Discrete Mathematics

Graph

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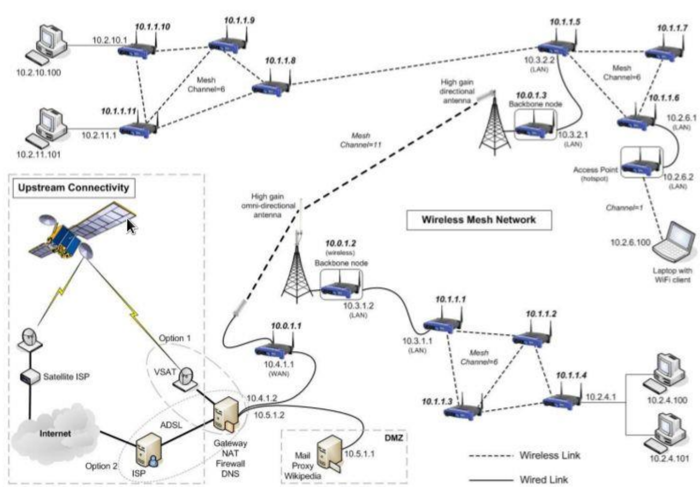
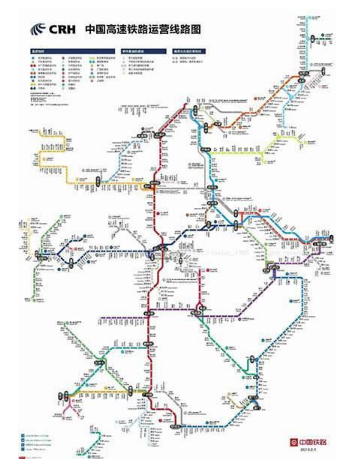
Applications of Graph

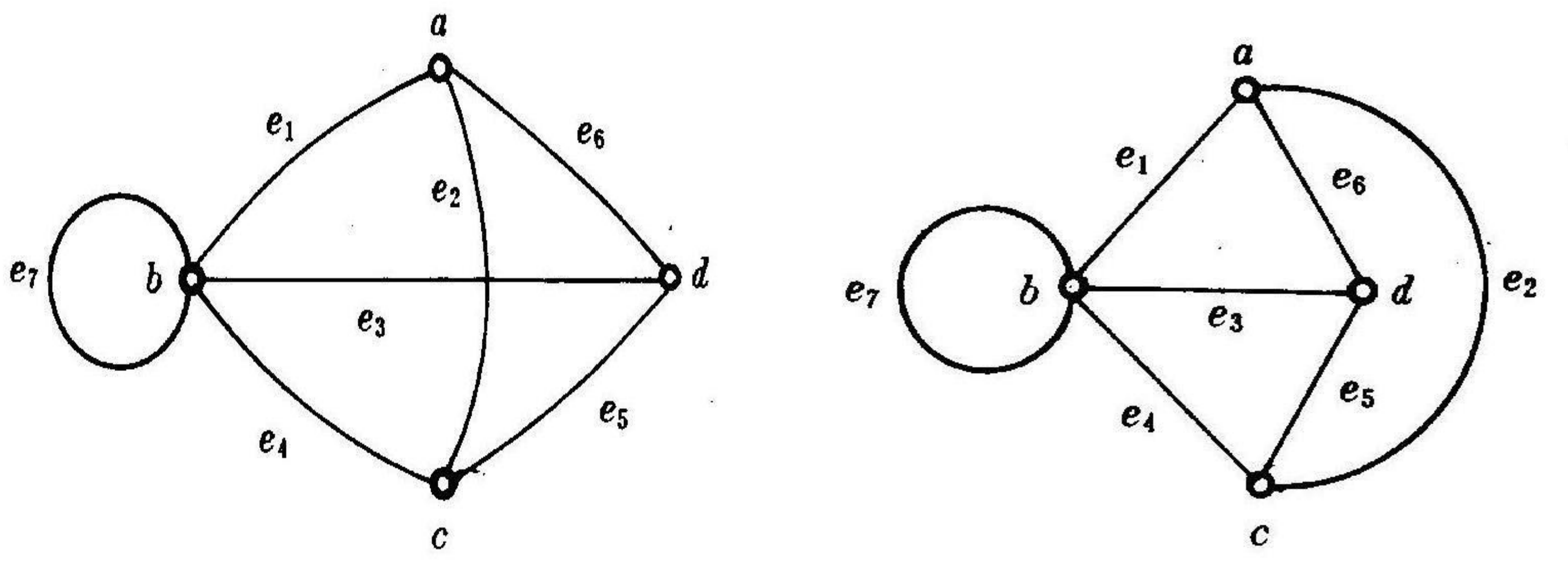
- Graph can be applied in many areas.
- Describe the webs, airline routes, call and so on
- Determine whether two computers are connected by a communications link
- Design the shortest water pipes for every house
- Get the shortest path from one city to another city
- Use graph to find the number of colors needed to color the regions of maps





Graph

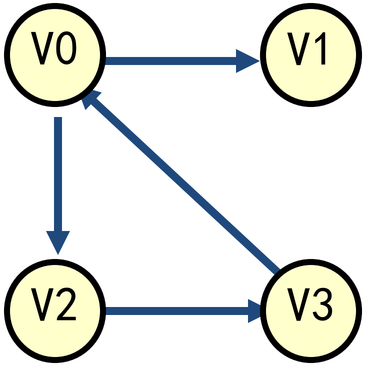
- A graph G consists of a set of vertices and a set of connections linking pairs of vertices. These pairs of vertices are called edges. That is:
- Graph G=(V,E)



(a) (b)

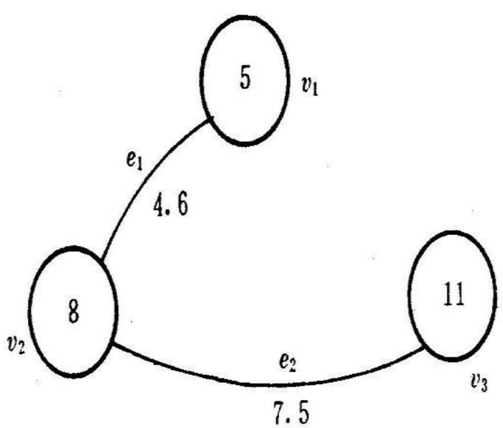
Types of graphs

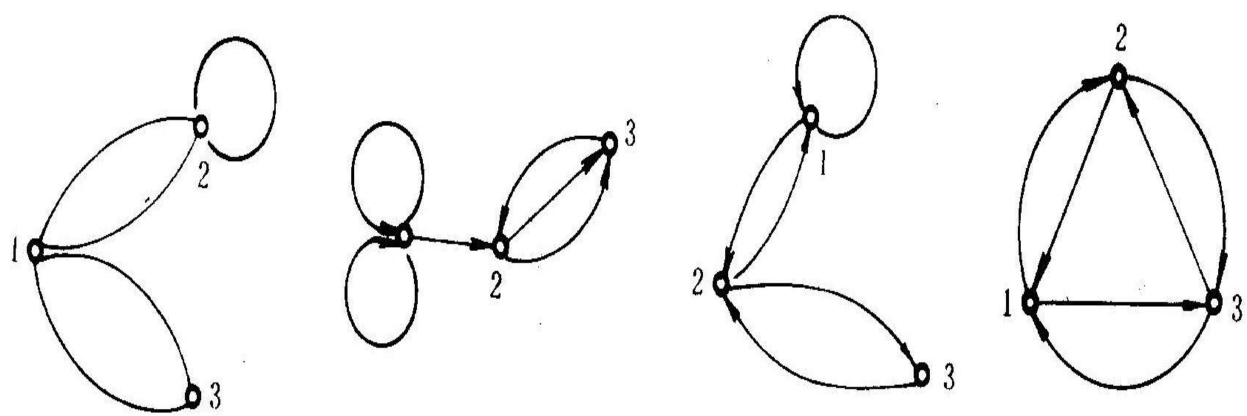
- A directed graph (V,E) consists of a nonempty set of vertices V and a set of directed edges E. Each directed edge associated with the ordered pair (u,v) is said to start at u and end at v.
- Undirected graph
- Mixed graph



Types of graphs

- Simple graph: each edge connects two different vertices and no two edges connect the same pair of vertices.
- Multigraph: Graphs that may have multiple edges connecting the same vertices are called multigraphs

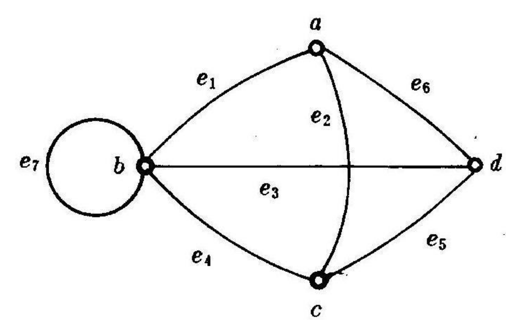




(a) (b) (c) (d)

Definitions for undirected graph

- Definition1: Two vertices u and v in an undirected graph G are called adjacent in G if u and v are endpoints of an edge e of G. such an edge is called incident with the vertices u and v, and e is said to connect u and v.
- Definition2: The set of all neighbors of a vertex v of G=(V,E) is called the neighborhood of v.
- Definition3: The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex, the degree of the vertex v is denoted by deg(v).



(a)

Handshaking theorem

- Let G=(V,E) be an undirected graph with m edges. Then

$$\sum_{i=1}^{n} \deg(v_i) = 2m$$

- The sum of the degrees of all vertices is twice the number of the edges.
- An undirected graph has an even number of vertices of odd degree.

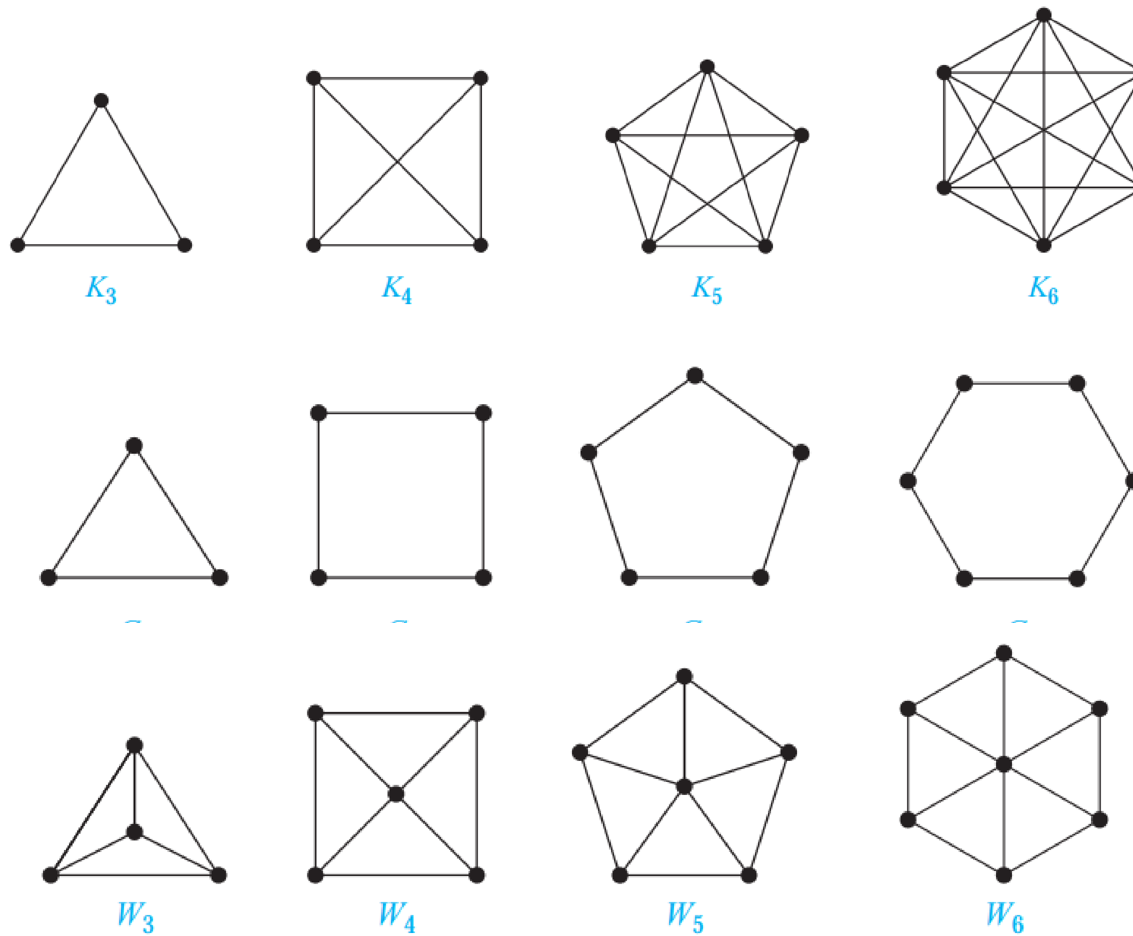
Definitions for directed graph

- Definition: when (u,v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called the initial vertex of (u,v) and v is the terminal or end vertex.
- In a graph with directed edges, the in-degree of a vertex v, denoted by $\deg^{-}(\mathrm{v})$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $\deg^{+}(\mathrm{v})$, is the number of edges with v as their initial vertex.

