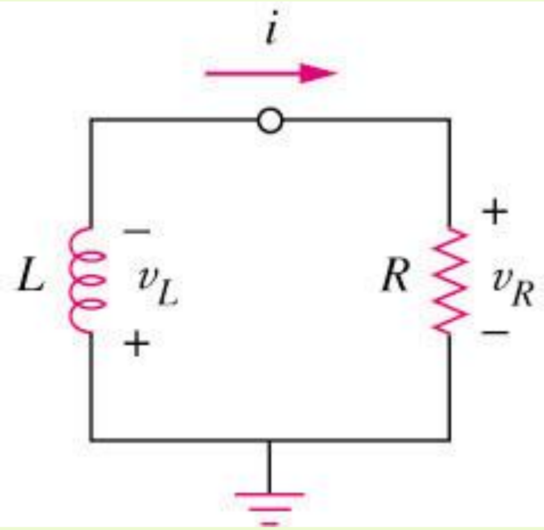
- A circuit reduced to a single equivalent capacitance *C* and a single equivalent resistance *R* will have a natural response given by $v(t) = V_0 e^{-t/\tau}$, where $\tau = RC$ is the circuit time constant.
- The unit-step function is a useful way to model the closing or opening of a switch, provided we are careful to keep an eye on the initial conditions.
- The complete response of an *RL* or *RC* circuit excited by a dc source will have the form

$$\boldsymbol{y(t) = y(\infty) + [y(0^+) - y(\infty)]e^{-t/\tau}}$$

Comparison between a RL and RC circuit

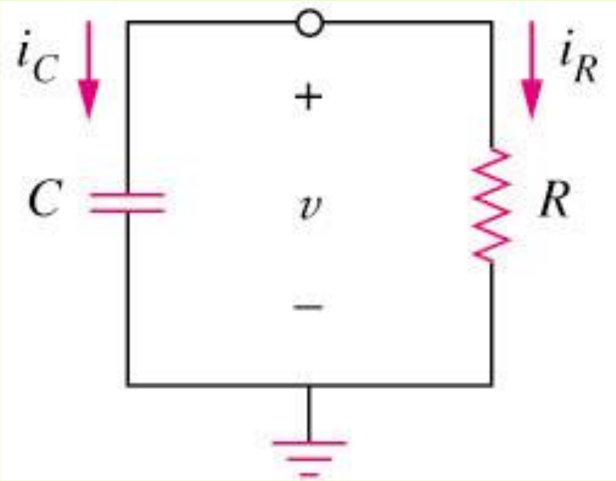
A RL source-free circuit

$$i(t) = I_0\, e^{-t/\tau} \quad \textbf{where} \quad \tau = \frac{L}{R}$$

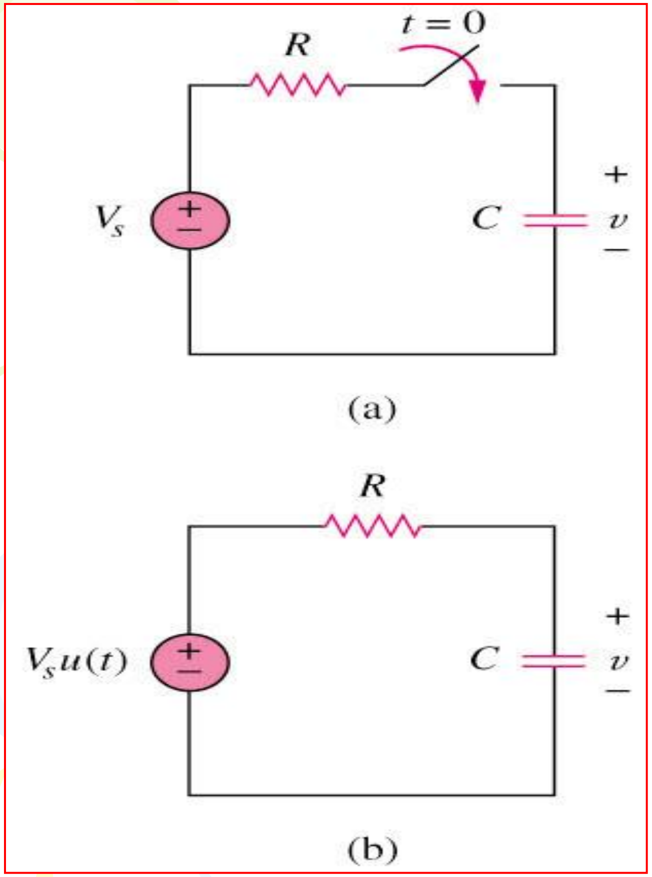


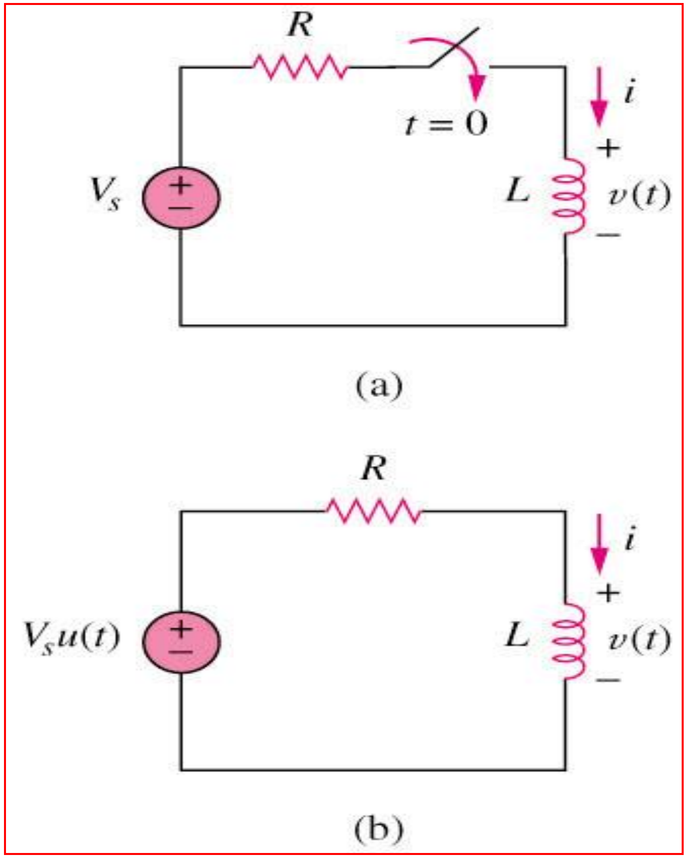
A RC source-free circuit

$$v(t) = V_0\, e^{-t/\tau} \quad \textbf{where} \quad \tau = RC$$



Step-Response





$v(0-) = v(0+) = V_0$

$v(\infty)$ $= \mathbf{V_s}$

$\tau = RC$

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$i(0^-) = i(0^+) = I_o$

$i(\infty) = V_s/R$

Time constant $\tau = L/R$

$$i(t) = i(\infty) + [i(0+) - i(\infty)]e^{-t/\tau}$$

First time assignment

7.7 Assuming that the switch in Fig. 7.87 has been in position A for a long time and is moved to position B at $t = 0$, Then at $t = 1$ second, the switch moves from B to C. Find $v_C(t)$ for $t \geq 0$.

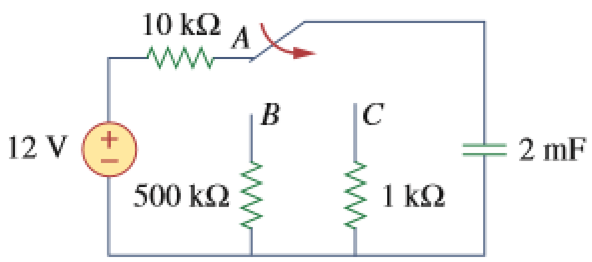


Figure 7.87
For Prob. 7.7.

7.9 The switch in Fig. 7.89 opens at $t = 0$. Find v_o for $t > 0$.

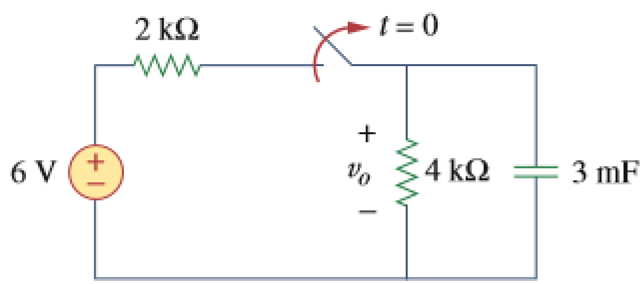


Figure 7.89
For Prob. 7.9.

Second time assignment

7.18 For the circuit in Fig. 7.98, determine $v_o(t)$ when $i(0) = 5$ A and $v(t) = 0$.

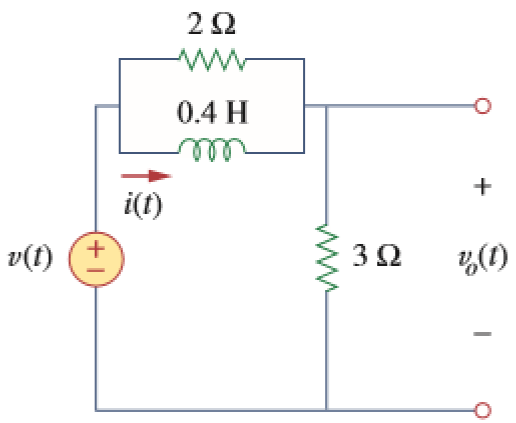


Figure 7.98
For Prob. 7.18.

7.19 In the circuit of Fig. 7.99, find $i(t)$ for $t > 0$ if $i(0) = 6$ A.

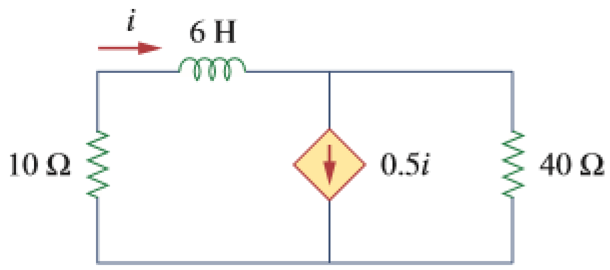


Figure 7.99
For Prob. 7.19.

7.26 Express the signals in Fig. 7.104 in terms of singularity functions.

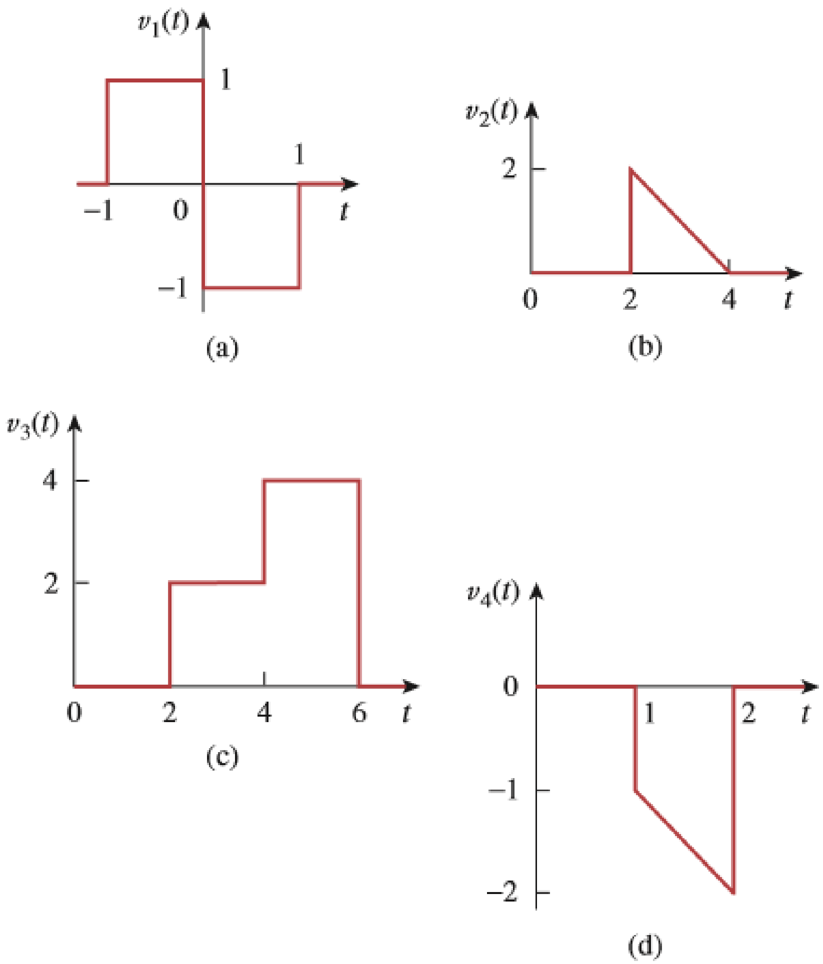


Figure 7.104
For Prob. 7.26.

7.39 Calculate the capacitor voltage for $t < 0$ and $t > 0$ for each of the circuits in Fig. 7.106.

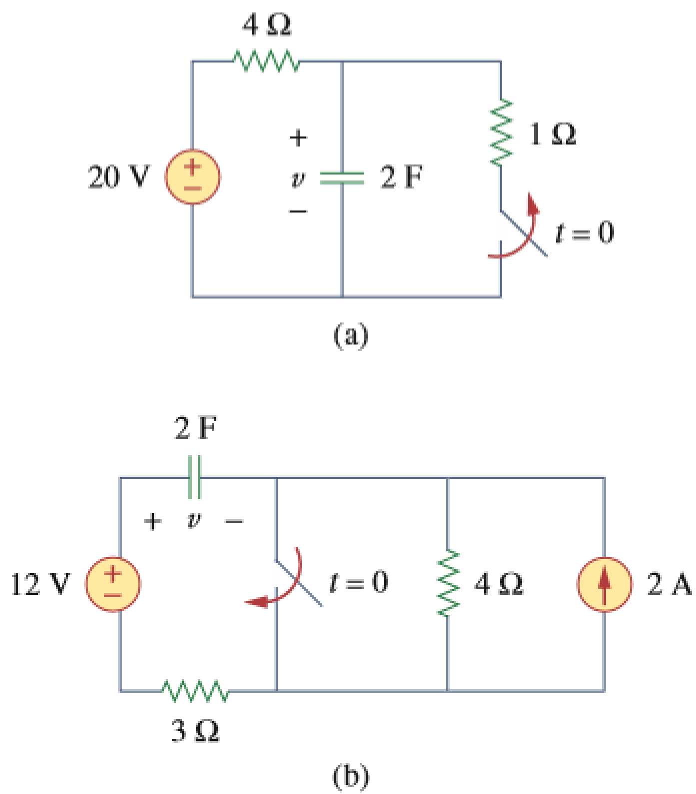


Figure 7.106
For Prob. 7.39.

7.53 Determine the inductor current $i(t)$ for both $t < 0$ and $t > 0$ for each of the circuits in Fig. 7.119.

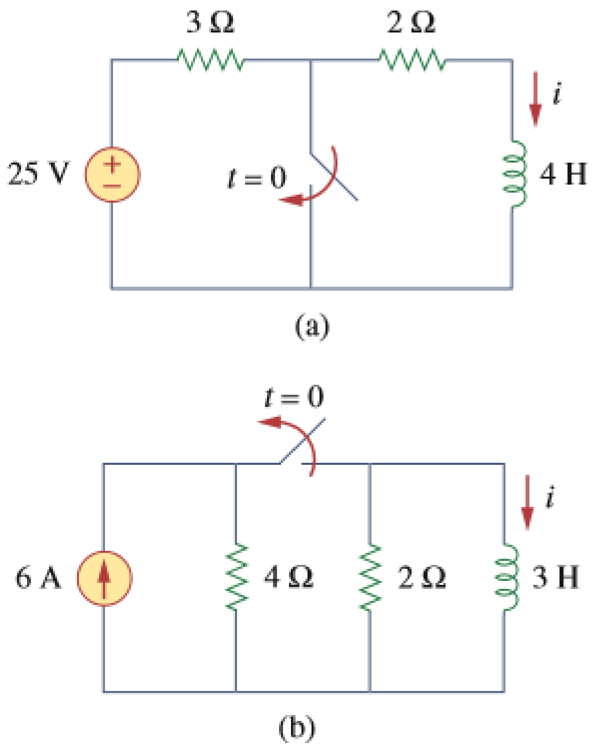


Figure 7.119
For Prob. 7.53.

7.65 If the input pulse in Fig. 7.130(a) is applied to the circuit in Fig. 7.130(b), determine the response $i(t)$.

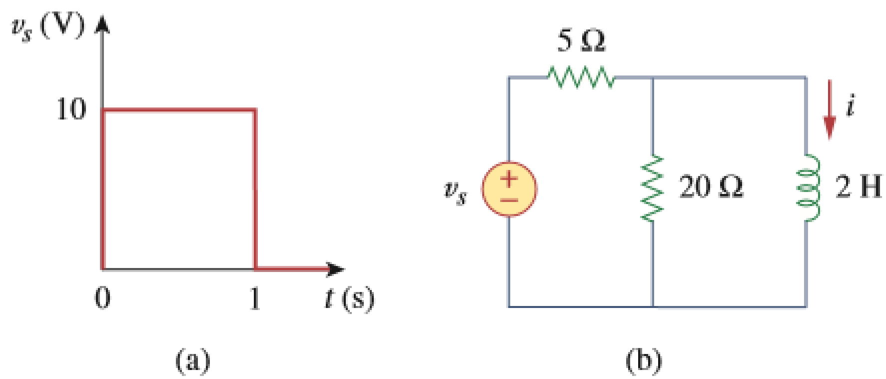


Figure 7.130
For Prob. 7.65.

Fundamentals of Electric Circuit 2021.4

Chapter 9

Sinusoids and Phasors

Chapter 9 Sinusoids and Phasors

9.1 Introduction

9.2 Sinusoids

9.3 Phasors

9.4 Phasors relationships for circuits elements

9.5 Impedance and admittance

9.6 Kirchhoff's Laws in the frequency domain

9.7 Impedance combinations

9.1 introduction

AC Analysis

DC Analysis: voltage and current are constant with respect to time.

AC Analysis: voltage and current vary with time.

AC can be sinusoidal, square waves, or arbitrary periodic waveforms.

Sinusoidal is particularly important

- ✓ Commonly used, e.g., power systems, communications, etc.
- ✓ Simple periodic function (e.g., derivative and anti-derivative of a sinusoidal is also a sinusoidal)
- ✓ Any periodic function can be represented as the sum of sinusoidal function

=> Fourier Series

- **We now begin the analysis of circuits in which the source voltage or current is sinusoid.**

A sinusoid is a signal that has the form of the sine or cosine function.

- **Circuits driven by sinusoidal current or voltage sources are called *ac circuits*.**

9.2 Sinusoids

$$v(t) = V_m \sin \omega t$$

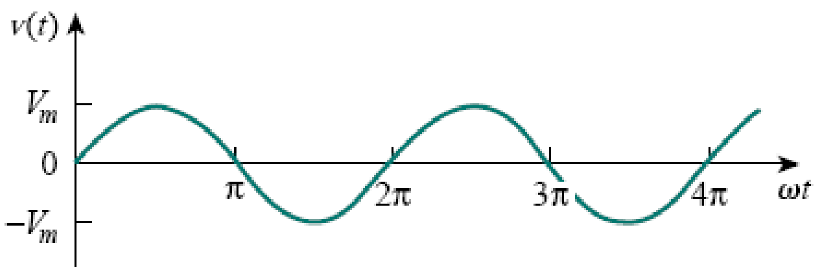
$$\boxed{T = \frac{2\pi}{\omega}}$$

V_m = the *amplitude* of the sinusoid

ω = the *angular frequency* in radians/s

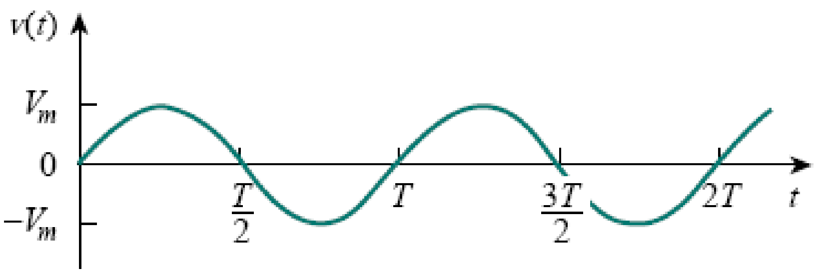
ωt = the *argument* of the sinusoid

$\omega T = 2\pi$



(a)

as a function of ωt



(b)

as a function of t

It is evident that the sinusoid repeats itself every ***T*** seconds; thus, *T* is called the ***period*** of the sinusoid.

period

$$v(t+T) = V_m \sin\omega(t+T) = V_m \sin\omega\left(t + \frac{2\pi}{\omega}\right)$$

$$= V_m \sin(\omega t + 2\pi) = V_m \sin\omega t = v(t)$$

$$\boxed{T = \frac{2\pi}{\omega}}$$

$$\boxed{v(t+T) = v(t)}$$

A periodic function is one that satisfies $f(t) = f(t + nT)$, for all t and for all integers n.

The reciprocal of this quantity is the number of cycles per second, known as the *cyclic frequency f* of the sinusoid.

$$\boxed{T = \frac{2\pi}{\omega}}$$

$$\boxed{f = \frac{1}{T}}$$

$$\omega = 2\pi f$$

Let us now consider a more general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \varphi)$$

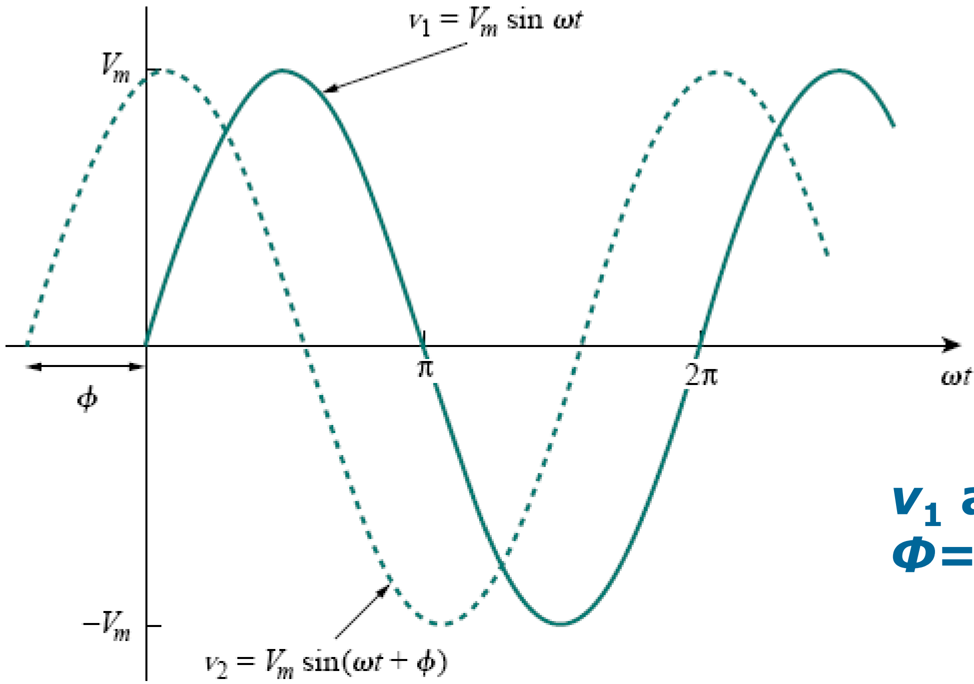
Three factors: V_m = the *amplitude* of the sinusoid

ω = the *angular frequency* in radians/s

φ is the *phase*.

Let us examine the two sinusoids

$$v_1(t) = V_m \sin\omega t \quad \text{and} \quad v_2(t) = V_m \sin(\omega t + \varphi)$$



v_2 leads v_1 by φ

v_1 lags v_2 by φ

v_1 and v_2 are *in phase:* $\Phi = 0$

A sinusoid can be expressed in either *sine* or *cosine* form.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t$$

Using these relationships, we can transform a sinusoid from *sine* form to *cosine* form or vice versa.

EXAMPLE 9.1

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10^\circ)$$

Solution:

The amplitude is $V_m = 12$ V.

The phase is $\phi = 10^\circ$.

The angular frequency is $\omega = 50$ rad/s.

The period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{50} = 0.1257$ s.

The frequency is $f = \dfrac{1}{T} = 7.958$ Hz.

PRACTICE PROBLEM 9.1

Given the sinusoid $5 \sin(4\pi t - 60^\circ)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Answer: 5, -60°, 12.57 rad/s, 0.5 s, 2 Hz.

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\cdots\cdots(1)$$

$$\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\cdots\cdots(2)$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\cdots\cdots(3)$$

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\cdots\cdots(4)$$

$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\sin\alpha\sin\beta=-\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$$

9.3 Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than *sine* and *cosine* functions.

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

A complex number z can be written in rectangular form as

$$z = x + jy$$

$j = \sqrt{-1}$ **x is the real part of z ; y is the imaginary part of z.**

The complex number z can also be written in polar or exponential form as

$$z = r\angle\phi = re^{j\phi}$$

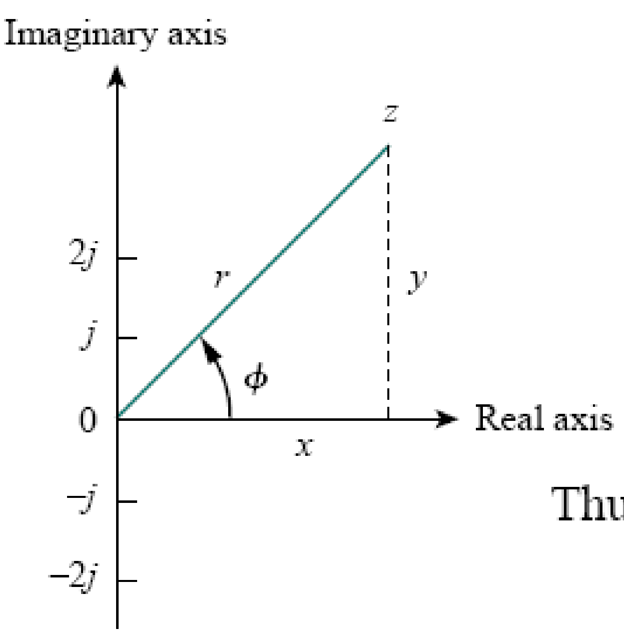
r is the magnitude of z
φ is the phase of z

Z can be represented in three ways:

$z = x + jy$ Rectangular form

$z = r\angle\phi$ Polar form

$z = re^{j\phi}$ Exponential form



$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi, \qquad y = r\sin\phi$$

Thus, z may be written as

$$\boxed{z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)}$$

The idea of phasor representation is based on Euler's identity.

$$\boxed{e^{\pm j\phi} = \cos\phi \pm j\sin\phi}$$

which shows that we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$; we may write

$$\cos\phi = \operatorname{Re}(e^{j\phi})$$

$$\sin\phi = \operatorname{Im}(e^{j\phi})$$

where Re and Im stand for the *real part of* and the *imaginary part of*.

Given a sinusoid ***$v(t)=V_m\cos(\omega t + \varphi)$***

$$v(t) = V_m\cos(\omega t + \phi) = \operatorname{Re}(V_m e^{j(\omega t+\phi)}) \quad \text{or} \quad v(t) = \operatorname{Re}(V_m e^{j\phi} e^{j\omega t})$$

$$\boxed{v(t) = \operatorname{Re}(\mathbf{V} e^{j\omega t})} \qquad \mathbf{V} = V_m e^{j\phi} = V_m \underline{/\phi}$$

V is thus the *phasor representation* of the sinusoid *v(t)*

A phasor is a complex representation of the magnitude and phase of a sinusoid.

By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$v(t) = \mathrm{Re}(\mathbf{V}e^{j\omega t})$$

$$v(t) = V_m \cos(\omega t + \phi) \quad \Longleftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

Note that in Eq. (9.25) the frequency (or time) factor $e^{j\omega t}$ is suppressed, and the frequency is not explicitly shown in the phasor-domain representation because ω is constant. However, the response depends on ω. For this reason, the phasor domain is also known as the *frequency domain.*

TABLE 9.1 Sinusoid-phasor transformation.

Time-domain representation	Phasor-domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

EXAMPLE 9.4

Transform these sinusoids to phasors:

(a) $v = -4\sin(30t + 50^\circ)$

(b) $i = 6\cos(50t - 40^\circ)$

$$v(t) = V_m\cos(\omega t + \phi) \quad \Longleftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

Solution:

(a) Since $-\sin A = \cos(A + 90^\circ)$,

$$v = -4\sin(30t + 50^\circ) = 4\cos(30t + 50^\circ + 90^\circ)$$

$$= 4\cos(30t + 140^\circ)$$

The phasor form of v is $\quad \mathbf{V} = 4\angle 140^\circ$

(b) $i = 6\cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\angle -40^\circ$$

$$v(t) = V_m \cos(\omega t + \phi) \quad \Longleftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)

PRACTICE PROBLEM 9.4

Express these sinusoids as phasors:

(a) $v = -7\cos(2t + 40°)$

(b) $i = 4\sin(10t + 10°)$

Answer: (a) $\mathbf{V} = 7\underline{/220°}$, (b) $\mathbf{I} = 4\underline{/-80°}$.

EXAMPLE 9.5

Find the sinusoids represented by these phasors:

(a) $\mathbf{V} = j8e^{-j20^\circ}$

(b) $\mathbf{I} = -3 + j4$

Solution:

(a) Since $j = 1\angle 90^\circ$,

$$\mathbf{V} = j8\angle -20^\circ = (1\angle 90^\circ)(8\angle -20^\circ)$$

$$= 8\angle 90^\circ - 20^\circ = 8\angle 70^\circ \text{ V}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70^\circ) \text{ V}$$

(b) $\mathbf{I} = -3 + j4 = 5\angle 126.87^\circ$.

$$i(t) = 5\cos(\omega t + 126.87^\circ) \text{ A}$$

EXAMPLE 9.6

Given $i_1(t) = 4\cos(\omega t + 30^\circ)$ and $i_2(t) = 5\sin(\omega t - 20^\circ)$, find their sum.

Solution:

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\angle 30^\circ$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20^\circ - 90^\circ) = 5\cos(\omega t - 110^\circ)$$

and its phasor is $\mathbf{I}_2 = 5\angle -110^\circ$

If we let $i = i_1 + i_2$, then

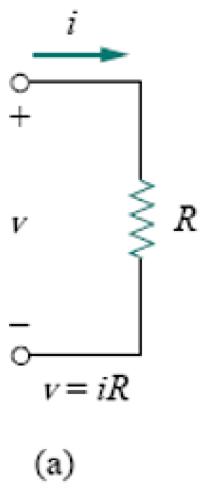
$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 4\angle 30^\circ + 5\angle -110^\circ$$

$$= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698$$

$$= 3.218\angle -56.97^\circ \text{ A}$$

$$i(t) = 3.218\cos(\omega t - 56.97^\circ) \text{ A}$$

9.4 Phasors relationships for circuits elements



For the resistor R.

If the current through a resistor R is $i = I_m \cos(\omega t + \varphi)$,

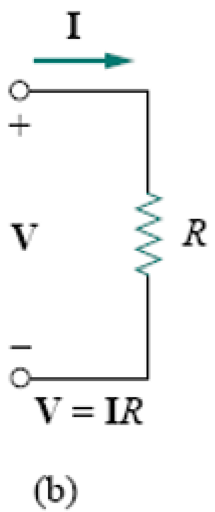
the voltage across it is given by Ohm's law as

$$v = iR = RI_m \cos(\omega t + \varphi)$$

Since $\mathbf{I} = I_m \angle \phi$. $\quad \mathbf{V} = RI_m \angle \phi$

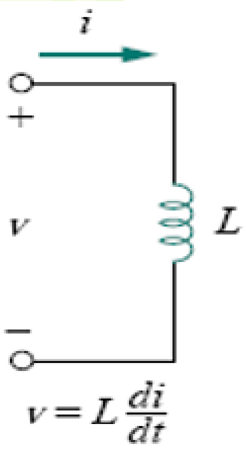
So $\quad \mathbf{V} = R\mathbf{I}$ the voltage-current relation for the resistor in the phasor domain

voltage and current are in phase



$$\mathbf{V} = R\mathbf{I}$$

For the inductor *L*



(a)

Assume the current through it is $i = I_m \cos(\omega t + \varphi)$.

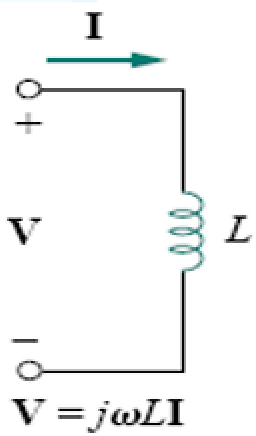
$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

Since $\mathbf{I} = I_m \angle \phi$ $\quad e^{j90^\circ} = j$

$$\mathbf{V} = \omega L I_m e^{j(\phi + 90^\circ)} = \omega L I_m e^{j\phi} e^{j90^\circ} = \omega L I_m \angle \phi\, e^{j90^\circ}$$

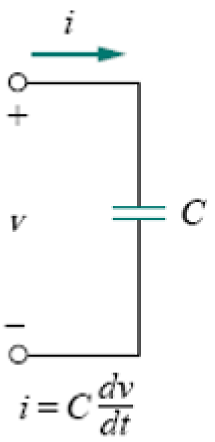
So $\mathbf{V} = j\omega L\mathbf{I}$

the current lags the voltage by 90°.



(b)

For the capacitor C



(a)

Aassume the voltage across it is $v = V_m \cos(\omega t + \varphi)$.

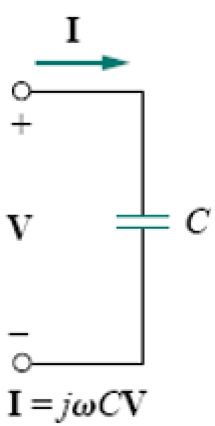
The current through the capacitor is

$$i = C\frac{dv}{dt}$$

$$\mathbf{I} = j\omega C\mathbf{V}$$

$$\Longrightarrow \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

the current leads the voltage by 90°.



(b)

Table 9.2 summarizes the time-domain and phasor-domain representations of the circuit elements.

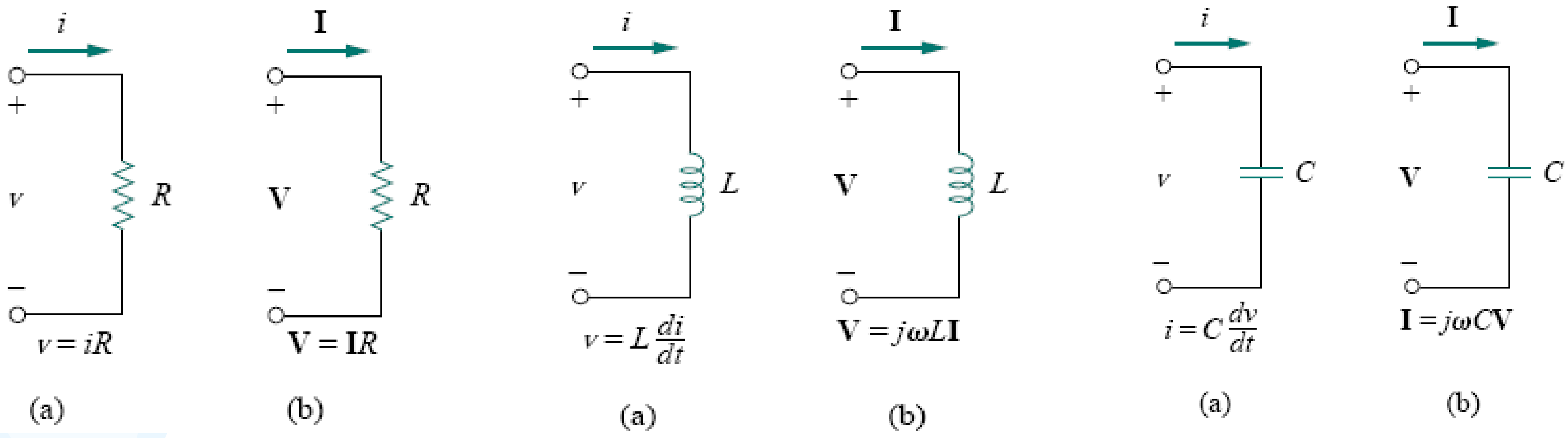


TABLE 9.2 Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

EXAMPLE 9.8

The voltage $v = 12\cos(60t + 45^\circ)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution:

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and

$$\mathbf{V} = 12\angle 45^\circ \text{ V.}$$

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\angle 45^\circ}{j60 \times 0.1} = \frac{12\angle 45^\circ}{6\angle 90^\circ} = 2\angle -45^\circ \text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45^\circ) \text{ A}$$

PRACTICE PROBLEM 9.8

If voltage $v = 6\cos(100t - 30^\circ)$ is applied to a 50 μF capacitor, calculate the current through the capacitor.

Answer: $30\cos(100t + 60^\circ)$ mA.

9.5 Impedance and Admittance

In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$\boxed{\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \qquad \text{or} \qquad \mathbf{V} = \mathbf{Z}\mathbf{I}}$$

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \qquad \text{or} \qquad \mathbf{V} = \mathbf{Z}\mathbf{I}$$

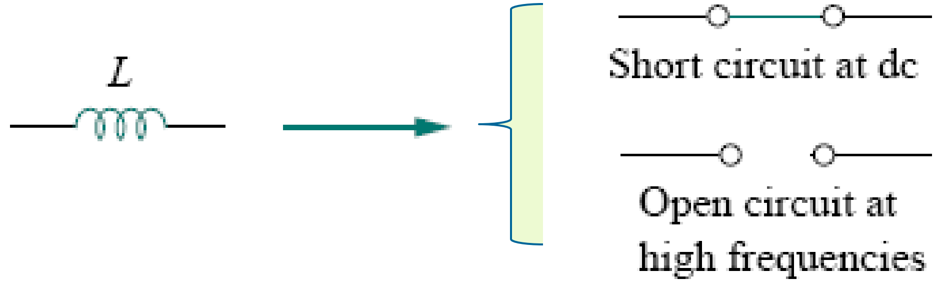
where **Z** is a frequency-dependent quantity known as *impedance*, measured in ohms.

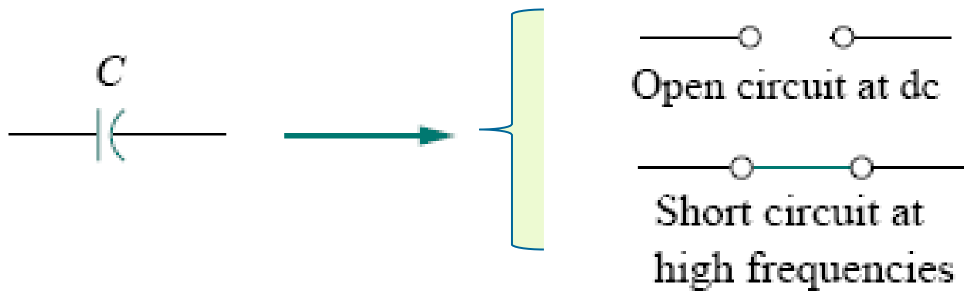
The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω).

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

Element	Impedance
R	$\mathbf{Z} = R$
L	$\mathbf{Z} = j\omega L$
C	$\mathbf{Z} = \frac{1}{j\omega C}$

Element	Impedance
R	$\mathbf{Z} = R$
L	$\mathbf{Z} = j\omega L$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$





Consider two extreme cases of angular frequency.

1. L: When $\omega = 0$, $\mathbf{Z}_L = 0$ (short circuit)
   When $\omega \rightarrow \infty$, $\mathbf{Z}_L \rightarrow \infty$(open circuit)
2. C: When $\omega = 0$, $\mathbf{Z}_C \rightarrow \infty$ (open circuit)
   When $\omega \rightarrow \infty$, $\mathbf{Z}_C = 0$ (short circuit)

As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R + jX$$

R = Re $\mathbf{Z}$ is the *resistance* and X = Im $\mathbf{Z}$ is the *reactance*.

The impedance may also be expressed in polar form as

$$\mathbf{Z} = |\mathbf{Z}|\angle\theta$$

$$\boxed{\mathbf{Z} = R + jX = |\mathbf{Z}|\angle\theta}$$

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R}$$

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta$$

It is sometimes convenient to work with the reciprocal of impedance, known as *admittance*.

The admittance **Y** is the reciprocal of impedance, measured in siemens (S).

$$\boxed{\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}}$$

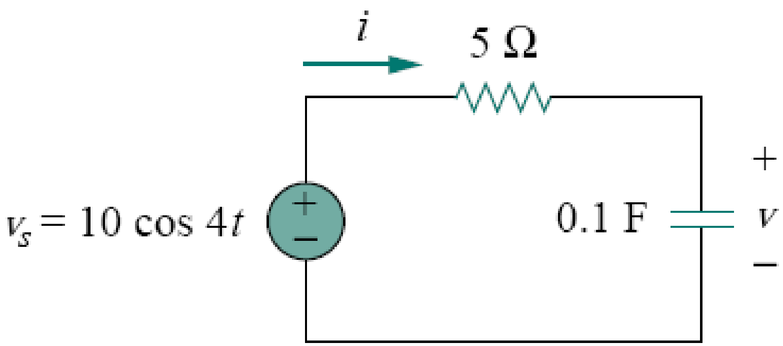
$$\boxed{\mathbf{Y} = G + jB}$$

G =Re**Y** is called the *conductance*

B =Im**Y** is called the *susceptance*.

EXAMPLE 9.9

Find *v(t)* and *i(t)* in the circuit shown in Fig. 9.16.



Solution:

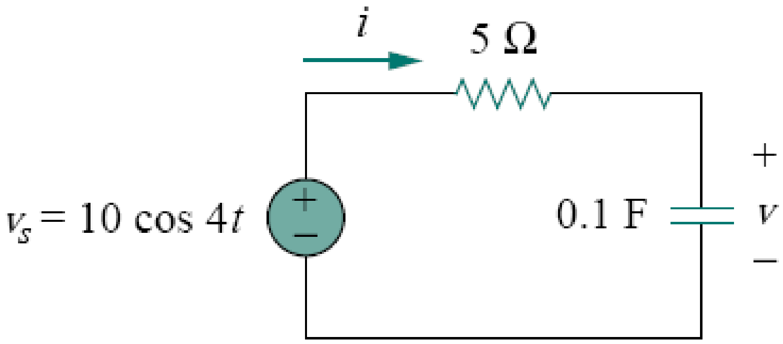
From the voltage source 10cos4*t* , *ω* = 4, $\mathbf{V}_s = 10\angle 0^\circ \text{ V}$

The impedance is $$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current $$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$
$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

EXAMPLE 9.9

Find *v(t)* and *i(t)* in the circuit shown in Fig. 9.16.



Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$
$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4 \times 0.1}$$
$$= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V}$$

$$i(t) = 1.789 \cos(4t + 26.57^\circ) \text{ A}$$

$$v(t) = 4.47 \cos(4t - 63.43^\circ) \text{ V}$$

9.6 Kirchhoff's Laws in the frequency domain

We cannot do circuit analysis in the frequency domain without Kirchhoff's current and voltage laws. Therefore, we need to express them in the frequency domain.

For KVL,

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdot \quad \cdot \quad \cdot + \mathbf{V}_n = 0$$

Kirchhoff's voltage law holds for phasors.

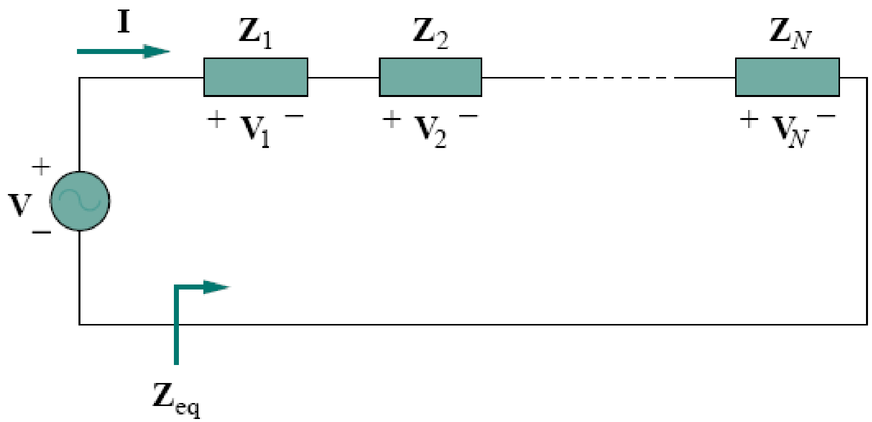
For KCL,

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdot \quad \cdot \quad \cdot + \mathbf{I}_n = 0$$

Kirchhoff's current law holds for phasors.

9.7 Impedance Combination

Consider the *N* series-connected impedances



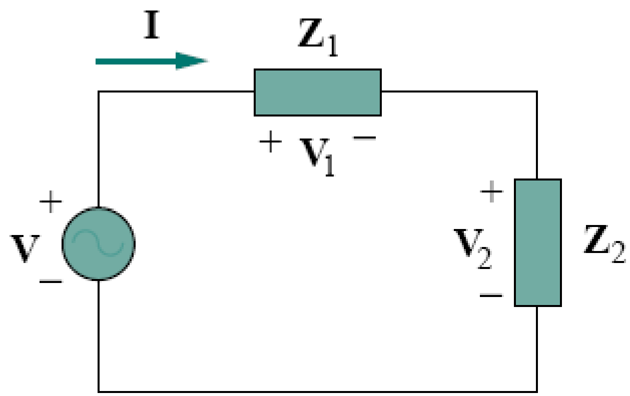
$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

The equivalent impedance at the input terminals is

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

$$\boxed{\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N}$$

The total or equivalent impedance of series-connected impedances is the sum of the individual impedances. This is similar to the series connection of resistances.



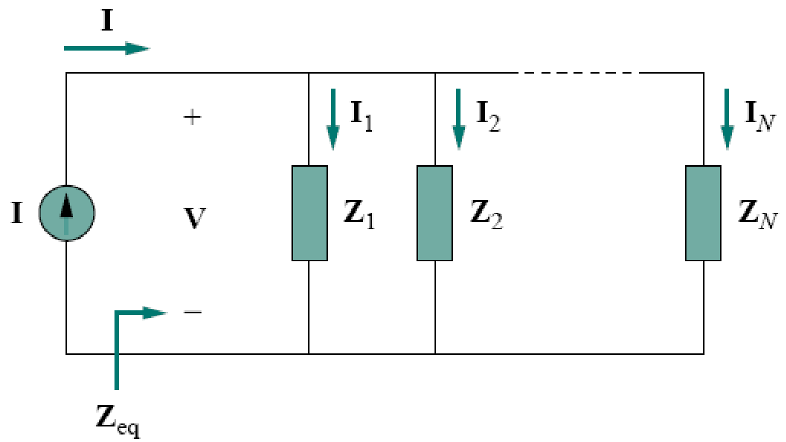
If N = 2, the current through the impedances is $\mathbf{I} = \dfrac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}$

Since $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$, then

$$\boxed{\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}}$$

which is the ***voltage-division*** relationship.

Consider the *N* parallel-connected impedances



$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right)$$

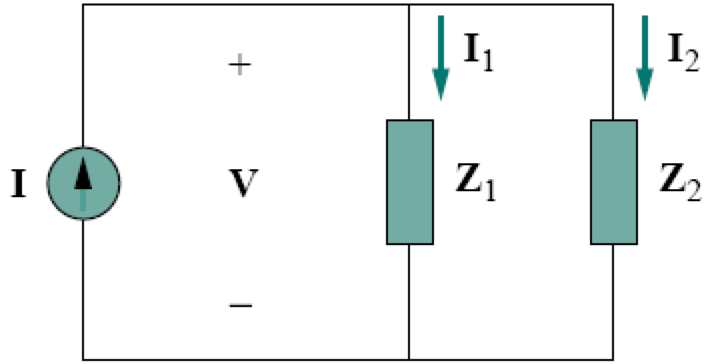
The equivalent impedance is

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

The equivalent admittance is

$$\boxed{\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N}$$

the equivalent admittance of a parallel connection of admittances is the sum of the individual admittances. This is similar to the parallel connection of resistances.



When N = 2, the equivalent impedance becomes

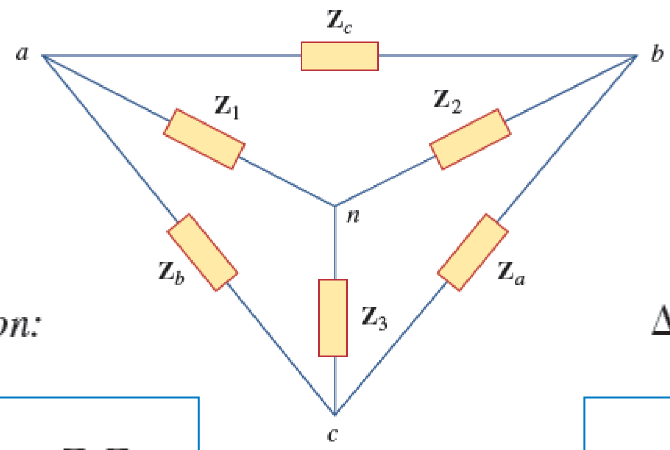
$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2$$

the currents in the impedances are

$$\boxed{\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}}$$

which is the ***current-division*** principle.



Y-Δ Conversion:

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2}$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

Δ-Y Conversion:

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

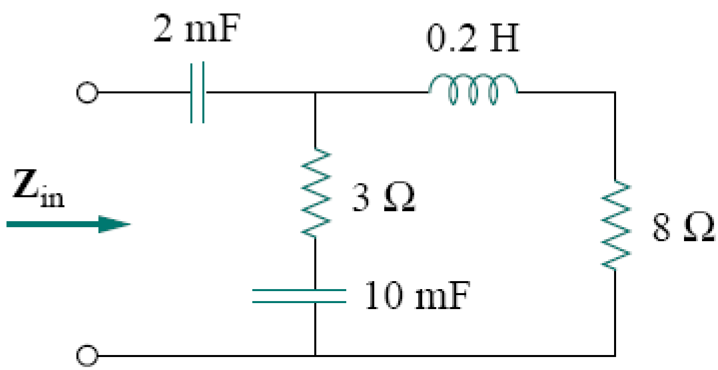
A delta or wye circuit is said to be **balanced** if it has equal impedances in all three branches.

$$\mathbf{Z}_Y = \mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 \text{ and } \mathbf{Z}_\Delta = \mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c.$$

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \qquad \text{or} \qquad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta$$

EXAMPLE 9.10

Find the input impedance of the circuit in this Fig. Assume that the circuit operates at ω = 50 rad/s.



The input impedance is

$$\mathbf{Z}_{\text{in}} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3$$

$$= 3.22 - j11.07\ \Omega$$

Solution:

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3- Ω resistor in series with the 10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

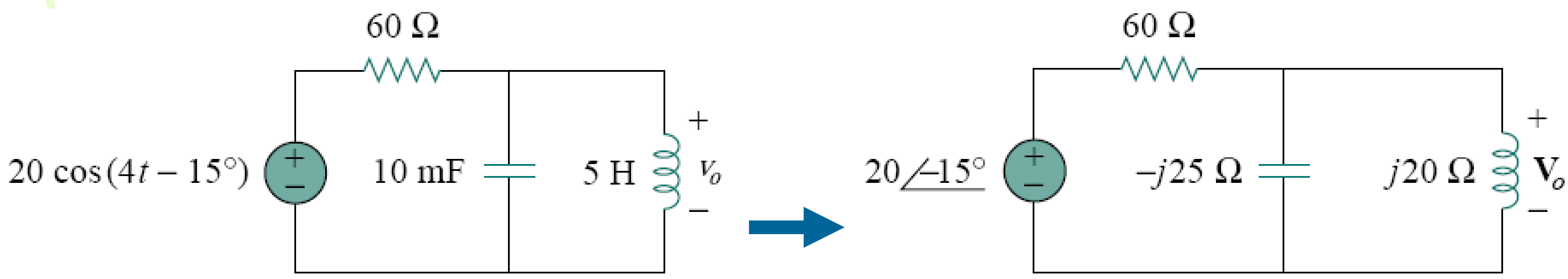
$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

EXAMPLE 9.11

Determine $v_o(t)$ in the circuit in Fig.25



Solution:

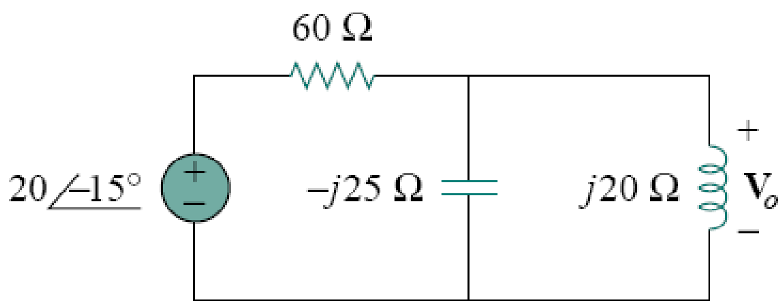
To do the analysis in the frequency domain, we must first transform the time-domain circuit in left Fig to the phasor-domain equivalent in right Fig. The transformation produces

$$v_s = 20\cos(4t - 15^\circ) \quad \Longrightarrow \quad \mathbf{V}_s = 20\underline{/-15^\circ}\ \text{V}, \qquad \omega = 4$$

$$10\ \text{mF} \quad \Longrightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5\ \text{H} \quad \Longrightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

EXAMPLE 9.11 Determine $v_o(t)$ in the circuit in Fig.25



$\mathbf{Z}_1$ = Impedance of the 60- resistor $\quad \mathbf{Z}_1 = 60\ \Omega$

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

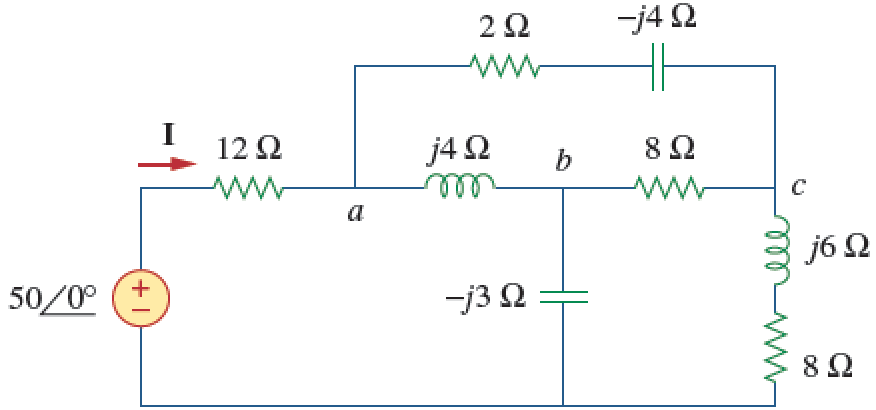
$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = 17.15\underline{/15.96^\circ}\ \text{V}.$$

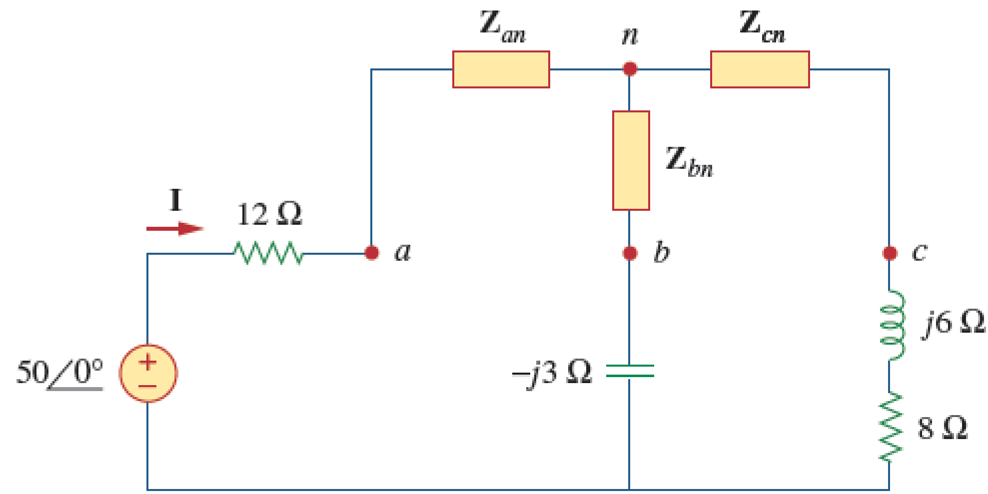
We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96^\circ)\text{V}$$

Example 9.12

Find current **I** in the circuit of Fig. 9.28.





Solution:

The delta network connected to nodes *a*, *b*, and *c* can be converted to the *Y* network of Fig. 9.29. We obtain the *Y* impedances as follows using Eq. (9.68):

$$\mathbf{Z}_{an} = \frac{j4(2 - j4)}{j4 + 2 - j4 + 8} = \frac{4(4 + j2)}{10} = (1.6 + j0.8)\ \Omega$$

$$\mathbf{Z}_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad \mathbf{Z}_{cn} = \frac{8(2 - j4)}{10} = (1.6 - j3.2)\ \Omega$$

The total impedance at the source terminals is

$$\mathbf{Z} = 12 + \mathbf{Z}_{an} + (\mathbf{Z}_{bn} - j3) \parallel (\mathbf{Z}_{cn} + j6 + 8) = 13.6 + j1 = 13.64\underline{/4.204^\circ}\ \Omega$$

The desired current is $\mathbf{I} = \dfrac{\mathbf{V}}{\mathbf{Z}} = \dfrac{50\underline{/0^\circ}}{13.64\underline{/4.204^\circ}} = 3.666\underline{/-4.204^\circ}$ A

Summary and Review

1. A sinusoid is a signal in the form of the sine or cosine function. It has the general form

$$v(t) = V_m \cos(\omega t + \phi)$$

where V_m is the amplitude, $\omega = 2\pi f$ is the angular frequency, $(\omega t + \phi)$ is the argument, and ϕ is the phase.

2. A phasor is a complex quantity that represents both the magnitude and the phase of a sinusoid. Given the sinusoid $v(t) = V_m \cos(\omega t + \phi)$, its phasor $\mathbf{V}$ is

$$\mathbf{V} = V_m \underline{/\phi}$$

3. In ac circuits, voltage and current phasors always have a fixed relation to one another at any moment of time. If $v(t) = V_m \cos(\omega t + \phi_v)$ represents the voltage through an element and $i(t) = I_m \cos(\omega t + \phi_i)$ represents the current through the element, then $\phi_i = \phi_v$ if the element is a resistor, ϕ_i leads ϕ_v by 90° if the element is a capacitor, and ϕ_i lags ϕ_v by 90° if the element is an inductor.

4. The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage across it to the phasor current through it:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R(\omega) + jX(\omega)$$

The admittance $\mathbf{Y}$ is the reciprocal of impedance:

$$\mathbf{Z} = \frac{1}{\mathbf{Y}} = G(\omega) + jB(\omega)$$

Impedances are combined in series or in parallel the same way as resistances in series or parallel; that is, impedances in series add while admittances in parallel add.

5. For a resistor $\mathbf{Z} = R$, for an inductor $\mathbf{Z} = jX = j\omega L$, and for a capacitor $\mathbf{Z} = -jX = 1/j\omega C$.

6. Basic circuit laws (Ohm's and Kirchhoff's) apply to ac circuits in the same manner as they do for dc circuits; that is,

$$\mathbf{V} = \mathbf{ZI}$$

$$\sum \mathbf{I}_k = 0 \quad \text{(KCL)}$$

$$\sum \mathbf{V}_k = 0 \quad \text{(KVL)}$$

7. The techniques of voltage/current division, series/parallel combination of impedance/admittance, circuit reduction, and Y-Δ transformation all apply to ac circuit analysis.

8. AC circuits are applied in phase-shifters and bridges.

First time homework

9.11 Find the phasors corresponding to the following signals:

(a) $v(t) = 21\cos(4t - 15^\circ)$ V

(b) $i(t) = -8\sin(10t + 70^\circ)$ mA

(c) $v(t) = 120\sin(10t - 50^\circ)$ V

(d) $i(t) = -60\cos(30t + 10^\circ)$ mA

9.16 Transform the following sinusoids to phasors:

(a) $-20\cos(4t + 135^\circ)$ (b) $8\sin(20t + 30^\circ)$

(c) $20\cos(2t) + 15\sin(2t)$

9.18 Obtain the sinusoids corresponding to each of the following phasors:

(a) $\mathbf{V}_1 = 60\underline{/15^\circ}$ V, $\omega = 1$

(b) $\mathbf{V}_2 = 6 + j8$ V, $\omega = 40$

(c) $\mathbf{I}_1 = 2.8e^{-j\pi/3}$ A, $\omega = 377$

(d) $\mathbf{I}_2 = -0.5 - j1.2$ A, $\omega = 10^3$

Second time homework

9.30 A voltage $v(t) = 100 \cos(60t + 20°)$ V is applied to a parallel combination of a 40-kΩ resistor and a 50-μF capacitor. Find the steady-state currents through the resistor and the capacitor.

9.35 Find current i in the circuit of Fig. 9.42, when $v_s(t) = 50 \cos 200t$ V.

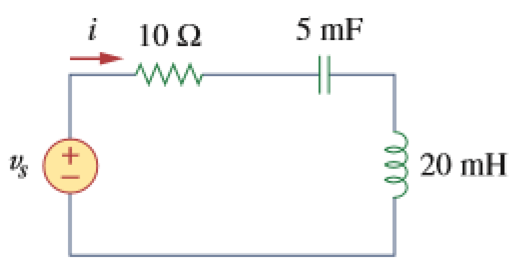


Figure 9.42
For Prob. 9.35.

9.34 What value of ω will cause the forced response, v_o, in Fig. 9.41 to be zero?

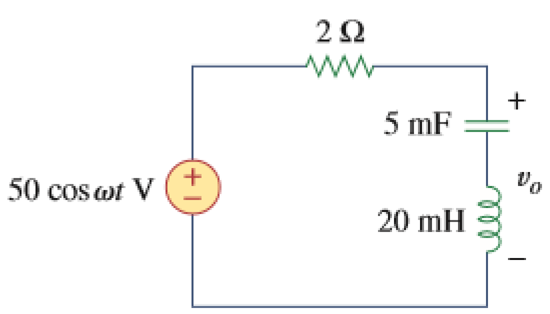


Figure 9.41
For Prob. 9.34.

Second time homework

9.42 Calculate $v_o(t)$ in the circuit of Fig. 9.49.

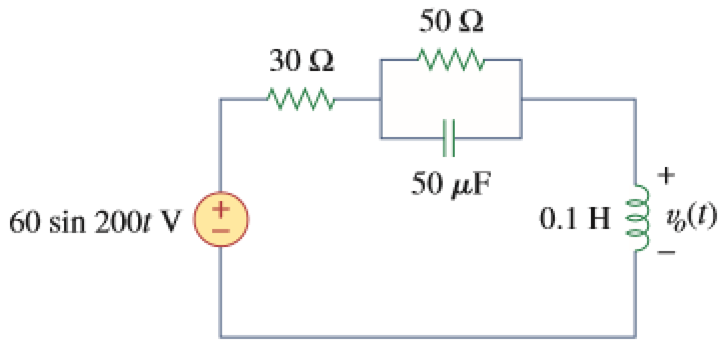


Figure 9.49
For Prob. 9.42.

9.56 At $\omega = 377$ rad/s, find the input impedance of the circuit shown in Fig. 9.63.

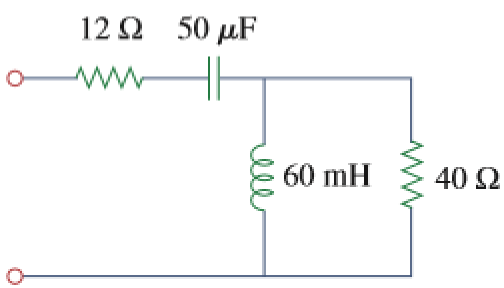


Figure 9.63
For Prob. 9.56.

Second time homework

9.61 Find $\mathbf{Z}_{eq}$ in the circuit of Fig. 9.68.

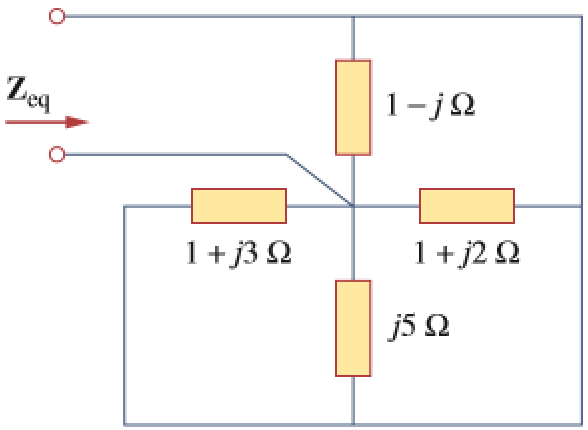


Figure 9.68
For Prob. 9.61.

9.63 For the circuit in Fig. 9.70, find the value of $\mathbf{Z}_T$.

ML

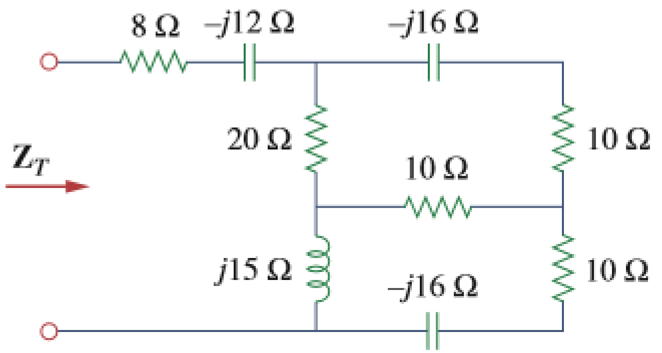


Figure 9.70
For Prob. 9.63.

Fundamentals of Electric Circuit 2021.5

Chapter 10
Sinusoidal Steady-State Analysis

Chapter 10 Sinusoidal Steady-State Analysis

10.1 Introduction

In Chapter 9, we learned that the forced or steady-state response of circuits to sinusoidal inputs can be obtained by using phasors. We also know that Ohm's and Kirchhoff's laws are applicable to ac circuits.

In this chapter, we want to see how nodal analysis, mesh analysis, Thevenin's theorem, Norton's theorem, superposition, and source transformations are applied in analyzing ac circuits. Since these techniques were already introduced for dc circuits, our major effort here will be to illustrate with examples.

- Analyzing ac circuits usually requires three steps:

1. Transform the circuit to the phasor or frequency domain.
2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
3. Transform the resulting phasor to the time domain.

- Step 1 is not necessary if the problem is specified in the frequency domain.
- In step 2, the analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved.
- Having read Chapter 9, we are adept at handling step 3.

10.2 Nodal analysis

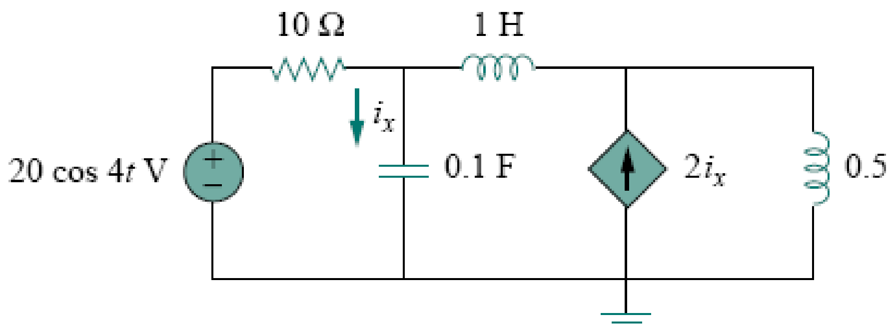
Steps to determine the node voltages:

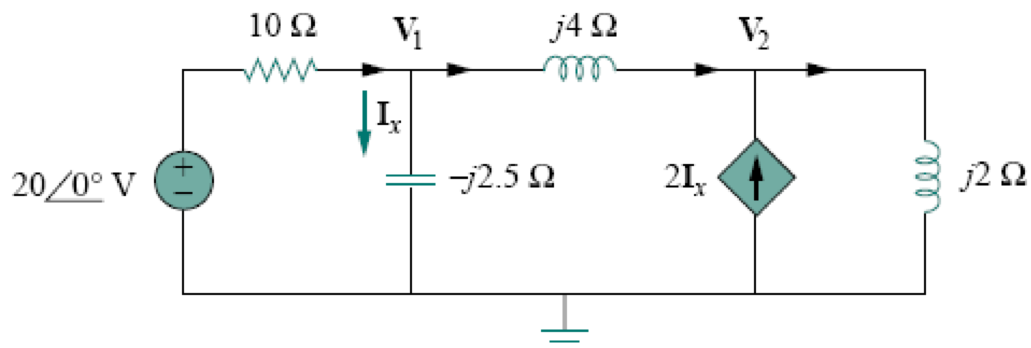
1. Select a node as the reference node.
2. Assign voltages $v_1, v_2, \ldots, v_{n-1}$ to the remaining n-1 nodes. The voltages are referenced with respect to the reference node.
3. Apply KCL to each of the n-1 non-reference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
4. Solve the resulting simultaneous equations to obtain the unknown node voltages.

10.2 Nodal analysis

EXAMPLE 10.1

Find i_x in the circuit of Fig. 10.1 using nodal analysis.



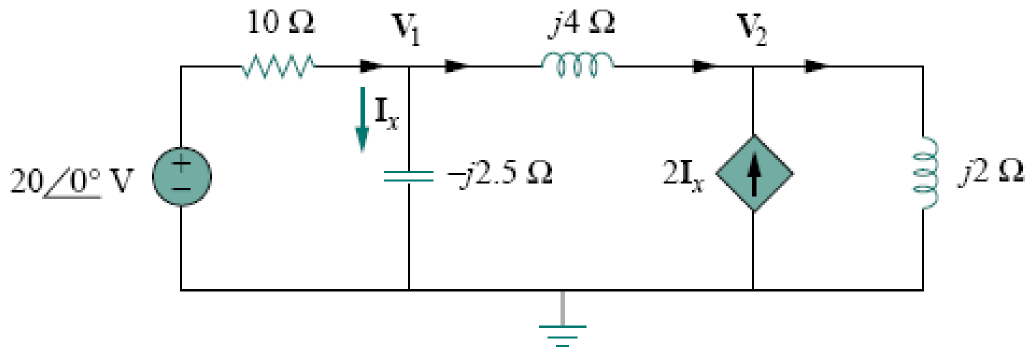


Solution:

We first convert the circuit to the frequency domain:

$$
\begin{aligned}
20\cos 4t \quad &\Longrightarrow \quad 20\angle 0^\circ, \qquad \omega = 4 \text{ rad/s} \\
1\text{ H} \quad &\Longrightarrow \quad j\omega L = j4 \\
0.5\text{ H} \quad &\Longrightarrow \quad j\omega L = j2 \\
0.1\text{ F} \quad &\Longrightarrow \quad \frac{1}{j\omega C} = -j2.5
\end{aligned}
$$

Thus, the frequency-domain equivalent circuit is as shown in Fig. 10.2.



Applying KCL at node 1,

$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

At node 2,

$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

$$\mathbf{I}_x = \mathbf{V}_1/-j2.5. \qquad \frac{2\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

$$\mathbf{V}_1 = 18.97\underline{/18.43^\circ} \text{ V}$$

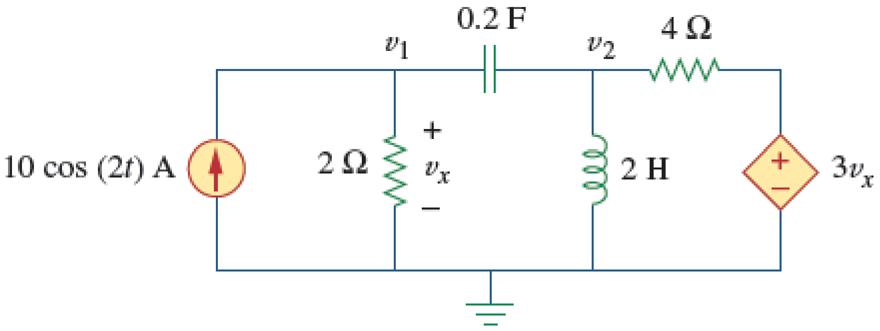
$$\mathbf{V}_2 = 13.91\underline{/198.3^\circ} \text{ V}$$

The current $\mathbf{I}_x$ is given by $\mathbf{I}_x = \dfrac{\mathbf{V}_1}{-j2.5} = 7.59\underline{/108.4^\circ}$ A

Transforming this to the time domain, $i_x = 7.59\cos(4t + 108.4^\circ)$ A

Practice Problem 10.1

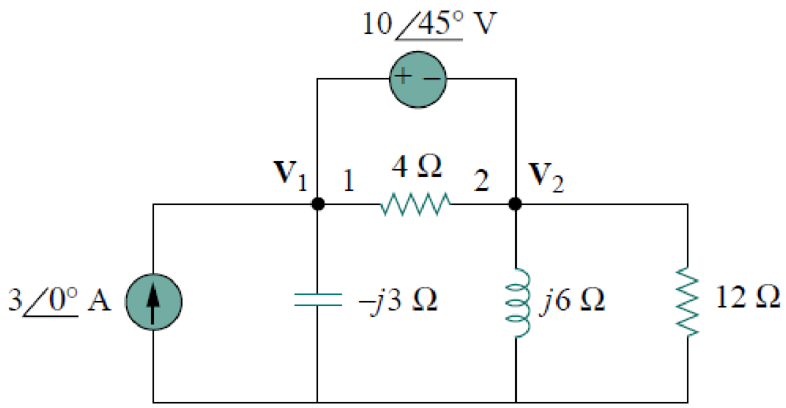
Using nodal analysis, find v_1 and v_2 in the circuit of Fig. 10.3.

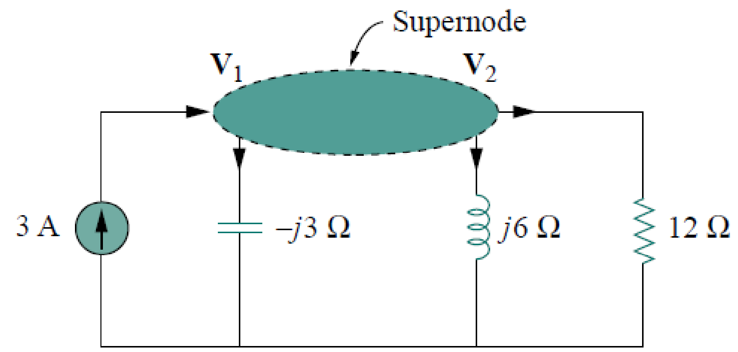


10.2 Nodal analysis

EXAMPLE 10.2

Compute $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit of Fig. 10.4.





Solution:

Nodes 1 and 2 form a supernode as shown in Fig. 10.5. Applying KCL at the supernode gives

$$3 = \frac{\mathbf{V}_1}{-j3} + \frac{\mathbf{V}_2}{j6} + \frac{\mathbf{V}_2}{12} \qquad 36 = j4\mathbf{V}_1 + (1 - j2)\mathbf{V}_2$$

But a voltage source is connected between nodes 1 and 2, so that $\mathbf{V}_1 = \mathbf{V}_2 + 10\underline{/45^\circ}$

$$36 - 40\underline{/135^\circ} = (1 + j2)\mathbf{V}_2 \quad \Longrightarrow \quad \mathbf{V}_2 = 31.41\underline{/-87.18^\circ}\text{ V}$$

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\underline{/45^\circ} = 25.78\underline{/-70.48^\circ}\text{ V}$$

PRACTICE PROBLEM 10.2

Calculate $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit shown in Fig. 10.6.

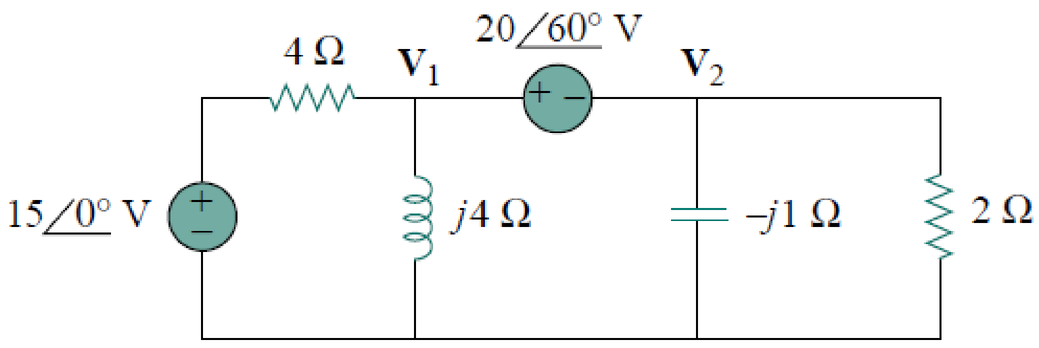


Figure 10.6 For Practice Prob. 10.2.

10.3 Mesh analysis

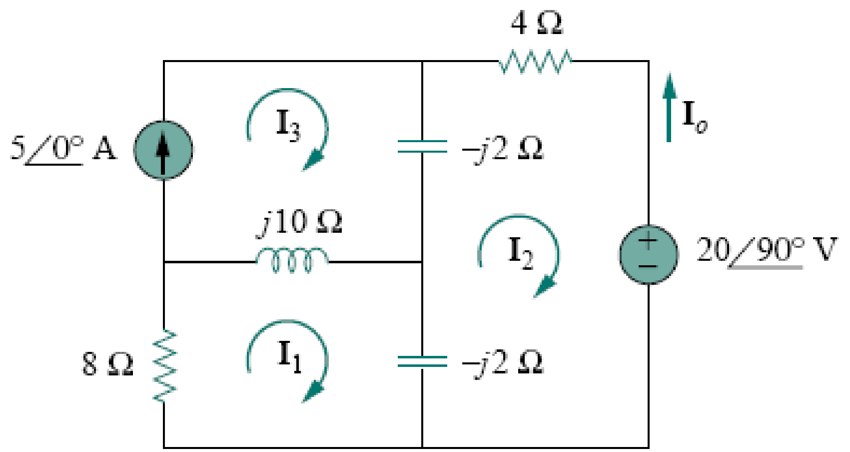
Steps to determine the mesh currents:

- *1.Assign* mesh currents i_1, i_2, …, i_n to the n meshes.
- *2. Apply* KVL to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.
- *3. Solve* the resulting n simultaneous equations to get the mesh currents.

10.3 Mesh analysis

EXAMPLE 10.3

Determine current I_o in the circuit using mesh analysis.



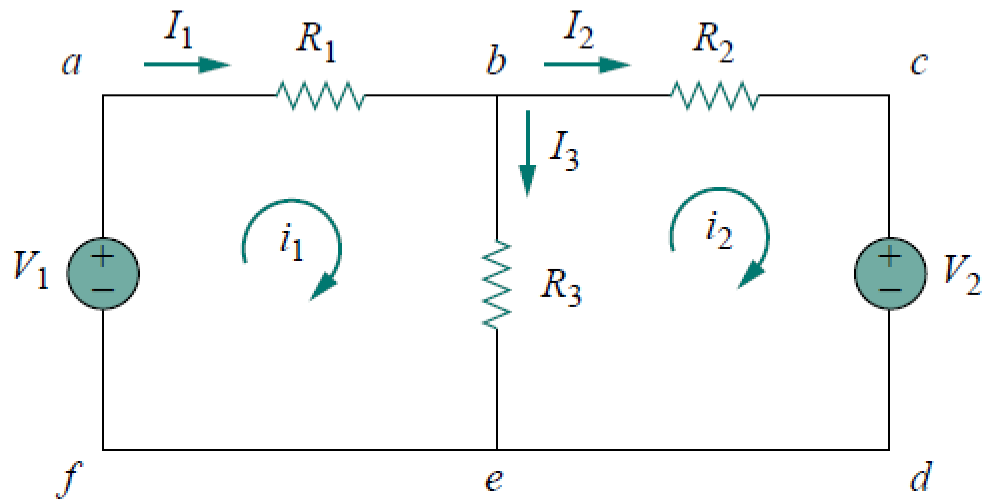
Solution:

Applying KVL to mesh 1, we obtain $$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0$$

For mesh 2, $$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\angle 90^\circ = 0$$

For mesh 3, $$\mathbf{I}_3 = 5.$$

The desired current is $$\mathbf{I}_o = -\mathbf{I}_2 = 6.12\angle 144.78^\circ \text{ A}$$



$$-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0 \qquad \text{or} \qquad (R_1 + R_3)i_1 - R_3 i_2 = V_1$$

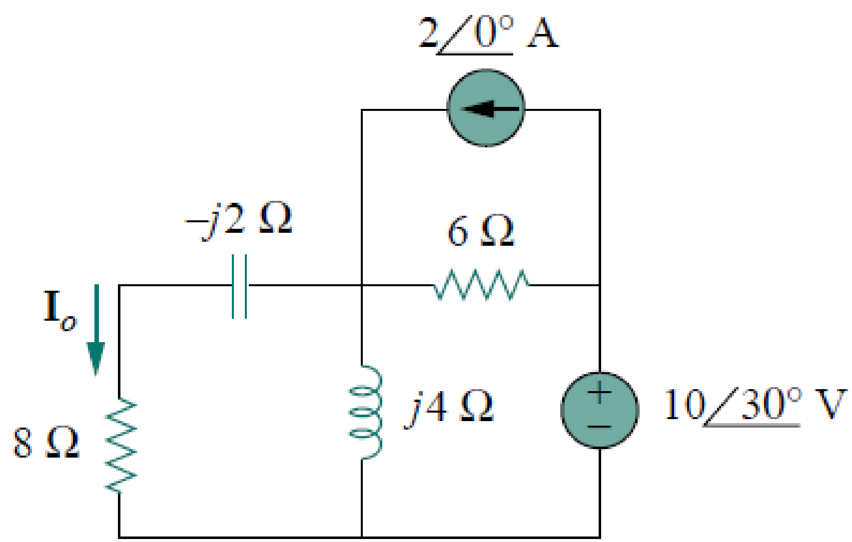
$$R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0 \qquad \text{or} \qquad -R_3 i_1 + (R_2 + R_3)i_2 = -V_2$$

Note in Eq. (3.13) that the coefficient of i_1 is the sum of the resistances in the first mesh, while the coefficient of i_2 is the negative of the resistance common to meshes 1 and 2. Now observe that the same is true in Eq. (3.14). This can serve as a shortcut way of writing the mesh equations.

The shortcut way will not apply if one mesh current is assumed clockwise and the other assumed anticlockwise, although this is permissible.

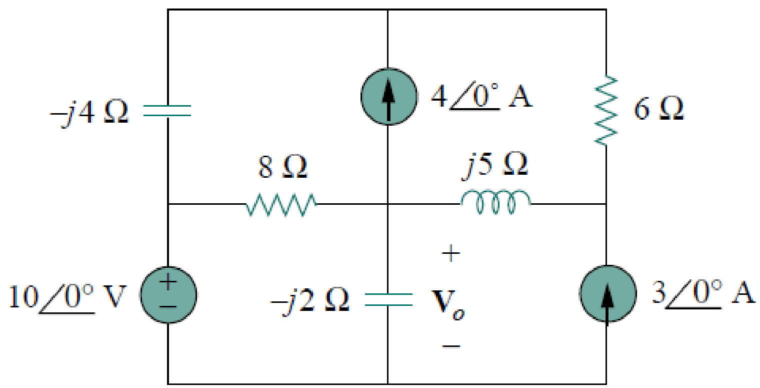
PRACTICE PROBLEM 10.3

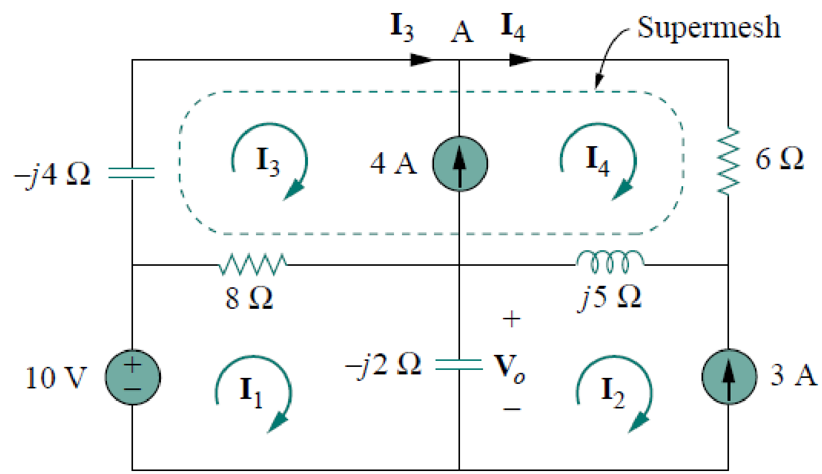
Find $\mathbf{I}_o$ in Fig. 10.8 using mesh analysis.



EXAMPLE 10.4

Solve for $\mathbf{V}_o$ in the circuit in Fig. 10.9 using mesh analysis.





Solution:

As shown in Fig. 10.10, meshes 3 and 4 form a supermesh due to the current source between the meshes. For mesh 1, KVL gives

$$-10 + (8 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - 8\mathbf{I}_3 = 0$$

For mesh 2, $\mathbf{I}_2 = -3$

For the supermesh, $(8 - j4)\mathbf{I}_3 - 8\mathbf{I}_1 + (6 + j5)\mathbf{I}_4 - j5\mathbf{I}_2 = 0$

Due to the current source between meshes 3 and 4, at node A, $\mathbf{I}_4 = \mathbf{I}_3 + 4$

$$\mathbf{V}_o = -j2(\mathbf{I}_1 - \mathbf{I}_2) = -j2(3.618\angle 274.5^\circ + 3)$$

$$= -7.2134 - j6.568 = 9.756\angle 222.32^\circ \text{ V}$$

Practice Problem 10.4

Calculate current $\mathbf{I}_o$ in the circuit of Fig. 10.11.

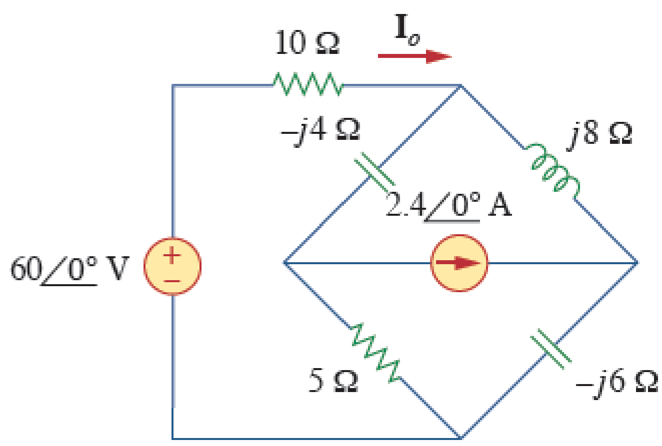


Figure 10.11
For Practice Prob. 10.4.

10.4 Superposition Theorem

- Since ac circuits are linear, the superposition theorem applies to ac circuits the same way it applies to dc circuits.

Steps to apply superposition principle

1. ***Turn off*** all independent sources except one source. Find the output (voltage or current) due to that active source using nodal or mesh analysis.
2. ***Repeat step 1*** for each of the other independent sources.
3. ***Find*** the total contribution by adding ***algebraically*** all the contributions due to the independent sources.

Superposition Theorem

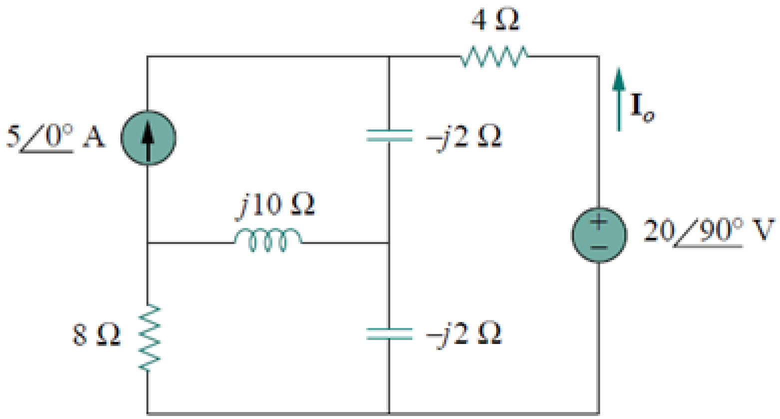
Two things have to be keep in mind:

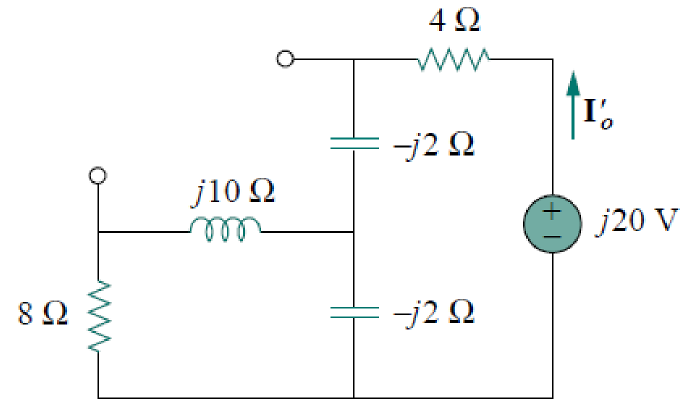
When we say turn off all other independent sources:

- Independent voltage sources are replaced by 0 V (short circuit) and
- Independent current sources are replaced by 0 A (open circuit).
- Dependent sources are left intact because they are controlled by circuit variables.

EXAMPLE 10.5

Use the superposition theorem to find $\mathbf{I}_o$ in the circuit in Fig. 10.7.





(a)

Solution:

Let

$$\mathbf{I}_o = \mathbf{I}'_o + \mathbf{I}''_o \tag{10.5.1}$$

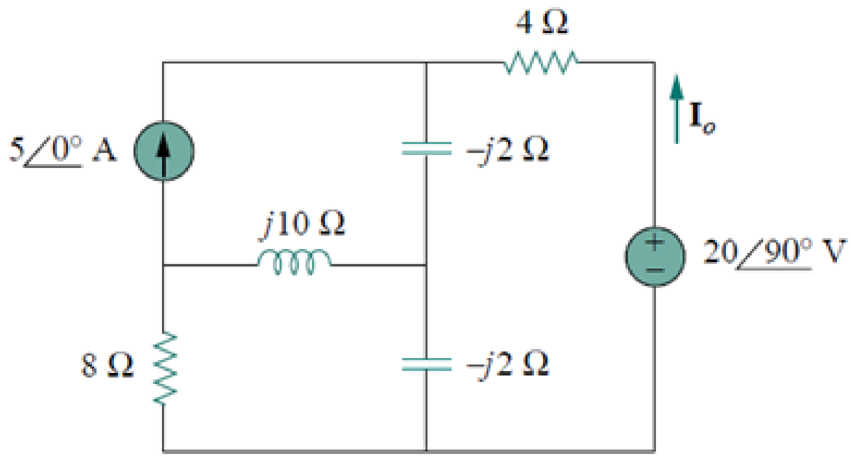
where $\mathbf{I}'_o$ and $\mathbf{I}''_o$ are due to the voltage and current sources, respectively. To find $\mathbf{I}'_o$, consider the circuit in Fig. 10.12(a). If we let $\mathbf{Z}$ be the parallel combination of $-j2$ and $8 + j10$, then

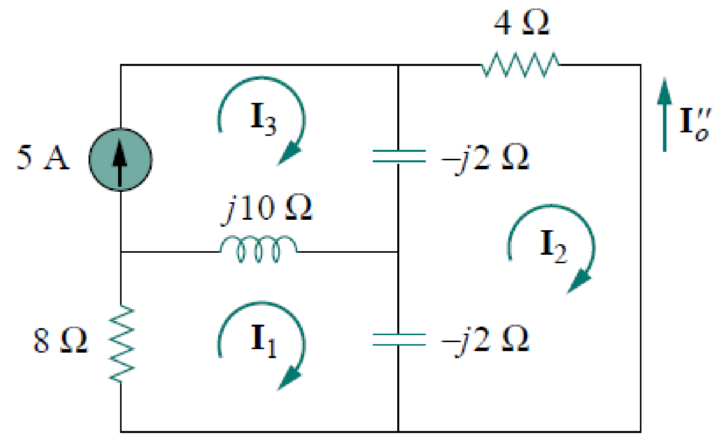
$$\mathbf{Z} = \frac{-j2(8 + j10)}{-2j + 8 + j10} = 0.25 - j2.25$$

and current $\mathbf{I}'_o$ is

$$\mathbf{I}'_o = \frac{j20}{4 - j2 + \mathbf{Z}} = \frac{j20}{4.25 - j4.25}$$

$$= -2.353 + j2.353$$





(b)

To get $\mathbf{I}_o''$, consider the circuit in Fig. 10.12(b). For mesh 1,

$$(8 + j8)\mathbf{I}_1 - j10\mathbf{I}_3 + j2\mathbf{I}_2 = 0$$

For mesh 2,

$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j2\mathbf{I}_3 = 0$$

For mesh 3,

$$\mathbf{I}_3 = 5$$

$$\mathbf{I}_2 = \frac{90 - j40}{34} = 2.647 - j1.176$$

Current $\mathbf{I}_o''$ is obtained as

$$\mathbf{I}_o'' = -\mathbf{I}_2 = -2.647 + j1.176$$

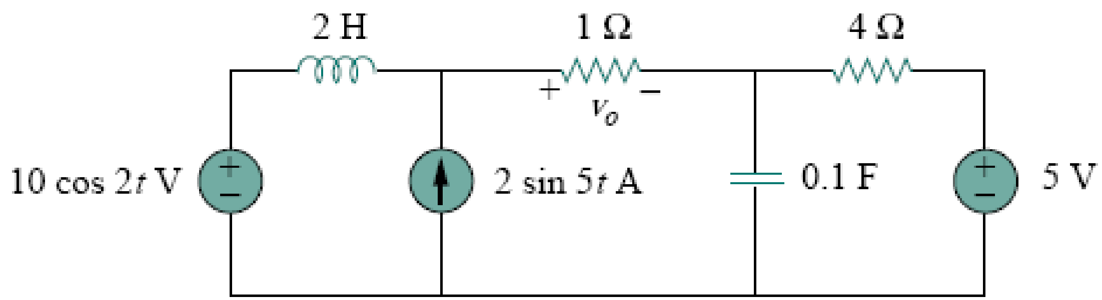
$$\mathbf{I}_o = \mathbf{I}_o' + \mathbf{I}_o'' = -5 + j3.529 = 6.12\angle 144.78^\circ \text{ A}$$

10.4 Superposition Theorem

- The theorem becomes important if the circuit has sources operating at *different frequencies*.
- In this case, since the impedances depend on frequency, we must have a different frequency-domain circuit for each frequency.
- The total response must be obtained by adding the individual responses in the time domain.
- When a circuit has sources operating at different frequencies, one must add the responses due to the individual frequencies in the time domain.

EXAMPLE 10.6

Find v_o in the circuit in Fig. 10.13 using the superposition theorem.



Solution:

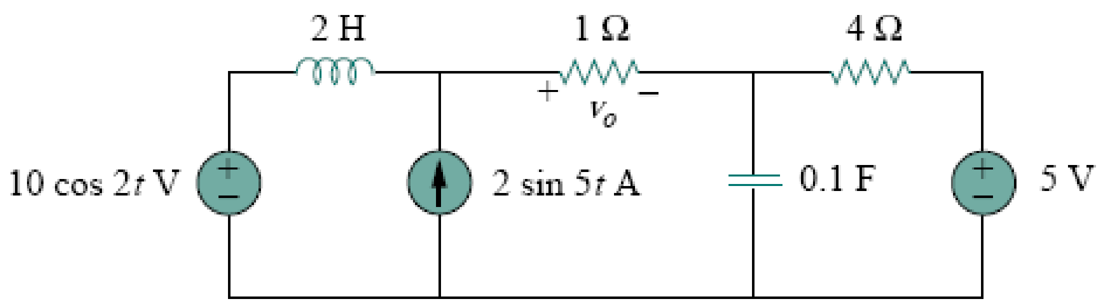
Since the circuit operates at three different frequencies ($\omega = 0$ for the dc voltage source), one way to obtain a solution is to use superposition, which breaks the problem into single-frequency problems. So we let

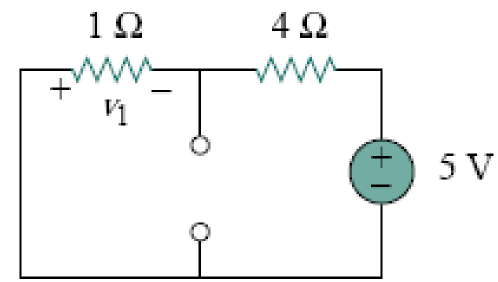
$$v_o = v_1 + v_2 + v_3$$

v_1 is due to the 5-V dc voltage source,

v_2 is due to the 10 cos 2t V voltage source,

v_3 is due to the 2 sin 5t A current source.





v_1 is due to the 5-V dc voltage source,

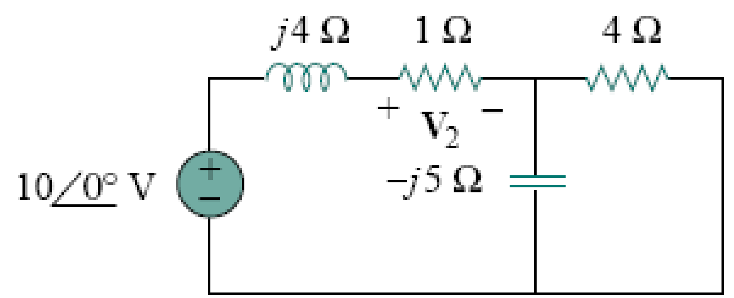
$$-v_1 = \frac{1}{1+4}(5) = 1\text{ V}$$

v_2 is due to the 10 cos 2*t* V voltage source,

$$10\cos 2t \implies 10\underline{/0^\circ}, \qquad \omega = 2\text{ rad/s}$$

$$2\text{ H} \implies j\omega L = j4\ \Omega$$

$$0.1\text{ F} \implies \frac{1}{j\omega C} = -j5\ \Omega$$

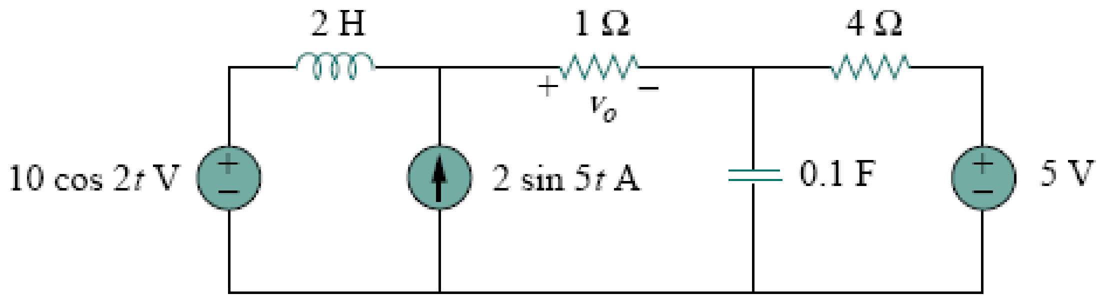


$$\mathbf{Z} = -j5 \parallel 4 = \frac{-j5 \times 4}{4 - j5} = 2.439 - j1.951$$

$$\mathbf{V}_2 = \frac{1}{1 + j4 + \mathbf{Z}}(10\underline{/0^\circ}) = \frac{10}{3.439 + j2.049} = 2.498\underline{/-30.79^\circ}$$

In the time domain,

$$v_2 = 2.498\cos(2t - 30.79^\circ)$$

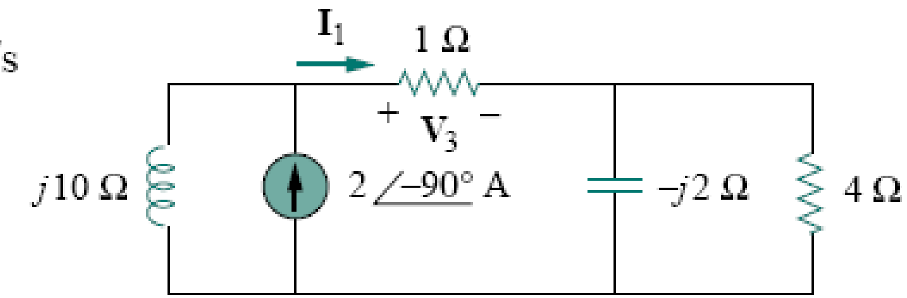


v_3 is due to the 2 sin 5*t* A current source.

$$2\sin 5t \quad\Longrightarrow\quad 2\angle -90^\circ, \qquad \omega = 5 \text{ rad/s}$$

$$2\text{ H} \quad\Longrightarrow\quad j\omega L = j10\ \Omega$$

$$0.1\text{ F} \quad\Longrightarrow\quad \frac{1}{j\omega C} = -j2\ \Omega$$



$$\mathbf{Z}_1 = -j2 \parallel 4 = \frac{-j2 \times 4}{4 - j2} = 0.8 - j1.6\ \Omega$$

By current division,

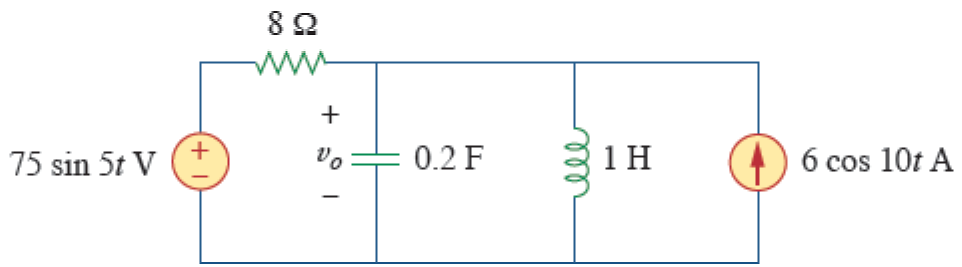
$$\mathbf{I}_1 = \frac{j10}{j10 + 1 + \mathbf{Z}_1}(2\angle -90^\circ)\ \text{A} \qquad \mathbf{V}_3 = \mathbf{I}_1 \times 1 = \frac{j10}{1.8 + j8.4}(-j2) = 2.328\angle -77.91^\circ\ \text{V}$$

In the time domain, $v_3 = 2.33\cos(5t - 80^\circ) = 2.33\sin(5t + 10^\circ)\ \text{V}$

$$v_o(t) = -1 + 2.498\cos(2t - 30.79^\circ) + 2.33\sin(5t + 10^\circ)\ \text{V}$$

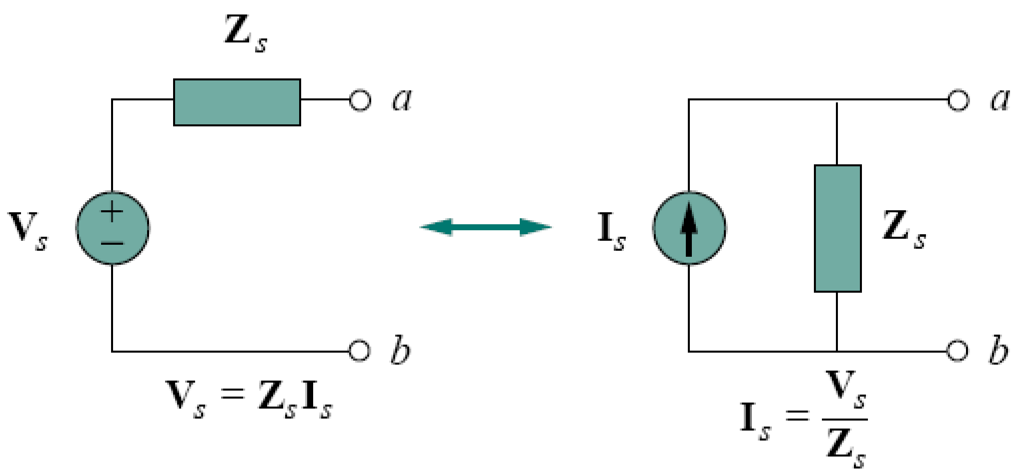
Practice Problem 10.6

Calculate v_o in the circuit of Fig. 10.15 using the superposition theorem.



