(1)

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Métodos Computacionales.
⇒ Fourier.

1. Si $f(t)$ es continua cuando $-T/2 \leq t \leq T/2$ con $f(-T/2) = f(T/2)$, y si la derivada $f'(t)$ es continua por tramos y diferenciable; entonces la serie de Fourier:

$$(1)\quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

se puede diferenciar término por término para obtener:

$$(2)\quad f'(t) = \sum_{n=1}^{\infty} n\omega_0(-a_n \sin(n\omega_0 t) + b_n \cos(n\omega_0 t))$$

Sea $f(t)$ continua por tramos en el intervalo $-T/2 \leq t \leq T/2$ y sea $f(t+T) = f(t)$. Demostrar que la serie de Fourier se puede integrar término por término para obtener:

$$(3)\quad \int_{t_1}^{t_2} f(t)\,dt = \frac{1}{2} a_0 (t_2 - t_1) + \sum_{n=1}^{\infty} \frac{1}{n\omega_0}\Big[-b_n(\cos(n\omega_0 t_2) - \cos(n\omega_0 t_1))$$
$$+ a_n(\operatorname{Sen}(n\omega_0 t_2) - \operatorname{Sen}(n\omega_0 t_1))\Big]$$

- Se quiere mostrar que $f'(t) = \sum_{n=1}^{K} f_n'(t)$; para esto es necesario comprobar que $\sum_{n=1}^{\infty} f_n(t)$ converge uniformemente

En este caso; que la $\underbrace{\sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))}_{g_n(t)}$ es convergente uniforme.

i.e. $\{\exists M_n \mid |g_n(t)| \leq M_n \forall n \geq 1\}$

$g_n(t) = C_n e^{in\omega_0 t}$ ⇝ Representación de Fourier

$\longrightarrow |g_n(t)| = |C_n|$

Dado que $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} \Rightarrow$ Converge y está acotado $\forall n \in \mathbb{N}$

Apoyados en la relación de Parseval:

$$\frac{1}{T}\int_{-T/2}^{T/2} \{f(t)\}^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2$$

$\rightarrow$ Si $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt \leq \left|\frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt\right|$

$$\leq \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|\,dt \leq \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2$$

↳ convergente

②

Si $M_n = |C_n|^2 \rightarrow |g_n(t)| = |C_n e^{in\omega_0 t}| \leq |C_n|^2$ //

y $\sum_{n=-\infty}^{\infty} |C_n|^2$ converge //

↦ Se cumplen las dos condiciones y $f(t)$ converge uniformemente; por lo tanto, se puede diferenciar término a término

$$f'(t) = \sum_{n=1}^{\infty} f_n'(t)$$

$$= \sum_{n=1}^{\infty} n\omega_0(-a_n \sin(n\omega_0 t) + b_n \cos(n\omega_0 t))$$

$$\rightarrow \int_{t_1}^{t_2} f(t)\,dt = \int_{t_1}^{t_2} \left(\sum_{n=1}^{\infty}(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))\right) + \frac{a_0}{2}\,dt$$

$$= \frac{a_0}{2}(t_2 - t_1) + \sum_{n=1}^{\infty} \int_{t_1}^{t_2} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\,dt$$

$$= \frac{a_0}{2}(t_2 - t_1) + \sum_{n=1}^{\infty} \left[\frac{1}{n\omega_0}\left(a_n \operatorname{sen}(n\omega_0 t) - b_n \cos(n\omega_0 t)\right)\right]_{t_1}^{t_2}$$

$$= \frac{a_0}{2}(t_2 - t_1) + \sum_{n=1}^{\infty} \frac{1}{n\omega_0}\left[a_n \operatorname{Sen}(n\omega_0 t_2) - a_n \operatorname{Sen}(n\omega_0 t_1) - b_n \cos(n\omega_0 t_2) + b_n \cos(n\omega_0 t_1)\right]$$

$$= \frac{a_0}{2}(t_2 - t_1) + \sum_{n=1}^{\infty} \frac{1}{n\omega_0}\left[a_n\left(\sin(n\omega_0 t_2) - \sin(n\omega_0 t_1)\right) - b_n\left(\cos(n\omega_0 t_2) - \cos(n\omega_0 t_1)\right)\right]$$