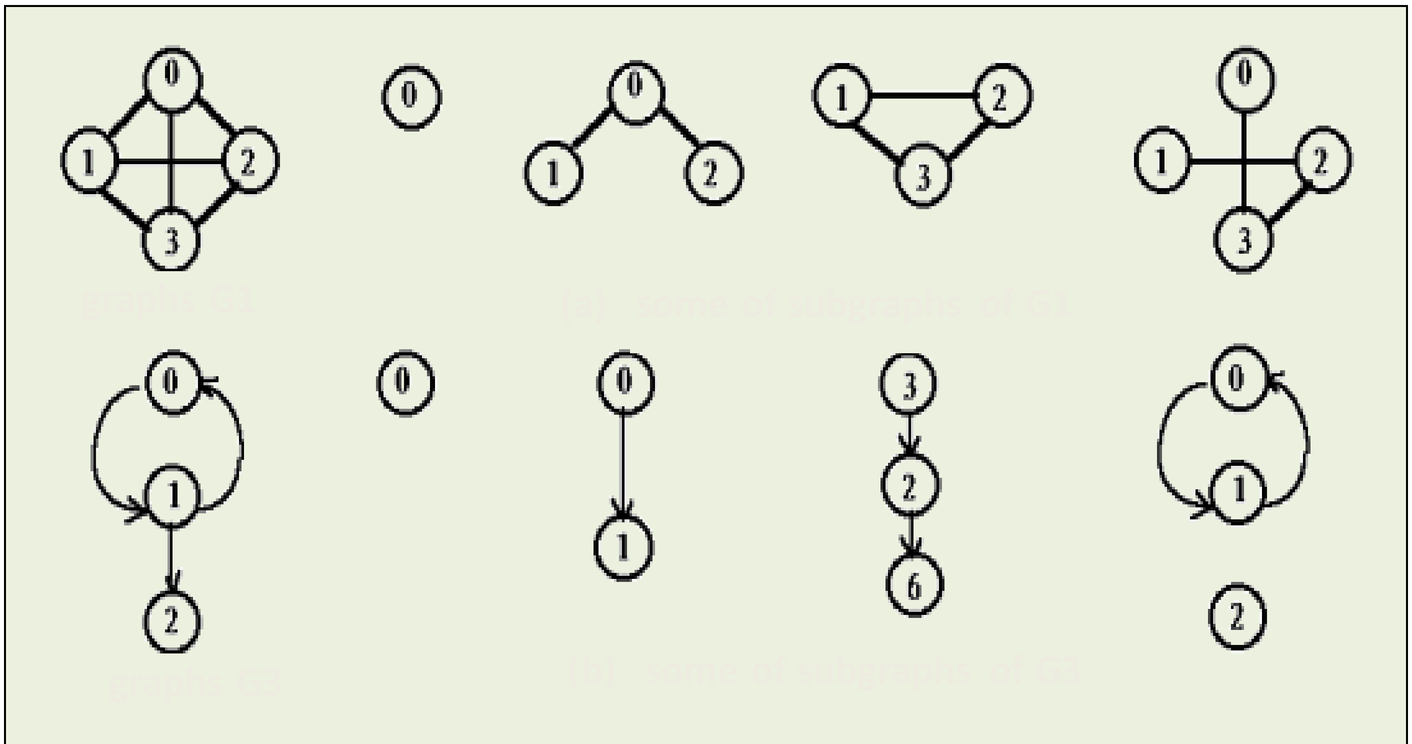
- Theorem
- Let G=(V,E) be a graph with directed edges, then
- $\sum_{v\in V} deg^{-}(\text{v}) = \sum_{v\in V} deg^{+}(\text{v})=|\text{E}|$
- The sum of the in degrees is same as the sum of the out degrees.
- Example 4

Special simple graphs

- Complete graph
- Cycles
- Wheels

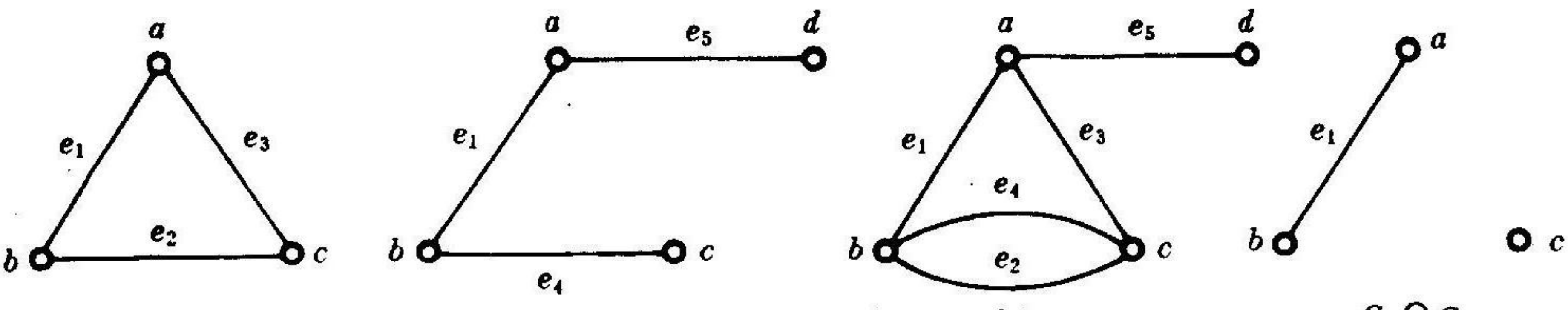


- ***subgraph :*** **Assume there are two graphs G=(V,E) and G'=(V',E'). if V'$\subseteq$V, and E'$\subseteq$E, G' is called subgraph of G.**



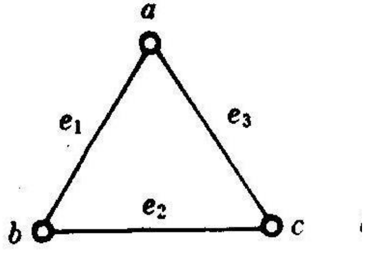
Graph operation

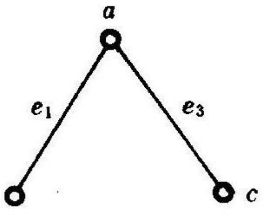
- $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ $G_3=(V_3,E_3)$
- **(1)** $G_3=G_1 \cup G_2$, where $V_3=V_1 \cup V_2, E_3=E_1 \cup E_2$
- **(2)** $G_3=G_1 \cap G_2$, where $V_3=V_1 \cap V_2, E_3=E_1 \cap E_2$



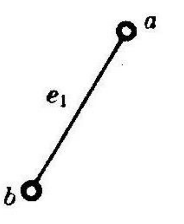
G_1 G_2 $G_1 \cup G_2$ $G_1 \cap G_2$

- G=(V,E)
- Removing edge
- G-e=(V, E-{e})
- Removing vertices from a graph
- G-v=(V-v, E') , where E' is the set of edges of G not incident to v.





G_1 删去边 e_2

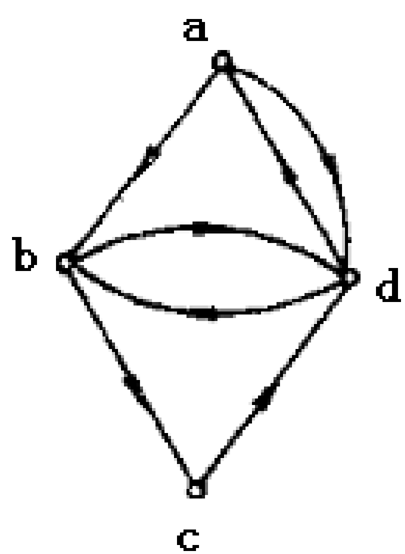


G_1 删去结点 c

Representing Graphs

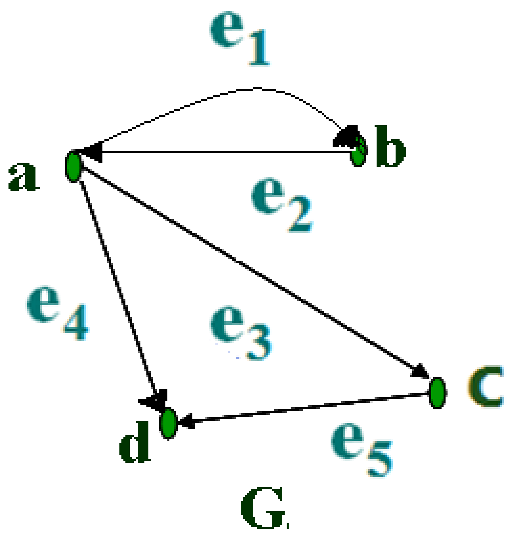
- Adjacency lists (table)
- Example 1 and 2
- Adjacency matrices
- Incidence matrices

- Adjacency matrices zero-one
- List all vertices in any order. $v_1, v_2, \ldots, v_n$
- $a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$
- It can be used to represent graphs with loops and multiple edges.
- Example 3, 5



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

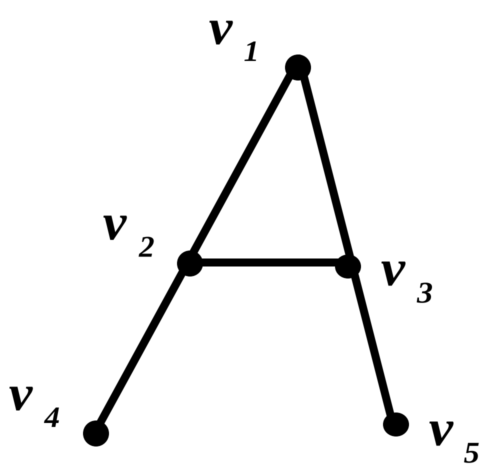
- Incidence matrices
- Suppose that $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_n$ are the vextices and $e_{1,}e_{2,\ldots\ldots,}e_{\mathrm{m}}$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the nXm matrix M.where
- $a_{ij}=\begin{cases} 1 & \text{when } e_{\mathrm{j}} \text{ is incident with } \boldsymbol{v}_{\mathbf{i}} \\ 0 & \text{otherwise} \end{cases}$
- Example 6   p671
- Example 7