10.5 Source Transformation

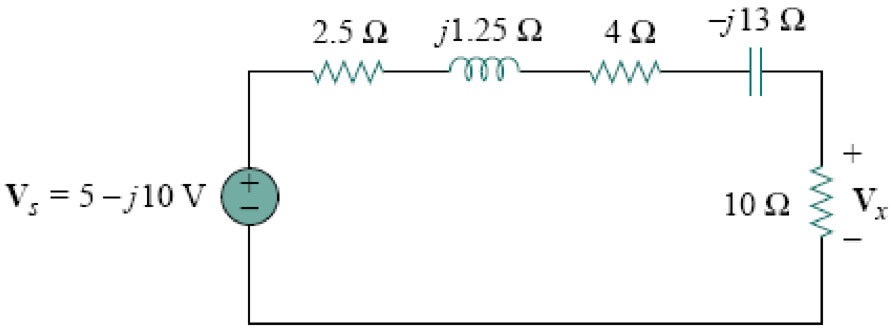
- Source transformation in the frequency domain involves transforming a voltage source in series with an impedance to a current source in parallel with an impedance, or vice versa.

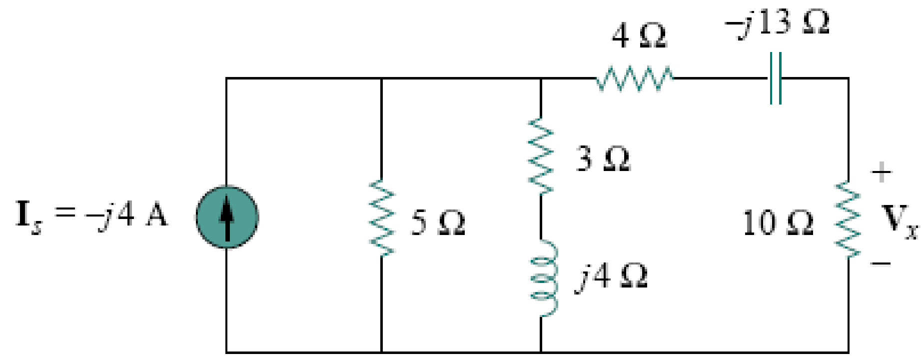


$$\mathbf{V}_s = \mathbf{Z}_s\mathbf{I}_s \quad \Longleftrightarrow \quad \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s}$$

EXAMPLE 10.7

Calculate $\mathbf{V}_x$ in the circuit of using the method of source transformation.





Solution:

We transform the voltage source to a current source and obtain the circuit

$$\mathbf{I}_s = \frac{20\angle -90^\circ}{5} = 4\angle -90^\circ = -j4 \text{ A}$$

The parallel combination of 5-Ω resistance and $(3 + j4)$ impedance gives

$$\mathbf{Z}_1 = \frac{5(3 + j4)}{8 + j4} = 2.5 + j1.25\ \Omega$$

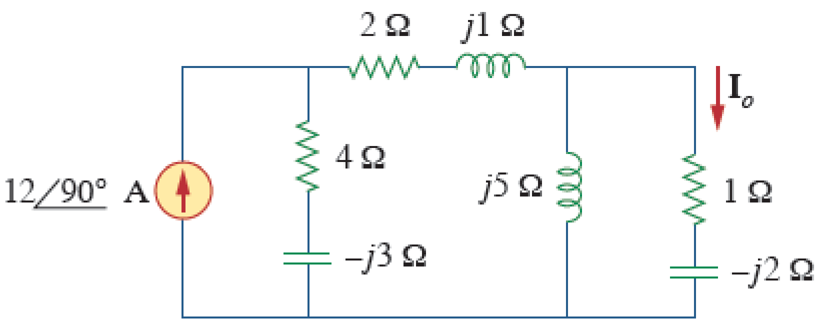
Converting the current source to a voltage source yields the circuit

$$\mathbf{V}_s = \mathbf{I}_s\mathbf{Z}_1 = -j4(2.5 + j1.25) = 5 - j10 \text{ V}$$

$$\mathbf{V}_x = \frac{10}{10 + 2.5 + j1.25 + 4 - j13}(5 - j10) = 5.519\angle -28^\circ \text{ V}$$

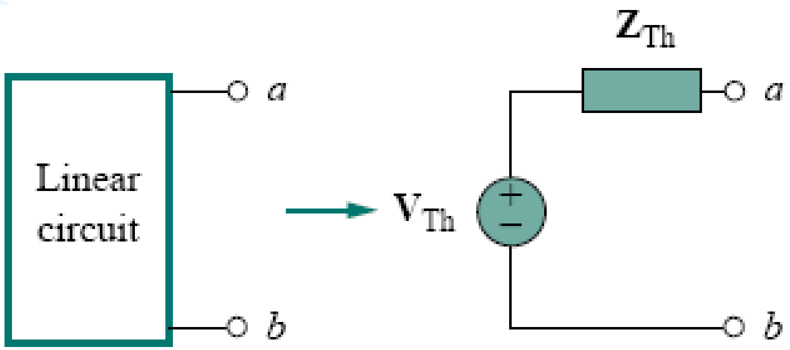
Practice Problem 10.7

Find $\mathbf{I}_o$ in the circuit of Fig. 10.19 using the concept of source transformation.



10.6 Thevenin and Norton equivalent circuits

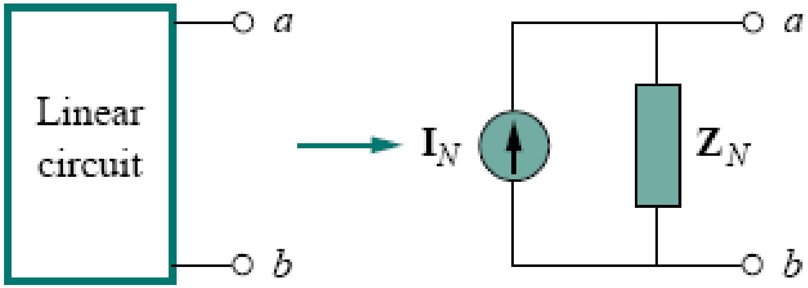
- Thevenin's and Norton's theorems are applied to ac circuits in the same way as they are to dc circuits. The only additional effort arises from the need to manipulate complex numbers.



$$\mathbf{V}_{\mathrm{Th}} = \mathbf{Z}_N \mathbf{I}_N, \qquad \mathbf{Z}_{\mathrm{Th}} = \mathbf{Z}_N$$

$\mathbf{V}_{\mathrm{Th}}$ is the open-circuit voltage

The frequency-domain version of a Thevenin equivalent circuit

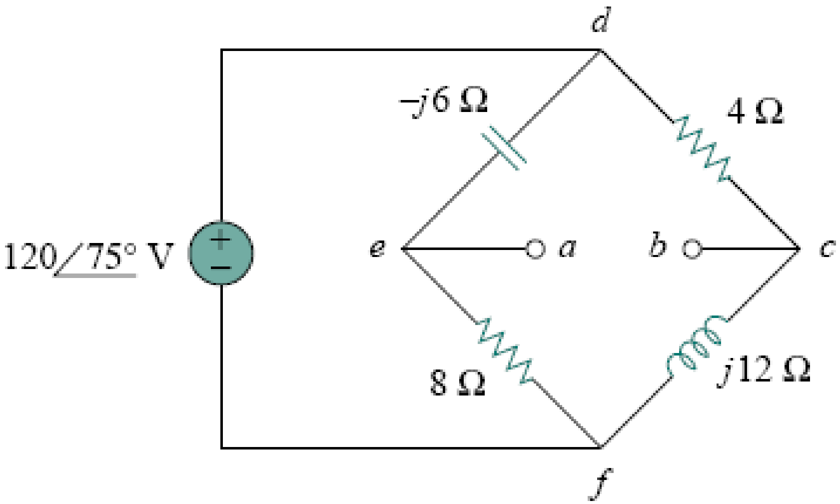


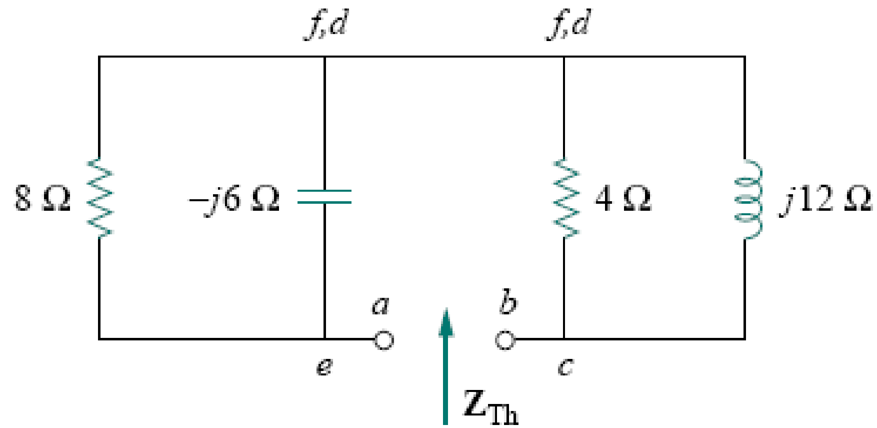
$\mathbf{I}_N$ is the short-circuit current

The frequency-domain version of a Norton equivalent circuit

EXAMPLE 10.8

Obtain the Thevenin equivalent at terminals *a*-*b* of the circuit.





Solution:

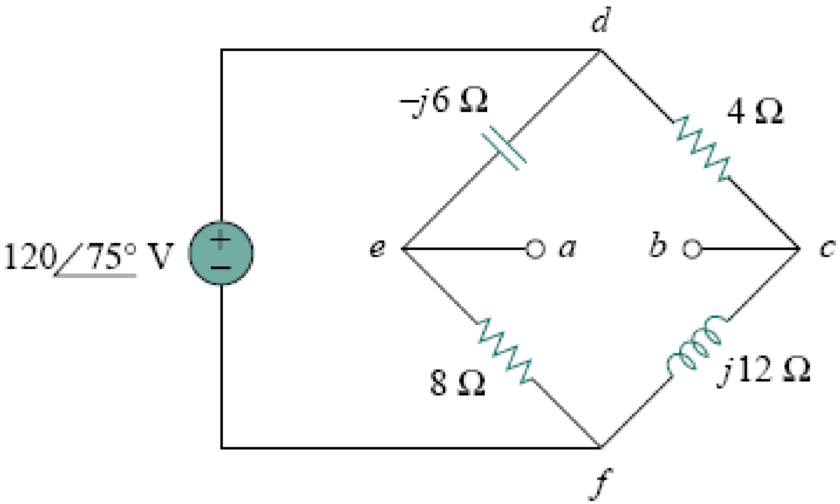
We find $\mathbf{Z}_{\text{Th}}$ by setting the voltage source to zero.

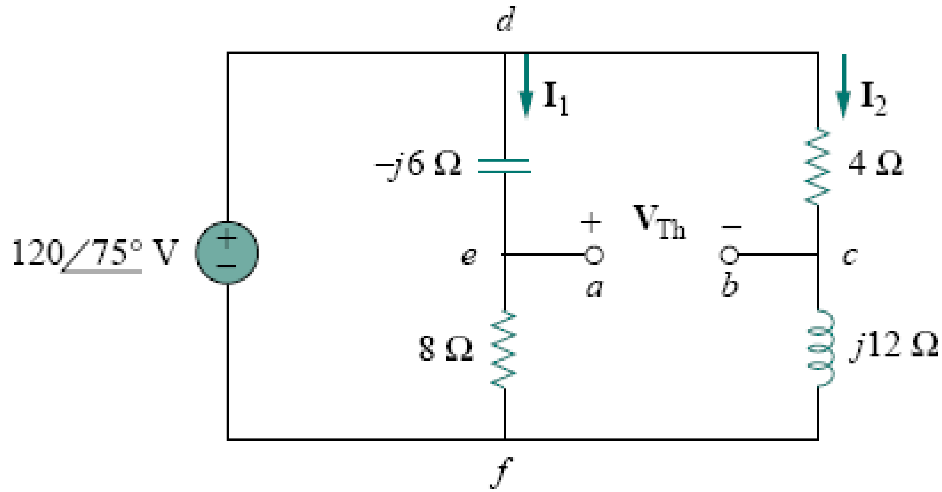
$$\mathbf{Z}_1 = -j6 \parallel 8 = \frac{-j6 \times 8}{8 - j6} = 2.88 - j3.84\ \Omega \qquad \mathbf{Z}_2 = 4 \parallel j12 = \frac{j12 \times 4}{4 + j12} = 3.6 + j1.2\ \Omega$$

$$\mathbf{Z}_{\text{Th}} = \mathbf{Z}_1 + \mathbf{Z}_2 = 6.48 - j2.64\ \Omega$$

EXAMPLE 10.8

Obtain the Thevenin equivalent at terminals *a*-*b* of the circuit.





To find $\mathbf{V}_{Th}$,

$$\mathbf{I}_1 = \frac{120\underline{/75^\circ}}{8 - j6}\ \text{A}, \qquad \mathbf{I}_2 = \frac{120\underline{/75^\circ}}{4 + j12}\ \text{A}$$

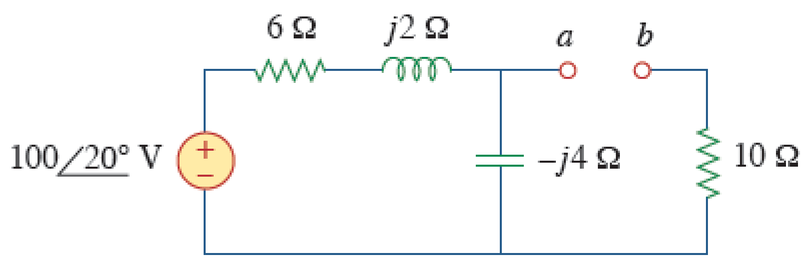
Applying KVL around loop *bcdeab*

$$\mathbf{V}_{Th} - 4\mathbf{I}_2 + (-j6)\mathbf{I}_1 = 0$$

$$\mathbf{V}_{Th} = 4\mathbf{I}_2 + j6\mathbf{I}_1 = 37.95\underline{/220.31^\circ}\ \text{V}$$

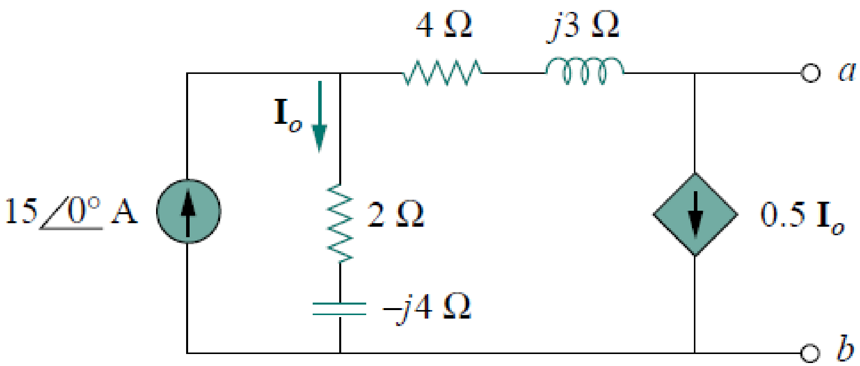
Practice Problem 10.8

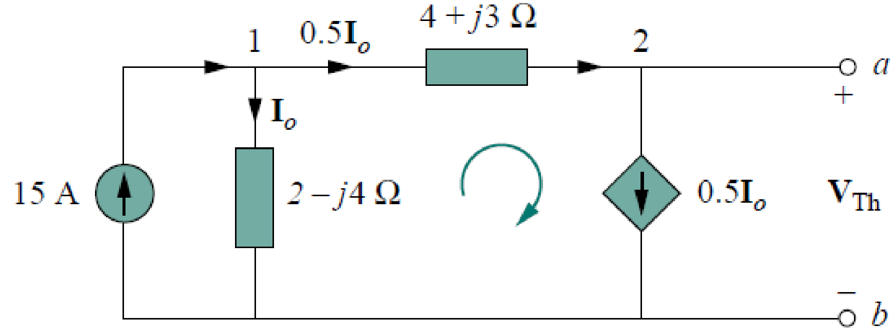
Find the Thevenin equivalent at terminals *a*-*b* of the circuit in Fig. 10.24.



EXAMPLE 10.9

Find the Thevenin equivalent of the circuit in Fig. 10.25 as seen from terminals *a*-*b*.





(a)

Solution:

To find $\mathbf{V}_{\mathrm{Th}}$, we apply KCL at node 1 in Fig. 10.26(a).

$$15 = \mathbf{I}_o + 0.5\mathbf{I}_o \qquad \Longrightarrow \qquad \mathbf{I}_o = 10 \text{ A}$$

Applying KVL to the loop on the right-hand side in Fig. 10.26(a), we obtain

$$-\mathbf{I}_o(2 - j4) + 0.5\mathbf{I}_o(4 + j3) + \mathbf{V}_{\mathrm{Th}} = 0$$

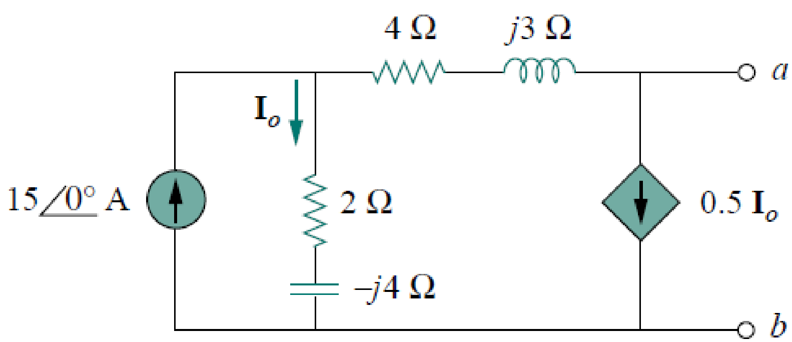
$$\mathbf{V}_{\mathrm{Th}} = 10(2 - j4) - 5(4 + j3) = -j55$$

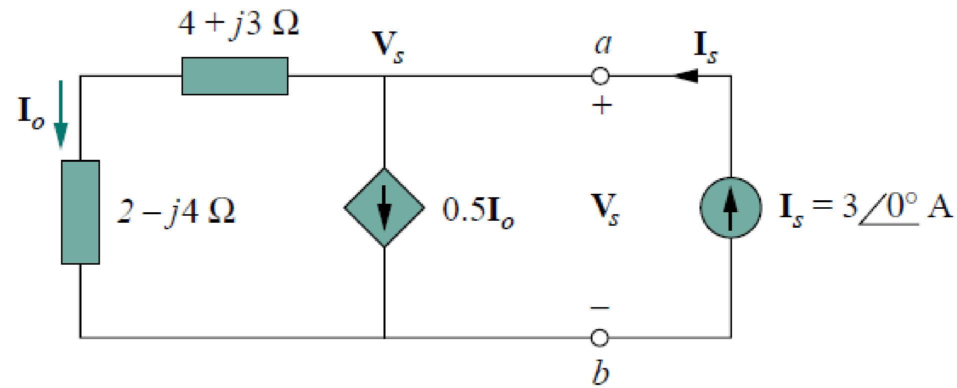
Thus, the Thevenin voltage is

$$\mathbf{V}_{\mathrm{Th}} = 55\underline{/-90^\circ} \text{ V}$$

EXAMPLE 10.9

Find the Thevenin equivalent of the circuit in Fig. 10.25 as seen from terminals *a*-*b*.





(b)

At the node, KCL gives

$$3 = \mathbf{I}_o + 0.5\mathbf{I}_o \quad \Longrightarrow \quad \mathbf{I}_o = 2 \text{ A}$$

Applying KVL to the outer loop in Fig. 10.26(b) gives

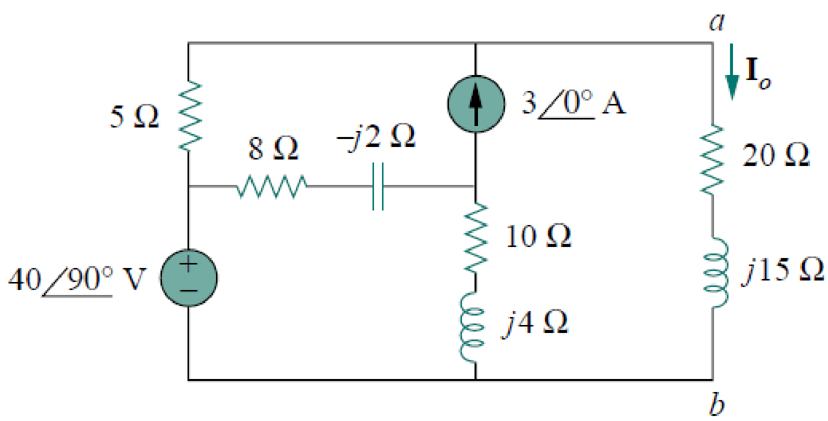
$$\mathbf{V}_s = \mathbf{I}_o(4 + j3 + 2 - j4) = 2(6 - j)$$

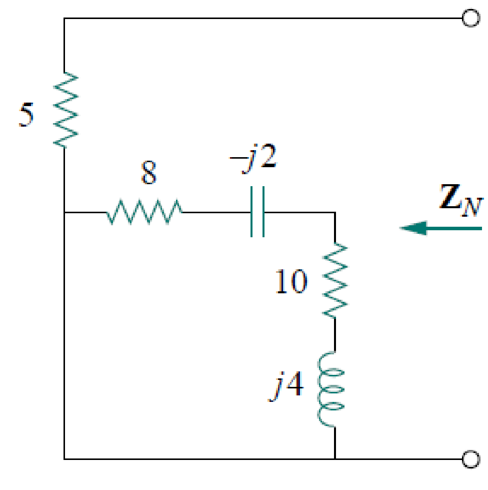
The Thevenin impedance is

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_s}{\mathbf{I}_s} = \frac{2(6 - j)}{3} = 4 - j0.6667\ \Omega$$

EXAMPLE 10.10

Obtain current $\mathbf{I}_o$ in Fig. 10.28 using Norton's theorem.



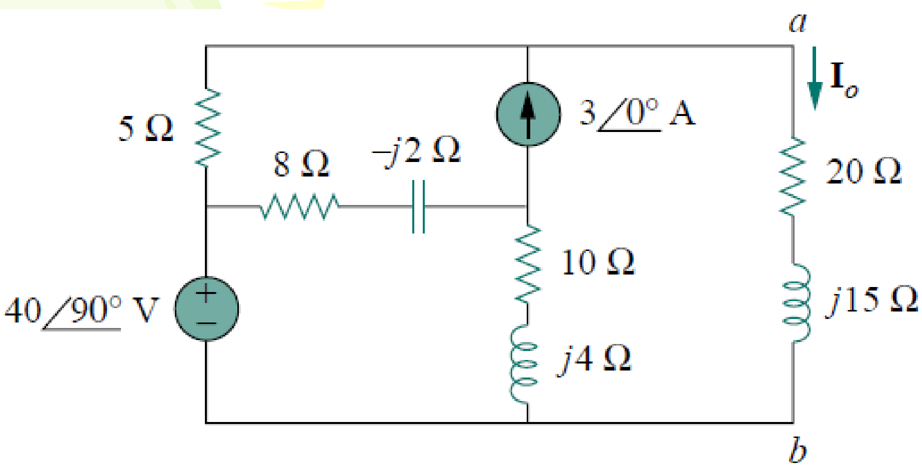


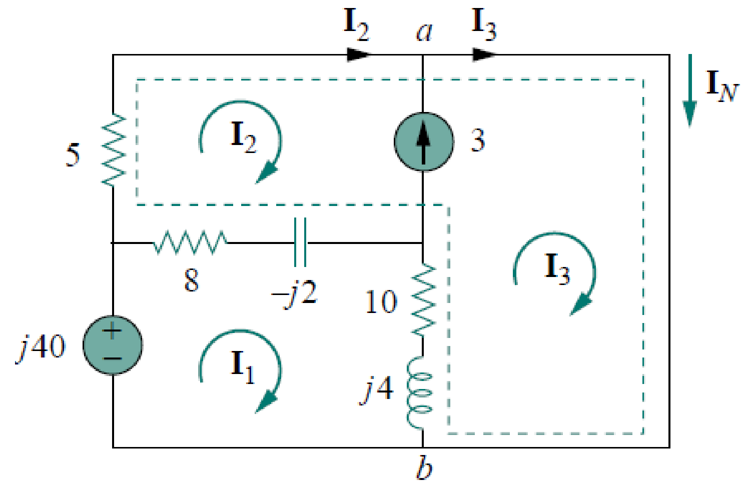
(a)

Solution:

Our first objective is to find the Norton equivalent at terminals a-b. $\mathbf{Z}_N$ is found in the same way as $\mathbf{Z}_{\mathrm{Th}}$. We set the sources to zero as shown in Fig. 10.29(a). As evident from the figure, the $(8 - j2)$ and $(10 + j4)$ impedances are short-circuited, so that

$$\mathbf{Z}_N = 5\ \Omega$$





(b)

To get $\mathbf{I}_N$, we short-circuit terminals a-b as in Fig. 10.29(b) and apply mesh analysis. Notice that meshes 2 and 3 form a supermesh because of the current source linking them. For mesh 1,

$$-j40 + (18 + j2)\mathbf{I}_1 - (8 - j2)\mathbf{I}_2 - (10 + j4)\mathbf{I}_3 = 0$$

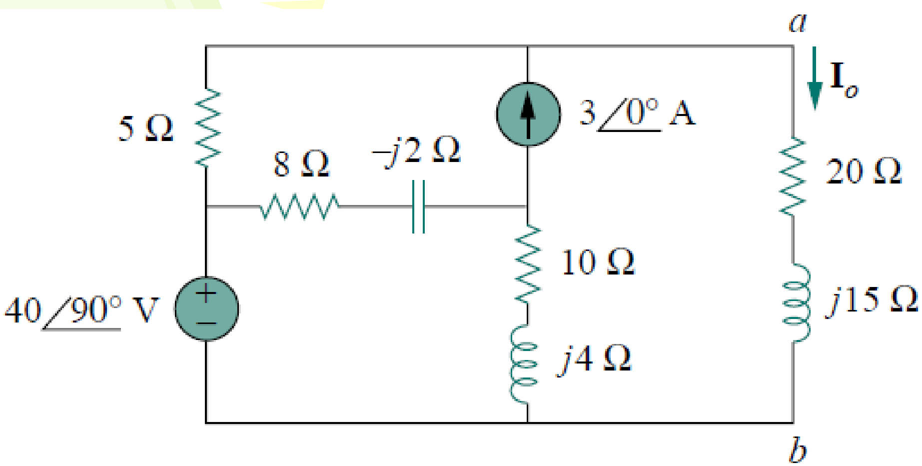
For the supermesh,

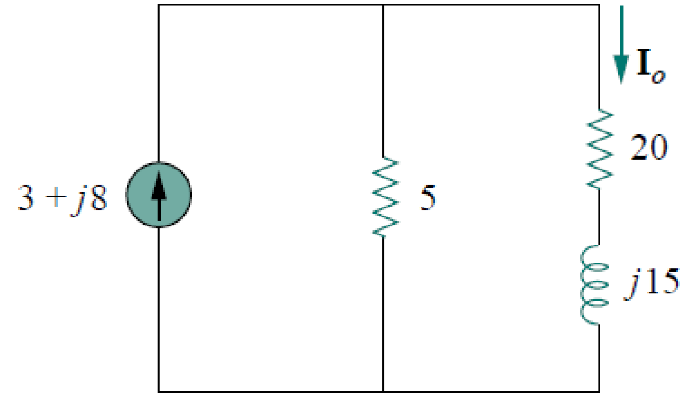
$$(13 - j2)\mathbf{I}_2 + (10 + j4)\mathbf{I}_3 - (18 + j2)\mathbf{I}_1 = 0$$

At node a, due to the current source between meshes 2 and 3,

$$\mathbf{I}_3 = \mathbf{I}_2 + 3$$

The Norton current is $$\mathbf{I}_N = \mathbf{I}_3 = (3 + j8)\text{ A}$$





(c)

$$\mathbf{Z}_N = 5\ \Omega \qquad\qquad \mathbf{I}_N = \mathbf{I}_3 = (3 + j8)\ \text{A}$$

Figure 10.29(c) shows the Norton equivalent circuit along with the impedance at terminals a-b. By current division,

$$\mathbf{I}_o = \frac{5}{5 + 20 + j15}\mathbf{I}_N = \frac{3 + j8}{5 + j3} = 1.465\underline{/38.48^\circ}\ \text{A}$$

10.7 Op Amp circuits

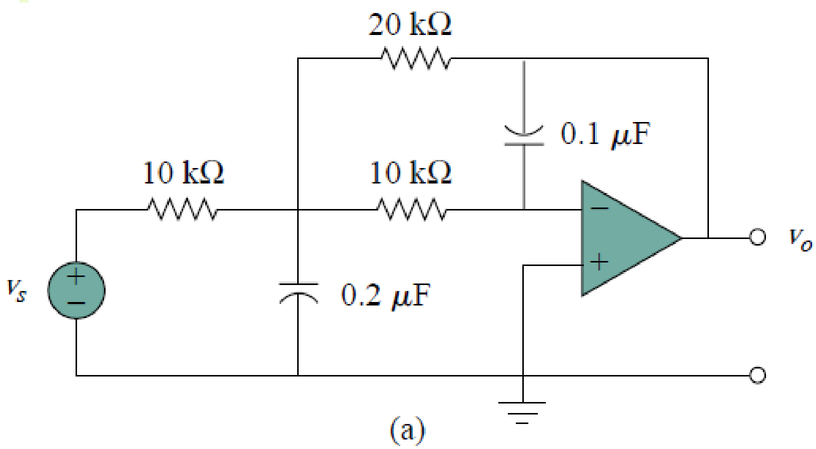
- The three steps stated in Section 10.1 also apply to op amp circuits, as long as the op amp is operating in the linear region. As usual, we will assume ideal op amps. (See Section 5.2.)
- As discussed in Chapter 5, the key to analyzing op amp circuits is to keep two important properties of an ideal op amp in mind:

- 1. No current enters either of its input terminals.

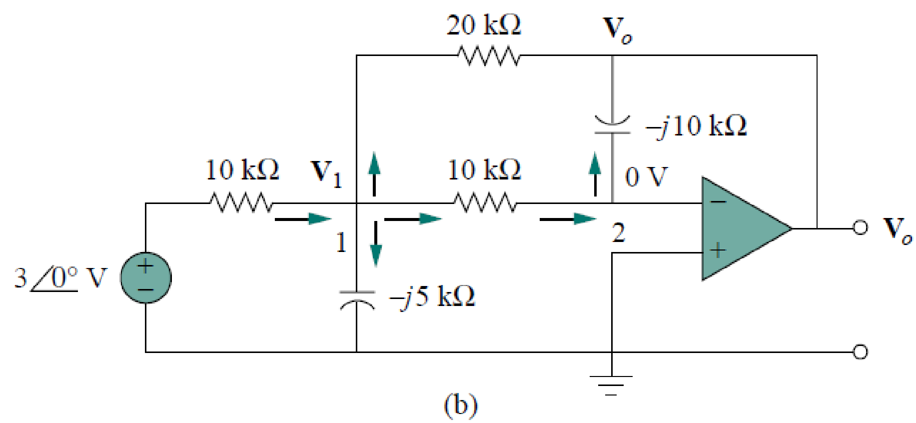
- 2. The voltage across its input terminals is zero.

EXAMPLE 10.11

Determine $v_o(t)$ for the op amp circuit in Fig. 10.31(a) if $v_s = 3\cos 1000t$ V.



(a)



(b)

Solution:

We first transform the circuit to the frequency domain, as shown in Fig. 10.31(b), where $\mathbf{V}_s = 3\angle 0°$, $\omega = 1000$ rad/s. Applying KCL at node 1,

$$\frac{3\angle 0° - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j5} + \frac{\mathbf{V}_1 - 0}{10} + \frac{\mathbf{V}_1 - \mathbf{V}_o}{20} \qquad 6 = (5 + j4)\mathbf{V}_1 - \mathbf{V}_o$$

At node 2, KCL gives $\frac{\mathbf{V}_1 - 0}{10} = \frac{0 - \mathbf{V}_o}{-j10}$ $\qquad \mathbf{V}_1 = -j\mathbf{V}_o$

$$\mathbf{V}_o = \frac{6}{3 - j5} = 1.029\angle 59.04° \qquad v_o(t) = 1.029\cos(1000t + 59.04°)\text{ V}$$

Summary and Review

1. We apply nodal and mesh analysis to ac circuits by applying KCL and KVL to the phasor form of the circuits.
2. In solving for the steady-state response of a circuit that has independent sources with different frequencies, each independent source must be considered separately.

   The most natural approach to analyzing such circuits is to apply the superposition theorem. A separate phasor circuit for each frequency must be solved independently, and the corresponding response should be obtained in the time domain.

   The overall response is the sum of the time-domain responses of all the individual phasor circuits.

3. **The concept of source transformation is also applicable in the frequency domain.**
4. **The Thevenin equivalent of an ac circuit consists of a voltage source V_{Th} in series with the Thevenin impedance Z_{Th}.**
5. **The Norton equivalent of an ac circuit consists of a current source I_N in parallel with the Norton impedance Z_N (= Z_{Th}).**

First time homework:

10.9 Use nodal analysis to find v_o in the circuit of Fig. 10.58.

ML

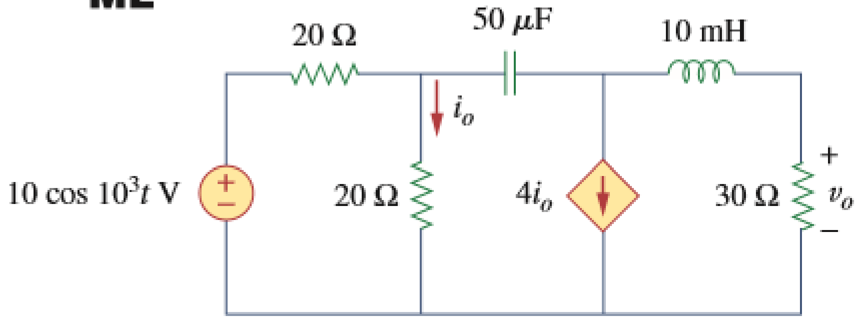


Figure 10.58
For Prob. 10.9.

10.19 Obtain $\mathbf{V}_o$ in Fig. 10.68 using nodal analysis.

ML

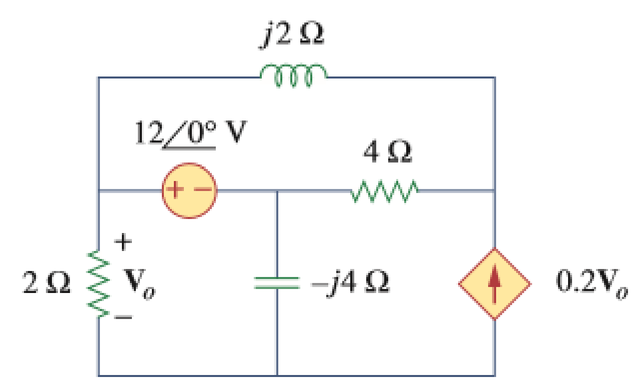


Figure 10.68
For Prob. 10.19.

10.32 Determine $\mathbf{V}_o$ and $\mathbf{I}_o$ in the circuit of Fig. 10.80 using mesh analysis.

ML

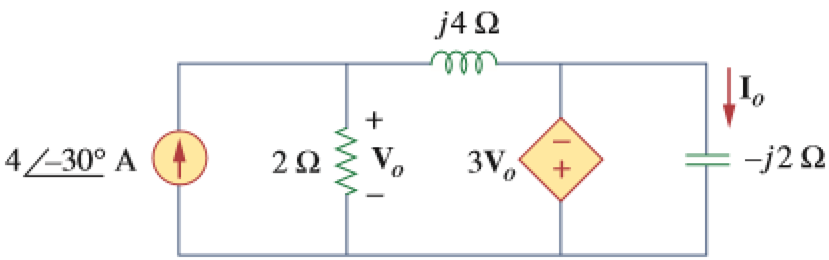


Figure 10.80
For Prob. 10.32.

10.37 Use mesh analysis to find currents $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ in the circuit of Fig. 10.82.

ML

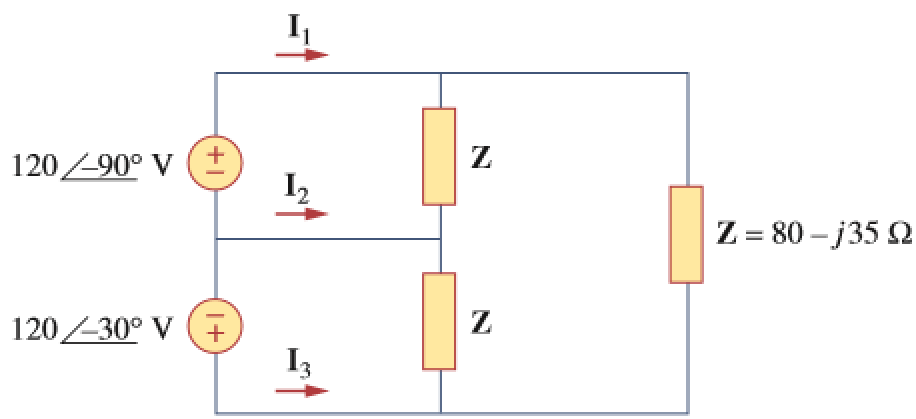


Figure 10.82
For Prob. 10.37.

Second time homework:

10.43 Using the superposition principle, find i_x in the circuit of Fig. 10.88.

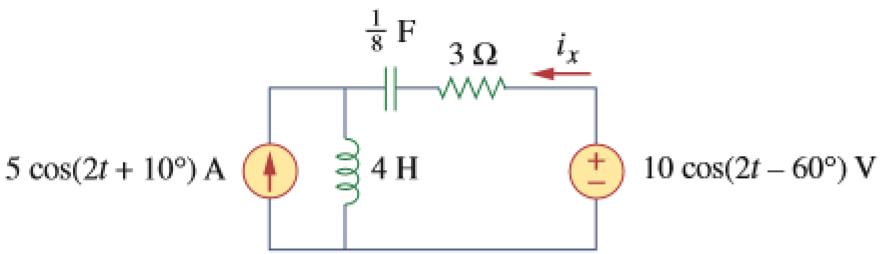


Figure 10.88
For Prob. 10.43.

10.45 Use superposition to find $i(t)$ in the circuit of Fig. 10.90.

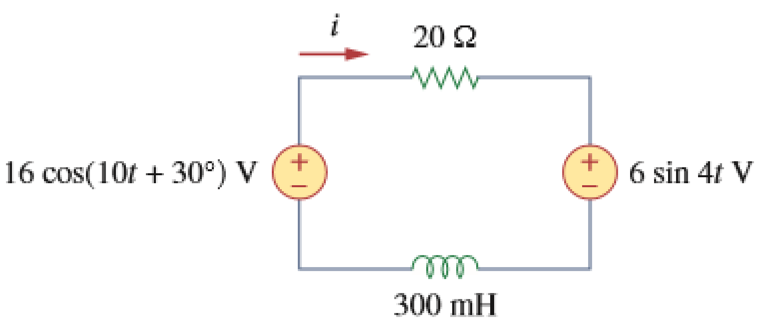


Figure 10.90
For Prob. 10.45.

10.53 Use the concept of source transformation to find $\mathbf{V}_o$ in the circuit of Fig. 10.97.

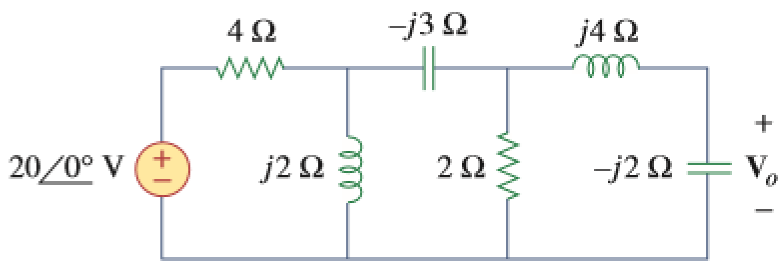


Figure 10.97
For Prob. 10.53.

10.56 For each of the circuits in Fig. 10.99, obtain Thevenin and Norton equivalent circuits at terminals *a-b*.

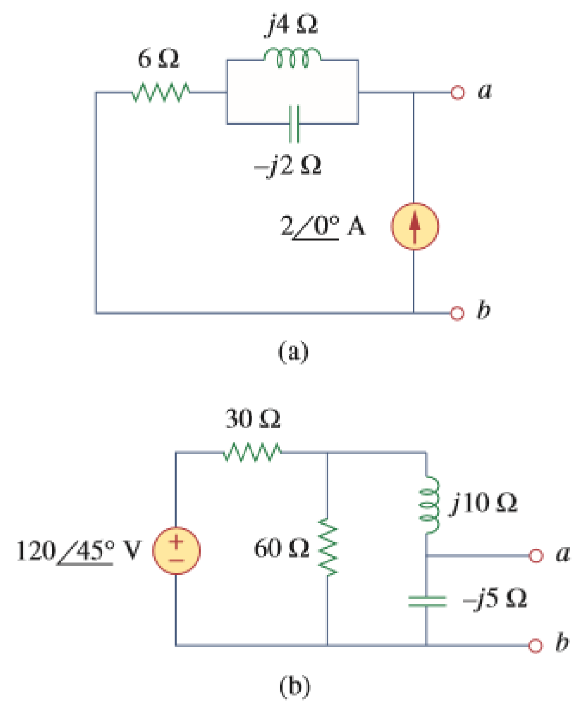


Figure 10.99
For Prob. 10.56.

10.71 Find v_o in the op amp circuit of Fig. 10.114.

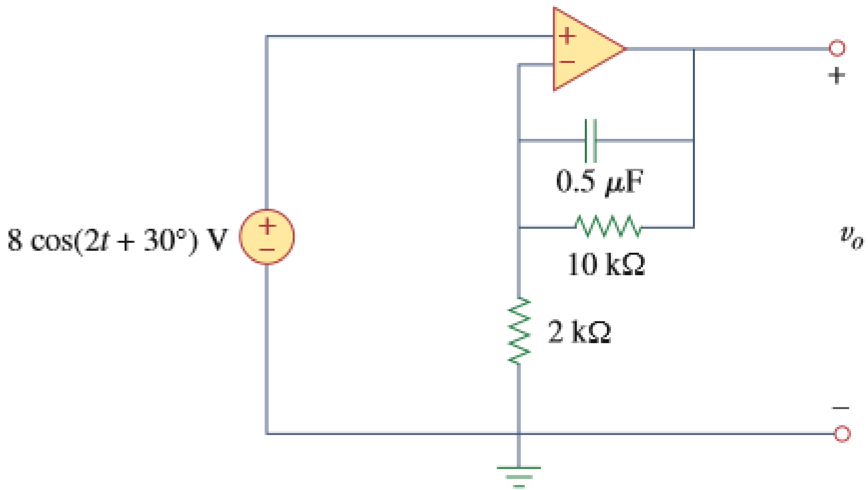


Figure 10.114
For Prob. 10.71.

10.74 Evaluate the voltage gain $\mathbf{A}_v = \mathbf{V}_o/\mathbf{V}_s$ in the op amp circuit of Fig. 10.117. Find $\mathbf{A}_v$ at $\omega = 0$, $\omega \to \infty$, $\omega = 1/R_1C_1$, and $\omega = 1/R_2C_2$.

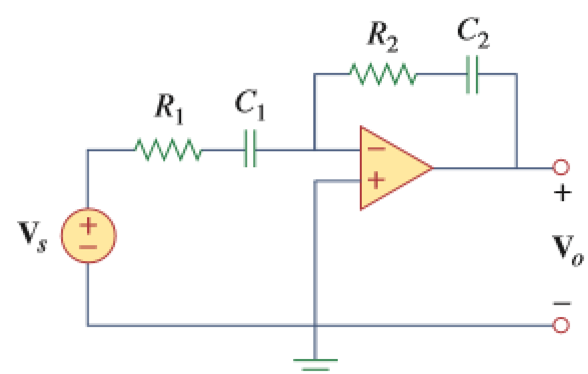


Figure 10.117
For Prob. 10.74.

Fundamentals of Electric Circuit 2021.5

Chapter 11
AC Power Analysis

Chapter 11 AC Power Analysis

11.1 Introduction

Our effort in ac circuit analysis so far has been focused mainly on calculating voltage and current. Our major concern in this chapter is power analysis.

Power analysis is very important. Power is the most important quantity in electric utilities, electronic, and communication systems, because such systems involve transmission of power from one point to another.

11.1 Introduction

Also, every industrial and household electrical device—every fan, motor, lamp, pressing iron, TV, personal computer—has a power rating that indicates how much power the equipment requires; exceeding the power rating can do permanent damage to an appliance.

The most common form of electric power is 50- or 60-Hz ac power. The choice of ac over dc allowed high-voltage power transmission from the power generating plant to the consumer.

11.2 INSTANTANEOUS AND AVERAGE POWER

The instantaneous power $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it.

$$\boxed{p(t) = v(t)i(t)}$$

The instantaneous power is the power at any instant of time.

$$v(t) = V_m \cos(\omega t + \theta_v)$$
$$i(t) = I_m \cos(\omega t + \theta_i)$$

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

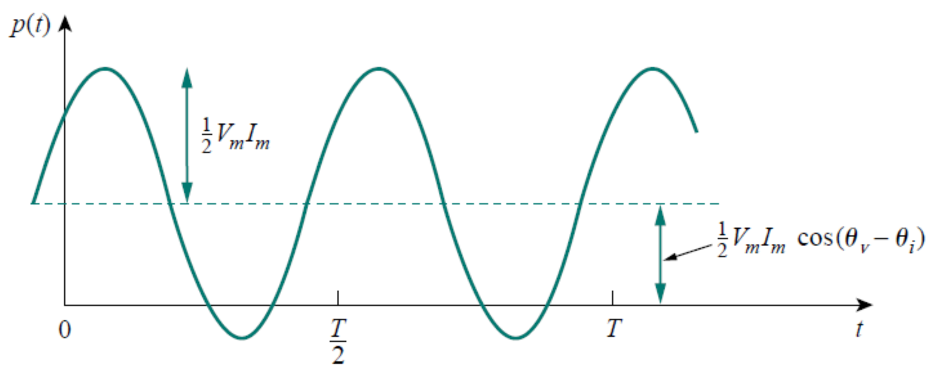
11.2 INSTANTANEOUS AND AVERAGE POWER

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

The instantaneous power has two parts.

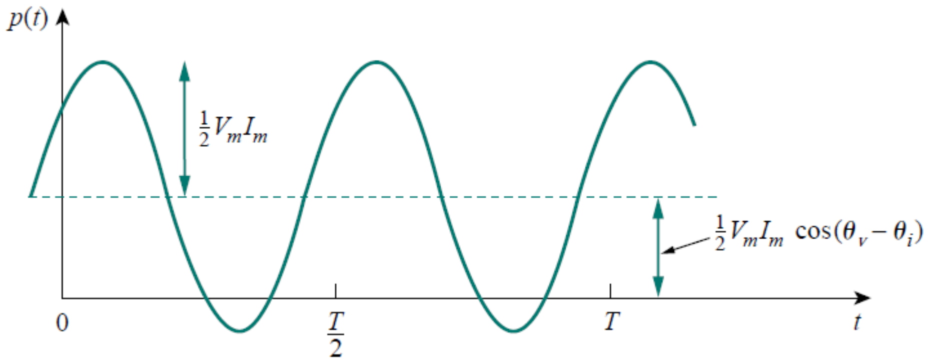
The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current.

The second part is a sinusoidal function whose frequency is 2ω, which is twice the angular frequency of the voltage or current.



11.2 INSTANTANEOUS AND AVERAGE POWER

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$



We observe that $p(t)$ is periodic, $p(t) = p(t + T_0)$, and has a period of $T_0 = T/2$, since its frequency is twice that of voltage or current. When p(t) is positive, power is absorbed by the circuit. When p(t) is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source.

This is possible because of the storage elements (capacitors and inductors) in the circuit.

11.2 INSTANTANEOUS AND AVERAGE POWER

The instantaneous power changes with time and is therefore difficult to measure. The average power is more convenient to measure.

The average power is the average of the instantaneous power over one period.

$$P = \frac{1}{T}\int_0^T p(t)\,dt \qquad p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

$$P = \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\frac{1}{T}\int_0^T dt + \frac{1}{2}V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \qquad \mathbf{V} = V_m \angle \theta_v \qquad \mathbf{I} = I_m \angle \theta_i$$

$$\frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2} V_m I_m \angle \theta_v - \theta_i$$

$$= \frac{1}{2} V_m I_m \left[\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)\right]$$

$$\boxed{P = \frac{1}{2} \operatorname{Re}\left[\mathbf{V}\mathbf{I}^*\right] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}$$

When $\theta_v = \theta_i$,

$$P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} |\mathbf{I}|^2 R$$

a purely resistive circuit absorbs power at all times

When $\theta_v - \theta_i = \pm 90^\circ$,

$$P = \frac{1}{2} V_m I_m \cos 90^\circ = 0$$

a purely reactive circuit absorbs no average power

A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

EXAMPLE 11.1

Given that

$$v(t) = 120\cos(377t + 45^\circ)\ \text{V} \qquad \text{and} \qquad i(t) = 10\cos(377t - 10^\circ)\ \text{A}$$

find the instantaneous power and the average power absorbed by the passive linear network of Fig. 11.1.

Solution:

The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45^\circ)\cos(377t - 10^\circ)$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$p = 600[\cos(754t + 35^\circ) + \cos 55^\circ]$$

$$p(t) = 344.2 + 600\cos(754t + 35^\circ)\ \text{W}$$

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}120(10)\cos[45^\circ - (-10^\circ)]$$

$$= 600\cos 55^\circ = 344.2\ \text{W}$$

EXAMPLE 11.2

Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\underline{/0^\circ}$ is applied across it.

Solution:

The current through the impedance is

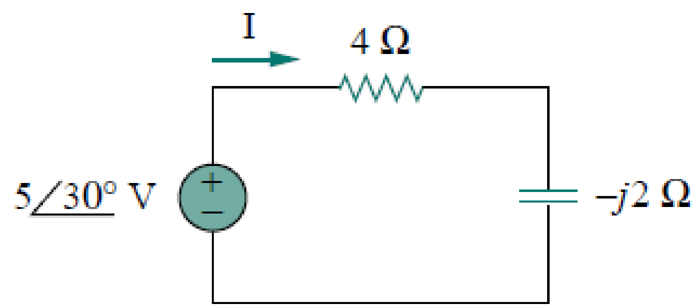
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\underline{/0^\circ}}{30 - j70} = \frac{120\underline{/0^\circ}}{76.16\underline{/-66.8^\circ}} = 1.576\underline{/66.8^\circ}\ \text{A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8^\circ) = 37.24\ \text{W}$$

EXAMPLE 11.3

For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.



Solution:

The current **I** is given by

$$\mathbf{I} = \frac{5\angle 30^\circ}{4 - j2} = \frac{5\angle 30^\circ}{4.472\angle -26.57^\circ} = 1.118\angle 56.57^\circ \text{ A}$$

The average power supplied by the voltage source is

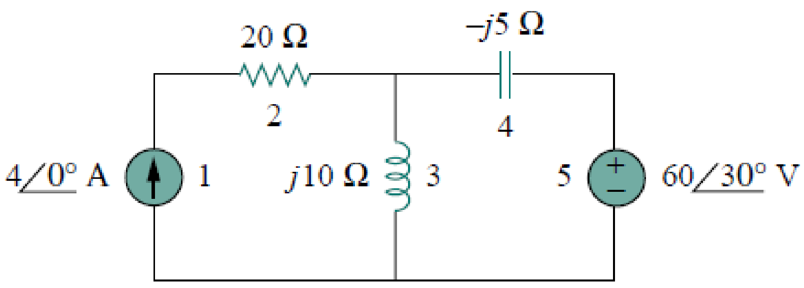
$$P = \frac{1}{2}(5)(1.118)\cos(30^\circ - 56.57^\circ) = 2.5 \text{ W}$$

$$\mathbf{I} = \mathbf{I}_R = 1.118\angle 56.57^\circ \text{ A} \qquad \mathbf{V}_R = 4\mathbf{I}_R = 4.472\angle 56.57^\circ \text{ V}$$

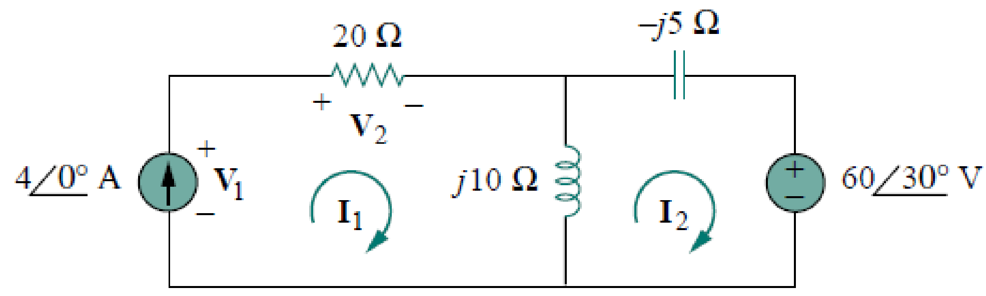
$$P = \frac{1}{2}(4.472)(1.118) = 2.5 \text{ W}$$

EXAMPLE 11.4

Determine the power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 11.5(a).



(a)



(b)

Solution:

We apply mesh analysis as shown in Fig. 11.5(b). For mesh 1,

$$\mathbf{I}_1 = 4 \text{ A}$$

For mesh 2,

$$(j10 - j5)\mathbf{I}_2 - j10\mathbf{I}_1 + 60\angle 30^\circ = 0, \qquad \mathbf{I}_1 = 4 \text{ A}$$

$$\mathbf{I}_2 = -12\angle -60^\circ + 8$$
$$= 10.58\angle 79.1^\circ \text{ A}$$

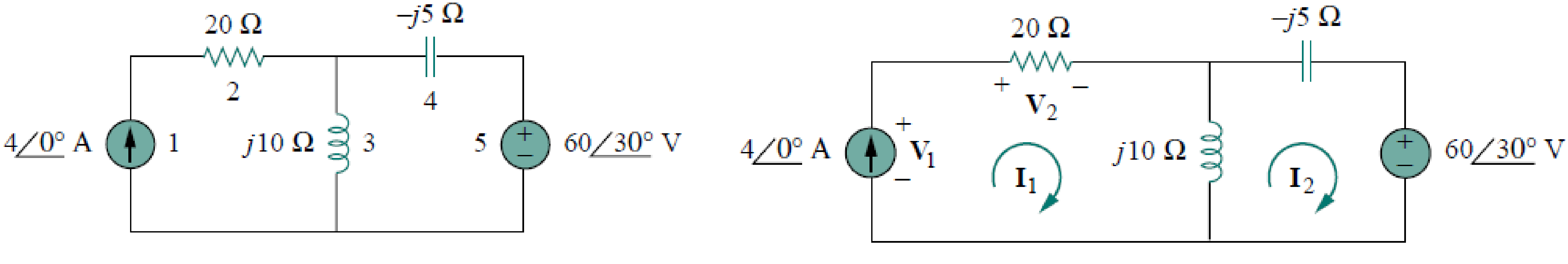
For the voltage source,

$$P_5 = \frac{1}{2}(60)(10.58)\cos(30^\circ - 79.1^\circ) = 207.8 \text{ W}$$

$$P_1 + P_2 + P_3 + P_4 + P_5 = -367.8 + 160 + 0 + 0 + 207.8 = 0$$

EXAMPLE 11.4

Determine the power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 11.5(a).



(a) (b)

For the current source, the current through it is $\mathbf{I}_1 = 4\angle 0^\circ$ and the voltage across it is

$$\mathbf{V}_1 = 20\mathbf{I}_1 + j10(\mathbf{I}_1 - \mathbf{I}_2) = 80 + j10(4 - 2 - j10.39)$$
$$= 183.9 + j20 = 184.984\angle 6.21^\circ \text{ V}$$

The average power supplied by the current source is

$$P_1 = -\frac{1}{2}(184.984)(4)\cos(6.21^\circ - 0) = -367.8 \text{ W}$$

For the resistor,

$$P_2 = \frac{1}{2}(80)(4) = 160 \text{ W}$$

For the capacitor,

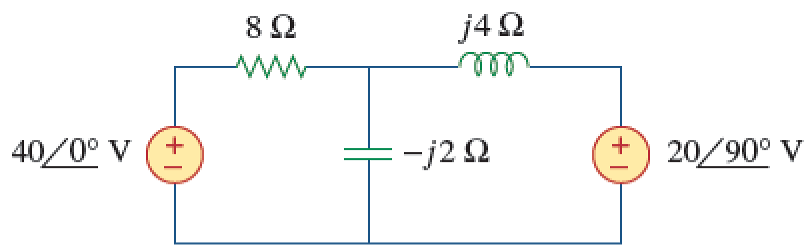
$$P_4 = \frac{1}{2}(52.9)(10.58)\cos(-90^\circ) = 0$$

For the inductor,

$$P_3 = \frac{1}{2}(105.8)(10.58)\cos 90^\circ = 0$$

Practice Problem 11.4

Calculate the average power absorbed by each of the five elements in the circuit of Fig. 11.6.

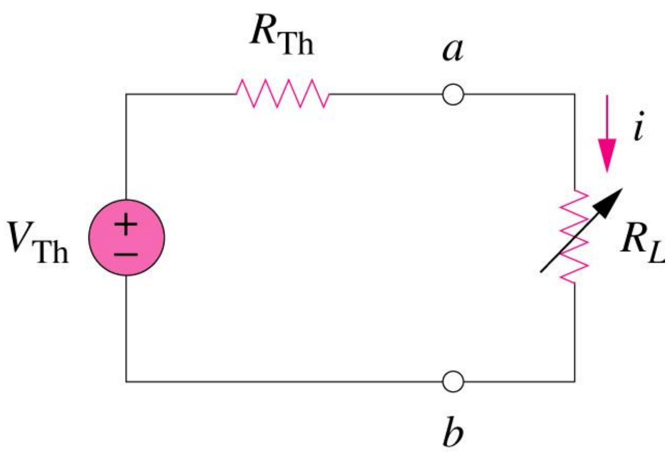


Answer: 40-V Voltage source: −60 W; j20-V Voltage source: −40 W; resistor: 100 W; others: 0 W.

11.3 MAXIMUM AVERAGE POWER TRANSFER

DC circuits

Maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$).



$$\boxed{R_L = R_{Th}}$$

$$\boxed{p_{max} = \frac{V_{Th}^2}{4R_{Th}}}$$

EXAMPLE 4.13

Find the value of R_L for maximum power transfer in the circuit of Fig. 4.50. Find the maximum power.

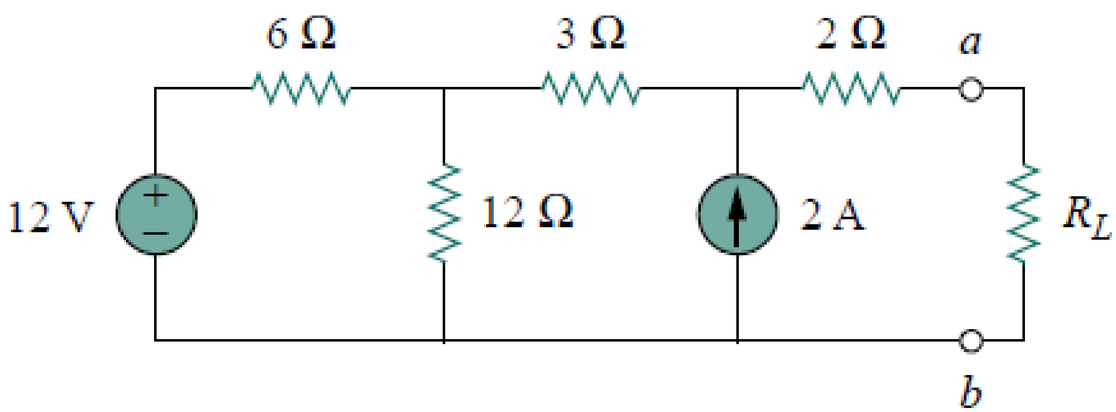
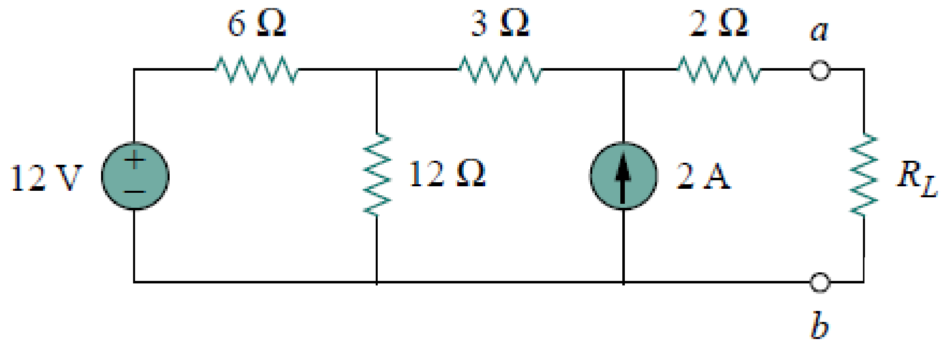
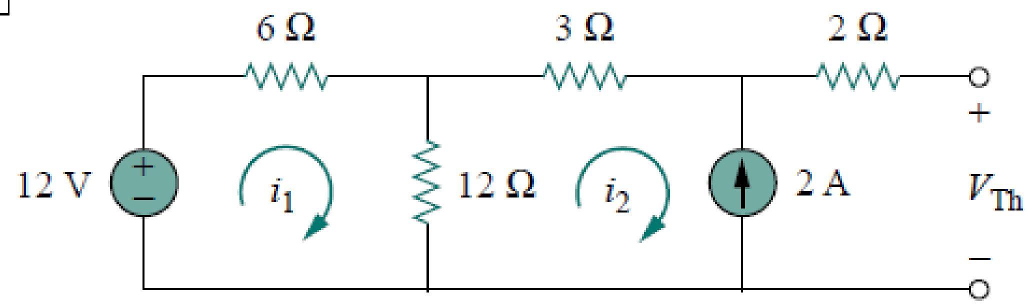


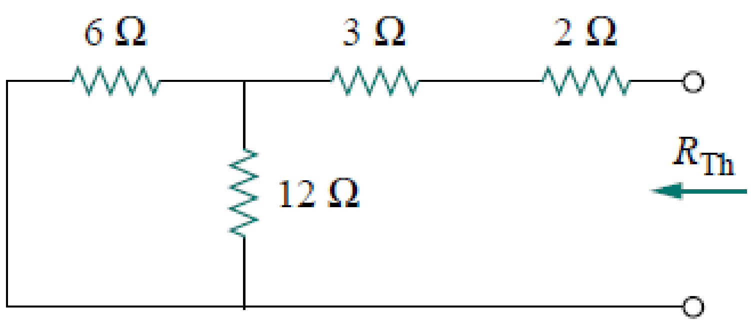
Figure 4.50 For Example 4.13.

Solution:

We need to find the Thevenin resistance R_{Th} and the Thevenin voltage V_{Th} across the terminals a-b. To get R_{Th}, we use the circuit in Fig. 4.51(a) and obtain







(a)

$$R_{\text{Th}} = 2 + 3 + 6 \parallel 12 = 5 + \frac{6 \times 12}{18} = 9\ \Omega$$

$$-12 + 18i_1 - 12i_2 = 0,$$

$$i_2 = -2\text{ A}$$

$$V_{\text{Th}} = 22\text{ V}$$

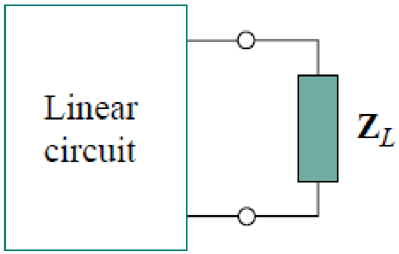
For maximum power transfer,

$$R_L = R_{\text{Th}} = 9\ \Omega$$

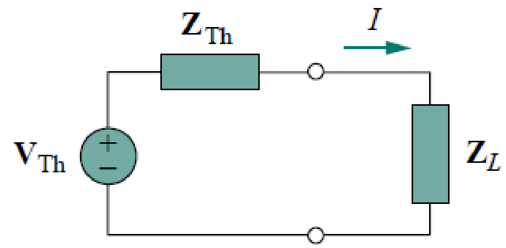
$$p_{\max} = \frac{V_{\text{Th}}^2}{4R_L} = \frac{22^2}{4 \times 9} = 13.44\text{ W}$$

11.3 MAXIMUM AVERAGE POWER TRANSFER

AC circuits



(a)



(b)

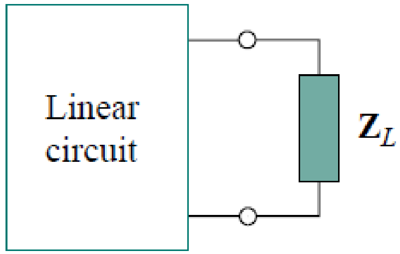
$$\mathbf{Z}_L = R_L + jX_L \qquad \mathbf{Z}_{\mathrm{Th}} = R_{\mathrm{Th}} + jX_{\mathrm{Th}}$$

$$\mathbf{I} = \frac{\mathbf{V}_{\mathrm{Th}}}{\mathbf{Z}_{\mathrm{Th}} + \mathbf{Z}_L} = \frac{\mathbf{V}_{\mathrm{Th}}}{(R_{\mathrm{Th}} + jX_{\mathrm{Th}}) + (R_L + jX_L)}$$

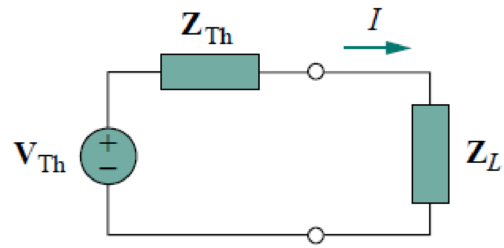
$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{\mathrm{Th}}|^2 R_L/2}{(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2}$$

Our objective is to adjust the load parameters R_L and X_L so that P is maximum. To do this we set $\partial P/\partial R_L$ and $\partial P/\partial X_L$ equal to zero. From

11.3 MAXIMUM AVERAGE POWER TRANSFER



(a)



(b)

$$\frac{\partial P}{\partial X_L} = -\frac{|\mathbf{V}_{\text{Th}}|^2 R_L (X_{\text{Th}} + X_L)}{[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2]^2}$$

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{\text{Th}}|^2[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2 - 2R_L(R_{\text{Th}} + R_L)]}{2[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2]^2}$$

Setting $\partial P/\partial X_L$ to zero gives

$$X_L = -X_{\text{Th}}$$

and setting $\partial P/\partial R_L$ to zero results in

$$R_L = \sqrt{R_{\text{Th}}^2 + (X_{\text{Th}} + X_L)^2}$$

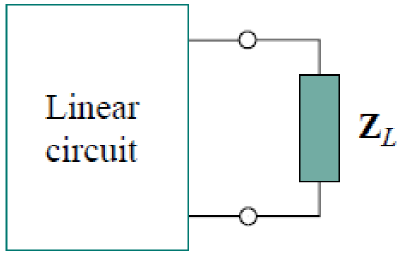
$$R_L = R_{\text{Th}}$$

$$\boxed{P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}}}$$

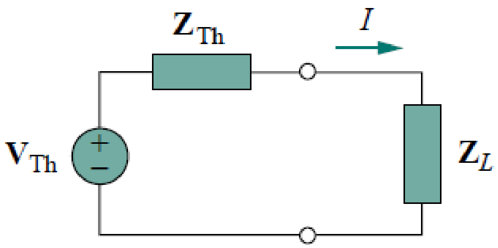
$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{\text{Th}} - jX_{\text{Th}} = \mathbf{Z}_{\text{Th}}^*}$$

For maximum average power transfer, the load impedance $\mathbf{Z}_L$ must be equal to the complex conjugate of the Thevenin impedance $\mathbf{Z}_{\text{Th}}$.

11.3 MAXIMUM AVERAGE POWER TRANSFER



(a)



(b)

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{\text{Th}} - jX_{\text{Th}} = \mathbf{Z}^*_{\text{Th}}}$$

$$\boxed{P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}}}$$

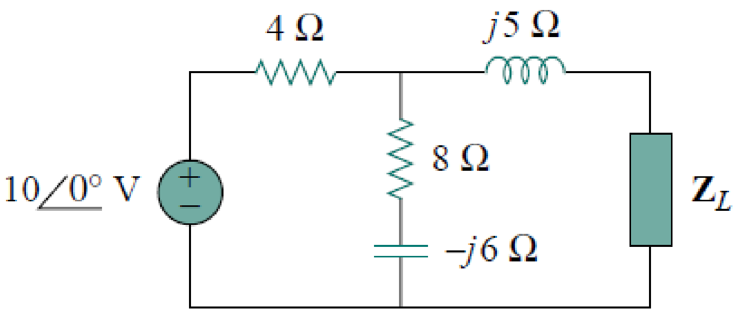
$X_L = 0$ the load is purely real（a purely resistive load）

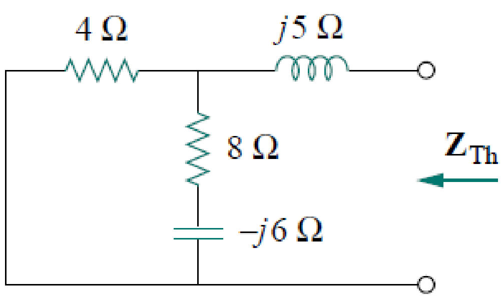
and setting $\partial P/\partial R_L$ to zero results in

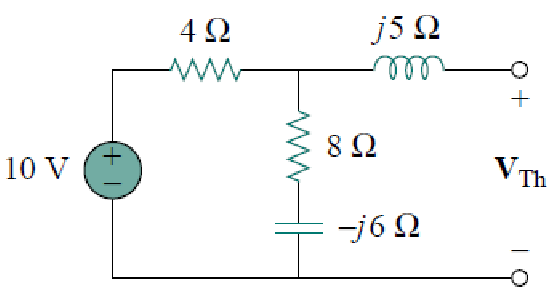
$$R_L = \sqrt{R^2_{\text{Th}} + (X_{\text{Th}} + X_L)^2} \qquad R_L = \sqrt{R^2_{\text{Th}} + X^2_{\text{Th}}} = |\mathbf{Z}_{\text{Th}}|$$

This means that for maximum average power transfer to a purely resistive load, the load impedance (or resistance) is equal to the magnitude of the Thevenin impedance.

Example 11.5 Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit of Fig. 11.8. What is the maximum average power?







Solution:

First we obtain the Thevenin equivalent at the load terminals. To get $\mathbf{Z}_{\text{Th}}$, consider the circuit shown in Fig. 11.9(a). We find

$$\mathbf{Z}_{\text{Th}} = j5 + 4 \parallel (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

To find $\mathbf{V}_{\text{Th}}$, consider the circuit in Fig. 11.8(b). By voltage division,

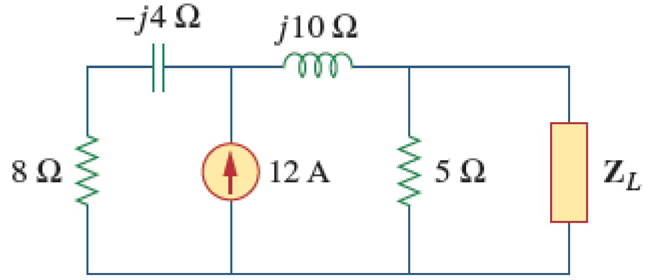
$$\mathbf{V}_{\text{Th}} = \frac{8 - j6}{4 + 8 - j6}(10) = 7.454\underline{/-10.3^\circ}\ \text{V}$$

The load impedance draws the maximum power from the circuit when

$$\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = 2.933 - j4.467\ \Omega \qquad P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}} = \frac{(7.454)^2}{8(2.933)} = 2.368\ \text{W}$$

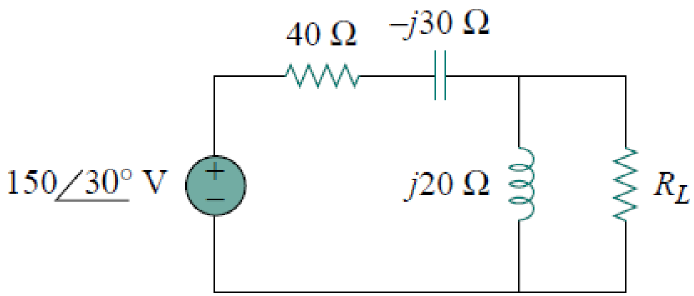
Practice Problem 11.5

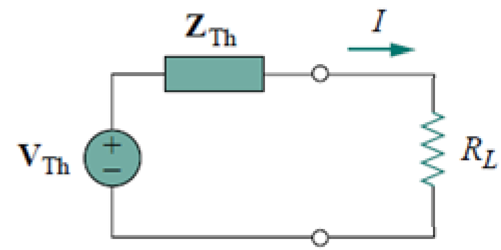
For the circuit shown in Fig. 11.10, find the load impedance $\mathbf{Z}_L$ that absorbs the maximum average power. Calculate that maximum average power.



Answer: $3.415 - j0.7317\ \Omega$, 51.47 W.

Example 11.6 In the circuit in Fig. 11.11, find the value of R_L that will absorb the maximum average power. Calculate that power.





Solution:

We first find the Thevenin equivalent at the terminals of R_L.

$$\mathbf{Z}_{\text{Th}} = (40 - j30) \parallel j20 = \frac{j20(40 - j30)}{j20 + 40 - j30} = 9.412 + j22.35\ \Omega$$

By voltage division,

$$\mathbf{V}_{\text{Th}} = \frac{j20}{j20 + 40 - j30}(150\angle 30^\circ) = 72.76\angle 134^\circ\ \text{V}$$

The value of R_L that will absorb the maximum average power is

$$R_L = |\mathbf{Z}_{\text{Th}}| = \sqrt{9.412^2 + 22.35^2} = 24.25\ \Omega$$

The current through the load is $\mathbf{I} = \dfrac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + R_L} = \dfrac{72.76\angle 134^\circ}{33.39 + j22.35} = 1.8\angle 100.2^\circ\ \text{A}$

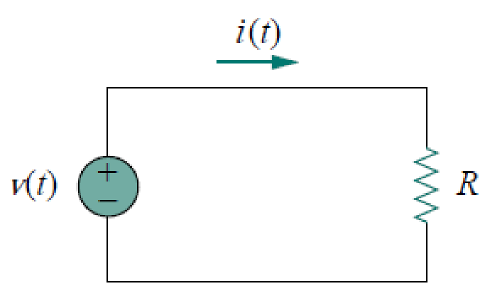
The maximum average power absorbed by R_L is $P_{\max} = \dfrac{1}{2}|\mathbf{I}|^2 R_L = \dfrac{1}{2}(1.8)^2(24.25) = 39.29\ \text{W}$

11.4 EFFECTIVE OR RMS VALUE

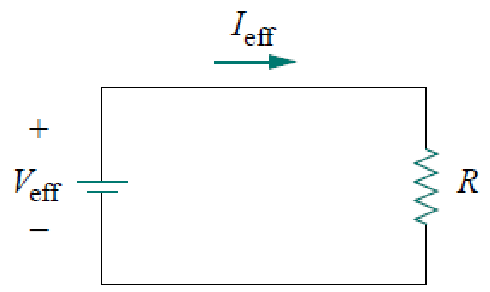
- The idea of effective value arises from the need to measure the effectiveness of a voltage or current source in delivering power to a resistive load.

> The effective value of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

In Fig. 11.13, the circuit in (a) is ac while that of (b) is dc. Our objective is to find I_{eff} that will transfer the same power to resistor R as the sinusoid i. The average power absorbed by the resistor in the ac circuit is

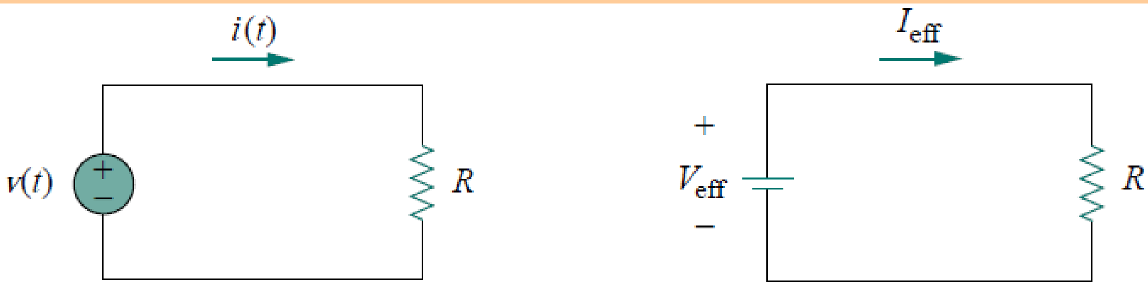


(a) ac circuit,



(b) dc circuit.

11.4 EFFECTIVE OR RMS VALUE



(a) ac circuit, (b) dc circuit.

$$P = \frac{1}{T}\int_0^T i^2 R\,dt = \frac{R}{T}\int_0^T i^2\,dt \qquad P = I_{\text{eff}}^2 R$$

$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} \qquad V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v^2\,dt}$$

the effective value is the (square) *root* of the *mean* (or average) of the *square* of the periodic signal.

the *root-mean-square* value, or *rms* value $I_{\text{eff}} = I_{\text{rms}}, \qquad V_{\text{eff}} = V_{\text{rms}}$

The effective value of a periodic signal is its root mean square (rms) value.

For any periodic function $x(t)$ in general, the rms value is given by

$$\boxed{X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}}$$

For the sinusoid $i(t) = I_m \cos \omega t$, the effective or rms value is

$$I_{\rm rms} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t \, dt} \quad = \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1}{2}(1+\cos 2\omega t)\, dt} = \frac{I_m}{\sqrt{2}}$$

Similarly, for $v(t) = V_m \cos \omega t$, $\quad V_{\rm rms} = \dfrac{V_m}{\sqrt{2}}$

The average power can be written in terms of the rms values.

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i) = V_{\rm rms} I_{\rm rms} \cos(\theta_v - \theta_i)$$

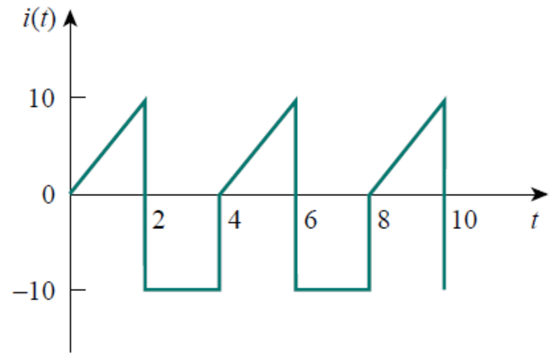
Similarly, the average power absorbed by a resistor R can be written as

$$P = I_{\rm rms}^2 R = \frac{V_{\rm rms}^2}{R}$$

- When a sinusoidal voltage or current is specified, it is often in terms of its maximum (or peak) value or its rms value, since its average value is zero.
- The power industries specify phasor magnitudes in terms of their rms values rather than peak values.
- For instance, the 220 V available at every household is the rms value of the voltage from the power company.

EXAMPLE 11.7

Determine the rms value of the current waveform in Fig. 11.14. If the current is passed through a 2-Ω resistor, find the average power absorbed by the resistor.



Solution:

The period of the waveform is $T = 4$. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

The rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{4}\left[\int_0^2 (5t)^2\,dt + \int_2^4 (-10)^2\,dt\right]}$$

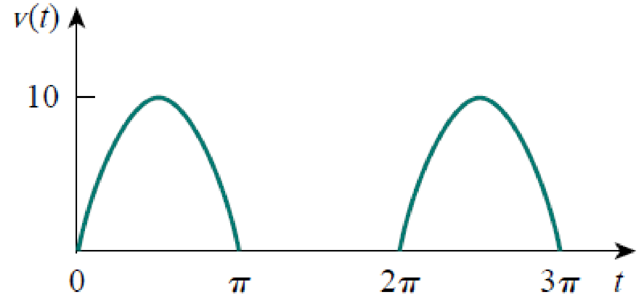
$$= \sqrt{\frac{1}{4}\left[25\frac{t^3}{3}\bigg|_0^2 + 100t\bigg|_2^4\right]} = \sqrt{\frac{1}{4}\left(\frac{200}{3} + 200\right)} = 8.165 \text{ A}$$

The power absorbed by a 2-Ω resistor is

$$P = I_{\text{rms}}^2 R = (8.165)^2(2) = 133.3 \text{ W}$$

EXAMPLE 11.8

The waveform shown in Fig. 11.16 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.



Solution:

The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10 \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi}(10\sin t)^2\,dt + \int_{\pi}^{2\pi} 0^2\,dt\right] = \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25,$$

The average power absorbed is

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{5^2}{10} = 2.5 \text{ W}$$

11.5 APPARENT POWER AND POWER FACTOR

$$v(t) = V_m \cos(\omega t + \theta_v) \qquad \text{and} \qquad i(t) = I_m \cos(\omega t + \theta_i)$$

or, in phasor form, $\mathbf{V} = V_m \angle \theta_v$ and $\mathbf{I} = I_m \angle \theta_i$, the average power is $P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i)$$

$$\boxed{S = V_{\text{rms}} I_{\text{rms}}}$$

- The average power is a product of two terms. The product $V_{\text{rms}} \times I_{\text{rms}}$ is known as the *apparent power S*. The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf).

The apparent power (in VA) is the product of the rms values of voltage and current.

$$\boxed{\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)}$$

The power factor is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

The angle $\theta_v - \theta_i$ is called the *power factor angle*,

11.5 APPARENT POWER AND POWER FACTOR

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

- For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and pf = 1. This implies that the apparent power is equal to the average power.
- For a purely reactive load, $\theta_v - \theta_i = \pm 90^\circ$ and pf = 0. In this case the average power is zero.
- In between these two extreme cases, pf is said to be leading or lagging.
- Leading power factor means that current leads voltage, which implies a capacitive load.
- Lagging power factor means that current lags voltage, implying an inductive load.

EXAMPLE 11.9

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10^\circ)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20^\circ)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

Solution:

The apparent power is

$$S = V_{\mathrm{rms}} I_{\mathrm{rms}} = \frac{120}{\sqrt{2}} \frac{4}{\sqrt{2}} = 240 \text{ VA}$$

The power factor is

$$\mathrm{pf} = \cos(\theta_v - \theta_i) = \cos(-20^\circ - 10^\circ) = 0.866 \qquad \text{(leading)}$$

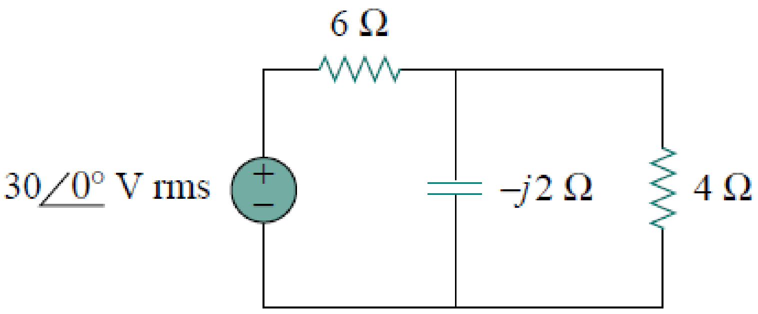
The pf is leading because the current leads the voltage.

The load impedance **Z** can be modeled by a 25.98-Ω resistor in series with a capacitor with

$$X_C = -15 = -\frac{1}{\omega C} \qquad C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

EXAMPLE 11.10

Determine the power factor of the entire circuit of Fig. 11.18 as seen by the source. Calculate the average power delivered by the source.



Solution:

The total impedance is

$$\mathbf{Z} = 6 + 4 \parallel (-j2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6 = 7\angle -13.24\ \Omega$$

The power factor is

$$\text{pf} = \cos(-13.24) = 0.9734 \qquad \text{(leading)} \qquad \text{since the impedance is capacitive.}$$

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} = \frac{30\angle 0^\circ}{7\angle -13.24^\circ} = 4.286\angle 13.24^\circ \text{ A}$$

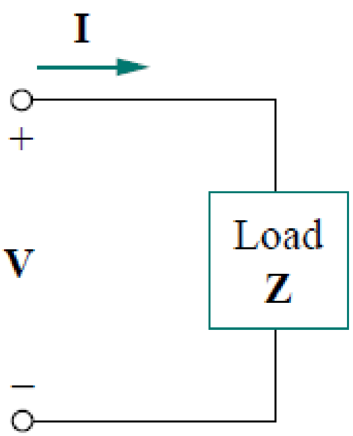
The average power supplied by the source is $P = V_{\text{rms}} I_{\text{rms}}\, \text{pf} = (30)(4.286)0.9734 = 125 \text{ W}$

$$P = I_{\text{rms}}^2 R = (4.286)^2 (6.8) = 125 \text{ W}$$

11.6 COMPLEX POWER

- Considerable effort has been expended over the years to express power relations as simply as possible. Power engineers have coined the term *complex power*, which they use to find the total effect of parallel loads.
- Complex power is important in power analysis because it contains *all* the information pertaining to the power absorbed by a given load.

$$\mathbf{V} = V_m \underline{/\theta_v} \text{ and } \mathbf{I} = I_m \underline{/\theta_i},$$



the *complex power* S absorbed by the ac load is the product of the voltage and the complex conjugate of the current, $\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^*$

In terms of the rms values, $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}} \underline{/\theta_v}$$

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}} \underline{/\theta_i}$$

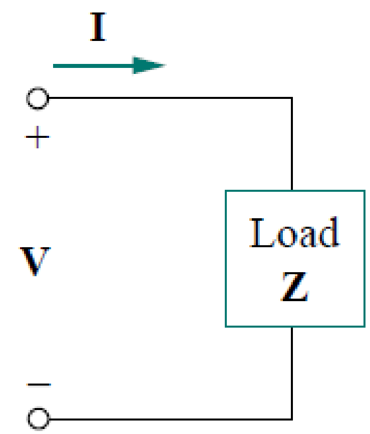
$$\begin{aligned}\mathbf{S} &= V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)\end{aligned}$$

11.6 COMPLEX POWER

$$\mathbf{V} = V_m \angle \theta_v \text{ and } \mathbf{I} = I_m \angle \theta_i,$$

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}} \angle \theta_v$$

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}} \angle \theta_i$$



$$\mathbf{S} = V_{\text{rms}} I_{\text{rms}} \angle \theta_v - \theta_i$$

$$= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

$$\mathbf{S} = I_{\text{rms}}^2 (R + jX) = P + jQ$$

$$P = \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X$$

P is the average or real power and it depends on the load's resistance R. Q depends on the load's reactance X and is called the *reactive* (or quadrature) power.

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

The real power P is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load. The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load. The unit of Q is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt.

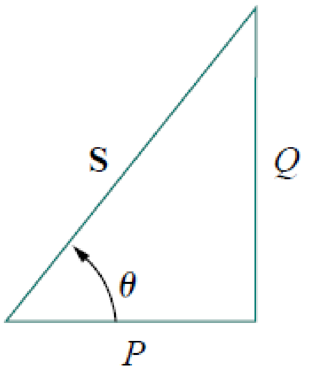
$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

Complex power (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power P and its imaginary part is reactive power Q.

Introducing the complex power enables us to obtain the real and reactive powers directly from voltage and current phasors.

$$\text{Complex Power} = \mathbf{S} = P + jQ = \frac{1}{2}\mathbf{V}\mathbf{I}^*$$
$$= V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i}$$
$$\text{Apparent Power} = S = |\mathbf{S}| = V_{\text{rms}} I_{\text{rms}} = \sqrt{P^2 + Q^2}$$
$$\text{Real Power} = P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i)$$
$$\text{Reactive Power} = Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i)$$
$$\text{Power Factor} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$



power triangle

EXAMPLE 11.11

The voltage across a load is $v(t) = 60\cos(\omega t - 10^\circ)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50^\circ)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Solution:

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\underline{/-10^\circ}, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\underline{/+50^\circ}$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = \left(\frac{60}{\sqrt{2}}\underline{/-10^\circ}\right)\left(\frac{1.5}{\sqrt{2}}\underline{/-50^\circ}\right) = 45\underline{/-60^\circ}\text{ VA}$$

The apparent power is $\quad S = |\mathbf{S}| = 45$ VA

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\underline{/-60^\circ} = 45[\cos(-60^\circ) + j\sin(-60^\circ)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is $P = 22.5$ W the reactive power is $Q = -38.97$ VAR

(c) The power factor is $\text{pf} = \cos(-60^\circ) = 0.5$ (leading) $\quad \mathbf{Z} = \dfrac{\mathbf{V}}{\mathbf{I}} = \dfrac{60\underline{/-10^\circ}}{1.5\underline{/+50^\circ}} = 40\underline{/-60^\circ}\ \Omega$

EXAMPLE 11.12

A load $\mathbf{Z}$ draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

Solution:

(a) Given that pf $= \cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13^\circ$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is

$$P = S\cos\theta = 12{,}000 \times 0.856 = 10.272 \text{ kW}$$

while the reactive power is

$$Q = S\sin\theta = 12{,}000 \times 0.517 = 6.204 \text{ kVA}$$

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + j6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$, we obtain

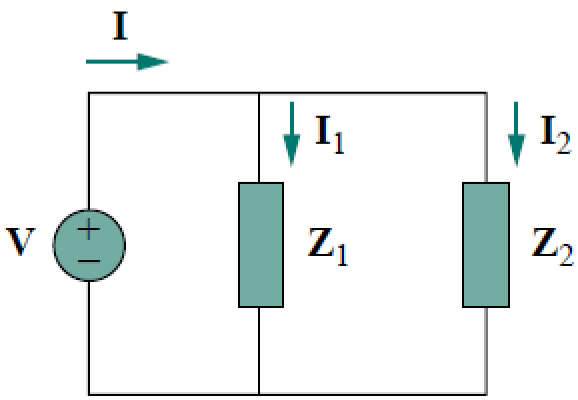
$$\mathbf{I}^*_{\text{rms}} = \frac{\mathbf{S}}{\mathbf{V}_{\text{rms}}} = \frac{10{,}272 + j6204}{120\angle 0^\circ} = 85.6 + j51.7 \text{ A} = 100\angle 31.13^\circ \text{ A}$$

Thus $\mathbf{I}_{\text{rms}} = 100\angle -31.13^\circ$ and the peak current is

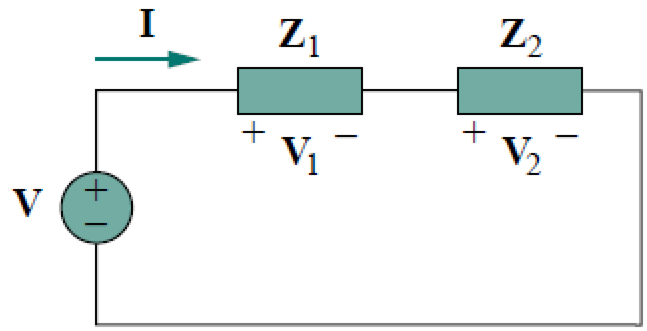
$$I_m = \sqrt{2}I_{\text{rms}} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance $\quad \mathbf{Z} = \dfrac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \dfrac{120\angle 0^\circ}{100\angle -31.13^\circ} = 1.2\angle 31.13^\circ \ \Omega$

11.7 CONSERVATION OF AC POWER



(a)



(b)

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2$$

$$\begin{aligned}\mathbf{S} &= \frac{1}{2}\mathbf{V}\mathbf{I}^* \\ &= \frac{1}{2}\mathbf{V}(\mathbf{I}_1^* + \mathbf{I}_2^*) \\ &= \frac{1}{2}\mathbf{V}\mathbf{I}_1^* + \frac{1}{2}\mathbf{V}\mathbf{I}_2^* \\ &= \mathbf{S}_1 + \mathbf{S}_2\end{aligned}$$

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$$

$$\begin{aligned}\mathbf{S} &= \frac{1}{2}\mathbf{V}\mathbf{I}^* \\ &= \frac{1}{2}(\mathbf{V}_1 + \mathbf{V}_2)\mathbf{I}^* \\ &= \frac{1}{2}\mathbf{V}_1\mathbf{I}^* + \frac{1}{2}\mathbf{V}_2\mathbf{I}^* \\ &= \mathbf{S}_1 + \mathbf{S}_2\end{aligned}$$

We conclude from Eqs. (11.53) and (11.55) that whether the loads are connected in series or in parallel (or in general), the total power *supplied* by the source equals the total power *delivered* to the load. Thus, in general, for a source connected to N loads,

$$\boxed{\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_1 + \cdots + \mathbf{S}_N}$$

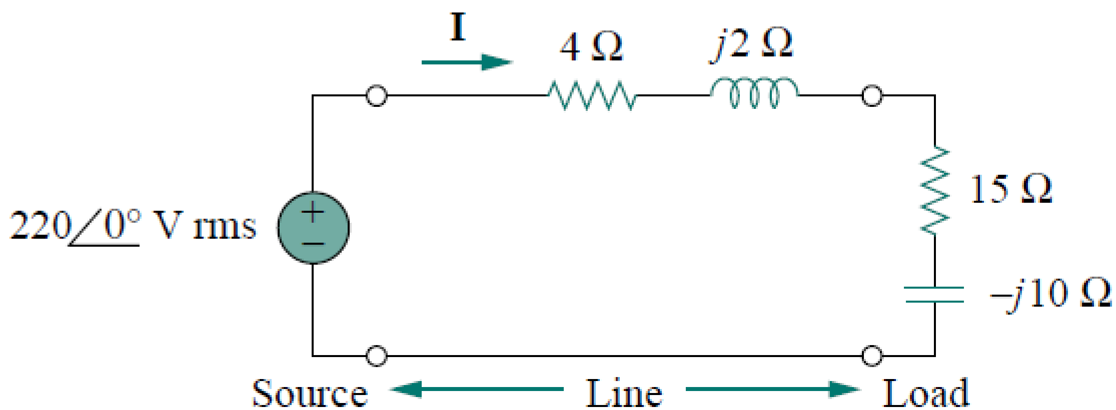
This means that the total complex power in a network is the sum of the complex powers of the individual components. (This is also true of real power and reactive power, but not true of apparent power.) This expresses the principle of conservation of ac power:

> The complex, real, and reactive powers of the sources equal the respective sums of the complex, real, and reactive powers of the individual loads.

From this we imply that the real (or reactive) power flow from sources in a network equals the real (or reactive) power flow into the other elements in the network.

EXAMPLE 11.13

Figure 11.24 shows a load being fed by a voltage source through a transmission line. The impedance of the line is represented by the $(4 + j2)\ \Omega$ impedance and a return path. Find the real power and reactive power absorbed by: (a) the source, (b) the line, and (c) the load.