//

1.2 Encontrar la serie de fourier de la función $f(t) = t$ para el intervalo $(-\pi, \pi)$ y $f(t) = f(t + 2\pi)$

La serie: $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$

pero; $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1.$

$$\rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt))$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos(nt)\,dt$$

$u = t \rightarrow du = dt$

$v = \frac{1}{n}\operatorname{Sen}(nt)$ $dv = \cos(nt)$

$$= \frac{1}{\pi}\left[\frac{t}{n}\sin(nt)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{n}\sin(nt)\,dt\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\sin(n\pi) + \frac{\pi}{n}\sin(-n\pi) + \frac{1}{n^2}\cos(nt)\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi}{n}\sin(n\pi) - \frac{\pi}{n}\sin(n\pi)} + \cancel{\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\underbrace{\cos(-n\pi)}_{n\pi}}\right]$$

$$\Rightarrow a_n = 0$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)\,dt$$

$u = t$
$du = dt$
$v = -\frac{1}{n}\cos($
$dv = \sin(nt)$

$$= \frac{1}{\pi}\left[-\frac{t}{n}\cos(nt)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{n}\cos(nt)\,dt\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos(\pi n) - \frac{\pi}{n}\underbrace{\cos(-\pi n)}_{\cos(\pi n)} + \frac{1}{n^2}\cancel{\sin(nt)}\Big|_{-\pi}^{\pi} \xrightarrow{} 0\right]$$

$$= \frac{1}{\pi}\left(-\frac{2\pi}{n}\cos(\pi n)\right)$$

$$= -\frac{2}{n}\cos(\pi n)$$

$$\rightarrow b_n = -\frac{2}{n}\cos(\pi n)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = \frac{1}{\pi}\frac{t^2}{2}\Big|_{-\pi}^{\pi} = 0$$

$$\Rightarrow a_0 = 0$$

$$\Rightarrow f(t) = \sum_{n=1}^{\infty} -\frac{2}{n}\overbrace{\cos(\pi n)}^{\substack{1 \text{ si } n \text{ es par} \\ -1 \text{ si } n \text{ es impar}}}\sin(nt)$$

$$= \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n\sin(nt) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin(nt)$$

1.3 Función de Riemann $\zeta(s)$

1. Integrar la serie de Fourier de $f(t)=t^2$ en $-\pi\leq t\leq \pi$ y $f(t+2\pi)=f(t)$.

1.a. Encontrar la serie para $f(t)=t^2$; $\omega_0=1$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nt)+b_n\sin(nt))$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}t^2\,dt=\frac{1}{\pi}\frac{t^3}{3}\Big|_{-\pi}^{\pi}=\frac{\pi^2}{3}+\frac{\pi^2}{3}=\frac{2}{3}\pi^2$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}t^2\cos(nt)\,dt \Rightarrow$$

u	dv
t^2	$\cos(nt)$
$2t$	$\frac{1}{n}\sin(nt)$
2	$-\frac{1}{n^2}\cos(nt)$
0	$-\frac{1}{n^3}\sin(nt)$

$$\Rightarrow a_n=\frac{1}{\pi}\left[\frac{t^2}{n}\sin(nt)\Big|_{-\pi\to 0}^{\pi\to 0}+\frac{2t}{n^2}\cos(nt)\Big|_{-\pi}^{\pi}+\frac{2}{n^3}\sin(nt)\Big|_{-\pi\to 0}^{\pi\to 0}\right]$$

$$=\frac{1}{\pi}\left[+\frac{2\pi}{n^2}\cos(\pi n)+\frac{2\pi}{n^2}\cos(\overset{-\pi n}{\pi n})\right]$$