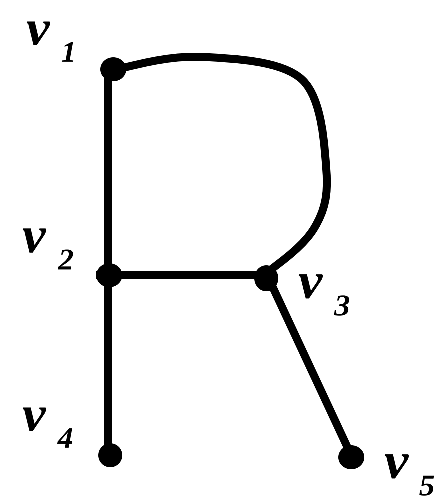


$$M(G) = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

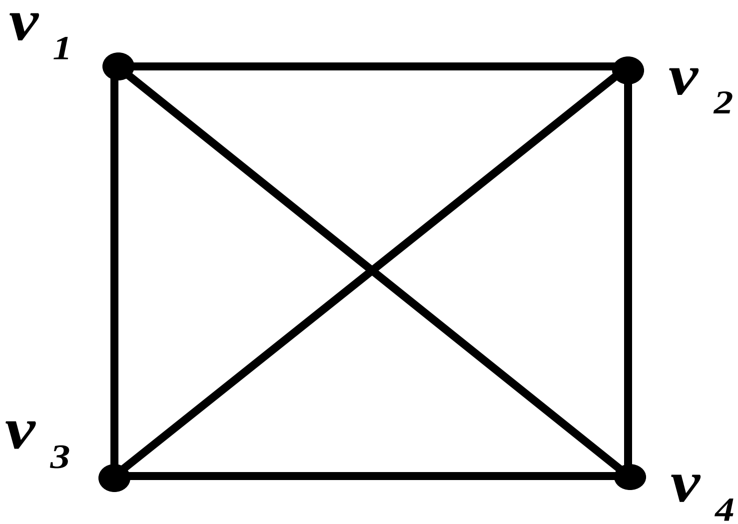
Isomorphism of graphs

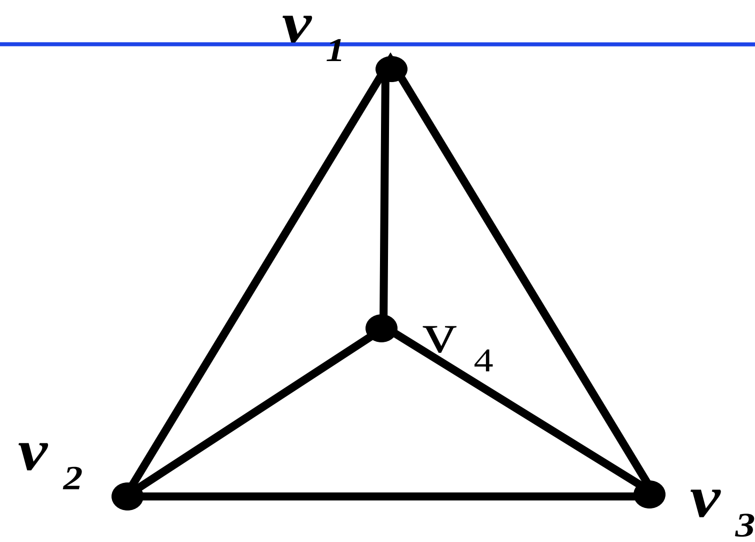
- Definition: the simple graph $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there exists a bijective function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2, for all a and b in V1. Such a function f is called an isomorphism.

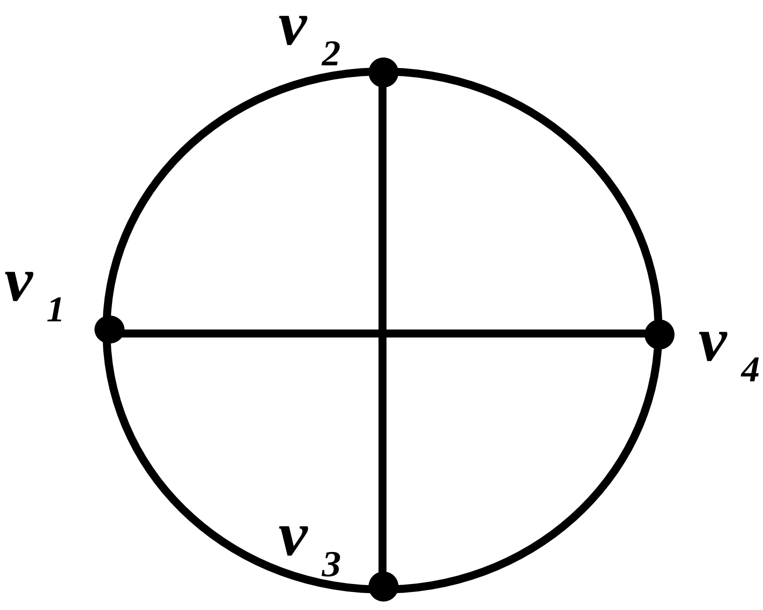


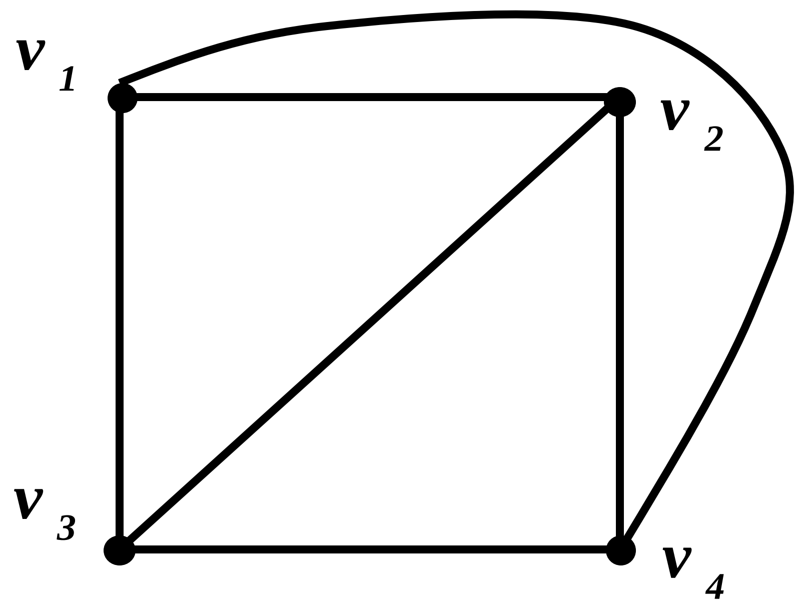


Isomorphism



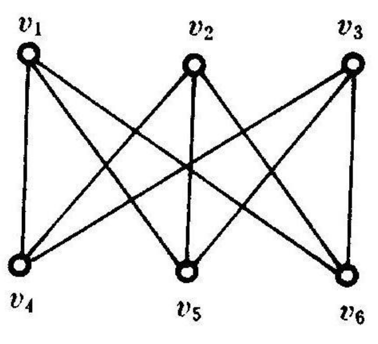


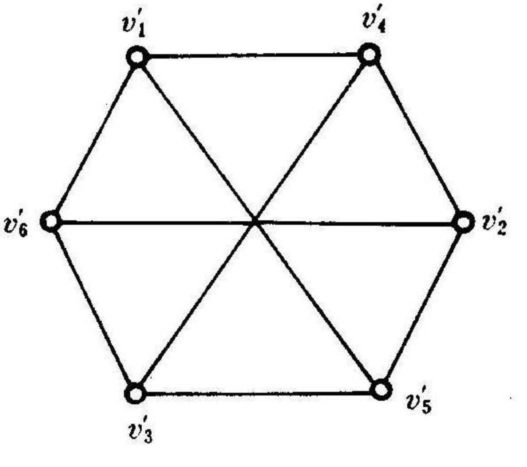




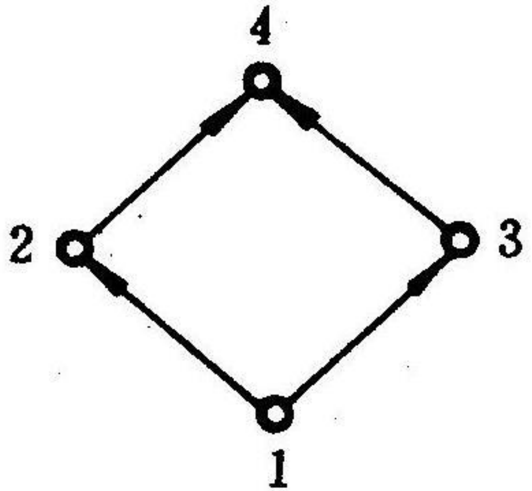
How to determine

- It is very difficult to determine
- We have design algorithms to determine
- Nauty software can determine whether two graphs with as many as 100 vertices are isomorphic in less than a second on a pc.
- Use degrees, the number of edges, subgraph and adjacency matrix

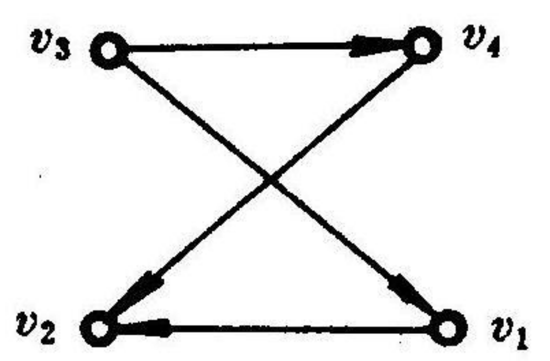




- Example 9
- Example 10
- Example 11



(a)



(b)

Homework

- 10.1 2
- 10.2 2, 8
- 10.3 11, 15, 17, 20,26, 27,

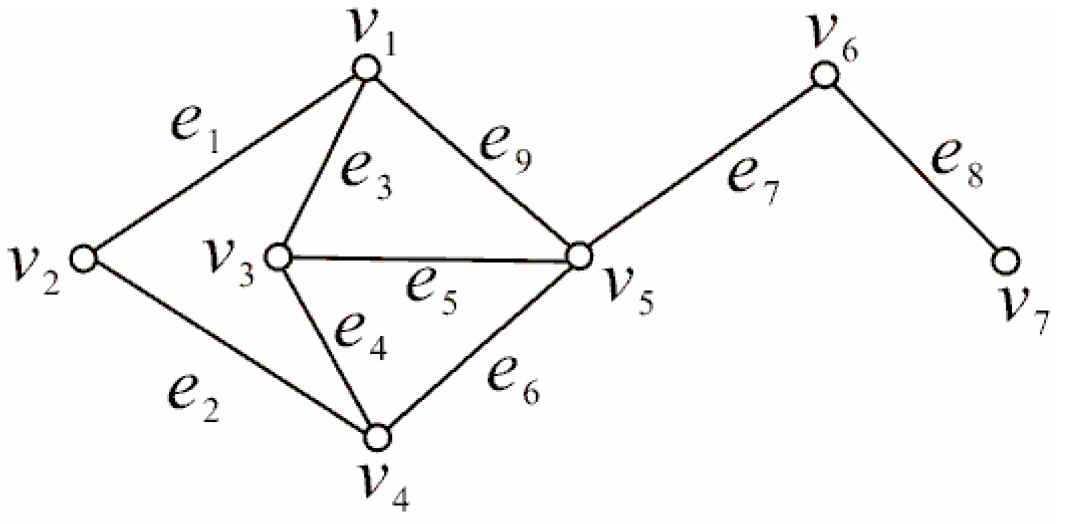
Paths

- Definition
- Let n be a nonnegative integer and G an undirected graph. A path of length n from u to v in G is a sequence of n edges e_1,… e_n of G for which there exists a sequence x_0=u, x_1, … x_n =v of vertices such that $e_i$has the endpoints $x_{i-1}$and x_i, when the graph is simple, we denote this path by its vertex sequence.