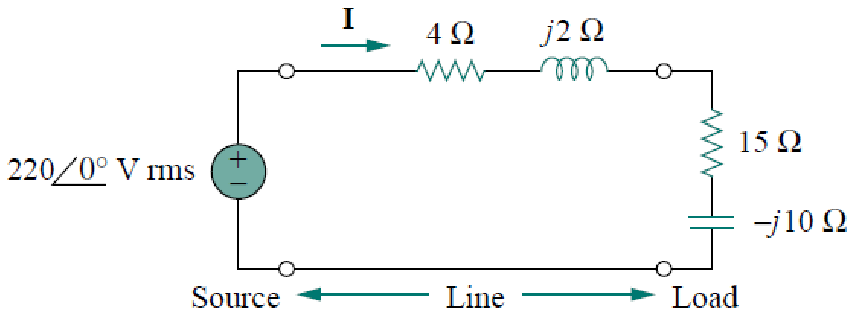


Solution:

The total impedance is

$$\mathbf{Z} = (4 + j2) + (15 - j10) = 19 - j8 = 20.62\angle -22.83^\circ\ \Omega$$

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{220\angle 0^\circ}{20.62\angle -22.83^\circ} = 10.67\angle 22.83^\circ \text{ A rms}$$



(a) For the source, the complex power is

$$\mathbf{S}_s = \mathbf{V}_s\mathbf{I}^* = (220\underline{/0^\circ})(10.67\underline{/-22.83^\circ})$$
$$= 2347.4\underline{/-22.83^\circ} = (2163.5 - j910.8) \text{ VA}$$

From this, we obtain the real power as 2163.5 W and the reactive power as 910.8 VAR (leading).

(b) For the line, the voltage is

$$\mathbf{V}_{\text{line}} = (4 + j2)\mathbf{I} = (4.472\underline{/26.57^\circ})(10.67\underline{/22.83^\circ})$$
$$= 47.72\underline{/49.4^\circ} \text{ V rms}$$

The complex power absorbed by the line is

$$\mathbf{S}_{\text{line}} = \mathbf{V}_{\text{line}}\mathbf{I}^* = (47.72\underline{/49.4^\circ})(10.67\underline{/-22.83^\circ})$$
$$= 509.2\underline{/26.57^\circ} = 455.4 + j227.7 \text{ VA}$$

(c) For the load, the voltage is

$$\mathbf{V}_L = (15 - j10)\mathbf{I} = (18.03\underline{/-33.7^\circ})(10.67\underline{/22.83^\circ})$$
$$= 192.38\underline{/-10.87^\circ} \text{ V rms}$$

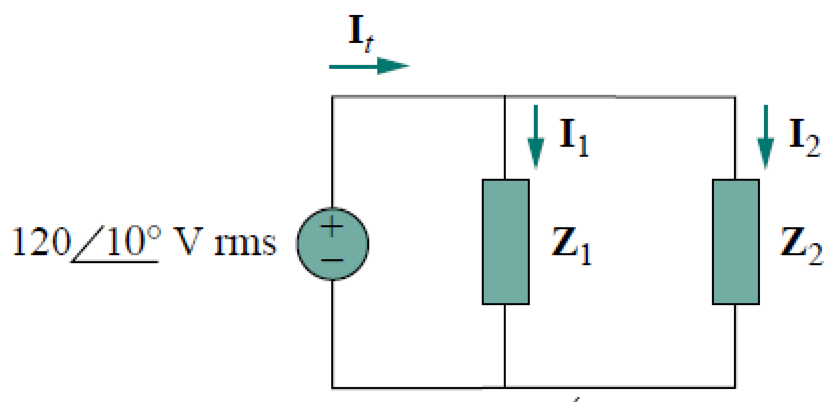
The complex power absorbed by the load is

$$\mathbf{S}_L = \mathbf{V}_L\mathbf{I}^* = (192.38\underline{/-10.87^\circ})(10.67\underline{/-22.83^\circ})$$
$$= 2053\underline{/-33.7^\circ} = (1708 - j1139) \text{ VA}$$

The real power is 1708 W and the reactive power is 1139 VAR (leading). Note that $\mathbf{S}_s = \mathbf{S}_{\text{line}} + \mathbf{S}_L$, as expected. We have used the rms values of voltages and currents.

EXAMPLE 11.14

In the circuit of Fig. 11.26, $\mathbf{Z}_1 = 60\angle -30^\circ\ \Omega$ and $\mathbf{Z}_2 = 40\angle 45^\circ\ \Omega$. Calculate the total: (a) apparent power, (b) real power, (c) reactive power, and (d) pf.



Solution:

The current through $\mathbf{Z}_1$ is
$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{120\angle 10^\circ}{60\angle -30^\circ} = 2\angle 40^\circ \text{ A rms}$$

while the current through $\mathbf{Z}_2$ is
$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{120\angle 10^\circ}{40\angle 45^\circ} = 3\angle -35^\circ \text{ A rms}$$

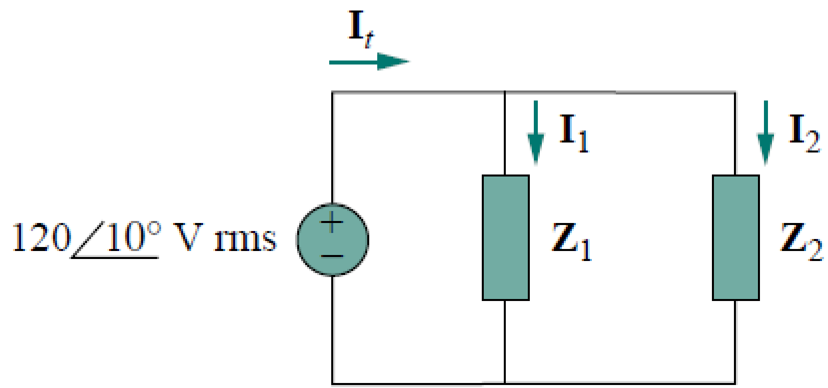
The complex powers absorbed by the impedances are

$$\mathbf{S}_1 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_1^*} = \frac{(120)^2}{60\angle 30^\circ} = 240\angle -30^\circ = 207.85 - j120 \text{ VA}$$

$$\mathbf{S}_2 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_2^*} = \frac{(120)^2}{40\angle -45^\circ} = 360\angle 45^\circ = 254.6 + j254.6 \text{ VA}$$

EXAMPLE 11.14

In the circuit of Fig. 11.26, $\mathbf{Z}_1 = 60\angle -30^\circ\ \Omega$ and $\mathbf{Z}_2 = 40\angle 45^\circ\ \Omega$. Calculate the total: (a) apparent power, (b) real power, (c) reactive power, and (d) pf.



The total complex power is $\mathbf{S}_t = \mathbf{S}_1 + \mathbf{S}_2 = 462.4 + j134.6 \text{ VA}$

(a) The total apparent power is $|\mathbf{S}_t| = \sqrt{462.4^2 + 134.6^2} = 481.6 \text{ VA}$.

(b) The total real power is $P_t = \text{Re}(\mathbf{S}_t) = 462.4 \text{ W}$ or $P_t = P_1 + P_2$.

(c) The total reactive power is $Q_t = \text{Im}(\mathbf{S}_t) = 134.6 \text{ VAR}$ or $Q_t = Q_1 + Q_2$.

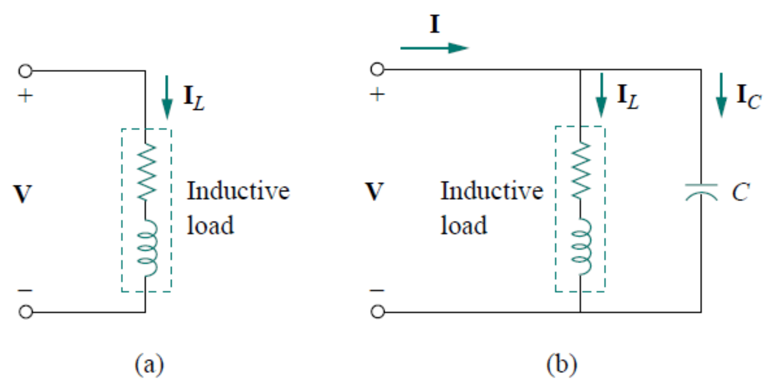
(d) The pf $= P_t/|\mathbf{S}_t| = 462.4/481.6 = 0.96$ (lagging).

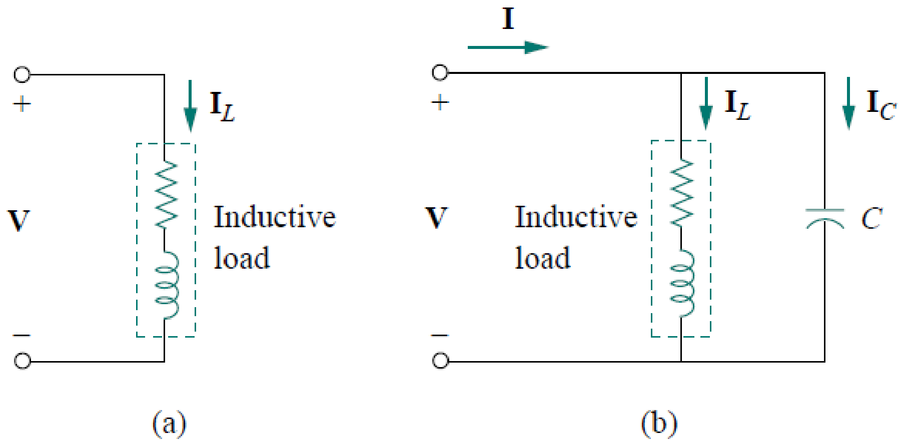
11.8 POWER FACTOR CORRECTION

1. Most domestic loads (such as washing machines, air conditioners, and refrigerators) and industrial loads (such as induction motors) are inductive and operate at a low lagging power factor. Although the inductive nature of the load cannot be changed, we can increase its power factor.

> The process of increasing the power factor without altering the voltage or current to the original load is known as power factor correction.

2. Since most loads are inductive, as shown in Fig. 11.27(a), a load's power factor is improved or corrected by deliberately installing a capacitor in parallel with the load, as shown in Fig. 11.27(b).





it is assumed that the circuit in Fig. 11.27(a) has a power factor of cosθ_1, while the one in Fig. 11.27(b) has a power factor of cos θ_2.

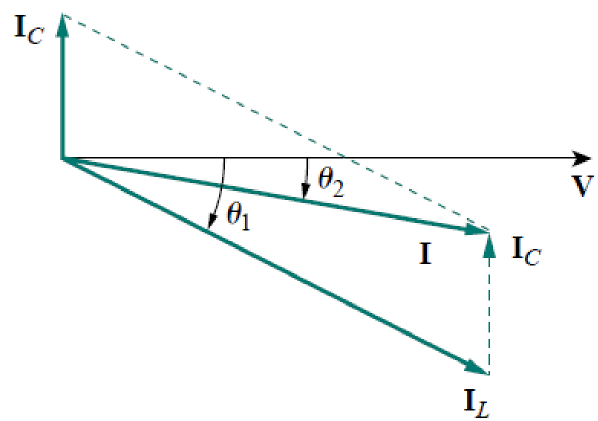
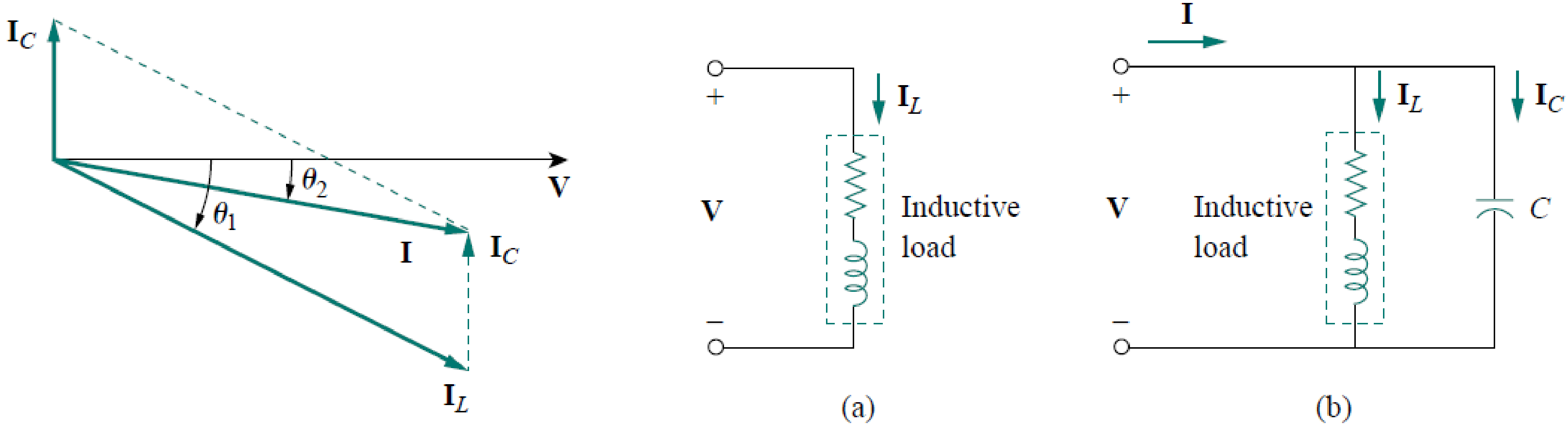


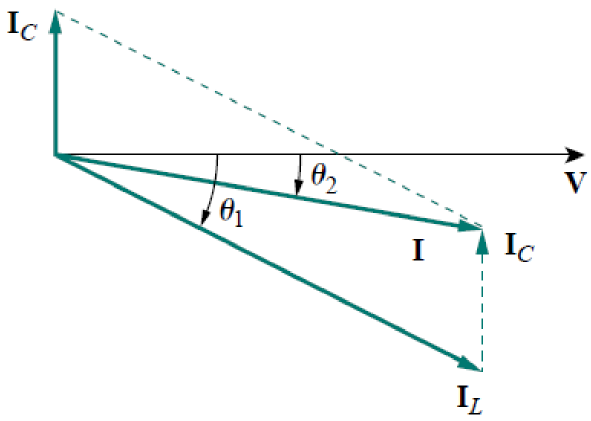
Fig. 11.28 shows that adding the capacitor has caused the phase angle between the supplied voltage and current to reduce from θ_1 to θ_2, thereby increasing the power factor.

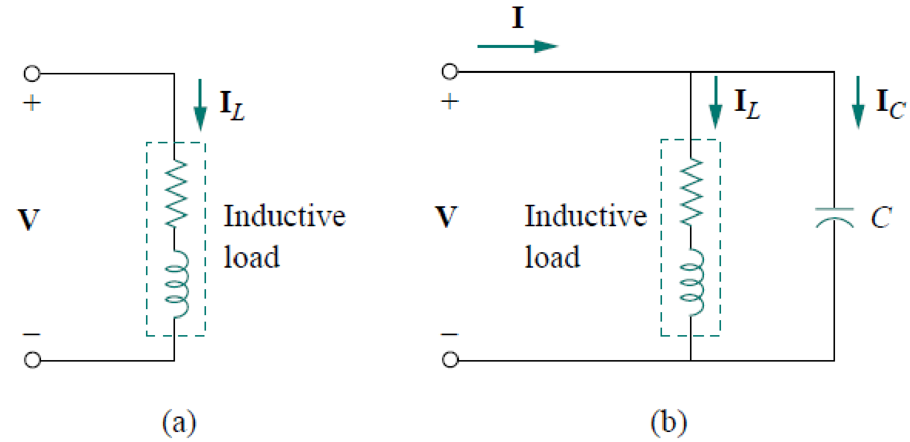


We also notice from the magnitudes of the vectors in Fig. 11.28 that with the same supplied voltage, the circuit in Fig. 11.27(a) draws larger current I_L than the current I drawn by the circuit in Fig. 11.27(b). Power companies charge more for larger currents, because they result in increased power losses (by a squared factor, since $\mathrm{P} = \mathrm{I}^2{}_L\mathrm{R}$).

Therefore, it is beneficial to both the power company and the consumer that every effort is made to minimize current level or keep the power factor as close to unity as possible.

By choosing a suitable size for the capacitor, the current can be made to be completely in phase with the voltage, implying unity power factor.





If we desire to increase the power factor from $\cos\theta_1$ to $\cos\theta_2$ without altering the real power (i.e., $P = S_2 \cos\theta_2$),

$$\boxed{C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2}}$$

Note that the real power P dissipated by the load is not affected by the power factor correction because the average power due to the capacitance is zero.

EXAMPLE 11.15

When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

$$\boxed{C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2}}$$

Solution:

If the pf $= 0.8$, then $\quad \cos\theta_1 = 0.8 \quad \Longrightarrow \quad \theta_1 = 36.87^\circ$

$$S_1 = \frac{P}{\cos\theta_1} = \frac{4000}{0.8} = 5000 \text{ VA} \qquad Q_1 = S_1 \sin\theta = 5000 \sin 36.87 = 3000 \text{ VAR}$$

When the pf is raised to 0.95, $\quad \cos\theta_2 = 0.95 \quad \Longrightarrow \quad \theta_2 = 18.19^\circ$

The real power P has not changed. But the apparent power has changed; its new value is

$$S_2 = \frac{P}{\cos\theta_2} = \frac{4000}{0.95} = 4210.5 \text{ VA} \qquad Q_2 = S_2 \sin\theta_2 = 1314.4 \text{ VAR}$$

EXAMPLE 11.15

When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

$$\boxed{C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2}}$$

The difference between the new and old reactive powers is due to the parallel addition of the capacitor to the load. The reactive power due to the capacitor is

$$Q_C = Q_1 - Q_2 = 3000 - 1314.4 = 1685.6 \text{ VAR}$$

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{1685.6}{2\pi \times 60 \times 120^2} = 310.5\ \mu\text{F}$$

Summary and Review

1. The instantaneous power absorbed by an element is the product of the element's terminal voltage and the current through the element: $p = vi$.

2. Average or real power P (in watts) is the average of instantaneous power p:

$$P = \frac{1}{T}\int_0^T p\, dt$$

If $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$, then $V_{\text{rms}} = V_m/\sqrt{2}$, $I_{\text{rms}} = I_m/\sqrt{2}$, and

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

Inductors and capacitors absorb no average power, while the average power absorbed by a resistor is $1/2\, I_m^2 R = I_{\text{rms}}^2 R$.

3. Maximum average power is transferred to a load when the load impedance is the complex conjugate of the Thevenin impedance as seen from the load terminals, $\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^*$.

4. The effective value of a periodic signal $x(t)$ is its root-mean-square (rms) value.

$$X_{\text{eff}} = X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\, dt}$$

For a sinusoid, the effective or rms value is its amplitude divided by $\sqrt{2}$.

5. The power factor is the cosine of the phase difference between voltage and current:

$$\text{pf} = \cos(\theta_v - \theta_i)$$

It is also the cosine of the angle of the load impedance or the ratio of real power to apparent power. The pf is lagging if the current lags voltage (inductive load) and is leading when the current leads voltage (capacitive load).

6. Apparent power S (in VA) is the product of the rms values of voltage and current:

$$S = V_{\text{rms}} I_{\text{rms}}$$

It is also given by $S = |\mathbf{S}| = \sqrt{P^2 + Q^2}$, where Q is reactive power.

7. Reactive power (in VAR) is:

$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

8. Complex power $\mathbf{S}$ (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. It is also the complex sum of real power P and reactive power Q.

$$\mathbf{S} = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* = V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} = P + jQ$$

Also,

$$\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*}$$

9. The total complex power in a network is the sum of the complex powers of the individual components. Total real power and reactive power are also, respectively, the sums of the individual real powers and the reactive powers, but the total apparent power is not calculated by the process.

10. Power factor correction is necessary for economic reasons; it is the process of improving the power factor of a load by reducing the overall reactive power.

Assignment (page 492)

Problems 11.15, 11.21, 11.51, 11.52

1、In the circuit of Fig. 1, load *A receives 4 kVA* at 0.8 pf leading. Load *B receives 2.4 kVA at 0.6 pf* lagging. Box *C is an inductive load that consumes* 1 kW and receives 500 VAR.

(a) Determine **I.**

(b) Calculate the power factor of the combination.

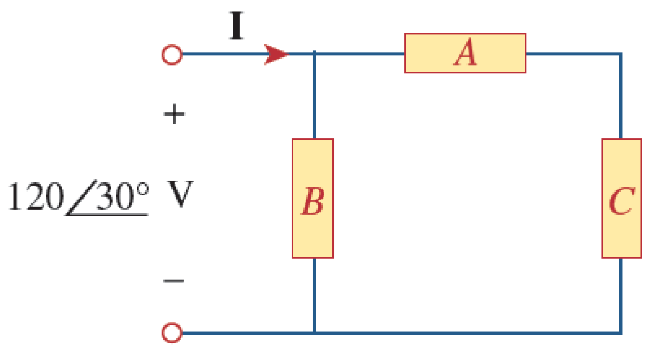


Fig. 1

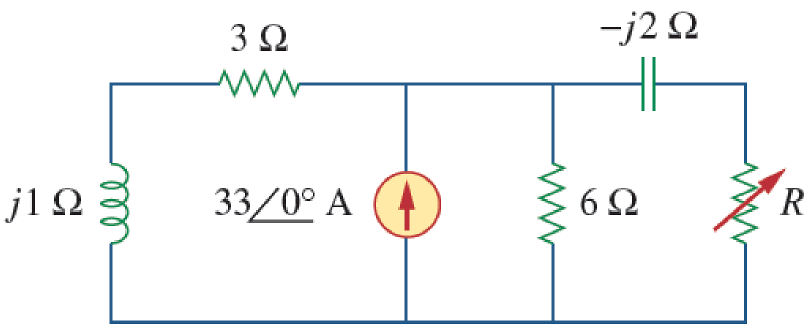


Fig. 2

2、The variable resistor *R* in the circuit of Fig. 2 is adjusted until it absorbs the maximum average power. Find *R* and the maximum average power absorbed.

11.4 EFFECTIVE OR RMS VALUE

The effective value of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

the effective value is the (square) *root* of the *mean* (or average) of the *square* of the periodic signal.

the *root-mean-square* value, or *rms* value $I_{\text{eff}} = I_{\text{rms}},\qquad V_{\text{eff}} = V_{\text{rms}}$

The effective value of a periodic signal is its root mean square (rms) value.

For any periodic function *x(t)* in general, the rms value is given by

$$X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$$

For the sinusoid $i(t) = I_m \cos \omega t$**, the effective or rms value is**

$$I_{\rm rms} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t \, dt} \quad = \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1}{2}(1 + \cos 2\omega t)\, dt} = \frac{I_m}{\sqrt{2}}$$

Similarly, for $v(t) = V_m \cos \omega t$**,** $\qquad V_{\rm rms} = \dfrac{V_m}{\sqrt{2}}$

The average power can be written in terms of the rms values.

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i) = V_{\rm rms} I_{\rm rms} \cos(\theta_v - \theta_i)$$

Similarly, the average power absorbed by a resistor R **can be written as**

$$P = I_{\rm rms}^2 R = \frac{V_{\rm rms}^2}{R}$$

11.5 APPARENT POWER AND POWER FACTOR

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i)$$

$$\boxed{S = V_{\text{rms}} I_{\text{rms}}}$$

$$\boxed{\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)}$$

- The average power is a product of two terms. The product $V_{rms} * I_{rms}$ is known as the *apparent power* S. The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf).

> The apparent power (in VA) is the product of the rms values of voltage and current.

> The power factor is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

The angle $\theta_v - \theta_i$ is called the *power factor angle*,

11.5 APPARENT POWER AND POWER FACTOR

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

- For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and pf = 1. This implies that the apparent power is equal to the average power.
- For a purely reactive load, $\theta_v - \theta_i = \pm 90^\circ$ and pf = 0. In this case the average power is zero.
- In between these two extreme cases, pf is said to be leading or lagging.
- Leading power factor means that current leads voltage, which implies a capacitive load.
- Lagging power factor means that current lags voltage, implying an inductive load.

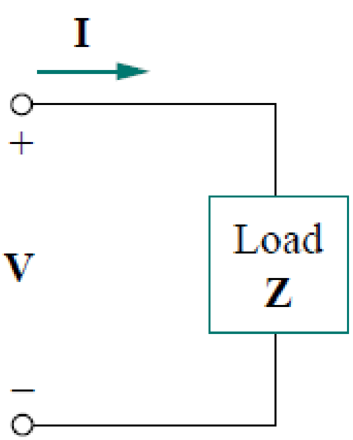
11.6 COMPLEX POWER

$$\mathbf{V} = V_m \angle \theta_v \text{ and } \mathbf{I} = I_m \angle \theta_i,$$

the *complex power* S absorbed by the ac load is the product of the voltage and the complex conjugate of the current,

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^*$$

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}} \angle \theta_v \qquad \mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}} \angle \theta_i$$



In terms of the rms values, $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$

$$\begin{aligned}\mathbf{S} &= V_{\text{rms}} I_{\text{rms}} \angle \theta_v - \theta_i \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)\end{aligned}$$

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

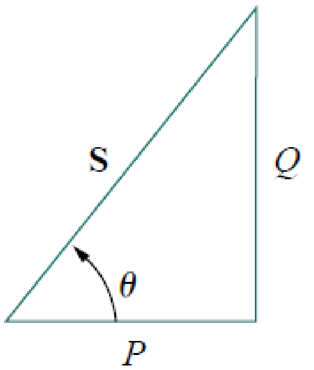
$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

Complex power (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power P and its imaginary part is reactive power Q.

Introducing the complex power enables us to obtain the real and reactive powers directly from voltage and current phasors.

$$\text{Complex Power} = \mathbf{S} = P + jQ = \frac{1}{2}\mathbf{V}\mathbf{I}^*$$
$$= V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i}$$
$$\text{Apparent Power} = S = |\mathbf{S}| = V_{\text{rms}} I_{\text{rms}} = \sqrt{P^2 + Q^2}$$
$$\text{Real Power} = P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i)$$
$$\text{Reactive Power} = Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i)$$
$$\text{Power Factor} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$



power triangle

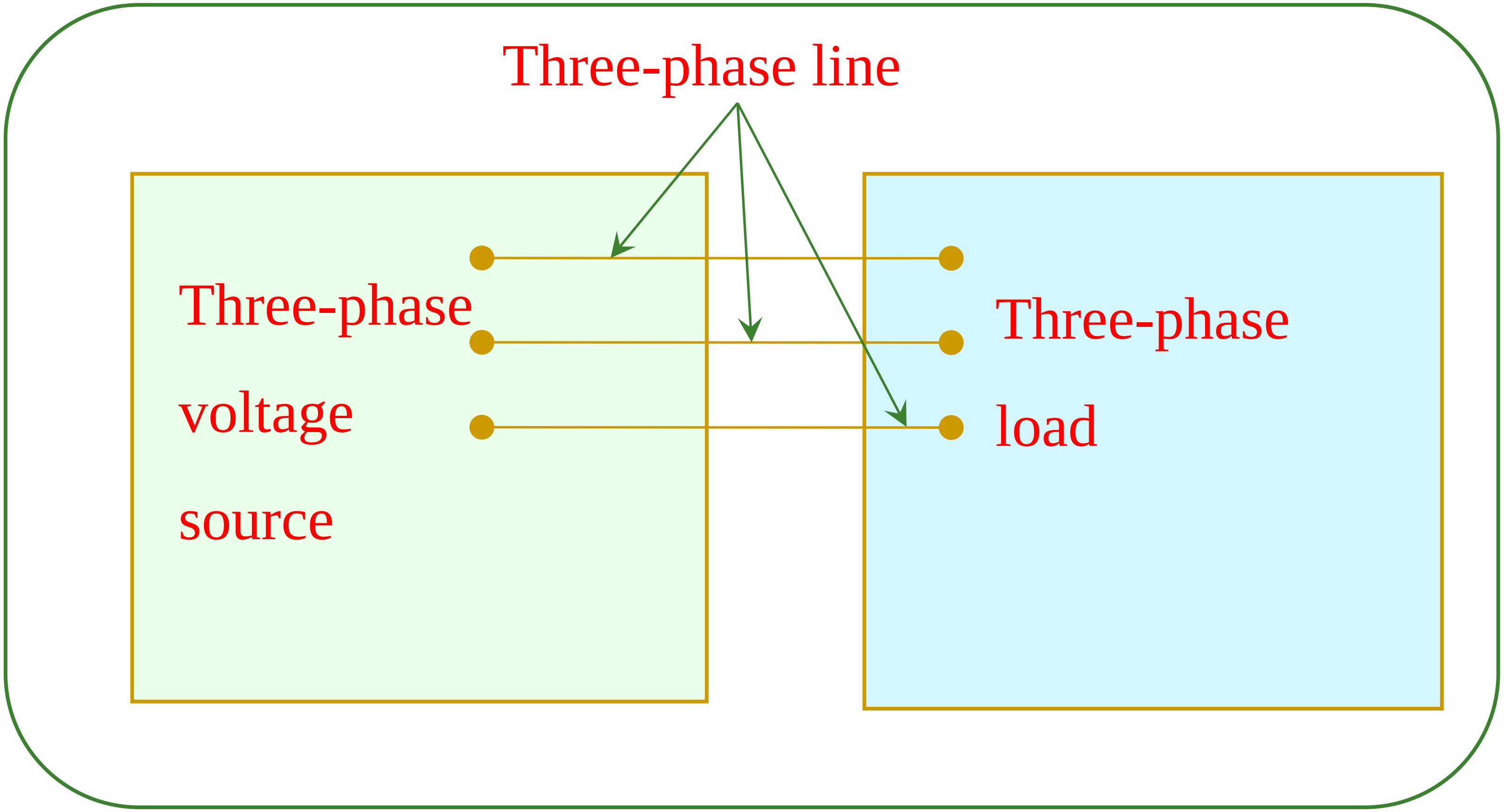
Fundamentals of Electric Circuit

2021.5

Chapter 12

Balanced Three-Phase Circuits

A basic three-phase circuit



Chapter 12
Balanced Three-Phase Circuits

12.1 Basic concepts of three-phase circuit

12.2 Analysis of the Wye-Wye (Y-Y) Circuit

12.3 Analysis of the Wye-Delta (Y-△) Circuit

12.4 Balance Delta-Delta (△-△) Connection

12.5 Balance Delta-Wye (△-Y) Connection

12.6 Summary of Balance Connection

12.7 Power in Balance System

12.8 UnBalance three-phase System

Chapter Contents

1. **Basic concepts of three-phase circuit**
2. **Analysis of the Wye-Wye (Y-Y) circuit**
3. **Analysis of the Wye-Delta (Y-△) circuit**

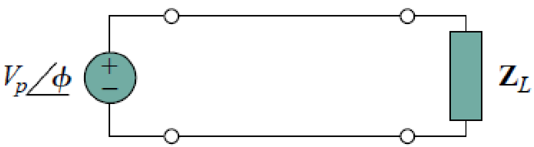
Chapter Objective

- Learning the distinction between single-phase and polyphase systems
- Becoming familiar with working with both Y- and △-connected three-phase sources
- Becoming familiar with working with both Y- and △-connected networks
- Mastering the technique of per-phase analysis of three-phase systems

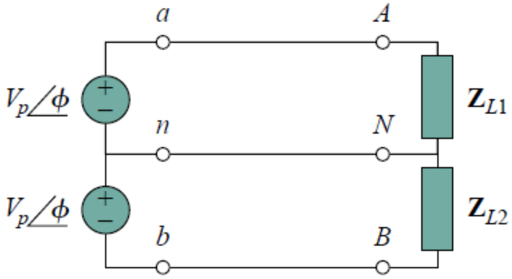
INTRODUCTION

So far in this text, we have dealt with single-phase circuits. A single phase ac power system consists of a generator connected through a pair of wires (a transmission line) to a load. Figure 12.1(a) depicts a single phase two-wire system, where V_p is the magnitude of the source voltage and φ is the phase.

What is more common in practice is a single-phase three-wire system, shown in Fig. 12.1(b). It contains two identical sources (equal magnitude and the same phase) which are connected to two loads by two outer wires and the neutral.



(a)



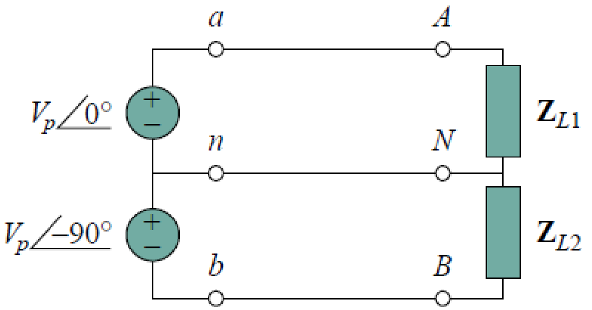
(b)

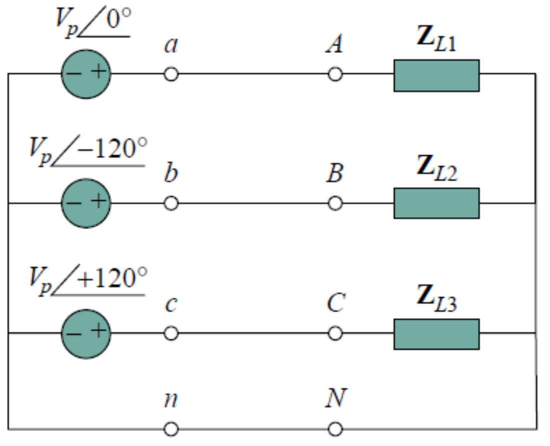
INTRODUCTION

Circuits or systems in which the ac sources operate at the same frequency but different phases are known as polyphase. Figure 12.2 shows a two-phase three-wire system, and Fig. 12.3 shows a three-phase four wire system.

As distinct from a single-phase system, a two-phase system is produced by a generator consisting of two coils placed perpendicular to each other so that the voltage generated by one lags the other by 90◦.

By the same token, a three-phase system is produced by a generator consisting of three sources having the same amplitude and frequency but out of phase with each other by 120◦.





INTRODUCTION

Three-phase systems are important for at least three reasons.

◆First, nearly all electric power is generated and distributed in three-phase, at the operating frequency of 60 Hz (or $\omega = 377$ rad/s) in the United States or 50 Hz (or $\omega = 314$ rad/s) in some other parts of the world.

When one phase or two-phase inputs are required, they are taken from the three phase system rather than generated independently.

Even when more than three phases are needed—such as in the aluminum industry, where 48 phases are required for melting purposes—they can be provided by manipulating the three phases supplied.

INTRODUCTION

- Second, the instantaneous power in a three-phase system can be constant (not pulsating), as we will see in Section 12.7. This results in uniform power transmission and less vibration of three-phase machines.
- Third, for the same amount of power, the three-phase system is more economical than the single-phase. The amount of wire required for a three-phase system is for less than that required an equivalent single-phase system.

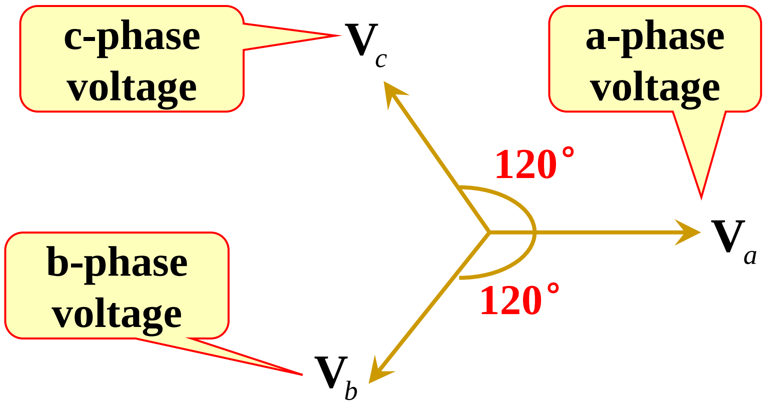
12.1 Basic concepts of three-phase circuit

1. Balanced three-phase voltages

$$v_a + v_b + v_c = 0$$

$\mathbf{V}_a$ $\mathbf{V}_b$ $\mathbf{V}_c$

$$\mathbf{V}_a + \mathbf{V}_b + \mathbf{V}_c = 0$$



Have equal amplitudes and frequencies

Each of the three voltages is 120° out of phase with each of the other two.

Choose a-phase as the reference phase.

$$\mathbf{V}_a = V_a \angle 0^o V$$

Phase sequence

- **abc (or positive) phase sequence: the b-phase voltage lags the a-phase voltage by 120º, and the c-phase voltage lags the b-phase voltage by 120º.**

$$\mathbf{V}_b = \mathbf{V}_a \angle -120^o V \qquad \mathbf{V}_c = \mathbf{V}_a \angle +120^o V$$

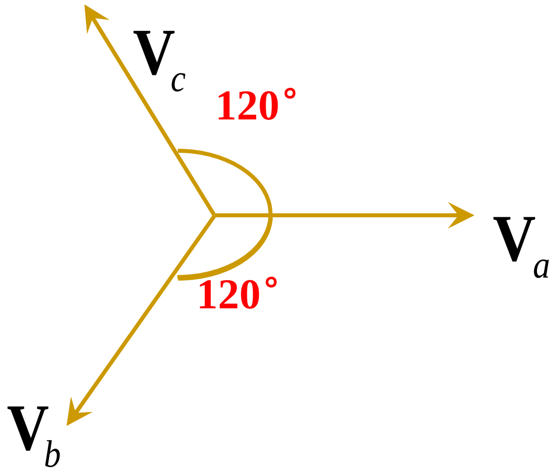
- **acb (or negative) phase sequence : the b-phase voltage leads the a-phase voltage by 120º, and the c-phase voltage leads the b-phase voltage by 120º.** $\mathbf{V}_b = \mathbf{V}_a \angle 120^o V$ $\qquad \mathbf{V}_c = \mathbf{V}_a \angle -120^o V$

Some terms

three-phase circuits

balanced three-phase circuits
three-phase source

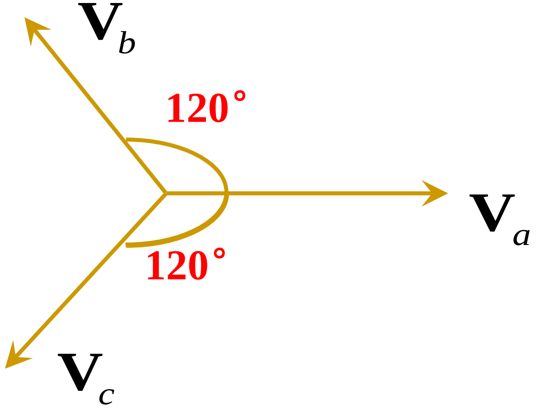
three-phase load



The phase sequence is the time order in which the voltages pass through their respective maximum values

abc (or positive) phase sequence

the b-phase voltage lags the a-phase voltage by 120º, and the c-phase voltage lags the b-phase voltage by 120º



acb (or negative) phase sequence

the b-phase voltage leads the a-phase voltage by 120º, and the c-phase voltage leads the b-phase voltage by 120º

Example 12.1

Determine the phase sequence of the set of voltages

$$v_{an} = 200\cos(\omega t + 10^\circ)$$

$$v_{bn} = 200\cos(\omega t - 230^\circ), \qquad v_{cn} = 200\cos(\omega t - 110^\circ)$$

Solution:

The voltages can be expressed in phasor form as

$$\mathbf{V}_{an} = 200\underline{/10^\circ}\text{ V}, \qquad \mathbf{V}_{bn} = 200\underline{/-230^\circ}\text{ V}, \qquad \mathbf{V}_{cn} = 200\underline{/-110^\circ}\text{ V}$$

We notice that $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$ by 120° and $\mathbf{V}_{cn}$ in turn leads $\mathbf{V}_{bn}$ by 120°. Hence, we have an *acb* sequence.

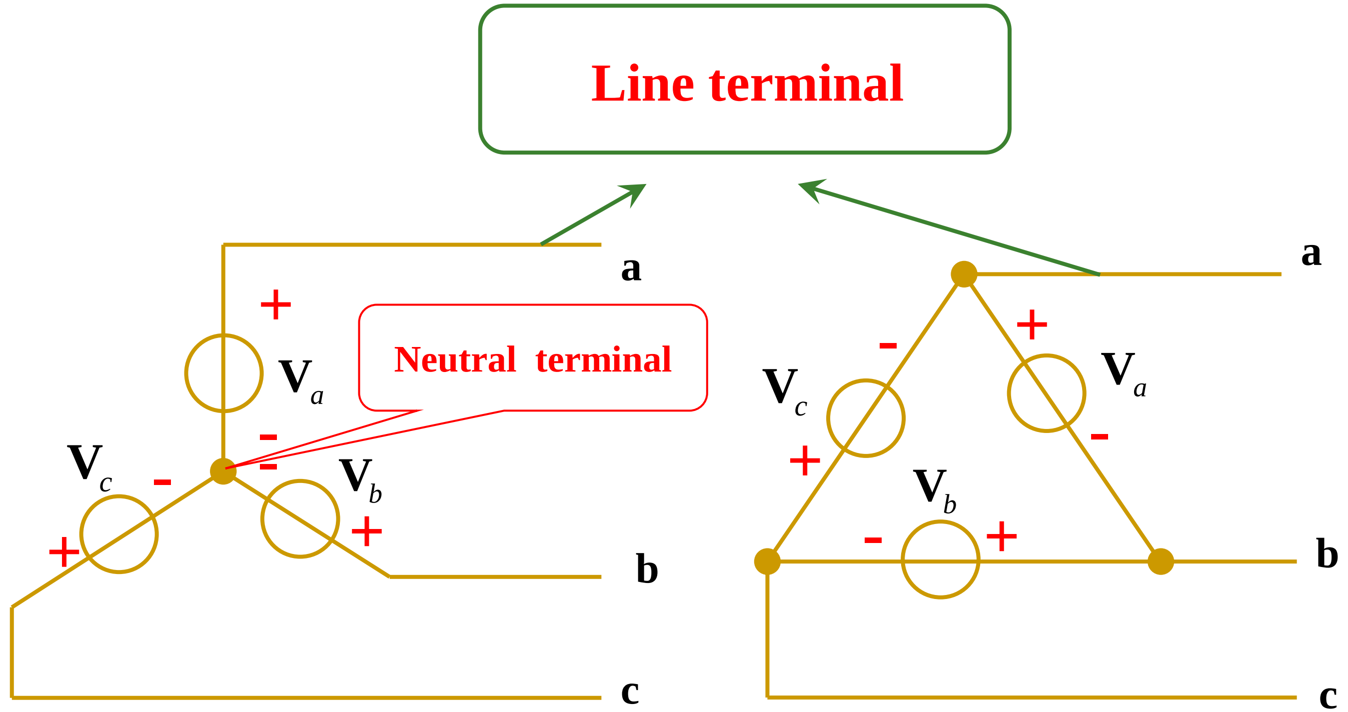
Practice Problem 12.1

Given that $\mathbf{V}_{bn} = 110\angle 30^\circ$ V, find $\mathbf{V}_{an}$ and $\mathbf{V}_{cn}$, assuming a positive (*abc*) sequence.

Answer: $110\angle 150^\circ$ V, $110\angle -90^\circ$ V.

2. Three-phase voltage sources

Two basic connection of an ideal three-phase source

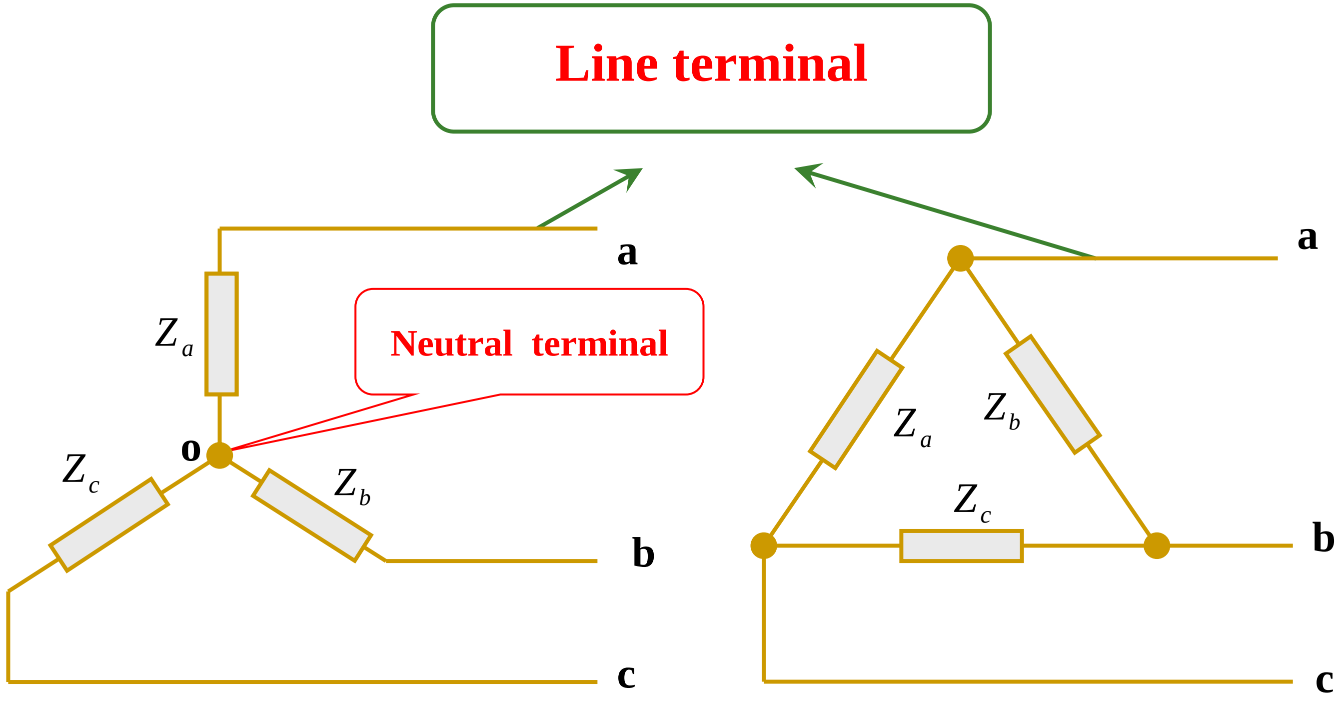


Y-connected source

△-connected source

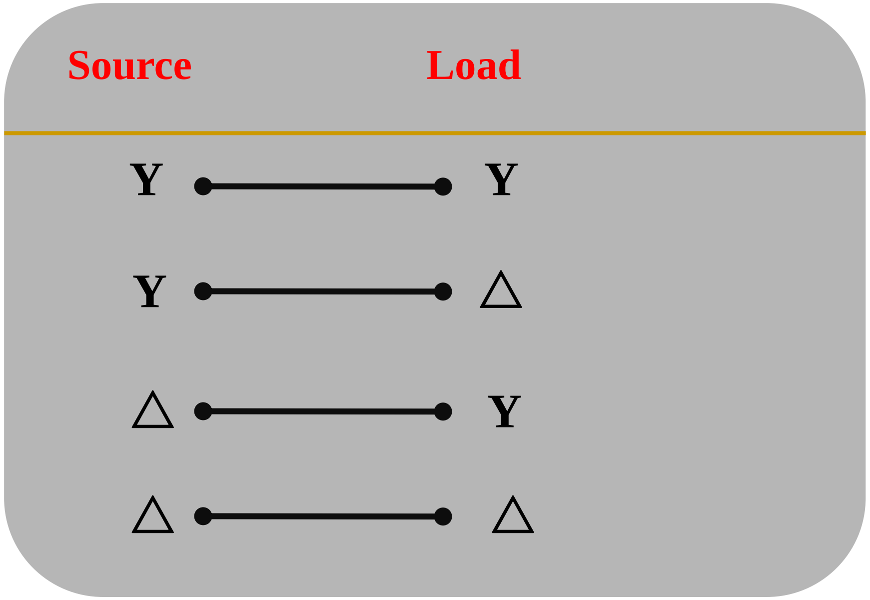
3. Three-phase loads

Two basic connection of three-phase loads



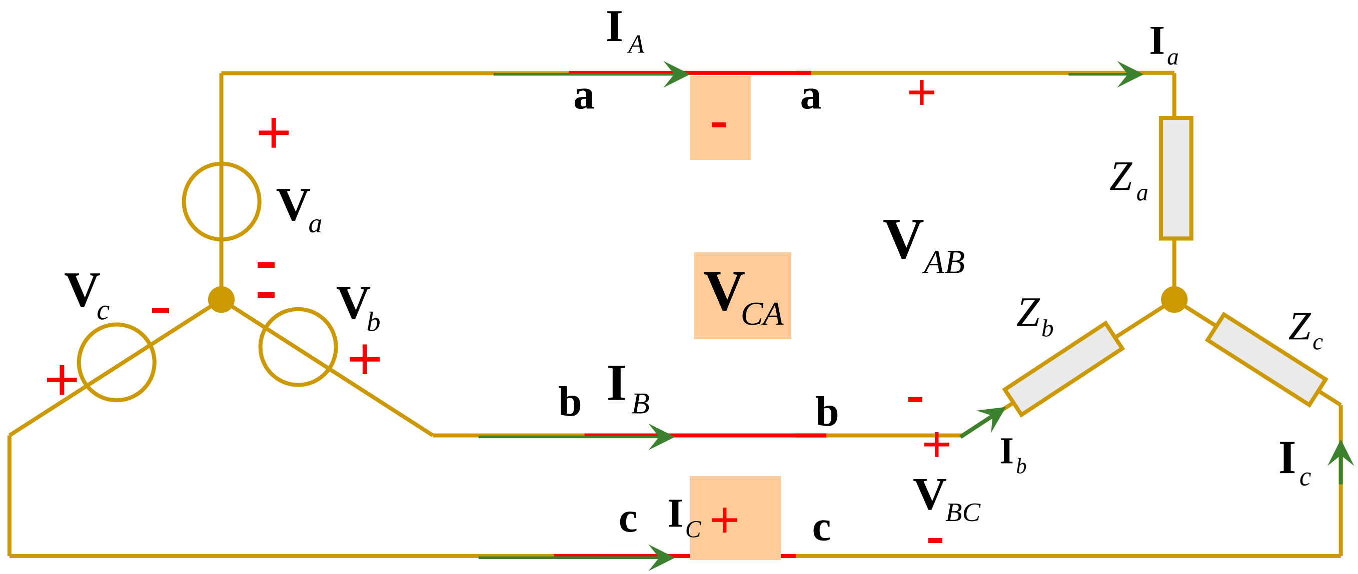
Y-connected loads **△-connected loads**

4. Three-phase circuit



Balanced Three-Phase Circuits： **balanced three-phase voltages,**

balanced three-phase loads



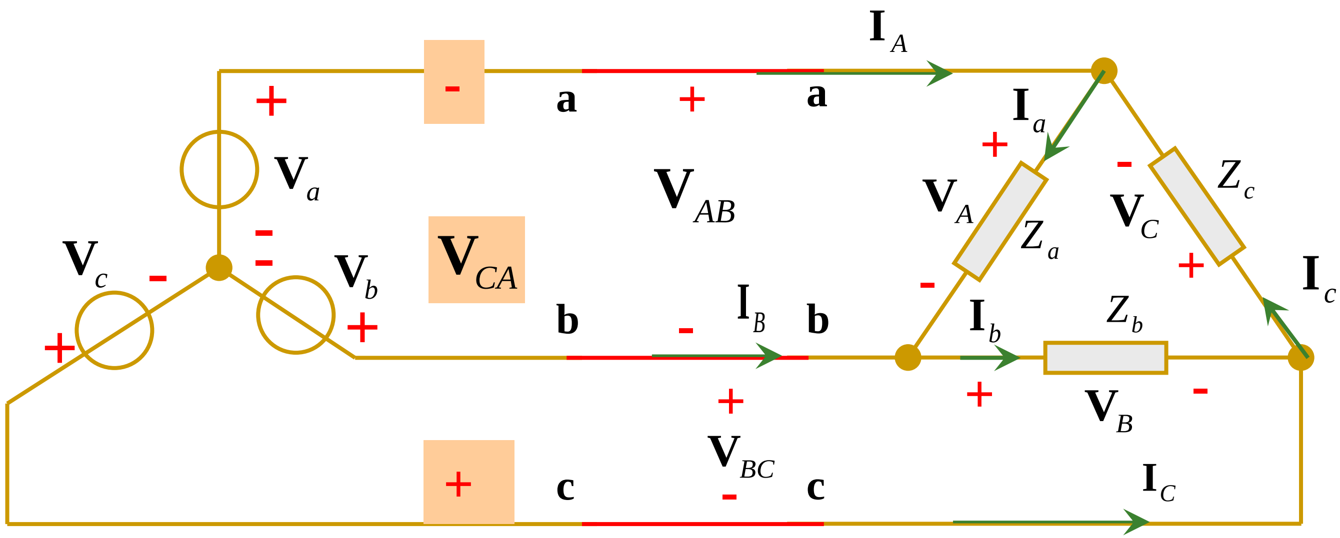
Y-Y connection

Line voltage: the voltage across any pair of lines. $\mathbf{V}_{AB}$ $\mathbf{V}_{BC}$ $\mathbf{V}_{CA}$

Phasor voltage: voltage across a single phase 。 $\mathbf{V}_a$ $\mathbf{V}_b$ $\mathbf{V}_c$

Line current: the current in a single line . $\mathbf{I}_A$ $\mathbf{I}_B$ $\mathbf{I}_C$

Phasor current : the current in a single phase 。 $\mathbf{I}_a$ $\mathbf{I}_b$ $\mathbf{I}_c$



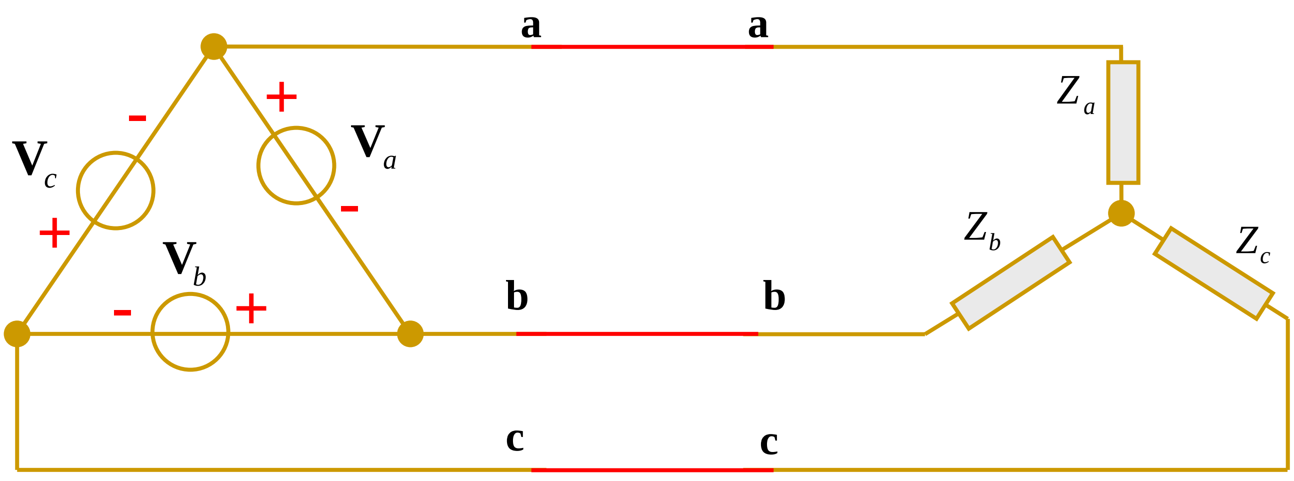
Y-△ connection

Phase current $\mathbf{I}_a$ $\mathbf{I}_b$ $\mathbf{I}_c$

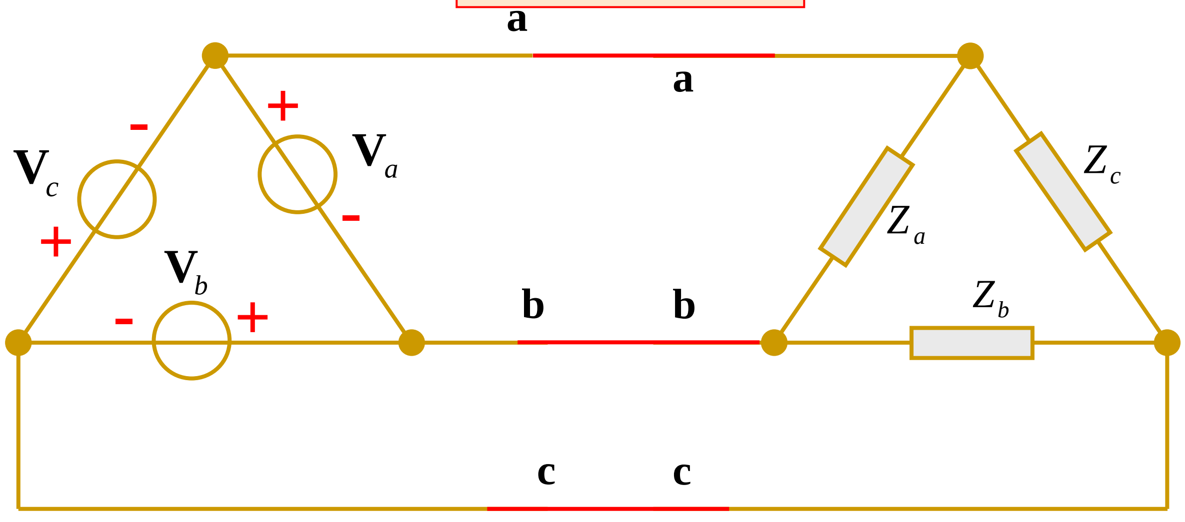
Line current $\mathbf{I}_A$ $\mathbf{I}_B$ $\mathbf{I}_C$

Phase voltage $\mathbf{V}_A$ $\mathbf{V}_B$ $\mathbf{V}_C$

Line voltage $\mathbf{V}_{AB}$ $\mathbf{V}_{BC}$ $\mathbf{V}_{CA}$



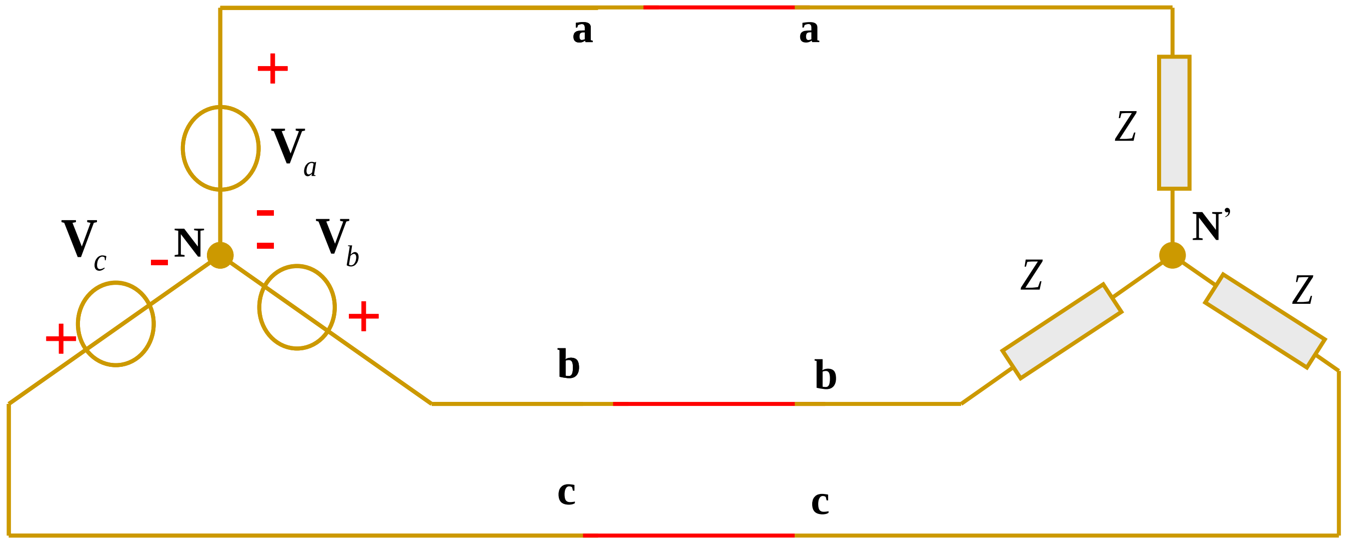
△ -Y connection



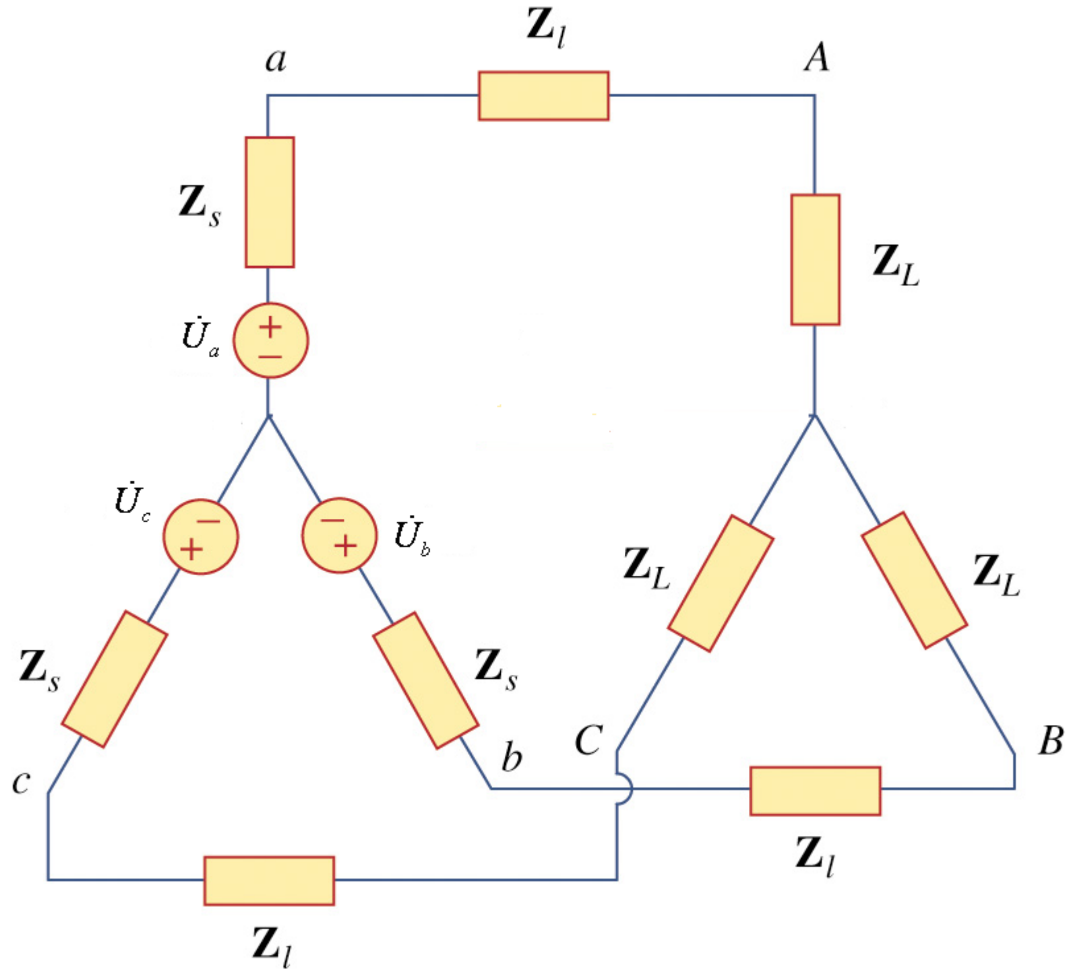
△ - △ connection

12.2 Analysis of the Wye-Wye (Y-Y) Circuit

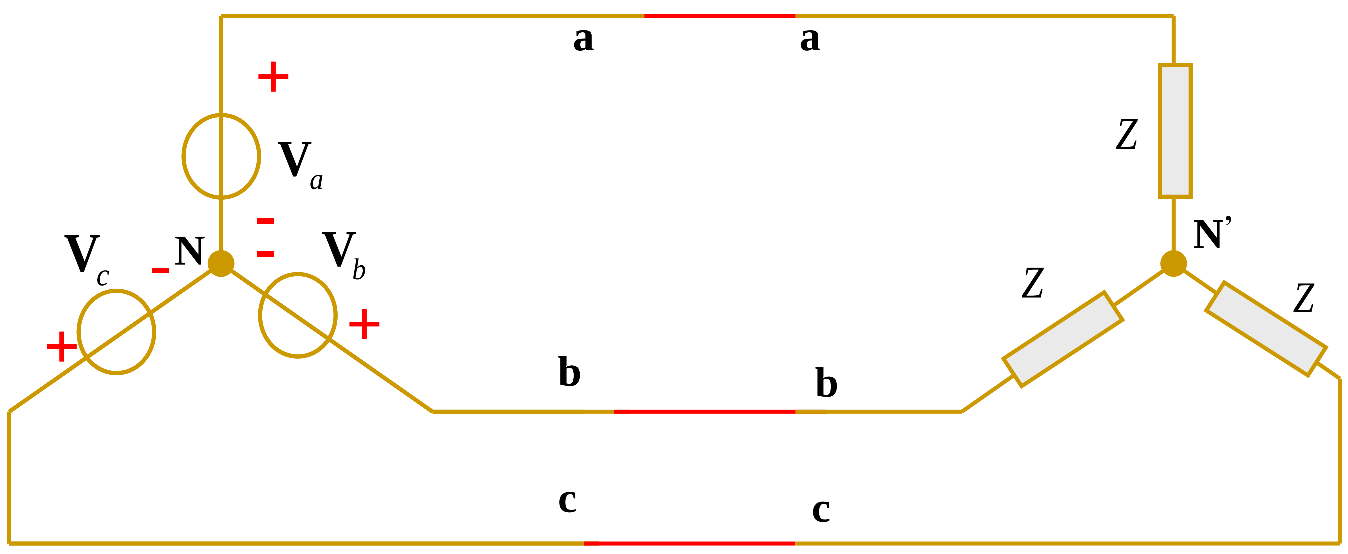
A balanced Y-Y system is a three-phase system with a balanced Y-connected source and a balanced Y-connected load.



Y-Y connection (three-phasor three-wire circuit)



$\mathbf{Z}_l$
a
A
$\mathbf{Z}_s$
$\mathbf{Z}_L$
$\dot{U}_a$
$\dot{U}_c$
$\dot{U}_b$
$\mathbf{Z}_L$
$\mathbf{Z}_L$
$\mathbf{Z}_s$
$\mathbf{Z}_s$
C
B
b
c
$\mathbf{Z}_l$
$\mathbf{Z}_l$

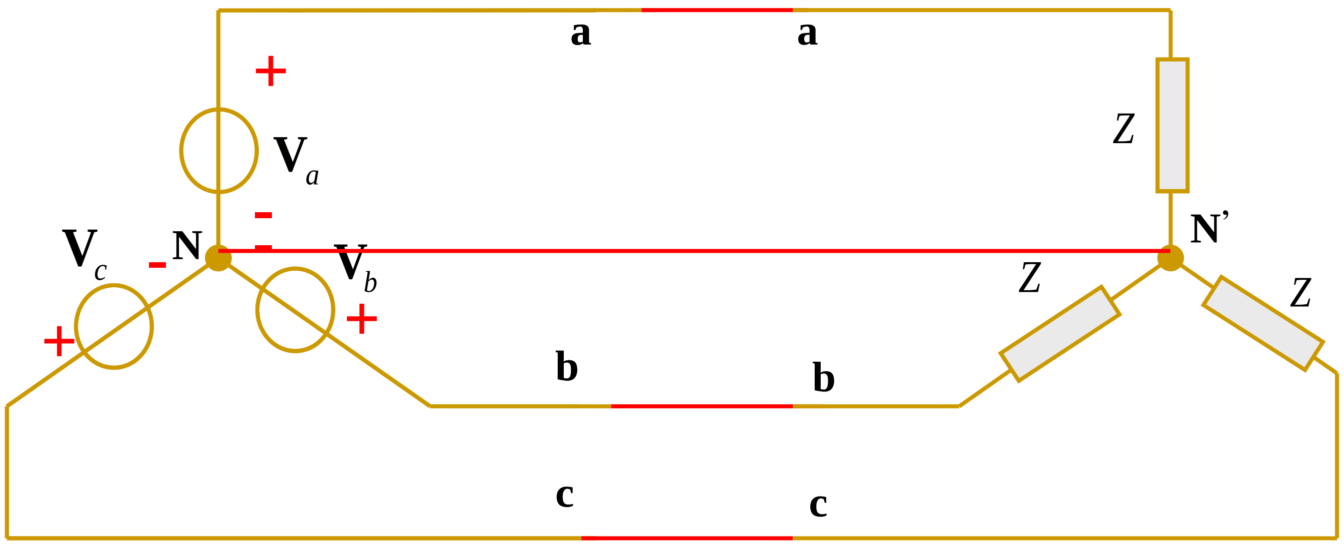


Apply the nodal analysis and select the node N' as reference node, we have

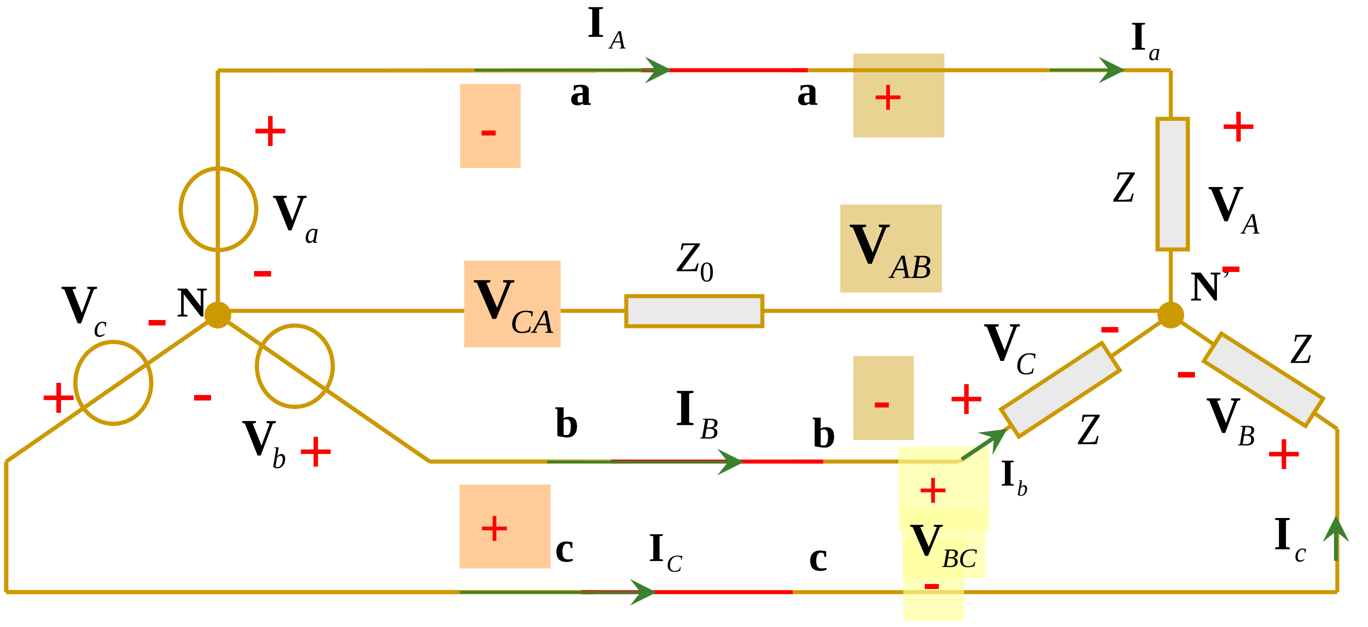
$$\mathbf{V}_{NN'} = -\frac{\mathbf{V}_a / Z + \mathbf{V}_b / Z + \mathbf{V}_c / Z}{1/Z + 1/Z + 1/Z} = -\frac{1}{3}(\mathbf{V}_a + \mathbf{V}_b + \mathbf{V}_c) = 0$$

Thus we can also get the following equivalent circuit

Thus we can get the following equivalent circuit



$$\mathbf{V}_{NN'} = -\frac{\mathbf{V}_a / Z + \mathbf{V}_b / Z + \mathbf{V}_c / Z}{1/Z + 1/Z + 1/Z} = -\frac{1}{3}(\mathbf{V}_a + \mathbf{V}_b + \mathbf{V}_c) = 0$$



Y-Y connection (three-phasor four-wire circuit)

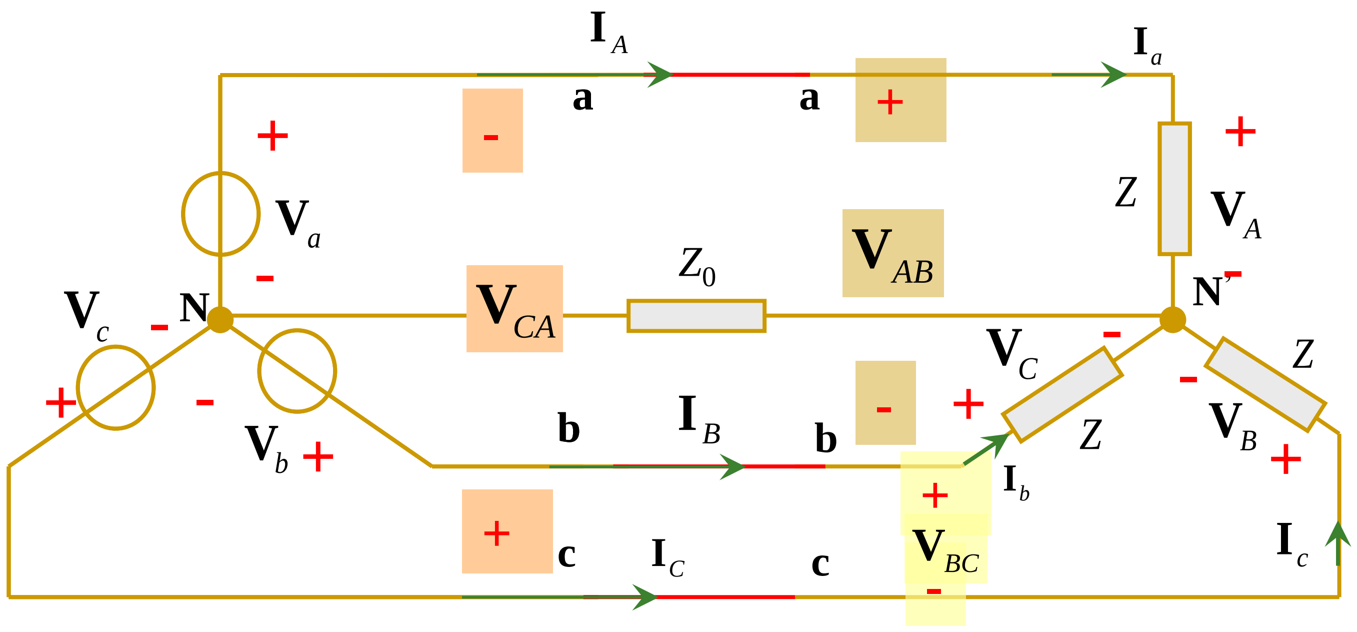
Suppose $\mathbf{V}_a$ $\mathbf{V}_b$ $\mathbf{V}_c$ are known and in the abc sequences, try to calculate

(1) Phasor currents $\mathbf{I}_a$ $\mathbf{I}_b$ $\mathbf{I}_c$

(2) Line currents $\mathbf{I}_A$ $\mathbf{I}_B$ $\mathbf{I}_C$

(3) Phasor voltage $\mathbf{V}_A$ $\mathbf{V}_B$ $\mathbf{V}_C$

(4) Line voltage $\mathbf{V}_{AB}$ $\mathbf{V}_{BC}$ $\mathbf{V}_{CA}$



Apply the nodal analysis, we have

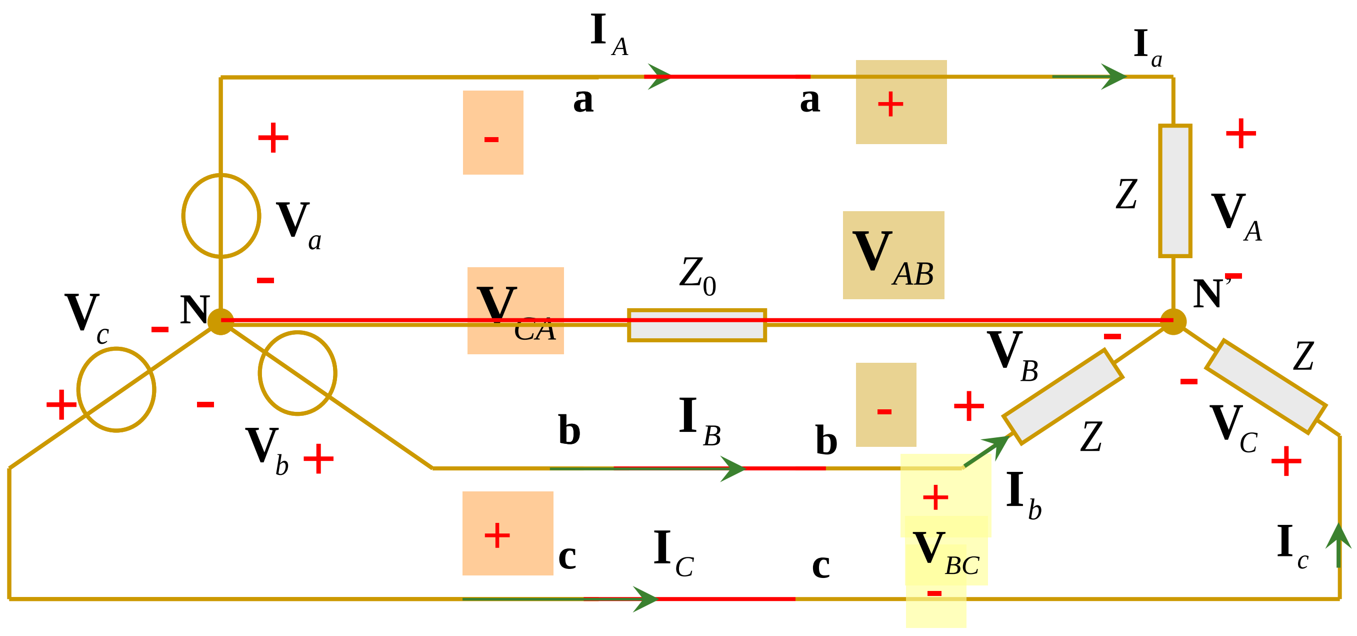
$$\mathbf{V}_{NN'} = -\frac{\mathbf{V}_a/Z+\mathbf{V}_b/Z+\mathbf{V}_c/Z}{1/Z+1/Z+1/Z+1/Z_0} = -\frac{\mathbf{V}_a+\mathbf{V}_b+\mathbf{V}_c}{1+1+1+Z/Z_0}$$

Since

$$\mathbf{V}_a+\mathbf{V}_b+\mathbf{V}_c=0 \qquad \mathbf{V}_{NN'}=0$$

Thus we can get the following equivalent circuit

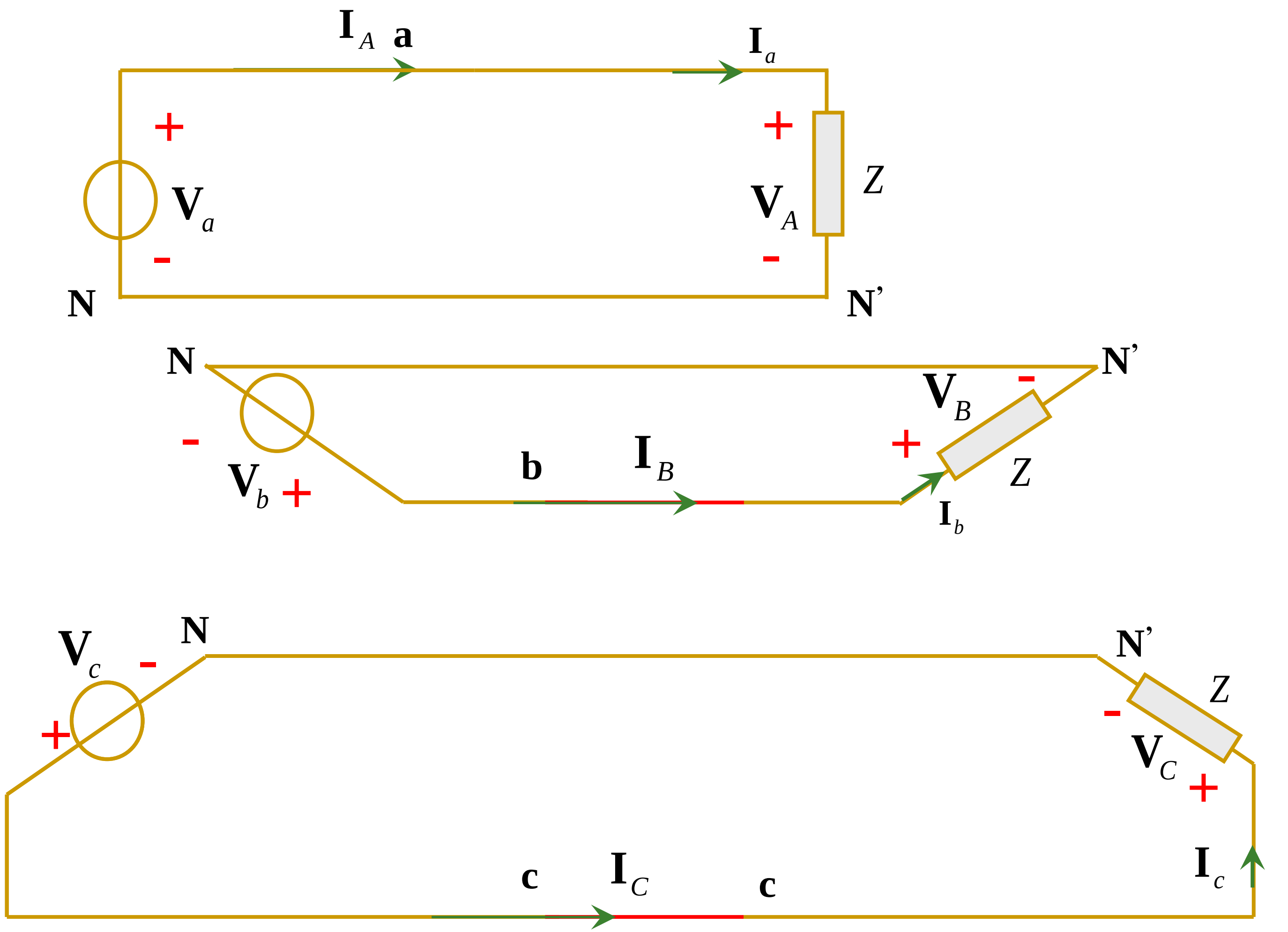
Thus we can get the following equivalent circuit



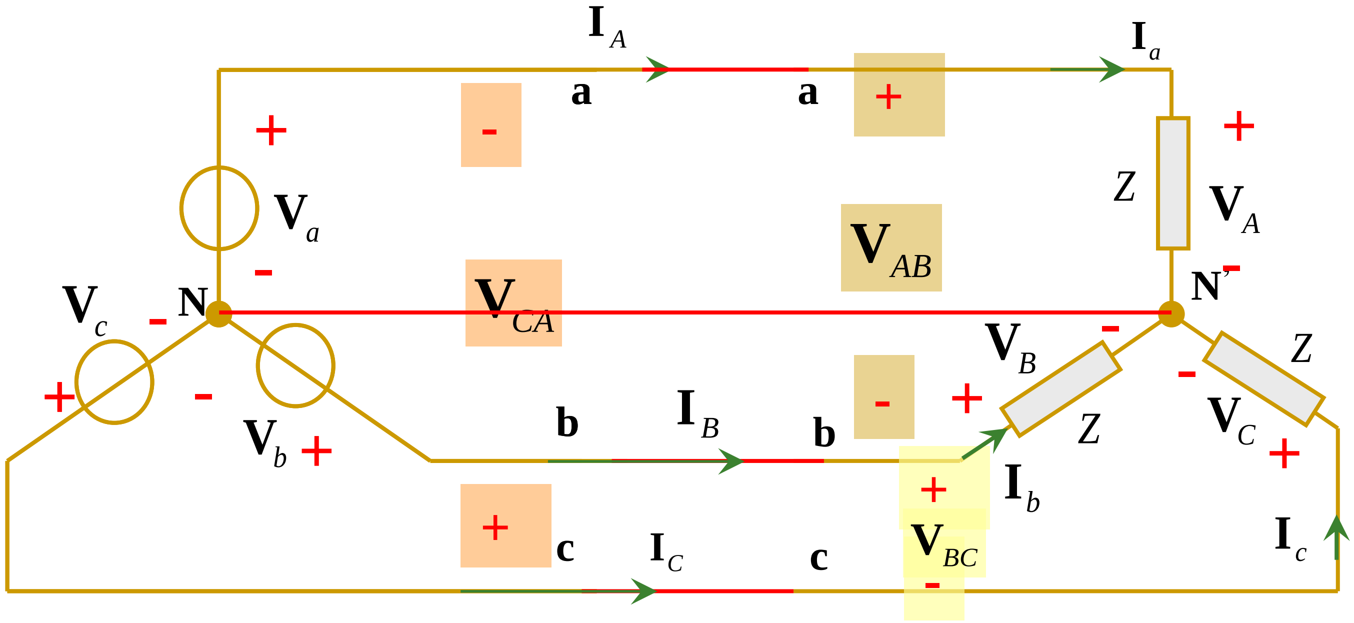
A three-phase circuit changes into three single phase circuits now.

It is easy to find phase currents and phase voltage.

$$\mathbf{V}_{NN'} = 0$$



I_A a
I_a
+
V_a
-
+
V_A
-
Z
N
N'
N
N'
V_B
-
-
+
V_b +
b
I_B
Z
I_b
V_c -
N
N'
Z
+
-
V_C
+
c
I_C
c
I_c



phase voltages:

$$\mathbf{V}_A = \mathbf{V}_a$$

$$\mathbf{V}_B = \mathbf{V}_b = \mathbf{V}_a\angle -120^0 = \mathbf{V}_A\angle -120^0$$

$$\mathbf{V}_C = \mathbf{V}_c = \mathbf{V}_a\angle +120^0 = \mathbf{V}_A\angle +120^0$$

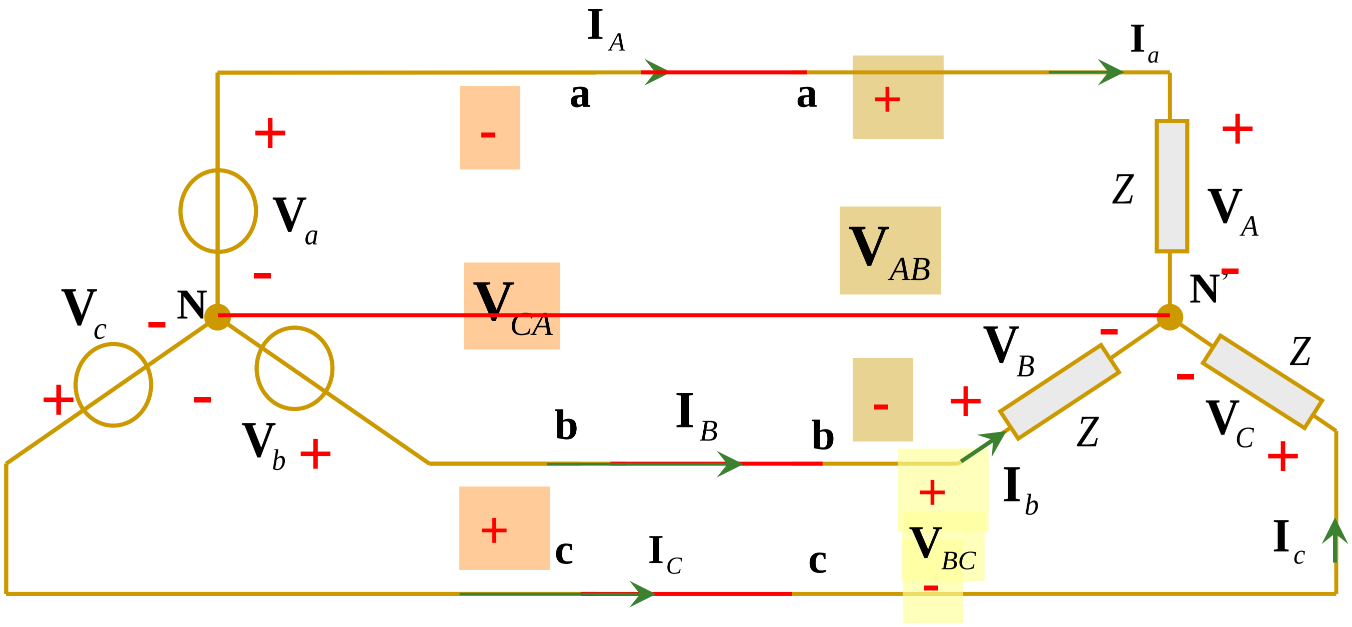
phase currents:

$$\mathbf{I}_a = \frac{\mathbf{V}_A}{Z}$$

$$\mathbf{I}_b = \frac{\mathbf{V}_B}{Z} = \frac{\mathbf{V}_A\angle -120^0}{Z} = \mathbf{I}_a\angle -120^0$$

$$\mathbf{I}_c = \frac{\mathbf{V}_C}{Z} = \frac{\mathbf{V}_A\angle +120^0}{Z} = \mathbf{I}_a\angle +120^0$$

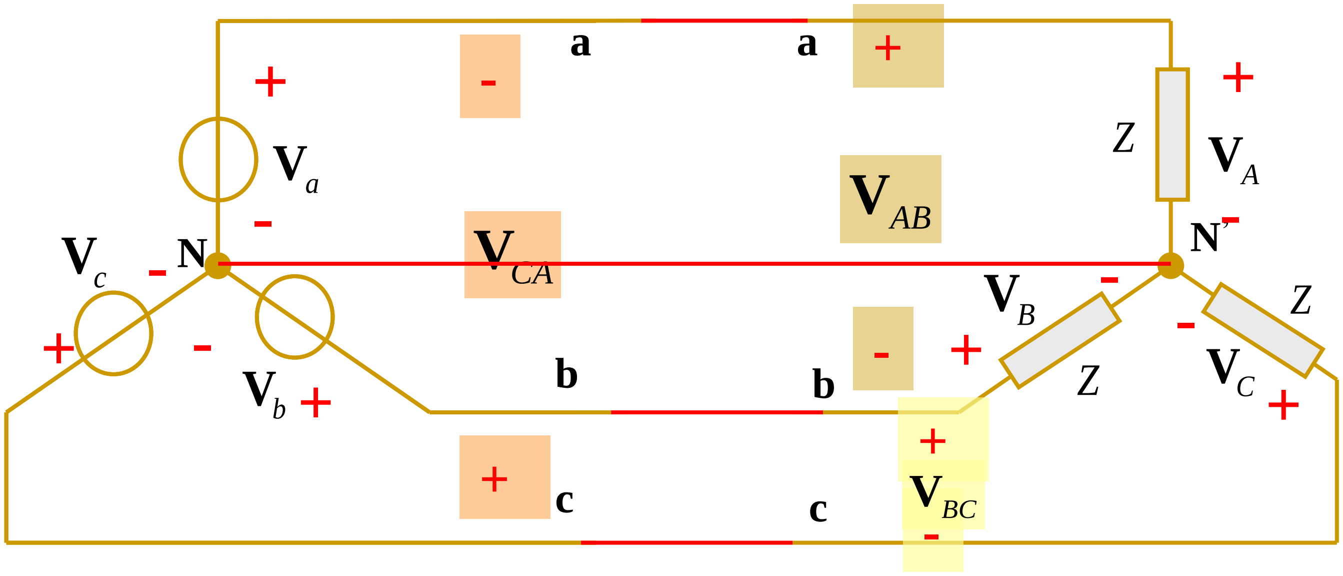
To get phasor voltage and phasor currents: ***just calculate one phase voltage and phase current***, then we can write other phasor voltages and phasor currents directly.

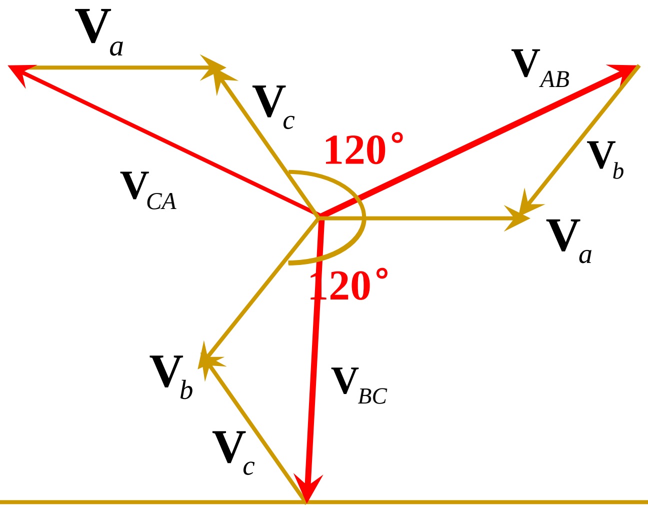


line currents: $\mathbf{I}_A = \mathbf{I}_a$ $\quad \mathbf{I}_B = \mathbf{I}_b = \mathbf{I}_A \angle -120^0$ $\quad \mathbf{I}_C = \mathbf{I}_c = \mathbf{I}_A \angle +120^0$

To get line currents: ***just calculate one phase current***, then we can write all of the line currents directly.

line voltages:





$$\mathbf{V}_{AB} = \mathbf{V}_a - \mathbf{V}_b \Rightarrow \mathbf{V}_a = \mathbf{V}_b + \mathbf{V}_{AB}$$

So $\mathbf{V}_{AB} = \sqrt{3}\mathbf{V}_a\angle 30^0 = \sqrt{3}\mathbf{V}_A\angle 30^0$

We can also obtain the following result in the similar way

$$\mathbf{V}_{BC} = \sqrt{3}\mathbf{V}_b\angle 30^0 = \sqrt{3}\mathbf{V}_B\angle 30^0$$

$$\mathbf{V}_{CA} = \sqrt{3}\mathbf{V}_c\angle 30^0 = \sqrt{3}\mathbf{V}_C\angle 30^0$$

Conclusion:

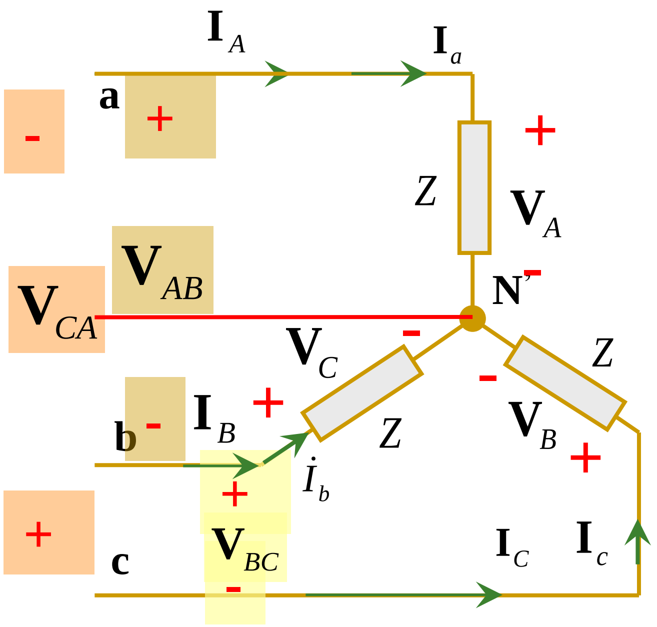
- **In the Y-Y circuit, $I_l = I_p$.**

- **In the balanced Y-Y circuit,** $V_l = \sqrt{3} V_p$

$$\varphi_{AB} = \varphi_A + 30^\circ$$

$$\varphi_{BC} = \varphi_B + 30^\circ$$

$$\varphi_{CA} = \varphi_C + 30^\circ$$



Steps for analysis the balanced Y-Y circuit

- Draw the single-phase equivalent circuit (usually a-phase circuit)
- Calculate its phase voltage and phase current.
- Write other phase voltages and phase currents.
- Write line current and line voltage.

Example 1. In one balanced three-phase Y-Y circuits, Z=6+j8Ω,

$$v_{AB}(t) = 380\cos(\omega t + 30°)V$$

Calculate the currents.

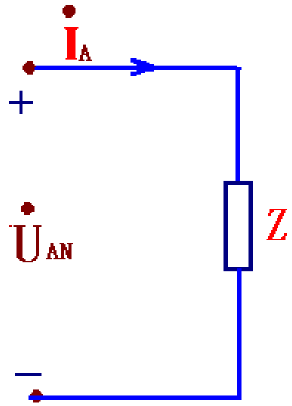
Solution:

Since

$$\mathbf{V}_{AB} = 380\angle 30° V$$

$$\mathbf{V}_{A} = \frac{380\angle 30°}{\sqrt{3}}\angle -30° = 220\angle 0° V$$

Draw the a-phase circuit



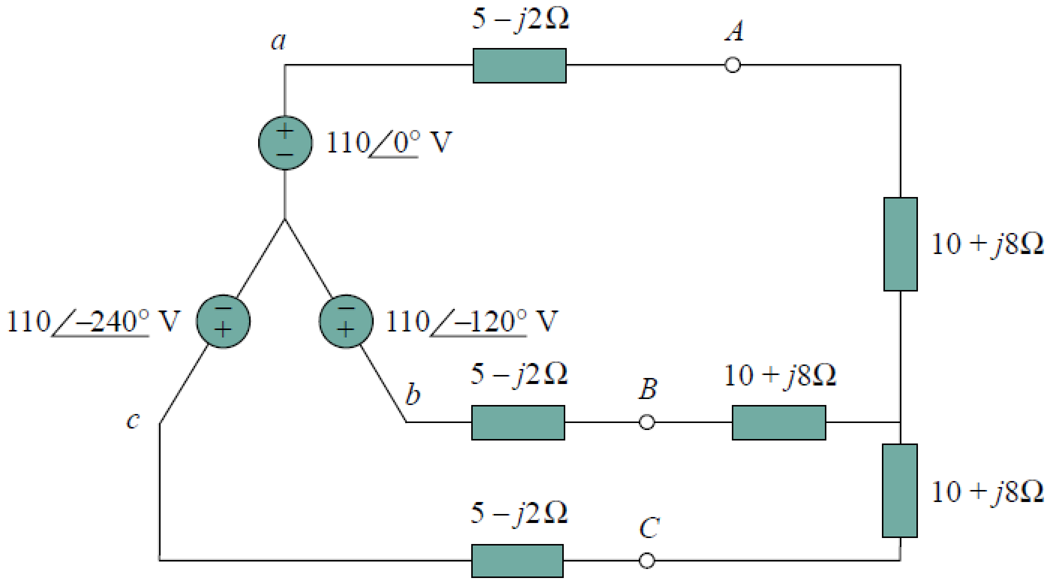
$$\therefore \quad \mathbf{I}_A = \frac{220\angle 0°}{6 + j8} = 22\angle -53.1° A$$

$$\mathbf{I}_B = 22\angle -173.1° A \qquad \mathbf{I}_C = 22\angle 66.9° A$$

How to get the voltage?

EXAMPLE 9.2

Calculate the line currents in the three-wire Y-Y system of Fig. 12.13.