$$=+\frac{4}{n^2}\cos(\pi n)$$

$(+)$	t^2	$\sin(nt)$
$(-)$	$2t$	$-\frac{1}{n}\cos(nt)$
	2	$-\frac{1}{n^2}\sin(nt)$
	0	$\frac{1}{n^3}\cos(nt)$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}t^2\sin(nt)\,dt$$

$$=\frac{1}{\pi}\left[-\frac{t^2}{n}\cos(nt)\Big|_{-\pi}^{\pi}+\frac{2t}{n^2}\sin(nt)\Big|_{-\pi\to 0}^{\pi\to 0}+\frac{2}{n^3}\cancel{\cos(nt)}\Big|_{-\pi}^{\pi}\right]$$

$\hookrightarrow 0$ Cos es par

$$=\frac{1}{\pi}\left(-\frac{\pi^2}{n}\cancel{\cos(\pi n)}+\frac{(-\pi)^2}{n}\cancel{\cos(\pi n)}\right) \quad \text{par}$$

$$=0$$

$$\Rightarrow f(t)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}+\frac{4}{n^2}\cos(\pi n)\cos(nt)$$

$$=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nt)$$

2do. Integral.

$$\int f(t) = \int_{-\pi}^{\pi} \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt)\, dt$$

$$\Rightarrow \int t^2 dt = \frac{\pi^2}{3} t + 4\sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \int \cos(nt)\, dt \right]$$

$$\frac{t^3}{3} = \frac{\pi^2}{3} t + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot \frac{1}{n} \sin(nt)$$

$$\Rightarrow \frac{t^3}{3} - \frac{\pi^2}{3} t = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \cdot \sin(nt)$$

$$\Rightarrow \boxed{\frac{t}{12}(t^2 - \pi^2) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nt)} \; *$$

2, De la identidad de Parseval:

$$\frac{1}{T}\int_{-T/2}^{T/2} [f(t)]^2 dt = \frac{1}{4} a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Note de $*$ que $b_n = \frac{(-1)^n}{n^3}$ y $a_0 = 0$, $T = 2\pi$

y $f(t) = \frac{t}{12}(t^2 - \pi^2)$

$$\Rightarrow$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\frac{t}{12}(t^2 - \pi^2)\right]^2 dt = \frac{1}{2}\sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^3}\right]^2$$

$$\Rightarrow \frac{1}{\cancel{2}\pi}\int_{-\pi}^{\pi} \frac{t^2}{144}(t^4 - 2t^2\pi^2 + \pi^4)\, dt = \frac{1}{\cancel{2}}\sum_{n=1}^{\infty} \frac{\overbrace{(-1)^{2n}}^{1}}{n^6}$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t^6}{144} - \frac{t^4}{72}\pi^2 + \frac{t^2}{144}\pi^4\, dt = \sum_{n=1}^{\infty} \frac{1}{n^6}$$

$$\Rightarrow \frac{1}{\pi}\left[\frac{t^7}{7\cdot 144}\Big|_{-\pi}^{\pi} - \frac{t^5}{5\cdot 72}\pi^2\Big|_{-\pi}^{\pi} + \frac{t^3}{3\cdot 144}\pi^4\Big|_{-\pi}^{\pi} \right] = \sum_{n=1}^{\infty} \frac{1}{n^6}$$

$$\rightarrow \frac{1}{\pi}\left[\frac{2\pi^7}{7\cdot 144} - \frac{2\cdot\pi^5}{5\cdot 72}\cdot\pi^2 + \frac{2\cdot\pi^6}{3\cdot 144}\right] = \sum_{n=1}^{\infty}\frac{1}{n^6}$$

$$\frac{\pi^6}{504} - \frac{\pi^6}{180} + \frac{\pi^6}{216} = \sum_{n=1}^{\infty}\frac{1}{n^6}$$

$$\rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}}$$