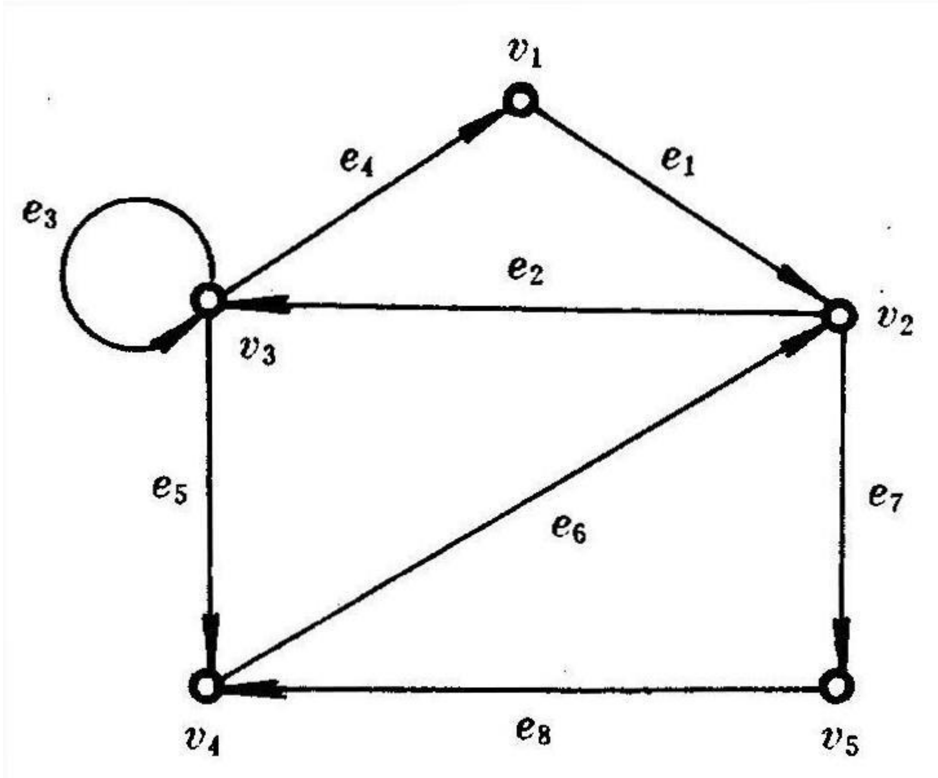
circuit ---- begins and ends at the same vertex

- Simple path ---- does not contain the same edge more than once.
- length of a path ---- the number of edges on the path

- How many simple paths from v2 to v5?
- What is the length of shortest path from v2 to v7?



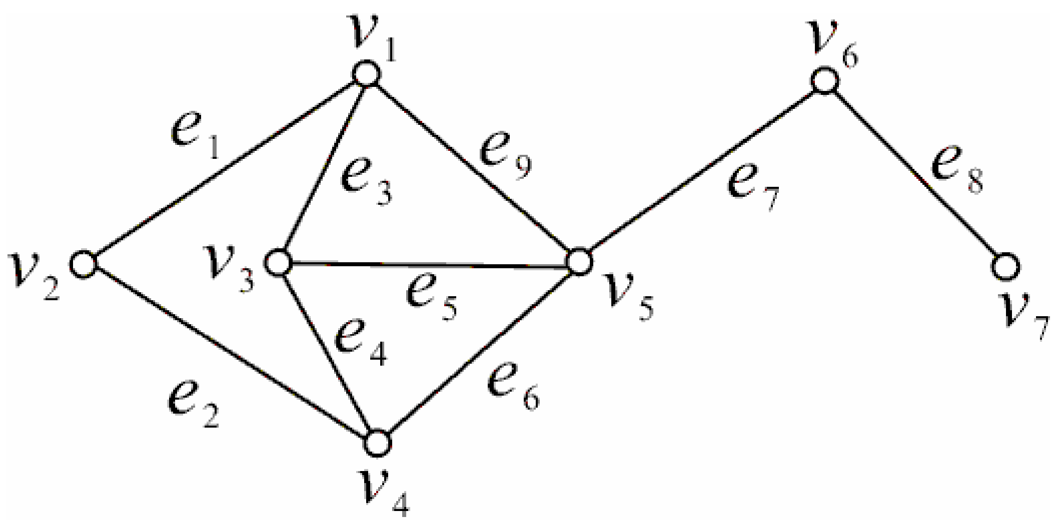
- Definition
- Let n be a nonnegative integer and G a directed graph. A path from a to b in G is a sequence of edges(x_0, x_1), (x_1, x_2),…(x_{n-1}, x_n) , where x_0=a, x_n =b. that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path.
- Using $e_1, e_2 \ldots e_n$ to denote path.
- A path can pass through a vertex more than once, can traverse an edge more than once.


v_1
e_4
e_1
e_3
e_2
v_3
v_2
e_5
e_6
e_7
v_4
e_8
v_5

Connectedness

- Definition:
- An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not connected is called disconnected. We say that we disconnect a graph when we remove vertices or edges, to produce a disconnected subgraph.

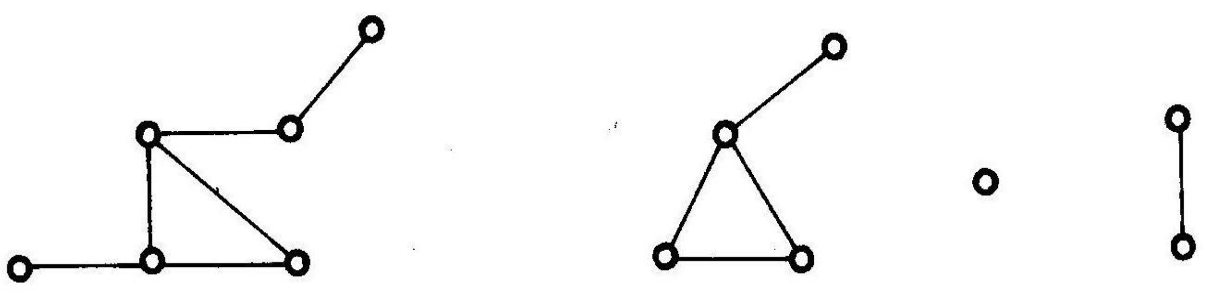
- How to remove one edge to make the graph to be disconnected.



- Theorem
- There is a simple path between every pair of distinct vertices of a connected undirected graph.

Connected components

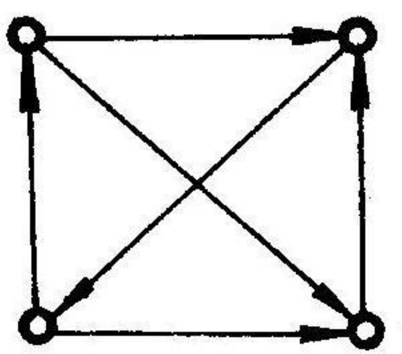
- A connected component of a graph G is a connected graph of G that is not a proper subgraph of another subgraph of G. That is, a connected component of a graph G is a maximal connected graph of G.
- A graph that is not connected has two or more connected component that are disjoint and have G as their union.



(a) (b)

Connectedness

- Definition:
- A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
- A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.



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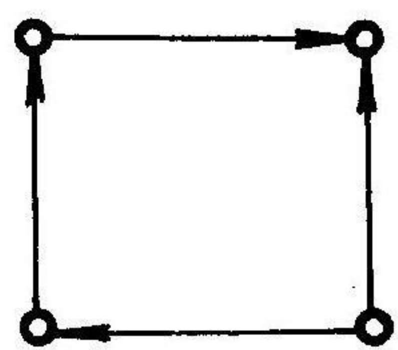
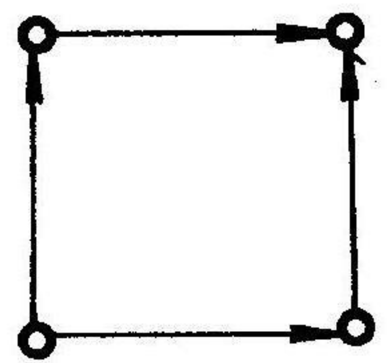


(b)



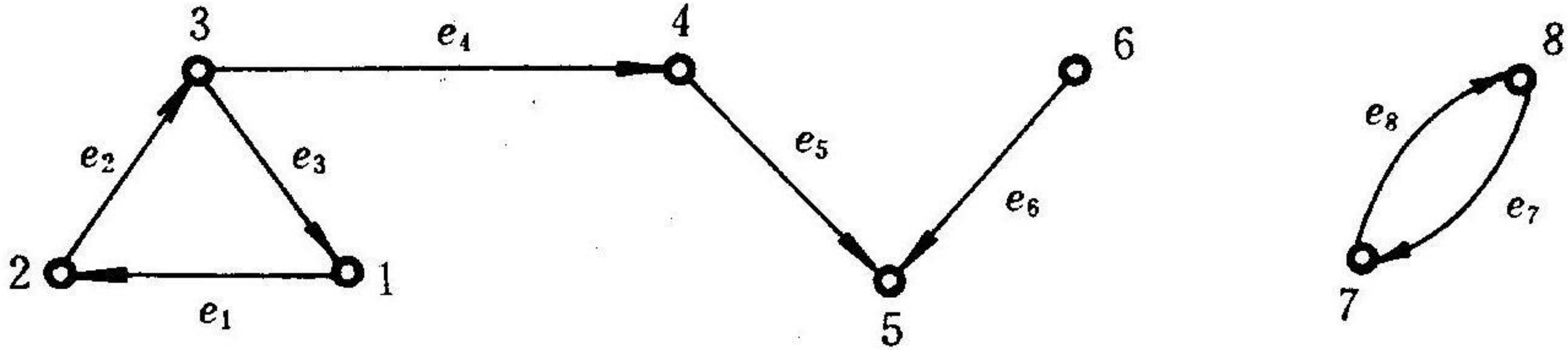
(c)

Strong components

- The subgraph of a directed graph G that are strongly connected but not contained in larger strongly connected subgraphs, that is, the maximal strongly connected subgraphs, are called the strongly connected components or strong components of G.

Strong components

- $\{\langle\{1,2,3\},\{e_1,e_2,e_3\}\rangle, \langle\{4\},\varphi\rangle, \langle\{5\},\varphi\rangle, \langle\{6\},\varphi\rangle, \langle\{7,8\},\{e_7,e_8\}\rangle\}$



Path and Isomorphism

- Path can be used to determine whether two graphs are isomorphic or not
- Example 13 and 14

Counting paths between vertices

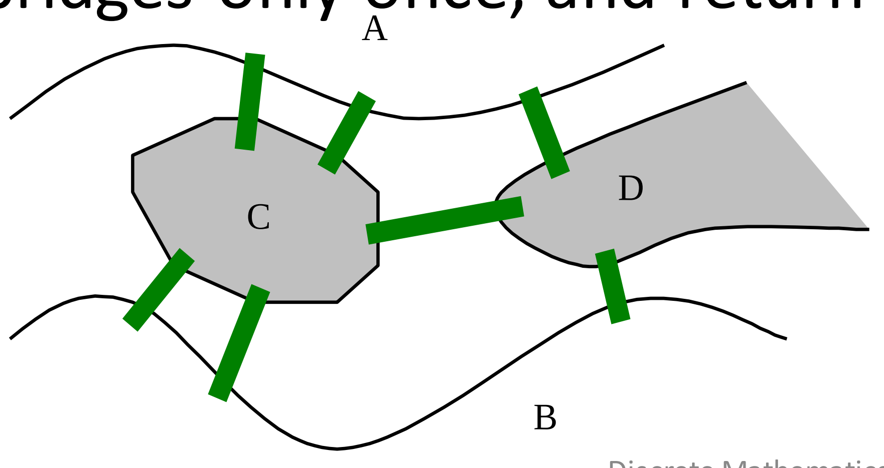
- Theorem
- Let G be a graph with adjacency matrix A with respect to the ordering of the vertices of the graph. The number of different paths of length r from $v_i\ to\ v_j$, where r is a positive integer , equals the (i,j)th entry of A^r .