Solution:

The three-phase circuit in Fig. 12.13 is balanced; we may replace it with its single-phase equivalent circuit such as in Fig. 12.12. We obtain $\mathbf{I}_a$ from the single-phase analysis as

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}$$

$$\mathbf{Z}_Y = (5 - j2) + (10 + j8) = 15 + j6 = 16.155\underline{/21.8^\circ}.$$

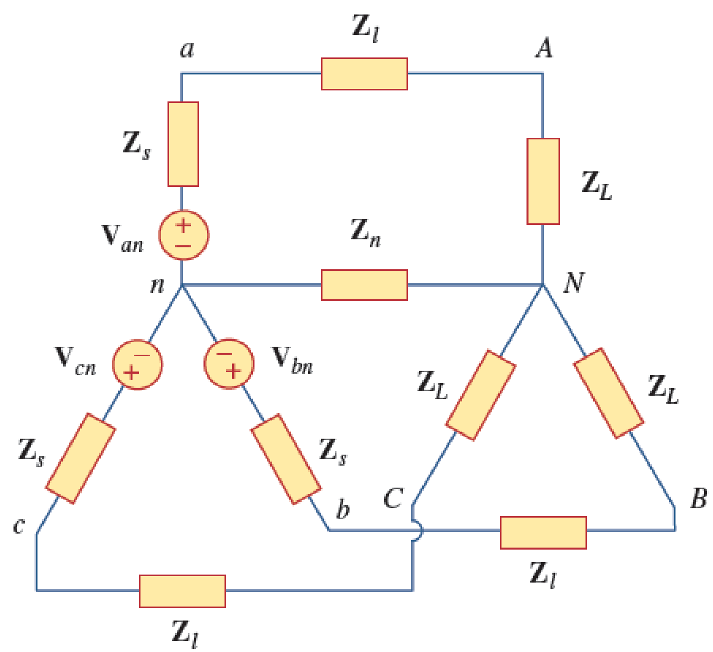
$$\mathbf{I}_a = \frac{110\underline{/0^\circ}}{16.155\underline{/21.8^\circ}} = 6.81\underline{/-21.8^\circ} \text{ A}$$

$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 6.81\underline{/-141.8^\circ} \text{ A}$$

$$\mathbf{I}_c = \mathbf{I}_a\underline{/-240^\circ} = 6.81\underline{/-261.8^\circ} \text{ A} = 6.81\underline{/98.2^\circ} \text{ A}$$

Practice Problem 12.2

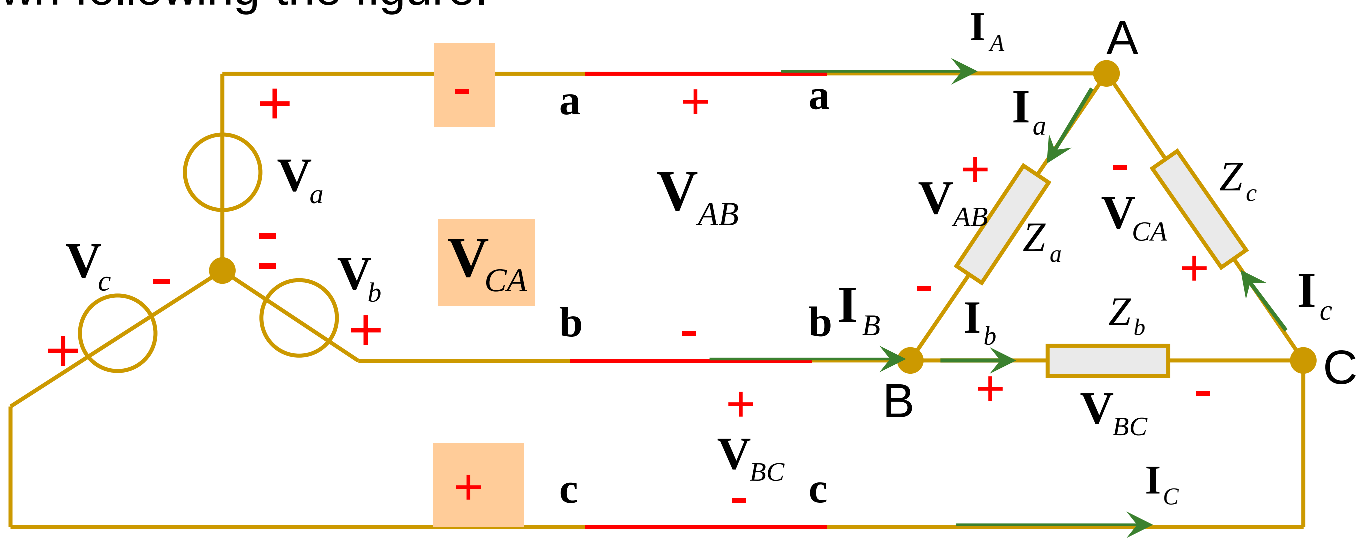
A Y-connected balanced three-phase generator with an impedance of $0.4 + j0.3\ \Omega$ per phase is connected to a Y-connected balanced load with an impedance of $24 + j19\ \Omega$ per phase. The line joining the generator and the load has an impedance of $0.6 + j0.7\ \Omega$ per phase. Assuming a positive sequence for the source voltages and that $\mathbf{V}_{an} = 120\underline{/30^\circ}$ V, find: (a) the line voltages, (b) the line currents.



Answer: (a) $207.8\underline{/60^\circ}$ V, $207.8\underline{/-60^\circ}$ V, $207.8\underline{/-180^\circ}$ V, (b) $3.75\underline{/-8.66^\circ}$ A, $3.75\underline{/-128.66^\circ}$ A, $3.75\underline{/111.34^\circ}$ A.

12.3 Analysis of the Wye-Delta (Y-△) Circuit

The balanced Y-Δ system consists of a balanced Y-connected source feeding a balanced Δ-connected load. The balanced Y-Δ system is shown following the figure.

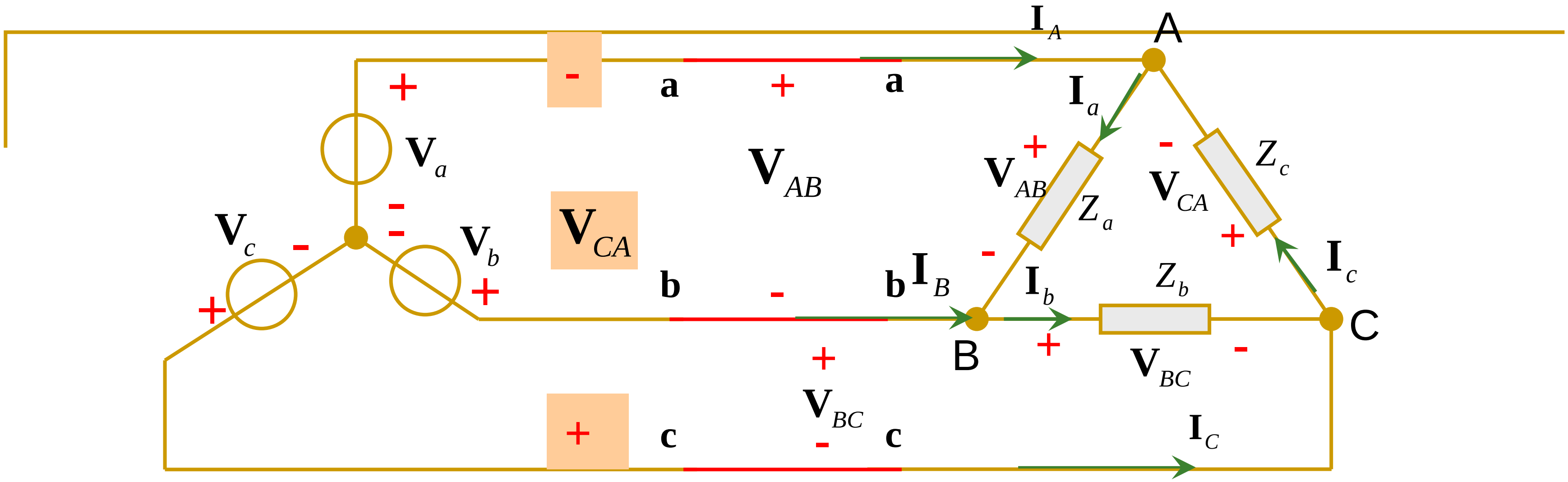


Assuming the positive phase sequence , the source voltages are again

$$\mathbf{V}_a = V_p \angle 0^o \qquad \mathbf{V}_b = \mathbf{V}_a \angle -120^0 \qquad \mathbf{V}_c = \mathbf{V}_a \angle +120^0$$

The line voltages and phase voltages are

$$\mathbf{V}_{ab} = \sqrt{3}\mathbf{V}_a \angle 30^o = \mathbf{V}_{AB} \qquad \mathbf{V}_{bc} = \sqrt{3}\mathbf{V}_b \angle 30^o = \mathbf{V}_{BC} \qquad \mathbf{V}_{ca} = \sqrt{3}\mathbf{V}_c \angle 30^o = \mathbf{V}_{CA}$$



showing that the line voltages are equal to the voltages across the load impedances for this system configuration.

From these voltages, we can obtain th phase currents as

$$\mathbf{I}_a = \frac{\mathbf{V}_{AB}}{Z_a} = \frac{\mathbf{V}_{AB}}{Z} \qquad \mathbf{I}_b = \frac{\mathbf{V}_{BC}}{Z_b} = \frac{\mathbf{V}_{AB}\angle -120^O}{Z} \qquad \mathbf{I}_c = \frac{\mathbf{V}_{CA}}{Z_c} = \frac{\mathbf{V}_{AB}\angle 120^O}{Z}$$

These currents have the same magnitude but are out of phase with each other by 120^o.

The line currents are obtained from the phase currents by applying KCL at nodes A,B, and C. Thus

$$\mathbf{I}_A = \mathbf{I}_a - \mathbf{I}_c \qquad \mathbf{I}_B = \mathbf{I}_b - \mathbf{I}_a \qquad \mathbf{I}_C = \mathbf{I}_c - \mathbf{I}_b$$

$$= \sqrt{3}\mathbf{I}_a \angle -30^o \qquad = \sqrt{3}\mathbf{I}_b \angle -30^o \qquad = \sqrt{3}\mathbf{I}_c \angle -30^O$$

$$\mathbf{I}_A = \sqrt{3}\mathbf{I}_a \angle -30^o \qquad \mathbf{I}_B = \sqrt{3}\mathbf{I}_b \angle -30^o \qquad \mathbf{I}_C = \mathbf{I}_c \angle -30^O$$

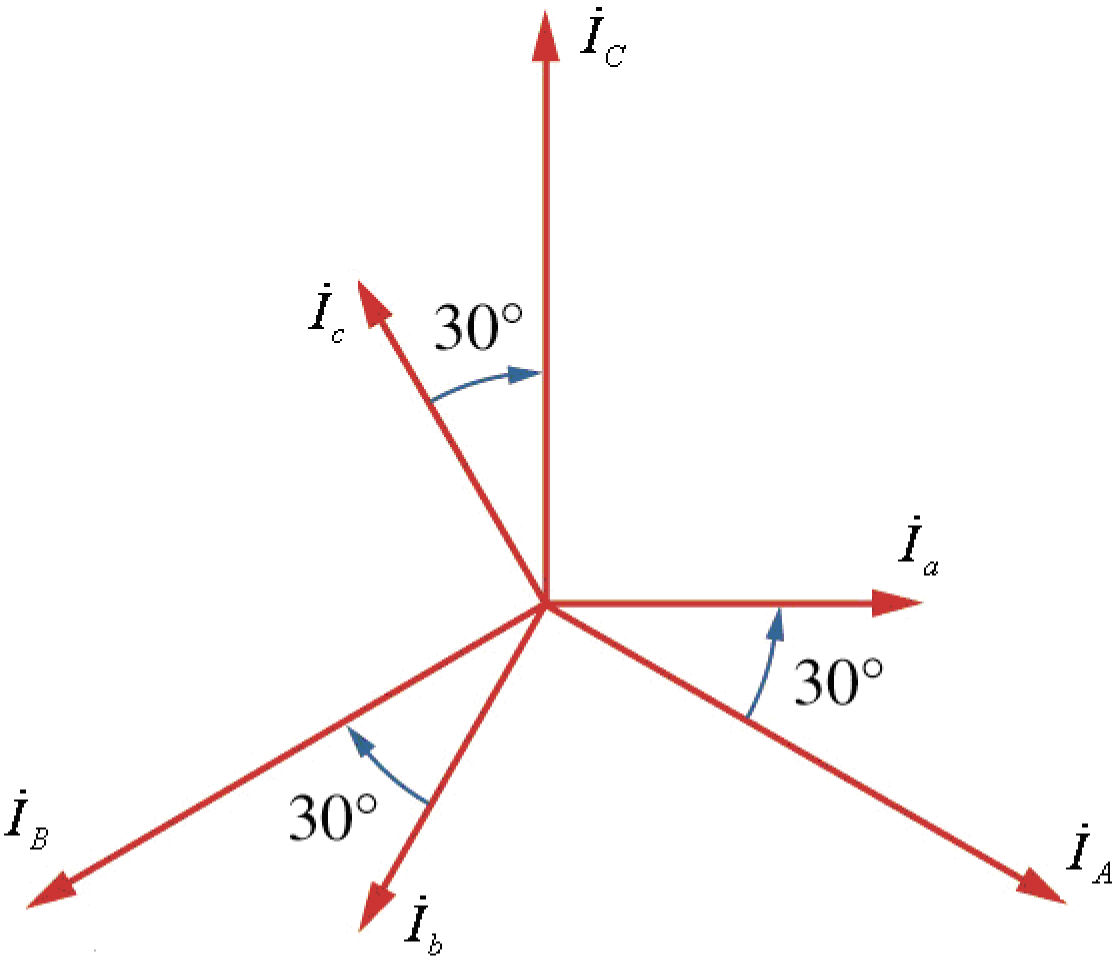
Showing that the magnitude of I_L of the line current is $\sqrt{3}$ times the magnitude I_p of the phase current, or

$$I_L = \sqrt{3} I_p$$

where

$$I_L = |\mathbf{I}_A| = |\mathbf{I}_B| = |\mathbf{I}_C| \qquad I_P = |\mathbf{I}_a| = |\mathbf{I}_b| = |\mathbf{I}_c|$$

Also, the line currents lag the corresponding phase currents by 30^o



Phasor diagram illustrating the relationship between phase and line currents.

EXAMPLE 12.3

A balanced abc-sequence Y-connected source with $\mathbf{V}_a = 100\angle 10^o$ is connected to Δ-connected balanced load $(8 + j4)\Omega$ per phase. Calculate the phase and line currents.

Solution:

This can be solved in two ways.

METHOD 1 The load impedance is $\mathbf{Z}_\Delta = 8 + j4 = 8.944\angle 26.57^\circ\ \Omega$

If the phase voltage $\mathbf{V}_{an} = 100\angle 10^\circ$, then the line voltage is

$$\mathbf{V}_{ab} = \mathbf{V}_{an}\sqrt{3}\angle 30^\circ = 100\sqrt{3}\angle 10^\circ + 30^\circ = \mathbf{V}_{AB} = 173.2\angle 40^\circ\ \text{V}$$

The phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{173.2\angle 40^\circ}{8.944\angle 26.57^\circ} = 19.36\angle 13.43^\circ\ \text{A}$$

$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\angle -120^\circ = 19.36\angle -106.57^\circ\ \text{A}$$

$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\angle +120^\circ = 19.36\angle 133.43^\circ\ \text{A}$$

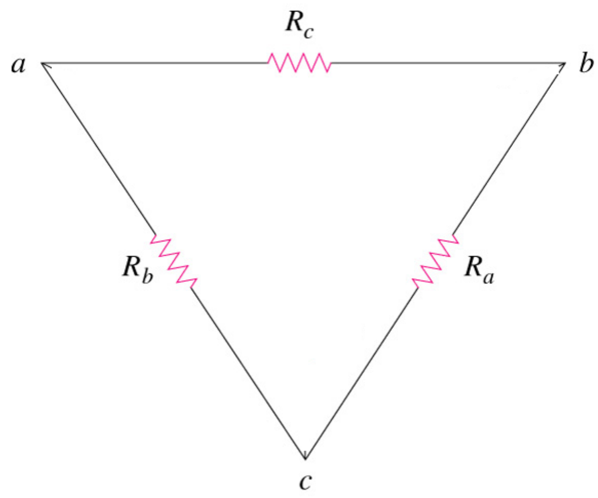
The line currents are

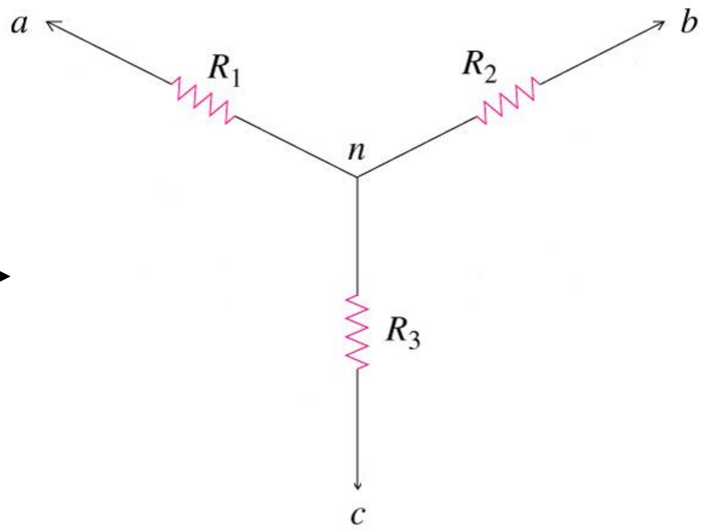
$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ = \sqrt{3}(19.36)\angle 13.43^\circ - 30^\circ$$
$$= 33.53\angle -16.57^\circ\ \text{A}$$

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 33.53\angle -136.57^\circ\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ = 33.53\angle 103.43^\circ\ \text{A}$$

Delta -> Wye





$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} \qquad R_2 = \frac{R_c R_a}{R_a + R_b + R_c} \qquad R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

EXAMPLE 12.3

A balanced abc-sequence Y-connected source with $\mathbf{V}_a = 100\angle 10^o$ is connected to Δ-connected balanced load $(8 + j4)\Omega$ per phase. Calculate the phase and line currents.

METHOD 2 Alternatively, using single-phase analysis,

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_\Delta/3} = \frac{100\angle 10^\circ}{2.981\angle 26.57^\circ} = 33.54\angle -16.57^\circ \text{ A}$$

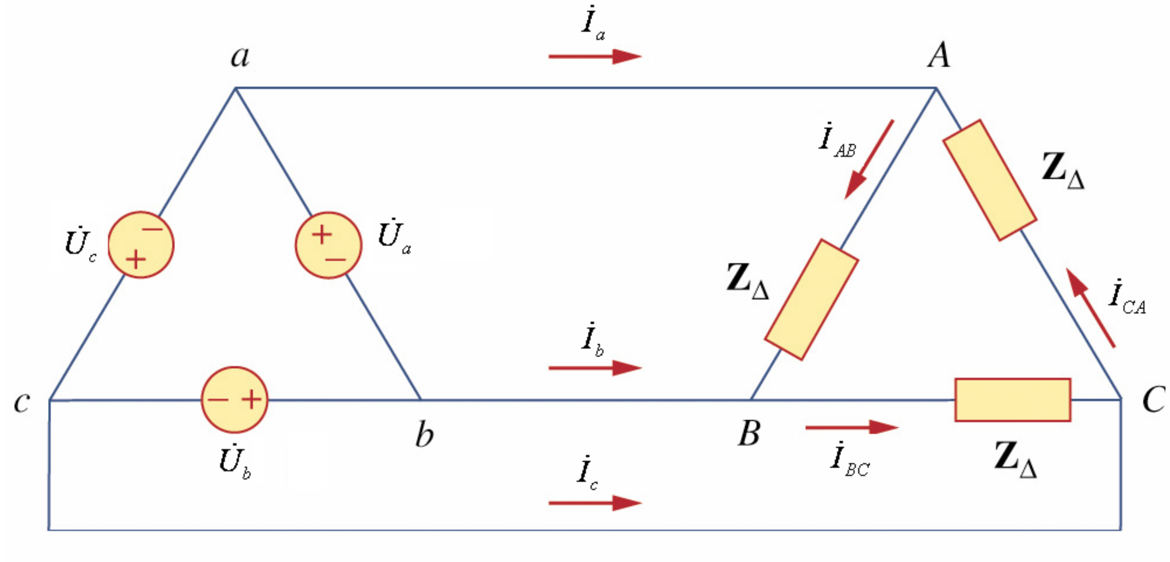
as above. Other line currents are obtained using the *abc* phase sequence.

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 33.53\angle -136.57^\circ \text{ A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ = 33.53\angle 103.43^\circ \text{ A}$$

12.4 Balance Delta-delta Connection

A balanced Δ- Δsystem is one in which both the balanced source and balanced load are Δ-connected



A balanced Δ- Δ connection

$$\mathbf{V}_{ab} = V_p\underline{/0^\circ}$$

$$\mathbf{V}_{bc} = V_p\underline{/-120^\circ}, \qquad \mathbf{V}_{ca} = V_p\underline{/+120^\circ}$$

$$\mathbf{V}_{ab} = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \mathbf{V}_{BC}, \qquad \mathbf{V}_{ca} = \mathbf{V}_{CA}$$

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_\Delta}$$

$$\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_\Delta}$$

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC}$$

$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\underline{/-30^\circ} \qquad I_L = \sqrt{3}I_p$$

Example 12.4

A balanced Δ-connected load having an impedance $20 - j15\ \Omega$ is connected to a Δ-connected, positive-sequence generator having $\mathbf{V}_{ab} = 330\underline{/0^\circ}$ V. Calculate the phase currents of the load and the line currents.

Solution:

The load impedance per phase is

$$\mathbf{Z}_\Delta = 20 - j15 = 25\underline{/-36.87^\circ}\ \Omega$$

Since $\mathbf{V}_{AB} = \mathbf{V}_{ab}$, the phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{330\underline{/0^\circ}}{25\underline{/-36.87}} = 13.2\underline{/36.87^\circ}\ \text{A}$$

$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\underline{/-120^\circ} = 13.2\underline{/-83.13^\circ}\ \text{A} \qquad \mathbf{I}_{CA} = \mathbf{I}_{AB}\underline{/+120^\circ} = 13.2\underline{/156.87^\circ}\ \text{A}$$

For a delta load, the line current always lags the corresponding phase current by 30° and has a magnitude $\sqrt{3}$ times that of the phase current. Hence, the line currents are

$$\begin{aligned}\mathbf{I}_a &= \mathbf{I}_{AB}\sqrt{3}\underline{/-30^\circ} = (13.2\underline{/36.87^\circ})(\sqrt{3}\underline{/-30^\circ}) \\ &= 22.86\underline{/6.87^\circ}\ \text{A}\end{aligned}$$

$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 22.86\underline{/-113.13^\circ}\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\underline{/+120^\circ} = 22.86\underline{/126.87^\circ}\ \text{A}$$

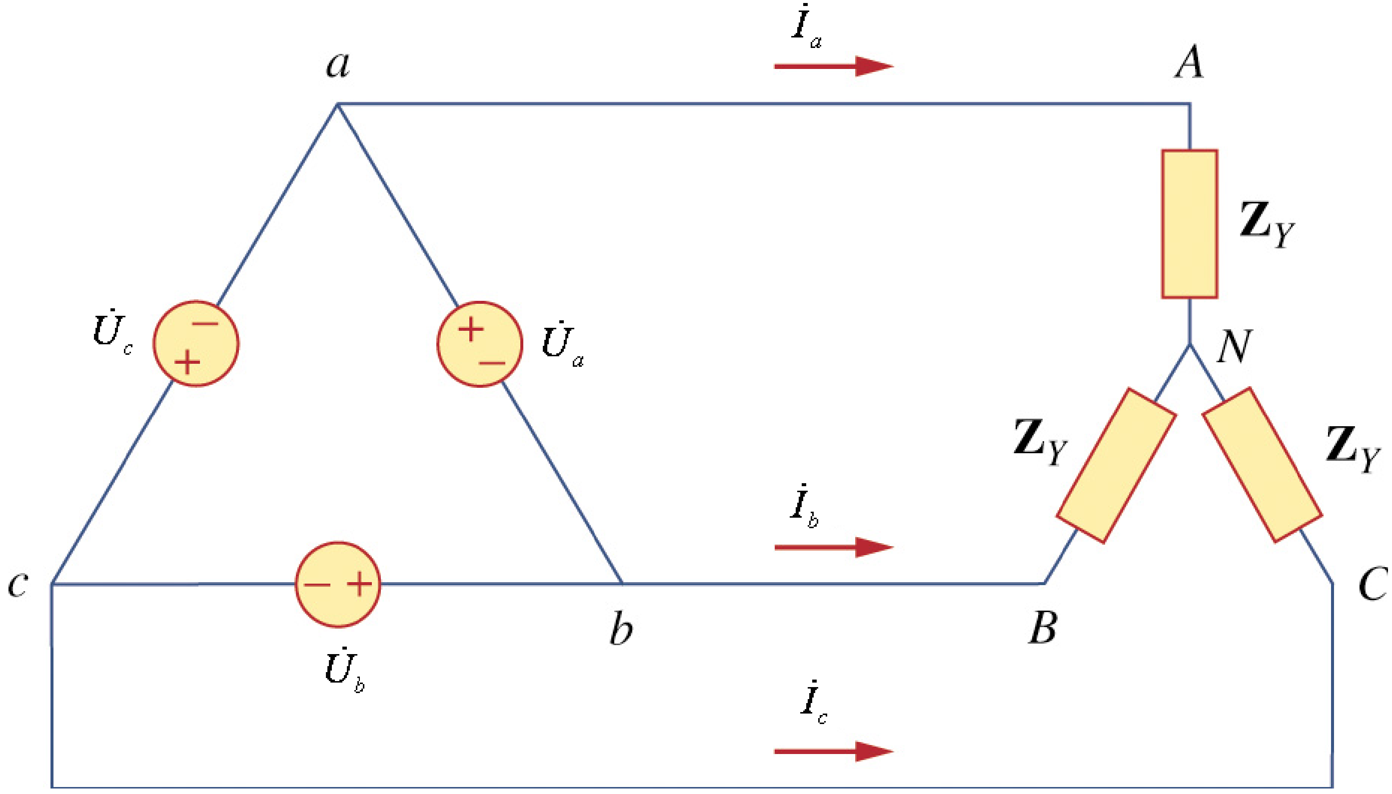
Practice Problem 12.4

A positive-sequence, balanced Δ-connected source supplies a balanced Δ-connected load. If the impedance per phase of the load is $18 + j12\ \Omega$ and $\mathbf{I}_a = 9.609\underline{/35^\circ}$ A, find $\mathbf{I}_{AB}$ and $\mathbf{V}_{AB}$.

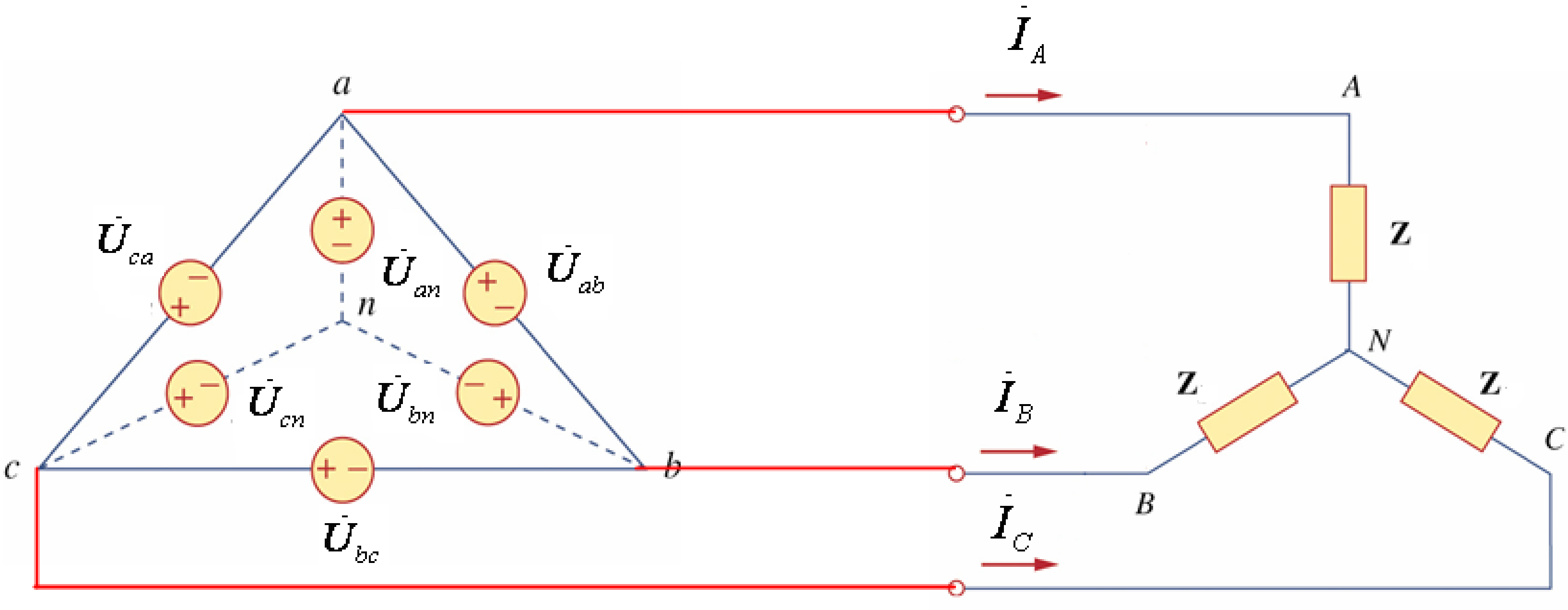
Answer: $5.548\underline{/65^\circ}$ A, $120\underline{/98.69^\circ}$ V.

12.5 Balance Delta-Wye Connection

A balanced Δ- Y system consists of a balanced Δ-connected source feeding a balanced Y-connected load.



A balanced Δ- Y connection



$$\mathbf{V}_{an} = \frac{\mathbf{V}_{ab}}{\sqrt{3}} \angle -30^o = \frac{\mathbf{V}_{ab}}{\sqrt{3}} \angle -30^o$$

(choose $\mathbf{V}_{ab}$ as a reference phasor)

$$\mathbf{V}_{AN} = \frac{\mathbf{V}_{ab}}{\sqrt{3}} \angle -30^o = \frac{\mathbf{V}_{ab}}{\sqrt{3}} \angle -30^o$$

$$\mathbf{V}_{bn} = \frac{\mathbf{V}_{bc}}{\sqrt{3}} \angle -30^o = \frac{\mathbf{V}_{ab}}{\sqrt{3}} \angle -150^o$$

$$\mathbf{V}_{BN} = \mathbf{V}_{AN} \angle -120^o$$

$$\mathbf{V}_{cn} = \frac{\mathbf{V}_{ca}}{\sqrt{3}} \angle -30^o = \frac{\mathbf{V}_{ab}}{\sqrt{3}} \angle 90^o$$

$$\mathbf{V}_{CN} = \mathbf{V}_{AN} \angle 120^o$$

$$\mathbf{I}_A = \frac{\mathbf{V}_{ab} / \sqrt{3} \angle -30^\circ}{Z}$$

Example 12.5

A balanced Y-connected load with a phase impedance of $40 + j25\ \Omega$ is supplied by a balanced, positive sequence Δ-connected source with a line voltage of 210 V. Calculate the phase currents. Use $\mathbf{V}_{ab}$ as a reference.

Solution:

The load impedance is

$$\mathbf{Z}_Y = 40 + j25 = 47.17\underline{/32^\circ}\ \Omega$$

and the source voltage is

$$\mathbf{V}_{ab} = 210\underline{/0^\circ}\text{ V}$$

When the Δ-connected source is transformed to a Y-connected source,

$$\mathbf{V}_{an} = \frac{\mathbf{V}_{ab}}{\sqrt{3}}\underline{/-30^\circ} = 121.2\underline{/-30^\circ}\text{ V}$$

The line currents are

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} = \frac{121.2\underline{/-30^\circ}}{47.12\underline{/32^\circ}} = 2.57\underline{/-62^\circ}\text{ A}$$

$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 2.57\underline{/-178^\circ}\text{ A}$$

$$\mathbf{I}_c = \mathbf{I}_a\underline{/120^\circ} = 2.57\underline{/58^\circ}\text{ A}$$

which are the same as the phase currents.

12.6 Summary of Balance Connection

Connection	Phase voltages/currents	Line voltages/currents
Y-Y	$\mathbf{V}_{an} = V_p\angle 0^\circ$ $\mathbf{V}_{bn} = V_p\angle -120^\circ$ $\mathbf{V}_{cn} = V_p\angle +120^\circ$ Same as line currents	$\mathbf{V}_{ab} = \sqrt{3}V_p\angle 30^\circ$ $\mathbf{V}_{bc} = \mathbf{V}_{ab}\angle -120^\circ$ $\mathbf{V}_{ca} = \mathbf{V}_{ab}\angle +120^\circ$ $\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_Y$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$
Y-Δ	$\mathbf{V}_{an} = V_p\angle 0^\circ$ $\mathbf{V}_{bn} = V_p\angle -120^\circ$ $\mathbf{V}_{cn} = V_p\angle +120^\circ$ $\mathbf{I}_{AB} = \mathbf{V}_{AB}/\mathbf{Z}_\Delta$ $\mathbf{I}_{BC} = \mathbf{V}_{BC}/\mathbf{Z}_\Delta$ $\mathbf{I}_{CA} = \mathbf{V}_{CA}/\mathbf{Z}_\Delta$	$\mathbf{V}_{ab} = \mathbf{V}_{AB} = \sqrt{3}V_p\angle 30^\circ$ $\mathbf{V}_{bc} = \mathbf{V}_{BC} = \mathbf{V}_{ab}\angle -120^\circ$ $\mathbf{V}_{ca} = \mathbf{V}_{CA} = \mathbf{V}_{ab}\angle +120^\circ$ $\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$

Connection	Phase voltages/currents	Line voltages/currents
Δ-Δ	$\mathbf{V}_{ab} = V_p\angle 0^\circ$ $\mathbf{V}_{bc} = V_p\angle -120^\circ$ $\mathbf{V}_{ca} = V_p\angle +120^\circ$	Same as phase voltages
	$\mathbf{I}_{AB} = \mathbf{V}_{ab}/\mathbf{Z}_\Delta$ $\mathbf{I}_{BC} = \mathbf{V}_{bc}/\mathbf{Z}_\Delta$ $\mathbf{I}_{CA} = \mathbf{V}_{ca}/\mathbf{Z}_\Delta$	$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$
Δ-Y	$\mathbf{V}_{ab} = V_p\angle 0^\circ$ $\mathbf{V}_{bc} = V_p\angle -120^\circ$ $\mathbf{V}_{ca} = V_p\angle +120^\circ$	Same as phase voltages
	Same as line currents	$\mathbf{I}_a = \dfrac{V_p\angle -30^\circ}{\sqrt{3}\mathbf{Z}_Y}$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$

12.7 Power in Balance System

Let us now consider the power in a balanced three-phase system. Examine the instantaneous power absorbed by the load.

For a Y-connected load, the phase voltages are

$$v_{AN}(t) = \sqrt{2}V_P \cos \omega t \qquad v_{BN}(t) = \sqrt{2}V_P \cos(\omega t - 120^\circ)$$

$$v_{CN}(t) = \sqrt{2}V_P \cos(\omega t + 120^\circ)$$

If $Z_Y = Z\angle\theta$, the phase currents lag behind their corresponding phase voltages by θ. Thus

$$i_A(t) = \sqrt{2}I_P \cos(\omega t - \theta) \quad i_B(t) = \sqrt{2}I_P \cos(\omega t - \theta - 120^\circ)$$

$$i_B(t) = \sqrt{2}I_P \cos(\omega t - \theta + 120^\circ)$$

Where I_p is the effective value of the phase current.

The total instantaneous power in the load is the sum of the instantaneous powers in the three phases; that is,

$$p = p_a + p_b + p_c = v_{AN} i_A + v_{BN} i_B + v_{CN} i_C$$

$$p = 2V_p I_p [\cos \omega t \cos(\omega t - \theta) + \cos(\omega t - 120^\circ) \cos(\omega t - \theta - 120^\circ)$$

$$+ \cos(\omega t + 120^\circ) \cos(\omega t - \theta + 120^\circ)]$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

gives

$$p = V_p I_p [3\cos\theta + \cos(2\omega t - \theta) + \cos(2\omega t - \theta - 240^\circ) + \cos(2\omega t + 240^\circ)]$$

$$= V_p I_p [3\cos\theta + \cos(2\omega t - \theta) + \cos(2\omega t - \theta)\cos 240^\circ + \sin(2\omega t - \theta)\sin 240^\circ$$

$$+ \cos(2\omega t - \theta)\cos 240^\circ - \sin(2\omega t - \theta)\sin 240^\circ]$$

$$p = V_p I_p[3\cos\theta + \cos(2\omega t - \theta) + 2\cos(2\omega t - \theta)\cos 240^\circ]$$

$$= V_p I_p[3\cos\theta + \cos(2\omega t - \theta) - 2(-\frac{1}{2})\cos(2\omega t - \theta)]$$

$$= 3V_p I_p \cos\theta = \sqrt{3}V_L I_L \cos\theta$$

1、average power

$$P = P_A + P_B + P_C \quad = V_A I_A \cos\theta_A + V_B I_B \cos\theta_B + V_C I_C \cos\theta_C$$

Balanced three phase system:

$$P = 3V_p I_p \cos\phi = \sqrt{3}V_l I_l \cos\phi$$

2、reactive power

$$Q = Q_A + Q_B + Q_C \quad = V_A I_A \sin\phi_A + V_B I_B \sin\phi_B + V_C I_C \sin\phi_C$$

Balanced three phase system:

$$Q = 3V_p I_p \sin\phi = \sqrt{3}V_l I_l \sin\phi$$

3、apparent power

$$S = \sqrt{P^2 + Q^2}$$

Balanced three phase system:

$$S = 3V_p I_p \quad = \sqrt{3} V_l I_l$$

4、complex power

$$\mathbf{S} = 3\mathbf{V}_p \mathbf{I}_p^* = P + jQ = \sqrt{3} V_L I_L \angle \theta$$

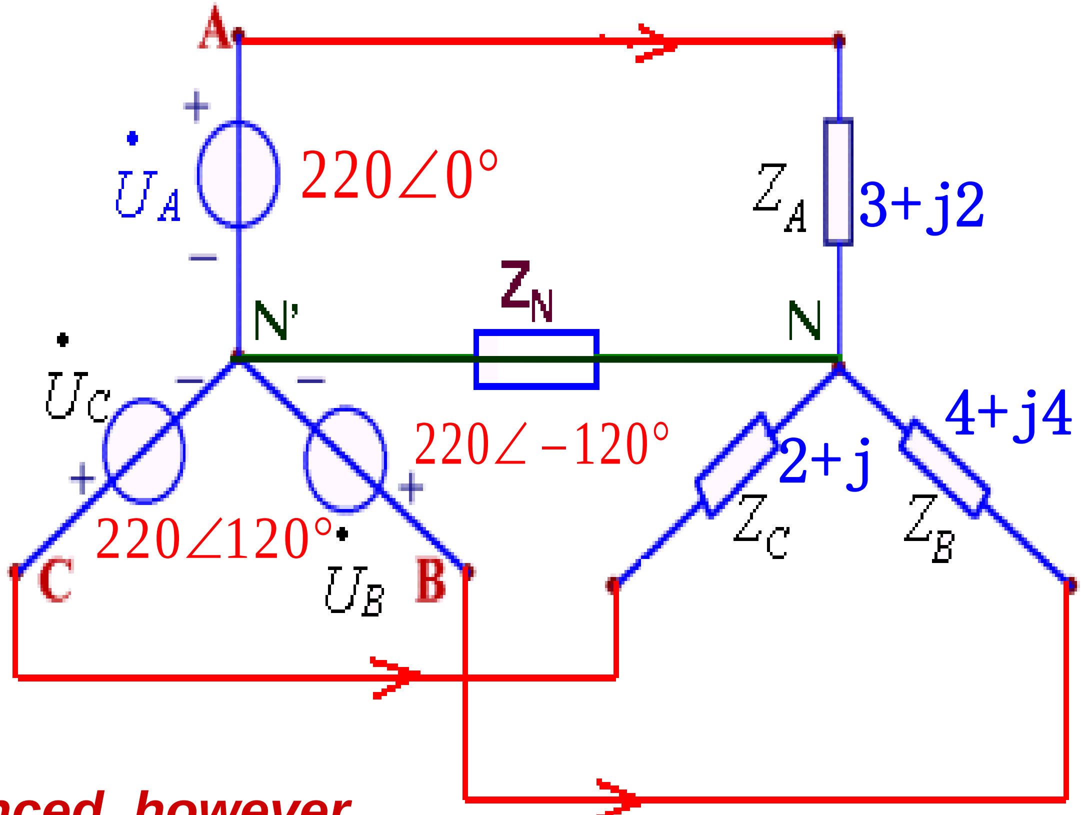
12.8 UnBalance three-phase System

An unbalanced system is due to unbalanced voltage sources or an unbalanced load. Generally, the source voltage is balanced, but the load is unbalanced.

Unbalanced three-phase system are solved by direct application of mesh and nodal analysis.

Example 12.4

In the circuit shown in the figure . $V_L = 380V$. Find line currents.



Solution

V_p=220V

1） when Z_N=0:

$$\mathbf{I}_A = 61\angle -33.7°$$

$$\mathbf{I}_B = 38.9\angle -165°$$

$$\mathbf{I}_C = 98.4\angle 93.4°$$

Load currents are unbalanced, however, they are independent , and do not impact each other.

2） when Z_N=4+j3:

$$\mathbf{V}_{NN'} = \frac{\mathbf{V}_A / Z_A + \mathbf{V}_B / Z_B + \mathbf{V}_C / Z_C}{1/Z_A + 1/Z_B + 1/Z_C + 1/Z_N}$$

$$\mathbf{V}_{NN'} = 54.16\angle 120°V$$

$$\mathbf{V}_{AN} = \mathbf{V}_A - \mathbf{V}_{NN'} \approx 232\angle -12°$$

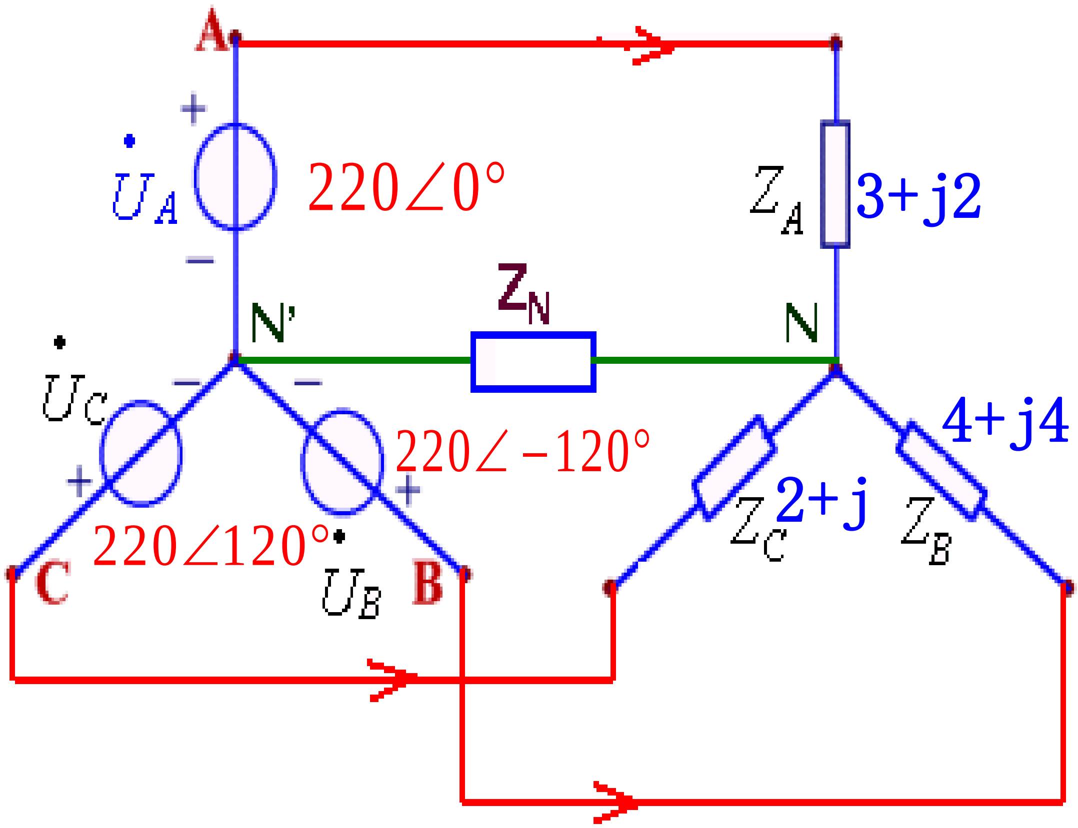
$$\mathbf{V}_{BN} = \mathbf{V}_B - \mathbf{V}_{NN'} \approx 257\angle -109°$$

$$\mathbf{V}_{CN} = \mathbf{V}_C - \mathbf{V}_{NN'} \approx 165\angle 120°$$

$$\mathbf{I}_A = \frac{\mathbf{V}_{AN}}{Z_A} \approx 64.4\angle -45.7°$$

$$\mathbf{I}_B = \frac{\mathbf{V}_{BN}}{Z_B} \approx 45.4\angle -154°$$

$$\mathbf{I}_C = \frac{\mathbf{V}_{CN}}{Z_C} \approx 73.8\angle 93.3°$$



Load currents and voltages are unbalanced, and they are dependent and impact each other.

3) when $Z_N=\infty$:

$$\mathbf{V}_{NN'} = \frac{\mathbf{V}_A / Z_A + \mathbf{V}_B / Z_B + \mathbf{V}_C / Z_C}{1/Z_A + 1/Z_B + 1/Z_C}$$

$$\mathbf{V}_{NN'} = 61.27\angle 115.76°V$$

$$\mathbf{V}_{AN} = \mathbf{V}_A - \mathbf{V}_{NN'}$$

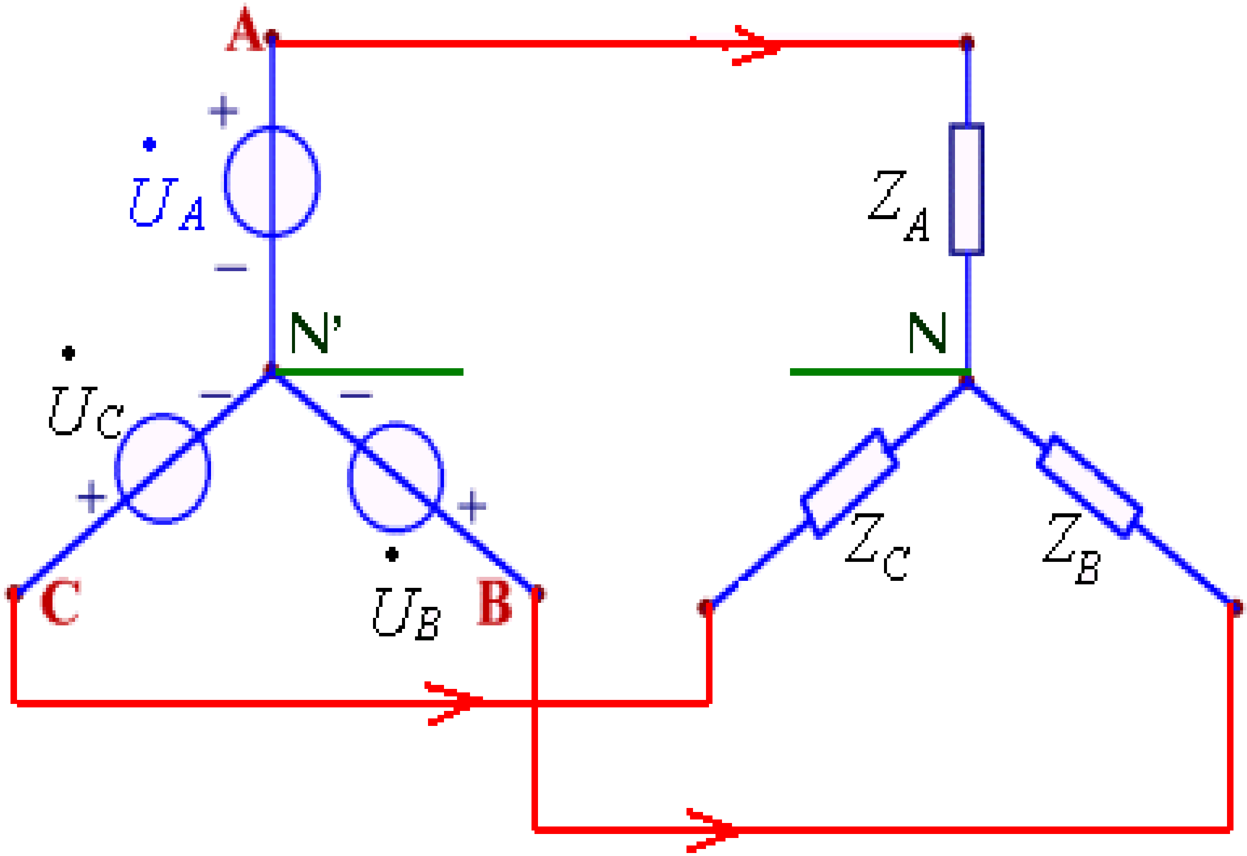
$$\approx 253\angle -13°$$

$$\mathbf{V}_{BN} = \mathbf{V}_B - \mathbf{V}_{NN'}$$

$$\approx 260\angle -109°$$

$$\mathbf{V}_{CN} = \mathbf{V}_C - \mathbf{V}_{NN'}$$

$$\approx 159\angle 122°$$



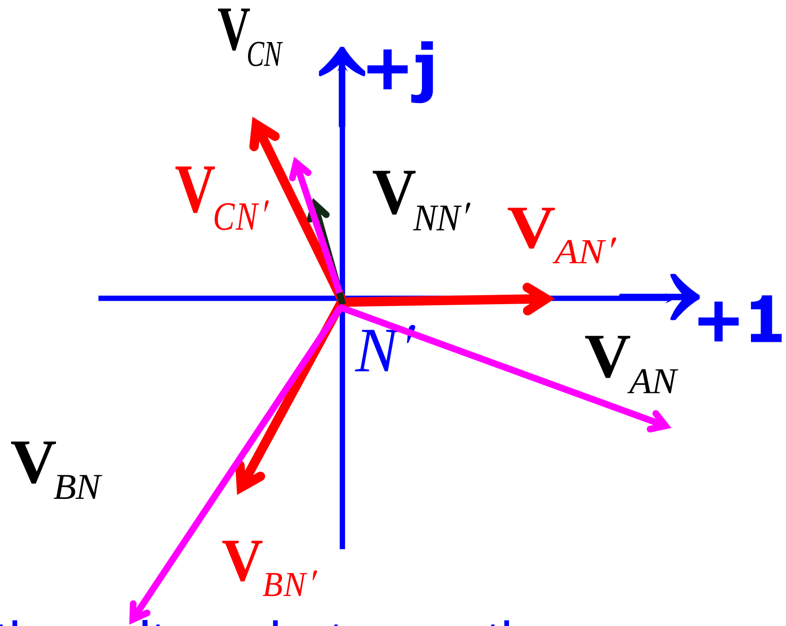
Phasor diagram

$$\mathbf{V}_{NN'} = 61.27\angle 115.76°V$$

$$\mathbf{V}_{AN} \approx 253\angle -13°$$

$$\mathbf{V}_{BN} \approx 260\angle -109°$$

$$\mathbf{V}_{CN} \approx 159\angle 122°$$



- From the phasor diagram, we know that the voltage between the neutral point of the source and the neutral of the load does not equal zero , because of the absence of neutral line, and results in unbalanced phase voltages.

- Therefore, in the three-phase four-wire system, the neutral line has large diameter, and it does not connect to the switch and fuse-element

Summary

- The phase sequence is the order in which the phase voltages of a three-phase generator occur with respect to time. In an *abc* sequence of balanced source voltages, $\mathbf{V}_{an}$ leads $\mathbf{V}_{bn}$ by 120°,which in turn leads $\mathbf{V}_{cn}$ by 120°. In an *acb* sequence of balanced source voltages, $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$ by 120°,which in turn leads $\mathbf{V}_{bn}$ by 120°.

- A balanced wye-or delta-connected load is one in which the three-phase impedances are equal.

- The easiest way to analyze a balanced three-phase circuit is to transform both the source and the load to Y-Y system and then analyze the single-phase equivalent circuit.

- The line current I_L is the current flowing from the generator to the load in each transmission line in a three-phase system. The line voltage V_L is the voltage between each pair of lines, excluding the neutral line if it exists. The phase voltage V_p is the voltage of each phase.

- For a wye-connected load,

$$V_L = \sqrt{3}V_P, \quad I_L = I_P$$

For a delta-connected load

$$V_L = V_P, \quad I_L = \sqrt{3}I_P$$

- The total instantaneous power in a balanced three-phase system is constant and equal to the average power.

- The total complex power absorbed by a balanced three-phase Y-connected or Δ-connected load is $\tilde{S} = P + jQ = \sqrt{3}V_L I_L \angle\theta$ where θis the angle of the load impedances.

- An unbalanced three-phase system can be analyzed using nodal or mesh analysis.

- The total real powr is measured in three-phases using either the three-wattmeter method or the two wattmeter method

12.25 In the circuit of Fig. 12.54, if $\mathbf{V}_{ab} = 440\angle 10°$, $\mathbf{V}_{bc} = 440\angle -110°$, $\mathbf{V}_{ca} = 440\angle 130°$ V, find the line currents.

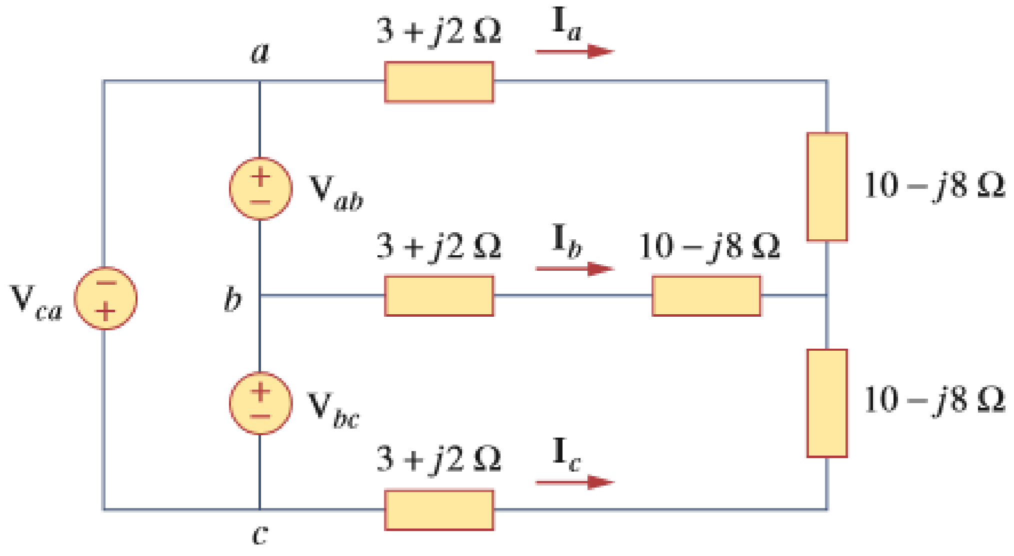


Figure 12.54
For Prob. 12.25.

12.26 Using Fig. 12.55, design a problem to help other students better understand balanced delta connected sources delivering power to balanced wye connected loads.

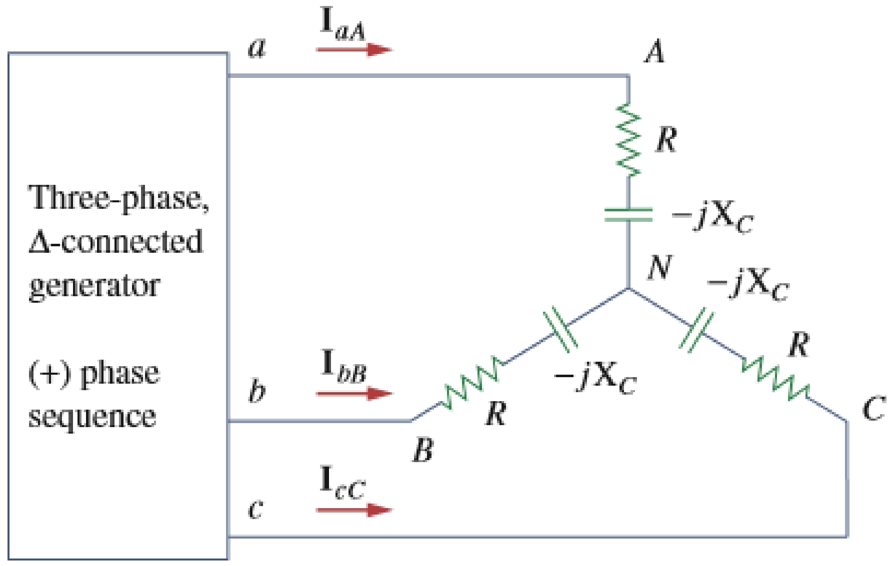


Figure 12.55
For Prob. 12.26.

12.30 In Fig. 12.56, the rms value of the line voltage is 208 V. Find the average power delivered to the load.

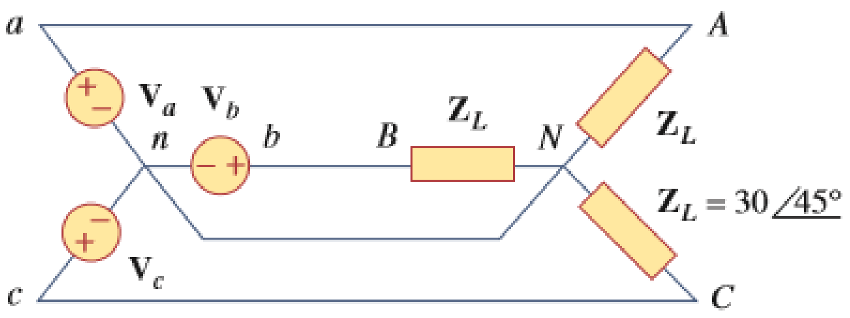


Figure 12.56
For Prob. 12.30.

12.39 Find the real power absorbed by the load in Fig. 12.58.

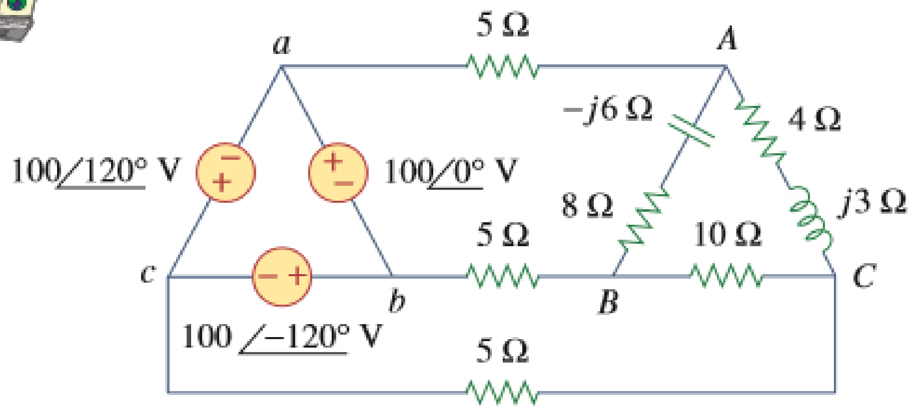


Figure 12.58
For Prob. 12.39.

12.40 For the three-phase circuit in Fig. 12.59, find the average power absorbed by the delta-connected load with $\mathbf{Z}_\Delta = 21 + j24\ \Omega$.

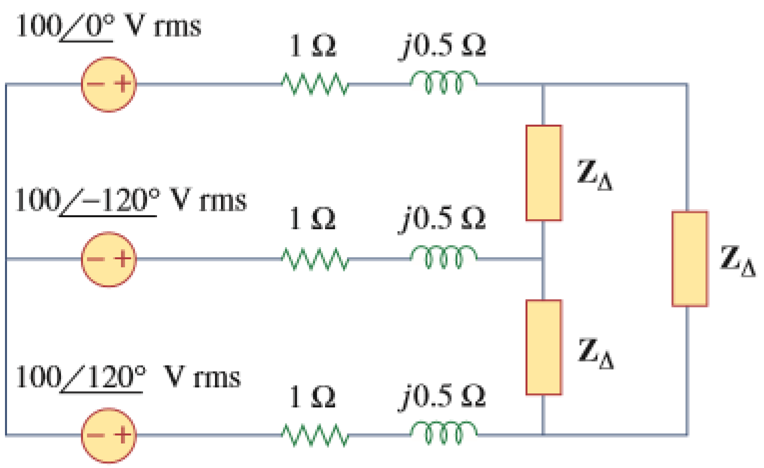


Figure 12.59
For Prob. 12.40.

Fundamentals of Electric Circuits 2021.6

Chapter13 Magnetically Coupled Circuits

Chapter13 Magnetically Coupled Circuits

13.1 Introduction

13.2 Mutual Inductance

13.3 Energy in a Coupled Circuit

13.4 Linear Transformers

13.5 Ideal Transformers

13.6 Ideal Autotransformers

13.1 Introduction

The circuits we have considered so far may be regarded as *conductively coupled*, because one loop affects the neighboring loop through current conduction.

When two loops with or without contacts between them affect each other through the magnetic field generated by one of them, they are said to be *magnetically coupled.*

13.1 Introduction

The transformer is an electrical device designed on the basis of the concept of magnetic coupling. It uses magnetically coupled coils to transfer energy from one circuit to another.

Transformers are key circuit elements.

They are used in power systems for stepping up or stepping down ac voltages or currents. They are used in electronic circuits such as radio and television receivers for such purposes as impedance matching, isolating one part of a circuit from another, and again for stepping up or down ac voltages and currents.

13.2 Mutual Inductance

When two inductors (or coils) are in a close proximity to each other, the magnetic flux caused by current in one coil links with the other coil, thereby inducing voltage in the latter. This phenomenon is known as *mutual inductance.*

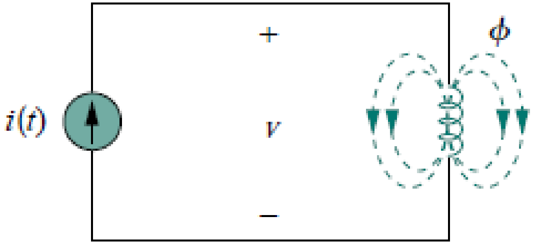


Figure 13.1 Magnetic flux produced by a single coil with N turns.

$$v = N\frac{d\phi}{dt} \qquad v = N\frac{d\phi}{di}\frac{di}{dt}$$

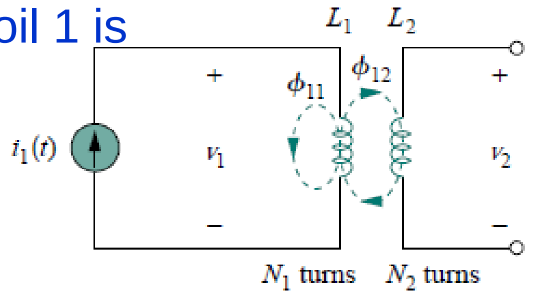
$$v = L\frac{di}{dt} \qquad L = N\frac{d\phi}{di}$$

This inductance is commonly called *self-inductance*, because it relates the voltage induced in a coil by a time-varying current in the same coil.

10.2 Mutual Inductance

For the sake of simplicity, assume that the second inductor carries no current. The magnetic flux φ_1 *emanating from coil 1* has two components: one component φ_{11} *links only coil 1, and another* component φ_{12} *links both coils. Hence,* $\varphi_1 = \varphi_{11} + \varphi_{12}$

Although the two coils are physically separated, they are said to be *magnetically coupled.* Since the entire flux φ_1 links coil 1, the voltage induced in coil 1 is



$$v_1 = N_1 \frac{d\phi_1}{dt} \qquad v_1 = N_1 \frac{d\phi_1}{di_1}\frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

$$v_2 = N_2 \frac{d\phi_{12}}{dt} \qquad v_2 = N_2 \frac{d\phi_{12}}{di_1}\frac{di_1}{dt} = M_{21} \frac{di_1}{dt}$$

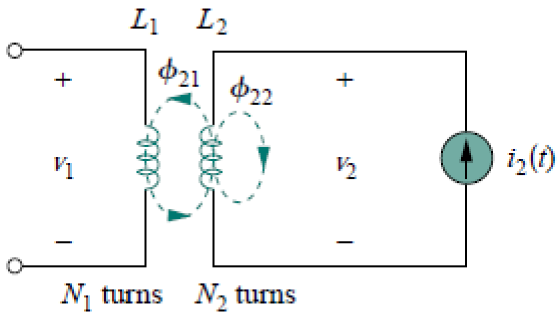
$$M_{21} = N_2 \frac{d\phi_{12}}{di_1}$$

M_{21} is known as the mutual inductance of coil 2 with respect to coil 1.

$$\boxed{v_2 = M_{21} \frac{di_1}{dt}}$$

10.2 Mutual Inductance

Suppose we now let current i_2 *flow in coil 2, while coil 1 carries no* current . The magnetic flux φ_2 *emanating from coil 2 comprises* flux φ_{22} *that links only coil 2 and flux* φ_{21} *that links both coils. Hence,* $\varphi_2 = \varphi_{21} + \varphi_{22}$



$$v_2 = N_2 \frac{d\phi_2}{dt} = N_2 \frac{d\phi_2}{di_2}\frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

$$v_1 = N_1 \frac{d\phi_{21}}{dt} = N_1 \frac{d\phi_{21}}{di_2}\frac{di_2}{dt} = M_{12} \frac{di_2}{dt}$$

$$\boxed{v_1 = M_{12} \frac{di_2}{dt}} \qquad \boxed{v_2 = M_{21} \frac{di_1}{dt}}$$

$$M_{12} = N_1 \frac{d\phi_{21}}{di_2}$$

which is the mutual inductance of coil 1 with respect to coil 2.

$$M_{12} = M_{21} = M$$

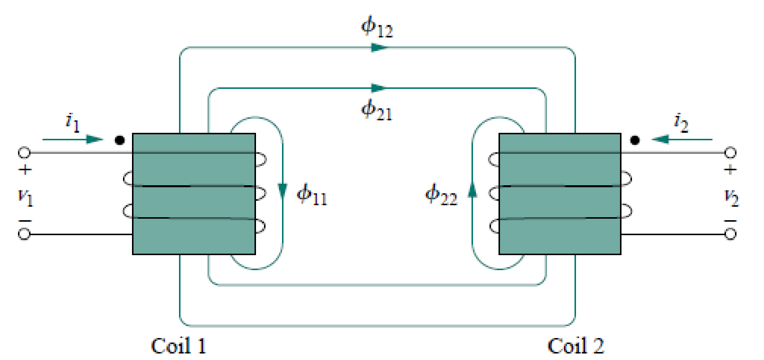
Mutual inductance is the ability of one inductor to induce a voltage across a neighboring inductor, measured in henrys (H).

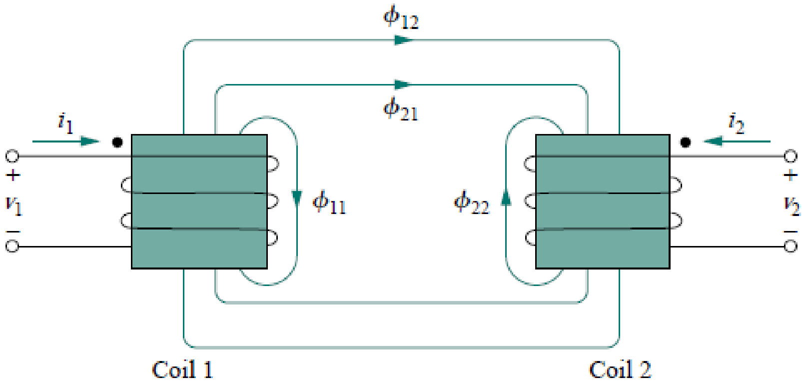
Although mutual inductance *M* is always a positive quantity, the mutual voltage *M di/dt* may be negative or positive, just like the self induced voltage *Ldi/dt.*

However, unlike the self-induced *Ldi/dt,* whose polarity is determined by the reference direction of the current and the reference polarity of the voltage (according to the passive sign convention), the polarity of mutual voltage *M di/dt* is not easy to determine, because four terminals are involved.

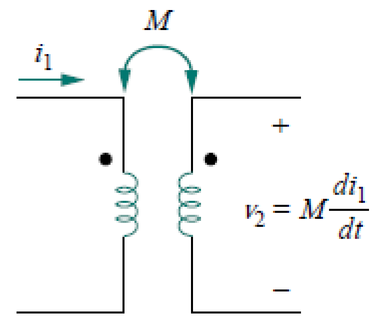
The choice of the correct polarity for *M di/dt* is made by examining the orientation or particular way in which both coils are physically wound and applying Lenz's law in conjunction with the *right-hand rule*.

Since it is inconvenient to show the construction details of coils on a circuit schematic, we apply the *dot convention* in circuit analysis. By this convention, a dot is placed in the circuit at one end of each of the two magnetically coupled coils to indicate the direction of the magnetic flux if current enters that dotted terminal of the coil.

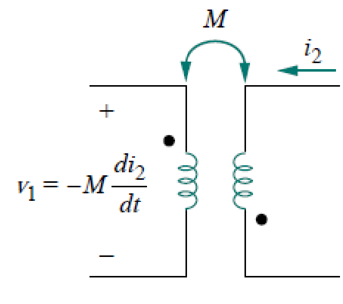




If a current **enters** the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is **positive** at the dotted terminal of the second coil.

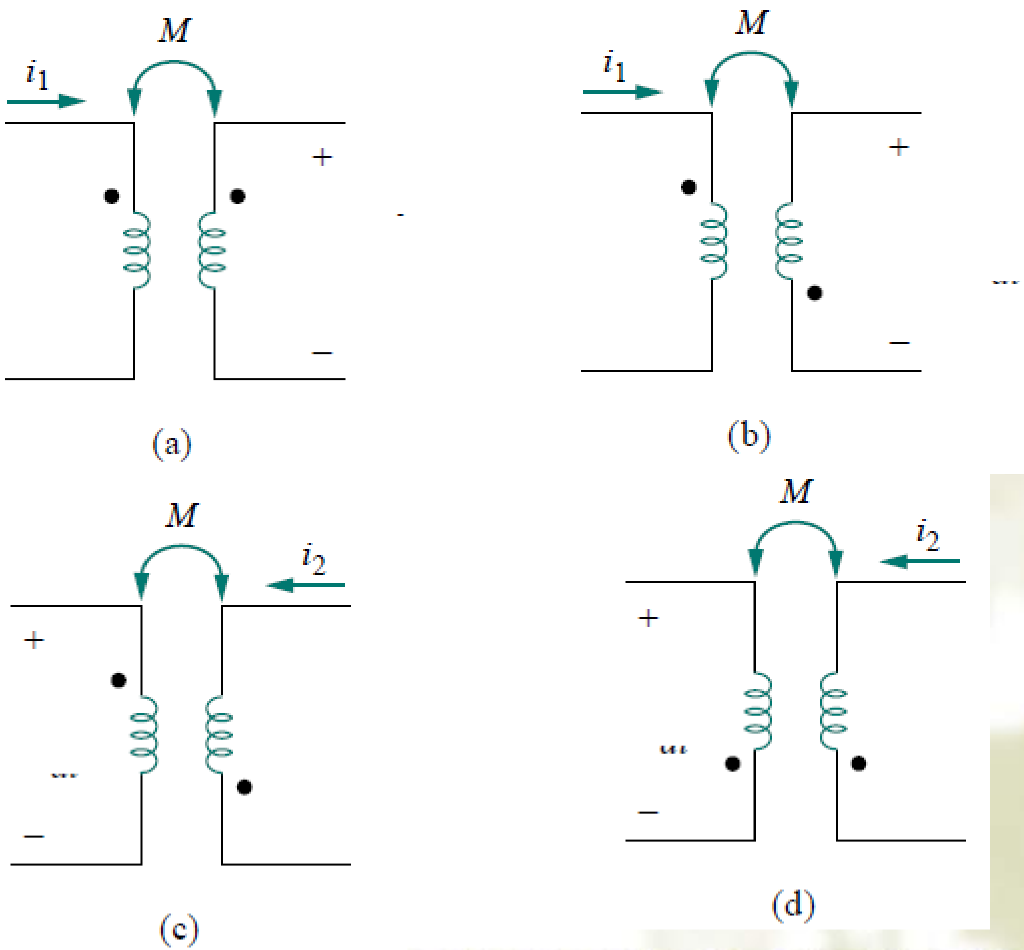


If a current **leaves** the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is **negative** at the dotted terminal of the second coil.

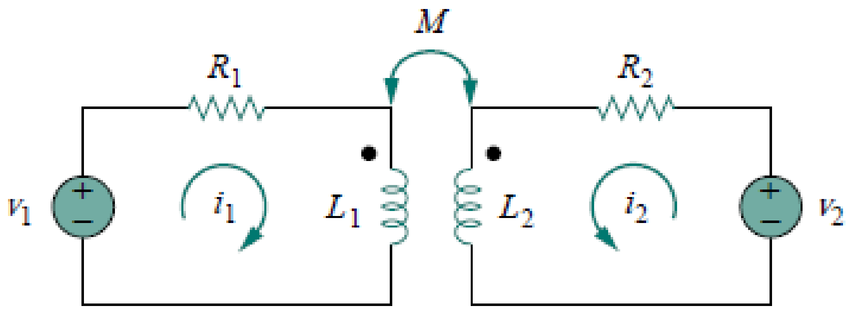


(c)

Thus, the reference polarity of the mutual voltage depends on the reference direction of the inducing current and the dots on the coupled coils.

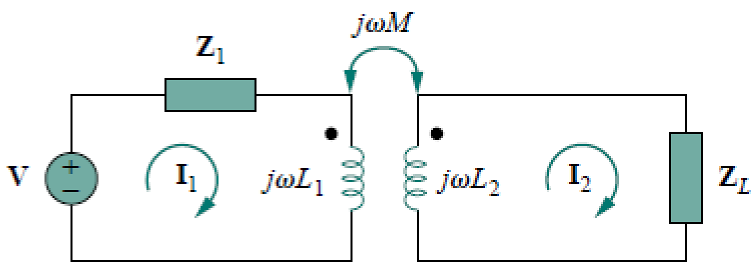


M
i1
+
–
(a)
M
i1
+
–
(b)
M
i2
+
–
(c)
M
i2
+
–
(d)

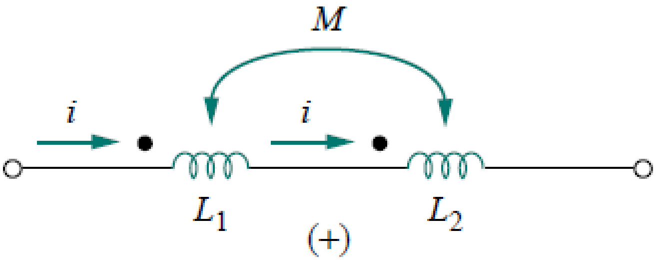


$$v_1 = i_1 R_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \qquad v_2 = i_2 R_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$\mathbf{V}_1 = (R_1 + j\omega L_1)\mathbf{I}_1 + j\omega M \mathbf{I}_2 \qquad \mathbf{V}_2 = j\omega M \mathbf{I}_1 + (R_2 + j\omega L_2)\mathbf{I}_2$$

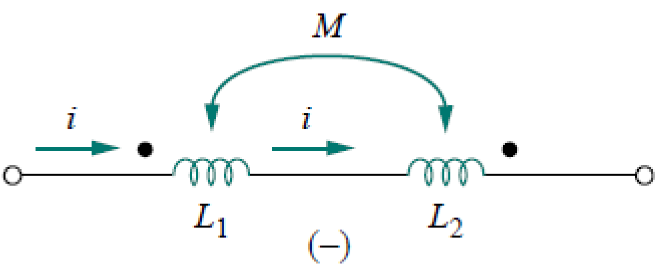


$$\mathbf{V} = (\mathbf{Z}_1 + j\omega L_1)\mathbf{I}_1 - j\omega M \mathbf{I}_2 \qquad 0 = -j\omega M \mathbf{I}_1 + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2$$



(a)
series-aiding connection,

$$L = L_1 + L_2 + 2M \qquad \text{(Series-aiding connection)}$$

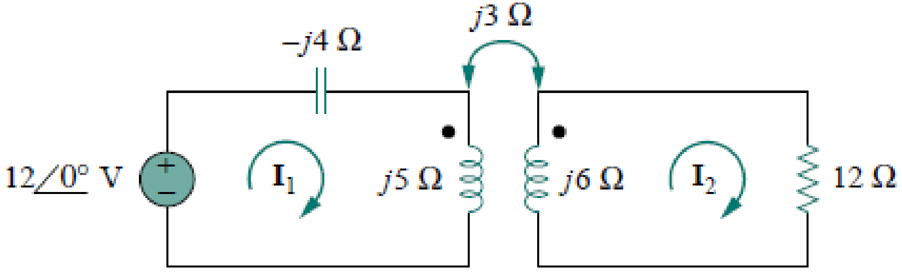


(b)
series-opposing connection.

$$L = L_1 + L_2 - 2M \qquad \text{(Series-opposing connection)}$$

EXAMPLE 13.1

Calculate the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit of Fig. 13.9.



Solution:

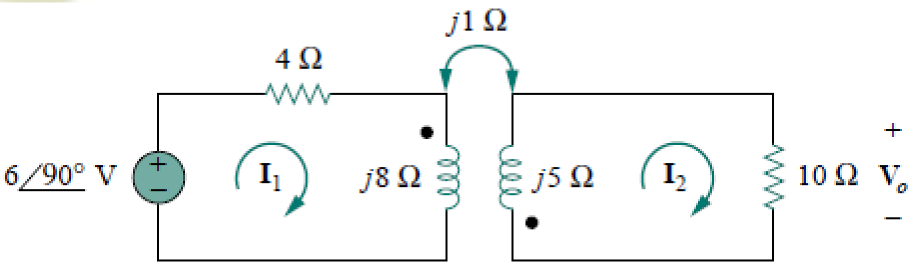
For coil 1, KVL gives $$-12 + (-j4 + j5)\mathbf{I}_1 - j3\mathbf{I}_2 = 0$$

For coil 2, KVL gives $$-j3\mathbf{I}_1 + (12 + j6)\mathbf{I}_2 = 0$$

$$\mathbf{I}_1 = 13.01\angle -49.39^\circ \text{ A} \qquad \mathbf{I}_2 = \frac{12}{4 - j} = 2.91\angle 14.04^\circ \text{ A}$$

PRACTICE PROBLEM 13.1

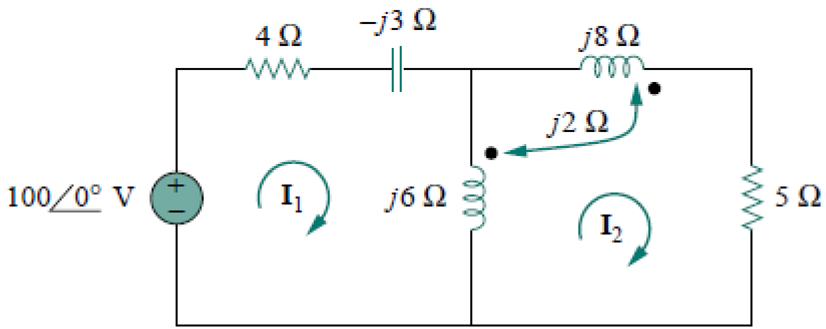
Determine the voltage $\mathbf{V}_o$ in the circuit of Fig. 13.10.



Answer: $0.6\angle -90^\circ$ V.

EXAMPLE 13.2

Calculate the mesh currents in the circuit of Fig. 13.11.



for mesh 1 in Fig. 13.11, KVL gives

$$-100 + \mathbf{I}_1(4 - j3 + j6) - j6\mathbf{I}_2 - j2\mathbf{I}_2 = 0$$

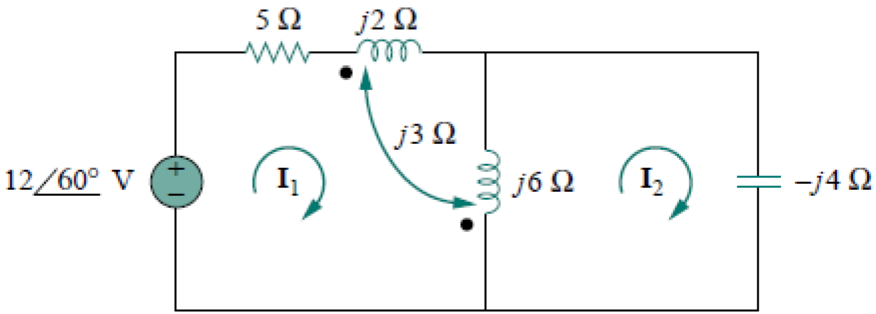
for mesh 2, KVL gives

$$0 = -2j\mathbf{I}_1 - j6\mathbf{I}_1 + (j6 + j8 + j2 \times 2 + 5)\mathbf{I}_2$$

$\mathbf{I}_1 = \; 20.3\angle 3.5^\circ$ A $\qquad \mathbf{I}_2 = \; 8.693\angle 19^\circ$ A

PRACTICE PROBLEM 13.2

Determine the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit of Fig. 13.13.



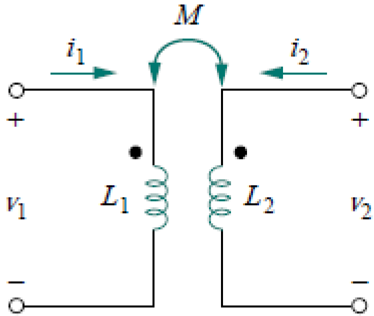
Answer: $2.15\angle 86.56^\circ$, $3.23\angle 86.56^\circ$ A.

10.3 ENERGY IN A COUPLED CIRCUIT

In Chapter 6, we saw that the energy stored in an inductor is given by

$$w = \frac{1}{2}Li^2$$

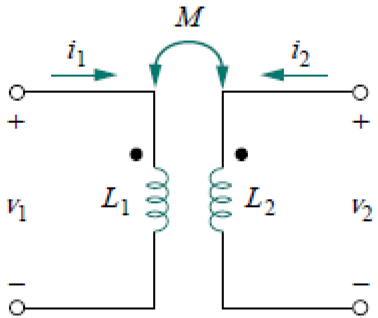
We now want to determine the energy stored in magnetically coupled coils



$$w_1 = \int p_1\,dt = L_1\int_0^{I_1} i_1\,di_1 = \frac{1}{2}L_1 I_1^2$$

Consider the circuit in Fig. 13.14. We assume that currents i_1 and i_2 are zero initially, so that the energy stored in the coils is zero. If we let i_1 increase from zero to I_1 while maintaining $i_2 = 0$, the power in coil 1 is

$$p_1(t) = v_1 i_1 = i_1 L_1 \frac{di_1}{dt} \qquad (13.24)$$



$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$$

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 - M I_1 I_2$$

If we now maintain $i_1 = I_1$ and increase i_2 from zero to I_2, the mutual voltage induced in coil 1 is $M_{12}\, di_2/dt$, while the mutual voltage induced in coil 2 is zero, since i_1 does not change. The power in the coils is now

$$p_2(t) = i_1 M_{12}\frac{di_2}{dt} + i_2 v_2 = I_1 M_{12}\frac{di_2}{dt} + i_2 L_2 \frac{di_2}{dt} \qquad (13.26)$$

$$w_2 = \int p_2\, dt = M_{12} I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2\, di_2 = M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

$$w = w_1 + w_2 = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2$$

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{21} I_1 I_2$$

$$M_{12} = M_{21} = M$$

$$\boxed{w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2}$$

The positive sign is selected for the mutual term if both currents enter or leave the dotted terminals of the coils; the negative sign is selected otherwise.

Coefficient of coupling k

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

$$\boxed{M = k\sqrt{L_1 L_2}}$$

where $0 \leq k \leq 1$

The coupling coefficient is the fraction of the total flux emanating from one coil that links the other coil.

$$k = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{12}}{\phi_{11} + \phi_{12}}$$

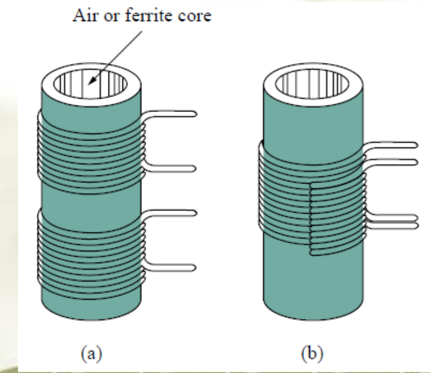
$$k = \frac{\phi_{21}}{\phi_2} = \frac{\phi_{21}}{\phi_{21} + \phi_{22}}$$

The coupling coefficient k is a measure of the magnetic coupling between two coils; $0 \leq k \leq 1$.

If the entire flux produced by one coil links another coil, then $k = 1$ and we have 100 percent coupling, or the coils are said to be *perfectly coupled*.

For $k < 0.5$, coils are said to be *loosely coupled;*
For $k > 0.5$, they are said to be *tightly coupled.*

We expect k to depend on the closeness of the two coils, their core, their orientation, and their windings.

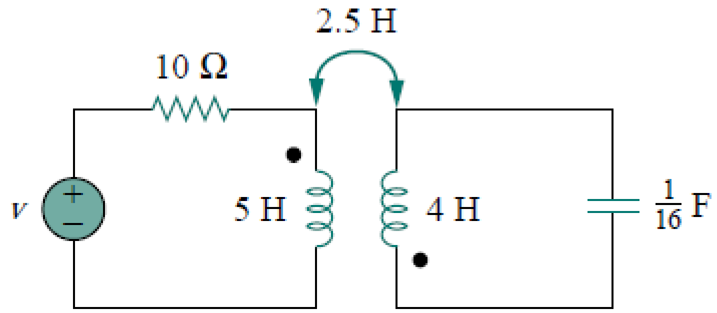


(a) loosely coupled,

(b) tightly coupled

EXAMPLE 13.3

Consider the circuit in Fig. 13.16. Determine the coupling coefficient. Calculate the energy stored in the coupled inductors at time $t = 1$ s if $v = 60\cos(4t + 30^\circ)$ V.

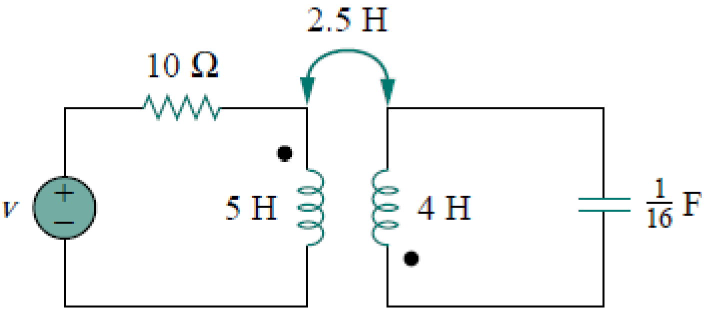


Solution:

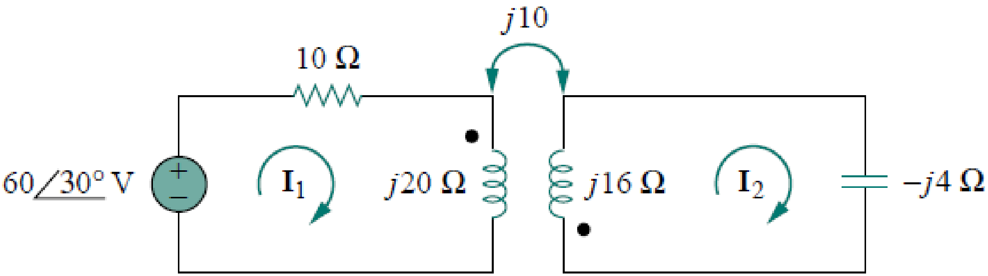
The coupling coefficient is

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

indicating that the inductors are tightly coupled. To find the energy stored, we need to obtain the frequency-domain equivalent of the circuit.



$$60\cos(4t + 30^\circ) \Longrightarrow 60\angle 30^\circ, \quad \omega = 4 \text{ rad/s}$$
$$5 \text{ H} \Longrightarrow j\omega L_1 = j20\ \Omega$$
$$2.5 \text{ H} \Longrightarrow j\omega M = j10\ \Omega$$
$$4 \text{ H} \Longrightarrow j\omega L_2 = j16\ \Omega$$
$$\frac{1}{16} \text{ F} \Longrightarrow \frac{1}{j\omega C} = -j4\ \Omega$$



For mesh 1, $(10 + j20)\mathbf{I}_1 + j10\mathbf{I}_2 = 60\angle 30^\circ$

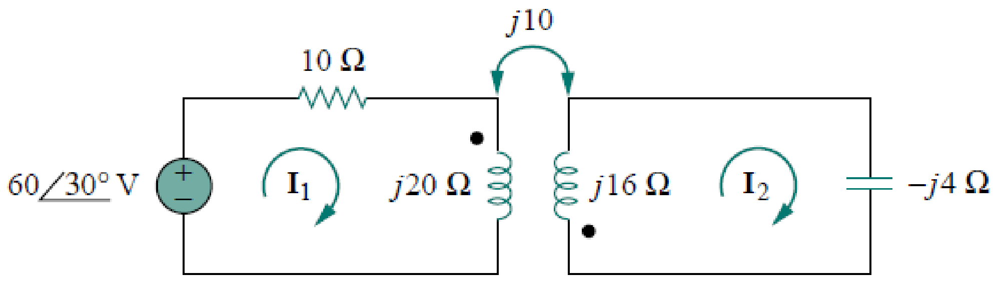
$\mathbf{I}_1 = 3.905\angle -19.4^\circ$ A

For mesh 2, $j10\mathbf{I}_1 + (j16 - j4)\mathbf{I}_2 = 0$

$\mathbf{I}_2 = 3.254\angle -160.6^\circ$ A

In the time-domain,

$$i_1 = 3.905\cos(4t - 19.4^\circ), \qquad i_2 = 3.254\cos(4t - 199.4^\circ)$$



In the time-domain,

$$i_1 = 3.905\cos(4t - 19.4°), \qquad i_2 = 3.254\cos(4t - 199.4°)$$

At time $t = 1$ s, $4t = 4$ rad $= 229.2°$, and

$$i_1 = 3.905\cos(229.2° - 19.4°) = -3.389\text{ A}$$

$$i_2 = 3.254\cos(229.2° + 160.6°) = 2.824\text{ A}$$

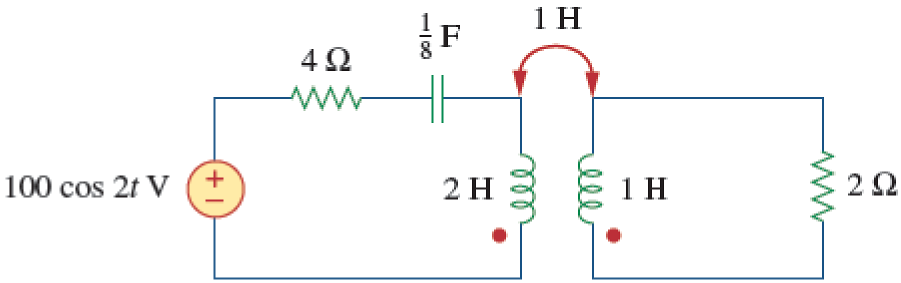
The total energy stored in the coupled inductors is

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2$$

$$= \frac{1}{2}(5)(-3.389)^2 + \frac{1}{2}(4)(2.824)^2 + 2.5(-3.389)(2.824) = 20.73\text{ J}$$

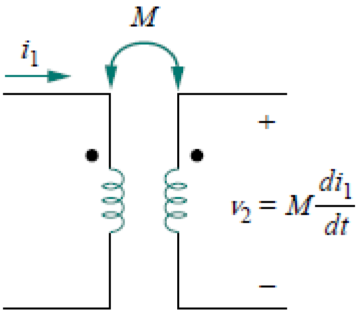
Practice Problem 13.3

For the circuit in Fig. 13.18, determine the coupling coefficient and the energy stored in the coupled inductors at $t = 1.5$ s.

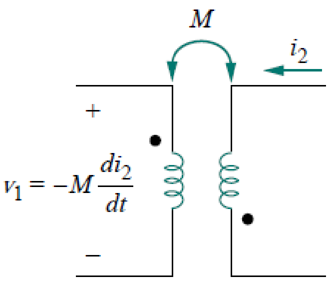


Mutual inductance is the ability of one inductor to induce a voltage across a neighboring inductor, measured in henrys (H).

If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.

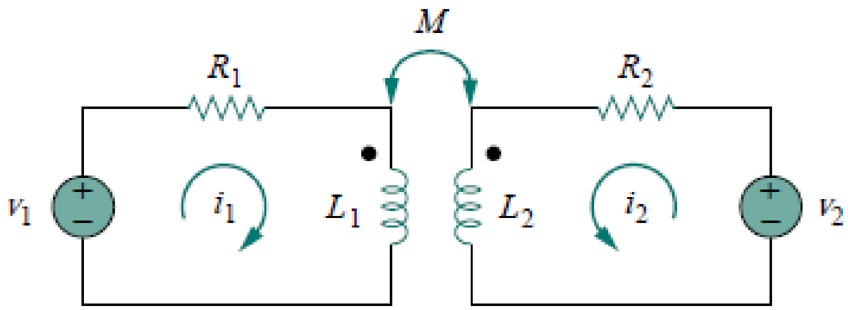


If a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.



(c)

Thus, the reference polarity of the mutual voltage depends on the reference direction of the inducing current and the dots on the coupled coils.

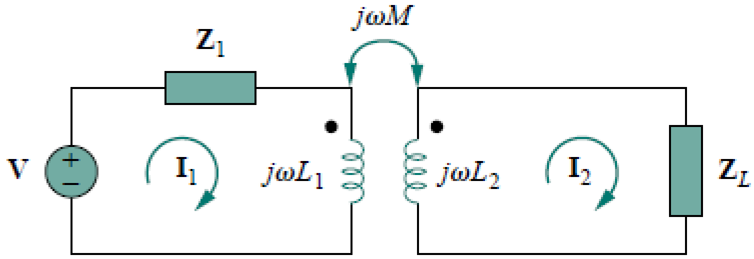


$$v_1 = i_1 R_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = i_2 R_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

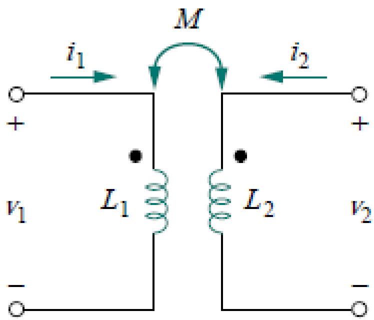
$$\mathbf{V}_1 = (R_1 + j\omega L_1)\mathbf{I}_1 + j\omega M \mathbf{I}_2$$

$$\mathbf{V}_2 = j\omega M \mathbf{I}_1 + (R_2 + j\omega L_2)\mathbf{I}_2$$



$$\mathbf{V} = (\mathbf{Z}_1 + j\omega L_1)\mathbf{I}_1 - j\omega M \mathbf{I}_2$$

$$0 = -j\omega M \mathbf{I}_1 + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2$$



$$\boxed{w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2}$$

Coefficient of coupling k

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

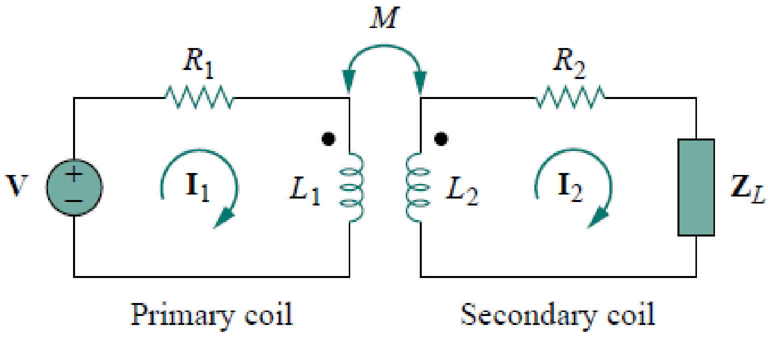
where $0 \leq k \leq 1$

$$\boxed{M = k\sqrt{L_1 L_2}}$$

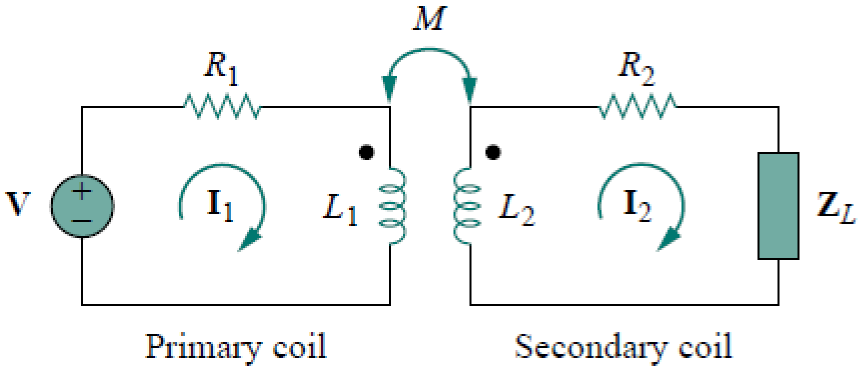
10.4 LINEAR TRANSFORMERS

Here we introduce the transformer as a new circuit element. A transformer is a magnetic device that takes advantage of the phenomenon of mutual inductance.

A **transformer** is generally a four-terminal device comprising two (or more) magnetically coupled coils.



air-core transformers,



We would like to obtain the input impedance $\mathbf{Z}_{in}$ as seen from the source, because $\mathbf{Z}_{in}$ governs the behavior of the primary circuit. Applying KVL to the two meshes in Fig. 13.19 gives

$$\mathbf{V} = (R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2$$

$$0 = -j\omega M\mathbf{I}_1 + (R_2 + j\omega L_2 + \mathbf{Z}_L)\mathbf{I}_2$$

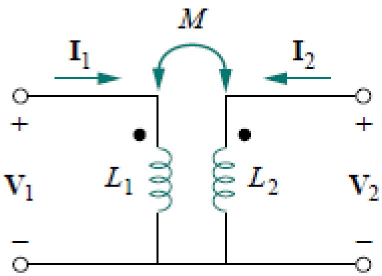
$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{V}}{\mathbf{I}_1} = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L}$$

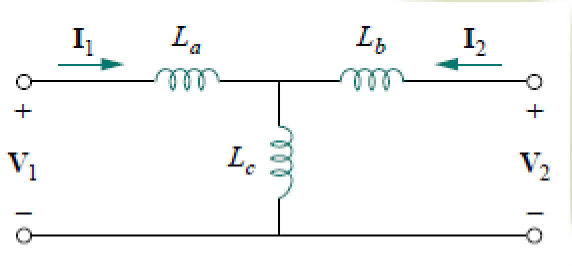
Primary impedance   Reflected impedance $\mathbf{Z}_R$

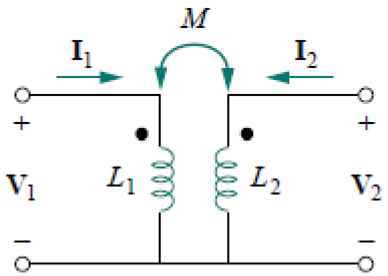
$$\boxed{\mathbf{Z}_R = \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L}}$$

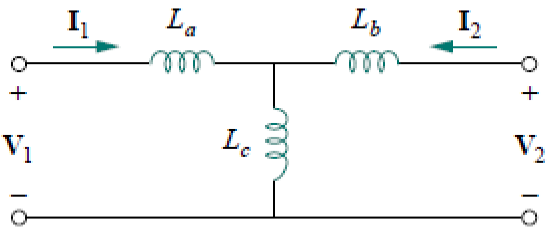
The little bit of experience gained in Sections 13.2 and 13.3 in analyzing magnetically coupled circuits is enough to convince anyone that analyzing these circuits is not as easy as circuits in previous chapters. For this reason, it is sometimes convenient to replace a magnetically coupled circuit by an equivalent circuit with no magnetic coupling.