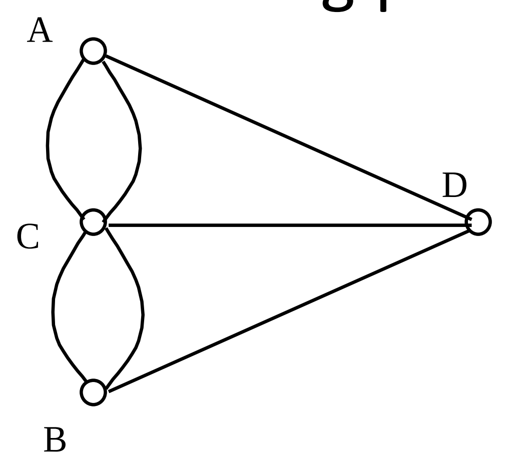
Homework

- 10.3 38, 39, 58
- 10.4 1, 2, 6, 11, 14(a), 14(b).

Euler Path

- Seven Bridges at Königsberg
- This town was divided into four sections by the branches of the river.
- Seven bridges connect these sections.
- The local people wondered whether it was possible to start from one location, travel across all the bridges only once, and return to the starting point.





- The Swiss mathematician Euler solved this problem using graph theory.
- This is the first use of graph theory.
- It is Euler that established the theory of graph.
- Euler path is named after him.

Euler Path and Euler Circuit

- Definition:
- An Euler path in G is a path which passes each edge in G exactly once.
- An Euler circuit in G is a circuit which passes each edge in G exactly once.
- An Euler graph is a graph which contains a Euler circuit.

How to determine

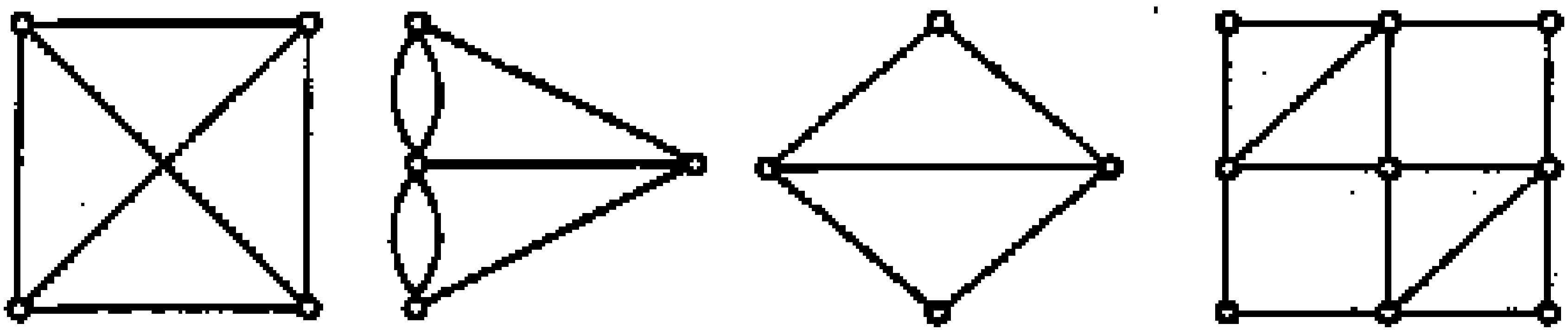
Theorem:

G is an Euler graph(has an Euler circuit) if and only if , for any vertex v in G, $d(v)$ is even.

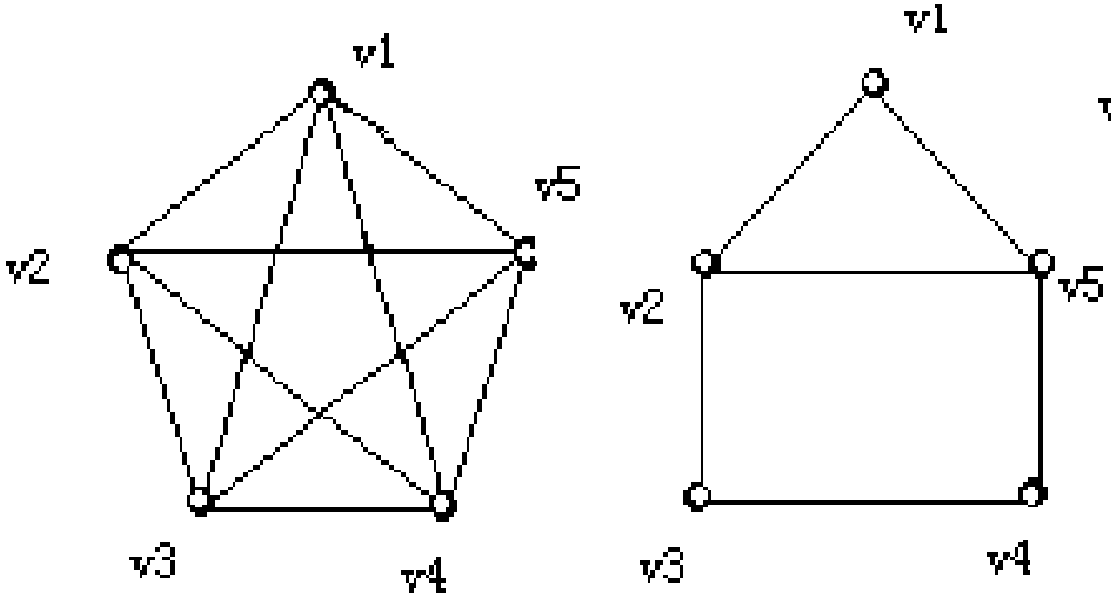
$\Rightarrow$There are three types of vertices: starting, terminal, other vertices passed through. The circuit enters via an edge incident with these vertices and leaves via another edge. Then it contributes two to the degree of these vertices. For circuit, starting and terminal are the same vertex, so this vertex's degree is even.

- $\Leftarrow$ when the degrees of all vertices are even, now, we can find a Euler circuit.
- Start from any vertex, because its degree is even, so there are at least two edges incident with this vertex, we come out via any of these edges, and reach the second vertex, then come out via another edge, different from the previous edge, until we entered the starting vertex via another edge, different from the first edge. If we walk all edges, this is circuit, if some edges are left, continue to add edges left using the above ways.

- Therorem
- A connected multigraph has an Euler path if and only if it has exactly two vertices of odd degree.



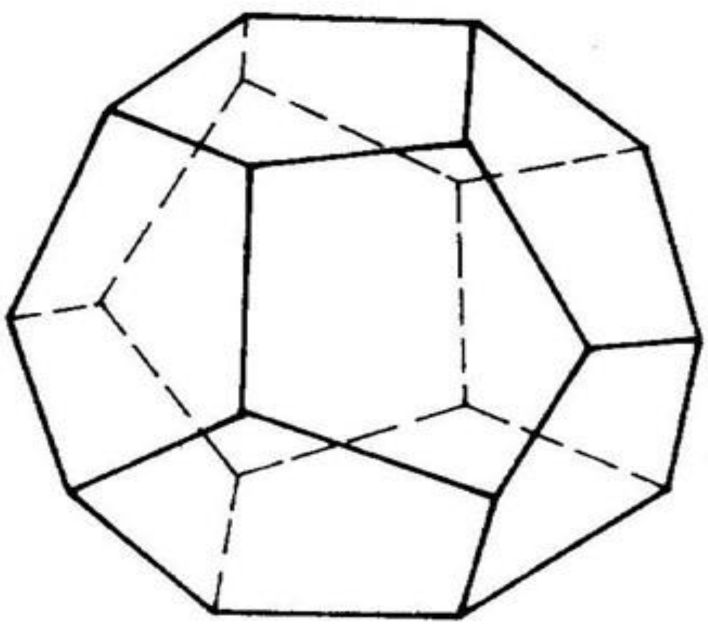
(1) (2) (3) (4)

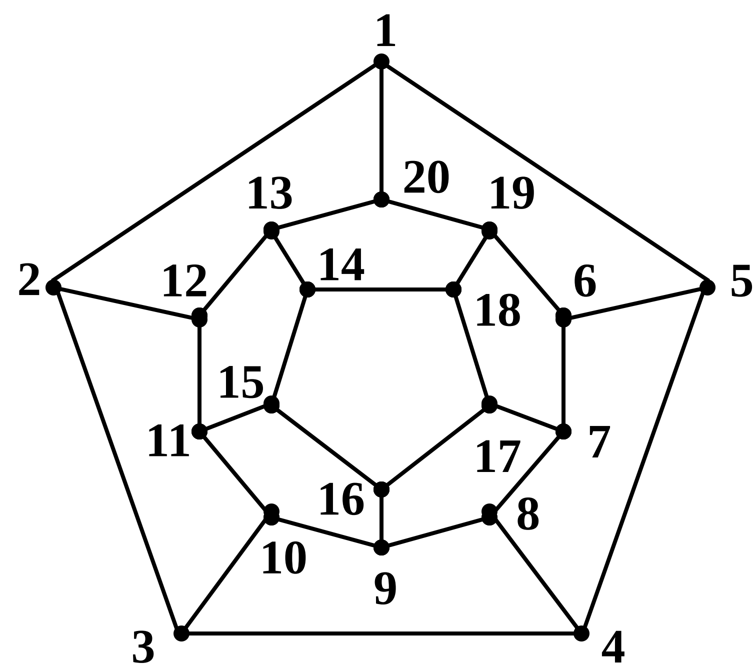


All Around the World

- In 1859, an Irish mathematician, Hamilton, introduced the game called “All around the world”

The player is required to start from a city, pass every city exactly once, along the edges, and return to the starting city finally.

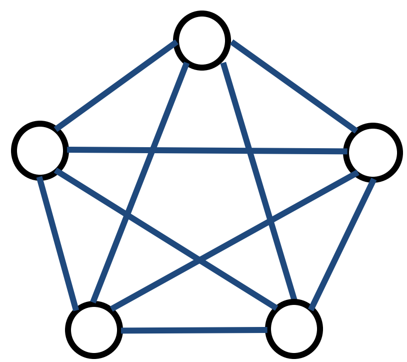




Hamilton Path and Circuit

- Definition:
- In a graph G, a circuit is called a ***Hamilton circuit*** if and only if it contains all the vertices in G exactly once. If G contains a Hamilton circuit, G itself is called a ***Hamilton graph***.
- A Hamilton path is a simple path which contains all vertices exactly once.
- If we pass an edge more than once, it cause us to pass vertex incident with this edge more than once.

There exists edges between any two vertices



The more edges the graph have, the more likely it is that this graph is ***Hamilton***

A Sufficient Condition for Hamilton Graph

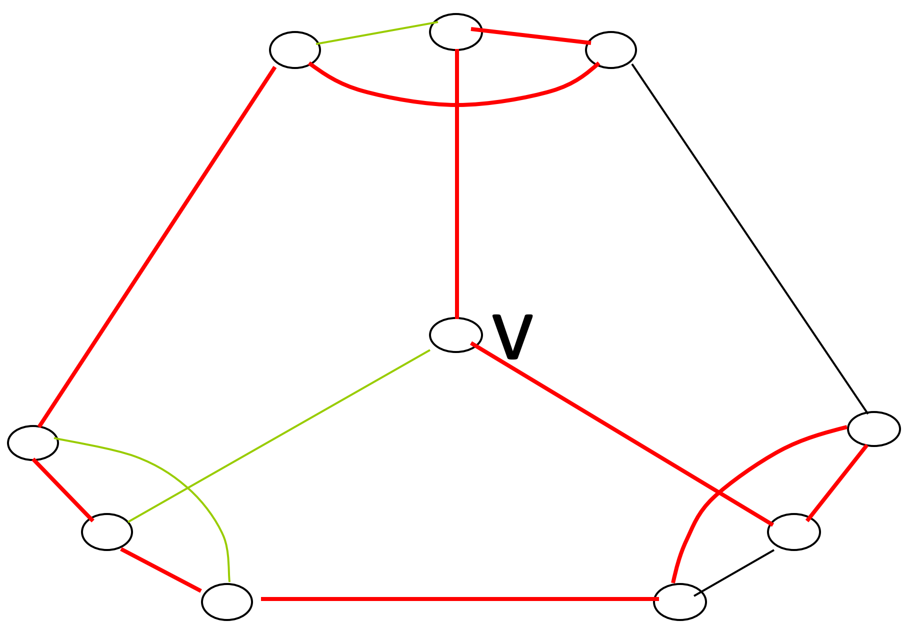
- G is a simple graph with n vertices with not less than 3. If for any vertices *u,v* not adjacent in G:

  $d(u)+d(v)\geq n$ (n is $|V_G|$)

  Then G is a Hamilton graph.