We want to replace the linear transformer in Fig. 13.19 by an equivalent T or Π circuit, a circuit that would have no mutual inductance. Ignore the resistances of the coils and assume that the coils have a common ground as shown in Fig. 13.21.





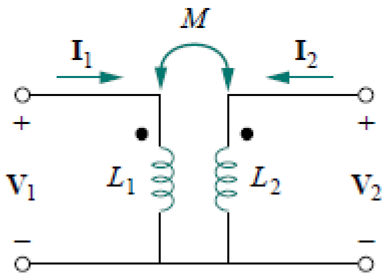


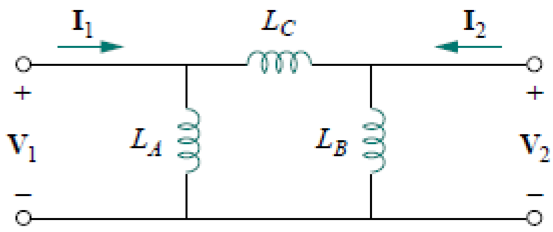


$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix}=\begin{bmatrix}j\omega L_1 & j\omega M\\ j\omega M & j\omega L_2\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix}=\begin{bmatrix}j\omega(L_a+L_c) & j\omega L_c\\ j\omega L_c & j\omega(L_b+L_c)\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix}$$

$$\boxed{L_a = L_1 - M, \qquad L_b = L_2 - M, \qquad L_c = M}$$





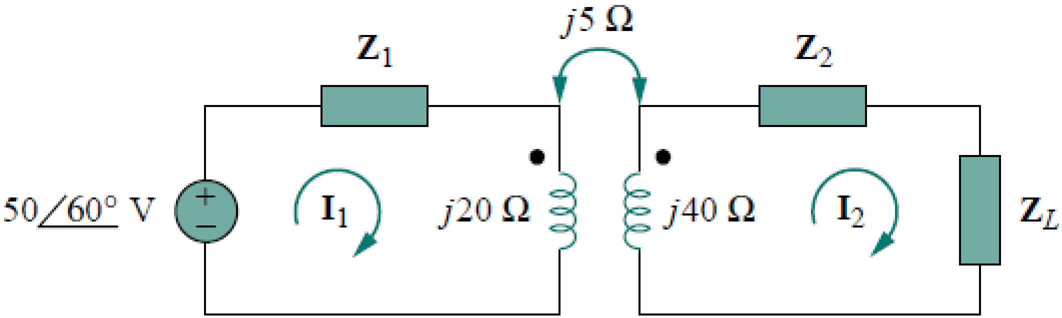
$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{L_2}{j\omega(L_1L_2 - M^2)} & \dfrac{-M}{j\omega(L_1L_2 - M^2)} \\ \dfrac{-M}{j\omega(L_1L_2 - M^2)} & \dfrac{L_1}{j\omega(L_1L_2 - M^2)} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{j\omega L_A} + \dfrac{1}{j\omega L_C} & -\dfrac{1}{j\omega L_C} \\ -\dfrac{1}{j\omega L_C} & \dfrac{1}{j\omega L_B} + \dfrac{1}{j\omega L_C} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$\boxed{\begin{gathered} L_A = \frac{L_1L_2 - M^2}{L_2 - M}, \qquad L_B = \frac{L_1L_2 - M^2}{L_1 - M} \\ L_C = \frac{L_1L_2 - M^2}{M} \end{gathered}}$$

EXAMPLE 13.4

In the circuit of Fig. 13.24, calculate the input impedance and current $\mathbf{I}_1$. Take $\mathbf{Z}_1 = 60 - j100\ \Omega$, $\mathbf{Z}_2 = 30 + j40\ \Omega$, and $\mathbf{Z}_L = 80 + j60\ \Omega$.

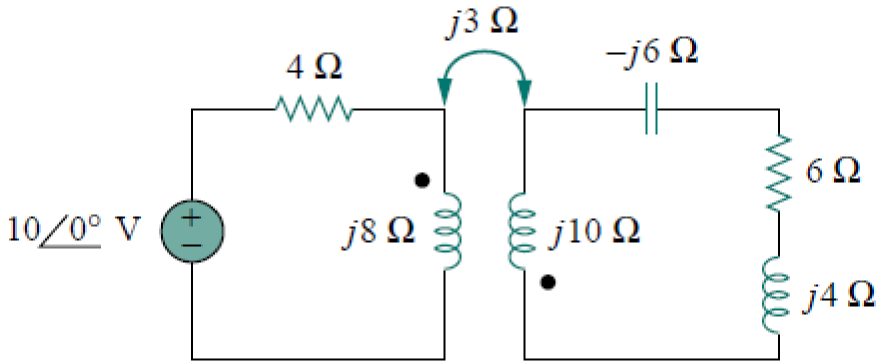


$$\mathbf{Z}_{in} = \mathbf{Z}_1 + j20 + \frac{(5)^2}{j40 + \mathbf{Z}_2 + \mathbf{Z}_L} = 60.09 - j80.11 = 100.14\angle -53.1^\circ\ \Omega$$

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_{in}} = \frac{50\angle 60^\circ}{100.14\angle -53.1^\circ} = 0.5\angle 113.1^\circ\ \text{A}$$

PRACTICE PROBLEM 13.4

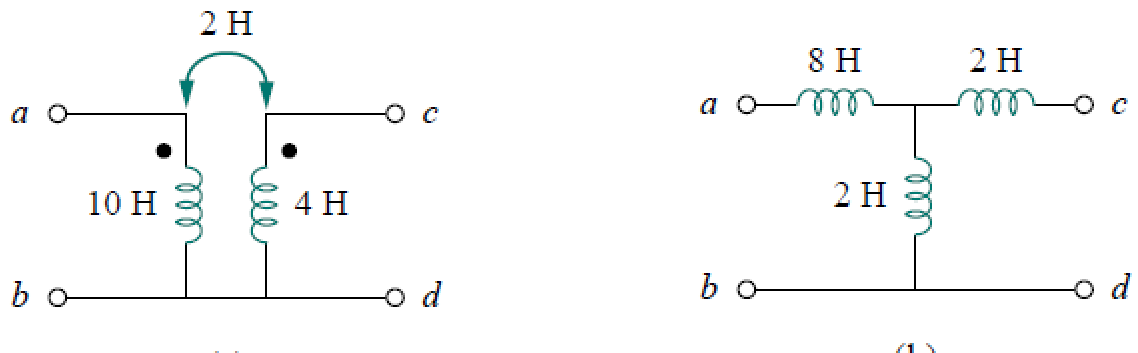
Find the input impedance of the circuit of Fig. 13.25 and the current from the voltage source.



Answer: $8.58\angle 58.05^\circ\ \Omega$, $1.165\angle -58.05^\circ$ A.

EXAMPLE 13.5

Determine the T-equivalent circuit of the linear transformer in Fig. 13.26(a).



(a) (b)

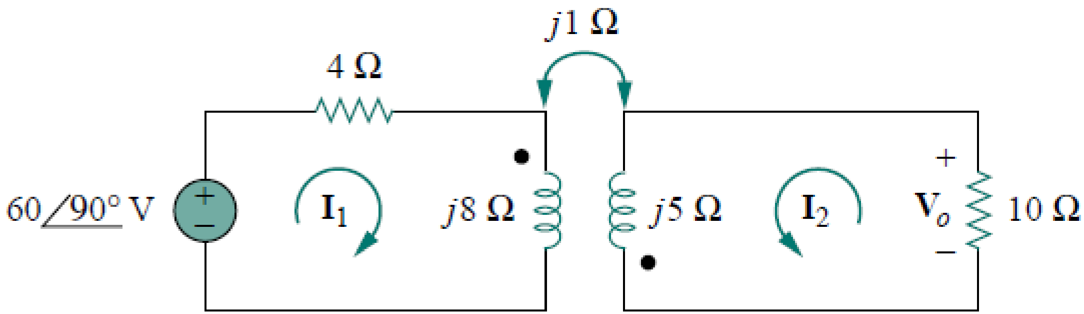
Given that $L_1 = 10$, $L_2 = 4$, and $M = 2$, the T equivalent network has the following parameters:

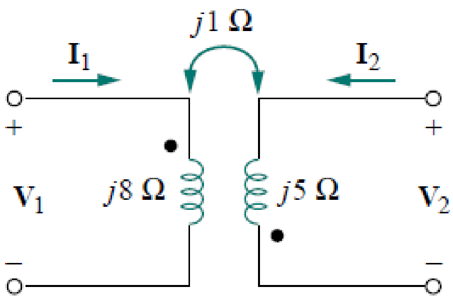
$$L_a = L_1 - M = 10 - 2 = 8 \text{ H}$$

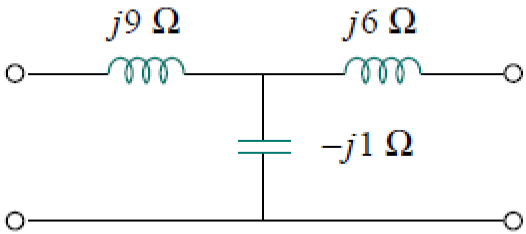
$$L_b = L_2 - M = 4 - 2 = 2 \text{ H}, \qquad L_c = M = 2 \text{ H}$$

EXAMPLE 13.6

Solve for $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{V}_o$ in Fig. 13.27 (the same circuit as for Practice Prob. 13.1) using the T-equivalent circuit for the linear transformer.

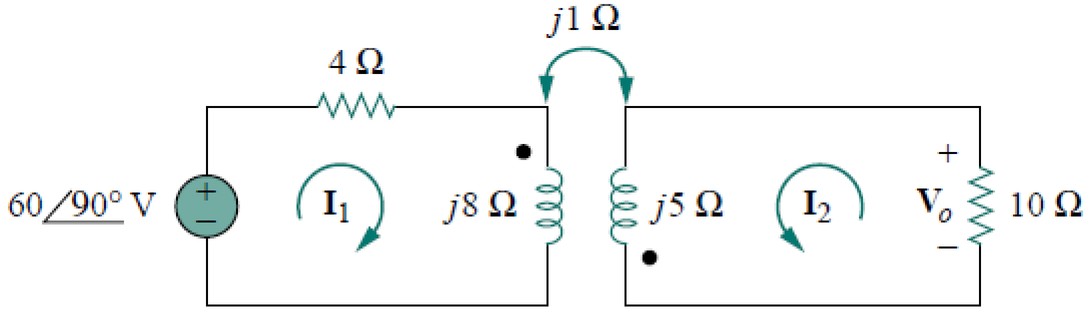


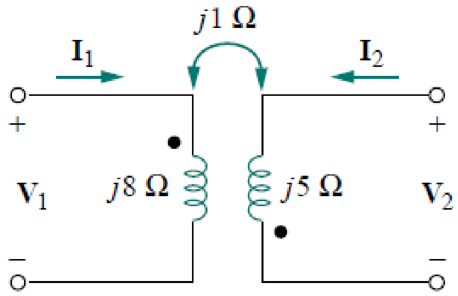


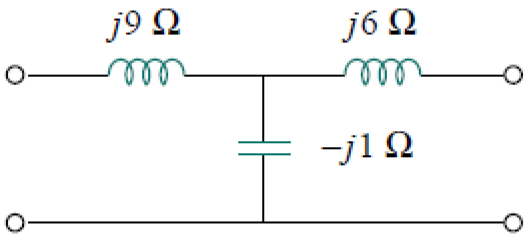


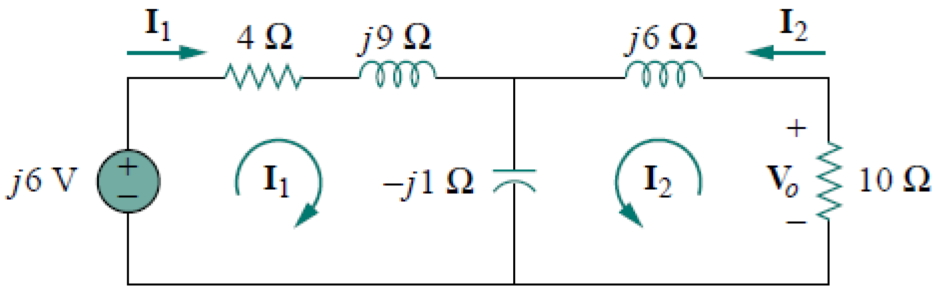
$$L_a = L_1 - (-M) = 8 + 1 = 9 \text{ H}$$

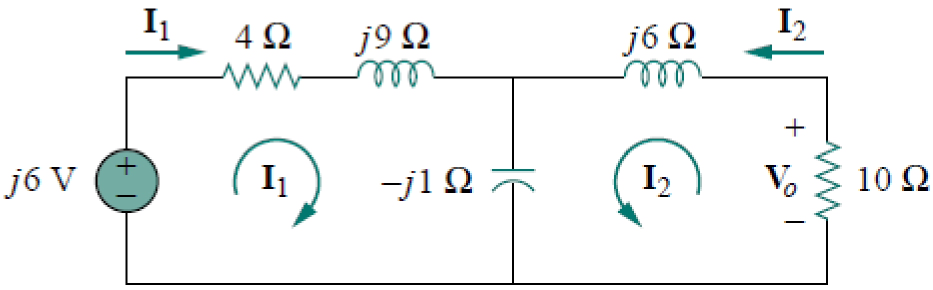
$$L_b = L_2 - (-M) = 5 + 1 = 6 \text{ H}, \qquad L_c = -M = -1 \text{ H}$$


j1 Ω
4 Ω
+
60∠90° V
I1
j8 Ω
j5 Ω
I2
Vo
10 Ω
–


j1 Ω
I1
I2
+
V1
j8 Ω
j5 Ω
V2
–


j9 Ω
j6 Ω
–j1 Ω


I1
4 Ω
j9 Ω
j6 Ω
I2
+
j6 V
I1
–j1 Ω
I2
Vo
10 Ω
–



$$j6 = \mathbf{I}_1(4 + j9 - j1) + \mathbf{I}_2(-j1)$$

$$0 = \mathbf{I}_1(-j1) + \mathbf{I}_2(10 + j6 - j1)$$

$$\mathbf{I}_2 = \frac{j6}{100} = j0.06 = 0.06\angle 90^\circ \text{ A}$$

$$\mathbf{I}_1 = (5 - j10)j0.06 = 0.6 + j0.3 \text{ A}$$

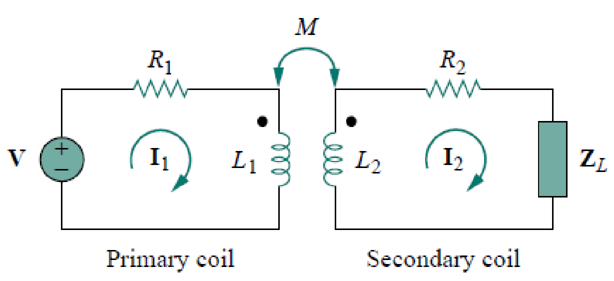
$$\mathbf{V}_o = -10\mathbf{I}_2 = -j0.6 = 0.6\angle -90^\circ \text{ V}$$

10.5 IDEAL TRANSFORMERS

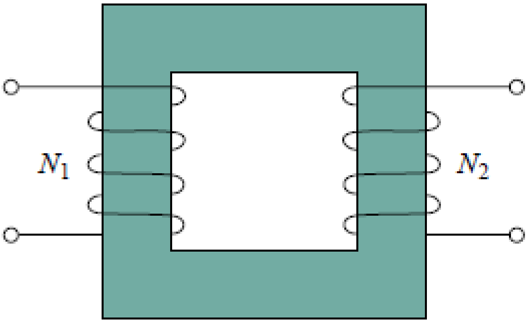
An ideal transformer is one with perfect coupling ($k = 1$). It consists of two (or more) coils with a large number of turns wound on a common core of high permeability. Because of this high permeability of the core, the flux links all the turns of both coils, thereby resulting in a perfect coupling.

A transformer is said to be ideal if it has the following properties:

1. Coils have very large reactances ($L_1, L_2, M \to \infty$).
2. Coupling coefficient is equal to unity ($k = 1$).
3. Primary and secondary coils are lossless ($R_1 = 0 = R_2$).

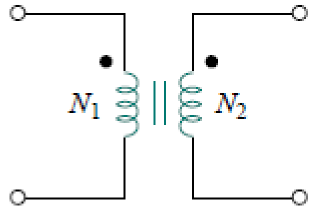


An ideal transformer is a unity-coupled, lossless transformer in which the primary and secondary coils have infinite self-inductances.



(a)

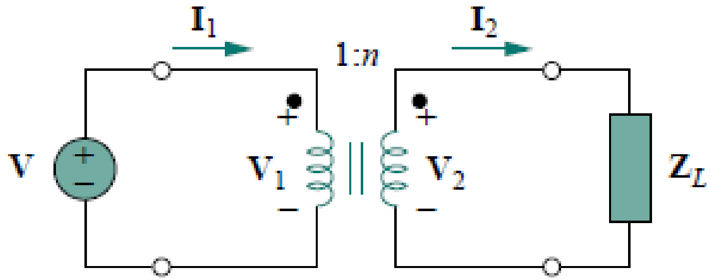
Ideal transformer,



(b)

circuit symbol for ideal transformers.

The vertical lines between the coils indicate an iron core as distinct from the air core used in linear transformers. The primary winding has N_1 turns; the secondary winding has N_2 turns.



When a sinusoidal voltage is applied to the primary winding as shown in above Fig, the same magnetic flux φ goes through both windings.

$$v_1 = N_1 \frac{d\phi}{dt} \qquad v_2 = N_2 \frac{d\phi}{dt} \qquad \frac{v_2}{v_1} = \frac{N_2}{N_1} = n$$

n is the turns ratio or transformation ratio

$$\boxed{\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{N_2}{N_1} = n} \qquad v_1 i_1 = v_2 i_2 \qquad \frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{\mathbf{V}_2}{\mathbf{V}_1} = n \qquad \boxed{\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2} = \frac{1}{n}}$$

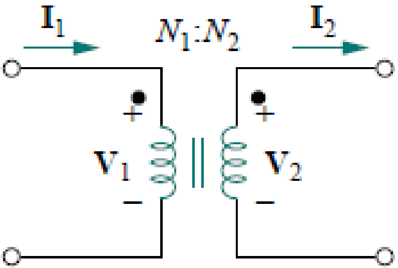
$$\frac{V_2}{V_1} = \frac{N_2}{N_1} = n$$

$$\frac{I_2}{I_1} = \frac{N_1}{N_2} = \frac{1}{n}$$

When n = 1, we generally call the transformer an ***isolation transformer***.
If n > 1, we have a ***step-up transformer***, as the voltage is increased from primary to secondary ($V_2 > V_1$).
On the other hand, if n < 1, the transformer is a ***step-down transformer***, since the voltage is decreased from primary to secondary ($V_2 < V_1$).

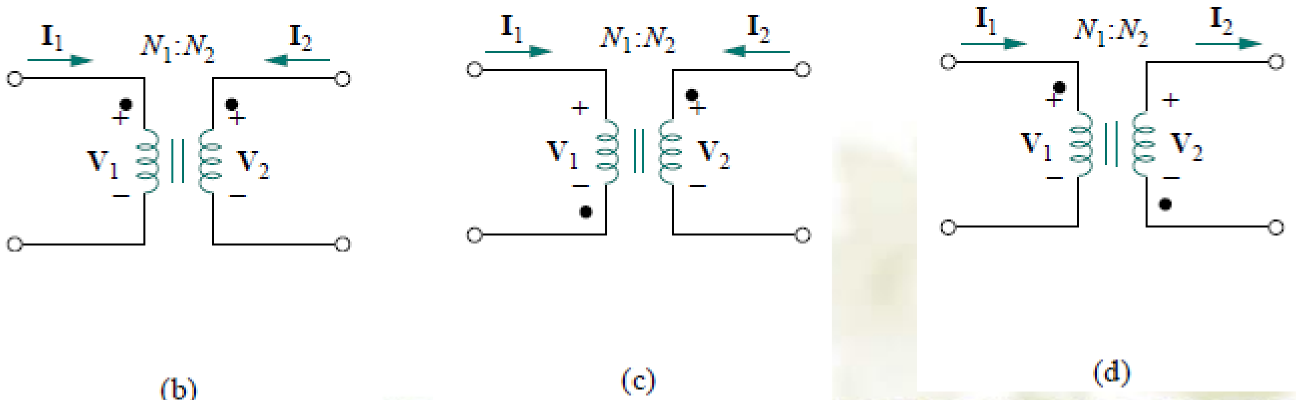
A step-up transformer is one whose secondary voltage is greater than its primary voltage.

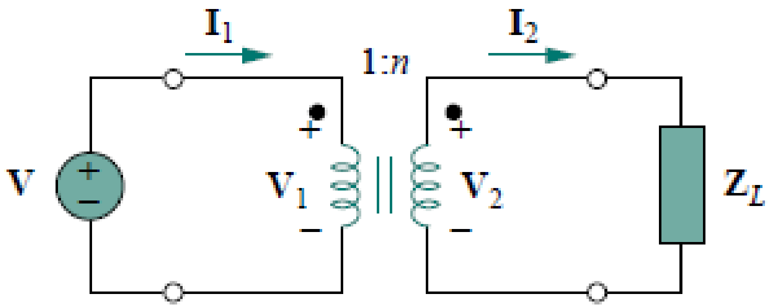
A step-down transformer is one whose secondary voltage is less than its primary voltage.



$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{N_2}{N_1} \qquad \frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2}$$

1. If V_1 and V_2 are both positive or both negative at the dotted terminals, use +n in Eq. Otherwise, use −n.
2. If I_1 and I_2 both enter into or both leave the dotted terminals, use −n in Eq. Otherwise, use +n.





$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{1}{n^2}\frac{\mathbf{V}_2}{\mathbf{I}_2}$$

$$\boxed{\mathbf{Z}_{\text{in}} = \frac{\mathbf{Z}_L}{n^2}}$$

EXAMPLE 13.7

An ideal transformer is rated at 2400/120 V, 9.6 kVA, and has 50 turns on the secondary side. Calculate: (a) the turns ratio, (b) the number of turns on the primary side, and (c) the current ratings for the primary and secondary windings.

Solution:

(a) This is a step-down transformer, since $V_1 = 2400$ V $> V_2 = 120$ V.

$$n = \frac{V_2}{V_1} = \frac{120}{2400} = 0.05$$

(b)

$$n = \frac{N_2}{N_1} \qquad \Longrightarrow \qquad 0.05 = \frac{50}{N_1}$$

or

$$N_1 = \frac{50}{0.05} = 1000 \text{ turns}$$

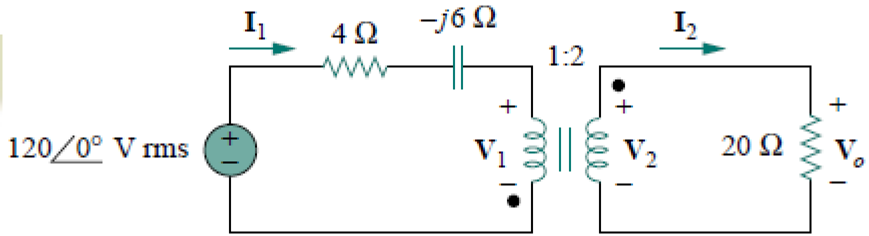
(c) $S = V_1 I_1 = V_2 I_2 = 9.6$ kVA. Hence,

$$I_1 = \frac{9600}{V_1} = \frac{9600}{2400} = 4 \text{ A}$$

$$I_2 = \frac{9600}{V_2} = \frac{9600}{120} = 80 \text{ A} \qquad \text{or} \qquad I_2 = \frac{I_1}{n} = \frac{4}{0.05} = 80 \text{ A}$$

EXAMPLE 13.8

For the ideal transformer circuit of Fig. 13.37, find: (a) the source current $\mathbf{I}_1$, (b) the output voltage $\mathbf{V}_o$, and (c) the complex power supplied by the source.



Solution:

(a) The 20-Ω impedance can be reflected to the primary side and we get

$$\mathbf{Z}_R = \frac{20}{n^2} = \frac{20}{4} = 5\ \Omega$$

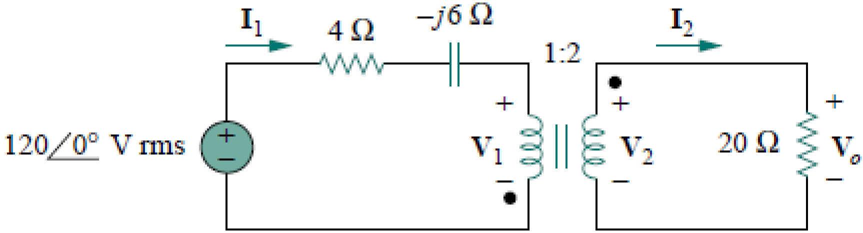
Thus,

$$\mathbf{Z}_{\text{in}} = 4 - j6 + \mathbf{Z}_R = 9 - j6 = 10.82\angle -33.69^\circ\ \Omega$$

$$\mathbf{I}_1 = \frac{120\angle 0^\circ}{\mathbf{Z}_{\text{in}}} = \frac{120\angle 0^\circ}{10.82\angle -33.69^\circ} = 11.09\angle 33.69^\circ\ \text{A}$$

EXAMPLE 13.8

For the ideal transformer circuit of Fig. 13.37, find: (a) the source current $\mathbf{I}_1$, (b) the output voltage $\mathbf{V}_o$, and (c) the complex power supplied by the source.



$$\mathbf{I}_1 = \frac{120\angle 0^\circ}{\mathbf{Z}_{in}} = \frac{120\angle 0^\circ}{10.82\angle -33.69^\circ} = 11.09\angle 33.69^\circ \text{ A}$$

(b) Since both $\mathbf{I}_1$ and $\mathbf{I}_2$ leave the dotted terminals,

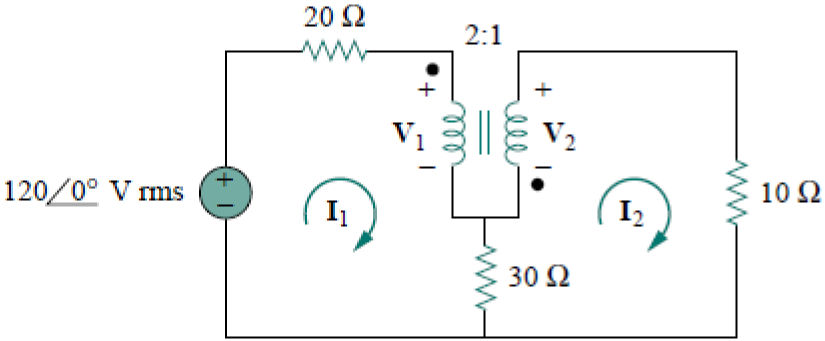
$$\mathbf{I}_2 = -\frac{1}{n}\mathbf{I}_1 = -5.545\angle 33.69^\circ \text{ A} \qquad \mathbf{V}_o = 20\mathbf{I}_2 = 110.9\angle 213.69^\circ \text{ V}$$

(c) The complex power supplied is

$$\mathbf{S} = \mathbf{V}_s\mathbf{I}_1^* = (120\angle 0^\circ)(11.09\angle -33.69^\circ) = 1330.8\angle -33.69^\circ \text{ VA}$$

EXAMPLE 13.9

Calculate the power supplied to the 10-Ω resistor in the ideal transformer circuit of Fig. 13.39.



Solution:

Reflection to the secondary or primary side cannot be done with this circuit: there is direct connection between the primary and secondary sides due to the 30-Ω resistor. We apply mesh analysis. For mesh 1,

$$-120 + (20 + 30)\mathbf{I}_1 - 30\mathbf{I}_2 + \mathbf{V}_1 = 0$$

For mesh 2,

$$-\mathbf{V}_2 + (10 + 30)\mathbf{I}_2 - 30\mathbf{I}_1 = 0$$

$$\mathbf{V}_2 = -\frac{1}{2}\mathbf{V}_1$$

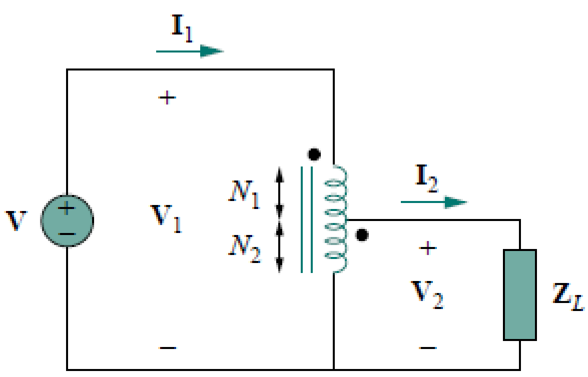
$$\mathbf{I}_2 = -2\mathbf{I}_1$$

$$\mathbf{I}_2 = -\frac{120}{165} = -0.7272 \text{ A} \qquad P = (-0.7272)^2(10) = 5.3 \text{ W}$$

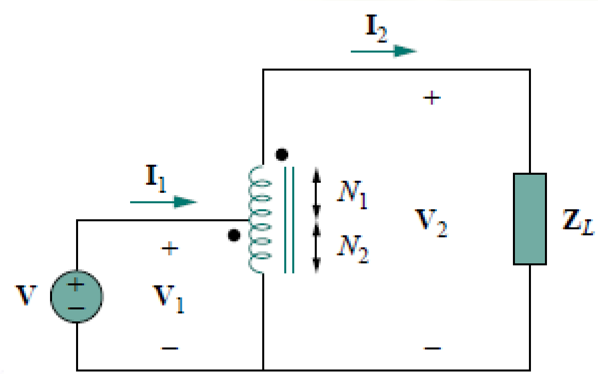
13.6 IDEAL AUTOTRANSFORMERS

Unlike the conventional two-winding transformer we have considered so far, an ***autotransformer*** has a single continuous winding with a connection point called a ***tap*** between the primary and secondary sides. The tap is often adjustable so as to provide the desired turns ratio for stepping up or stepping down the voltage. This way, a variable voltage is provided to the load connected to the autotransformer.

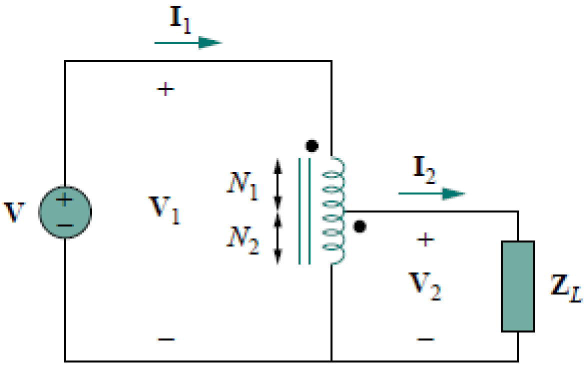
An autotransformer is a transformer in which both the primary and the secondary are in a single winding.



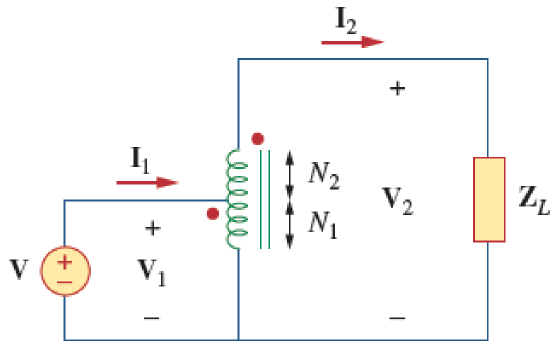
Step-down autotransformer,



step-up autotransformer.



Step-down autotransformer,



step-up autotransformer.

$$\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{N_1 + N_2}{N_2} = 1 + \frac{N_1}{N_2}$$

$$\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{N_1}{N_1 + N_2}$$

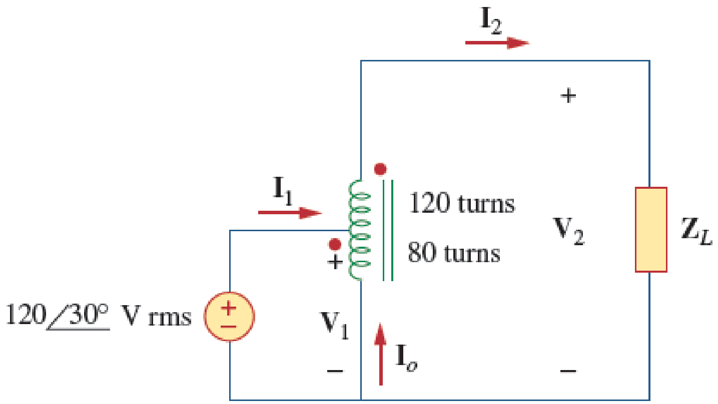
$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{N_2}{N_1 + N_2}$$

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{N_1 + N_2}{N_1} = 1 + \frac{N_2}{N_1}$$

A major difference between conventional transformers and autotransformers is that the primary and secondary sides of the autotransformer are not only coupled magnetically but also coupled conductively. The autotransformer can be used in place of a conventional transformer when electrical isolation is not required.

EXAMPLE 13.11

Refer to the autotransformer circuit in Fig. 13.44. Calculate: (a) $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_o$ if $\mathbf{Z}_L = 8 + j6\ \Omega$, and (b) the complex power supplied to the load.



Solution:

(a) This is a step-up autotransformer with $N_1 = 80$, $N_2 = 120$, $\mathbf{V}_1 = 120\underline{/30^\circ}$, so Eq. (13.67) can be used to find $\mathbf{V}_2$ by

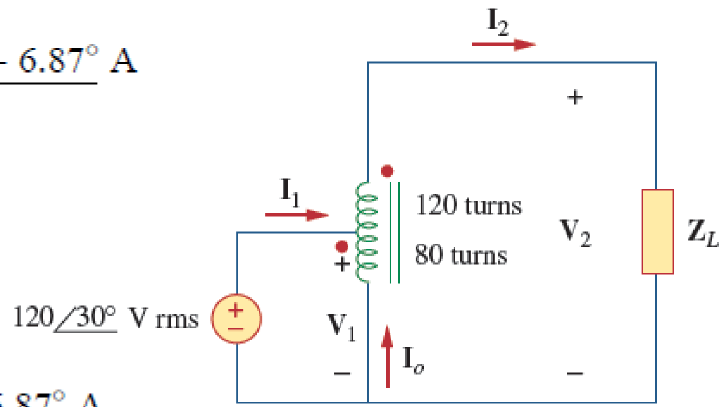
$$\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{N_1}{N_1 + N_2} = \frac{80}{200} \qquad \mathbf{V}_2 = \frac{200}{80}\mathbf{V}_1 = \frac{200}{80}(120\underline{/30^\circ}) = 300\underline{/30^\circ}\ \text{V}$$

$$\mathbf{I}_2 = \frac{\mathbf{V}_2}{\mathbf{Z}_L} = \frac{300\underline{/30^\circ}}{8 + j6} = \frac{300\underline{/30^\circ}}{10\underline{/36.87^\circ}} = 30\underline{/-6.87^\circ}\ \text{A}$$

EXAMPLE 13.11

Refer to the autotransformer circuit in Fig. 13.44. Calculate: (a) $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_o$ if $\mathbf{Z}_L = 8 + j6\ \Omega$, and (b) the complex power supplied to the load.

$$\mathbf{I}_2 = \frac{\mathbf{V}_2}{\mathbf{Z}_L} = \frac{300\underline{/30^\circ}}{8 + j6} = \frac{300\underline{/30^\circ}}{10\underline{/36.87^\circ}} = 30\underline{/-6.87^\circ}\ \text{A}$$



$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{N_1 + N_2}{N_1} = \frac{200}{80}$$

$$\mathbf{I}_1 = \frac{200}{80}\mathbf{I}_2 = \frac{200}{80}(30\underline{/-6.87^\circ}) = 75\underline{/-6.87^\circ}\ \text{A}$$

$$\mathbf{I}_1 + \mathbf{I}_o = \mathbf{I}_2 \qquad \mathbf{I}_o = \mathbf{I}_2 - \mathbf{I}_1 = 30\underline{/-6.87^\circ} - 75\underline{/-6.87^\circ} = 45\underline{/173.13^\circ}\ \text{A}$$

(b) The complex power supplied to the load is

$$\mathbf{S}_2 = \mathbf{V}_2\mathbf{I}_2^* = |\mathbf{I}_2|^2\mathbf{Z}_L = (30)^2(10\underline{/36.87^\circ}) = 9\underline{/36.87^\circ}\ \text{kVA}$$

Summary and Review

- Mutual inductance describes the voltage induced at the ends of a coil due to the magnetic field generated by a second coil.
- The dot convention allows a sign to be assigned to the mutual inductance term.
- According to the dot convention, a current entering the dotted terminal of one coil produces an open-circuit voltage with a positive voltage reference at the dotted terminal of the second coil.
- The total energy stored in a pair of coupled coils has three separate terms: the energy stored in each self-inductance ($1/2Li^2$), and the energy stored in the mutual inductance (Mi_1i_2).

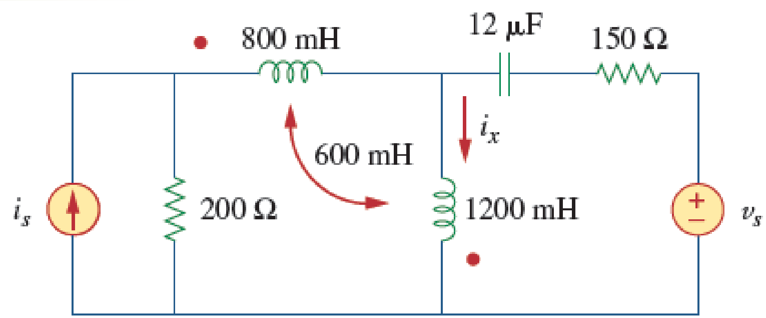
- A linear transformer consists of two coupled coils: the primary winding and the secondary winding.
- The coupling coefficient is restricted to values between 0 and 1.
- Ideal transformer is a useful approximation for practical iron-core transformers. The coupling coefficient is taken to be unity, and the inductance values are assumed to be infinite.

Assignment (page 599)

Problems 13.7, 13.15, 13.33, 13.41,
13.53, 13.68.

1 Use mesh analysis to find i_x in Fig. 1 , where

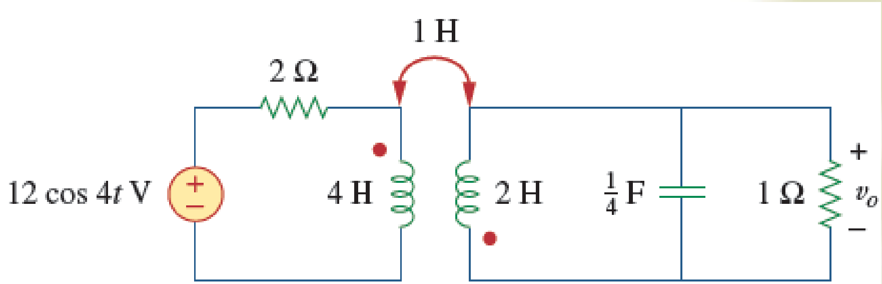
$i_s = 4\cos(600t)$ A and $v_s = 110\cos(600t + 30°)$



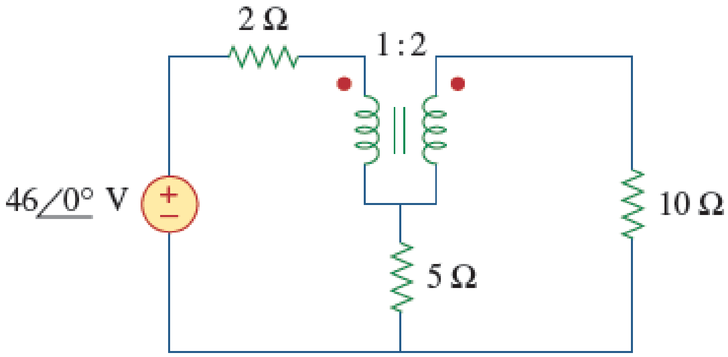
2 In the circuit of Fig. 2 ,

(a) find the coupling coefficient,

(b) calculate v_o



3 Find the power absorbed by the 10-Ω resistor in the ideal transformer circuit of Fig. 3.



Fundamentals of Electric Circuits 2021.6

Chapter14

Frequency Response

Chapter14 Frequency Response

14.1 Introduction

14.2 Transfer Function

14.5 Series Resonance

14.6 Parallel Resonance

13.7 Passive Filters

13.8 Active Filters

14.1 Introduction

In our sinusoidal circuit analysis, we have learned how to find voltages and currents in a circuit with a constant frequency source. If we let the amplitude of the sinusoidal source remain constant and vary the frequency, we obtain the circuit's *frequency response*.

> The **frequency response** of a circuit is the variation in its behavior with change in signal frequency.

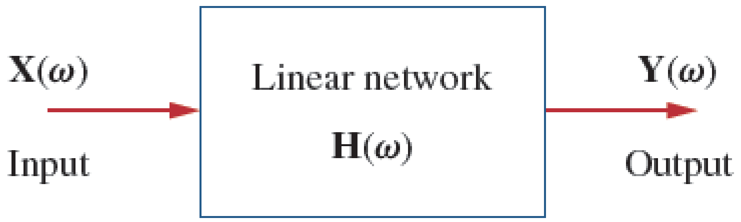
14.1 Introduction

- The sinusoidal steady-state frequency responses of circuits are of significance in many applications, especially in communications and control systems.
- A specific application is in electric filters that block out or eliminate signals with unwanted frequencies and pass signals of the desired frequencies.
- Filters are used in radio, TV, and telephone systems to separate one broadcast frequency from another.

14.2 Transfer Function

The transfer function $\mathbf{H}(\omega)$(also called the network function) is a useful analytical tool for finding the frequency response of a circuit.

A transfer function is the frequency-dependent ratio of a forced function to a forcing function (or of an output to an input).



The **transfer function** $\mathbf{H}(\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $\mathbf{Y}(\omega)$ (an element voltage or current) to a phasor input $\mathbf{X}(\omega)$ (source voltage or current).

$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}}$$

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)}$$

$\mathbf{H}(\omega)$ has a magnitude $H(\omega)$ and a phase ϕ; $\mathbf{H}(\omega) = H(\omega)\underline{/\phi}$

The transfer function $\mathbf{H}(\omega)$ can be expressed in terms of its numerator polynomial $\mathbf{N}(\omega)$ and denominator polynomial $\mathbf{D}(\omega)$ as

$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{N}(\omega)}{\mathbf{D}(\omega)}}$$

The roots of $\mathbf{N}(\omega)=0$ are called the zeros of $\mathbf{H}(\omega)$ and are usually represented as $j\omega = z_1, z_2, \ldots$

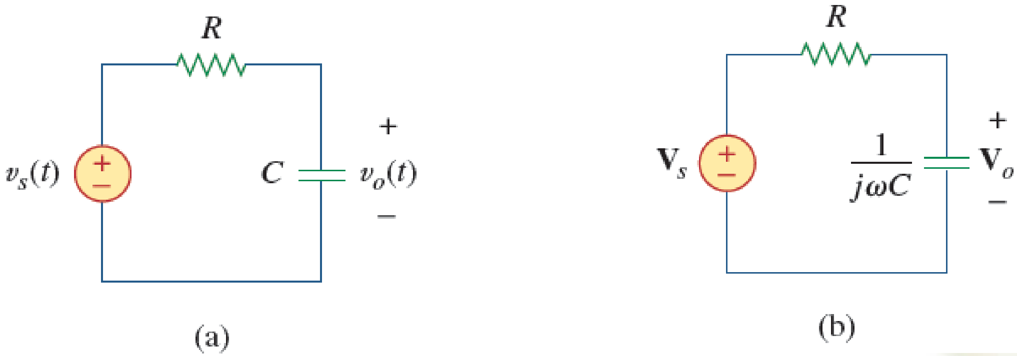
Similarly, the roots of $\mathbf{D}(\omega)=0$ are the poles of $\mathbf{D}(\omega)$ and are represented as $j\omega = p_1, p_2, \ldots..$

A **zero,** as a *root* of the numerator polynomial, is a value that results in a zero value of the function. A **pole,** as a *root* of the denominator polynomial, is a value for which the function is infinite.

To avoid complex algebra, it is expedient to replace $j\omega$ temporarily with s when working with $\mathbf{H}(\omega)$ and replace s with $j\omega$ at the end.

Example 14.1

For the *RC* circuit in Fig. 14.2(a), obtain the transfer function $\mathbf{V}_o/\mathbf{V}_s$ and its frequency response. Let $v_s = V_m \cos \omega t$.



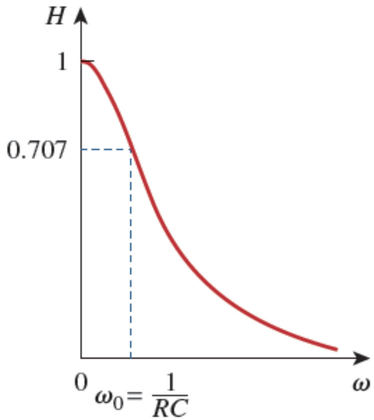
Solution:

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

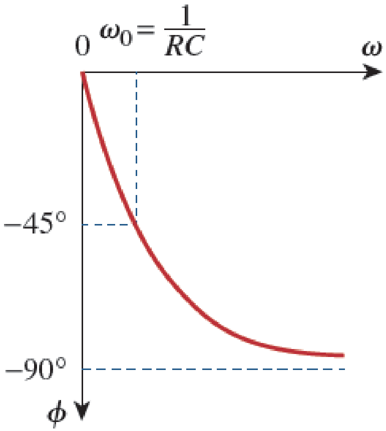
$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

where $\omega_0 = 1/RC$. To plot H and ϕ for $0 < \omega < \infty$, we obtain their values at some critical points and then sketch.

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC} \qquad H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$



(a)



(b)

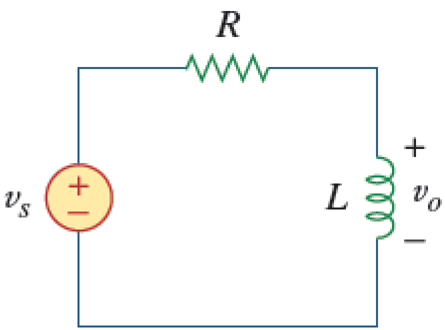
TABLE 14.1

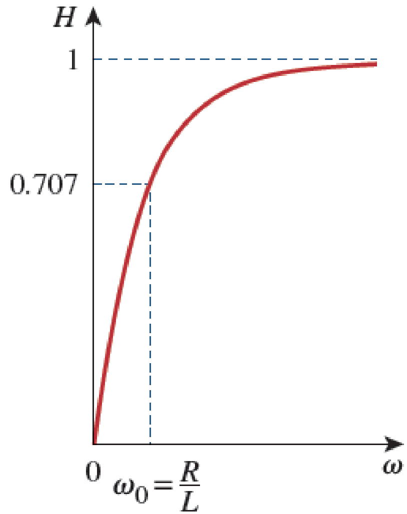
For Example 14.1.

ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	−84°
1	0.71	−45°	20	0.05	−87°
2	0.45	−63°	100	0.01	−89°
3	0.32	−72°	∞	0	−90°

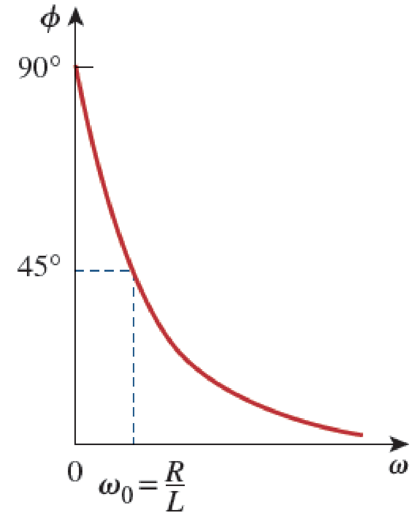
Practice Problem 14.1

Obtain the transfer function $\mathbf{V}_o/\mathbf{V}_s$ of the RL circuit in Fig. 14.4, assuming $v_s = V_m \cos \omega t$. Sketch its frequency response.





(a)

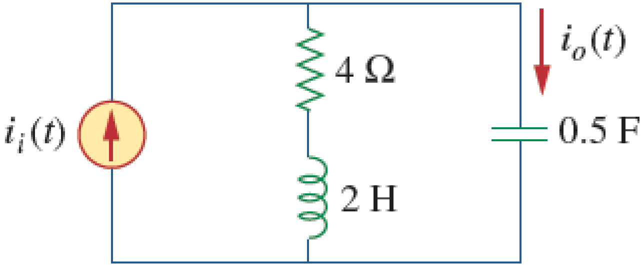


(b)

Answer: $j\omega L/(R + j\omega L)$;

Example 14.2

For the circuit in Fig. 14.6, calculate the gain $\mathbf{I}_o(\omega)/\mathbf{I}_i(\omega)$ and its poles and zeros.



Solution:

By current division,

$$\mathbf{I}_o(\omega) = \frac{4 + j2\omega}{4 + j2\omega + 1/j0.5\omega}\mathbf{I}_i(\omega)$$

$$\frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)} = \frac{j0.5\omega(4 + j2\omega)}{1 + j2\omega + (j\omega)^2} = \frac{s(s+2)}{s^2 + 2s + 1}, \qquad s = j\omega$$

The zeros are at $\quad s(s+2) = 0 \quad \Rightarrow \quad z_1 = 0, z_2 = -2$

The poles are at $\quad s^2 + 2s + 1 = (s+1)^2 = 0$

Thus, there is a repeated pole (or double pole) at $p = -1$.

14.5 Series Resonance

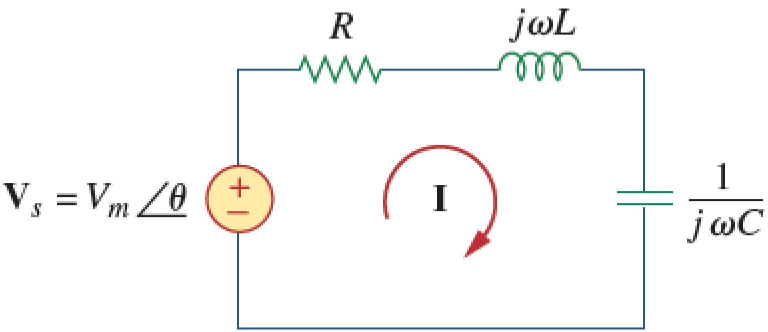
The most prominent feature of the frequency response of a circuit may be the sharp peak (or *resonant peak)* exhibited in its amplitude characteristic.

Resonance occurs in any circuit that has at least one inductor and one capacitor.

Resonance is a condition in an *RLC* circuit in which the capacitive and inductive reactances are equal in magnitude, thereby resulting in a purely resistive impedance.

Resonant circuits (series or parallel) are useful for constructing filters, as their transfer functions can be highly frequency selective. They are used in many applications such as selecting the desired stations in radio and TV receivers.

Consider the series *RLC circuit shown in the frequency* domain.



$$\mathbf{Z} = \mathbf{H}(\omega) = \frac{\mathbf{V}_s}{\mathbf{I}} = R + j\omega L + \frac{1}{j\omega C}$$

$$\mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

Resonance results when the imaginary part of the transfer function is zero.

$$\mathrm{Im}(\mathbf{Z}) = \omega L - \frac{1}{\omega C} = 0$$

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}}$$

The value of ω that satisfies this condition is called the *resonant frequency* ω_0

Note that at resonance:

$$\mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}}$$

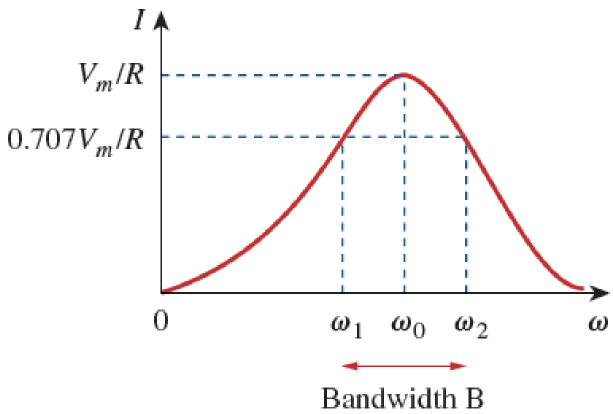
1. The impedance is purely resistive, thus, $\mathbf{Z} = R$. In other words, the LC series combination acts like a short circuit, and the entire voltage is across R.
2. The voltage $\mathbf{V}_s$ and the current $\mathbf{I}$ are in phase, so that the power factor is unity.
3. The magnitude of the transfer function $\mathbf{H}(\omega) = \mathbf{Z}(\omega)$ is minimum.
4. The inductor voltage and capacitor voltage can be much more than the source voltage.

$$|\mathbf{V}_L| = \frac{V_m}{R}\omega_0 L = QV_m \qquad |\mathbf{V}_C| = \frac{V_m}{R}\frac{1}{\omega_0 C} = QV_m$$

where Q is the quality factor,

The frequency response of the circuit's current magnitude

$$I = |\mathbf{I}| = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$



$$P(\omega) = \frac{1}{2} I^2 R$$

The highest power dissipated occurs at resonance, when $I = V_m/R$, so that

$$P(\omega_0) = \frac{1}{2}\frac{V_m^2}{R} \qquad P(\omega_1) = P(\omega_2) = \frac{(V_m/\sqrt{2})^2}{2R} = \frac{V_m^2}{4R}$$

Hence, ω_1 and ω_2 are called the *half-power frequencies*.

The half-power frequencies are obtained by setting Z equal to $\sqrt{2}R$, and writing

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}R$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

Although the height of the curve in Fig. 14.22 is determined by R, the width of the curve depends on other factors. The width of the response curve depends on the *bandwidth* B, which is defined as the difference between the two half-power frequencies, $B = \omega_2 - \omega_1$

The "sharpness" of the resonance in a resonant circuit is measured quantitatively by the *quality factor Q*. At resonance, the reactive energy in the circuit oscillates between the inductor and the capacitor. The quality factor relates the maximum or peak energy stored to the energy dissipated in the circuit per cycle of oscillation:

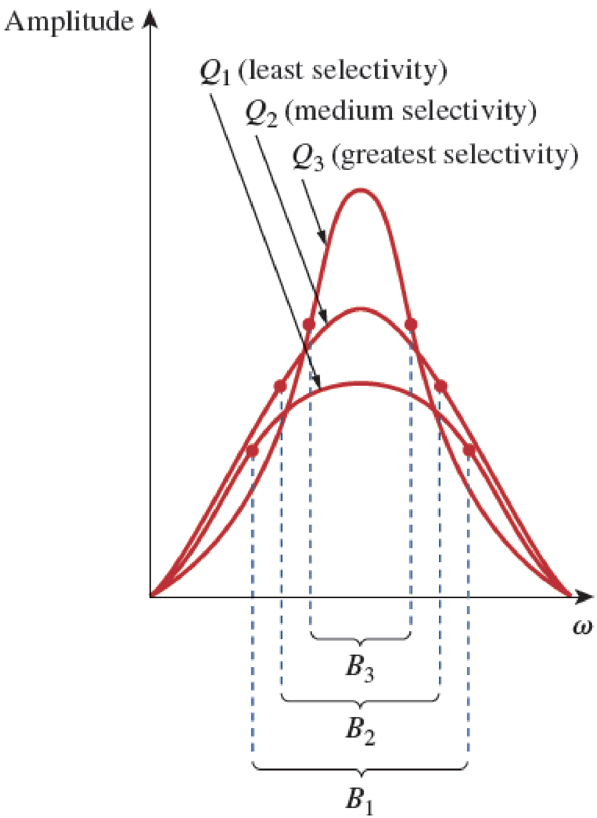
$$Q = 2\pi \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}}$$

$$Q = 2\pi \frac{\frac{1}{2}LI^2}{\frac{1}{2}I^2R(1/f_0)} = \frac{2\pi f_0 L}{R}$$

$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}}$$

$$\boxed{B = \frac{R}{L} = \frac{\omega_0}{Q}}$$

The **quality factor** of a resonant circuit is the ratio of its resonant frequency to its bandwidth.



The higher the circuit Q, the smaller the bandwidth.

The quality factor is a measure of the selectivity (or "sharpness" of resonance) of the circuit.

The higher the value of *Q*, the more selective the circuit is but the smaller the bandwidth.

The *selectivity* of an RLC circuit is the ability of the circuit to respond to a certain frequency and discriminate against all other frequencies.

$$\boxed{B = \frac{R}{L} = \frac{\omega_0}{Q}}$$

A resonant circuit is designed to operate at or near its resonant frequency. It is said to be a *high-Q circuit* when its quality factor is equal to or greater than 10. For high-Q circuits ($Q \geq 10$), the half-power frequencies are, for all practical purposes, symmetrical around the resonant frequency and can be approximated as

$$\boxed{\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}}$$

High-Q circuits are used often in communications networks.

We see that a resonant circuit is characterized by five related parameters: the two half-power frequencies ω_1 and ω_2, the resonant frequency ω_0, the bandwidth B, and the quality factor Q.

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$B = \frac{R}{L} = \frac{\omega_0}{Q}$$

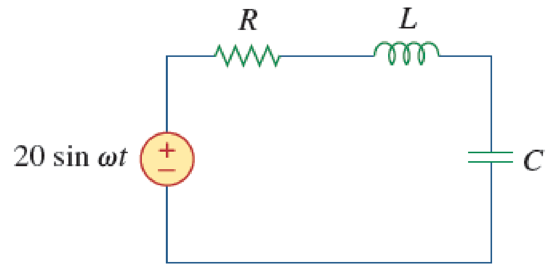
$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}$$

High-Q circuits

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$

Example 14.7

In the circuit of Fig. 14.24, $R = 2\ \Omega$, $L = 1$ mH, and $C = 0.4\ \mu$F. (a) Find the resonant frequency and the half-power frequencies. (b) Calculate the quality factor and bandwidth. (c) Determine the amplitude of the current at ω_0, ω_1, and ω_2.



(a) The resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s}$$

METHOD 1 The lower half-power frequency is

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$= -\frac{2}{2 \times 10^{-3}} + \sqrt{(10^3)^2 + (50 \times 10^3)^2}$$

$$= -1 + \sqrt{1 + 2500} \text{ krad/s} = 49 \text{ krad/s}$$

Similarly, the upper half-power frequency is

$$\omega_2 = 1 + \sqrt{1 + 2500} \text{ krad/s} = 51 \text{ krad/s}$$

(b) The bandwidth is

$$B = \omega_2 - \omega_1 = 2 \text{ krad/s}$$

The quality factor is

$$Q = \frac{\omega_0}{B} = \frac{50}{2} = 25$$

METHOD 2 Alternatively, we could find

$$Q = \frac{\omega_0 L}{R} = \frac{50 \times 10^3 \times 10^{-3}}{2} = 25$$

From Q, we find

$$B = \frac{\omega_0}{Q} = \frac{50 \times 10^3}{25} = 2 \text{ krad/s}$$

Since $Q > 10$, this is a high-Q circuit and we can obtain the half-power frequencies as

$$\omega_1 = \omega_0 - \frac{B}{2} = 50 - 1 = 49 \text{ krad/s}$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 50 + 1 = 51 \text{ krad/s}$$

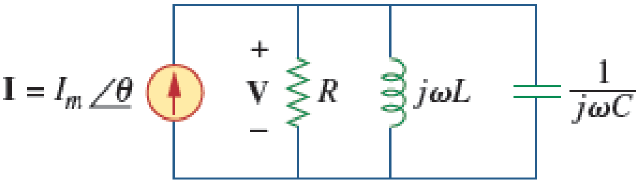
(c) At $\omega = \omega_0$,

$$I = \frac{V_m}{R} = \frac{20}{2} = 10 \text{ A}$$

At $\omega = \omega_1, \omega_2$,

$$I = \frac{V_m}{\sqrt{2}R} = \frac{10}{\sqrt{2}} = 7.071 \text{ A}$$

14.6 Parallel Resonance

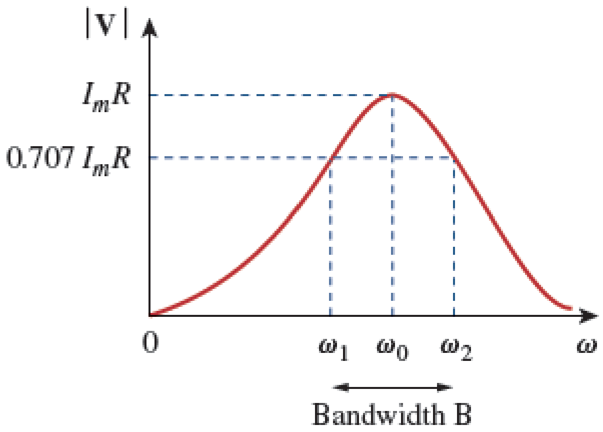


$$\mathbf{Y} = H(\omega) = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

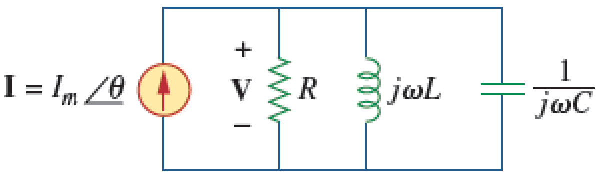
Resonance occurs when the imaginary part of **Y** is zero,

$$\omega C - \frac{1}{\omega L} = 0$$



$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}\ \text{rad/s}}$$

14.6 Parallel Resonance



$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

$$\omega C - \frac{1}{\omega L} = 0$$

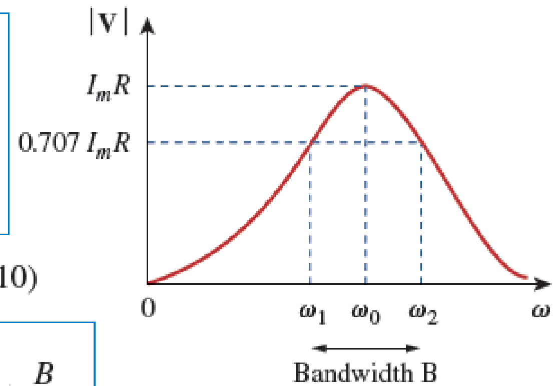
Notice that at resonance, the parallel LC combination acts like an open circuit, so that the entire current flows through R. Also, the inductor and capacitor current can be much more than the source current at resonance.

$$|\mathbf{I}_L| = \frac{I_m R}{\omega_0 L} = Q I_m$$

$$|\mathbf{I}_C| = \omega_0 C I_m R = Q I_m$$

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$



$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

Again, for high-Q circuits ($Q \geq 10$)

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$$

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$

TABLE 14.4

Summary of the characteristics of resonant *RLC* circuits.

Characteristic	Series circuit	Parallel circuit
Resonant frequency, ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Quality factor, Q	$\dfrac{\omega_0 L}{R}$ or $\dfrac{1}{\omega_0 RC}$	$\dfrac{R}{\omega_0 L}$ or $\omega_0 RC$
Bandwidth, B	$\dfrac{\omega_0}{Q}$	$\dfrac{\omega_0}{Q}$
Half-power frequencies, ω_1, ω_2	$\omega_0\sqrt{1+\left(\dfrac{1}{2Q}\right)^2} \pm \dfrac{\omega_0}{2Q}$	$\omega_0\sqrt{1+\left(\dfrac{1}{2Q}\right)^2} \pm \dfrac{\omega_0}{2Q}$
For $Q \geq 10$, ω_1, ω_2	$\omega_0 \pm \dfrac{B}{2}$	$\omega_0 \pm \dfrac{B}{2}$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

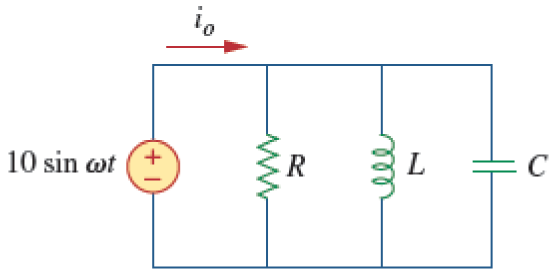
$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

Example 14.8

In the parallel RLC circuit of Fig. 14.27, let $R = 8\ \text{k}\Omega$, $L = 0.2$ mH, and $C = 8\ \mu$F. (a) Calculate ω_0, Q, and B. (b) Find ω_1 and ω_2. (c) Determine the power dissipated at ω_0, ω_1, and ω_2.



Solution:

(a)

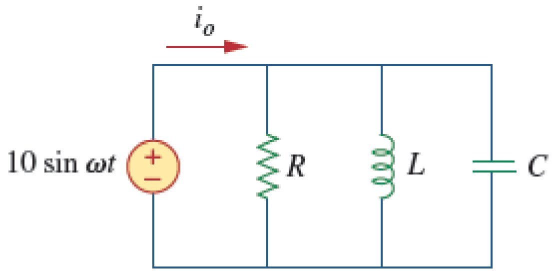
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2 \times 10^{-3} \times 8 \times 10^{-6}}} = \frac{10^5}{4} = 25\ \text{krad/s}$$

$$Q = \frac{R}{\omega_0 L} = \frac{8 \times 10^3}{25 \times 10^3 \times 0.2 \times 10^{-3}} = 1{,}600$$

$$B = \frac{\omega_0}{Q} = 15.625\ \text{rad/s}$$

Example 14.8

In the parallel RLC circuit of Fig. 14.27, let $R = 8\ \text{k}\Omega$, $L = 0.2$ mH, and $C = 8\ \mu\text{F}$. (a) Calculate ω_0, Q, and B. (b) Find ω_1 and ω_2. (c) Determine the power dissipated at ω_0, ω_1, and ω_2.



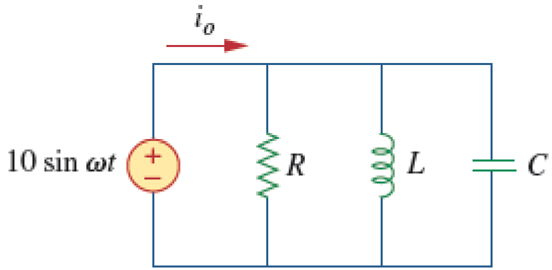
(b) Due to the high value of Q, we can regard this as a high-Q circuit, Hence,

$$\omega_1 = \omega_0 - \frac{B}{2} = 25{,}000 - 7.812 = 24{,}992 \text{ rad/s}$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 25{,}000 + 7.812 = 25{,}008 \text{ rad/s}$$

Example 14.8

In the parallel *RLC* circuit of Fig. 14.27, let $R = 8\ \text{k}\Omega$, $L = 0.2\ \text{mH}$, and $C = 8\ \mu\text{F}$. (a) Calculate ω_0, Q, and B. (b) Find ω_1 and ω_2. (c) Determine the power dissipated at ω_0, ω_1, and ω_2.



(c) At $\omega = \omega_0$, $\mathbf{Y} = 1/R$ or $\mathbf{Z} = R = 8\ \text{k}\Omega$. Then

$$\mathbf{I}_o = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{10\underline{/-90^\circ}}{8{,}000} = 1.25\underline{/-90^\circ}\ \text{mA}$$

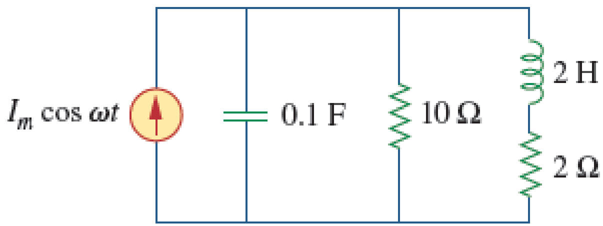
Since the entire current flows through R at resonance, the average power dissipated at $\omega = \omega_0$ is

$$P = \frac{1}{2}|\mathbf{I}_o|^2 R = \frac{1}{2}(1.25 \times 10^{-3})^2(8 \times 10^3) = 6.25\ \text{mW}$$

At $\omega = \omega_1, \omega_2,$ $$P = \frac{V_m^2}{4R} = 3.125\ \text{mW}$$

Example 14.9

Determine the resonant frequency of the circuit in Fig. 14.28.



Solution:

The input admittance is

$$\mathbf{Y} = j\omega 0.1 + \frac{1}{10} + \frac{1}{2 + j\omega 2} = 0.1 + j\omega 0.1 + \frac{2 - j\omega 2}{4 + 4\omega^2}$$

At resonance, Im(**Y**) = 0 and $\omega_0 0.1 - \dfrac{2\omega_0}{4 + 4\omega_0^2} = 0 \quad \Rightarrow \quad \omega_0 = 2 \text{ rad/s}$

The **transfer function** $\mathbf{H}(\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $\mathbf{Y}(\omega)$ (an element voltage or current) to a phasor input $\mathbf{X}(\omega)$ (source voltage or current).

$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}}$$

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$$

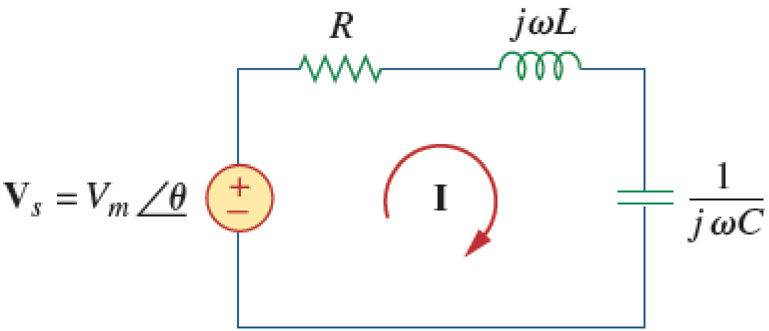
$$\mathbf{H}(\omega) = \text{Transfer Impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)}$$

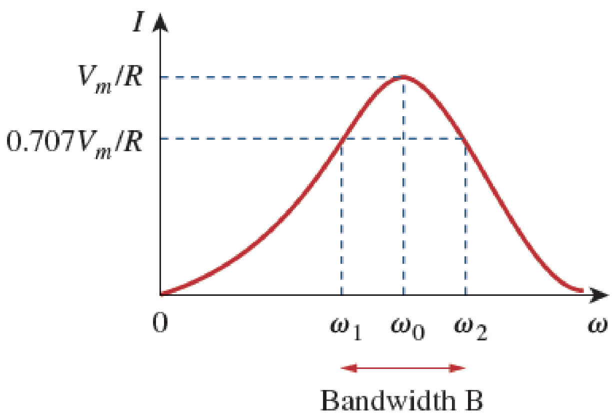
$$\mathbf{H}(\omega) = \text{Transfer Admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)}$$

$\mathbf{H}(\omega)$ has a magnitude $H(\omega)$ and a phase ϕ; $\mathbf{H}(\omega) = H(\omega)\angle\phi$

Resonance is a condition in an *RLC* circuit in which the capacitive and inductive reactances are equal in magnitude, thereby resulting in a purely resistive impedance.



series *RLC circuit*



$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

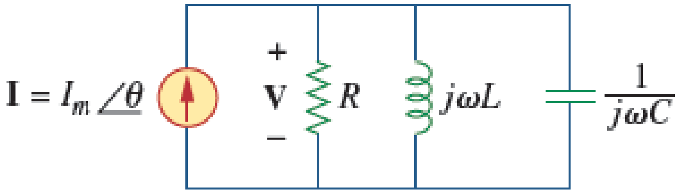
$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

$$B = \frac{R}{L} = \frac{\omega_0}{Q}$$

High-Q circuits

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$



$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

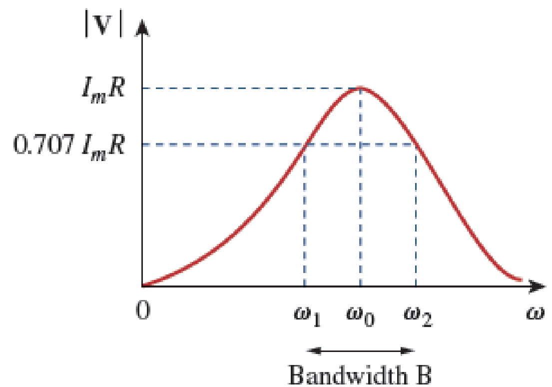
Parallel *RLC circuit*

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$$

$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$



Again, for high-Q circuits ($Q \geq 10$)

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$

$$|\mathbf{I}_L| = \frac{I_m R}{\omega_0 L} = Q I_m$$

$$|\mathbf{I}_C| = \omega_0 C I_m R = Q I_m$$

14.7 Passive Filters

A **filter** is a circuit that is designed to pass signals with desired frequencies and reject or attenuate others.

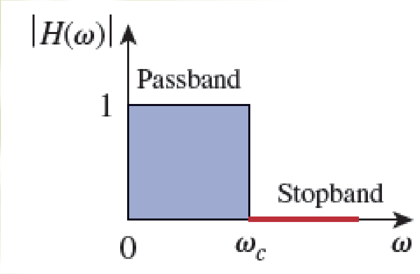
As a frequency-selective device, a filter can be used to limit the frequency spectrum of a signal to some specified band of frequencies.

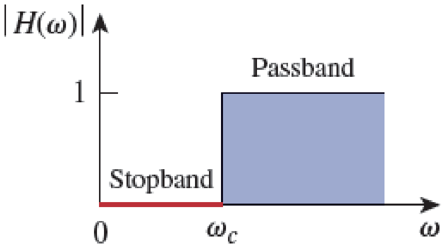
Filters are the circuits used in radio and TV receivers to allow us to select one desired signal out of a multitude of broadcast signals in the environment.

A filter is a *passive filter* if it consists of only passive elements R, L, and C. It is said to be an *active filter* if it consists of active elements (such as transistors and op amps) in addition to passive elements R, L, and C.

There are four types of filters whether passive or active:

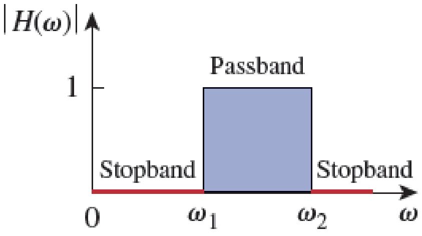
1. A lowpass filter passes low frequencies and stops high frequencies.

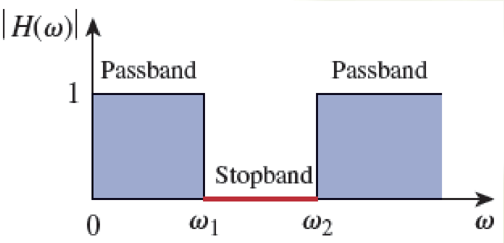




2. A highpass filter passes high frequencies and rejects low frequencies.

3. A bandpass filter passes frequencies within a frequency band and blocks or attenuates frequencies outside the band.



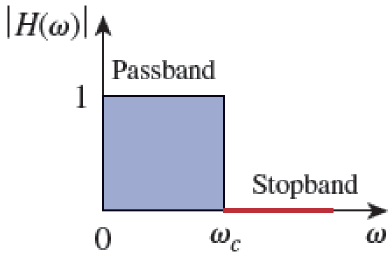


4. A bandstop filter passes frequencies outside a frequency band and blocks or attenuates frequencies within the band.

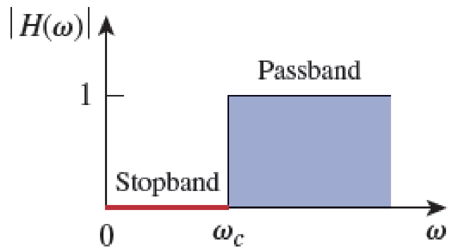
TABLE 14.5

Summary of the characteristics of ideal filters.

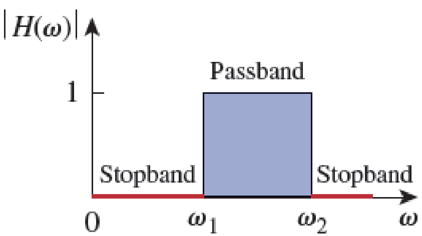
Type of Filter	$H(0)$	$H(\infty)$	$H(\omega_c)$ or $H(\omega_0)$
Lowpass	1	0	$1/\sqrt{2}$
Highpass	0	1	$1/\sqrt{2}$
Bandpass	0	0	1
Bandstop	1	1	0



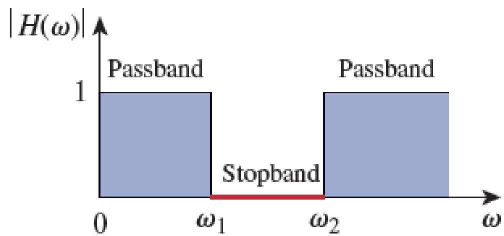
lowpass filter



highpass filter



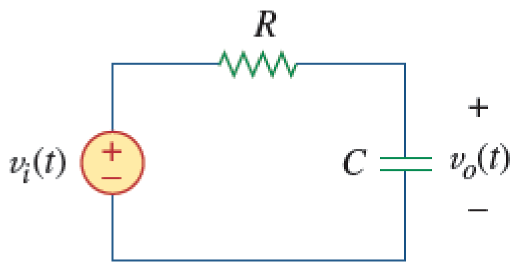
bandpass filter



bandstop filter

14.7.1 Lowpass Filter

A typical lowpass filter is formed when the output of an RC circuit is taken off the capacitor.

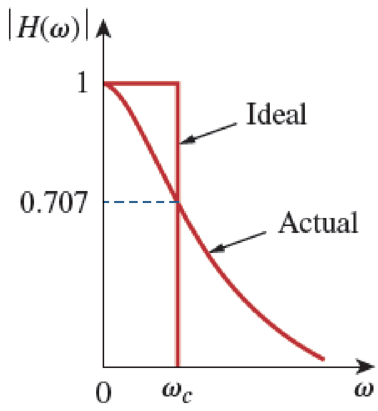


$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{1/j\omega C}{R + 1/j\omega C}$$

$$\mathbf{H}(\omega) = \frac{1}{1 + j\omega RC}$$

$$\mathbf{H}(0) = 1, \ \mathbf{H}(\infty) = 0$$

$$H(\omega_c) = \frac{1}{\sqrt{1 + \omega_c^2 R^2 C^2}} = \frac{1}{\sqrt{2}}$$

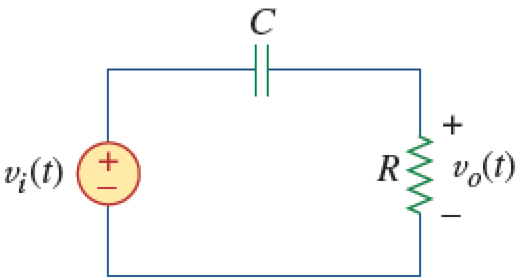


cutoff frequency $\omega_c = \frac{1}{RC}$ (rolloff frequency)

A **lowpass filter** is designed to pass only frequencies from dc up to the cutoff frequency ω_c.

14.7.2 Highpass Filter

A highpass filter is formed when the output of an RC circuit is taken off the resistor.



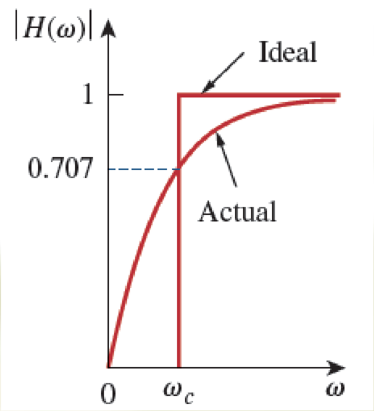
$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{R}{R + 1/j\omega C}$$

$$\mathbf{H}(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

$$\mathbf{H}(0) = 0, \ \mathbf{H}(\infty) = 1$$

the corner or cutoff frequency is

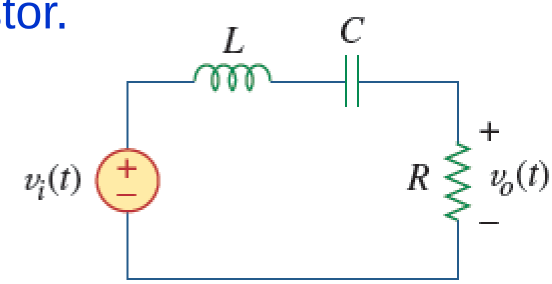
$$\omega_c = \frac{1}{RC}$$



A highpass filter is designed to pass all frequencies above its cutoff frequency ω_c.

14.7.3 Bandpass Filter

The RLC series resonant circuit provides a bandpass filter when the output is taken off the resistor.



$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{R}{R + j(\omega L - 1/\omega C)}$$

$$\mathbf{H}(0) = 0, \ \mathbf{H}(\infty) = 0$$

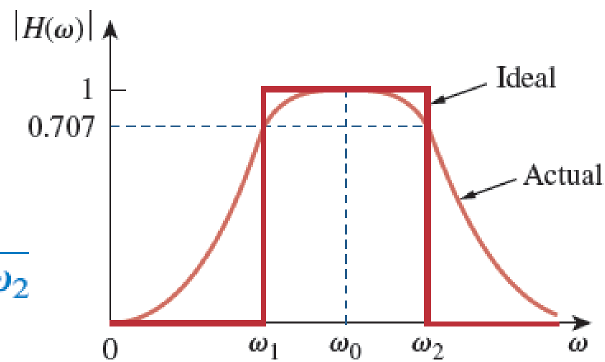
center frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

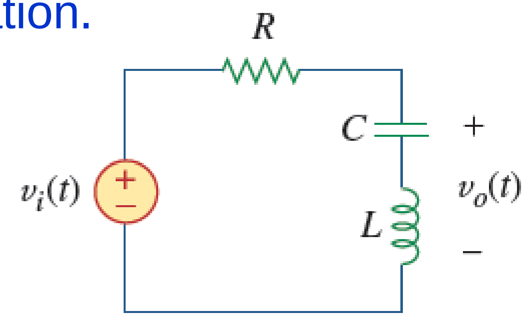
$$\omega_0 = \sqrt{\omega_1 \omega_2}$$



A bandpass filter is designed to pass all frequencies within a band of frequencies, $\omega_1 < \omega < \omega_2$.

14.7.4 Bandstop Filter

A bandstop filter is formed when the output RLC series resonant circuit is taken off the LC series combination.



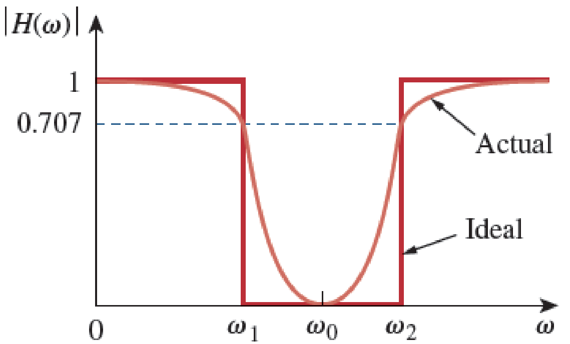
$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{j(\omega L - 1/\omega C)}{R + j(\omega L - 1/\omega C)}$$

$$\mathbf{H}(0) = 1, \quad \mathbf{H}(\infty) = 1$$

the center frequency is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

ω_0 is called the *frequency of rejection*



corresponding bandwidth ($B = \omega_2 - \omega_1$) is known as the *bandwidth of rejection*. Thus,

A **bandstop filter** is designed to stop or eliminate all frequencies within a band of frequencies, $\omega_1 < \omega < \omega_2$.

14.8 Active Filters

There are three major limitations to the passive filters considered in the previous section.

First, they cannot generate gain greater than 1; passive elements cannot add energy to the network.

Second, they may require bulky and expensive inductors.

Third, they perform poorly at frequencies below the audio frequency range (300 Hz < *f < 3,000 Hz).*

Nevertheless, passive filters are useful at high frequencies.

14.8 Active Filters

Active filters consist of combinations of resistors, capacitors, and op amps. They offer some advantages over passive *RLC filters.*

First, they are often smaller and less expensive, because they do not require inductors. This makes feasible the integrated circuit realizations of filters.

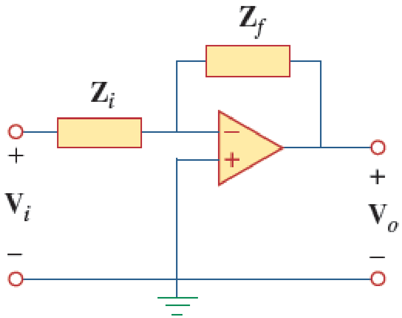
Second, they can provide amplifier gain in addition to providing the same frequency response as *RLC filters.*

Third, active filters can be combined with buffer amplifiers (voltage followers) to isolate each stage of the filter from source and load impedance effects.

However, active filters are less reliable and less stable. The practical limit of most active filters is about 100 kHz-most active filters operate well below that frequency.

14.8.1 First-Order Lowpass Filter

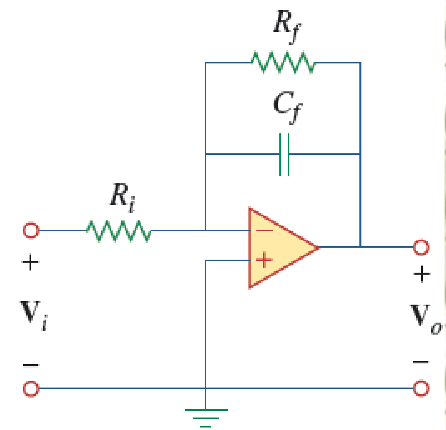
The components selected for Z_i and Z_f determine whether the filter is lowpass or highpass, but one of the components must be reactive.



$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = -\frac{\mathbf{Z}_f}{\mathbf{Z}_i}$$

where $\mathbf{Z}_i = R_i$ and

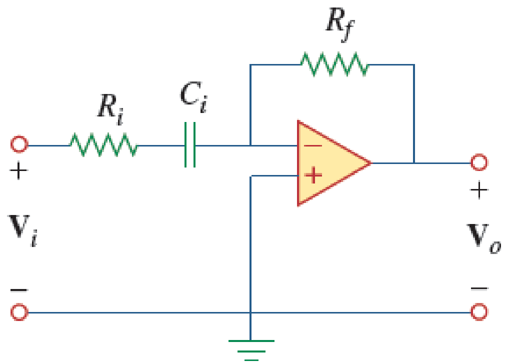
$$\mathbf{Z}_f = R_f \left\| \frac{1}{j\omega C_f} = \frac{R_f/j\omega C_f}{R_f + 1/j\omega C_f} = \frac{R_f}{1 + j\omega C_f R_f}\right.$$



$$\mathbf{H}(\omega) = -\frac{R_f}{R_i}\frac{1}{1 + j\omega C_f R_f} \qquad \omega_c = \frac{1}{R_f C_f}$$

14.8.2 First-Order Highpass Filter

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = -\frac{\mathbf{Z}_f}{\mathbf{Z}_i}$$



where $\mathbf{Z}_i = R_i + 1/j\omega C_i$ and $\mathbf{Z}_f = R_f$ so that

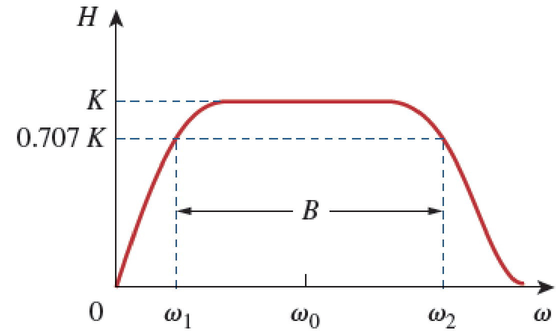
$$\mathbf{H}(\omega) = -\frac{R_f}{R_i + 1/j\omega C_i} = -\frac{j\omega C_i R_f}{1 + j\omega C_i R_i}$$

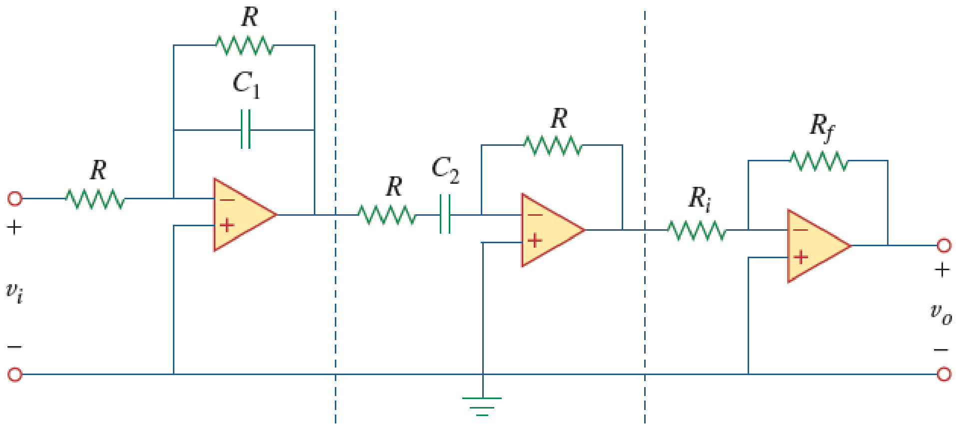
This is similar to Eq. (14.52), except that at very high frequencies ($\omega \rightarrow \infty$), the gain tends to $-R_f/R_i$. The corner frequency is

$$\omega_c = \frac{1}{R_i C_i}$$

14.8.3 Bandpass Filter

(a)





Stage 1
Low-pass filter
sets ω_2 value

Stage 2
High-pass filter
sets ω_1 value

Stage 3
An inverter
provides gain

upper corner frequency

$$\omega_2 = \frac{1}{RC_1}$$

lower corner frequency

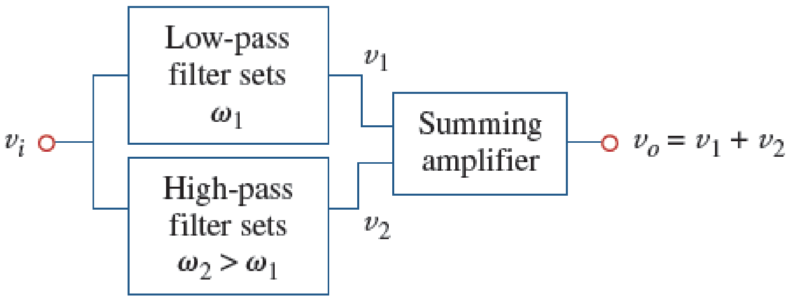
$$\omega_1 = \frac{1}{RC_2}$$

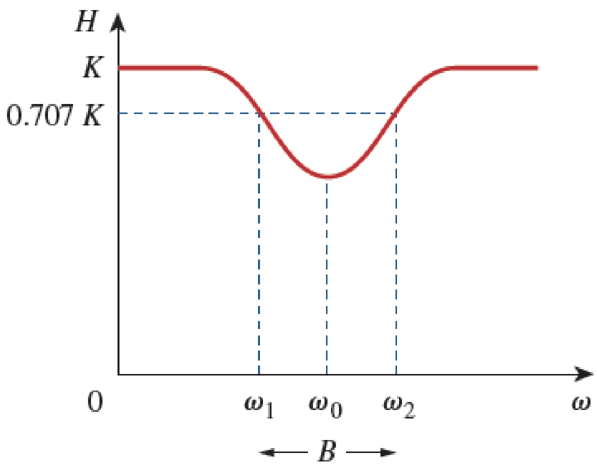
the passband gain is

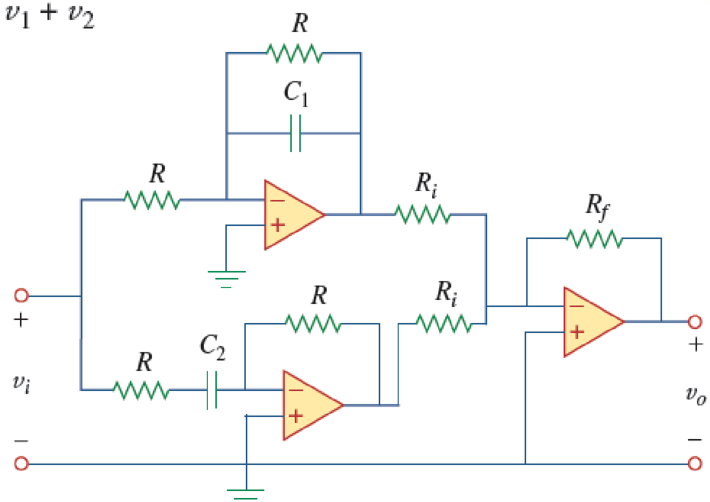
$$K = \frac{R_f}{R_i}\frac{\omega_2}{\omega_1 + \omega_2}$$

14.8.4 Bandreject Filter

A bandreject filter may be constructed by parallel combination of a lowpass filter and a highpass filter and a summing amplifier







$$K = \frac{R_f}{R_i}$$

Summary and Review

1. The transfer function $\mathbf{H}(\omega)$ is the ratio of the output response $\mathbf{Y}(\omega)$ to the input excitation $\mathbf{X}(\omega)$; that is, $\mathbf{H}(\omega) = \mathbf{Y}(\omega)/\mathbf{X}(\omega)$.
2. The frequency response is the variation of the transfer function with frequency.

6. The resonant frequency is that frequency at which the imaginary part of a transfer function vanishes. For series and parallel *RLC* circuits.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

7. The half-power frequencies (ω_1, ω_2) are those frequencies at which the power dissipated is one-half of that dissipated at the resonant frequency. The geometric mean between the half-power frequencies is the resonant frequency, or

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

8. The bandwidth is the frequency band between half-power frequencies:

$$B = \omega_2 - \omega_1$$

9. The quality factor is a measure of the sharpness of the resonance peak. It is the ratio of the resonant (angular) frequency to the bandwidth,

$$Q = \frac{\omega_0}{B}$$

10. A filter is a circuit designed to pass a band of frequencies and reject others. Passive filters are constructed with resistors, capacitors, and inductors. Active filters are constructed with resistors, capacitors, and an active device, usually an op amp.
11. Four common types of filters are lowpass, highpass, bandpass, and bandstop. A lowpass filter passes only signals whose frequencies are below the cutoff frequency ω_c. A highpass filter passes only signals whose frequencies are above the cutoff frequency ω_c. A bandpass filter passes only signals whose frequencies are within a prescribed range ($\omega_1 < \omega < \omega_2$). A bandstop filter passes only signals whose frequencies are outside a prescribed range ($\omega_1 > \omega > \omega_2$).

Assignment (page 667)

Problems 14.30, 14.34, 14.40, 14.45,

Fundamentals of Electric Circuits

2021.6

Chapter 19
Two-Port Networks

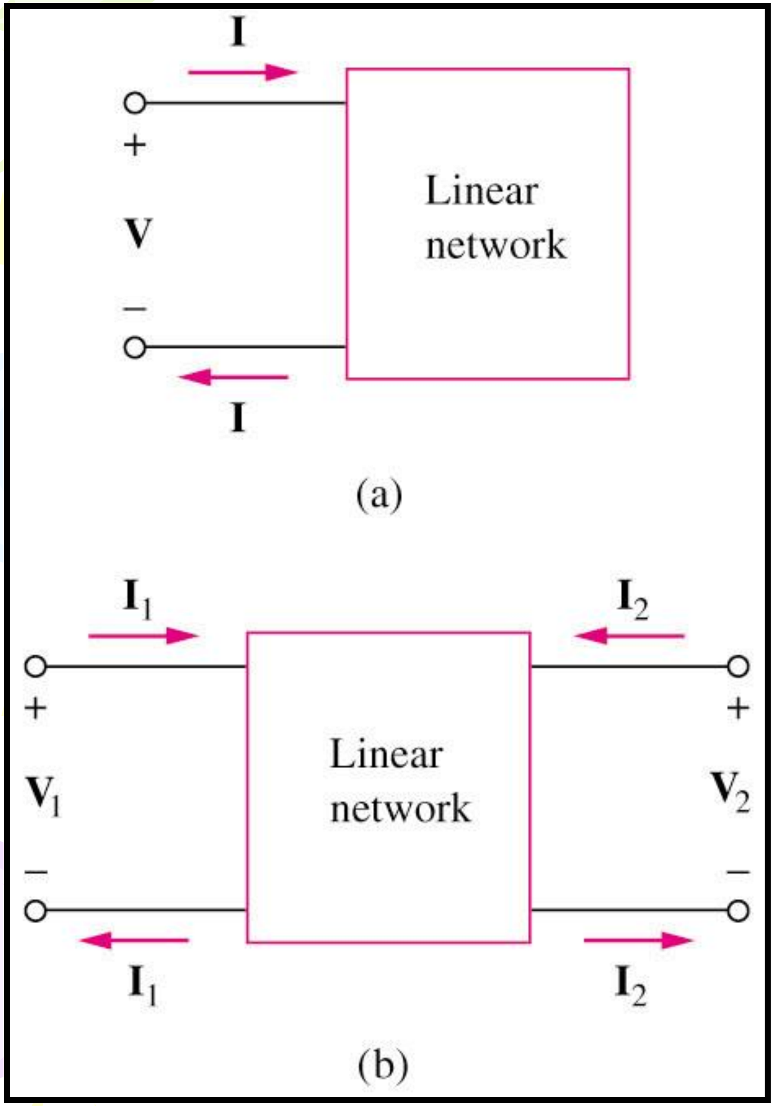
Chapter 19 Two Port Networks

19.1 Introduction

What is a port?

It is a pair of terminals through which a *current* may enter or leave a network.

The current entering one terminal of a pair leaves the other terminal in the pair.



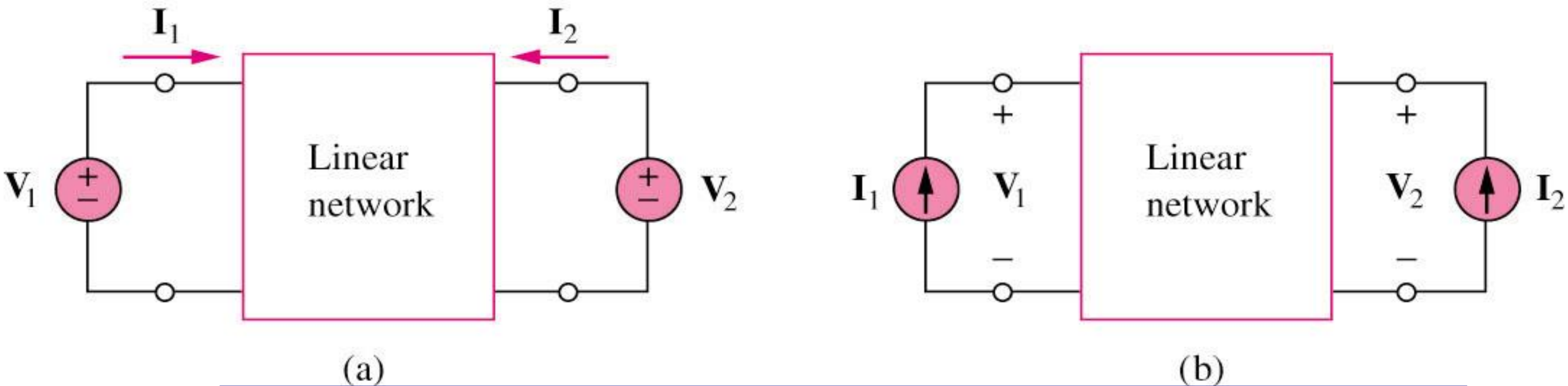
One port or two terminal circuit

Two port or four terminal circuit

- **It is an electrical network with two separate ports for input and output.**
- **No independent sources.**

- Our study of two-port networks is for at least two reasons.
- First, such networks are useful in communications, control systems, power systems, and electronics. For example, they are used in electronics to model transistors and to facilitate cascaded design.
- Second, knowing the parameters of a two-port network enables us to treat it as a “black box” when embedded within a larger network.

19.2 Impedance parameters (1)



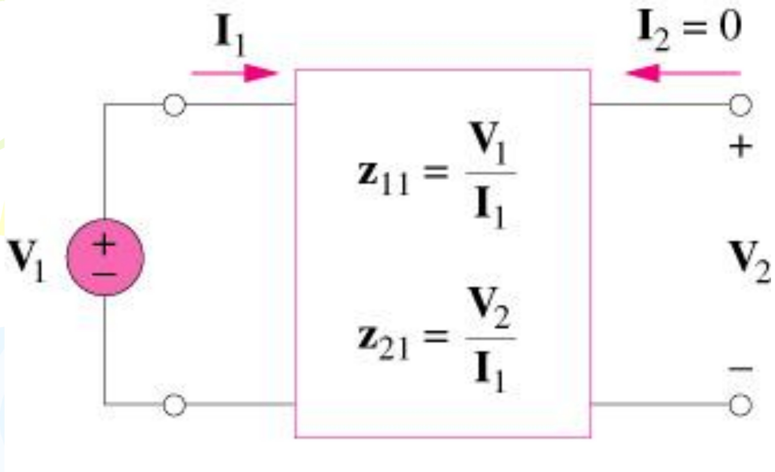
Assume no independent source in the network

$$\begin{aligned} V_1 &= z_{11}I_1 + z_{12}I_2 \\ V_2 &= z_{21}I_1 + z_{22}I_2 \end{aligned}$$

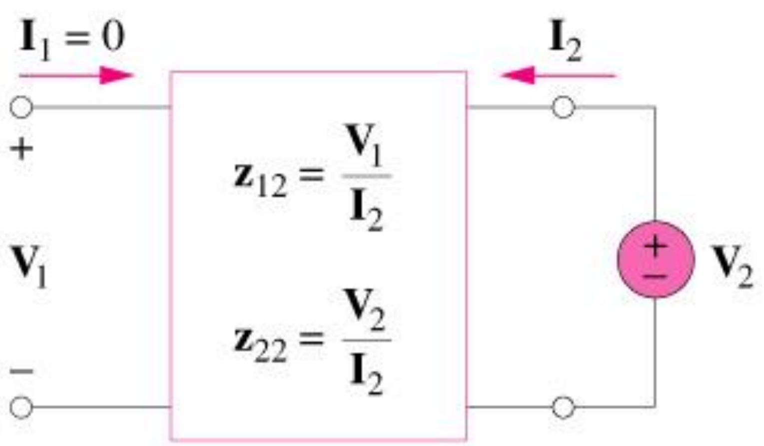
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [\mathbf{z}] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

where the z terms are called the ***impedance parameters***, or simply z parameters, and have units of ohms.

19.2 Impedance parameters (2a)



(a)



(b)

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \quad \text{and} \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

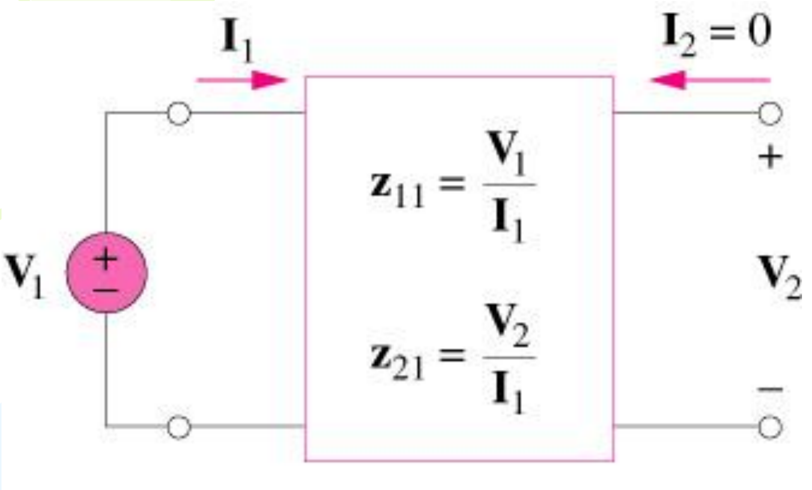
z_{11} = Open-circuit input impedance
z_{21} = Open-circuit transfer impedance from port 1 to port 2

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \quad \text{and} \quad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

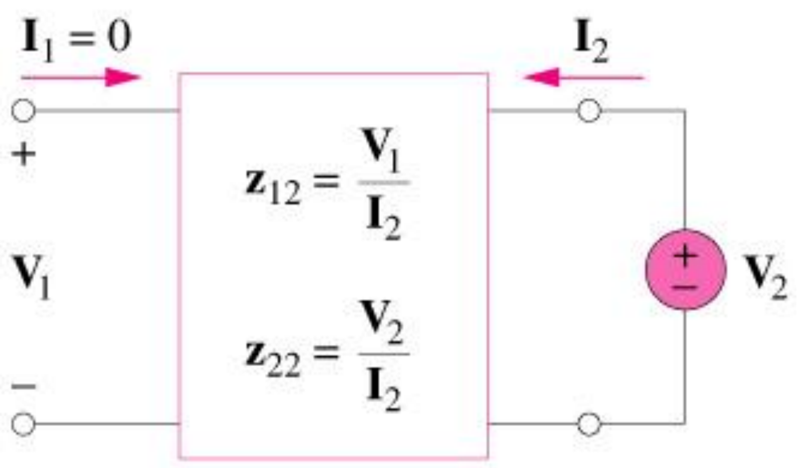
z_{12} = Open-circuit transfer impedance from port 2 to port 1

z_{22} = Open-circuit output impedance

19.2 Impedance parameters (2b)



(a)



(b)

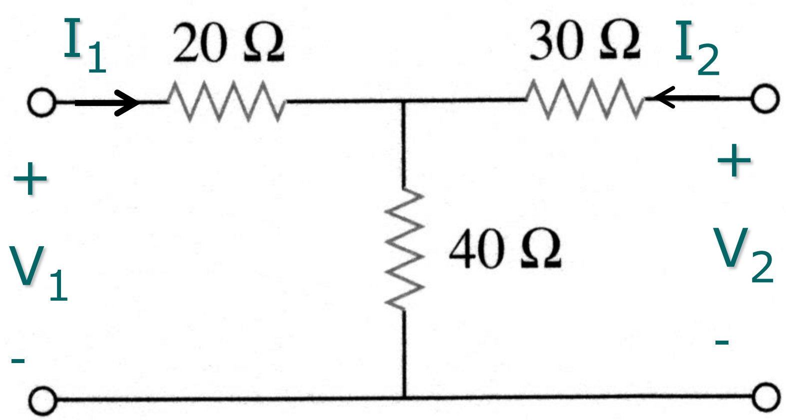
$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \quad \text{and} \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \quad \text{and} \quad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

- When $z_{11} = z_{22}$, the two-port network is said to be **symmetrical**.
- When the two-port network is **linear** and has **no dependent sources**, the transfer impedances are equal ($z_{12} = z_{21}$), and the two-port is said to be **reciprocal**.

Example 1

Determine the Z-parameters of the following circuit.



$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \quad \text{and} \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \quad \text{and} \quad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

$$z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \Omega$$

Answer: $z = \begin{bmatrix} 60 & 40 \\ 40 & 70 \end{bmatrix} \Omega$

PRACTICE PROBLEM 18.1

Find the z parameters of the two-port network in Fig. 18.9.

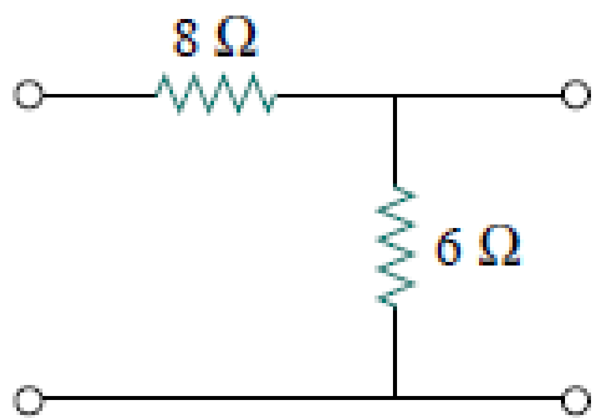
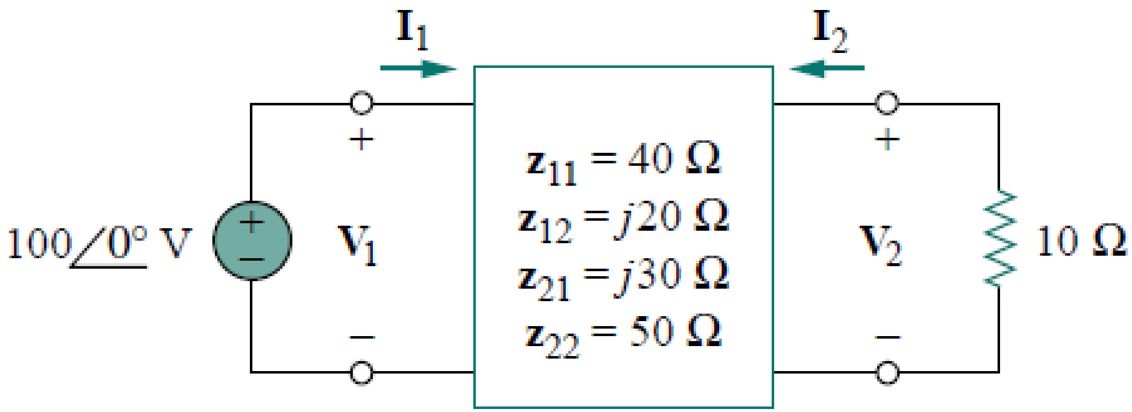


Figure 18.9 For Practice Prob. 18.1.

Answer: $\mathbf{z}_{11} = 14$, $\mathbf{z}_{12} = \mathbf{z}_{21} = \mathbf{z}_{22} = 6\ \Omega$.

Example 19.2

Find $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit in Fig. 19.10.



$$\mathbf{V}_1 = 40\mathbf{I}_1 + j20\mathbf{I}_2$$
$$\mathbf{V}_2 = j30\mathbf{I}_1 + 50\mathbf{I}_2$$

$$\mathbf{V}_1 = 100\angle 0^\circ, \qquad \mathbf{V}_2 = -10\mathbf{I}_2$$

$$100 = 40\mathbf{I}_1 + j20\mathbf{I}_2$$
$$-10\mathbf{I}_2 = j30\mathbf{I}_1 + 50\mathbf{I}_2$$

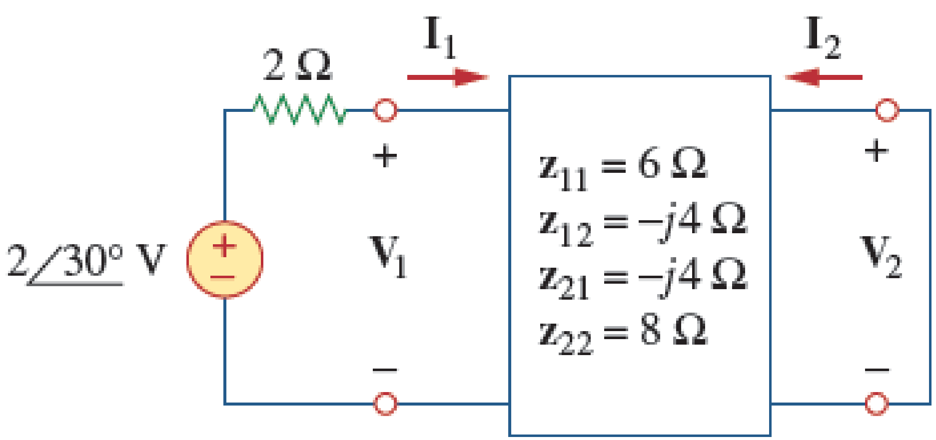
$$\mathbf{I}_2 = \frac{100}{j100} = -j$$

$$\mathbf{I}_1 = j2(-j) = 2.$$

$$\mathbf{I}_1 = 2\angle 0^\circ \text{ A}, \qquad \mathbf{I}_2 = 1\angle -90^\circ \text{ A}$$

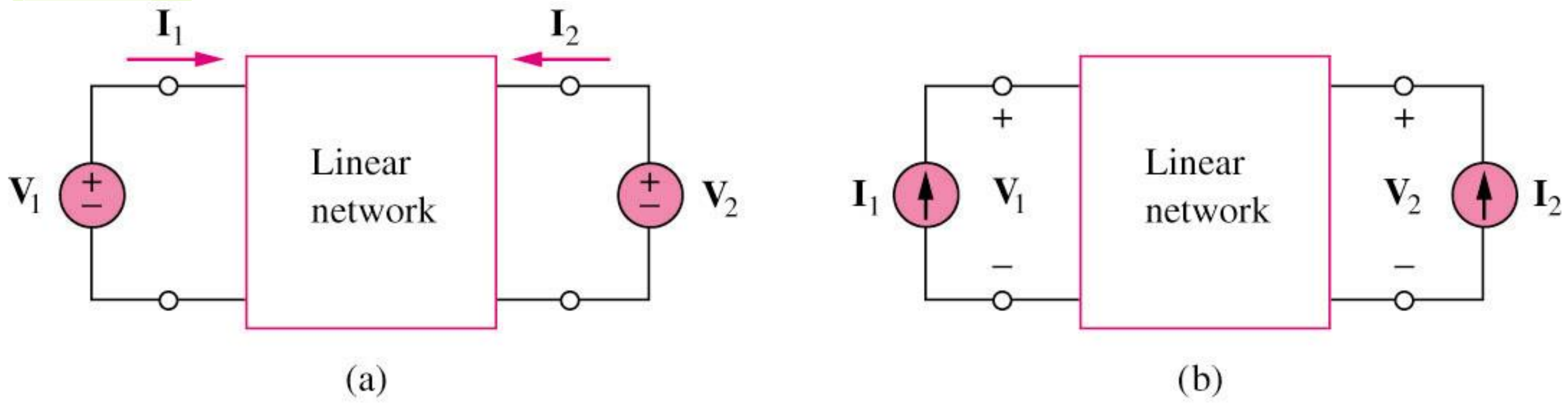
Practice Problem 19.2

Calculate $\mathbf{I}_1$ and $\mathbf{I}_2$ in the two-port of Fig. 19.11.



Answer: $200\underline{/30^\circ}$ mA, $100\underline{/120^\circ}$ mA.

19.3 Admittance parameters (1)



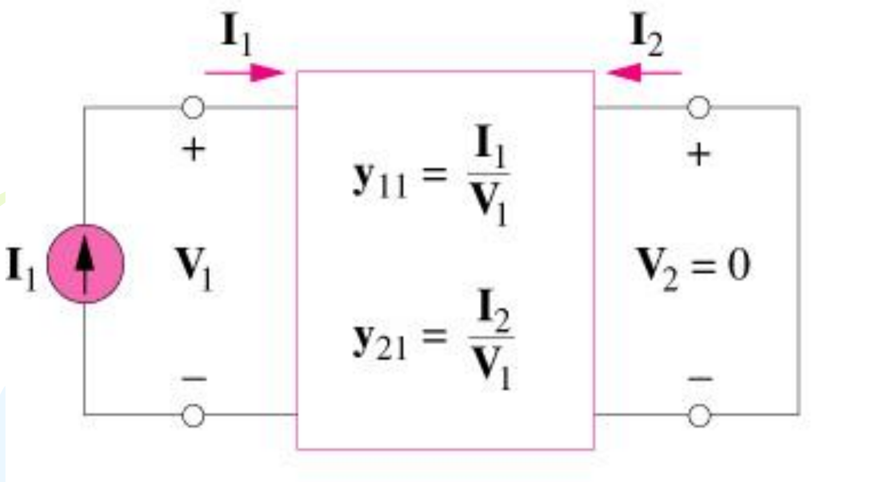
Assume no independent source in the network

$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = [y] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where the **y** terms are called the ***admittance parameters,*** or simply y parameters, and they have units of **Siemens**.

19.3 Admittance parameters (2)

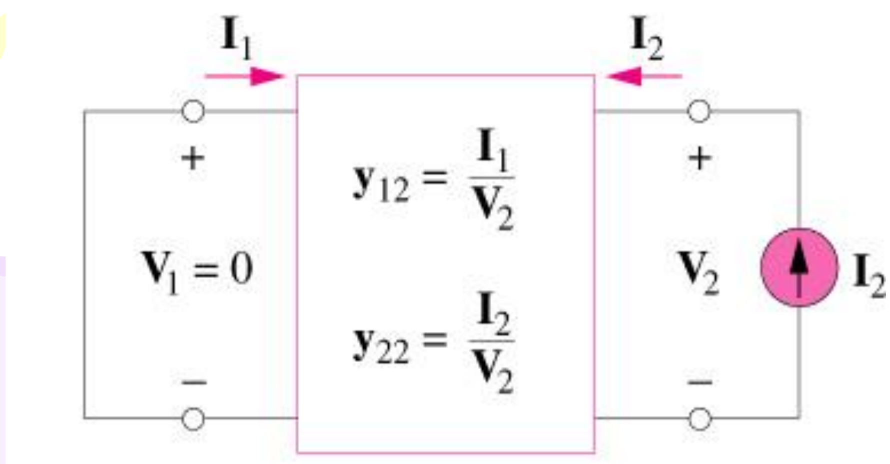


(a)

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \quad \text{and} \quad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

y_{11} = Short-circuit input admittance

y_{21} = Short-circuit transfer admittance from port 1 to port 2



(b)

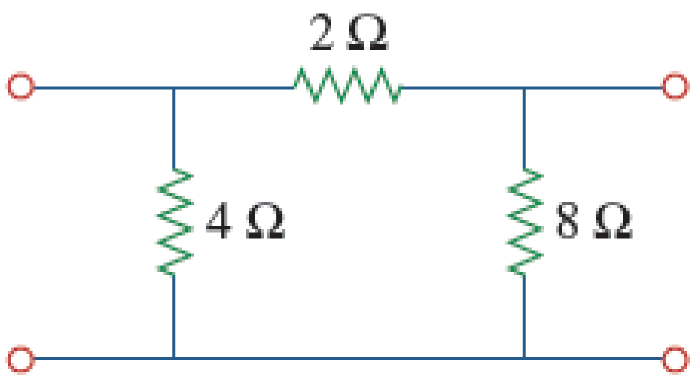
$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \quad \text{and} \quad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

y_{12} = Short-circuit transfer admittance from port 2 to port 1

y_{22} = Short-circuit output admittance

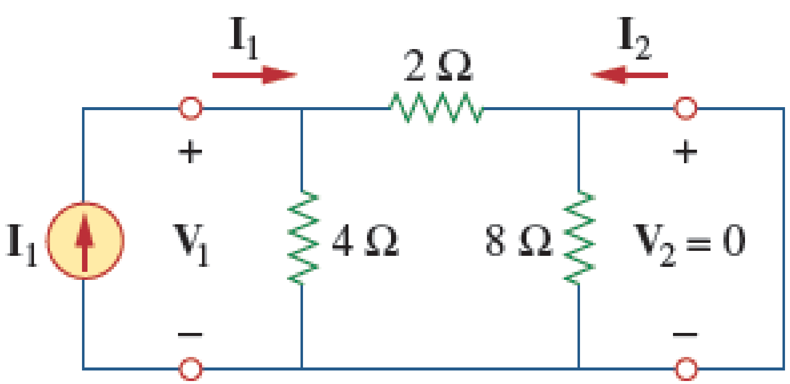
Example 19.3

Obtain the y parameters for the Π network shown in Fig. 19.14.

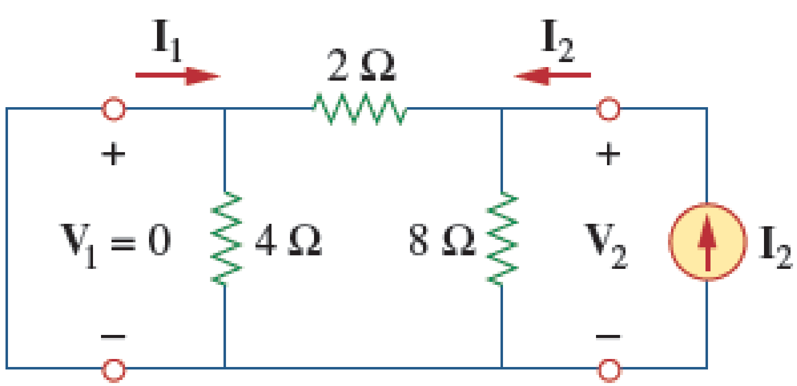


$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \quad \text{and} \quad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

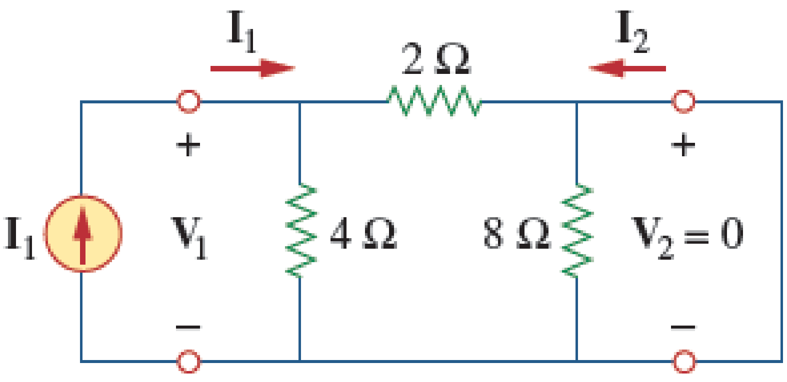
$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \quad \text{and} \quad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$



(a)



(b)



(a)

$$y_{11} = \frac{I_1}{V_1}\bigg|_{V_2=0} \quad \text{and} \quad y_{21} = \frac{I_2}{V_1}\bigg|_{V_2=0}$$

$$y_{12} = \frac{I_1}{V_2}\bigg|_{V_1=0} \quad \text{and} \quad y_{22} = \frac{I_2}{V_2}\bigg|_{V_1=0}$$

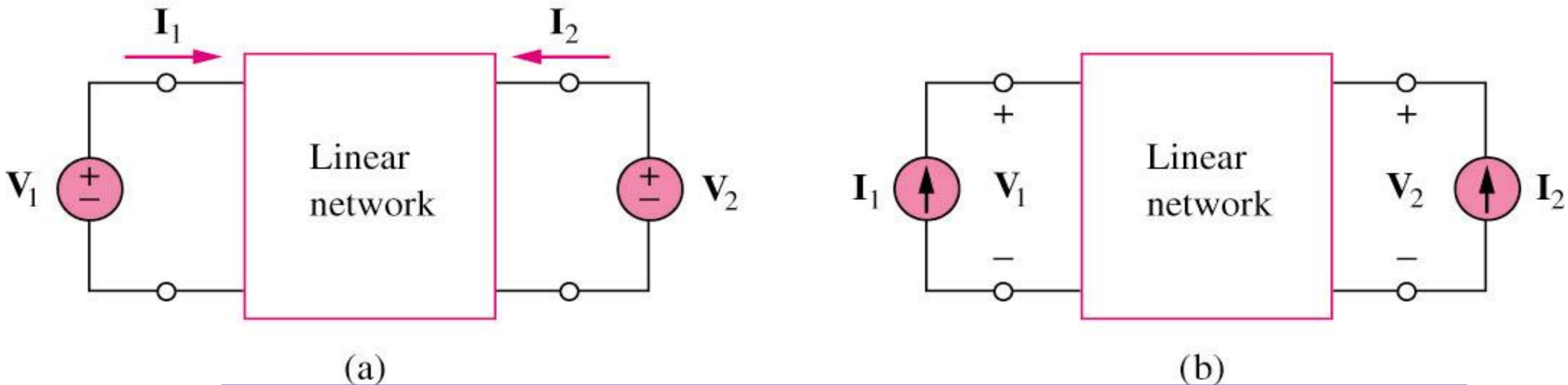
$$\mathbf{V}_1 = \mathbf{I}_1(4 \parallel 2) = \frac{4}{3}\mathbf{I}_1, \qquad y_{11} = \frac{\mathbf{I}_1}{\mathbf{V}_1} = \frac{\mathbf{I}_1}{\frac{4}{3}\mathbf{I}_1} = 0.75 \text{ S}$$

$$-\mathbf{I}_2 = \frac{4}{4+2}\mathbf{I}_1 = \frac{2}{3}\mathbf{I}_1, \qquad y_{21} = \frac{\mathbf{I}_2}{\mathbf{V}_1} = \frac{-\frac{2}{3}\mathbf{I}_1}{\frac{4}{3}\mathbf{I}_1} = -0.5 \text{ S}$$

$$\mathbf{V}_2 = \mathbf{I}_2(8 \parallel 2) = \frac{8}{5}\mathbf{I}_2, \qquad y_{22} = \frac{\mathbf{I}_2}{\mathbf{V}_2} = \frac{\mathbf{I}_2}{\frac{8}{5}\mathbf{I}_2} = \frac{5}{8} = 0.625 \text{ S}$$

$$-\mathbf{I}_1 = \frac{8}{8+2}\mathbf{I}_2 = \frac{4}{5}\mathbf{I}_2, \qquad y_{12} = \frac{\mathbf{I}_1}{\mathbf{V}_2} = \frac{-\frac{4}{5}\mathbf{I}_2}{\frac{8}{5}\mathbf{I}_2} = -0.5 \text{ S}$$

Impedance parameters



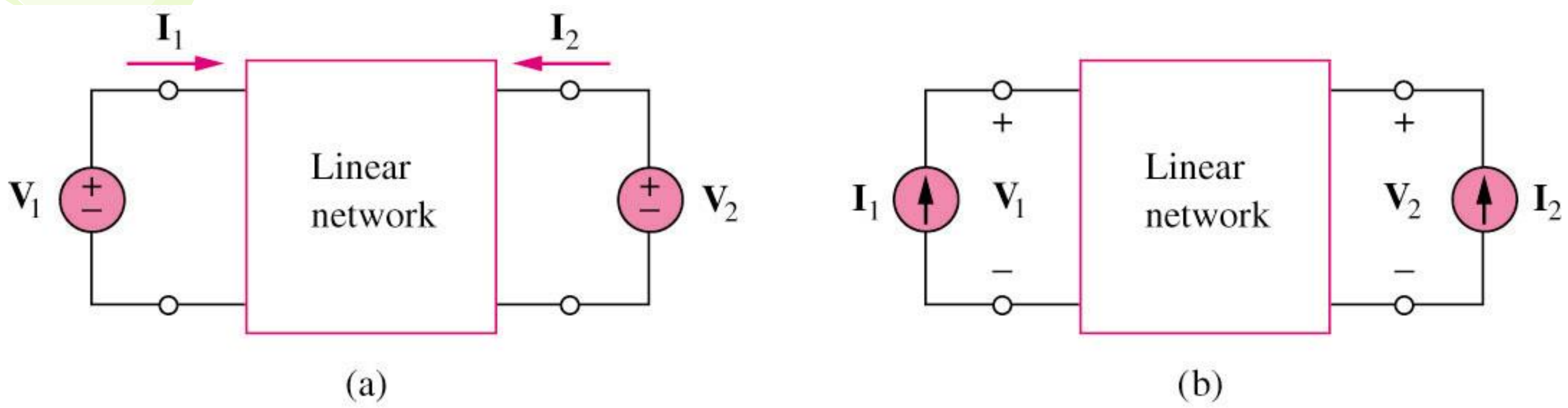
Assume no independent source in the network

$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [\mathbf{z}] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

where the z terms are called the ***impedance parameters***, or simply z parameters, and have units of ohms.

Admittance parameters



Assume no independent source in the network

$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

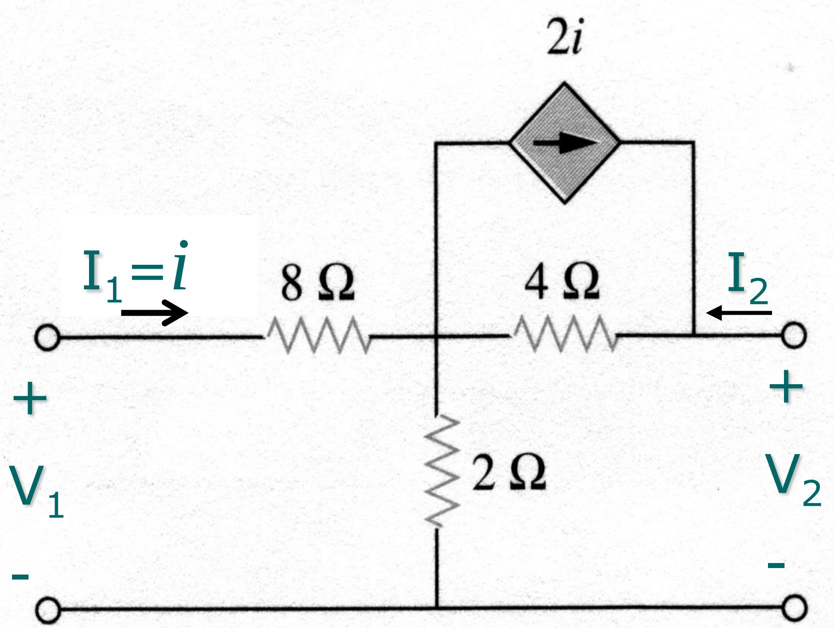
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = [y] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where the **y** terms are called the ***admittance parameters,*** or simply y parameters, and they have units of **Siemens**.

Example 19.4

Determine the y-parameters of the following circuit.

$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$



Apply KVL

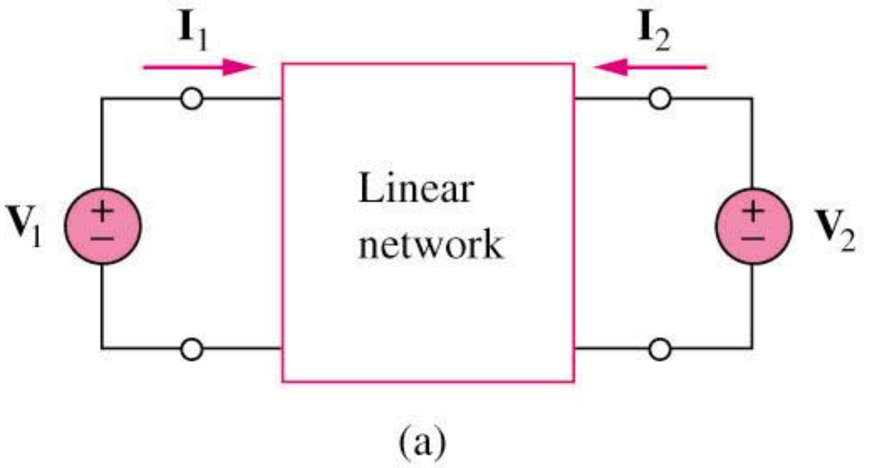
$$V_1 = 8I_1 + 2(I_1 + I_2)$$
$$V_2 = 4(2i + I_2) + 2(I_1 + I_2)$$

$$I_1 = 0.15V_1 - 0.05V_2$$
$$I_2 = -0.25V_1 + 0.25V_2$$

Answer:

$$y = \begin{bmatrix} 0.15 & -0.05 \\ -0.25 & 0.25 \end{bmatrix} S$$

19.4 Hybrid parameters (1)



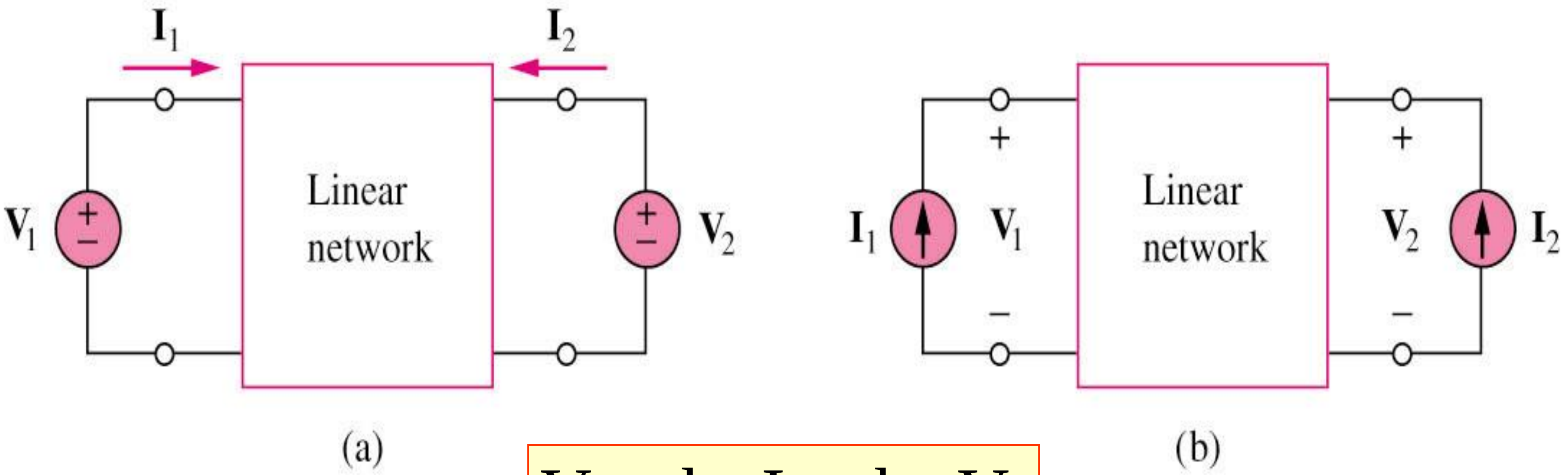
Assume no independent source in the network

$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = [h] \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

where the **h** terms are called the ***hybrid parameters***, or simply h parameters, and each parameter has **different units**, refer above.

19.4 Hybrid parameters (2)



$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

h_{11}= short-circuit input impedance (Ω)

h_{21} = short-circuit forward current gain

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

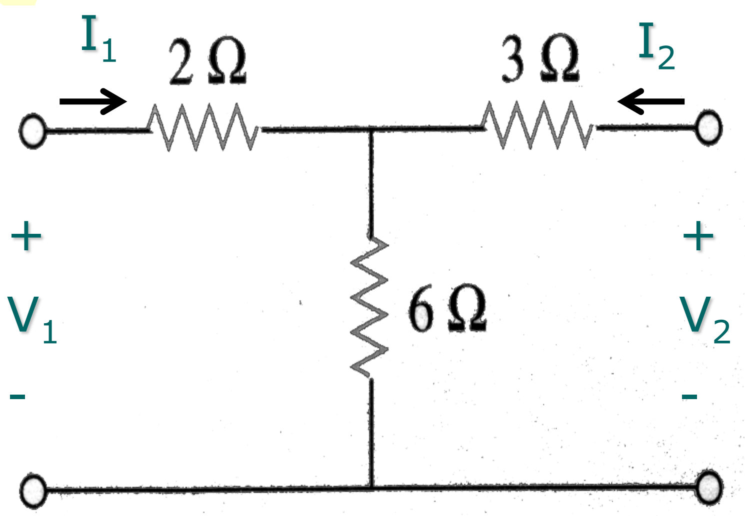
$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

h_{12} = open-circuit reverse voltage-gain

h_{22} = open-circuit output admittance (S)

Example 4

Determine the h-parameters of the following circuit.



$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} \quad \text{and} \quad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} \quad \text{and} \quad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

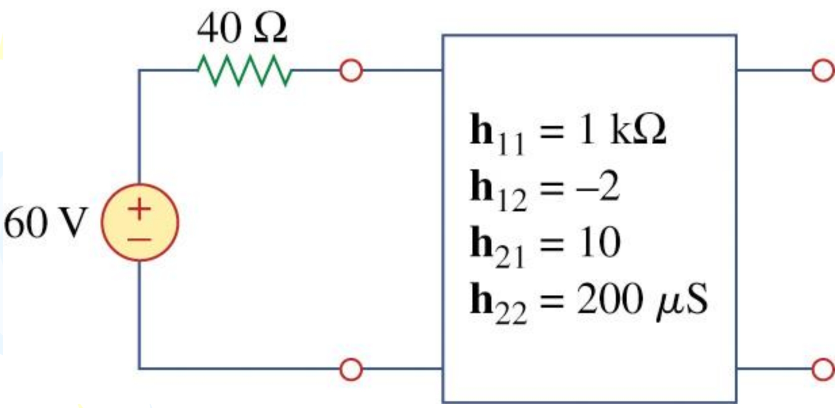
$$h = \begin{bmatrix} h_{11}\Omega & h_{12} \\ h_{21} & h_{22}S \end{bmatrix}$$

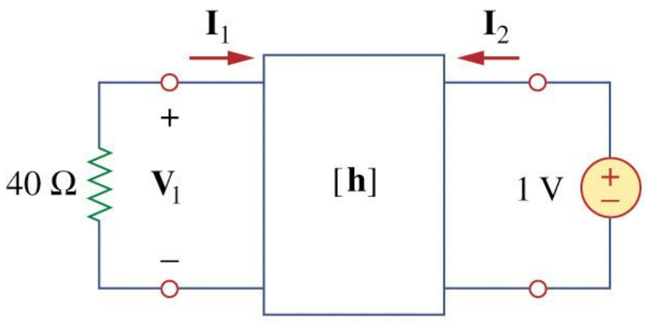
Answer:

$$h = \begin{bmatrix} 4\Omega & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{9}S \end{bmatrix}$$

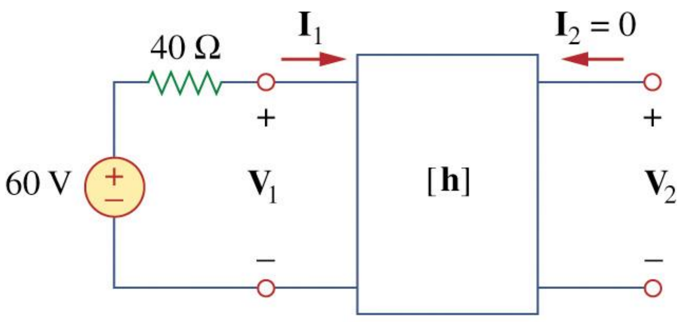
Example 5

Determine the Thevenin equivalent at the output port of the circuit shown in the following Fig.





(a)

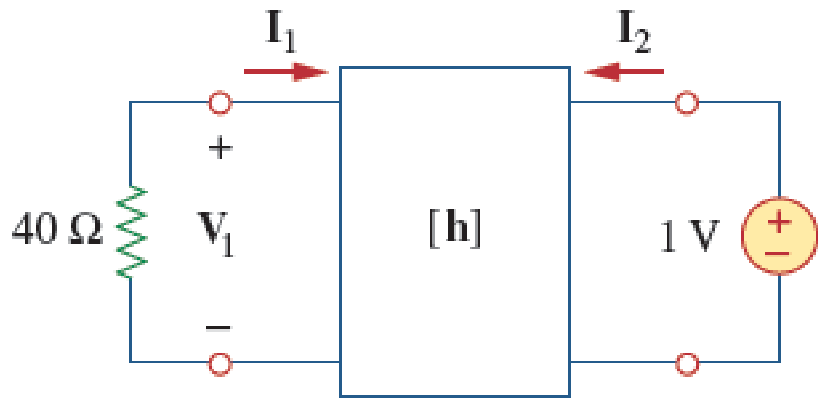


(b)

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2$$

$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2$$



(a)

$$\mathbf{V}_2 = 1, \text{ and } \mathbf{V}_1 = -40\mathbf{I}_1$$

$$-40\mathbf{I}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12} \quad \Rightarrow \quad \mathbf{I}_1 = -\frac{\mathbf{h}_{12}}{40 + \mathbf{h}_{11}}$$

$\mathbf{h}_{11} = 1\ \text{k}\Omega$
$\mathbf{h}_{12} = -2$
$\mathbf{h}_{21} = 10$
$\mathbf{h}_{22} = 200\ \mu\text{S}$

$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}$$

$$\mathbf{I}_2 = \mathbf{h}_{22} - \frac{\mathbf{h}_{21}\mathbf{h}_{12}}{\mathbf{h}_{11} + 40} = \frac{\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{21}\mathbf{h}_{12} + \mathbf{h}_{22}40}{\mathbf{h}_{11} + 40}$$

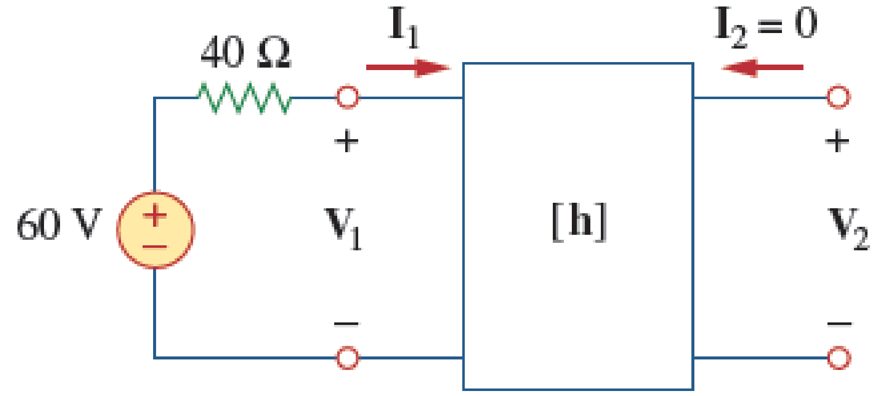
$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_2}{\mathbf{I}_2} = \frac{1}{\mathbf{I}_2} = \frac{\mathbf{h}_{11} + 40}{\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{21}\mathbf{h}_{12} + \mathbf{h}_{22}40}$$

$$\mathbf{Z}_{\text{Th}} = \frac{1000 + 40}{10^3 \times 200 \times 10^{-6} + 20 + 40 \times 200 \times 10^{-6}}$$

$$= \frac{1040}{20.21} = 51.46\ \Omega$$

$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2$$

$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2$$



(b)

$$-60 + 40\mathbf{I}_1 + \mathbf{V}_1 = 0 \quad \Rightarrow \quad \mathbf{V}_1 = 60 - 40\mathbf{I}_1$$

$$\mathbf{I}_2 = 0$$

$$60 - 40\mathbf{I}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2 \qquad 60 = (\mathbf{h}_{11} + 40)\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2$$

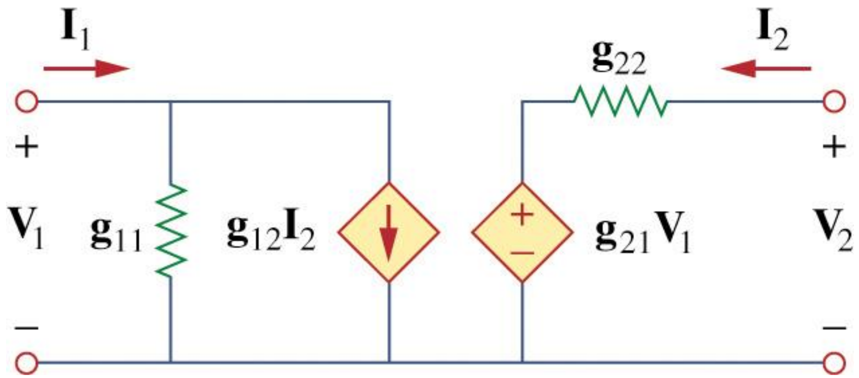
$$0 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2 \quad \Rightarrow \quad \mathbf{I}_1 = -\frac{\mathbf{h}_{22}}{\mathbf{h}_{21}}\mathbf{V}_2$$

$$60 = \left[-(\mathbf{h}_{11} + 40)\frac{\mathbf{h}_{22}}{\mathbf{h}_{21}} + \mathbf{h}_{12}\right]\mathbf{V}_2$$

$$\mathbf{V}_{\mathrm{Th}} = \mathbf{V}_2 = \frac{60}{-(\mathbf{h}_{11} + 40)\mathbf{h}_{22}/\mathbf{h}_{21} + \mathbf{h}_{12}}$$

$$= \frac{60 \times 10}{-20.21} = -29.69 \text{ V}$$

19.5 Inverse hybrid parameters



$$I_1 = g_{11}V_1 + g_{12}I_2$$
$$V_2 = g_{21}V_1 + g_{22}I_2$$

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = [g] \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0}$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}$$

g_{11}=open-circuit input admittance(S)

g_{21} = open- circuit forward voltage gain

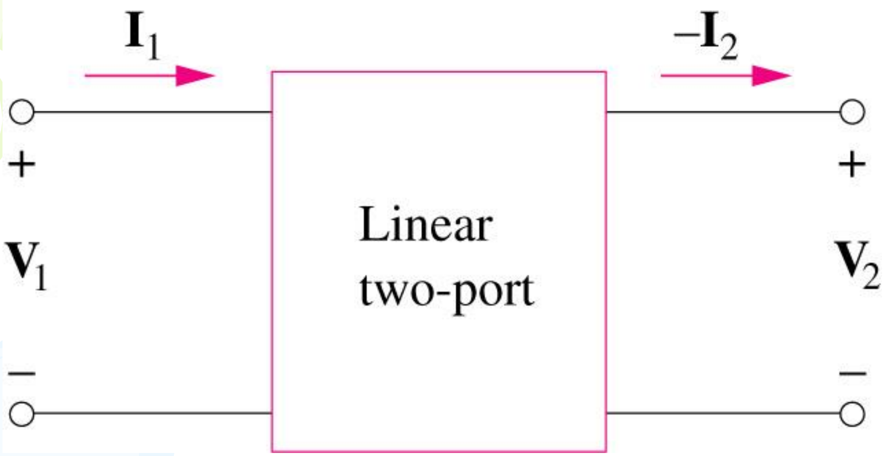
$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0}$$

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0}$$

g_{12} = Short-circuit reverse current-gain

g_{22} = short-circuit output impedance (Ω)

19.6 Transmission parameters (1)



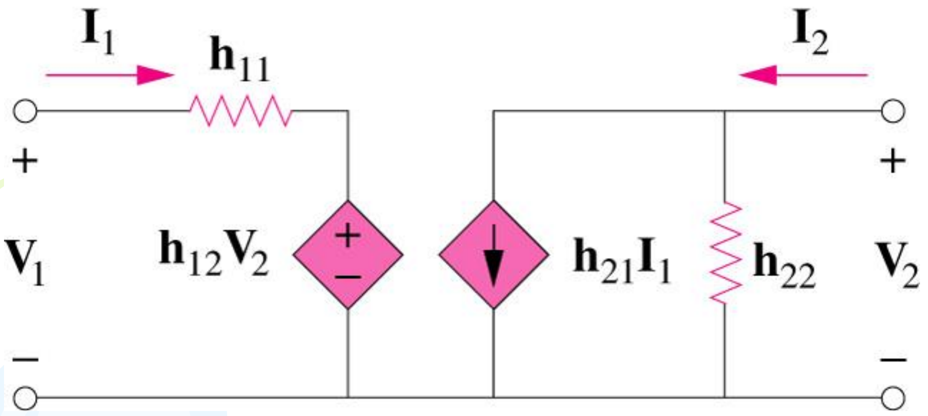
Assume no independent source in the network

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = [T] \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

where the **T** terms are called the ***transmission parameters,*** or simply T or ***ABCD parameters,*** and each parameter has different units.

19.6 Transmission parameters (2)



$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

A=open-circuit voltage ratio

C= open-circuit transfer admittance (S)

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0}$$

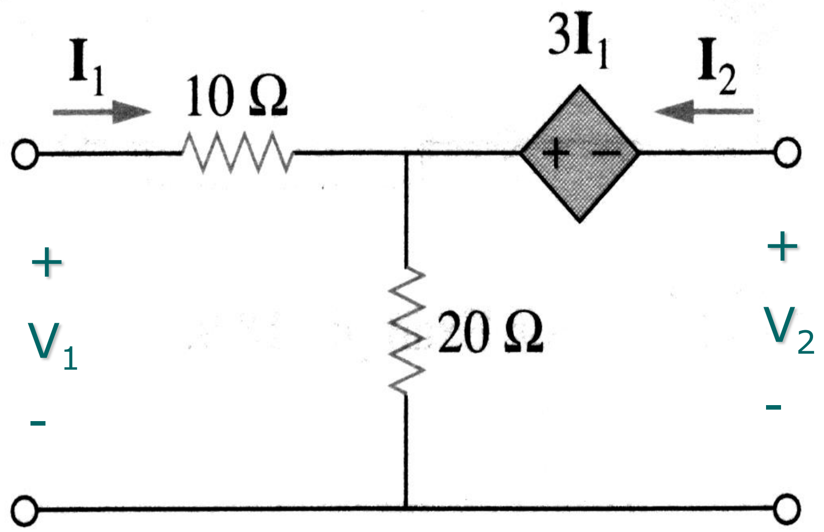
$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0}$$

B= negative short-circuit transfer impedance (Ω)

D=negative short-circuit current ratio

Example 6

Determine the T-parameters of the following circuit.



$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

Apply KVL

$$V_1 = 10I_1 + 20(I_1 + I_2)$$
$$V_2 = -3I_1 + 20(I_1 + I_2)$$

$$V_1 = \frac{30}{17}V_2 - \frac{260}{17}I_2$$
$$I_1 = \frac{1}{17}V_2 - \frac{20}{17}I_2$$

Answer:

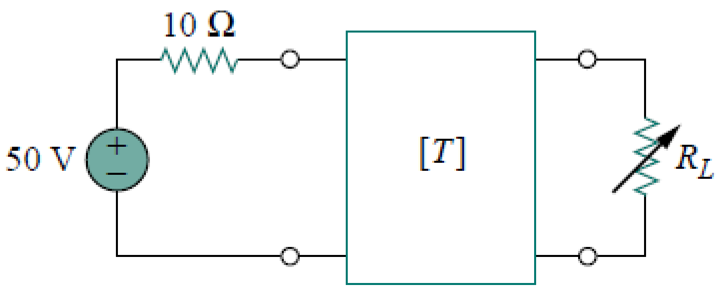
$$T = \begin{bmatrix} 1.765 & 15.294\Omega \\ 0.059\text{S} & 1.176 \end{bmatrix}$$

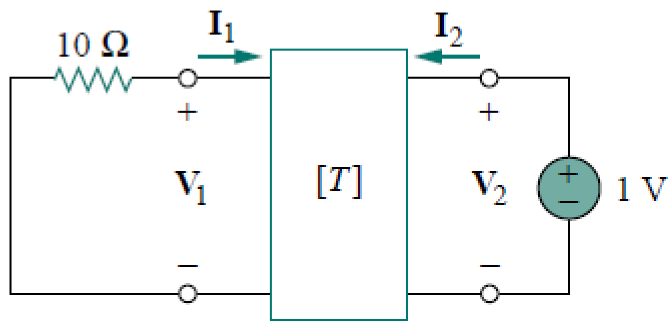
EXAMPLE 18.9

The **ABCD** parameters of the two-port network in Fig. 18.34 are

$$\begin{bmatrix} 4 & 20\ \Omega \\ 0.1\ \text{S} & 2 \end{bmatrix}$$

The output port is connected to a variable load for maximum power transfer. Find R_L and the maximum power transferred.





(a)

Solution:

What we need is to find the Thevenin equivalent ($\mathbf{Z}_{\text{Th}}$ and $\mathbf{V}_{\text{Th}}$) at the load or output port. We find $\mathbf{Z}_{\text{Th}}$ using the circuit in Fig. 18.35(a). Our goal is to get $\mathbf{Z}_{\text{Th}} = \mathbf{V}_2/\mathbf{I}_2$. Substituting the given **ABCD** parameters into Eq. (18.22), we obtain

$$\mathbf{V}_1 = 4\mathbf{V}_2 - 20\mathbf{I}_2$$

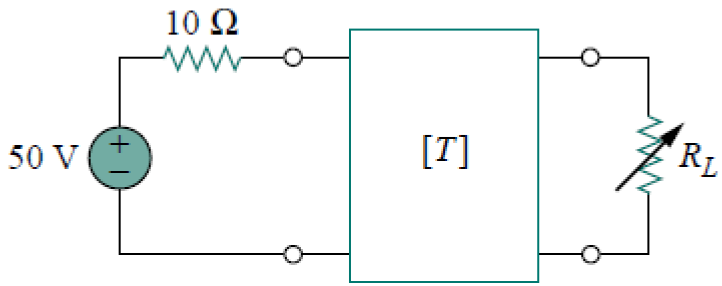
$$\mathbf{I}_1 = 0.1\mathbf{V}_2 - 2\mathbf{I}_2$$

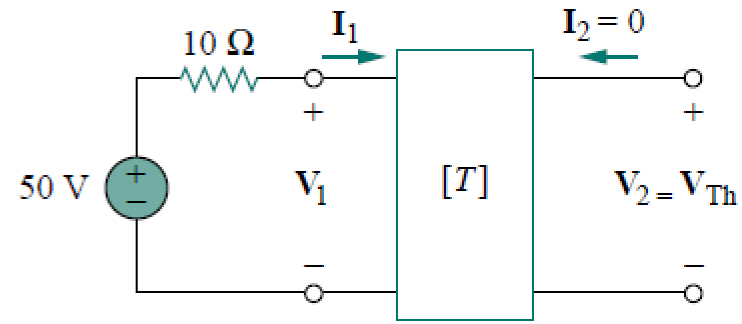
$$\mathbf{V}_1 = -10\mathbf{I}_1.$$

$$-10\mathbf{I}_1 = 4\mathbf{V}_2 - 20\mathbf{I}_2$$

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_2}{\mathbf{I}_2} = \frac{4}{0.5} = 8\ \Omega$$

EXAMPLE 18.9



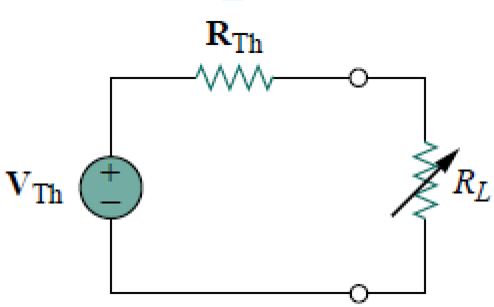


To find $\mathbf{V}_{Th}$, we use the circuit in Fig. 18.35(b). At the output port $\mathbf{I}_2 = 0$ and at the input port $\mathbf{V}_1 = 50 - 10\mathbf{I}_1$. Substituting these into Eqs.

$$\mathbf{V}_1 = 4\mathbf{V}_2 - 20\mathbf{I}_2$$
$$\mathbf{I}_1 = 0.1\mathbf{V}_2 - 2\mathbf{I}_2$$

$$50 - 10\mathbf{I}_1 = 4\mathbf{V}_2$$
$$\mathbf{I}_1 = 0.1\mathbf{V}_2$$

$$50 - \mathbf{V}_2 = 4\mathbf{V}_2 \quad \Longrightarrow \quad \mathbf{V}_2 = 10 \qquad \mathbf{V}_{Th} = \mathbf{V}_2 = 10 \text{ V}$$



The equivalent circuit is shown in Fig. 18.35(c). For maximum power transfer,

$$R_L = \mathbf{Z}_{Th} = 8\ \Omega$$

$$P = I^2 R_L = \left(\frac{\mathbf{V}_{Th}}{2R_L}\right)^2 R_L = \frac{\mathbf{V}_{Th}^2}{4R_L} = \frac{100}{4 \times 8} = 3.125 \text{ W}$$

19.7 Inverse Transmission parameters

Inverse transmission parameters are defined by expressing the variables at the output port in terms of the variables at the input port.

$$V_2 = aV_1 - bI_1$$
$$I_2 = cV_1 - dI_1$$

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix} = [t] \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$$

$$a = \left.\frac{V_2}{V_1}\right|_{I_1=0}$$
$$c = \left.\frac{I_2}{V_1}\right|_{I_1=0}$$

a=open-circuit voltage gian

c= open-circuit transfer admittance (S)

$$b = -\left.\frac{V_2}{I_1}\right|_{V_1=0}$$
$$d = -\left.\frac{I_2}{I_1}\right|_{V_1=0}$$

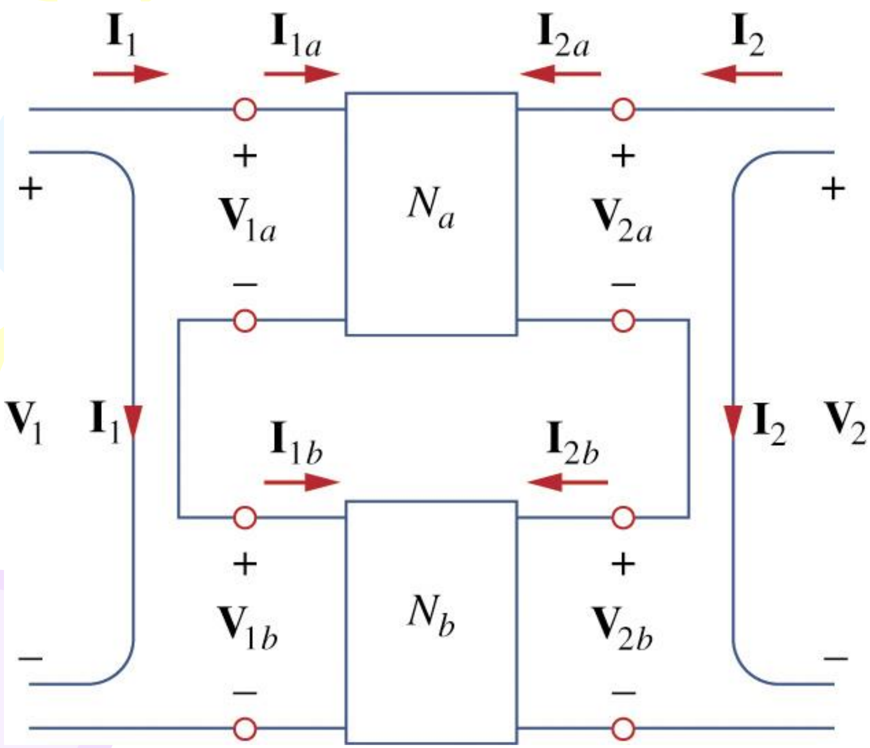
b= negative short-circuit transfer impedance (Ω)

d=negative short-circuit current gain

19.8 Interconnection of Networks(1)

1. The series connection of two-port networks

The series connection of two-port networks is shown in following Fig.. For the series connection, the input currents of the ports are the same and their voltage add.



For network $N_{a,}$

$$V_{1a} = z_{11a}I_{1a} + z_{12a}I_{2a}$$
$$V_{2a} = z_{21a}I_{1a} + z_{22a}I_{2a}$$

For network $N_{b,}$

$$V_{1b} = z_{11b}I_{1b} + z_{12b}I_{2b}$$
$$V_{2b} = z_{21b}I_{1b} + z_{22b}I_{2b}$$

19.8 Interconnection of Networks(2)

We notice that from above Fig.

$$I_1 = I_{1a} = I_{1b}, I_2 = I_{2a} = I_{2b}$$

and that

$$V_1 = V_{1a} + V_{1b} = (Z_{11a} + Z_{11b})I_1 + (Z_{12a} + Z_{12b})I_2$$

$$V_2 = V_{2a} + V_{2b} = (Z_{21a} + Z_{21b})I_1 + (Z_{22a} + Z_{22b})I_2$$

Thus, the z parameters for the overall network are

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} z_{11a} + z_{11b} & z_{12a} + z_{12b} \\ z_{21a} + z_{21b} & z_{22a} + z_{22b} \end{bmatrix}$$

or

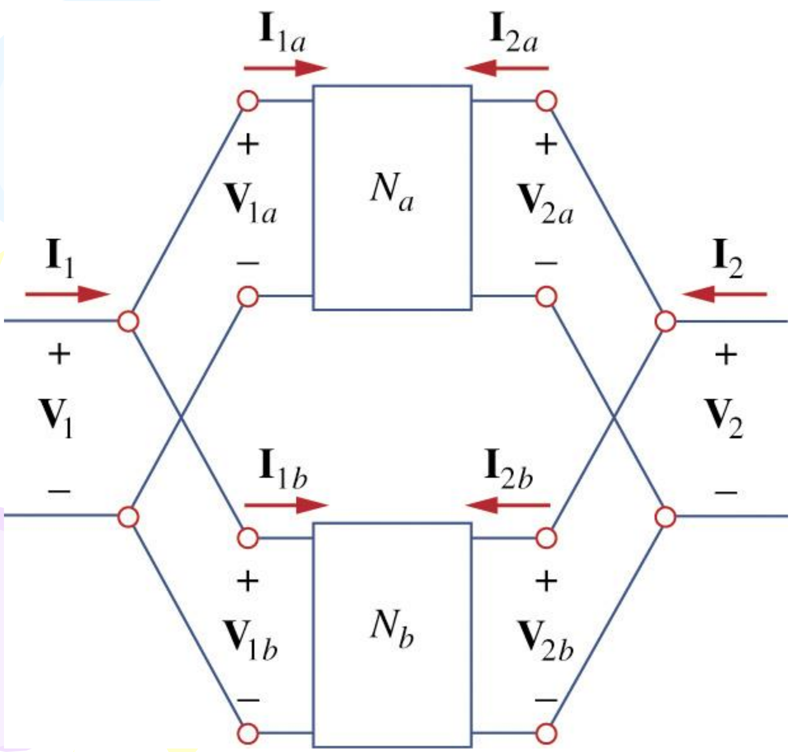
$$[\mathbf{z}] = [\mathbf{z}_a + \mathbf{z}_b]$$

Showing that the z parameters for the overall network are the sum of the z parameters for the individual network. This can be extended to n networks in series.

19.8 Interconnection of Networks(3)

2. The parallel connection of two-port networks

Two two-port networks are in parallel when their port voltages are equal and the port currents of the lager network are the sums of the individual port currents. The parallel connection of two two-port networks is shown in following Fig.



For network $N_{a,}$

$$\begin{aligned} I_{1a} &= y_{11a}V_{1a} + y_{12a}V_{2a} \\ I_{2a} &= y_{21a}V_{1a} + y_{22a}V_{2a} \end{aligned}$$

For network $N_{b,}$

$$\begin{aligned} I_{1b} &= y_{11b}V_{1b} + y_{12b}V_{2b} \\ I_{2b} &= y_{21b}V_{1b} + y_{22b}V_{2b} \end{aligned}$$

19.8 Interconnection of Networks(4)

But from above Fig., we can get

$$V_1 = V_{1a} = V_{1b}, \quad V_2 = V_{2a} = V_{2b}$$
$$I_1 = I_{1a} + I_{1b}, \quad I_2 = I_{2a} + I_{2b}$$

From these equations, we obtain

$$\mathrm{I}_1 = (\mathrm{y}_{11a} + \mathrm{y}_{11b})\mathrm{V}_{1a} + (\mathrm{y}_{12a} + \mathrm{y}_{12b})\mathrm{V}_{2a}$$
$$\mathrm{I}_2 = (\mathrm{y}_{21a} + \mathrm{y}_{21b})\mathrm{V}_{1a} + (\mathrm{y}_{22a} + \mathrm{y}_{22b})\mathrm{V}_{2a}$$

Thus, the y parameters for the overall network are

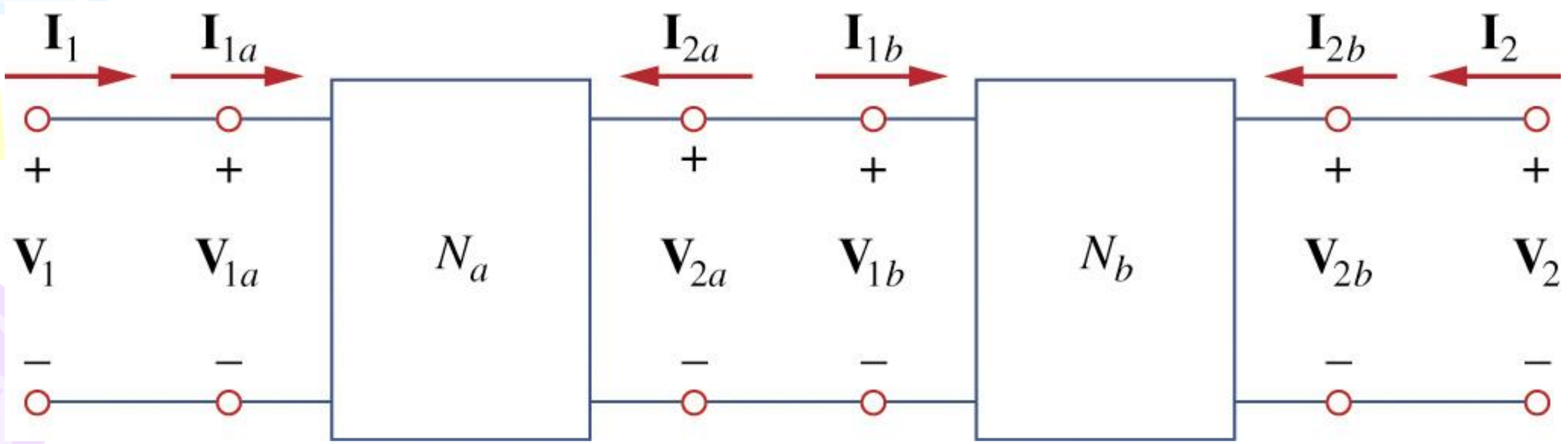
$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} y_{11a} + y_{11b} & y_{12a} + y_{12b} \\ y_{21a} + y_{21b} & y_{22a} + y_{22b} \end{bmatrix}$$

3. The ***cascaded connection*** of two-port networks

Two networks are said to cascaded when the output of one is the input of the other. The connection of two two-port networks in cascade is shown in following Fig. For the two networks,

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix}$$

$$\begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix}$$



From above Fig.,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix}, \quad \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} = \begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix}, \quad \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix} = \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

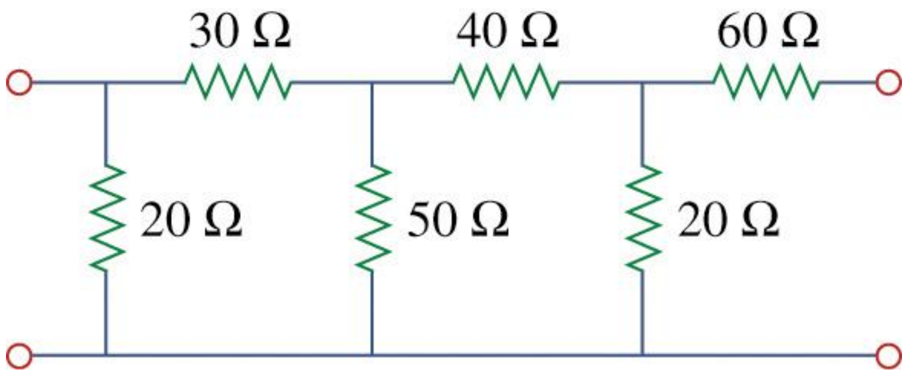
From these equations , we obtain

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Thus, the transmission parameters for the overall network are the product the transmission parameters for the individual transmission parameters:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}$$

Practice Find **the ABCD parameters** in the circuit in following Fig.



Answer :

$$[T] = \begin{bmatrix} 29.5 & 2200\Omega \\ 0.425S & 32 \end{bmatrix}$$

SUMMARY(1)

- A two-port network is one with two ports (two pairs of access terminals), known as input and output ports.
- The six parameters used to model a two-port network are the impedance **[z]**, admittance **[y]**, hybrid **[h]**, inverse hybrid **[g]**, transmission **[T]**, and inverse transmission **[t]** parameters.
- The parameters can be calculated or measured by short-circuiting or open-circuiting the appropriate input or output port.
- A two-port network is reciprocal if $z_{12}=z_{21}$, $y_{12}=y_{21}$, $h_{12}=-h_{21}$, $g_{12}=-g_{21}$, $\Delta_T=1$, $\Delta_t=1$. Network containing dependent sources are generally not reciprocal.

SUMMARY(2)

- Two –port network may be connecte in series, in parallel,or in cascade. In the series connection the z parameters are added, in the parallel connection the y parameters added, and in the cascade connection the transmission parameters are multiplied in the correct order.

Assignment (page 895)

Problems 19.3, 19.13, 19.15, 19.31, 19.45

Main methods for DC circuit analysis

2021.5

Overview of DC Analysis

1. What is the task for circuit analysis?

   Given a circuit, determine branch's voltages, currents, power.

2. How to do circuit analysis?

   Write equations, and solve these equations.

3. How to write equations? (3 methods)

   (1)use two basis directly : V-I relationship for elements; KCL,KVL(chapter 2)

   (2)use basic methods directly: Nodal analysis, Mesh Analysis (chapter 3)

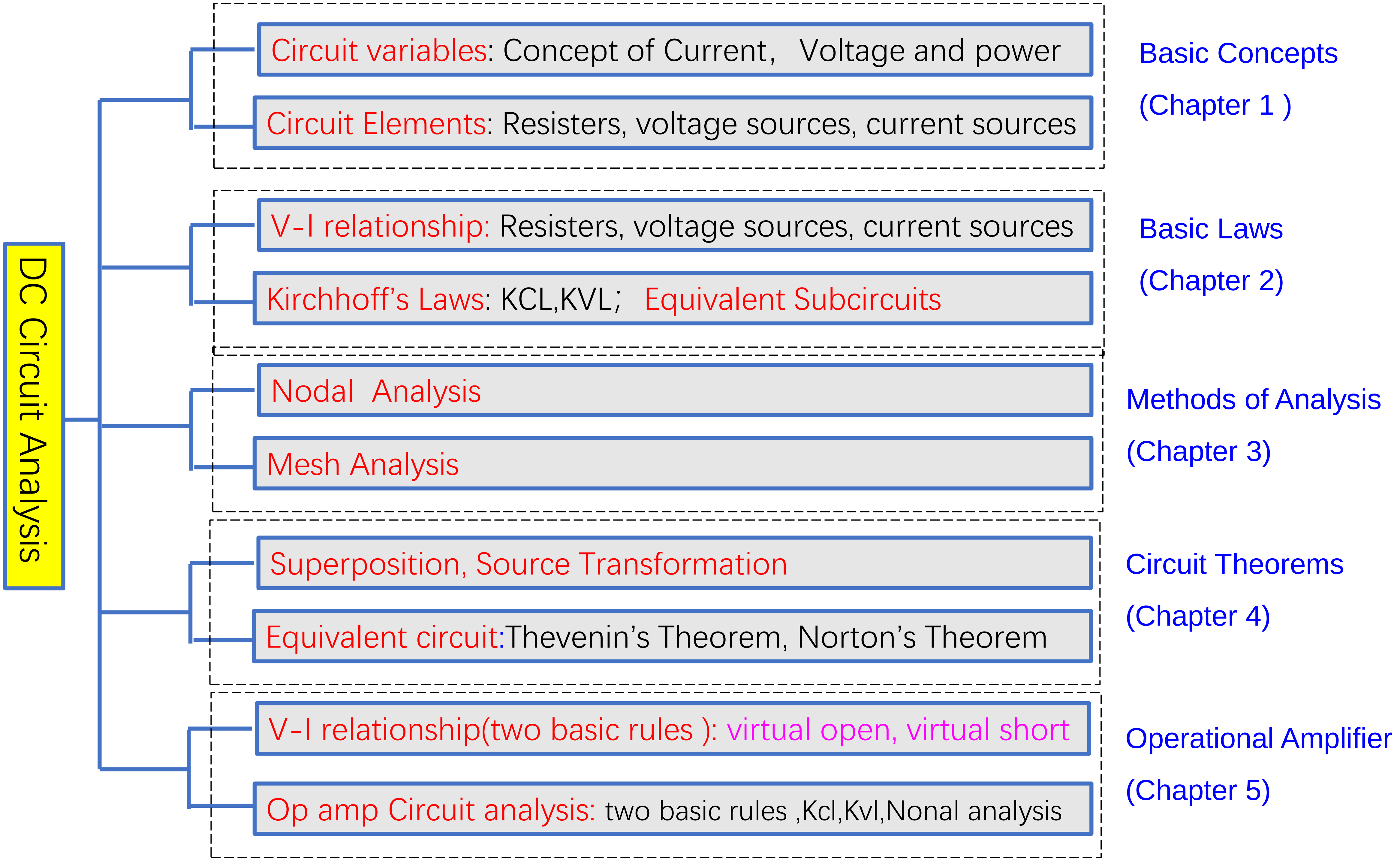
   (3)first use circuit Theorem to simplify the circuit ,then use two basis or basic methods (chapter 4).

4. What are the relationship of these 5 chapters?

From chapter 1-5, first, variables and elements of DC circuit are introduced, then DC analysis methods are introduced. (1) In chapter 1, circuit variables for circuit equations are introduced. (2) V-I relationship for DC circuits elements are introduced in chapter 1 and chapter 5: resisters, voltage sources, current sources, OP amp. (3) circuit analysis by two basis are introduced in chapter 2; circuit analysis by Nodal analysis and mesh analysis are introduced in chapter 3; circuit simplify methods are introduced in chapter 4.

DC ANALYSIS

1. Basic Concepts(Chapter 1)
2. Basic Laws(Chapter 2)
3. Methods of Analysis(Chapter 3)
4. Circuit Theorems(Chapter 4)
5. Operational Amplifier(Chapter 5)
6. Summary for DC analysis


DC Circuit Analysis
Circuit variables: Concept of Current， Voltage and power
Circuit Elements: Resisters, voltage sources, current sources
Basic Concepts
(Chapter 1)
V-I relationship: Resisters, voltage sources, current sources
Kirchhoff's Laws: KCL,KVL； Equivalent Subcircuits
Basic Laws
(Chapter 2)
Nodal Analysis
Mesh Analysis
Methods of Analysis
(Chapter 3)
Superposition, Source Transformation
Equivalent circuit:Thevenin's Theorem, Norton's Theorem
Circuit Theorems
(Chapter 4)
V-I relationship(two basic rules): virtual open, virtual short
Op amp Circuit analysis: two basic rules ,Kcl,Kvl,Nonal analysis
Operational Amplifier
(Chapter 5)

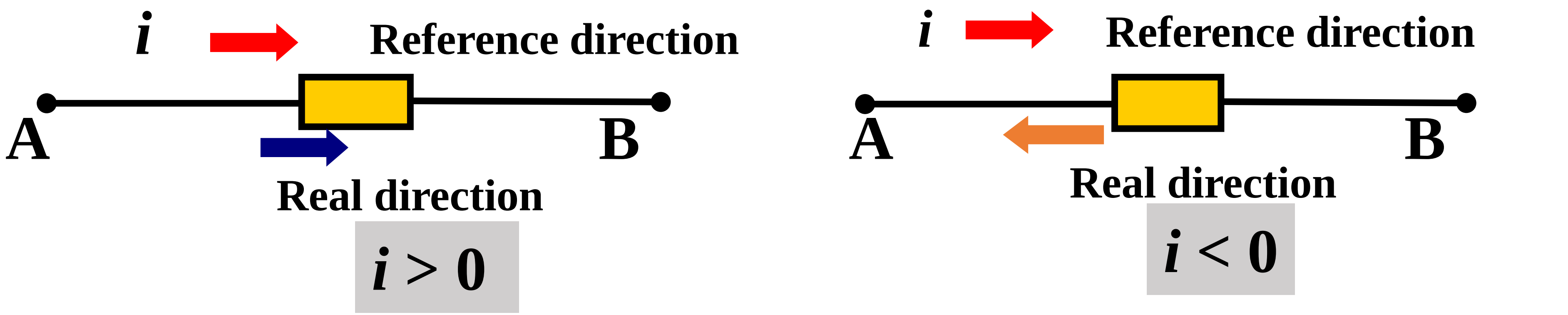
Chapter 1 Basic Concepts

1. Circuit variables: Concept of Current， Voltage and power

（1） In this course, all the current and voltage directions are reference directions.

(2) Each current has two factors: reference direction and value.

Reference current direction



When we start to analyze a circuit, we can select each branche's reference current direction, and then we can decide the real current direction according to whether the value of current is positive or negative.

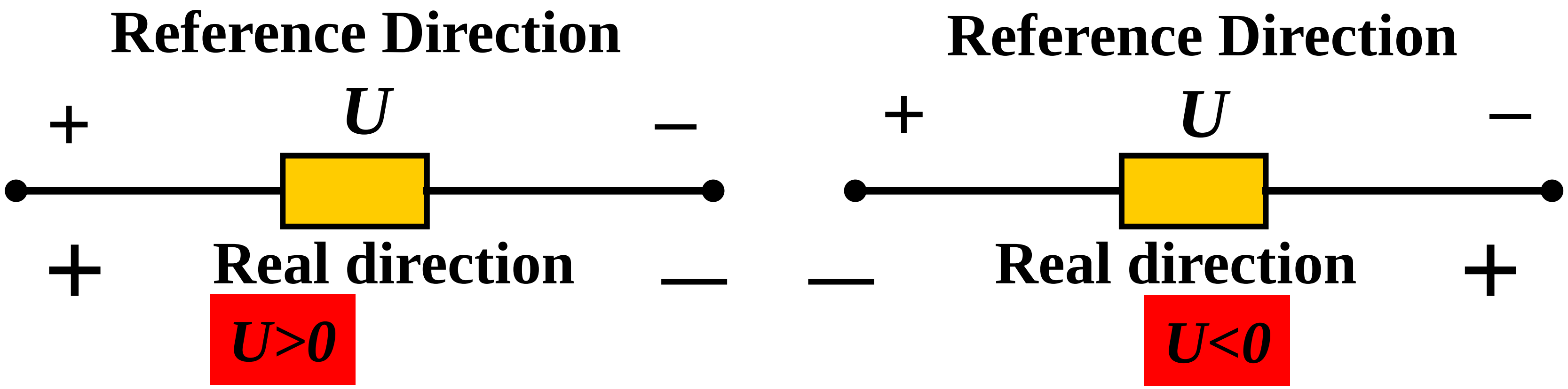
Notice: the selection for reference current direction will not influence the real direction and value of circuit variables.

Chapter 1 Basic Concepts

1. Circuit variables: Concept of Current， Voltage and power

（3） Each voltage has two factors: reference direction and value.

Reference voltage direction



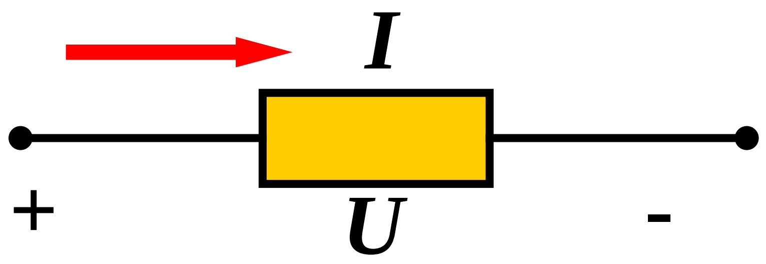
When we start to analyze a circuit, we can select each branches reference voltage direction, and then we can decide the real voltage direction according to whether the value of voltage is positive or negative.

Notice: the selection for reference voltage direction will not influence the real direction and value of circuit variables.

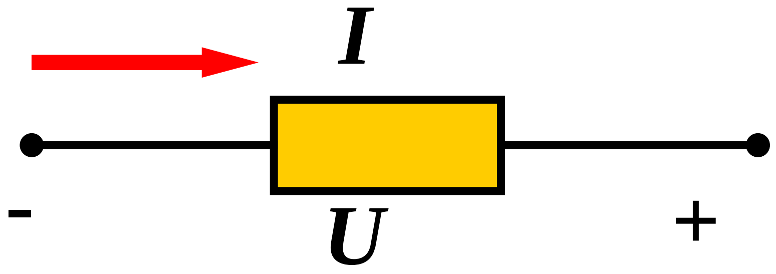
Chapter 1 Basic Concepts

1. Circuit variables: Concept of Current, Voltage and power

（4） Passive sign convention and active sign convention



Passive sign convention
Associate Reference Direction

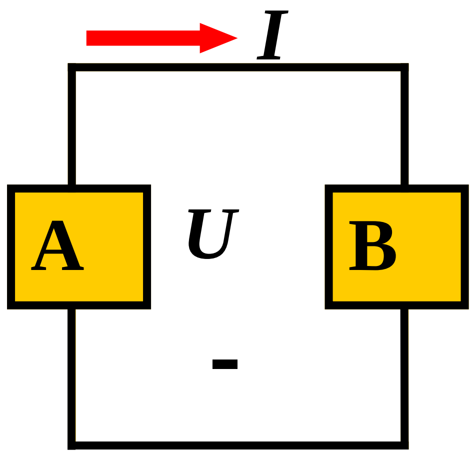


Active sign convention
Non-associate Reference Direction

For one circuit element or one branch, current(reference direction) enters through the positive polarity of the voltage (reference direction).

For one circuit element or one branch, current(reference direction) enters through the negtive polarity of the voltage (reference direction).

Example:



Solution:

For A:U and I is Active sign convention, or Non-associate Reference Direction

For B: U and I is Passive sign convention,or Associate Reference Direction

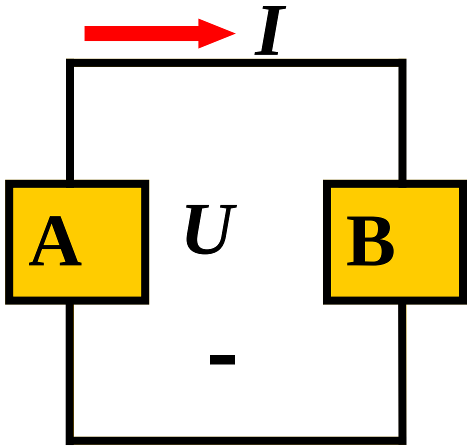
Chapter 1 Basic Concepts

1. Circuit variables: Concept of Current， Voltage and power

(4) Power

For one circuit element, we can calculate its supplied power or its absorbed power.

Example:



For A:U and I is Active sign convention, or Non-associate Reference Direction

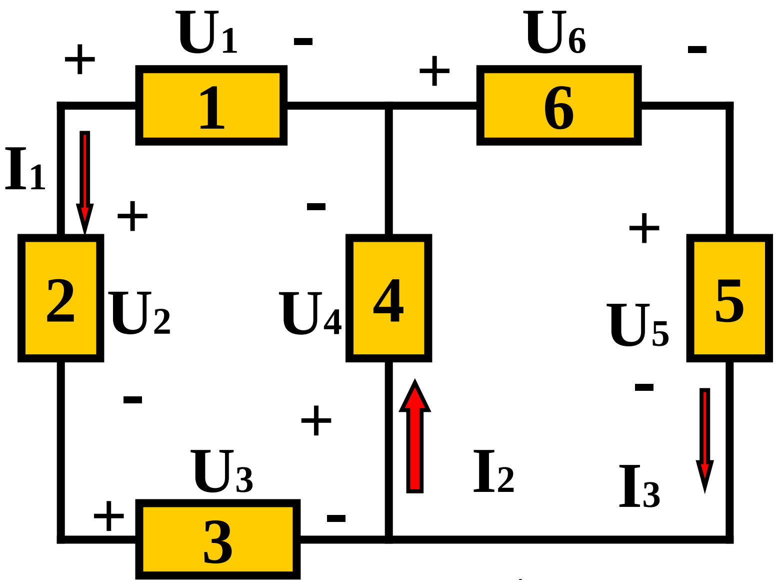
$$P_{Absord} = UI \qquad P_{Supply} = -UI$$

For B: U and I is Passive sign convention,or Associate Reference Direction

$$P_{Absord} = -UI \qquad P_{Supply} = UI$$

Notice: During circuit analysis, we are usually required to calculate sources‘’ supplied power, and passive elements’ absorbed power(R).

Example 1. Assume: U_1=1V, U_2= -3V,U_3=8V, U_4= -4V,U_5=7V, U_6= -3V, I1=2A, I2=1A, I3= -1A. Try to calculate power absorbed by each element.



$$P_1 = -U_1 I_1 = -1 \times 2 = -2W < 0$$ (In fact, the elemet is supplying power)

$$P_2 = U_2 I_1 = (-3) \times 2 = -6W < 0$$ (In fact, the elemet is supplying power)

Solution:

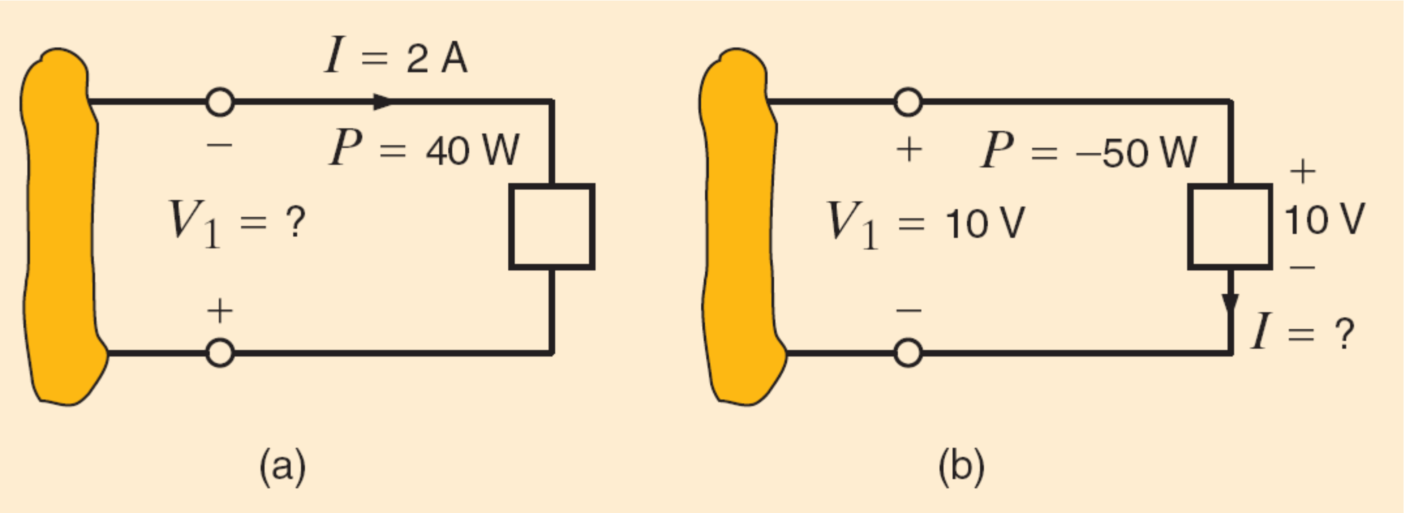
$$P_3 = U_3 I_1 = 8 \times 2 = 16W > 0$$ (In fact, the elemet is absorbing power)

$$P_4 = U_4 I_2 = (-4) \times 1 = -4W < 0$$ (In fact, the elemet is supplying power)

$$P_5 = U_5 I_3 = 7 \times (-1) = -7W < 0$$ (In fact, the elemet is supplying power)

$$P_6 = U_6 I_3 = (-3) \times (-1) = 3W > 0$$ (In fact, the elemet is absorging power)

Example 2. Determine the unknown variable in each circuit? （Suppose the power in the Figure is absorbed power）



Solution:

For (a):

$P=-V_1 I=40$

So $V_1=-20V$

For (b):

$P=V_1 I=-50$

So $I=-5A$

Chapter 1 Basic Concepts

2. Circuit Elements: Resistors, voltage sources, current sources

(1) Circuit Elements in DC circuit:

- Active elements
  - Sources
    - Voltage source
    - Current source
      - Independent source
      - Dependent source
        - CCCS
        - CCVS
        - VCVS
        - VCCS

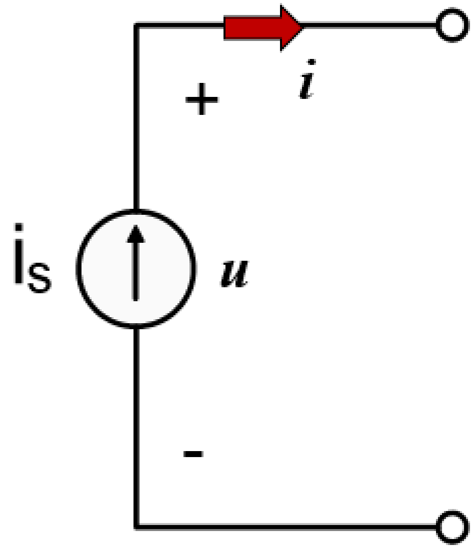
    (Chapter 1)
  - the operational amplifier(Chapter 5)
- Passive elements
  - Resistors (Chapter 2)
  - Inductors（short circuit）
  - Capacitors （open circuit）

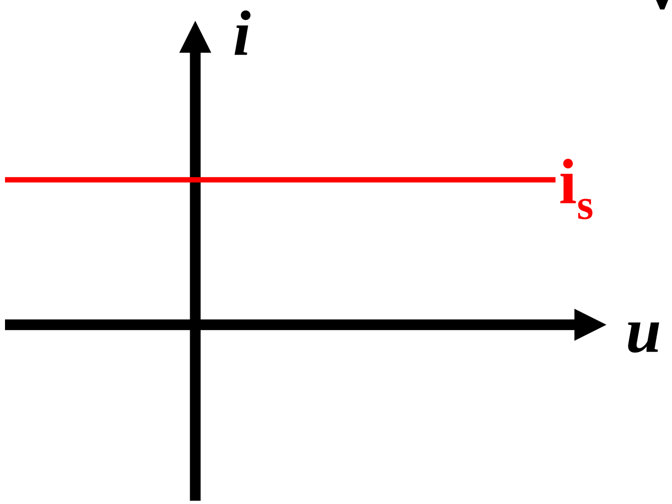
Chapter 1 Basic Concepts

2. Circuit Elements: Resistors, voltage sources, current sources

V-I relationship

Independent Current Sources



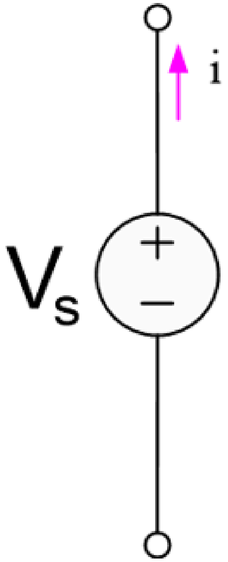


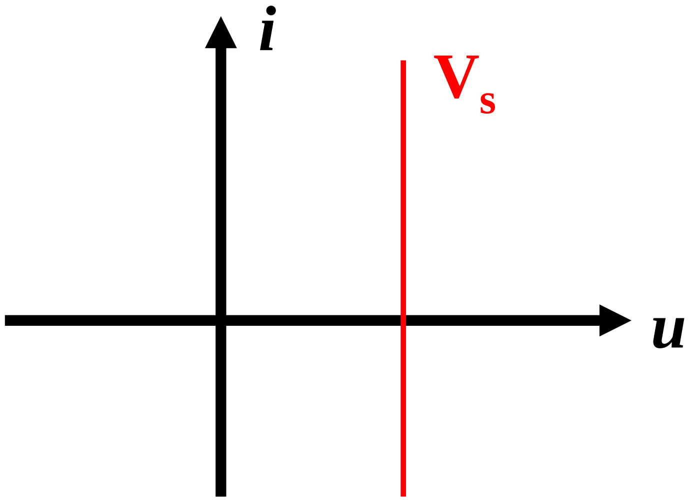
$i = i_s$

u is decided by the circuit

Independent Sources

Independent Voltage Sources





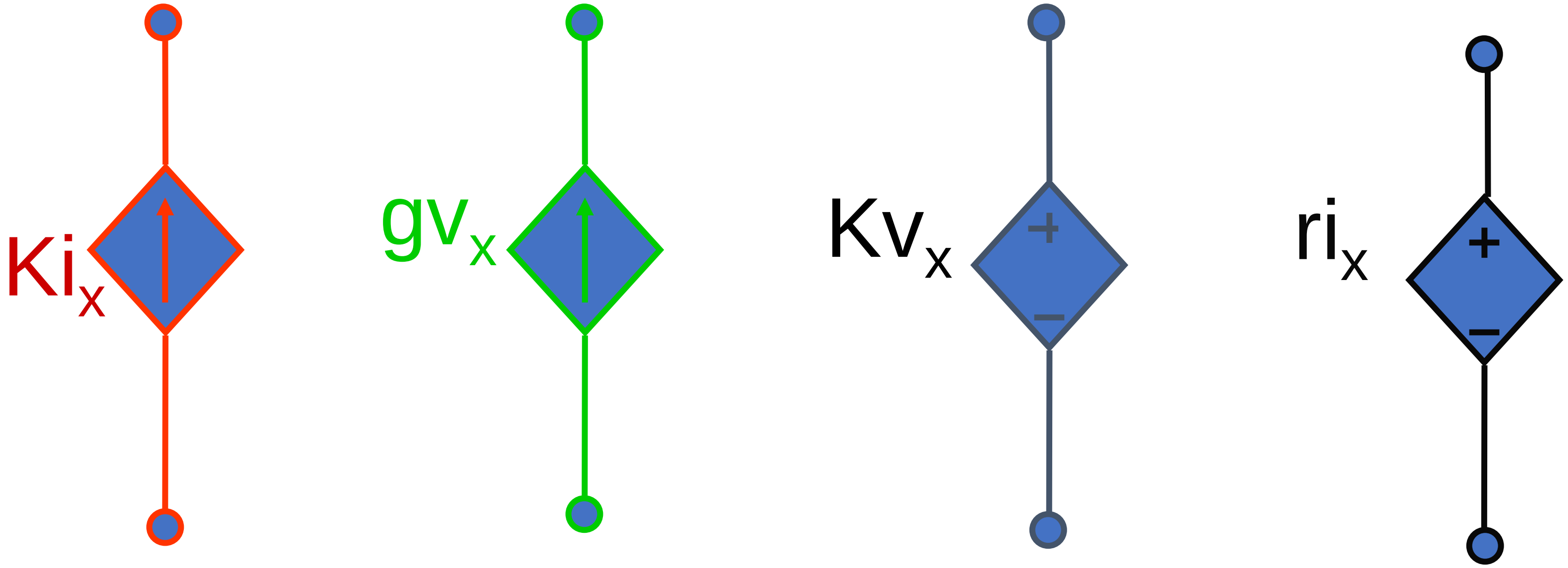
$u = V_s$

i is decided by the circuit

Chapter 1 Basic Concepts

2. Circuit Elements: Resistors, voltage sources, current sources

Dependent Sources



(a) **CCCS**

$$\begin{cases} i = k\mathbf{i_x} \\ \text{u is decided by the circuit} \end{cases}$$

(b) **VCCS**

$$\begin{cases} i = g\mathbf{v_x} \\ \text{u is decided by the circuit} \end{cases}$$

(c) **VCVS**

$$\begin{cases} u = K\mathbf{V_x} \\ \text{i is decided by the circuit} \end{cases}$$

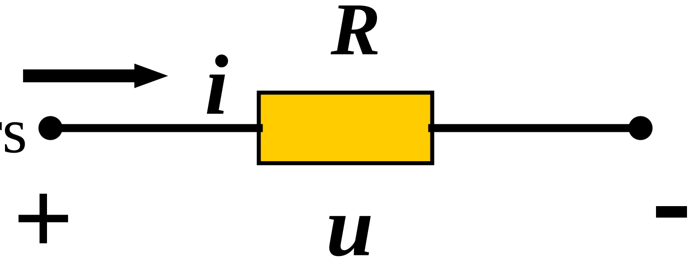
(d) **ICVS**

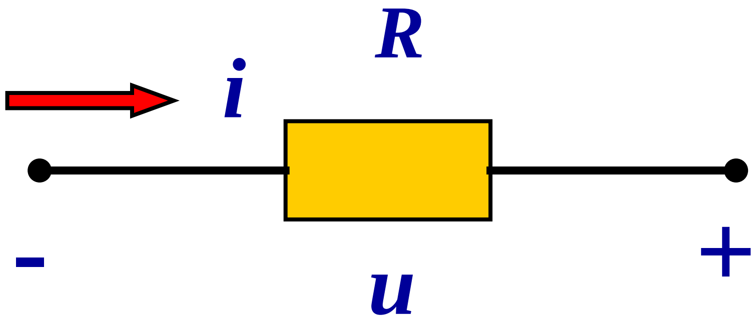
$$\begin{cases} u = r\mathbf{i_x} \\ \text{i is decided by the circuit} \end{cases}$$

Chapter 2 Basic Laws

1. V-I relationship: Resistors, voltage sources, current sources， OP amp

（1） Resistors

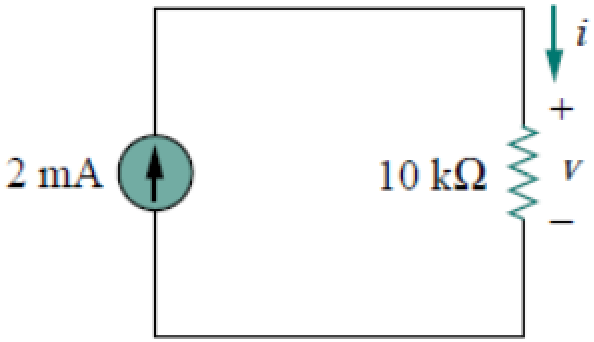




V-I relationship $u = R\,i$ $\quad i = G\,u$ $\qquad u = -R\,i$ $\quad i = -G\,u$

$p_{absorbed} = u\,i = i^2R = u^2/R$ $\qquad p_{absorbed} = -u\,i = i^2R = u^2/R$

Example 1. calculate the voltage v, and the power p.



Solution:

$v = Ri = 10 \times 10^3 \times 2 \times 10^{-3} = 20V$

$p = vi = 20 \times 2 \times 10^{-3} = 40\text{m}W$ (Absorbing power)

（2） voltage sources, current sources （Chapter 1)

（3） OP amp (Chapter 5)

Chapter 2 Basic Laws

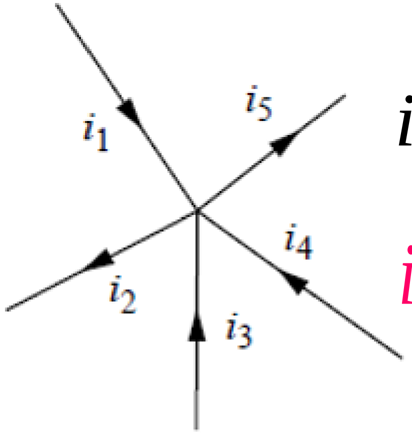
2. Kirchhoff's Laws: KCL,KVL

KCL

Kirchhoff's current law (KCL) states that the algebraic sum of currents entering a node (or a closed boundary) is zero.

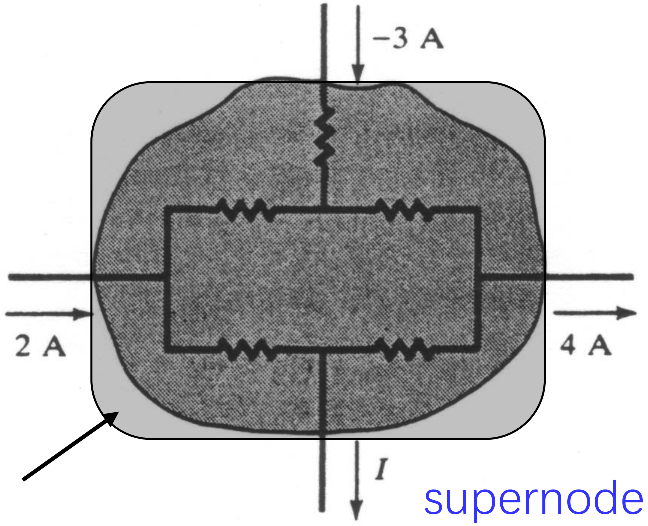
The sum of the currents entering a node is equal to the sum of the currents leaving the node.

node"



$$i_1 - i_2 + i_3 + i_4 - i_5 = 0$$

$$i_1 + i_3 + i_4 = i_2 + i_5$$



supernode

-I – 4+(-3)+2 = 0

I + 4 = (-3)+2

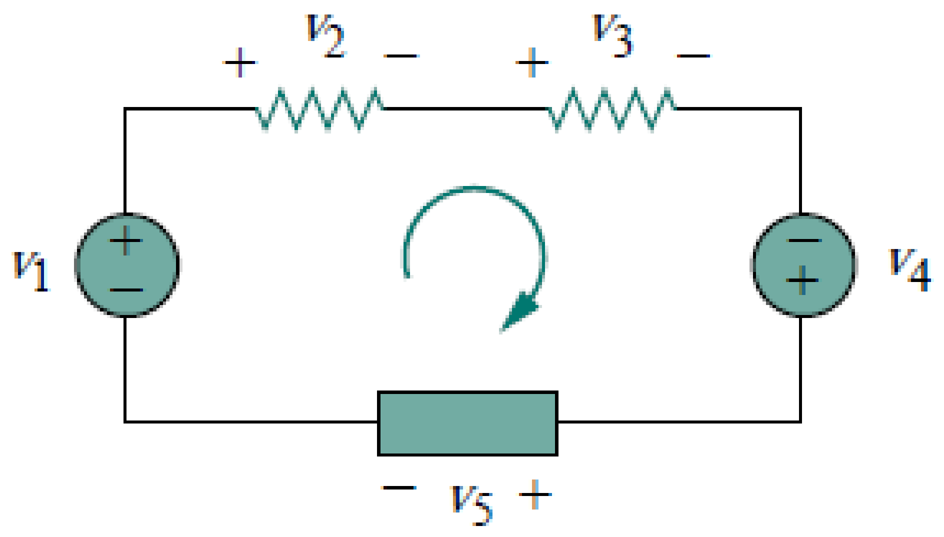
Chapter 2 Basic Laws

2. Kirchhoff's Laws: KCL,KVL

KVL

Kirchhoff's voltage law (KVL) states that the **algebraic sum** of all voltages around a closed path (or loop) is zero.