- G is a simple graph with n vertices with not less than 3. **$d(v)\geq n/2$,** then G is a Hamilton graph.

- G is Hamilton graph , but d(v) = 3 < 10/2



Homework

- P703
- 1-8, 30-32

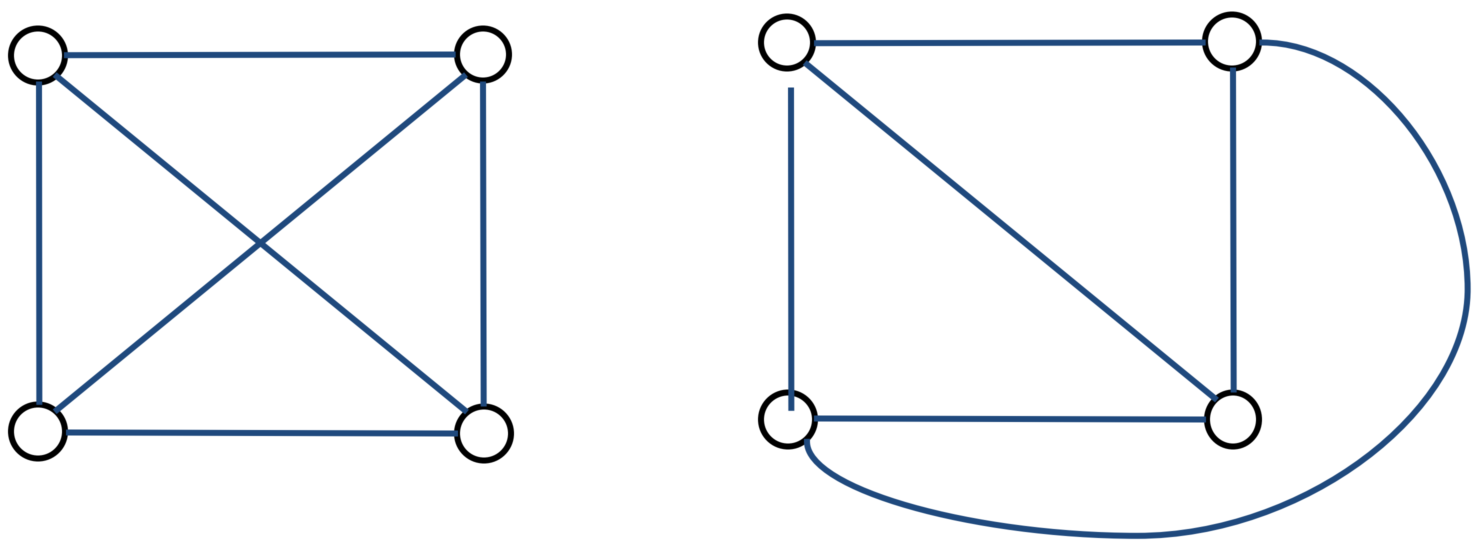
Planar Graphs

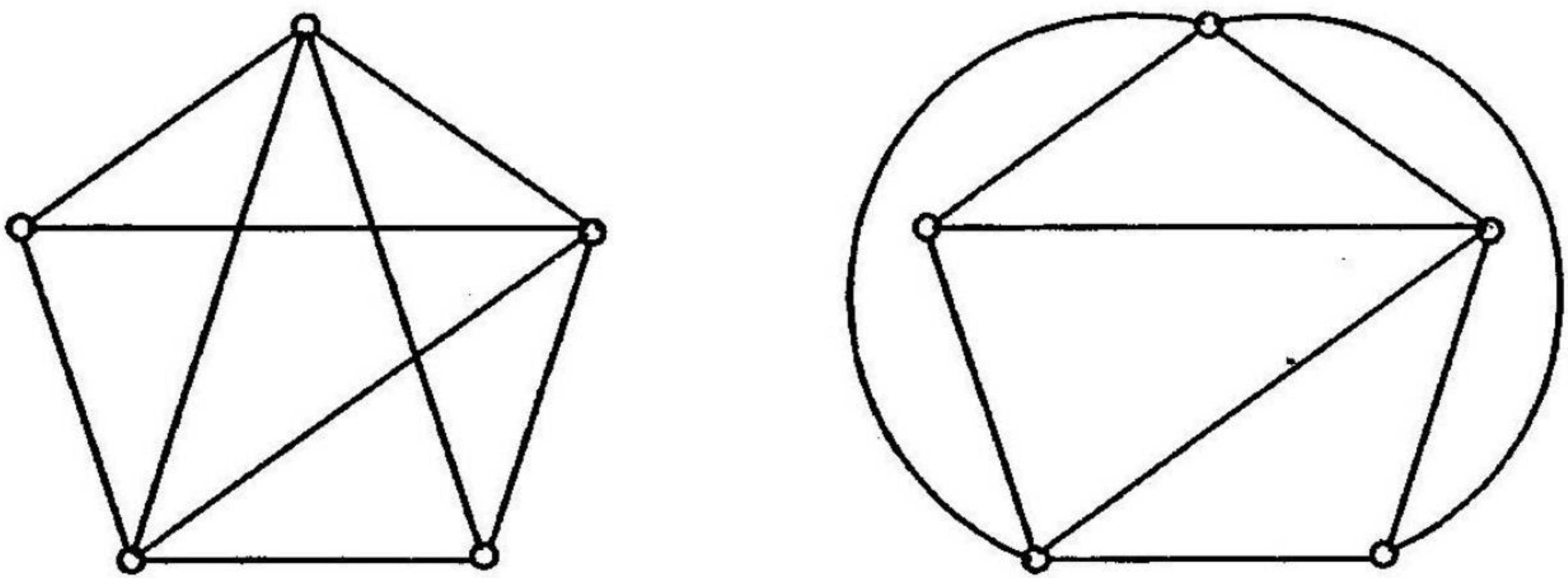
- Driveway design in a big factory: transporting goods from one storehouse to another, try to design such way without crossing in order to avoid trafic accidents.
- Circuit design: try to design such way without crossing in order to avoid mutual effect between different elements

Planar Graphs

- Definition: A graph is called ***planar*** if it can be drawn in the plane without any edges crossing. Such a drawing is called a *planar representation* of the graph

K_4



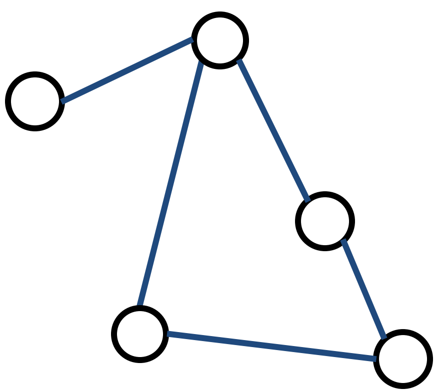


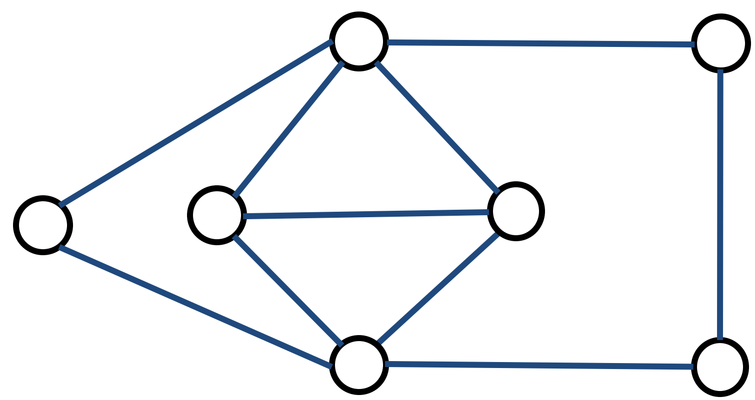
(a) (b)

Regions, faces

- A planar representation of a graph splits the plane into regions that we call ***faces***, including an unbounded region.

two faces

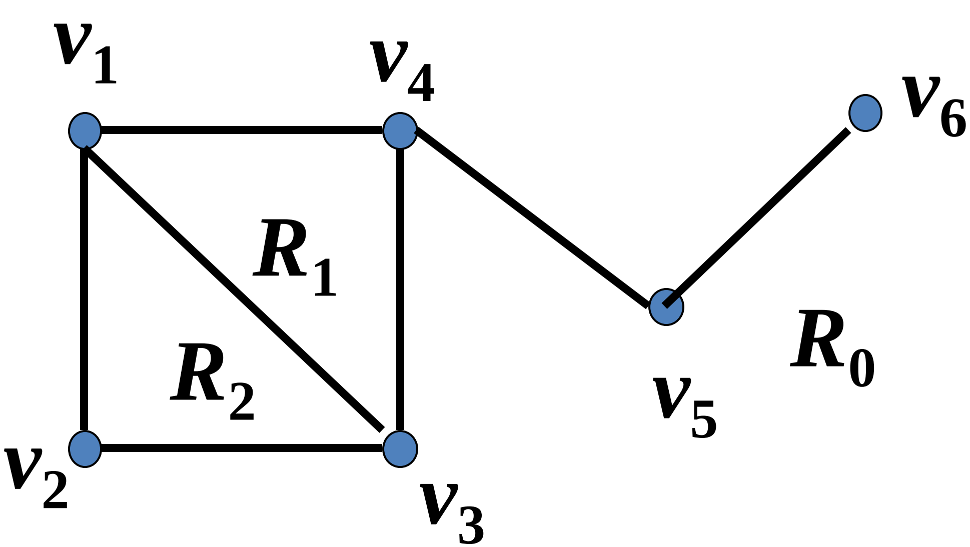




The degree of a region

- The number of edges on the boundary of a region is called the degree of this region.
- The boundary of every region must be a cycle.

Including unbounded region



R_0, R_1, R_2；

R_1： $v_1v_3v_4v_1$， $\deg(R_1) = 3$;

R_2： $v_1v_2v_3v_1$， $\deg(R_2) = 3$；

R_0： $v_1v_2v_3v_4v_5v_6v_5v_4v_1$， $\deg(R_0) = 8$。

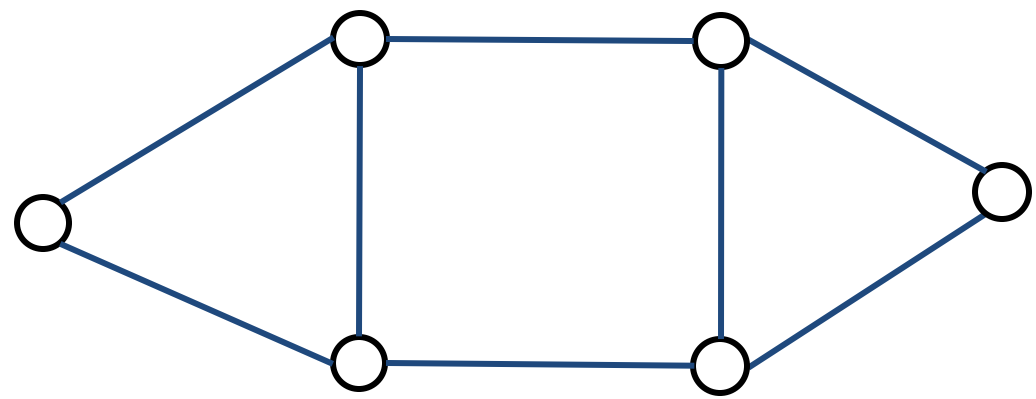
Important conclusion

- The sum of the degree of the regions is exactly twice the number of edges in the planar graph.

Euler Theorem

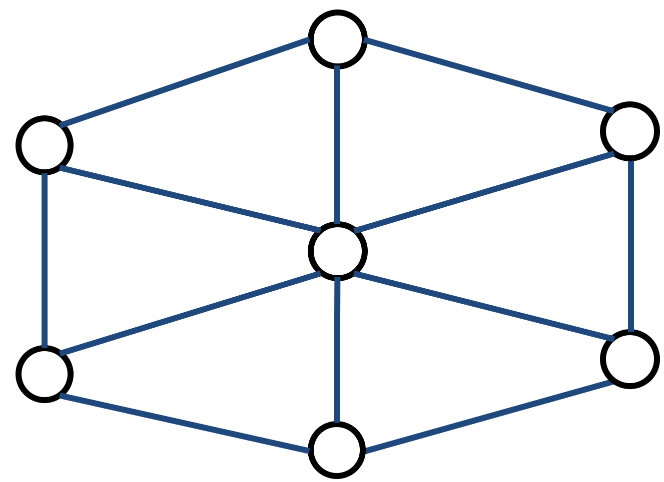
- Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then
- $r = e - v + 2$.

- This graph has
  - 6 vertices
  - 8 edges and
  - 4 faces



vertices – edges + faces = 2

- This graph has
  - 7 vertices
  - 12 edges and
  - 7 faces



- vertices – edges + faces = 2

Corollary 1

- If G is a connected planar simple graph with e edges and v vertices, where v >= 3, then
- e<=3v-6
- Proof: the degree of each region is at least three. No multiple edges produce regions of degree two, no loops produce regions of degree one.
- 2*e*>=3*r*

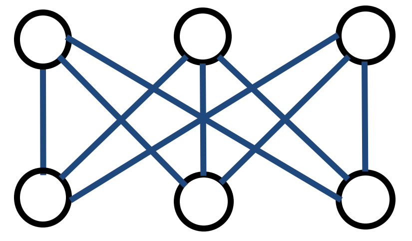
$$v - e + r = 2$$

$$3v - 3e + 3r = 6$$

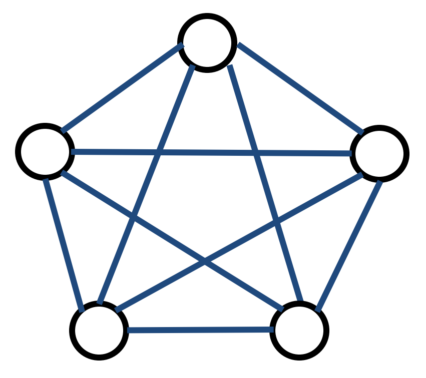
$$3v - 3e + 2e \geq 6$$

$$e \leq 3v - 6$$

$K_{3,3}$



K_5



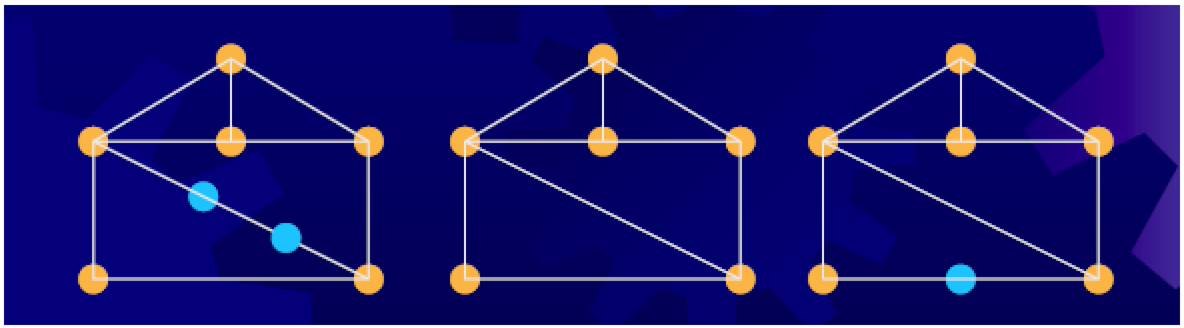
Corollary 2 and 3

- If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.
- If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuit of length three, then $e \leq 2v-4$

Kuratowski's theorem

- Elementary subdivision(初等细分)
- Removing an edge {u, v} and adding a new vertex w together with edges {u, w} and {w, v}

- The graphs G1 and G2 are called homeomorphic(同胚) if they can be obtained from the same graph by a sequence of elementary subdivisions.



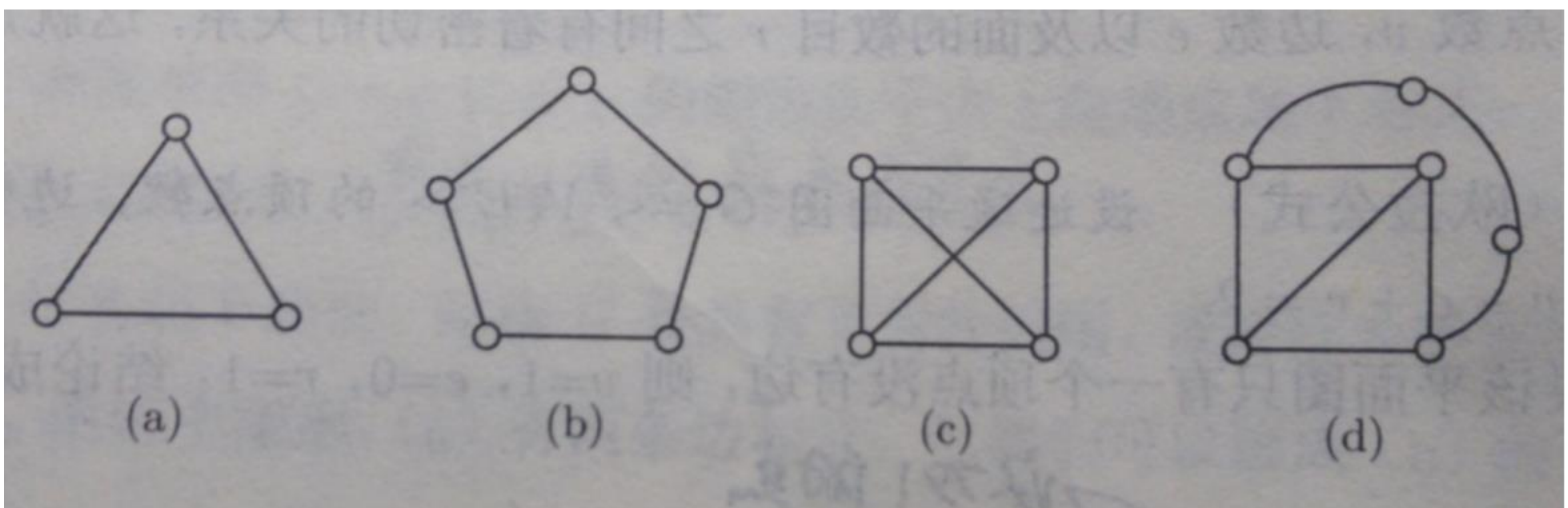
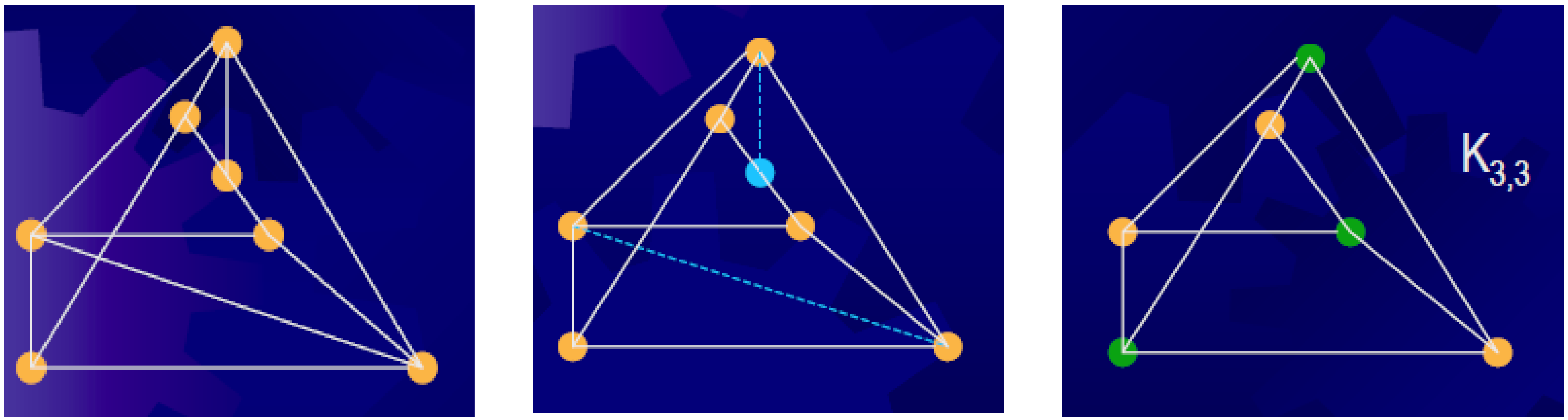


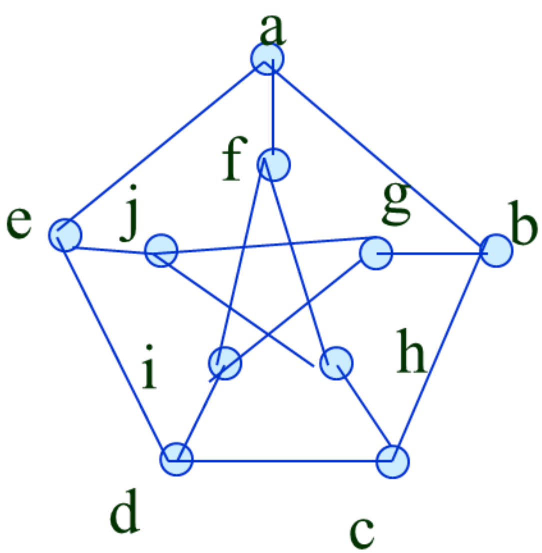
图 5.23

Conclusions

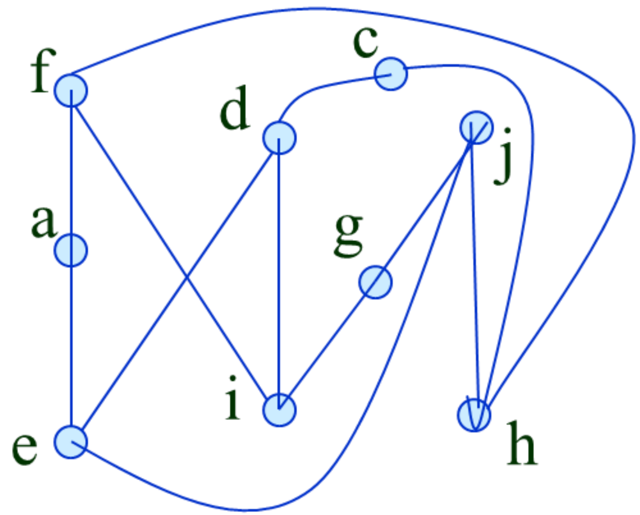
- If two graphs are homeomorphic, both of them are either planar or nonplanar.