Sum of voltage drops = Sum of voltage rises



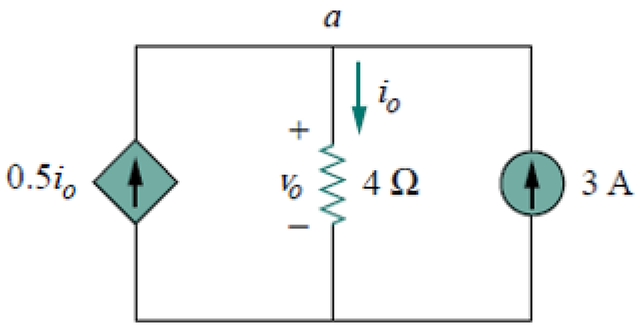
$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$

$$v_2 + v_3 + v_5 = v_1 + v_4$$

Chapter 2 Basic Laws

2. Kirchhoff's Laws: KCL,KVL

Example 2. Determine current i_0 and voltage v_0 in the following circuit.



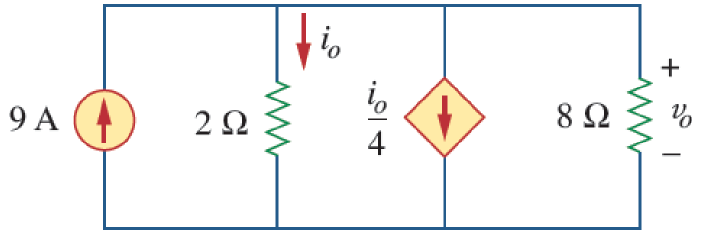
Solution:

Applying KCL to node a, we obtain

$$3 + 0.5i_0 = i_0 \quad \Rightarrow \quad i_0 = 6\ \text{A}$$

$$so \quad v_0 = 4i_0 = 24\text{V}$$

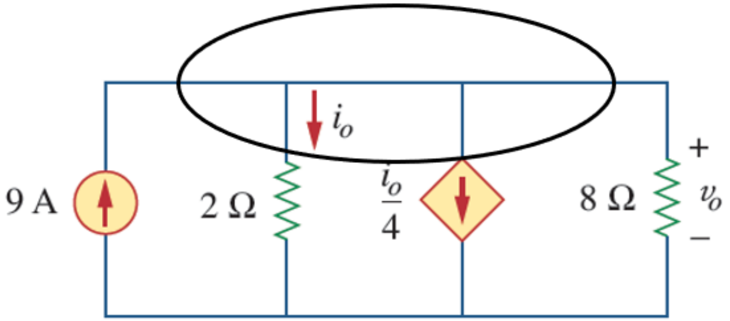
Example 3. Determine current i_0 and voltage v_0 in the following circuit.



Solution:

Applying KCL to node a, we obtain

$$\frac{v_0}{8} + \frac{i_0}{4} + i_0 = 9$$



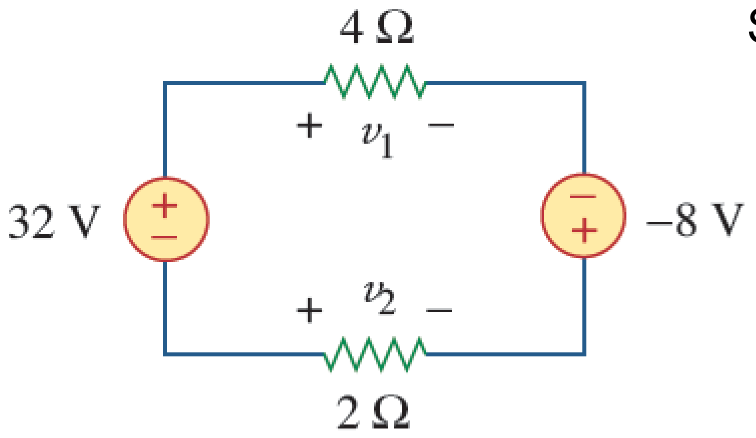
Applying Ohm's law: $v_0 = 2\, i_0$

So $i_0 = 6\ \text{A}$ $\quad v_0 = 2i_0 = 12\text{V}$

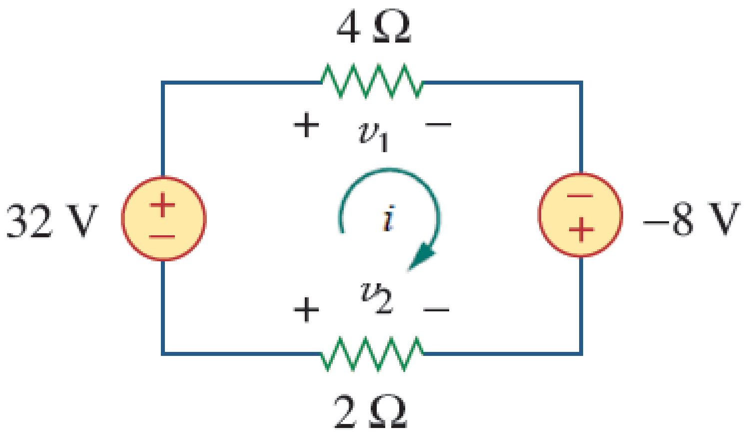
Chapter 2 Basic Laws

2. Kirchhoff's Laws: KCL,KVL

Example 4. Determine current v_1 and voltage v_2 in the following circuit.



Solution:

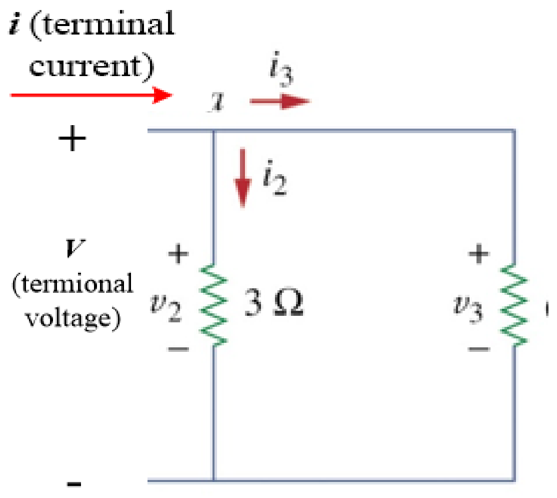


Applying Ohm's law: v_1=4i, v_2=-2i

Applying KVL : v_1=-8+v_2 +32

So v_1= 16V, v_2= -8V

Chapter 2 Basic Laws

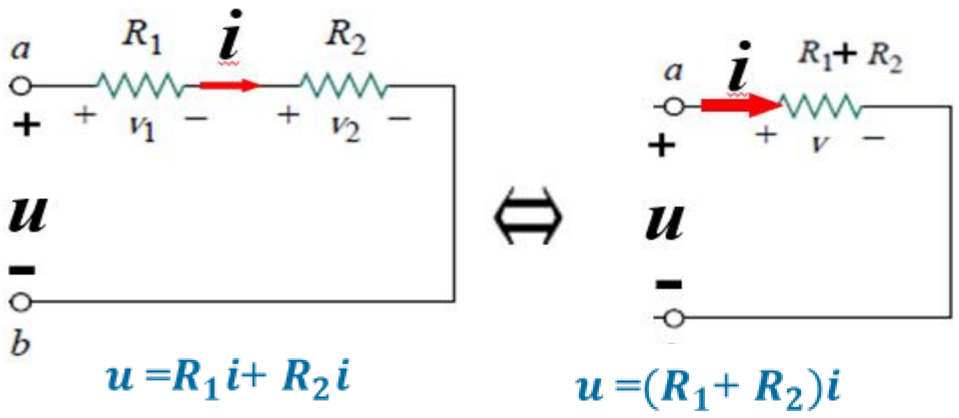


3. Equivalent Subcircuits

（1） **Terminal law** : the voltage-current relationship of a two-terminal subcircuit.

$$v = f(i) \quad \text{or} \quad i = g(v)$$

（2） **Equivalent subcircuits:** Two two-terminal subcircuit are said to be equivalent if they have same terminal law.



（3） **Why equivalent subcircuits?**

To simplify the circuit and then solve it.

(4) **How to simplify a circuit by equivalent subcircuits?**

Series resistors and parallel resistors (this chapter)

Superposition, source transformation, Thevenin’s Theorem, Norton’s Theorem(Chapter 4)

Chapter 2 Basic Laws

3. Equivalent Subcircuits

Resistors in series behave as a single resistor whose resistance is equal to the sum of the resistances of the individual resistors.

$$R_{eq} = R_1 + R_2 + \cdots + R_N = \sum_{n=1}^{N} R_n$$

voltage division:

$$v_n = \frac{R_n}{R_1 + R_2 + \cdots + R_N} v$$

The equivalent conductance of resistors connected in parallel is the sum of their individual conductances.

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}$$

$$\boxed{G_{eq} = G_1 + G_2 + G_3 + \cdots + G_N}$$

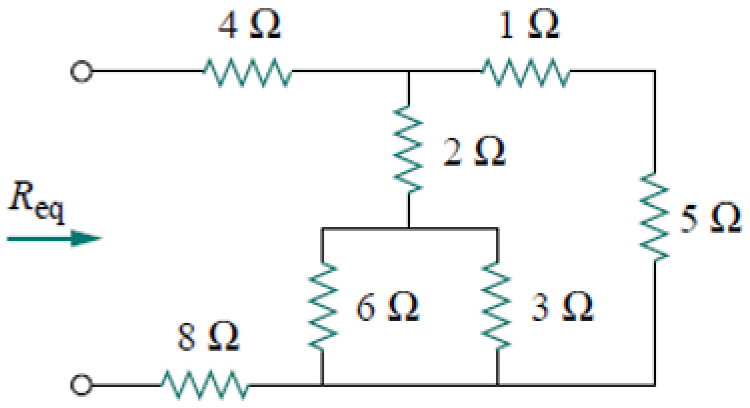
The equivalent resistance of two parallel resistors is equal to the product of their resistances divided by their sum.

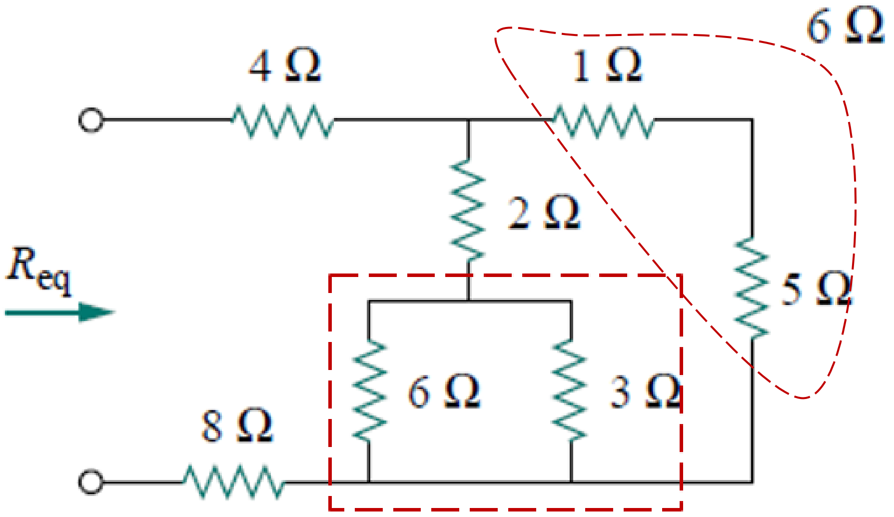
$$\boxed{R_{eq} = \frac{R_1 R_2}{R_1 + R_2}}$$

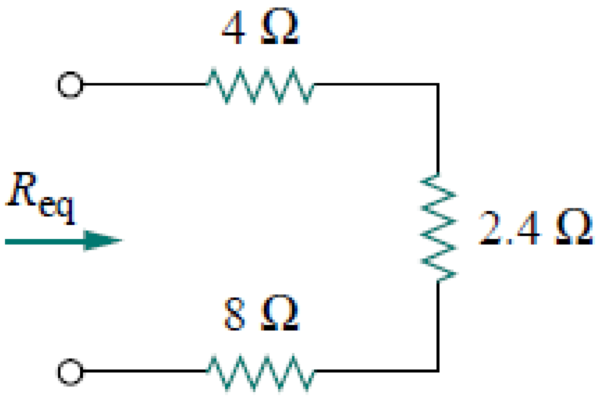
$$\boxed{i_1 = \frac{R_2\, i}{R_1 + R_2}, \qquad i_2 = \frac{R_1\, i}{R_1 + R_2}}$$

3. Equivalent Subcircuits

Example 5   Find R_{eq} for the circuit shown in the following circuit.

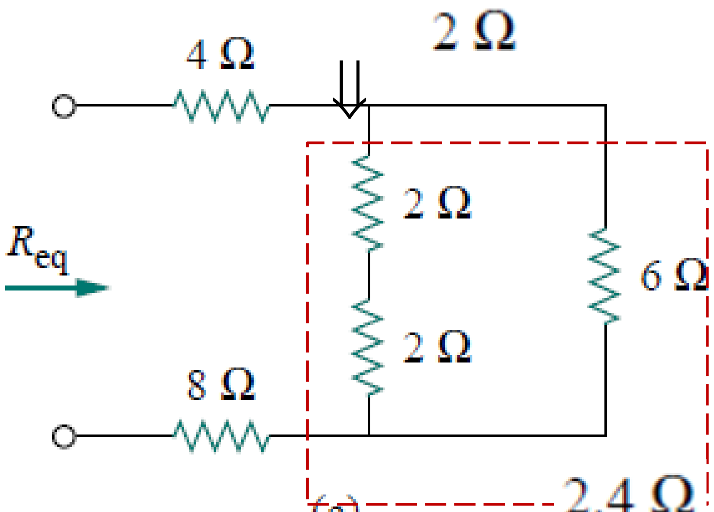






(b)

⇐

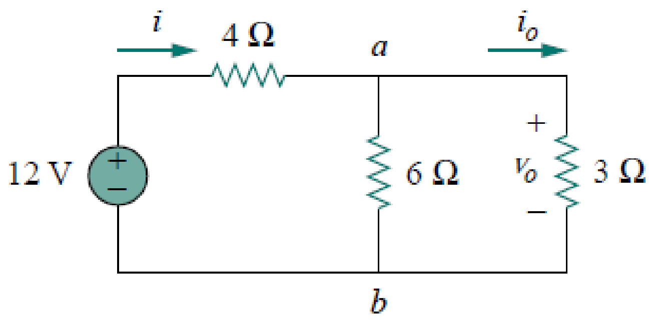


(a)

Chapter 2 Basic Laws

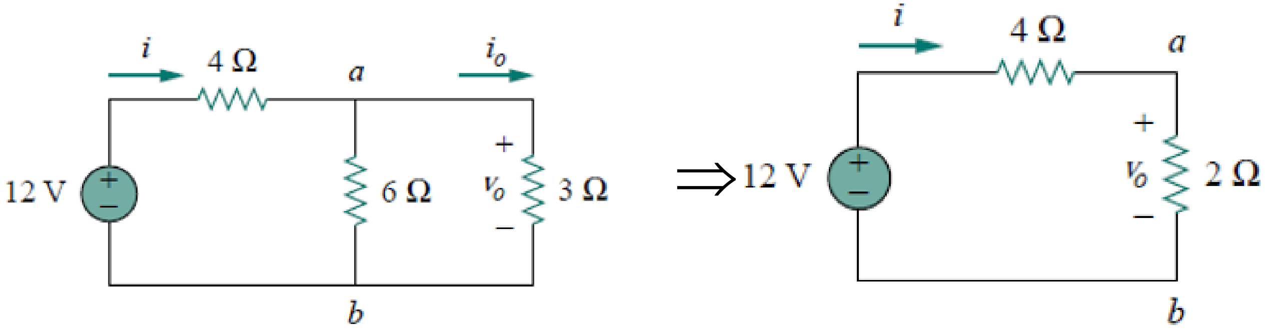
3. Equivalent Subcircuits

Example 6 Find i_0 and v_0 in the following circuit. calculate the power dissipated in the 3-Ω resistor.



(a)

Solution:



$$i = \frac{12}{4+2} = 2\text{ A} \qquad v_o = \frac{2}{2+4}(12\text{ V}) = 4\text{ V}$$

$$v_o = 3i_o = 4 \quad \Longrightarrow \quad i_o = \frac{4}{3}\text{ A} \quad \text{or} \quad i_o = \frac{6}{6+3}i = \frac{2}{3}(2\text{ A}) = \frac{4}{3}\text{ A}$$

$$p_o = v_o i_o = 4\left(\frac{4}{3}\right) = 5.333\text{ W}$$

Chapter 3 Methods of Analysis

1. Nodal analysis

(1) Circuit variables： node voltages

Notice:if a circuit has n nodes, select one node as reference node, the next n-1 nodes are independent nodes and their voltages are the variables of node voltage.

(2)How to write Equations?

KCL for independent nodes, replace current with node voltages

(3) **Steps to determine the node voltages:**

①Select the reference node.

②Assign node voltages $v_1, v_2, \ldots, v_{n-1}$ to the remaining n-1 nodes.

③Apply KCL to each of the n-1 non-reference nodes.

Use Ohm's law to express the branch currents in terms of node voltages.

④Solve the equations to obtain the unknown node voltages.

⑤Compute branch(element) currents and voltages.

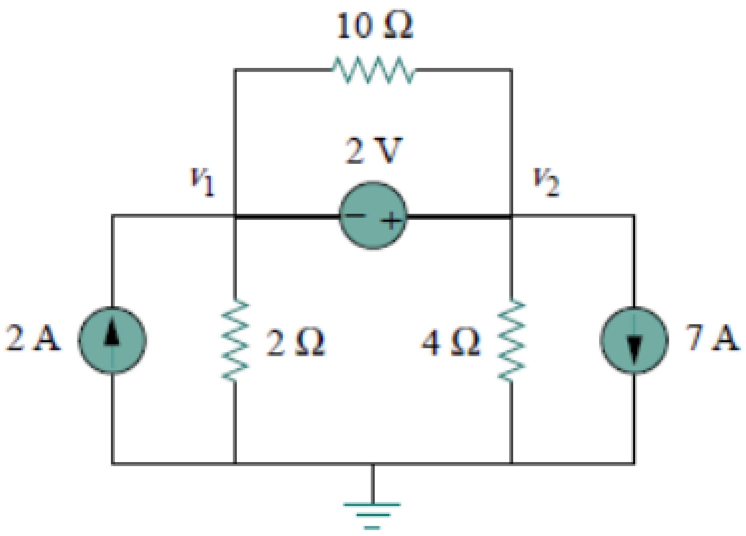
Chapter 3 Methods of Analysis

1. Nodal analysis

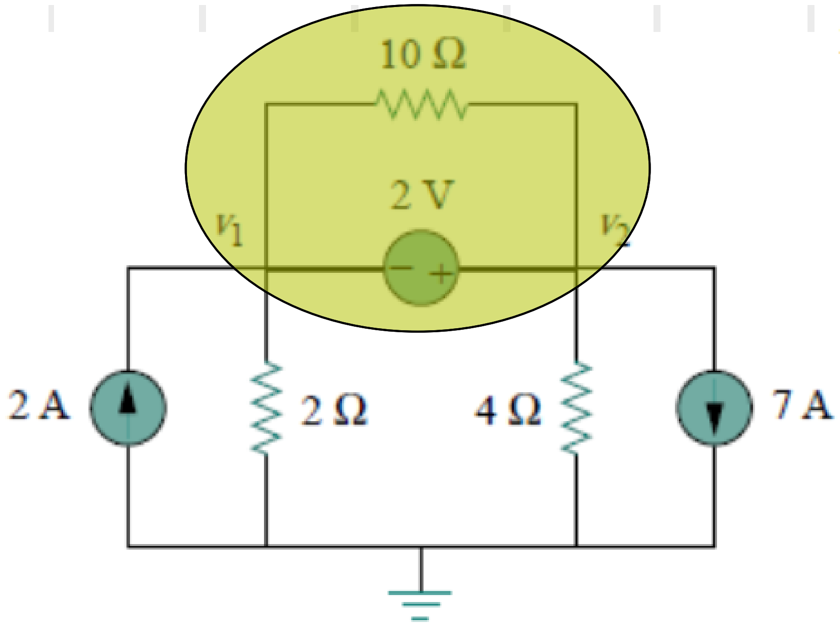
(4) Two special cases for Nodal analysis

Case 1: If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source.

Example 7– In the following circuit, find the node voltages.



Solution:



$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7$$

$v_2 - v_1 = 2$

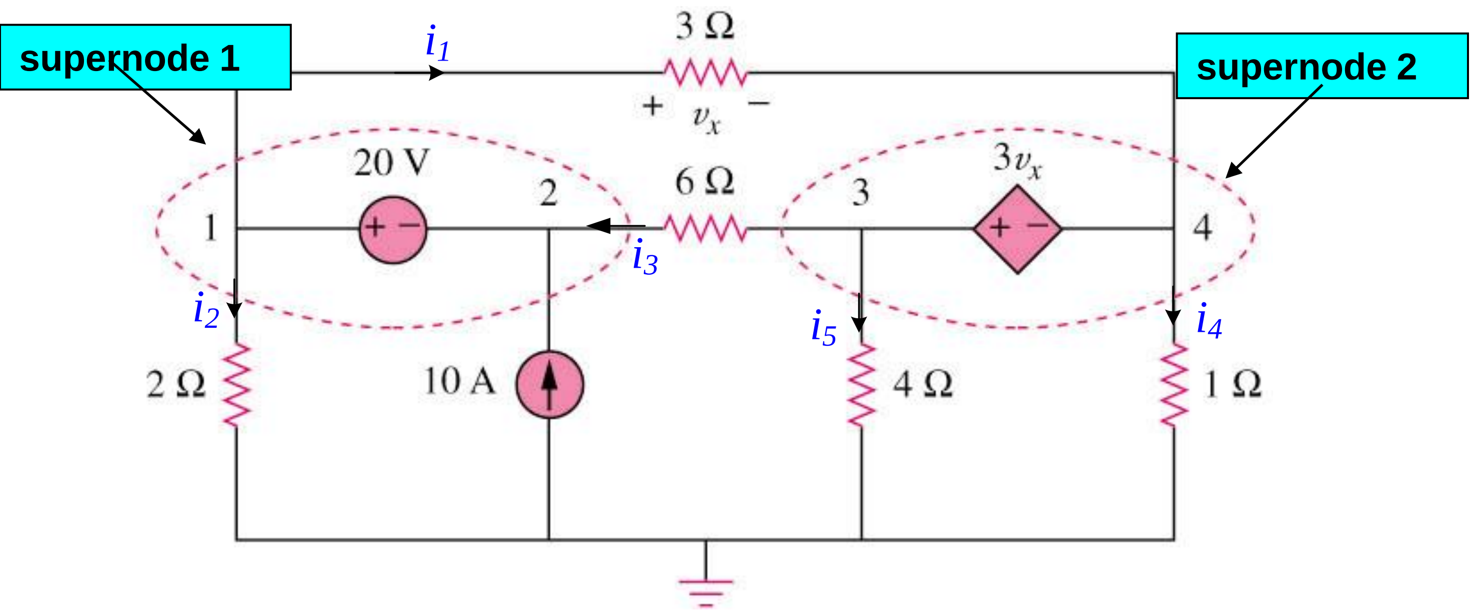
Chapter 3 Methods of Analysis

1. Nodal analysis

(4) Two special cases for Nodal analysis

CASE 2 If the voltage source (dependent or independent) is connected between two nonreference nodes, the two nonreference nodes form a *generalized node* or *supernode*; we apply both KCL and KVL to determine the node voltages.

Example 8 – circuit with one independent voltage source and one dependent voltage source



Supernode 1, $\frac{v_3 - v_2}{6} + 10 = \frac{v_1 - v_4}{3} + \frac{v_1}{2}$

Supernode 2, $\frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + \frac{v_4}{1} + \frac{v_3}{4}$

$v_1 - v_2 = 20$ $\quad$ $v_3 - v_4 = 3v_x$

$v_1 - v_4 = v_x$ (Controled source)

2. Mesh analysis

(1) Circuit variables： Mesh current

Notice: A mesh is a loop which does not contain any other loops within it,and we define a mesh current as a current that flows only around the perimeter of a mesh.

(2)How to write Equations?

KVL for meshes, replace voltage with mesh current.

(3) **Steps to determine the node voltages:**

①Assign mesh currents i1, i2, …, in to the n meshes.

②Apply KVL to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.

③Solve the resulting 2 simultaneous equations to get the mesh currents.

④Compute branch(element) currents and voltages.

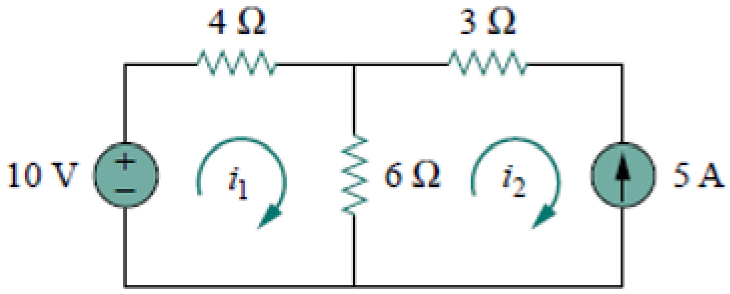
2. Mesh analysis

(4) Two special cases for Mesh analysis

Case 1. When a current source exists only in one mesh.

For the mesh containing the current source, its mesh current is known.

Set $i_2 = -5\ A$



Case 1

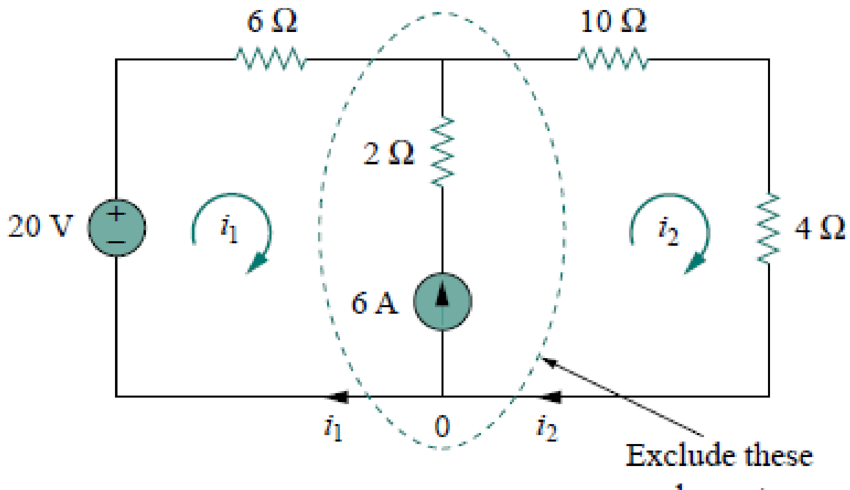
For other meshes, write the mesh current equation using KVL.

$4i_1 + 6(i_1 - i_2) = 10$

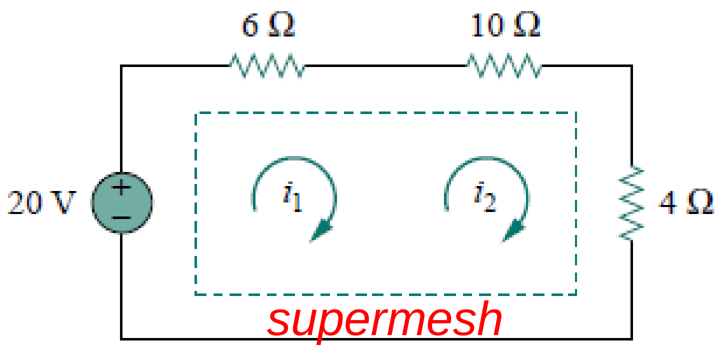
2. Mesh analysis

(4) Two special cases for Mesh analysis

Case 2. A current source exists between two meshes .



(a)



(b)

A supermesh results when two meshes have a (dependent or independent) current source in common.

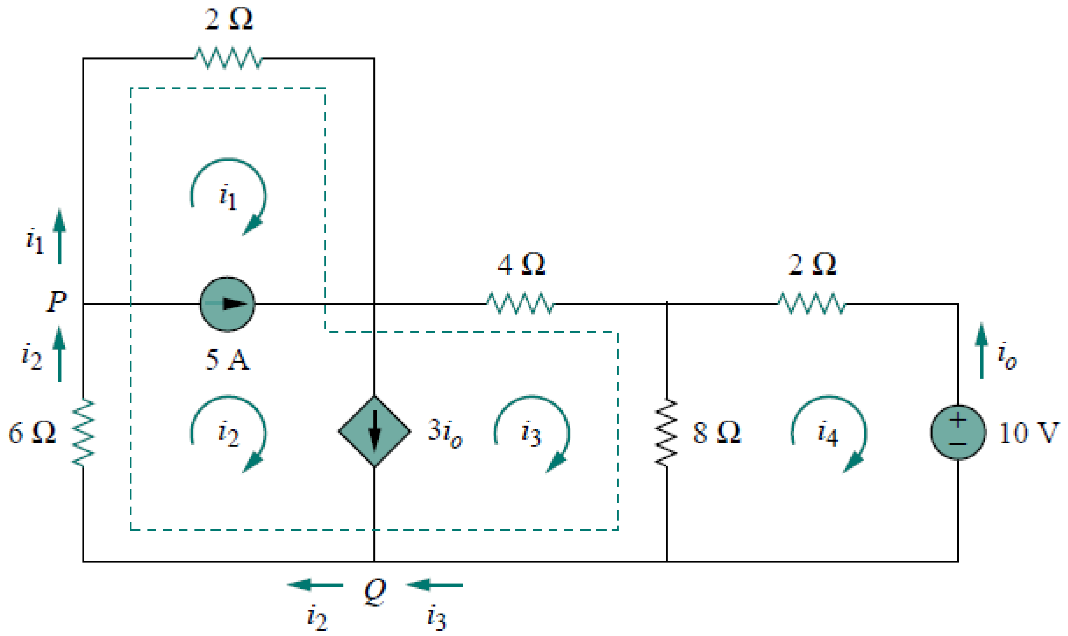
We create a *supermesh by excluding* the current source and any elements connected in series with it, as shown in above circuit(b).

Applying KVL to the supermesh $-20 + 6i_1 + 10i_2 + 4i_2 = 0$

Applying KCL to node 0, $i_2 = i_1 + 6$

2. Mesh analysis

Example 9 – For the circuit, find i_1 to i_4 using mesh analysis.



Applying KVL to the larger supermesh, $2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0$

Applying KCL to node *P:* $i_2 = i_1 + 5$

Applying KCL to node *Q:* $i_2 = i_3 + 3i_o$

Applying KVL in mesh 4, $2i_4 + 8(i_4 - i_3) + 10 = 0$

$$i_o = -i_4$$

$i_1 = -7.5$ A, $\quad i_2 = -2.5$ A, $\quad i_3 = 3.93$ A, $\quad i_4 = 2.143$ A

Chapter 4 Circuit Theorems

1. Superposition Theorem

（1）Problems: If a circuit has two or more independent sources

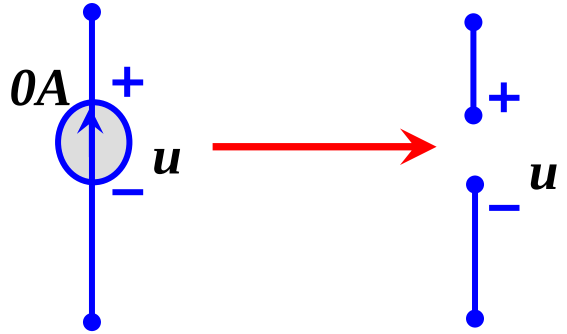
Method 1: Nodal or mesh analysis method.

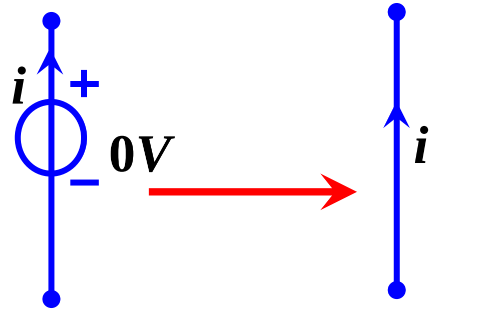
Method 2: Superposition.

(2) Superposition: If a linear circuit with more than one independent source, its response is the algebraic sum of the contribution of each independent source acting alone.

(3) One independent source acting alone, means all other independent sources are turned off. This implies that we replace

- every voltage source by 0 V (or a short circuit),
- every current source by 0 A (or an open circuit).
- Dependent sources are left.





Chapter 4 Circuit Theorems

1. Superposition Theorem

（4）Steps to apply superposition principle

①Let one independent source acting alone, and find the output (voltage or current) due to that active source using nodal or mesh analysis.

②**Repeat step 1** for each of the other independent sources.

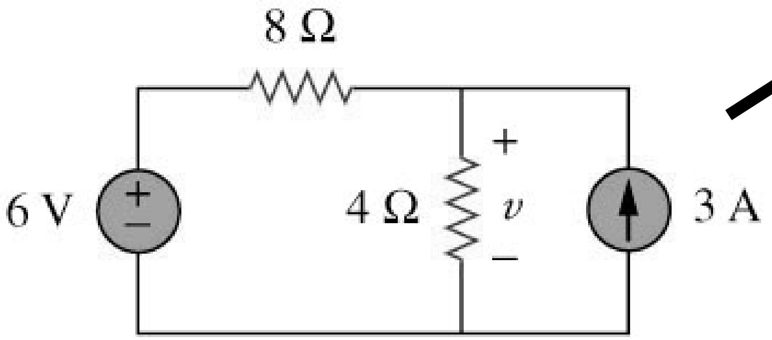
③**Find** the total contribution by adding **algebraically** all the contributions.

Pay attention to the reference direction.

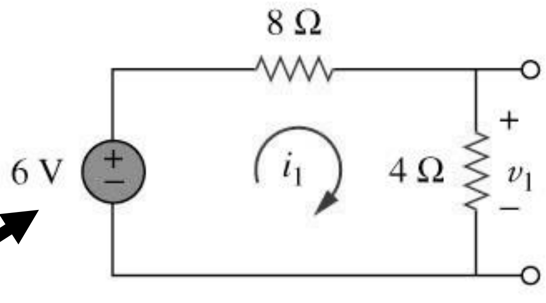
Chapter 4 Circuit Theorems

1. Superposition Theorem

Example 1. Use the superposition theorem to find *v* in the circuit shown below.



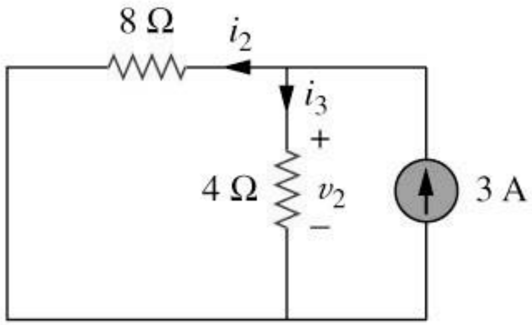
3A is discarded
by open-circuit



(a)

$v_1 = 2\text{V}$

6V is discarded
by short-circuit



(b)

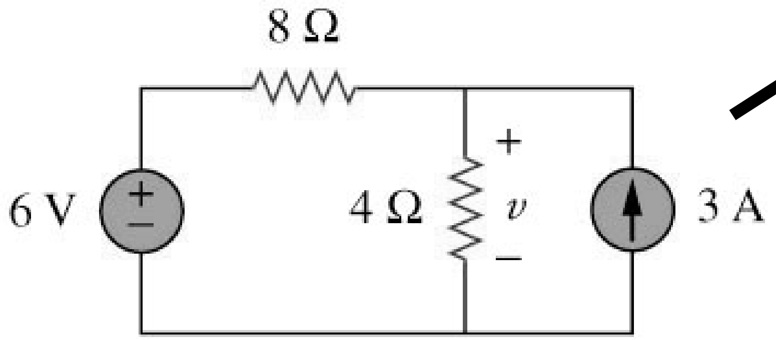
$v_2 = 3 \times \frac{8}{8+4} \times 4 = 8\text{V}$

$v = v_1 + v_2 = 10\text{V}$

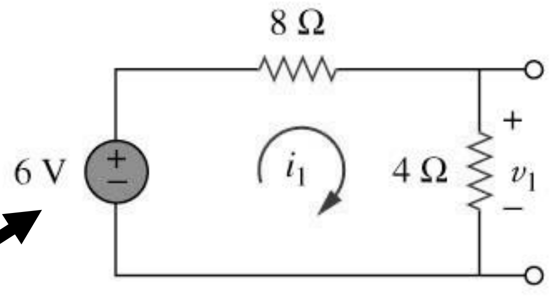
Chapter 4 Circuit Theorems

1. Superposition Theorem

Example 1. Use the superposition theorem to find *v* in the circuit shown below.



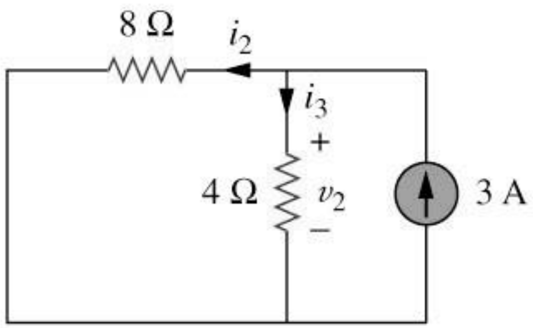
3A is discarded by open-circuit



(a)

$v_1 = 2\text{V}$

6V is discarded by short-circuit



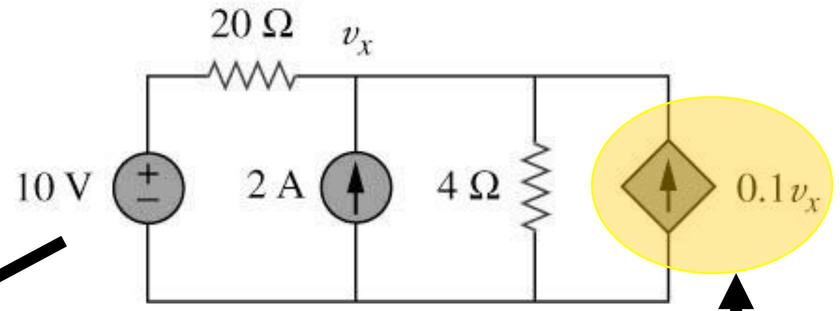
(b)

$v_2 = 3 \times \frac{8}{8+4} \times 4 = 8\text{V}$

$v = v_1 + v_2 = 10\text{V}$

1. Superposition Theorem

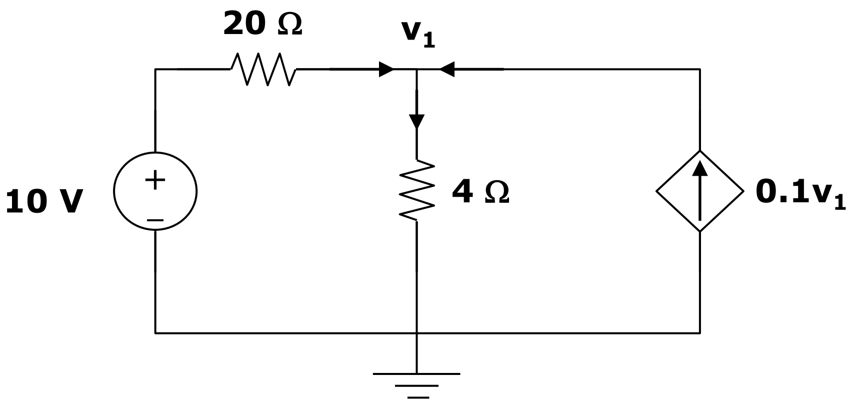
Example 2. Use superposition to find v_x in the circuit below.



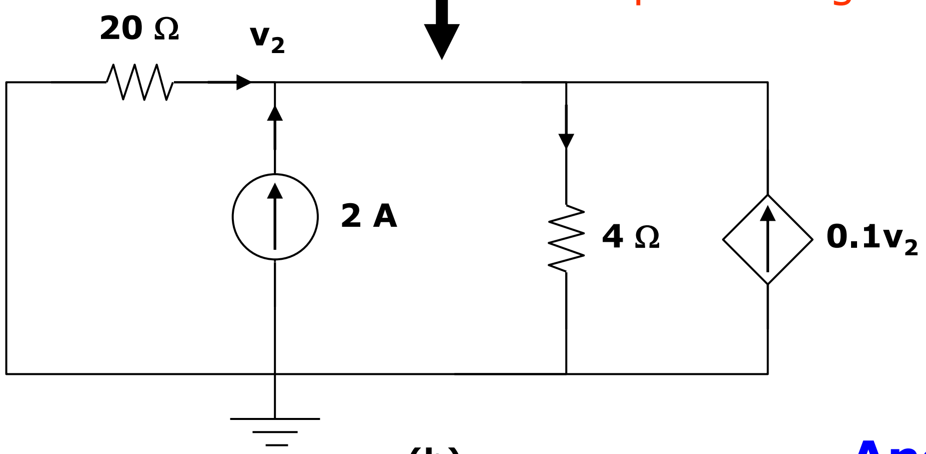
2A is discarded by open-circuit

10V is discarded by short-circuit

Dependant source keep unchanged



(a)



(b)

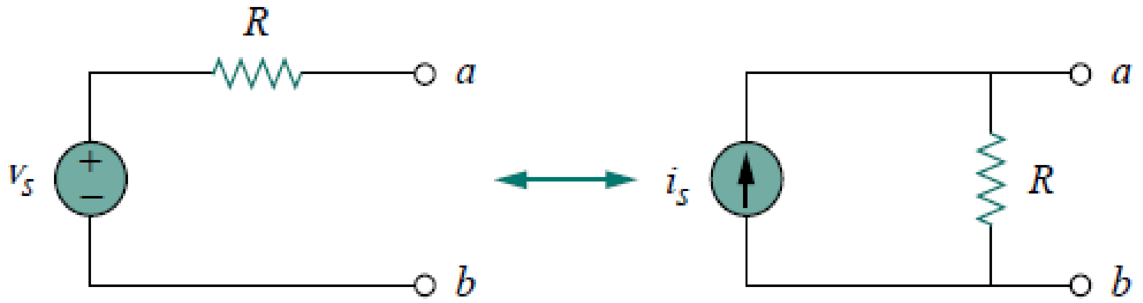
Answer Vx = 12.5V

Notice: The circuit involves a dependent source, which must be left intact.

Chapter 4 Circuit Theorems

2. Source Transformation

（1）



(a) (b)

Steps to replace (a) by (b):

1.Structure:

before : ***a voltage source v_s in series with a resistor R***

Now: ***a current source i_s in parallel with a resistor R***

2. Value: R the same, $i_s = \frac{v_s}{R}$

3. Source direction:

the '-' to '+' direction of voltage source= current source direction.

Steps to replace (b) by (a):

1.Structure:

before : ***a current source i_s in parallel with a resistor R***

Now: ***a voltage source v_s in series with a resistor R***

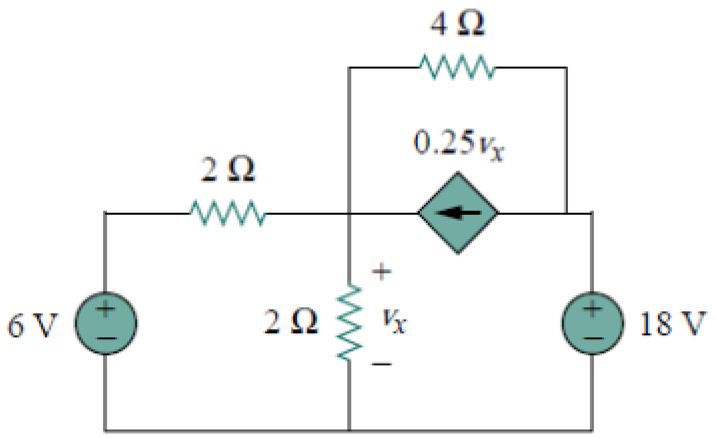
2. Value: R the same, $i_s = \frac{v_s}{R}$

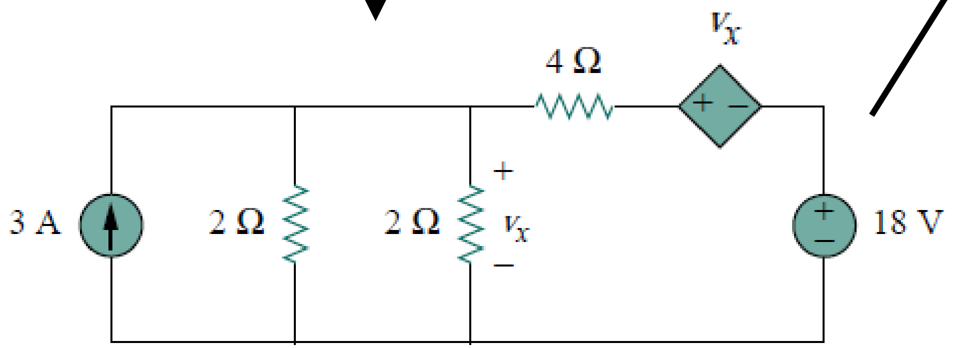
3. Source direction:

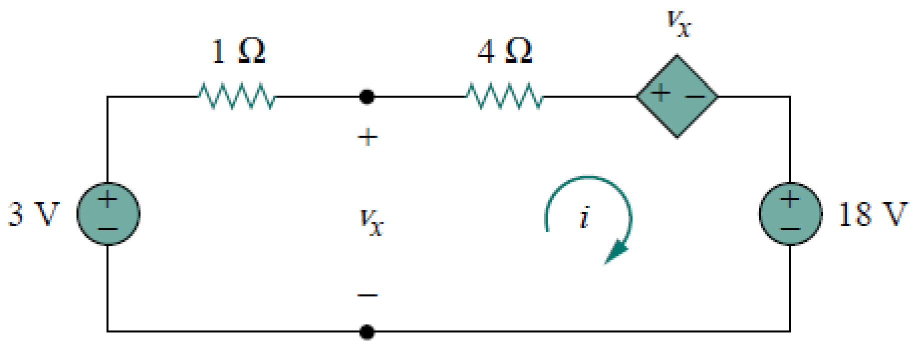
current source direction= the '-' to '+' direction of voltage source.

2. Source Transformation

Example 3. Find vx in the following circuit using source transformation.







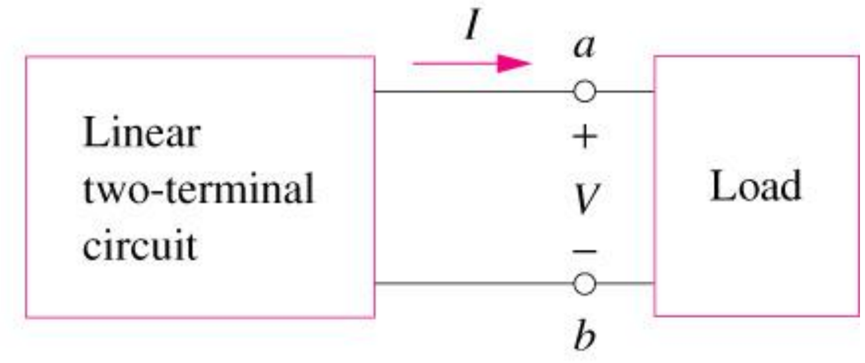
$$-3 + 5i + v_x + 18 = 0$$

$$-3 + 1i + v_x = 0$$

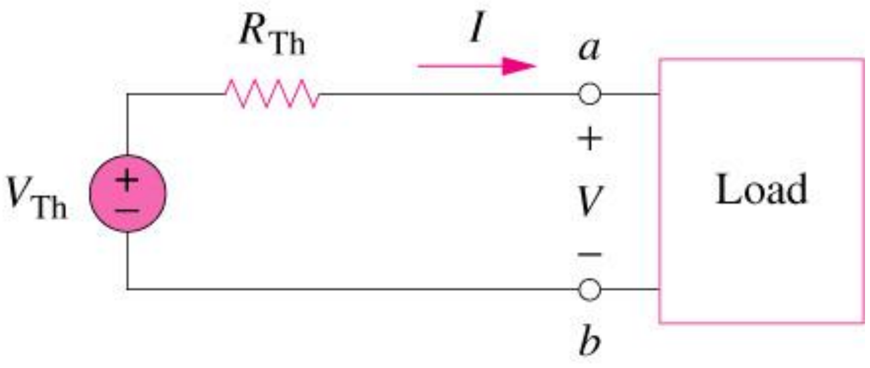
$$i = -4.5 \text{ A}$$

3. Thevenin's Theorem

(1)Conclusion:

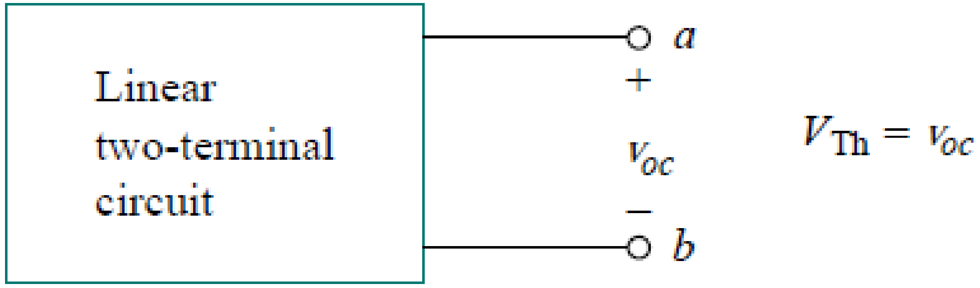


(a)

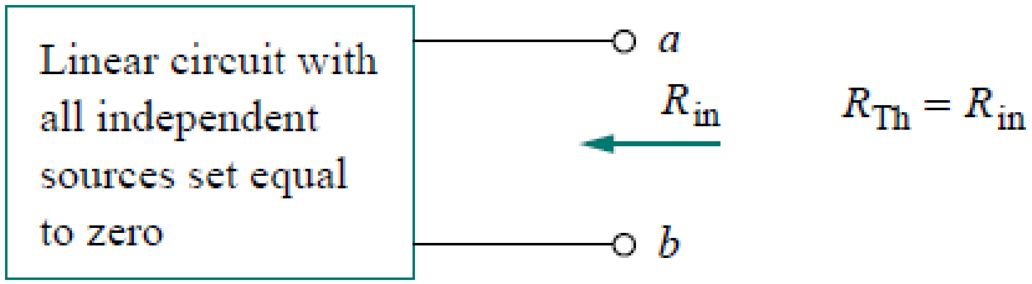


(b)

- V_{TH} is the open-circuit voltage at the terminals.



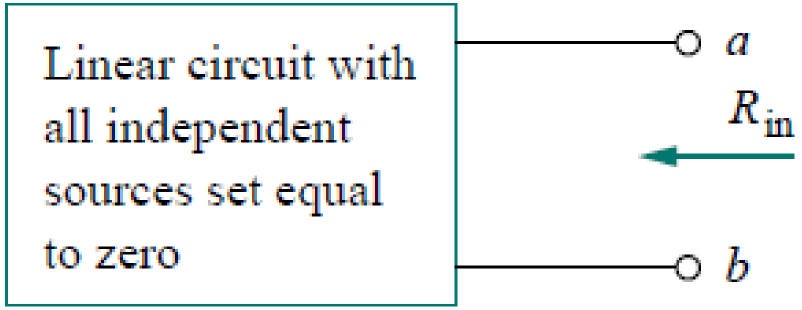
R_{TH} is the input or equivalent resistance at the terminals when the independent sources are turned off.



3. Thevenin's Theorem

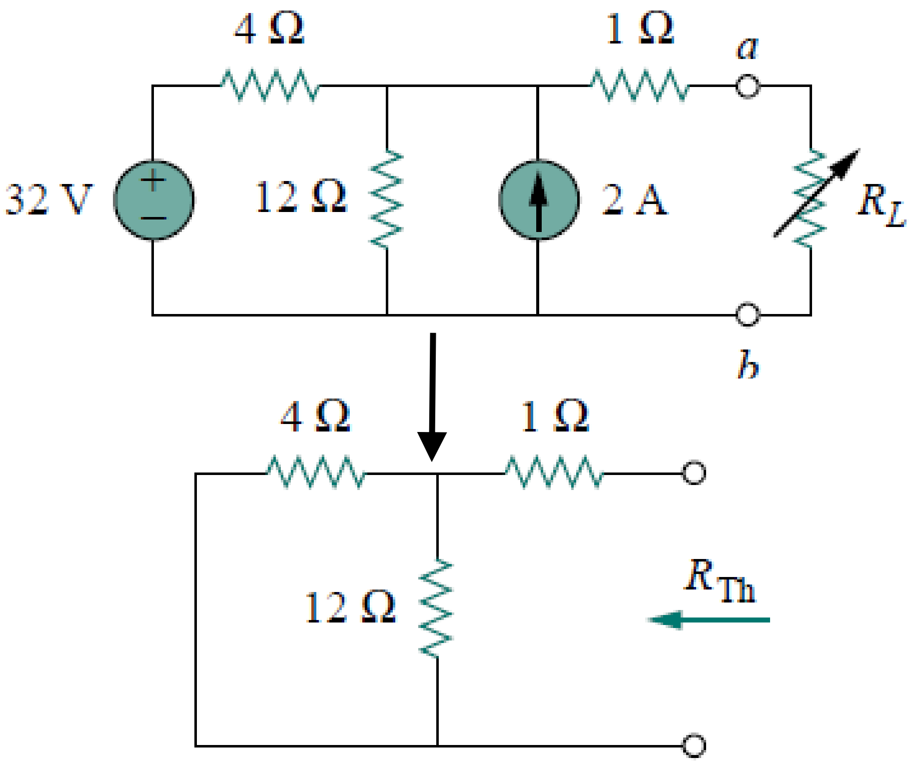
(2) To apply this idea in finding the Thevenin resistance R_{Th}, we need to consider two cases.

CASE I If the network has no dependent sources, we turn off all independent sources. R_{Th} is the input resistance of the network looking between terminals a and b



$R_{Th} = R_{in}$

(b)

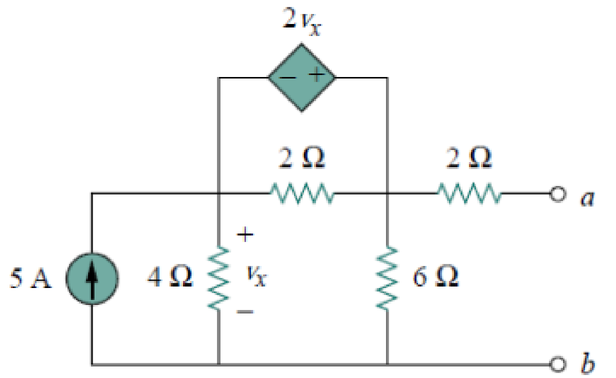


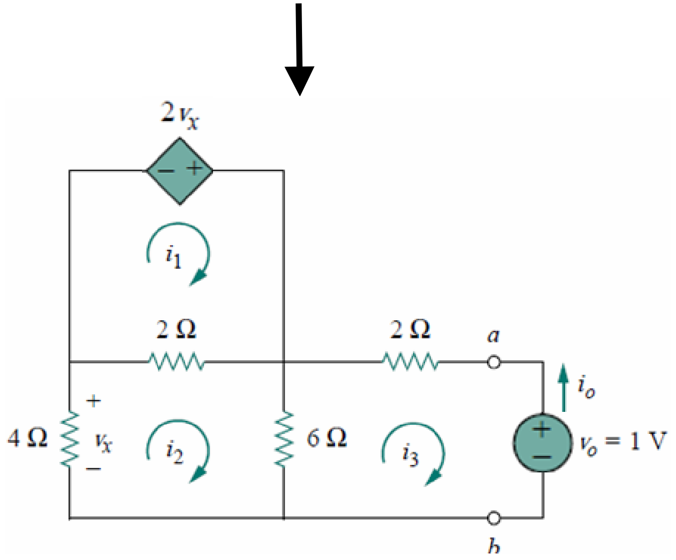
$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$

3. Thevenin's Theorem

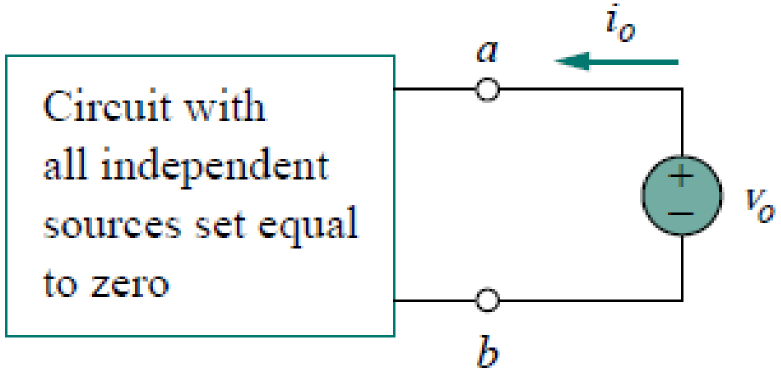
(2) To apply this idea in finding the Thevenin resistance R_{Th}, we need to consider two cases.

CASE 2 If the network has dependent sources, we turn off all independent sources. As with superposition, dependent sources are not to be turned off because they are controlled by circuit variables. We apply a voltage source v_o at terminals a and b and determine the resulting current i_o. Then $R_{Th} = v_o / i_o$

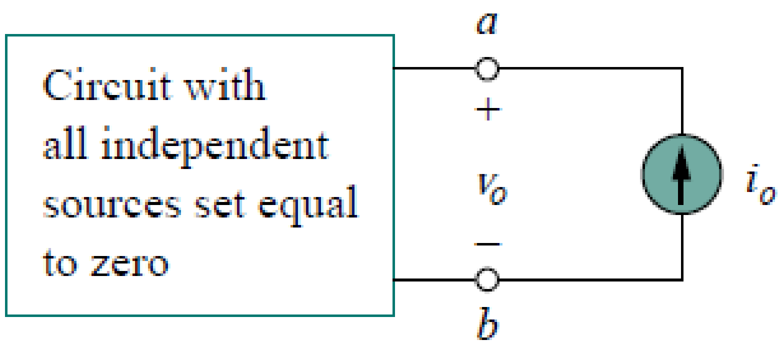




$$R_{Th} = \frac{1\text{ V}}{i_o} = 6\ \Omega$$



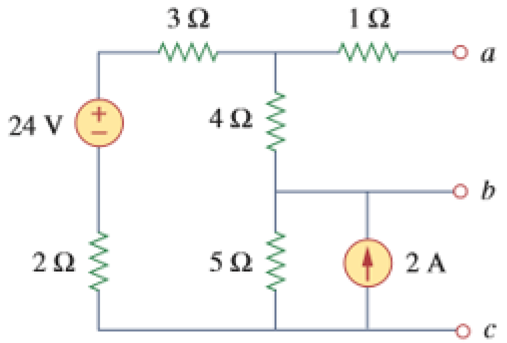
$$R_{Th} = \frac{v_o}{i_o}$$



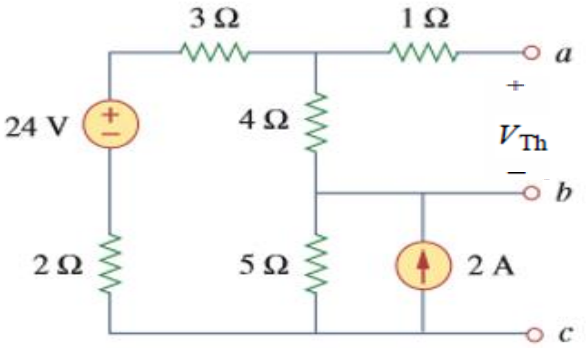
$$R_{Th} = \frac{v_o}{i_o}$$

3. Thevenin's Theorem

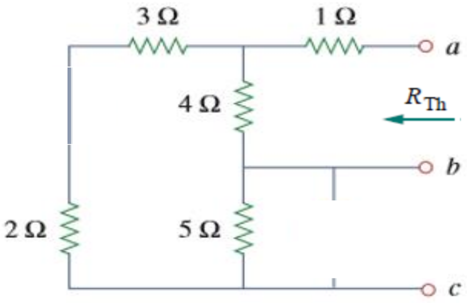
Example 4. For the following circuit, obtain Thevenin equivalent as seen from terminals： (a)a-b (b)b-c



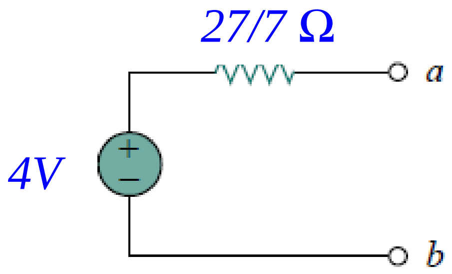
（a） from terminal a-b:



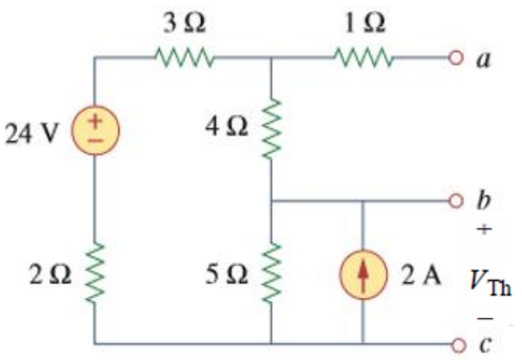
V_{TH} =4V



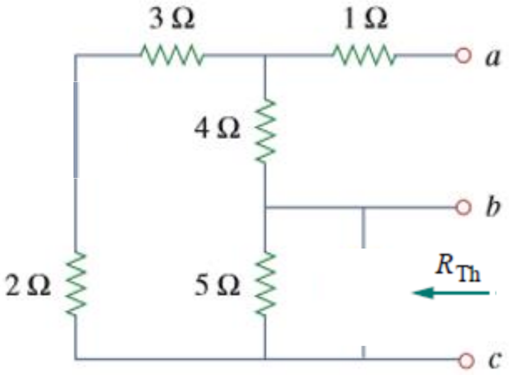
R_{TH} =27/7Ω



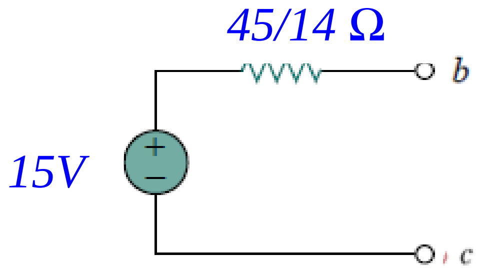
（b） from terminal b-c:



V_{TH} =15V

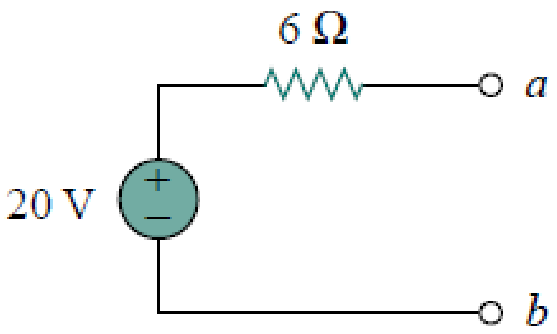


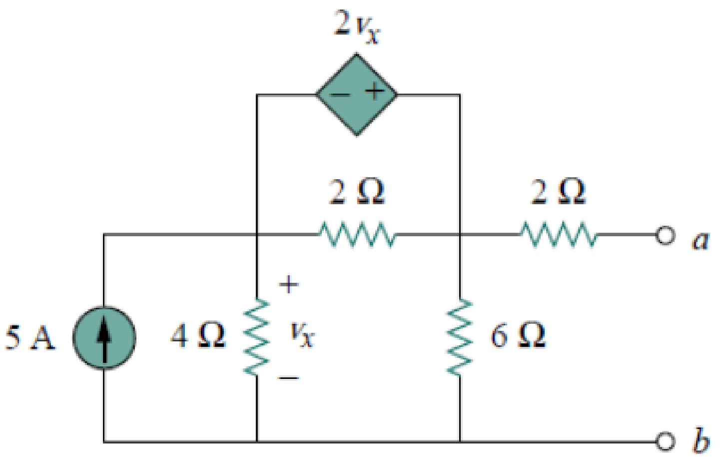
R_{TH} =45/14 Ω

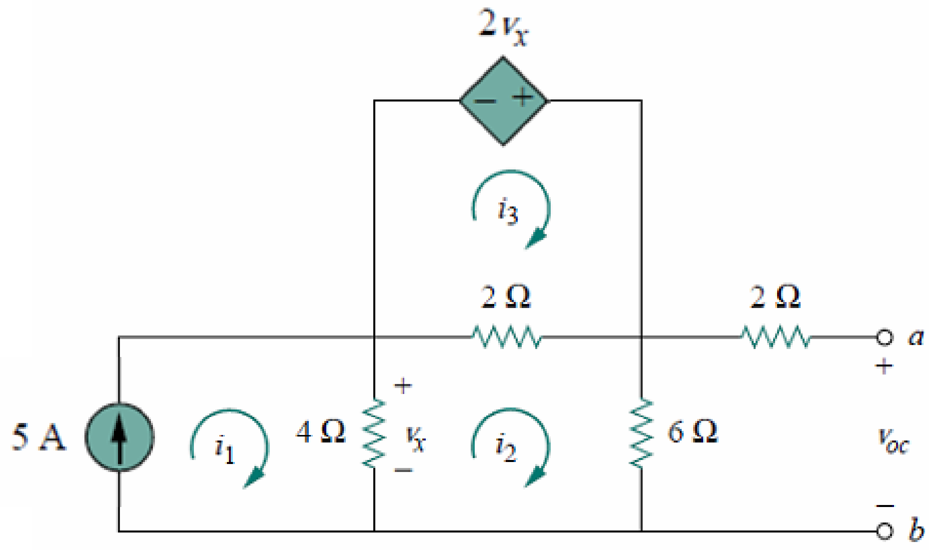


3. Thevenin's Theorem

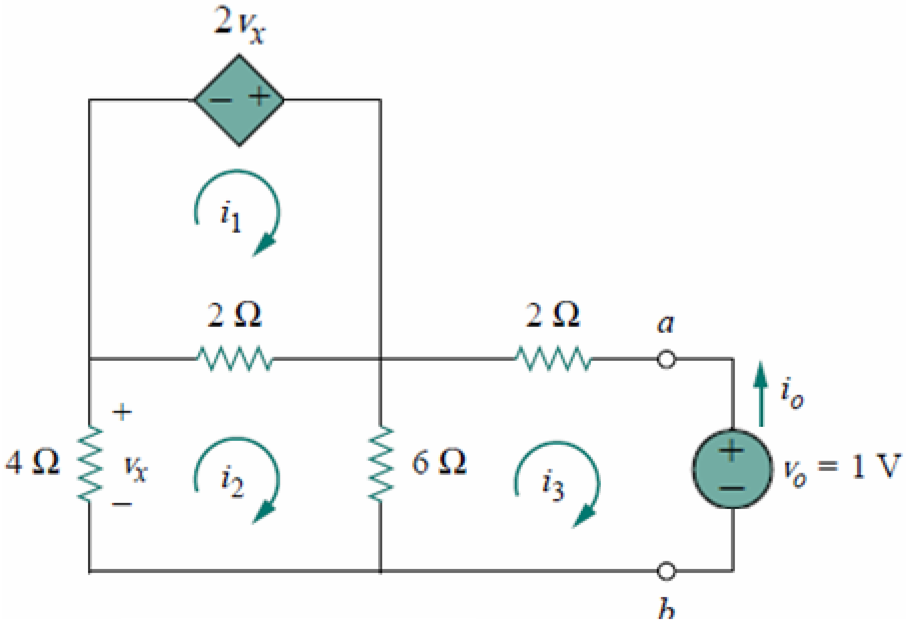
Example 5. Find the Thevenin equivalent circuit of the circuit shown below.



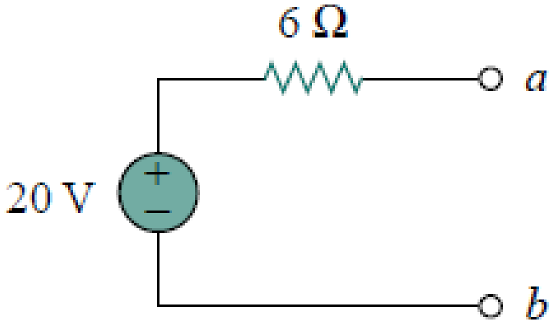




$$V_{\mathrm{Th}} = v_{oc} = 6i_2 = 20\ \mathrm{V}$$

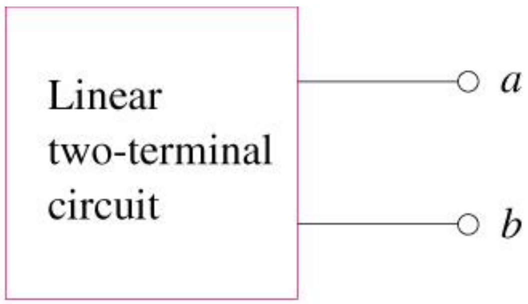


$$R_{\mathrm{Th}} = \frac{1\ \mathrm{V}}{i_o} = 6\ \Omega$$

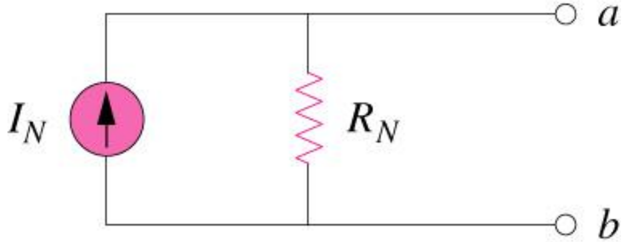


4. Norton's Theorem

(1)Conclusion:

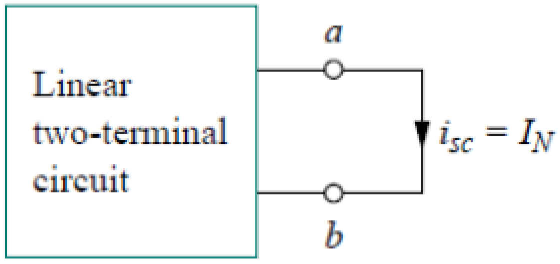


(a)



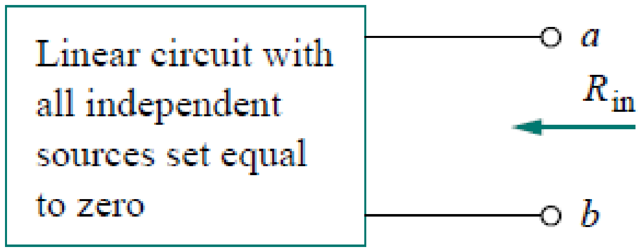
(b)

I_N is the short circuit current through the terminals.



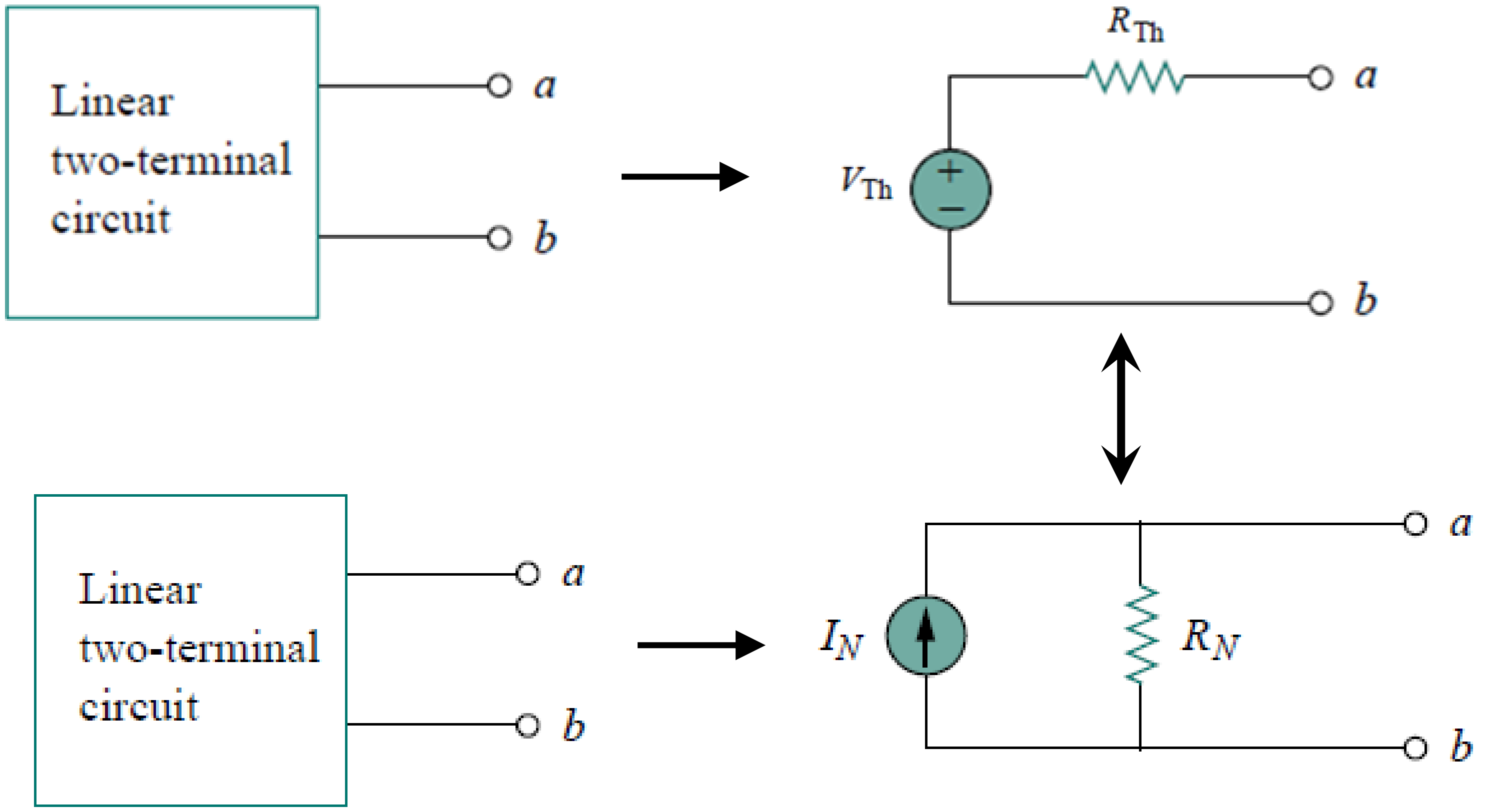
$$I_N = i_{sc}$$

R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.



4. Norton's Theorem

(2) The Thevenin's and Norton equivalent circuits are related by a source transformation.

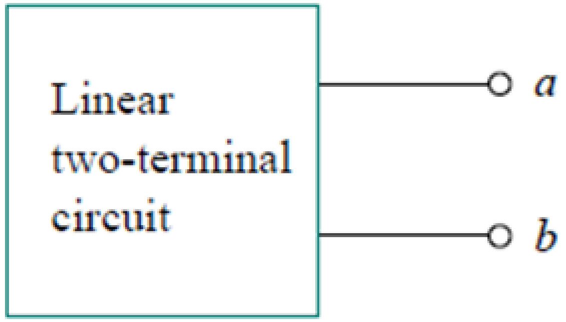


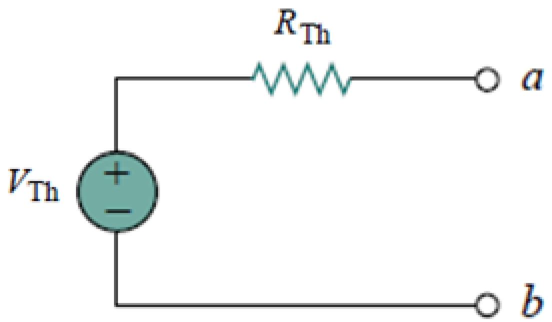
$$\boxed{R_N = R_{Th}}$$

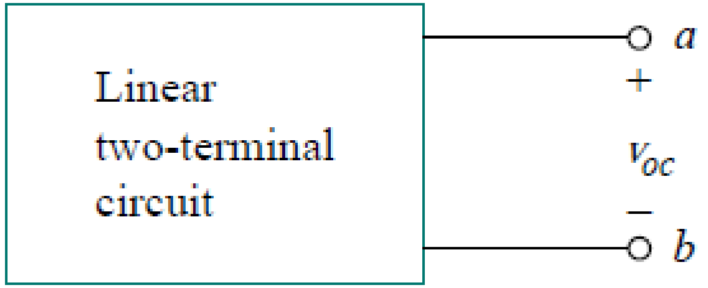
$$\boxed{I_N = \frac{V_{Th}}{R_{Th}}}$$

4. Norton's Theorem

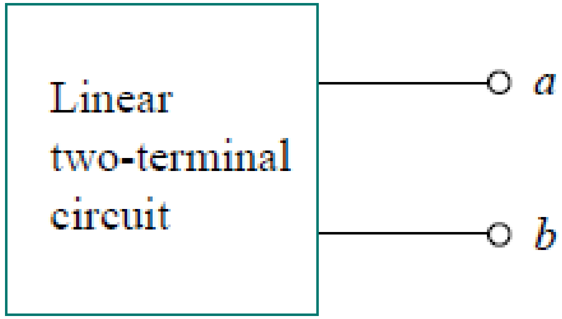
(3) Another method to calculate R_{TH} and R_N

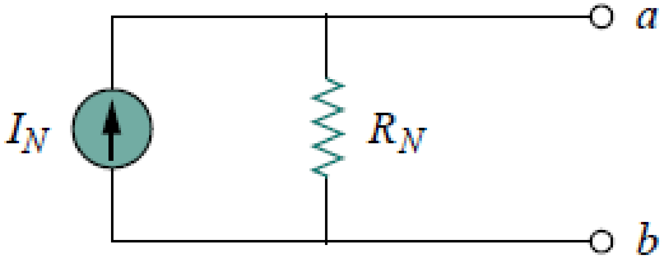


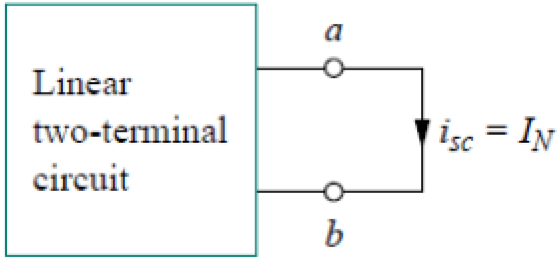




$V_{Th} = v_{oc}$







$I_N = i_{sc}$

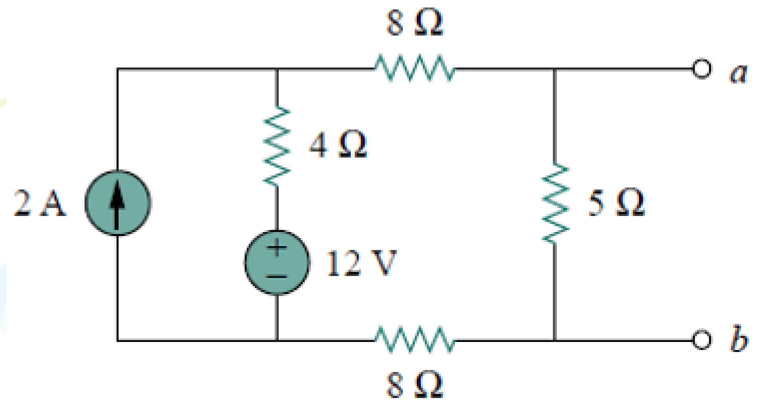
$$\boxed{R_N = R_{Th}}$$

$$\boxed{I_N = \frac{V_{Th}}{R_{Th}}}$$

$$R_{Th} = \frac{v_{oc}}{i_{sc}} = R_N$$

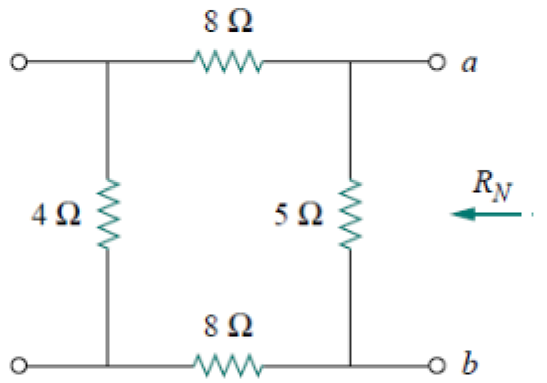
4. Norton's Theorem

Example 6. Find the Norton equivalent circuit for the circuit at terminals *a*-*b*.



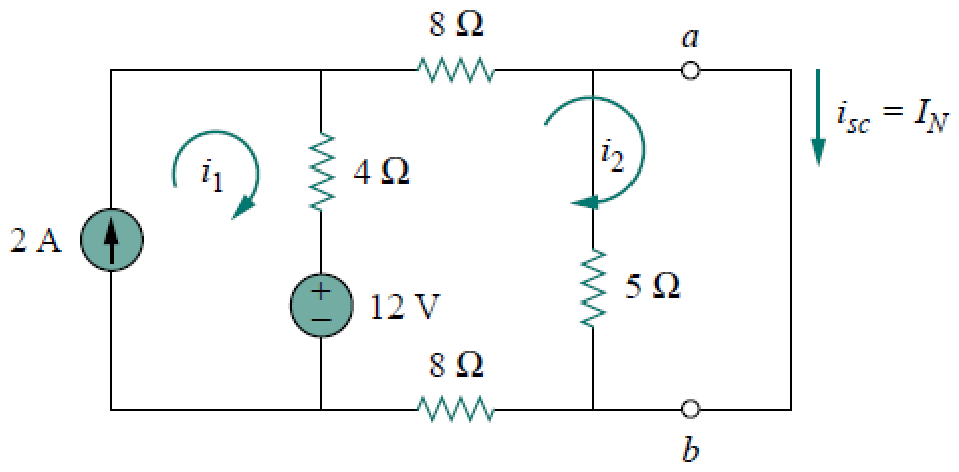
Method 1

find R_N



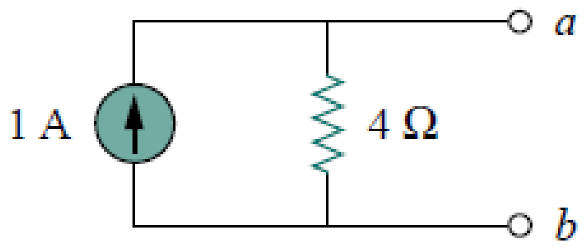
$$R_N = 5 \parallel (8+4+8) = 5 \parallel 20 = \frac{20 \times 5}{25} = 4\ \Omega$$

find I_N



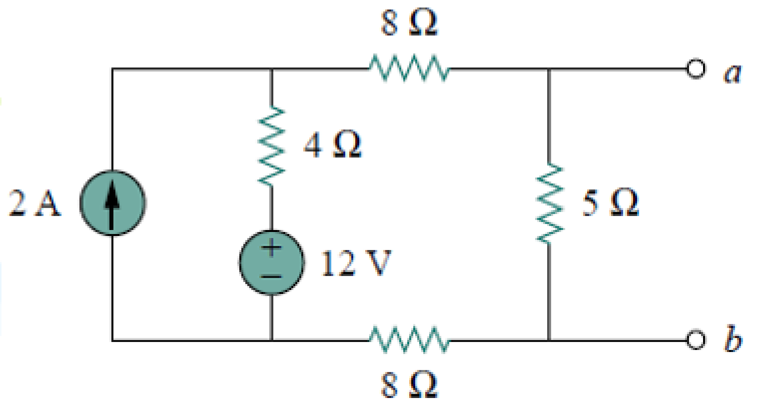
$$i_1 = 2\ \text{A}, \quad 20i_2 - 4i_1 - 12 = 0$$

From these equations, we obtain $i_2 = 1\ \text{A} = i_{sc} = I_N$



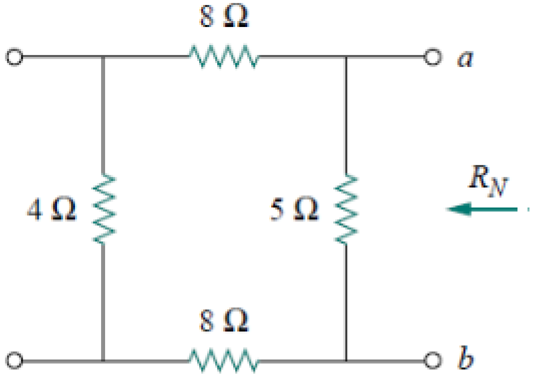
4. Norton's Theorem

Example 6. Find the Norton equivalent circuit for the circuit at terminals *a*-*b*.



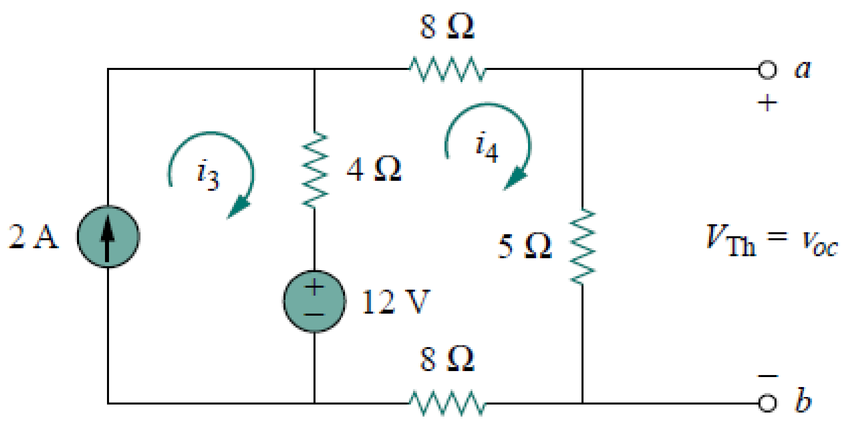
Method 2

find R_N



$$R_N = 5 \parallel (8+4+8) = 5 \parallel 20 = \frac{20 \times 5}{25} = 4\ \Omega$$

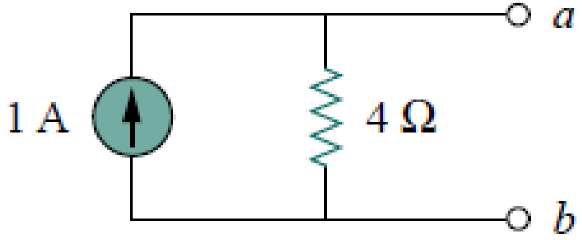
Find V_{TH}



$$v_{oc} = V_{Th} = 5i_4 = 4\ \text{V}$$

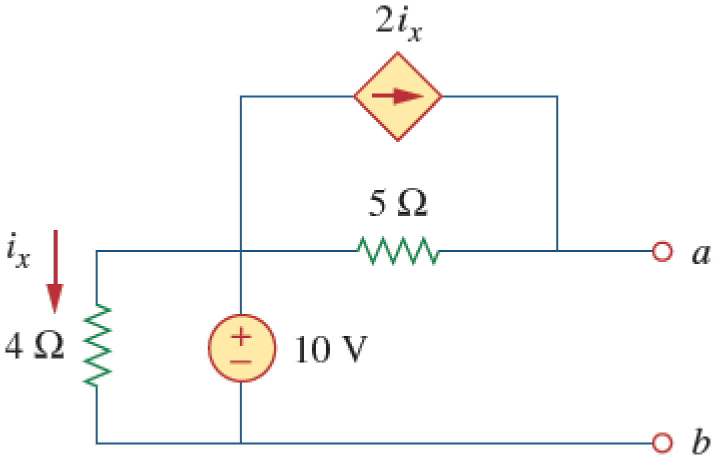
$$I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1\ \text{A}$$

$$I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1\ \text{A}$$

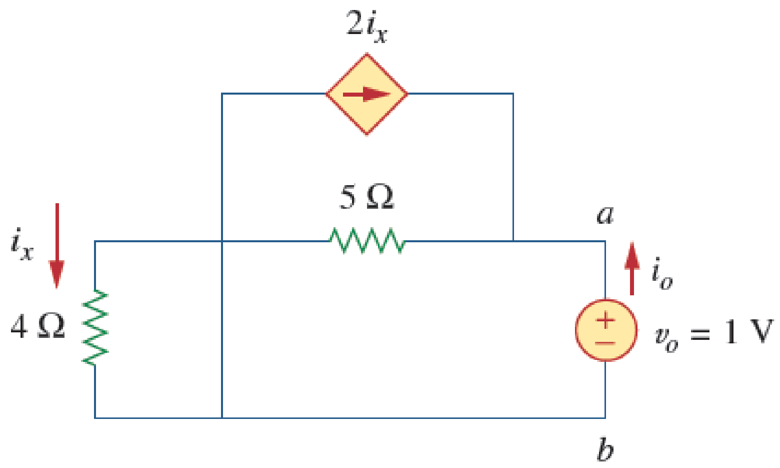


4. Norton's Theorem

Example 7. Find the Norton equivalent circuit for the circuit at terminals *a*-*b*.

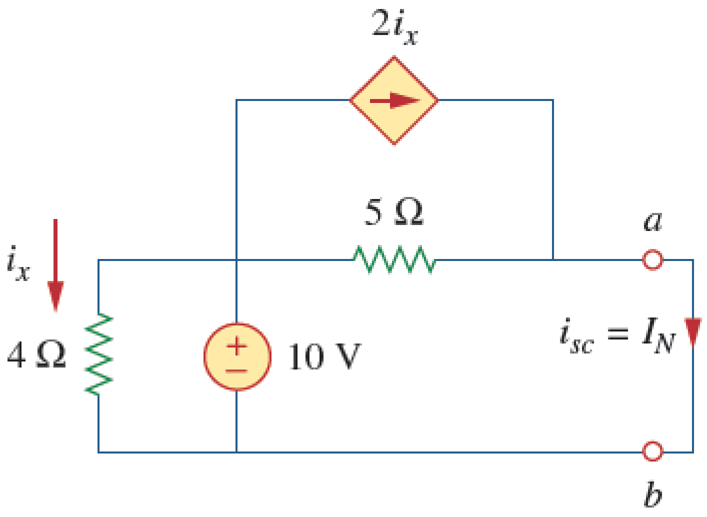


Method 1



(a)

$$R_N = \frac{v_o}{i_o} = \frac{1}{0.2} = 5\ \Omega$$

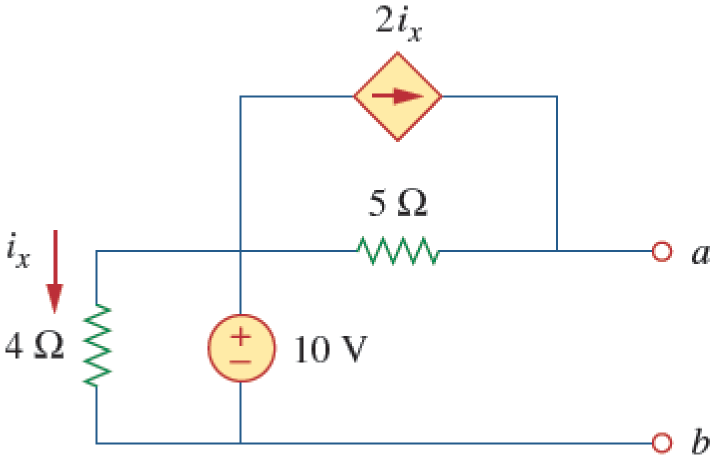


(b)

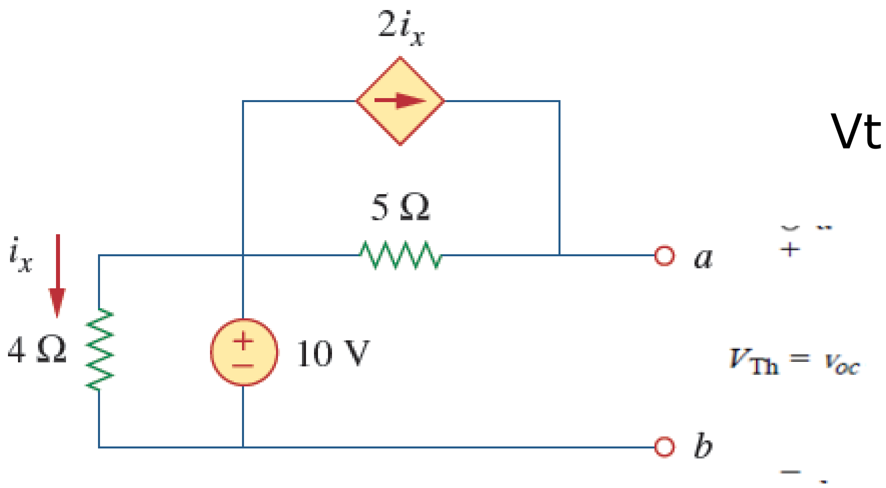
$$i_{sc} = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7\ \text{A}$$

4. Norton's Theorem

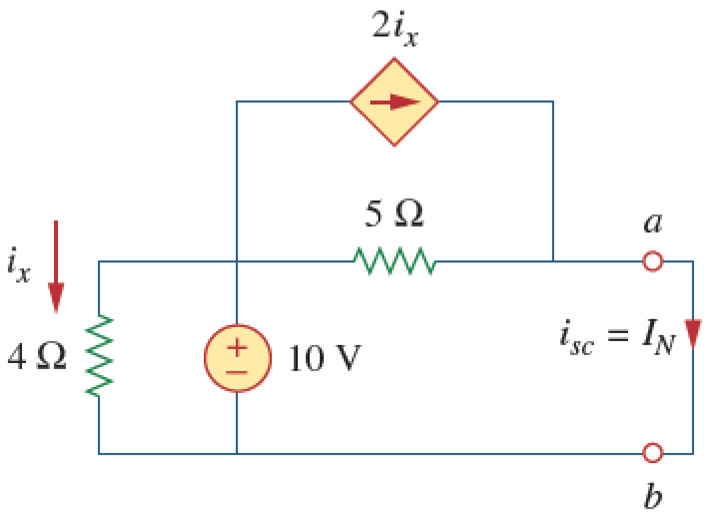
Example 7. Find the Norton equivalent circuit for the circuit at terminals *a*-*b*.



Method 2



Vth=10+25=35V



(b)

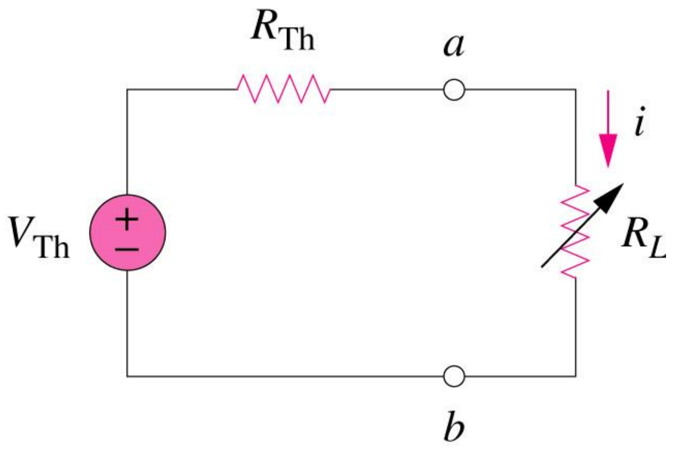
$$i_{sc} = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7\text{ A}$$

R_N = 35/7=5Ω,

Chapter 4 Circuit Theorems

5. Maximum Power Transfer

(1)Conclusion:



Maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$).

$$\boxed{R_L = R_{Th}}$$

$$\boxed{p_{max} = \frac{V_{Th}^2}{4R_{Th}}}$$

(2) Steps to use Maximum Power Transfer:

step 1: Find V_{Th}

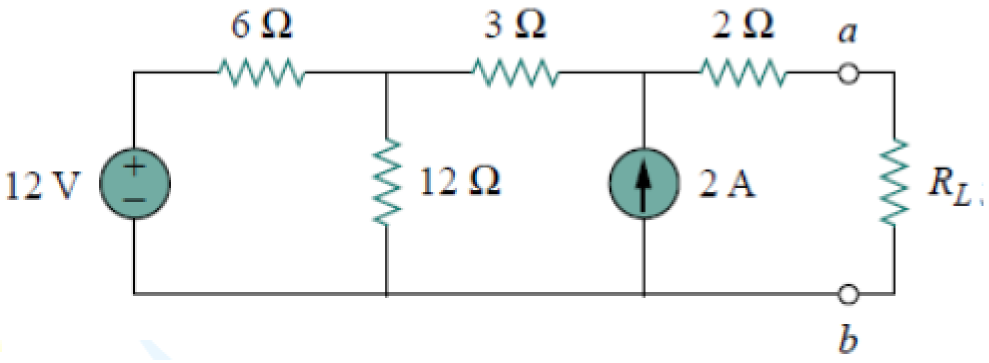
step 2: Find R_{Th}

step 3: $\boxed{R_L = R_{Th}}$

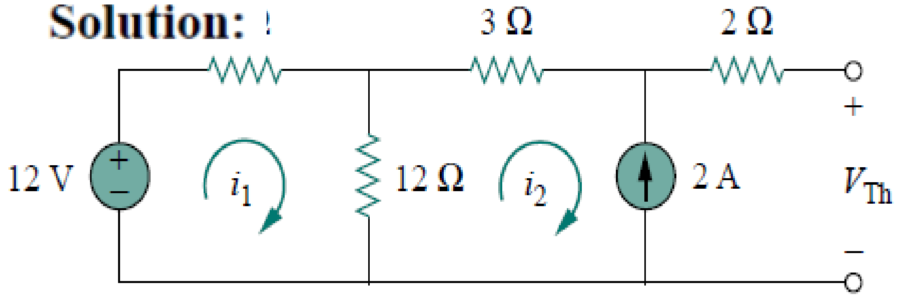
$$\boxed{p_{max} = \frac{V_{Th}^2}{4R_{Th}}}$$

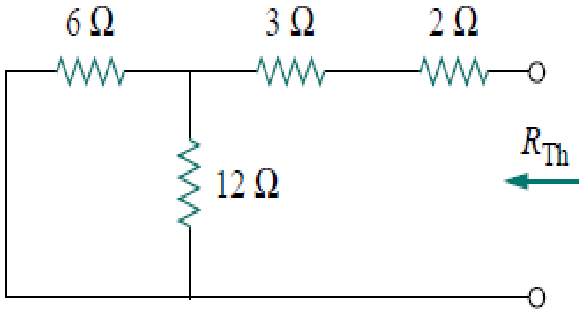
5. Maximum Power Transfer

Example 8. Find the value of R_L for maximum power transfer in the circuit



Solution:





(a)

$$-12 + 18i_1 - 12i_2 = 0,$$

$$i_2 = -2 \text{ A}$$

$$V_{Th} = 22 \text{ V}$$

$$R_{Th} = 2 + 3 + 6 \parallel 12 = 5 + \frac{6 \times 12}{18} = 9\ \Omega$$

For maximum power transfer,

$$R_L = R_{Th} = 9\ \Omega$$

$$p_{max} = \frac{V_{Th}^2}{4R_L} = \frac{22^2}{4 \times 9} = 13.44 \text{ W}$$

Chapter 5 Operational Amplifier

1. V-I relationship(two basic rules): virtual open, virtual short

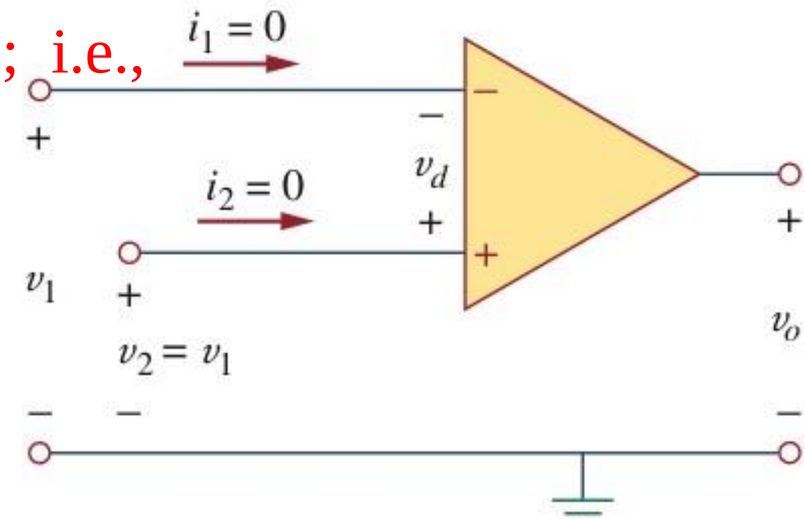
> An op amp is an active circuit element designed to perform mathematical operations of addition, subtraction, multiplication, division, differentiation, and integration.