- THEOREM A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

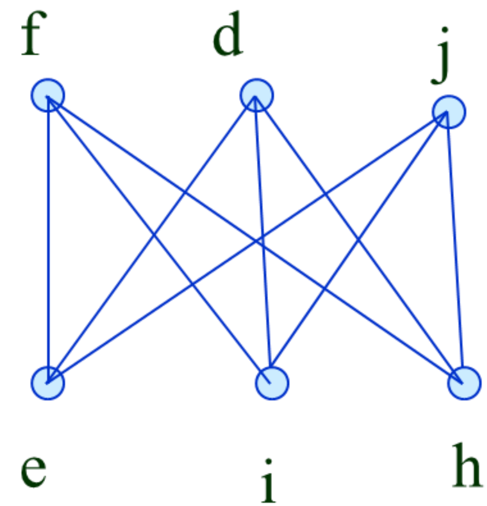

$K_{3,3}$



(a)



(b)H



homework

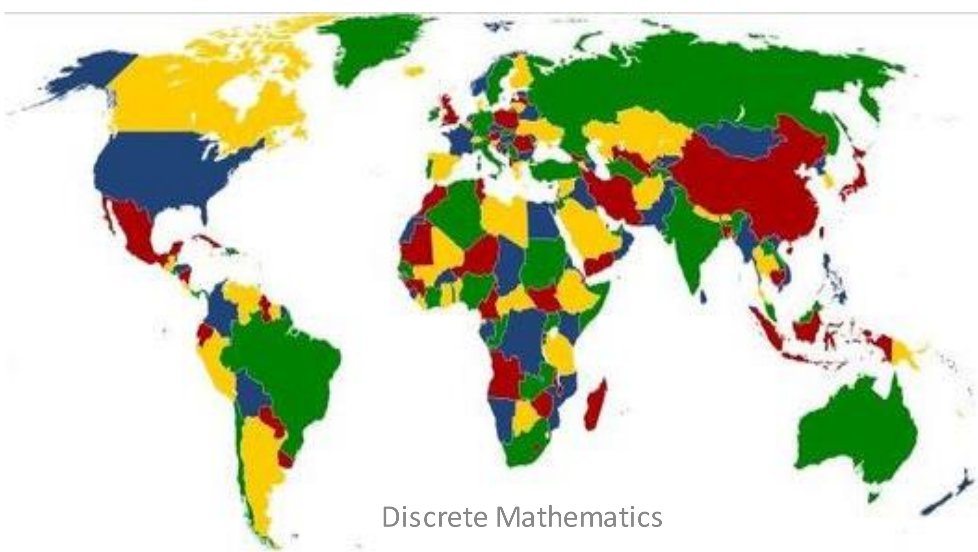
- P725
- 2-4
- 13,14,20,21,23,24

Graph Coloring

Four Colors Conjecture

Four colors are always enough to color any map drawn on a plane

- This conjecture was proved to be true in 1976 with the aid of computer computations performed on almost 2,000 configurations of graphs. There is still no proof known that does not depend on computer checking.



- Each map in the plane can be denoted by a graph
- Each region of the map is represented by a vertex
- if this regions represented by these vertices have a common border, use edges connect two vertices.
- The problem of coloring map becomes the problem of coloring the vertices so that no two adjacent vertices have the same color.

Definitions

- A coloring of a simple graph is the assignment of a color to each vertex if any two adjacent vertices *v* and *u* have different colors.
- The smallest number of colors needed to produce a coloring of a graph *G* is called the *chromatic number of G*(*G*的着色数), denoted by $\chi(G)$.

The four color theorem

- The chromatic number of a planar graph is no greater than four.

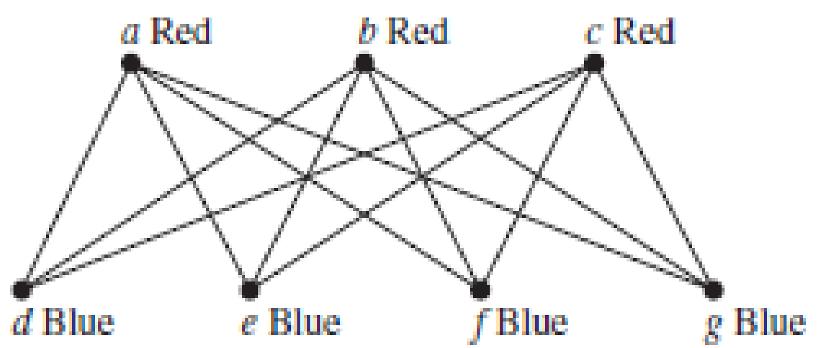
Note that the four color theorem applies only to planar graphs. Nonplanar graphs can have arbitrarily large chromatic numbers.

Example 1 p729

Example 2 p730

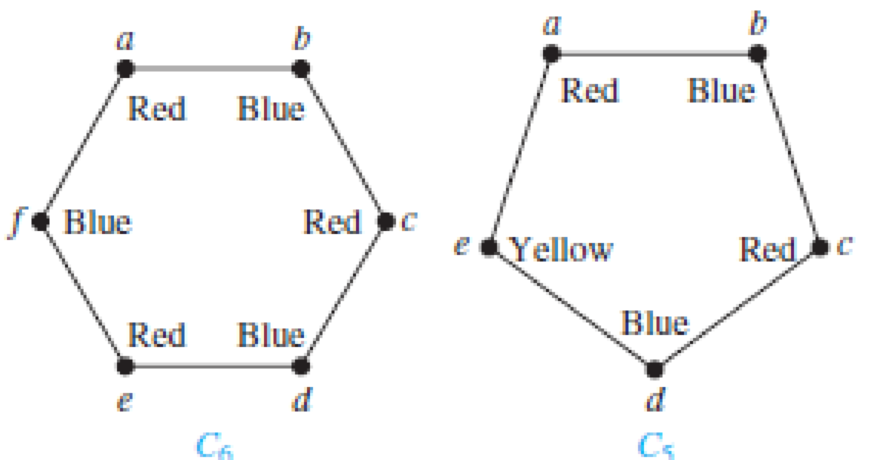
Example

- What is the chromatic number of the complete bipartite graph $k_{m,n}$ where m and n are positive integers?



Example

- What is the chromatic number of the graph C_n?
- If n is an even positive integer, $\chi(Cn)$ =2
- If n is an odd positive integer, $\chi(Cn)$ =3



Example

- Fifteen different foods are to be held in refrigerated compartments within the same refrigerator.
- Construct a graph *G* as follows.
  - Construct one vertex for each food and connect two with an edge if they must be kept in separate compartments in the refrigerator.
  - Then $\chi(G)$ is the smallest number of separate containers needed to store the 15 foods properly.

- How can the final exams at a university be scheduled so that no student has two exams at the same time? p731

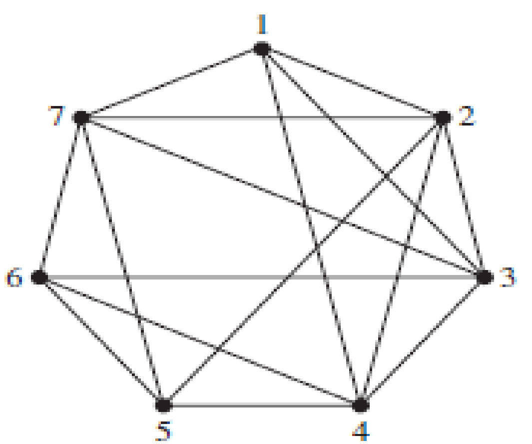
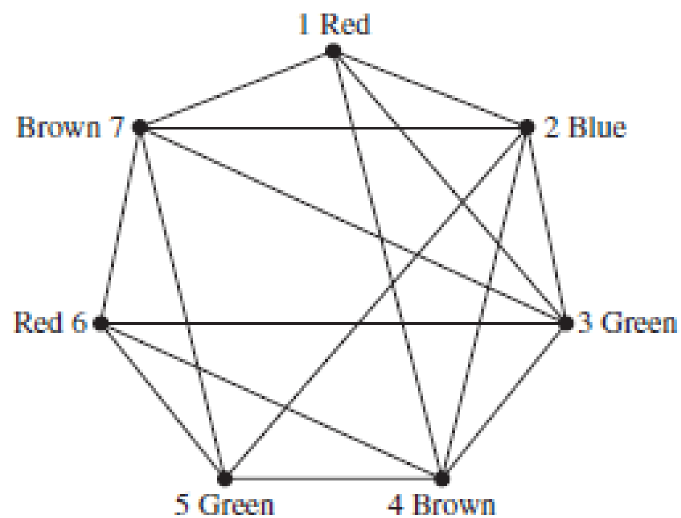


FIGURE 8 **The Graph Representing the Scheduling of Final Exams.**



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

FIGURE 9 **Using a Coloring to Schedule Final Exams.**

Shortest-path problems

- In many applications, each edge of a graph has an associated numerical value, called a weight. The weight of an edge is often referred to as the "cost" of the edge.
- In applications, the weight may be a measure of the length of a route, the capacity of a line, the energy required to move between locations along a route, etc.
- Usually, the edge weights are nonnegative integers.

Shortest Paths

- shortest-path: Given two vertices A and B, there are more than one paths from A to B. The path with minimum cost is called shortest path.
- There are several different algorithms to find a shortest path.

Dijkstra's Algorithm

- We assume that there is a path from the source vertex v0 to every other vertex in the graph.
- Let S be the set of vertices whose minimum distance from the source vertex has been found. Initially S contains only the source vertex.
- The algorithm is iterative, adding one vertex to S on each pass.
- We maintain an array D such that for each vertex v, D[v] is the minimum distance from the source vertex to v via vertices that are already in S.
- Every subpath is the shortest path in this whole shortest path.

- STEPS:

1. let S={v0}, compute D[i] for each vertex vi as following:

$$D[i] \begin{cases} \mathbf{0} & \textbf{if } i=0 \\ \mathbf{w_{0i}} & \textbf{if } i\neq 0\textbf{, and } \langle v_0, v_i\rangle \textbf{ is an edge, } w_{si} \textbf{ is the weight} \\ \boldsymbol{\infty} & \textbf{if } i\neq 0\textbf{, and } \langle v_0, v_i\rangle \textbf{ is not an edge} \end{cases}$$

2. choose the vertex vj such that:

   $$D[j] = \min \{ D[k] \mid vk \notin S \}$$

   then the vj is starting of the next shortest path and D[j] is its cost.

STEPS:

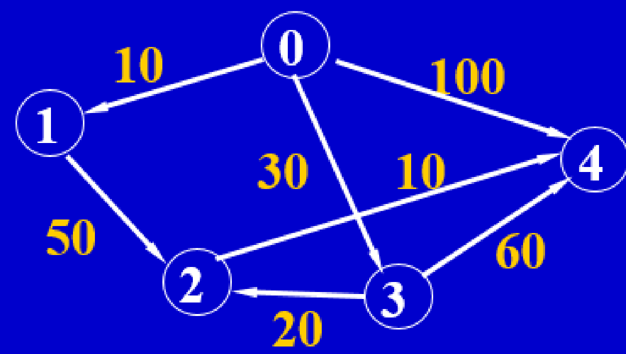
3. Place vj in S. That is

$$S = S \cup \{vj\}$$

4. For each vk$\notin$S, modify the D[k]:

$$D[k] = \min \{ D[k],\ D[j]+weight(<vj,vk>)\}$$

5. Repeat 2---4 until all vertices have been added in S.



Steps	S	D[0]	D[1]	D[2]	D[3]	D[4]
begin	{ 0 }	0	10	∞	30	100
1	{ 0, 1 }	0	10	60	30	100
2	{ 0, 1, 3 }	0	10	50	30	90
3	{ 0, 1, 3, 2 }	0	10	50	30	60
4	{ 0, 1, 3, 2, 4 }	0	10	50	30	60

homework

- P716
- 2， 3
- P732
- 2，3， 6，7