(1) The currents into both input terminals are zero:

$$i_1 = 0, \qquad i_2 = 0$$

(virtual open)

(2). The voltage across the input terminals is negligibly small; i.e.,



$$v_d = v_2 - v_1 \approx 0$$

or

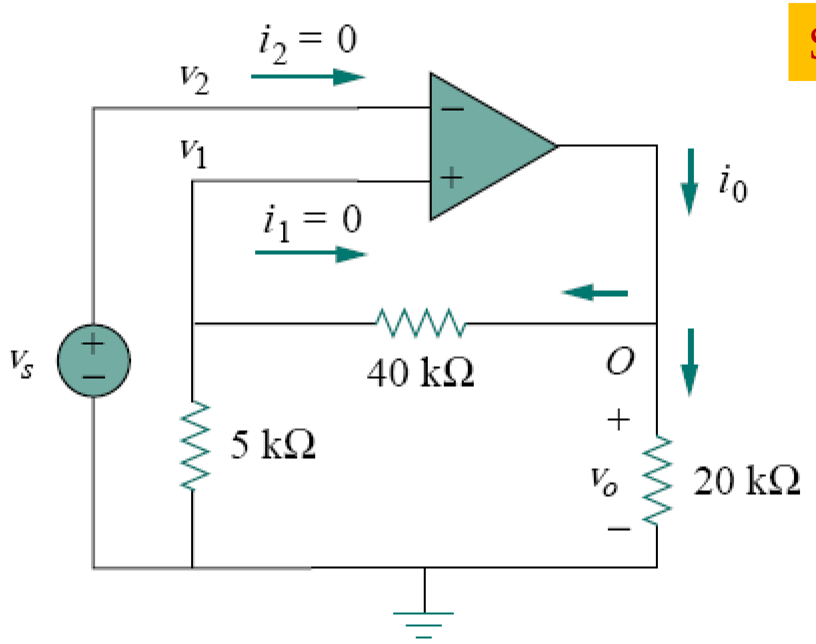
$$v_2 = v_1$$

(virtual short)

The virtual short and virtual open are extremely important and should be regarded as the key handles to analyzing op amp circuits

2. Op amp Circuit analysis: two basic rules ,KCL,KVL,Nonal analysis

Example 1. **If the ideal op amp is used in the circuit, calculate the closed-loop gain v_o/v_s , Find i_o when v_s = 1 V.**



Solution:

$$\boxed{i_1 = 0, \quad i_2 = 0}$$ (virtual open)

So $$v_1 = \frac{5}{5+40} v_o = \frac{v_o}{9}$$

$$\boxed{v_1 = v_2}$$ (virtual short)

$$v_2 = v_s = v_1 = \frac{v_o}{9} \implies \frac{v_o}{v_s} = 9$$

At node O, $$i_o = \frac{v_o}{40+5} + \frac{v_o}{20} \text{ mA}$$

$$v_o = 9 \text{ V} \qquad i_o = 0.2 + 0.45 = 0.65 \text{ mA}$$

Summary of DC analysis

1. How many kinds of analysis methods ?

(1)use two basis directly : V-I relationship for elements; KCL,KVL(chapter 2)

(2)use basic methods directly: Nodal analysis, Mesh Analysis (chapter 3)

(3)first use circuit Theorems to simplify the circuit ,then use two basis or basic methods (chapter 4).

2. For Nodal and Mesh analysis method, you should pay attention to special cases.

(1)Nodal analysis method:

CASE 1

If a voltage source is connected between the reference node and a nonreference node

Method:

Set the voltage at the nonreference node equal to the voltage of the voltage source.

CASE 2

If a voltage source is connected between two nonreference node

Method:

The nonreference nodes forms a supernode

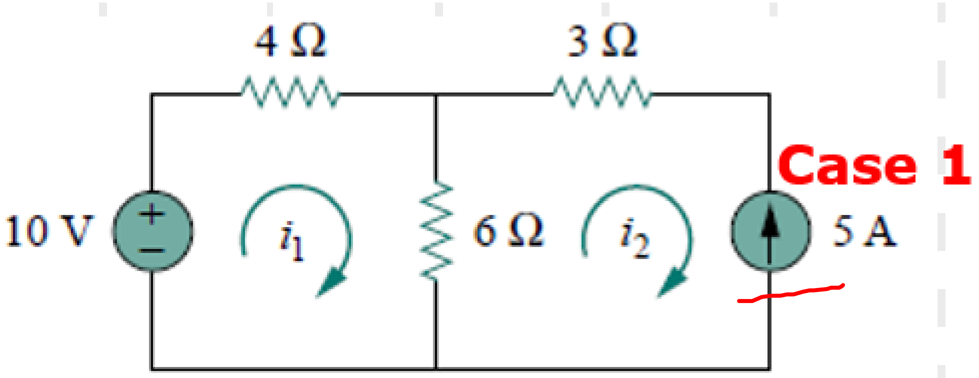
(1) Apply KCL to super-nodes;

(2) Write the equation that defines the voltage relationship between the two nonreference nodes as a result of the presences of the voltage source.

Summary of DC analysis

(2)Mesh analysis method:

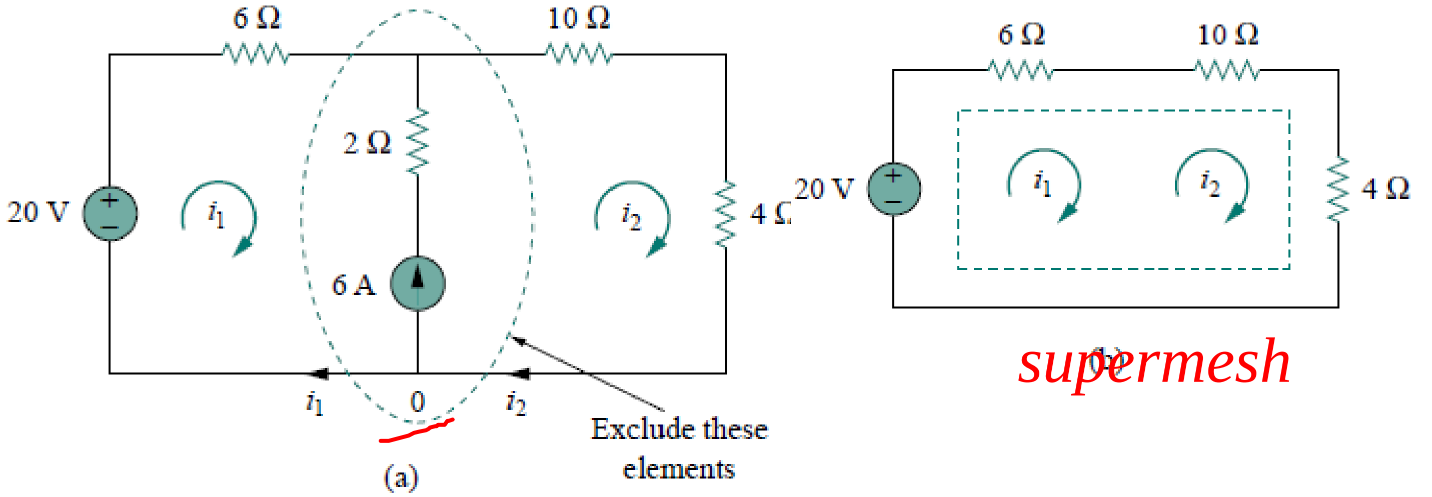
Case 1. When a current source exists only in one mesh.



For the mesh containing the current source, its mesh current is known.

Set $i_2 = -5\ A$

Case 2. A current source exists between two meshes .



We create a *supermesh by excluding* the current source and any elements connected in series with it

Applying KVL to the supermesh $-20 + 6i_1 + 10i_2 + 4i_2 = 0$

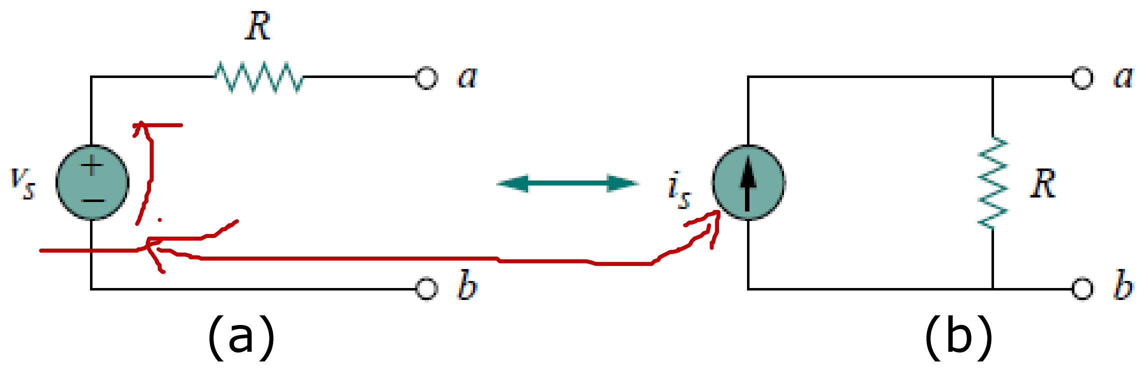
Apply KCL to a node in the branch where the two meshes intersect.

$$i_2 = i_1 + 6$$

Summary of DC analysis

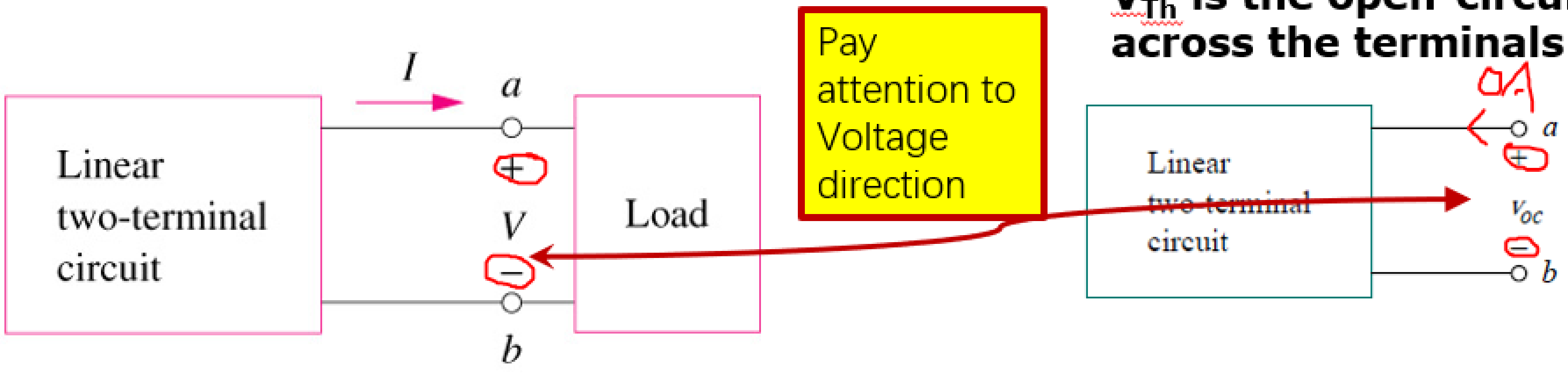
3. Pay attention: Superposition Theorems only work for independent sources.

4. Pay attention: Reference direction of Source transformation.



$$i_s = \frac{v_s}{R}$$

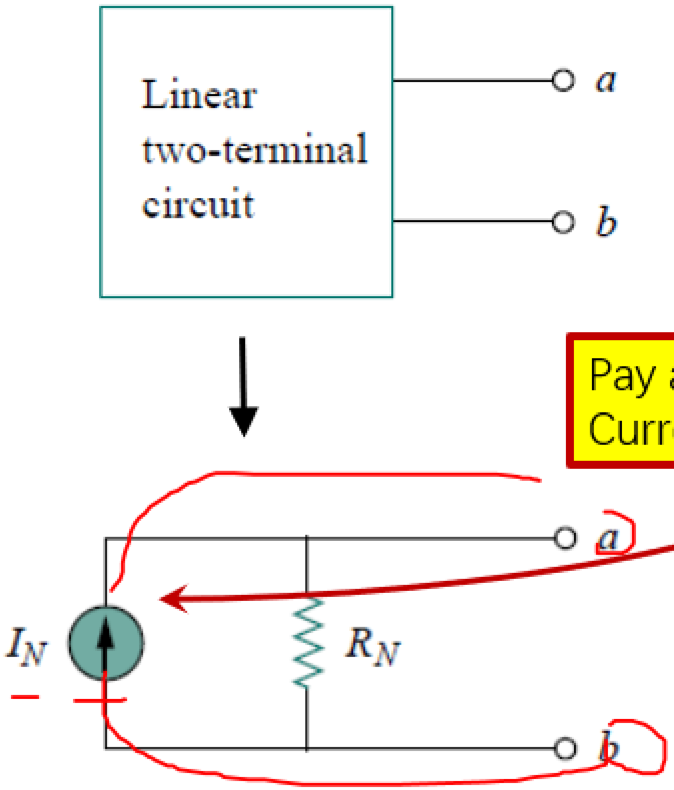
5. Pay attention: Reference direction of Thevenin's Theorem.



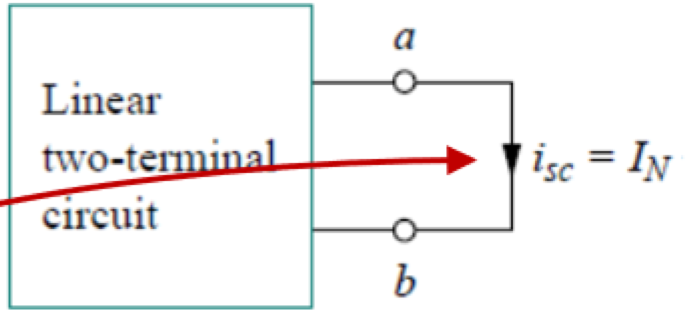
$V_{Th} = v_{oc}$

Summary of DC analysis

6. Pay attention: Reference direction of Norton's Theorem.



Pay attention to
Current direction



$$I_N = i_{sc}$$

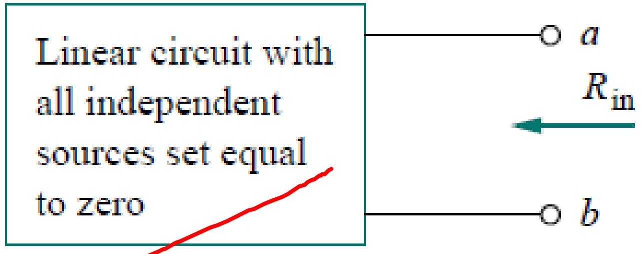
I_N is the short circuit current through the terminals

Summary of DC analysis

7. Pay attention: How to compute Thevenin resistance and Norton resistance?

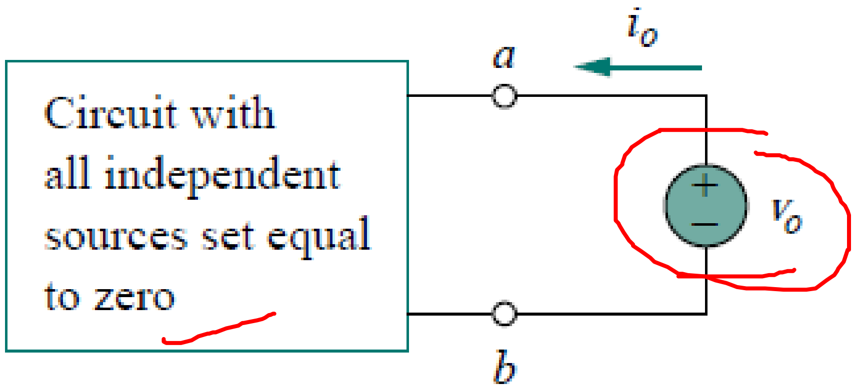
Method 1: turn off all the independent sources of the subcircuit

CASE 1 If the network has no dependent sources.

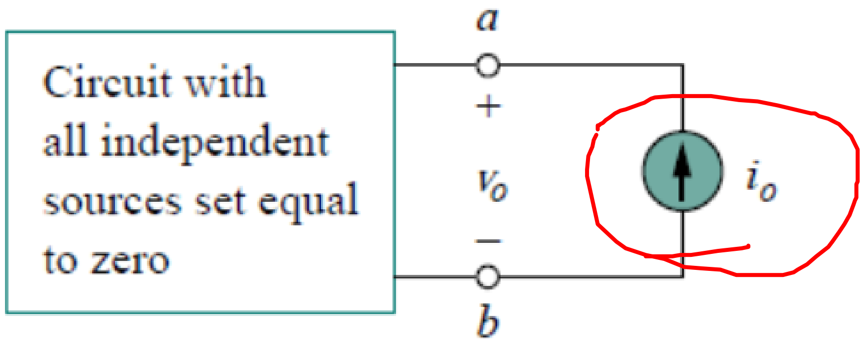


$R_N = R_{Th} = R_{in}$

CASE 2 If the network has dependent sources.



$R_{Th} = \frac{v_o}{i_o}$

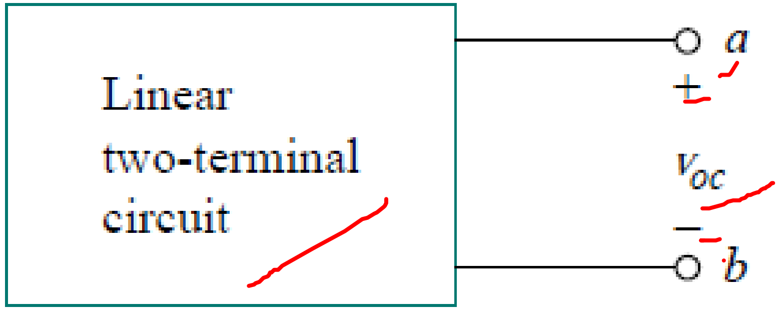


$R_{Th} = \frac{v_o}{i_o}$

Summary of DC analysis

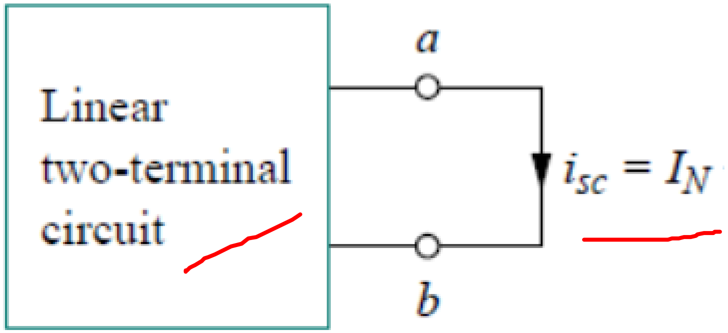
7. Pay attention: How to compute Thevenin resistance and Norton resistance?

Method 2: Do not turn off all the independent sources of the subcircuit



$$V_{\mathrm{Th}} = v_{oc}$$

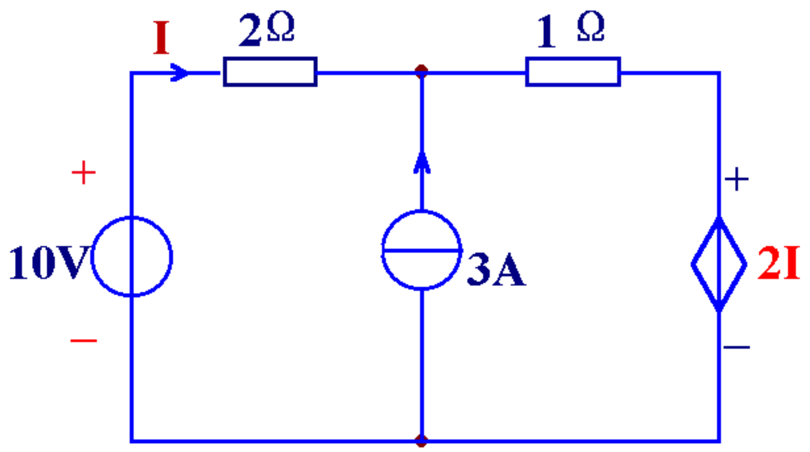
$$\boxed{R_{\mathrm{Th}} = \frac{v_{oc}}{i_{sc}} = R_N}$$



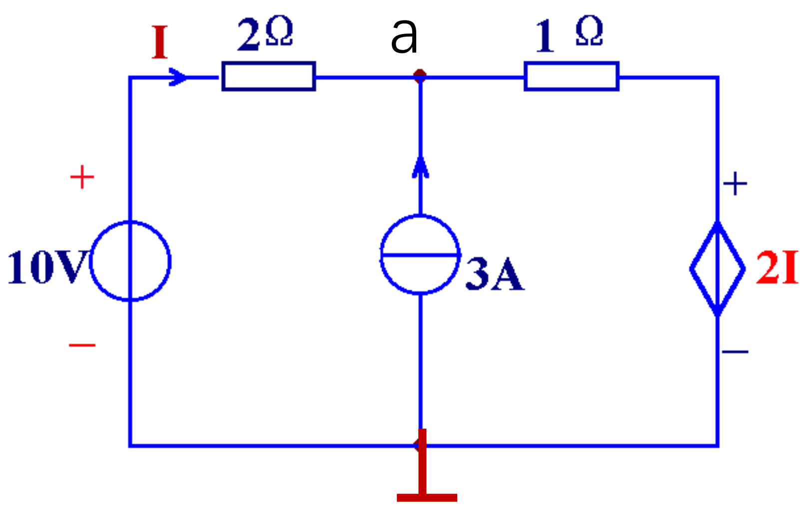
$$I_N = i_{sc}$$

Open-circuit short-circuit method

Example. Find I.

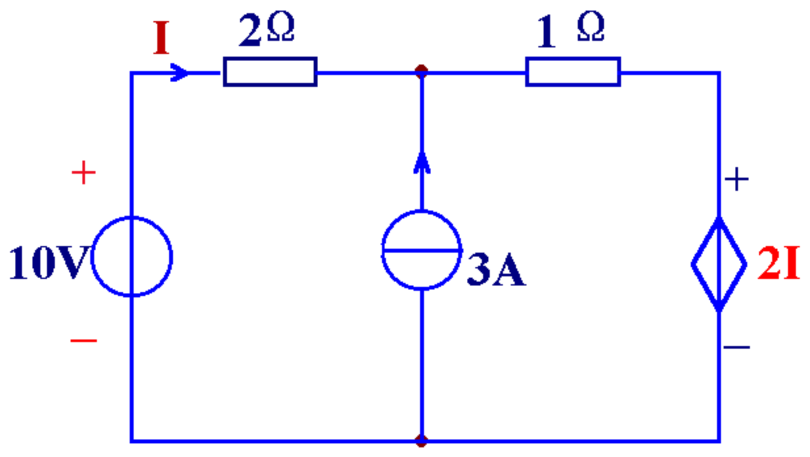


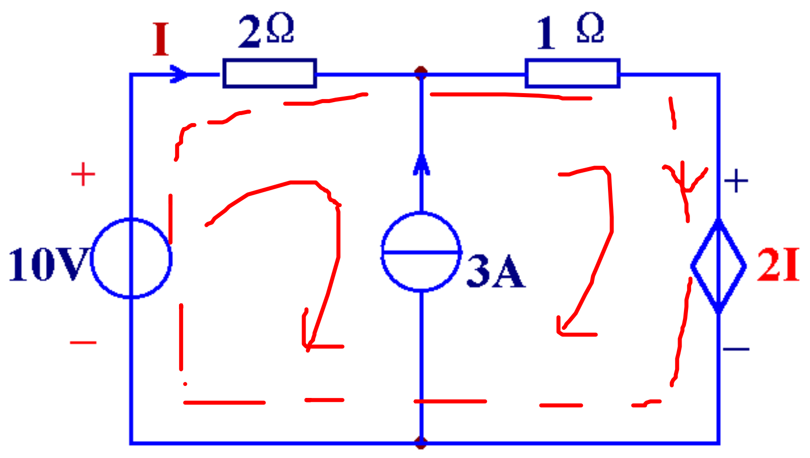
Method 1:Nodal analysis



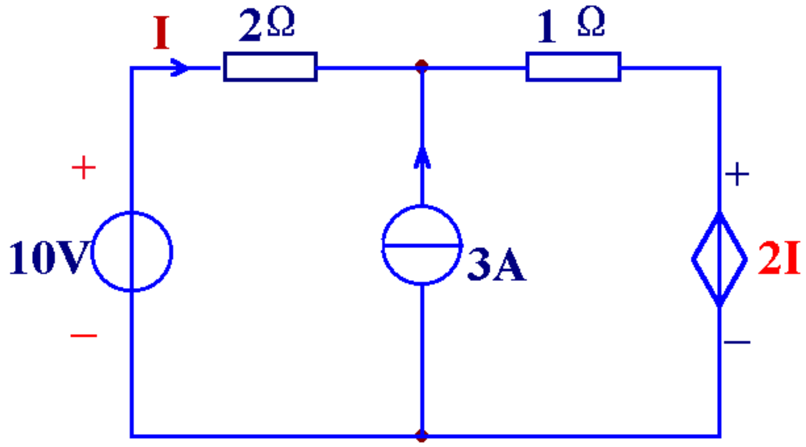
Example. Find I.

Method 2:Mesh analysis



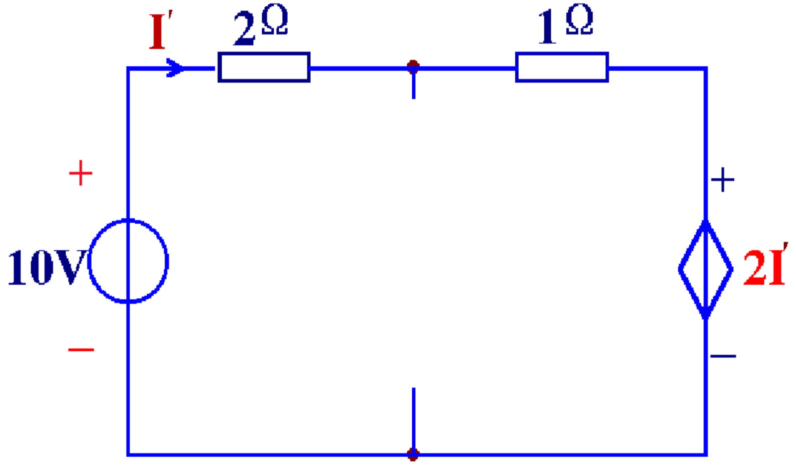


Example. Find I.



Method 3:Superposition theorem

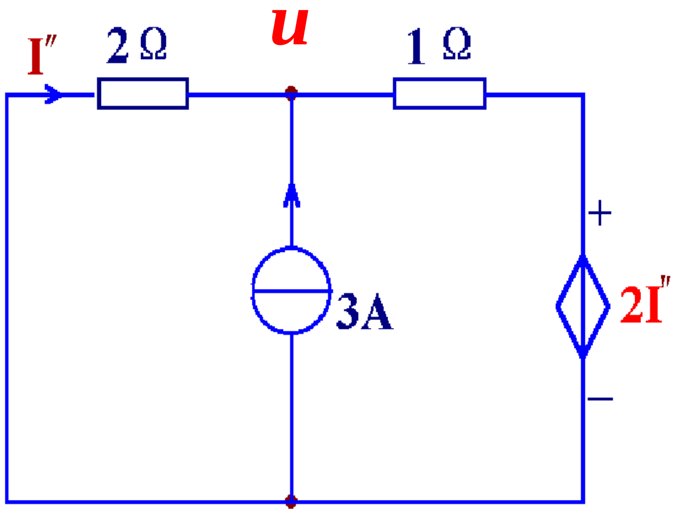
Voltage source works alone:



$$I' = \frac{10-2I'}{2+1}$$

$$I' = 2A$$

Current source works alone:



$$-\frac{u}{2} + 3 = \frac{u - 2I''}{1}$$

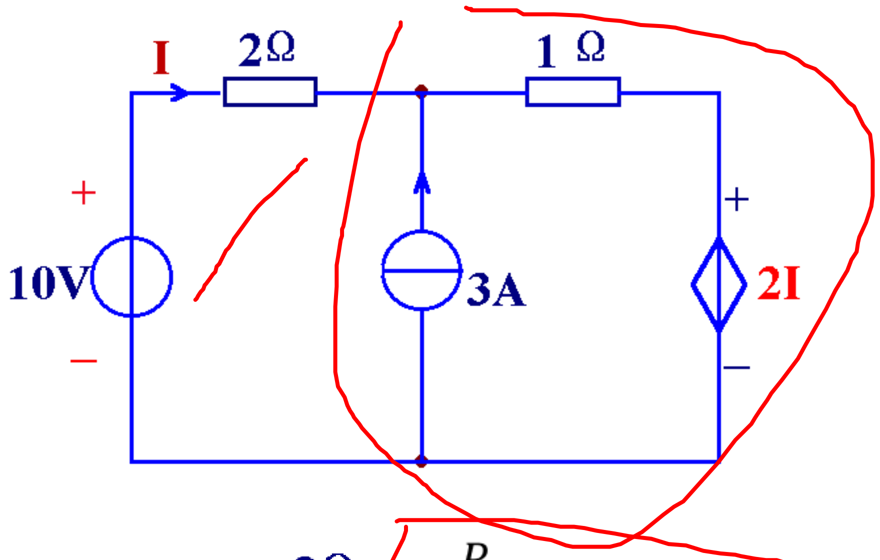
$$I'' = -\frac{u}{2}$$

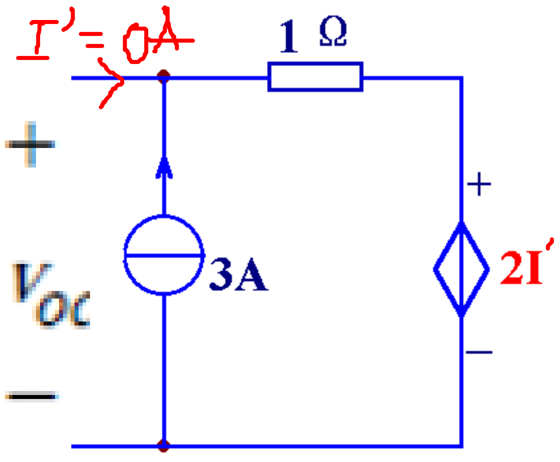
$$I'' = -\frac{3}{5}A$$

$$I = I' + I'' = \frac{7}{5}A$$

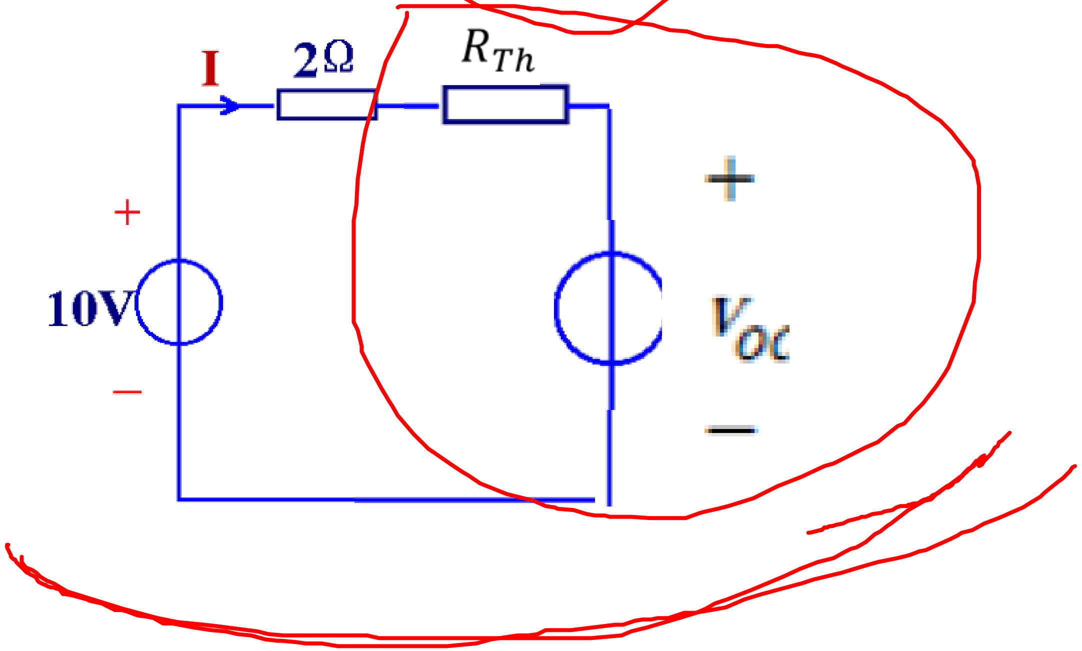
Example. Find I.

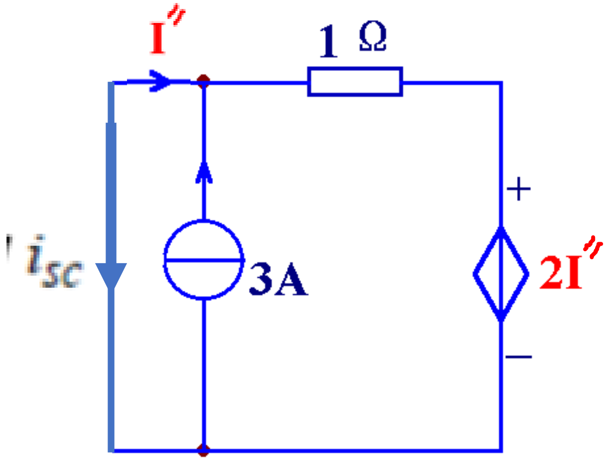
Method 4:Thevenin's theorem





$V_{OC} = 3V$





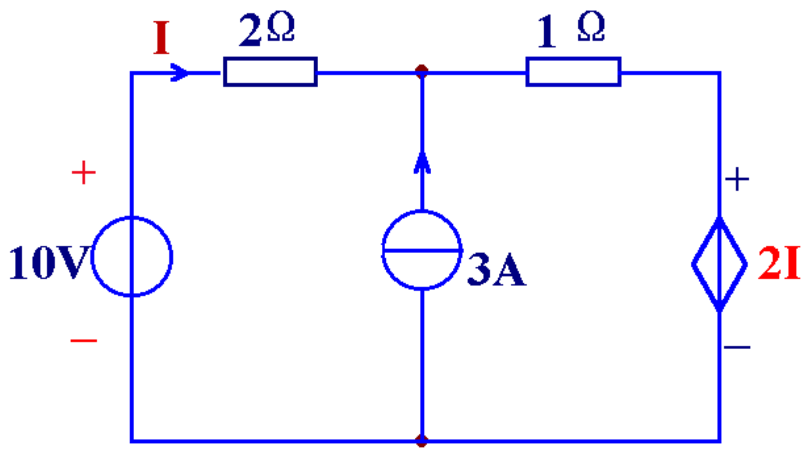
$1 \times (3 + I'') + 2I'' = 0$

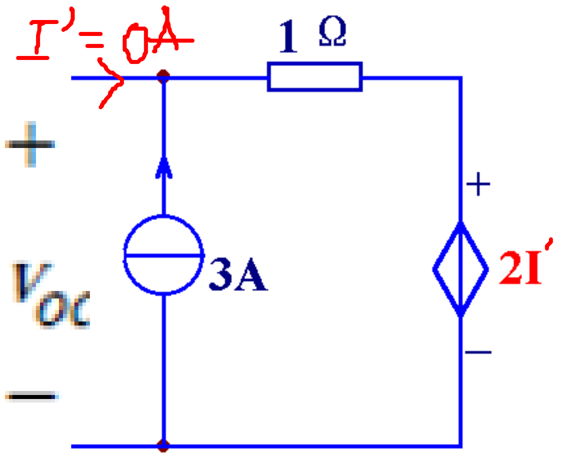
$i_{SC} = -I'' = 1A$

$$R_{Th} = \frac{V_{OC}}{i_{SC}} = 3\Omega$$

Example. Find I.

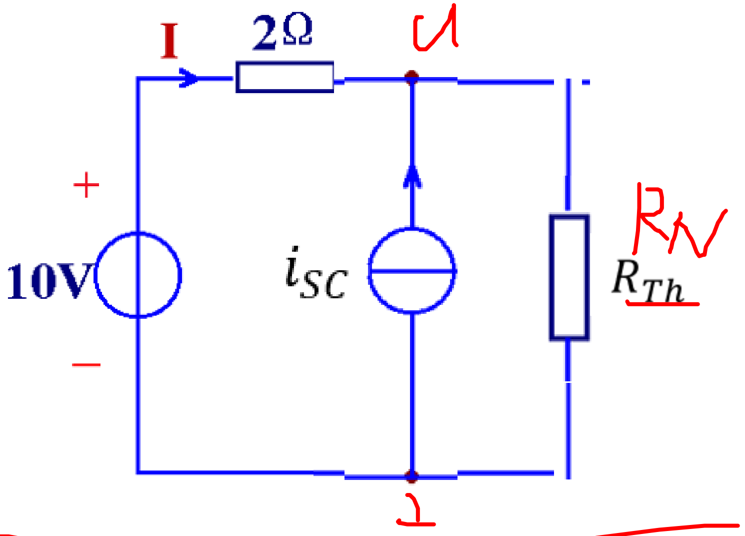
Method 5:Norton's theorem

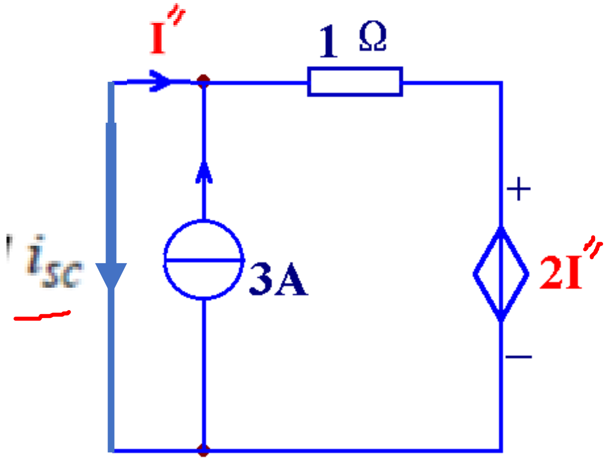




$V_{OC} = 3V$

R_{Th}





$$1 \times (3 + I'') + 2I'' = 0$$

$$i_{SC} = -I'' = 1A$$

$$R_{Th} = \frac{V_{OC}}{i_{SC}} = 3\Omega$$

Main methods for AC circuit analysis

2021.5

Overview of AC Analysis

1. Why analyze circuit in phasor domain?

Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources; solutions of such circuits would be intractable otherwise.

2. How to do AC circuit analysis?

Steps to Analyze AC Circuits:

1. Transform the circuit to the phasor or frequency domain.
2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
3. Transform the resulting phasor to the time domain.

DC ANALYSIS

1. Sinusoids and Phasors(Chapter 9)
2. Sinusoidal Steady-State Analysis(Chapter 10)
3. AC Power Analysis(Chapter 11)

AC Circuit Analysis

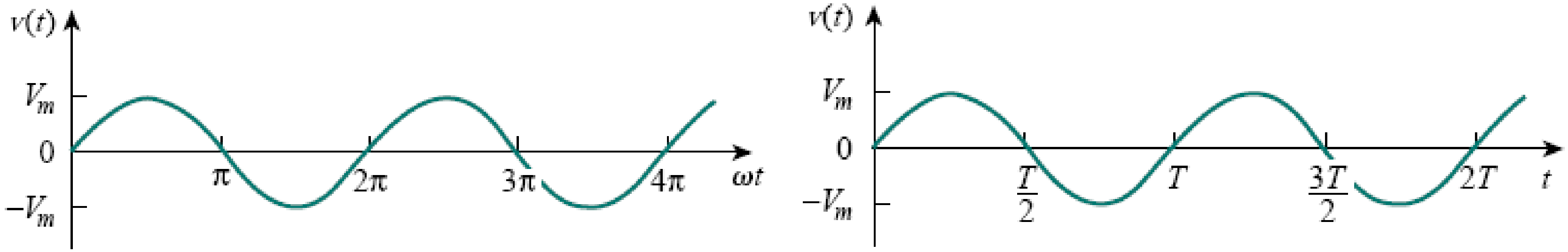
- **Sinusoids and Phasors (Chapter 9)**
  - Sinusoids and Phasors: Concept, Transformation
  - V-I relationship in phasor domain: R,L,C
  - Kirchhoff's Laws in phasor domain:KCL,KVL
- **Sinusoidal Steady-State Analysis (Chapter 10)**
  - Nodal Analysis
  - Mesh Analysis
  - Superposition, Source Transformation
  - Equivalent circuit:Thevenin's Theorem, Norton's Theorem
  - Op amp Circuit analysis: two basic rules ,Kcl,Kvl,Nonal analysis
- **AC Power Analysis (Chapter 11)**
  - AC Power: Instantaneous power,Average Power, Apparent Power, Complex Power
  - Effective or RMS Value, Maximum Average Power Transfer, Power Factor Correction

Chapter 9 Sinusoids and Phasors

1. Sinusoids:

A sinusoid is a signal that has the form of the sine or cosine function.

2. AC circuits: **Circuits driven by sinusoidal current or voltage sources.**



(a) as a function of ωt

(b) as a function of t

$$v(t) = V_m \sin(\omega t + \varphi)$$

Three factors: V_m = the *amplitude* of the sinusoid

ω = the *angular frequency* in radians/s

φ is the *phase*.

3. Phasors:

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

4. Sinusoid-phasor transformation:

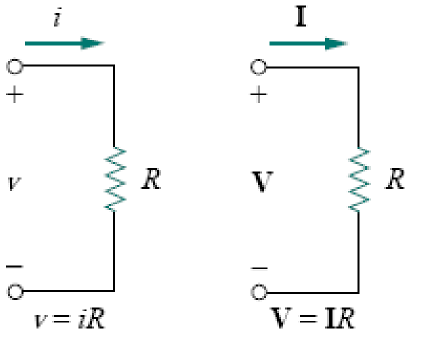
$$v(t) = V_m \cos(\omega t + \phi) \iff \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)

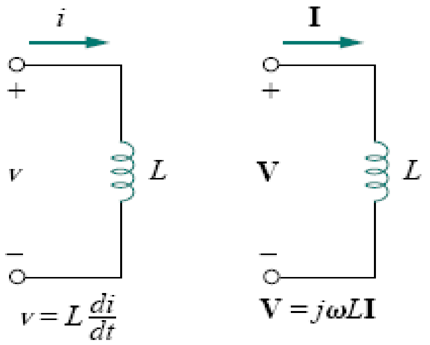
TABLE 9.1 Sinusoid-phasor transformation.

Time-domain representation	Phasor-domain representation
$V_m \cos(\omega t + \phi)$	$V_m \underline{/\phi}$
$V_m \sin(\omega t + \phi)$	$V_m \underline{/\phi - 90^\circ}$
$I_m \cos(\omega t + \theta)$	$I_m \underline{/\theta}$
$I_m \sin(\omega t + \theta)$	$I_m \underline{/\theta - 90^\circ}$

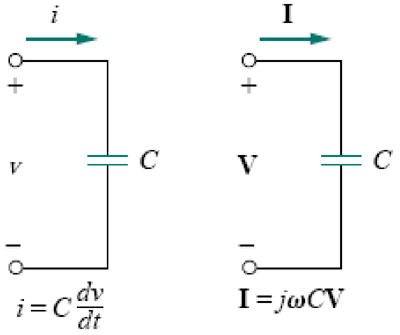
5. V-I relationship for circuit element in phasor domain:



(a) (b)



(a) (b)



(a) (b)

TABLE 9.2 Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

$$\mathbf{V} = \mathbf{Z}\mathbf{I}$$

Chapter 9 Sinusoids and Phasors

6. Kirchhoff's Laws in phasor domain :

$$\sum \mathbf{I}_k = 0 \quad \text{(KCL)}$$
$$\sum \mathbf{V}_k = 0 \quad \text{(KVL)}$$

$I = 3 + j4 = 5\angle 53.13^o$

$I = 3 - j4 = 5\angle -53.13^o$

$I = -3 + j4 = 5\angle 126.87^o$

$I = -3 - j4 = 5\angle -126.87^o$

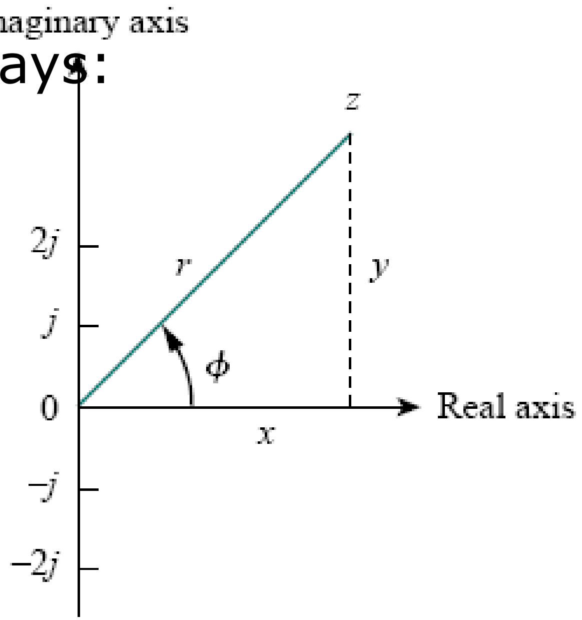
7. A complex number:

Z can be represented in three ways:

$z = x + jy$ Rectangular form

$z = r\angle\phi$ Polar form

$z = re^{j\phi}$ Exponential form



$I = 3 + j4 = 5\angle 53.13^o$

$I = 3 - j4 = 5\angle -53.13^o$

$I = -3 + j4 = 5\angle 126.87^o$

$I = -3 - j4 = 5\angle -126.87^o$

Chapter 9 Sinusoids and Phasors

8. Computation of complex number:

$$z = x + jy \qquad r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$z = r\underline{/\phi} \qquad x = r\cos\phi, \qquad y = r\sin\phi$$

$$z = re^{j\phi}$$

$$\boxed{z = x + jy = r\underline{/\phi} = r(\cos\phi + j\sin\phi)}$$

$$z = x + jy = r\underline{/\phi}, \qquad z_1 = x_1 + jy_1 = r_1\underline{/\phi_1}$$

$$z_2 = x_2 + jy_2 = r_2\underline{/\phi_2}$$

$$\boxed{e^{\pm j\phi} = \cos\phi \pm j\sin\phi}$$

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Subtraction: $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Multiplication: $z_1 z_2 = r_1 r_2 \underline{/\phi_1 + \phi_2}$

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\underline{/\phi_1 - \phi_2}$

$$4\underline{/30^\circ} + 5\underline{/-110^\circ} = 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698 = 3.218\underline{/-56.97^\circ}$$

$$\frac{12\underline{/45^\circ}}{j60 \times 0.1} = \frac{12\underline{/45^\circ}}{6\underline{/90^\circ}} = 2\underline{/-45^\circ}\text{ A}$$

$$\frac{10\underline{/0^\circ}}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} = 1.6 + j0.8 = 1.789\underline{/26.57^\circ}$$

Chapter 10 Sinusoidal Steady-State Analysis

10.1 Introduction

10.2 Nodal analysis

10.3 Mesh analysis

10.4 Superposition Theorem

10.5 Source Transformation

10.6 Thevenin and Norton equivalent circuits

10.7 Op Amp circuits

Chapter 10 Sinusoidal Steady-State Analysis

1. 3 steps to analyze ac circuits:

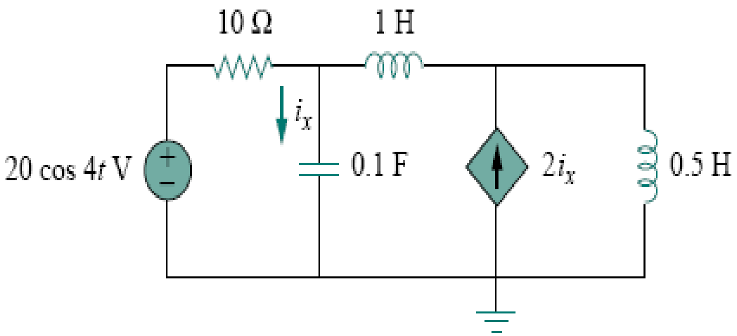
(1) Transform the circuit to the phasor domain.

(2) Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.) in phasor domain.(Regard the circuit as DC ciecuit)

(3) Transform the resulting phasor to the time domain.

2. 3 steps to transform a time domain circuit to phasor domain:

(1)construction, (2)parameter, (3)variables

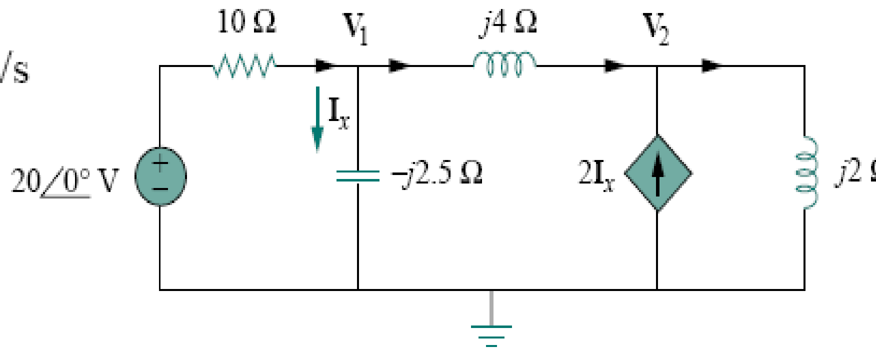


$$20\cos 4t \quad \Longrightarrow \quad 20\angle 0^\circ, \qquad \omega = 4 \text{ rad/s}$$

$$1\text{ H} \quad \Longrightarrow \quad j\omega L = j4$$

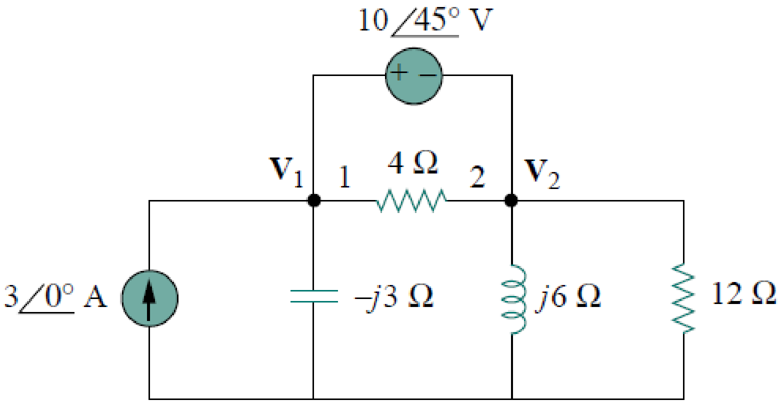
$$0.5\text{ H} \quad \Longrightarrow \quad j\omega L = j2$$

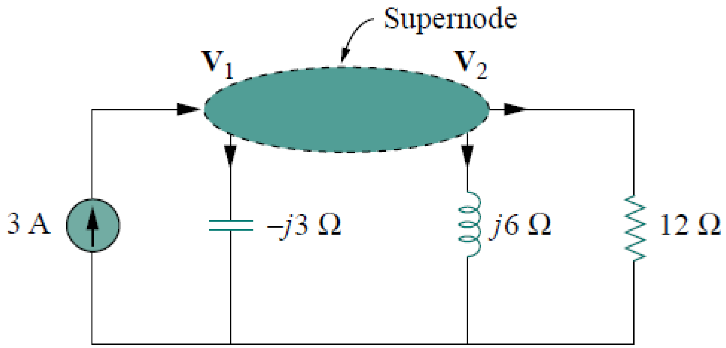
$$0.1\text{ F} \quad \Longrightarrow \quad \frac{1}{j\omega C} = -j2.5$$



Chapter 10 Sinusoidal Steady-State Analysis

EXAMPLE 10.2 Compute $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit of Fig. 10.4. EXAMPLE 10.2





Solution:

Nodes 1 and 2 form a supernode as shown in Fig. 10.5. Applying KCL at the supernode gives

$$3 = \frac{\mathbf{V}_1}{-j3} + \frac{\mathbf{V}_2}{j6} + \frac{\mathbf{V}_2}{12} \qquad\qquad 36 = j4\mathbf{V}_1 + (1 - j2)\mathbf{V}_2$$

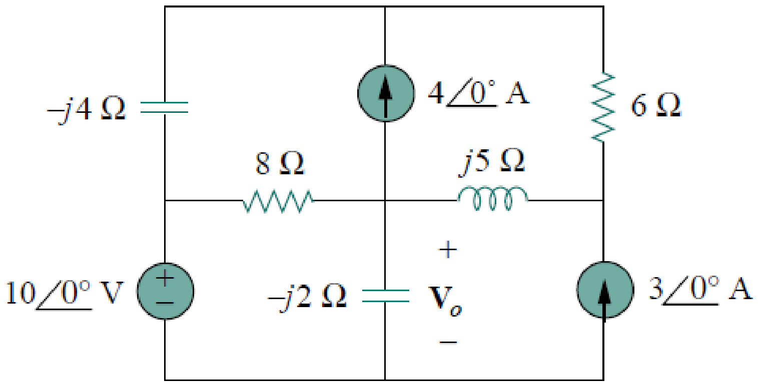
But a voltage source is connected between nodes 1 and 2, so that $\mathbf{V}_1 = \mathbf{V}_2 + 10\underline{/45^\circ}$

$$36 - 40\underline{/135^\circ} = (1 + j2)\mathbf{V}_2 \quad \Longrightarrow \quad \mathbf{V}_2 = 31.41\underline{/-87.18^\circ}\text{ V}$$

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\underline{/45^\circ} = 25.78\underline{/-70.48^\circ}\text{ V}$$

Chapter 10 Sinusoidal Steady-State Analysis

EXAMPLE 10.4



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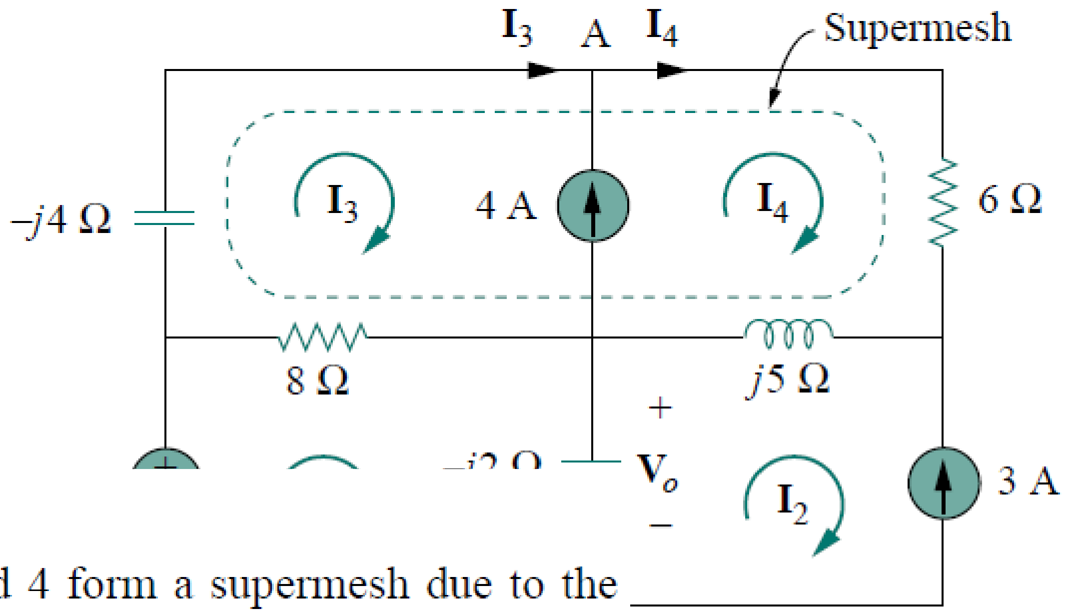


Solution:

As shown in Fig. 10.10, meshes 3 and 4 form a supermesh due to the current source between the meshes. For mesh 1, KVL gives

$$-10 + (8 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - 8\mathbf{I}_3 = 0$$

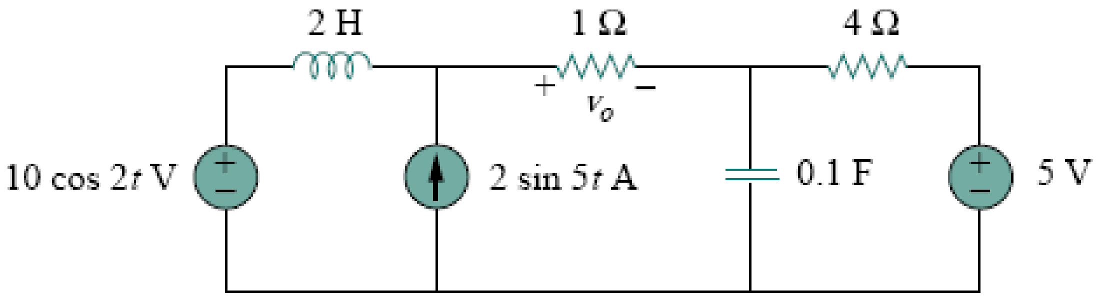
For mesh 2, $\mathbf{I}_2 = -3$

For the supermesh, $(8 - j4)\mathbf{I}_3 - 8\mathbf{I}_1 + (6 + j5)\mathbf{I}_4 - j5\mathbf{I}_2 = 0$

Due to the current source between meshes 3 and 4, at node A, $\mathbf{I}_4 = \mathbf{I}_3 + 4$

$$\mathbf{V}_o = -j2(\mathbf{I}_1 - \mathbf{I}_2) = -j2(3.618\angle 274.5^\circ + 3)$$
$$= -7.2134 - j6.568 = 9.756\angle 222.32^\circ \text{ V}$$

EXAMPLE 10.6 Find v_o in the circuit in Fig. 10.13 using the superposition theorem.



Solution:

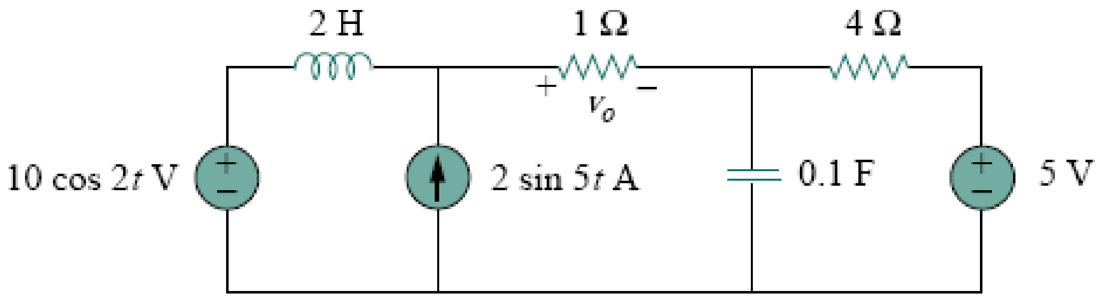
Since the circuit operates at three different frequencies (ω = 0 for the dc voltage source), one way to obtain a solution is to use superposition, which breaks the problem into single-frequency problems. So we let

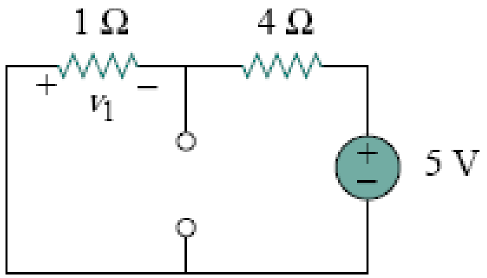
$$v_o = v_1 + v_2 + v_3$$

v_1 is due to the 5-V dc voltage source,

v_2 is due to the 10 cos $2t$ V voltage source,

v_3 is due to the 2 sin $5t$ A current source.





v_1 is due to the 5-V dc voltage source,

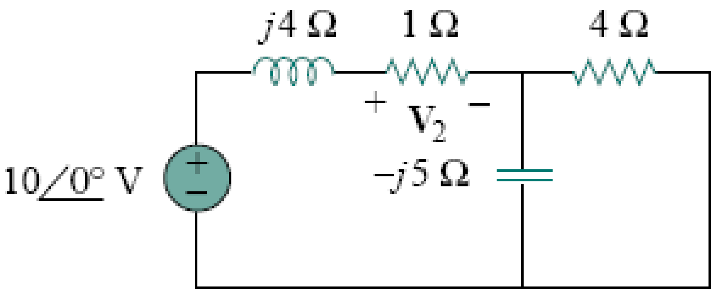
$$-v_1 = \frac{1}{1+4}(5) = 1 \text{ V}$$

v_2 is due to the 10 cos 2*t* V voltage source,

$$10\cos 2t \quad \Longrightarrow \quad 10\angle 0^\circ, \qquad \omega = 2 \text{ rad/s}$$

$$2 \text{ H} \quad \Longrightarrow \quad j\omega L = j4\ \Omega$$

$$0.1 \text{ F} \quad \Longrightarrow \quad \frac{1}{j\omega C} = -j5\ \Omega$$

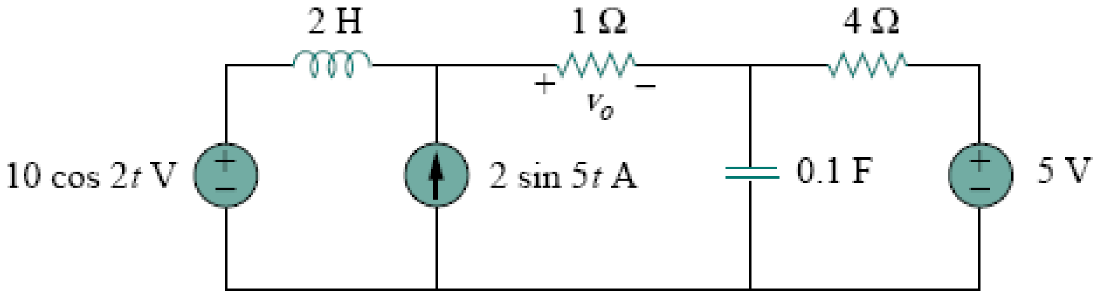


$$\mathbf{Z} = -j5 \parallel 4 = \frac{-j5 \times 4}{4 - j5} = 2.439 - j1.951$$

$$\mathbf{V}_2 = \frac{1}{1 + j4 + \mathbf{Z}}(10\angle 0^\circ) = \frac{10}{3.439 + j2.049} = 2.498\angle -30.79^\circ$$

In the time domain,

$$v_2 = 2.498\cos(2t - 30.79^\circ)$$

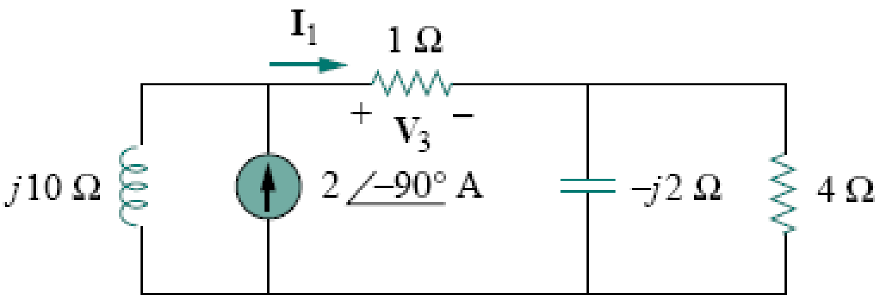


v_3 is due to the 2 sin 5*t* A current source.

$$2\sin 5t \quad \Longrightarrow \quad 2\angle -90^\circ\ , \qquad \omega = 5 \text{ rad/s}$$

$$2\text{ H} \quad \Longrightarrow \quad j\omega L = j10\ \Omega$$

$$0.1\text{ F} \quad \Longrightarrow \quad \frac{1}{j\omega C} = -j2\ \Omega$$



$$\mathbf{Z}_1 = -j2 \parallel 4 = \frac{-j2 \times 4}{4 - j2} = 0.8 - j1.6\ \Omega$$

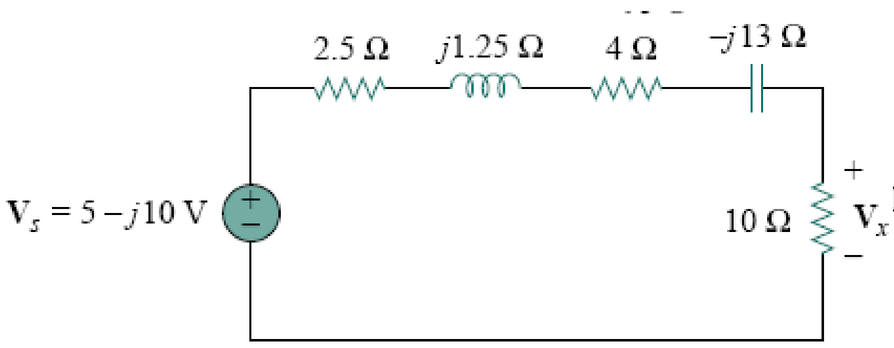
By current division,

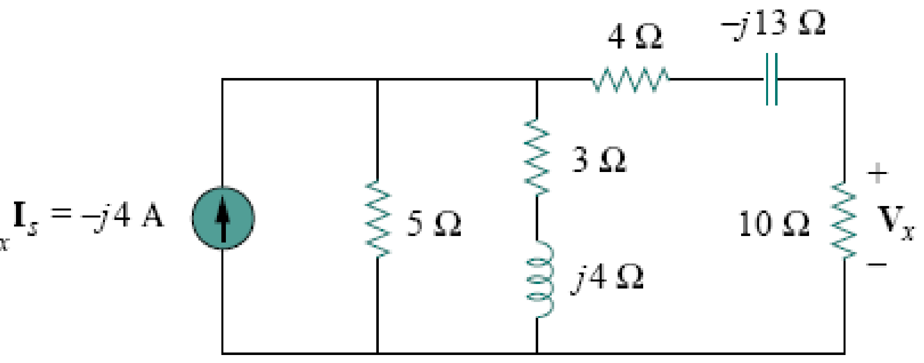
$$\mathbf{I}_1 = \frac{j10}{j10 + 1 + \mathbf{Z}_1}(2\angle -90^\circ)\text{ A} \qquad \mathbf{V}_3 = \mathbf{I}_1 \times 1 = \frac{j10}{1.8 + j8.4}(-j2) = 2.328\angle -77.91^\circ\text{ V}$$

In the time domain, $\quad v_3 = 2.33\cos(5t - 80^\circ) = 2.33\sin(5t + 10^\circ)\text{ V}$

$$v_o(t) = -1 + 2.498\cos(2t - 30.79^\circ) + 2.33\sin(5t + 10^\circ)\text{ V}$$

Calculate $\mathbf{V}_x$ in the circuit of using the method of source transformation.





Solution:

We transform the voltage source to a current source and obtain the circuit

$$\mathbf{I}_s = \frac{20\angle -90^\circ}{5} = 4\angle -90^\circ = -j4 \text{ A}$$

The parallel combination of 5-Ω resistance and $(3 + j4)$ impedance gives

$$\mathbf{Z}_1 = \frac{5(3 + j4)}{8 + j4} = 2.5 + j1.25\ \Omega$$

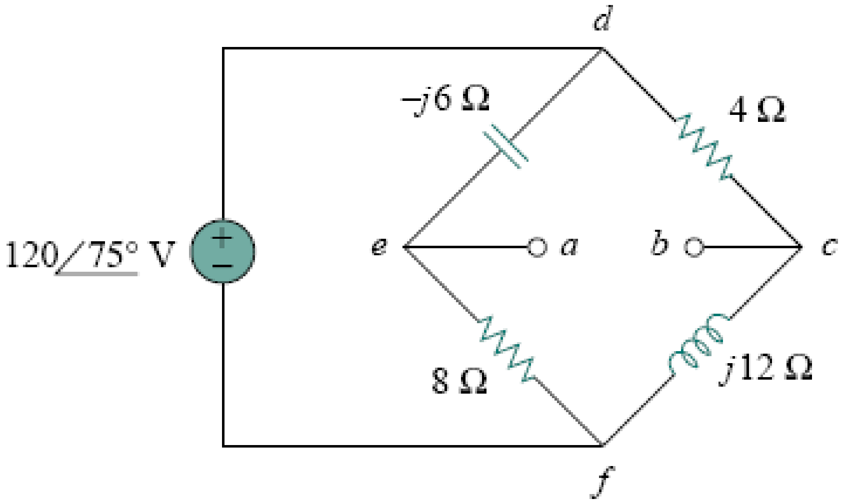
Converting the current source to a voltage source yields the circuit

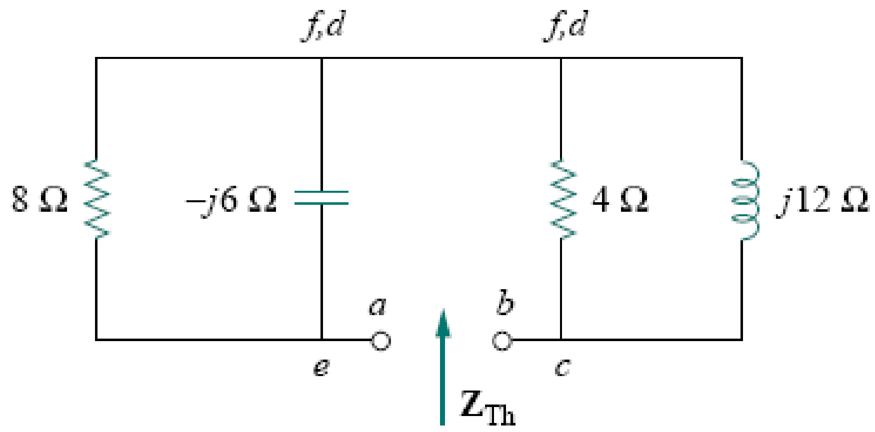
$$\mathbf{V}_s = \mathbf{I}_s\mathbf{Z}_1 = -j4(2.5 + j1.25) = 5 - j10 \text{ V}$$

$$\mathbf{V}_x = \frac{10}{10 + 2.5 + j1.25 + 4 - j13}(5 - j10) = 5.519\angle -28^\circ \text{ V}$$

EXAMPLE 10.8

Obtain the Thevenin equivalent at terminals *a*-*b* of the circuit.





Solution:

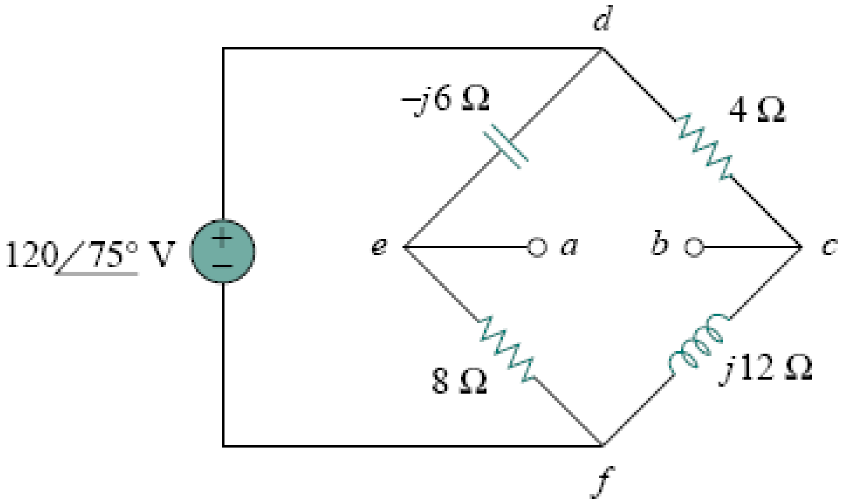
We find $\mathbf{Z}_{\mathrm{Th}}$ by setting the voltage source to zero.

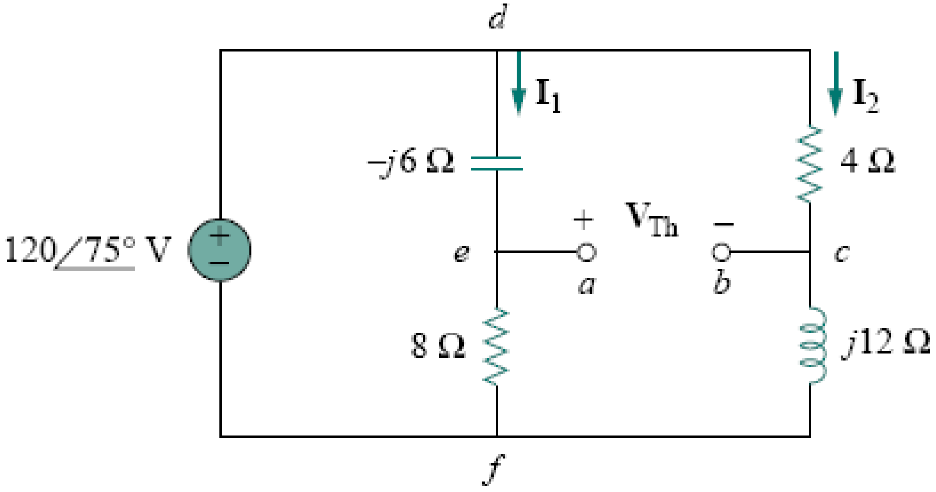
$$\mathbf{Z}_1 = -j6 \parallel 8 = \frac{-j6 \times 8}{8 - j6} = 2.88 - j3.84\ \Omega \qquad \mathbf{Z}_2 = 4 \parallel j12 = \frac{j12 \times 4}{4 + j12} = 3.6 + j1.2\ \Omega$$

$$\mathbf{Z}_{\mathrm{Th}} = \mathbf{Z}_1 + \mathbf{Z}_2 = 6.48 - j2.64\ \Omega$$

EXAMPLE 10.8

Obtain the Thevenin equivalent at terminals *a*-*b* of the circuit.





To find $\mathbf{V}_{\mathrm{Th}}$,

$$\mathbf{I}_1 = \frac{120\angle 75^\circ}{8 - j6}\ \mathrm{A}, \qquad \mathbf{I}_2 = \frac{120\angle 75^\circ}{4 + j12}\ \mathrm{A}$$

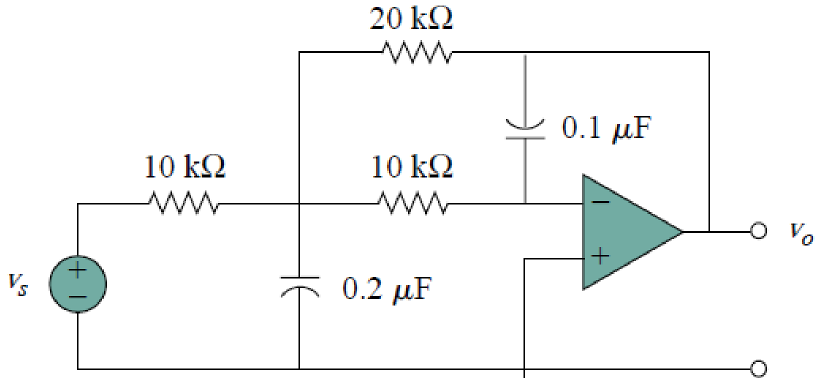
Applying KVL around loop *bcdeab*

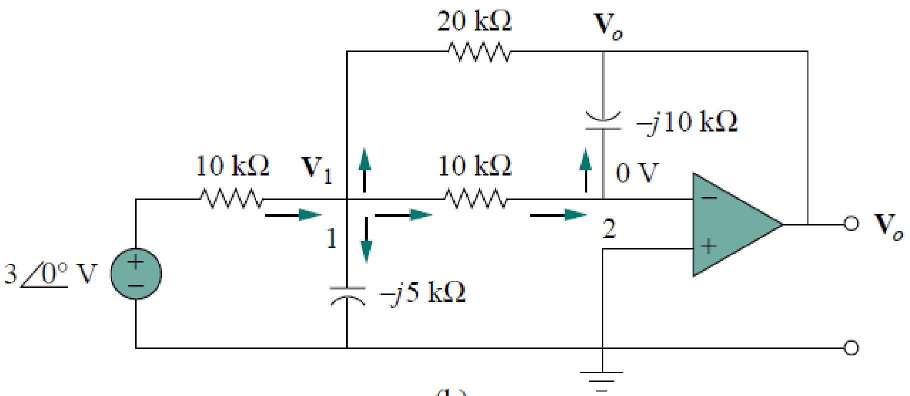
$$\mathbf{V}_{\mathrm{Th}} - 4\mathbf{I}_2 + (-j6)\mathbf{I}_1 = 0$$

$$\mathbf{V}_{\mathrm{Th}} = 4\mathbf{I}_2 + j6\mathbf{I}_1 = 37.95\angle 220.31^\circ\ \mathrm{V}$$

EXAMPLE 10.11

Determine $v_o(t)$ for the op amp circuit in Fig. 10.31(a) if $v_s = 3 \cos 1000t$ V.





(b)

Solution:

We first transform the circuit to the frequency domain, as shown in Fig. 10.31(b), where $\mathbf{V}_s = 3\angle 0^\circ$, $\omega = 1000$ rad/s. Applying KCL at node 1,

$$\frac{3\angle 0^\circ - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j5} + \frac{\mathbf{V}_1 - 0}{10} + \frac{\mathbf{V}_1 - \mathbf{V}_o}{20} \qquad 6 = (5 + j4)\mathbf{V}_1 - \mathbf{V}_o$$

At node 2, KCL gives $\frac{\mathbf{V}_1 - 0}{10} = \frac{0 - \mathbf{V}_o}{-j10}$ $\qquad \mathbf{V}_1 = -j\mathbf{V}_o$

$$\mathbf{V}_o = \frac{6}{3 - j5} = 1.029\angle 59.04^\circ \qquad v_o(t) = 1.029 \cos(1000t + 59.04^\circ) \text{ V}$$

Chapter 11 AC Power Analysis

11.1 Introduction

11.2 Instantaneous and Average Power

11.3 Maximum Average Power Transfer

11.4 Effective or RMS Value

11.5 Apparent Power and Power Factor

11.6 Complex Power

11.7 Conservation of AC Power

11.8 Power Factor Correction

Chapter 11 AC Power Analysis

1. The instantaneous power $p(t)$

$$\boxed{p(t) = v(t)i(t)}$$

$$v(t) = V_m \cos(\omega t + \theta_v)$$
$$i(t) = I_m \cos(\omega t + \theta_i)$$

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

2. The average power P:

The average power is the average of the instantaneous power over one period.

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$
$$= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

$$\boxed{P = \frac{1}{2}\operatorname{Re}[\mathbf{VI}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}$$

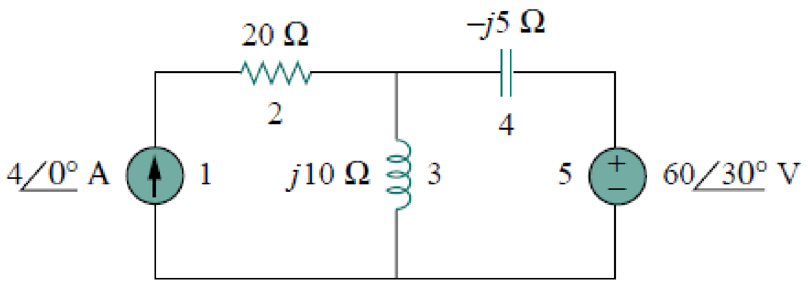
the average power absorbed by a resistor *R* can be written as $P = I_{\text{rms}}^2 R = \dfrac{V_{\text{rms}}^2}{R}$

Chapter 11 AC Power Analysis

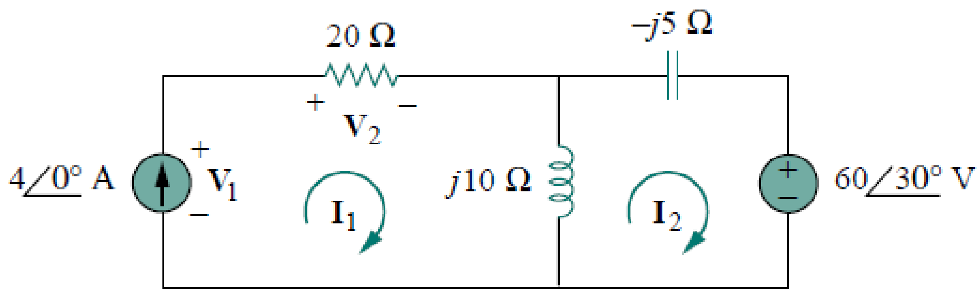
EXAMPLE 11.4

$$P_1 + P_2 + P_3 + P_4 + P_5 = -367.8 + 160 + 0 + 0 + 207.8 = 0$$

Determine the power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 11.5(a).



(a)



(b)

For the current source, the current through it is $\mathbf{I}_1 = 4\angle 0^\circ$ and the voltage across it is

$$\mathbf{V}_1 = 20\mathbf{I}_1 + j10(\mathbf{I}_1 - \mathbf{I}_2) = 80 + j10(4 - 2 - j10.39)$$
$$= 183.9 + j20 = 184.984\angle 6.21^\circ \text{ V}$$

The average power supplied by the current source is

$$P_1 = -\frac{1}{2}(184.984)(4)\cos(6.21^\circ - 0) = -367.8 \text{ W}$$

For the resistor,

$$P_2 = \frac{1}{2}(80)(4) = 160 \text{ W}$$

For the capacitor,

$$P_4 = \frac{1}{2}(52.9)(10.58)\cos(-90^\circ) = 0$$

For the inductor,

$$P_3 = \frac{1}{2}(105.8)(10.58)\cos 90^\circ = 0$$

Chapter 11 AC Power Analysis

3. Apparent power S:

The apparent power (in VA) is the product of the rms values of voltage and current.

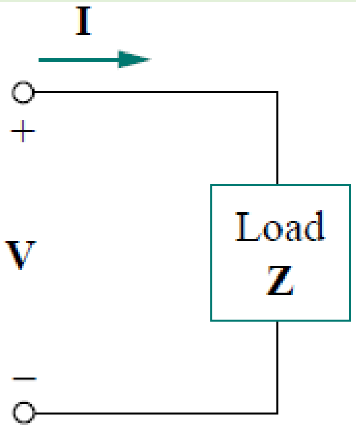
$$S = V_{\text{rms}} I_{\text{rms}}$$

4. The complex power :

the *complex power* S absorbed by the ac load is the product of the voltage and the complex conjugate of the current

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^*$$

In terms of the rms values, $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$



$$\mathbf{S} = V_{\text{rms}} I_{\text{rms}} \angle \theta_v - \theta_i$$

$$= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

$$\mathbf{S} = I^2_{\text{rms}}(R + jX) = P + jQ$$

$$P = \text{Re}(\mathbf{S}) = I^2_{\text{rms}} R$$ Real power

$$Q = \text{Im}(\mathbf{S}) = I^2_{\text{rms}} X$$ Reactive power

Complex power (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power P and its imaginary part is reactive power Q.

EXAMPLE 11.12

A load $\mathbf{Z}$ draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

Solution:

(a) Given that pf $= \cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13°$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is $P = S\cos\theta = 12{,}000 \times 0.856 = 10.272$ kW

while the reactive power is $Q = S\sin\theta = 12{,}000 \times 0.517 = 6.204$ kVA

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + j6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$, we obtain

$$\mathbf{I}^*_{\text{rms}} = \frac{\mathbf{S}}{\mathbf{V}_{\text{rms}}} = \frac{10{,}272 + j6204}{120\underline{/0°}} = 85.6 + j51.7 \text{ A} = 100\underline{/31.13°} \text{ A}$$

Thus $\mathbf{I}_{\text{rms}} = 100\underline{/-31.13°}$ and the peak current is

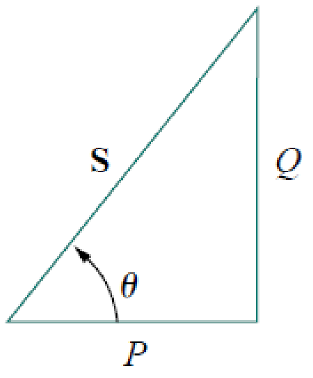
$$I_m = \sqrt{2}I_{\text{rms}} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance $\mathbf{Z} = \dfrac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \dfrac{120\underline{/0°}}{100\underline{/-31.13°}} = 1.2\underline{/31.13°}\ \Omega$

Chapter 11 AC Power Analysis

5. Power triangle:

$$\text{Complex Power} = \mathbf{S} = P + jQ = \frac{1}{2}\mathbf{V}\mathbf{I}^*$$
$$= V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i}$$
$$\text{Apparent Power} = S = |\mathbf{S}| = V_{\text{rms}} I_{\text{rms}} = \sqrt{P^2 + Q^2}$$
$$\text{Real Power} = P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i)$$
$$\text{Reactive Power} = Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i)$$
$$\text{Power Factor} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$



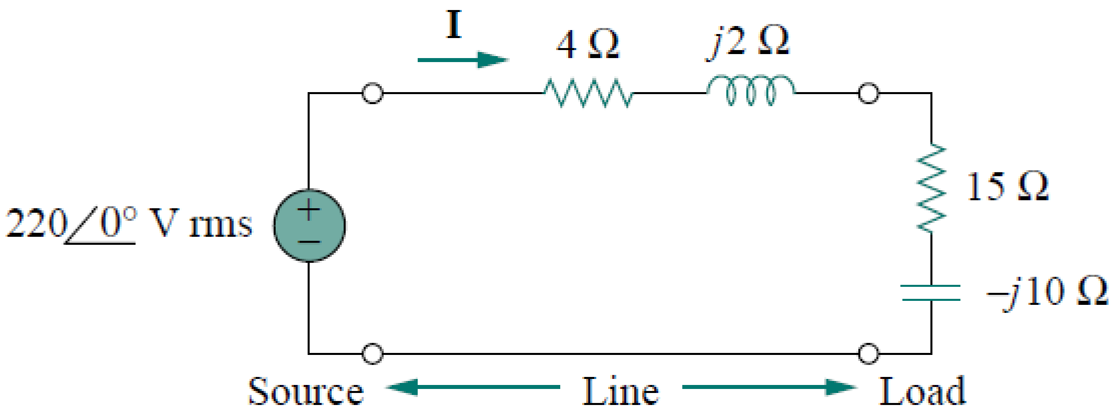
power triangle

6. Conservation of AC Power:

The complex, real, and reactive powers of the sources equal the respective sums of the complex, real, and reactive powers of the individual loads.

EXAMPLE 11.13

Figure 11.24 shows a load being fed by a voltage source through a transmission line. The impedance of the line is represented by the $(4 + j2)\ \Omega$ impedance and a return path. Find the real power and reactive power absorbed by: (a) the source, (b) the line, and (c) the load.

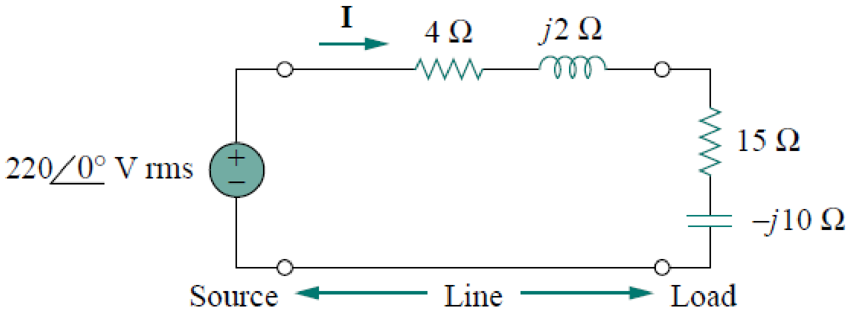


Solution:

The total impedance is

$$\mathbf{Z} = (4 + j2) + (15 - j10) = 19 - j8 = 20.62\angle -22.83^\circ\ \Omega$$

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{220\angle 0^\circ}{20.62\angle -22.83^\circ} = 10.67\angle 22.83^\circ \text{ A rms}$$



(a) For the source, the complex power is

$$\mathbf{S}_s = \mathbf{V}_s\mathbf{I}^* = (220\angle 0^\circ)(10.67\angle -22.83^\circ)$$

$$= 2347.4\angle -22.83^\circ = (2163.5 - j910.8)\text{ VA}$$

From this, we obtain the real power as 2163.5 W and the reactive power as 910.8 VAR (leading).

(b) For the line, the voltage is

$$\mathbf{V}_{\text{line}} = (4 + j2)\mathbf{I} = (4.472\angle 26.57^\circ)(10.67\angle 22.83^\circ)$$

$$= 47.72\angle 49.4^\circ \text{ V rms}$$

The complex power absorbed by the line is

$$\mathbf{S}_{\text{line}} = \mathbf{V}_{\text{line}}\mathbf{I}^* = (47.72\angle 49.4^\circ)(10.67\angle -22.83^\circ)$$

$$= 509.2\angle 26.57^\circ = 455.4 + j227.7\text{ VA}$$

(c) For the load, the voltage is

$$\mathbf{V}_L = (15 - j10)\mathbf{I} = (18.03\underline{/-33.7^\circ})(10.67\underline{/22.83^\circ})$$
$$= 192.38\underline{/-10.87^\circ}\text{ V rms}$$

The complex power absorbed by the load is

$$\mathbf{S}_L = \mathbf{V}_L\mathbf{I}^* = (192.38\underline{/-10.87^\circ})(10.67\underline{/-22.83^\circ})$$
$$= 2053\underline{/-33.7^\circ} = (1708 - j1139)\text{ VA}$$

The real power is 1708 W and the reactive power is 1139 VAR (leading). Note that $\mathbf{S}_s = \mathbf{S}_{\text{line}} + \mathbf{S}_L$, as expected. We have used the rms values of voltages and currents.

Chapter 11 AC Power Analysis

7. Power factor:

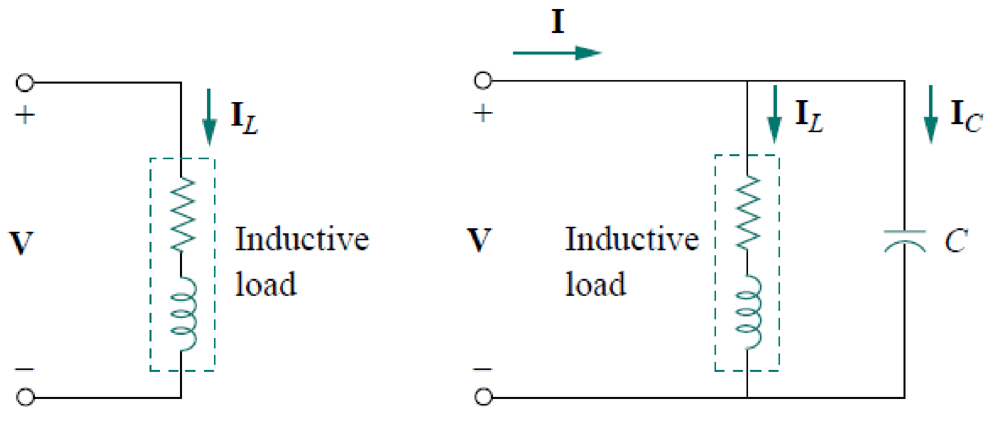
$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

Leading power factor means that current leads voltage, which implies a capacitive load.

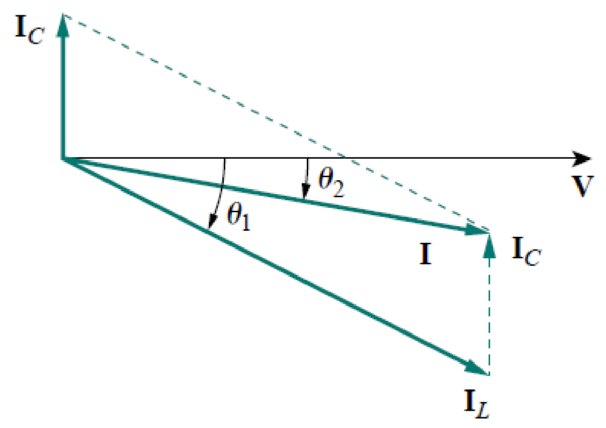
Lagging power factor means that current lags voltage, implying an inductive load.

8. Power factor correction:

The process of increasing the power factor without altering the voltage or current to the original load is known as power factor correction.

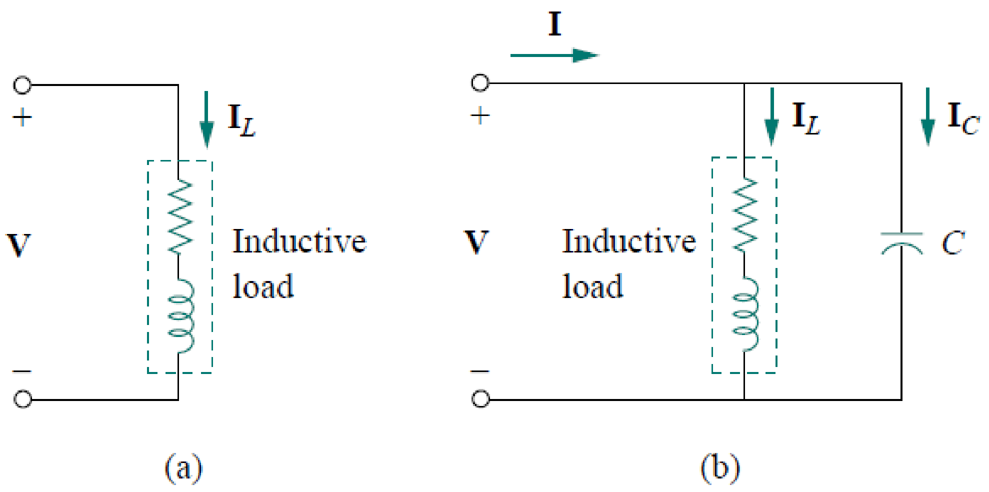


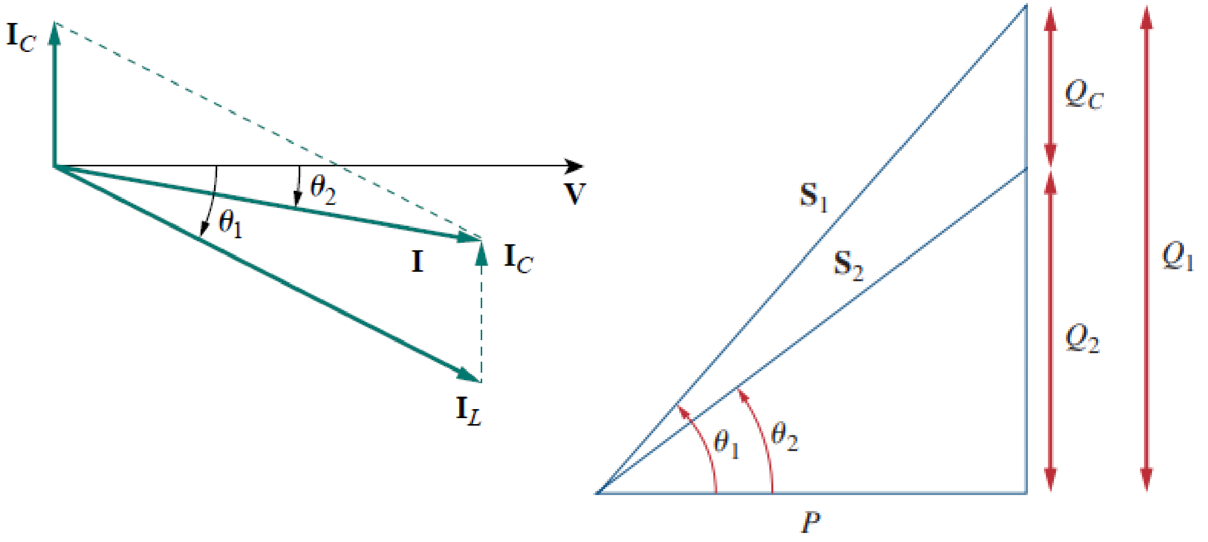
(a) (b)



Chapter 11 AC Power Analysis

8. Power factor correction:





$P = S_1 \cos\theta_1, \qquad Q_1 = S_1 \sin\theta_1 = P \tan\theta_1$

If we desire to increase the power factor from $\cos\theta_1$ to $\cos\theta_2$ without altering the real power (i.e., $P = S_2 \cos\theta_2$), then the new reactive power is $Q_2 = P \tan\theta_2$

The reduction in the reactive power is caused by the shunt capacitor $Q_C = Q_1 - Q_2 = P(\tan\theta_1 - \tan\theta_2)$

$Q_C = V_{\text{rms}}^2 / X_C = \omega C V_{\text{rms}}^2.$

$$\boxed{C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2}}$$

EXAMPLE 11.15

When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

$$\boxed{C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2}}$$

Solution:

If the pf $= 0.8$, then

$$\cos\theta_1 = 0.8 \quad \Longrightarrow \quad \theta_1 = 36.87^\circ$$

$$S_1 = \frac{P}{\cos\theta_1} = \frac{4000}{0.8} = 5000\text{ VA} \qquad Q_1 = S_1 \sin\theta = 5000 \sin 36.87 = 3000\text{ VAR}$$

When the pf is raised to 0.95, $\cos\theta_2 = 0.95 \quad \Longrightarrow \quad \theta_2 = 18.19^\circ$

The real power P has not changed. But the apparent power has changed; its new value is

$$S_2 = \frac{P}{\cos\theta_2} = \frac{4000}{0.95} = 4210.5\text{ VA} \qquad Q_2 = S_2 \sin\theta_2 = 1314.4\text{ VAR}$$

Chapter 11 AC Power Analysis

EXAMPLE 11.15

When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2}$$

The difference between the new and old reactive powers is due to the parallel addition of the capacitor to the load. The reactive power due to the capacitor is

$$Q_C = Q_1 - Q_2 = 3000 - 1314.4 = 1685.6 \text{ VAR}$$

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{1685.6}{2\pi \times 60 \times 120^2} = 310.5\ \mu\text{F}$$