We have a simple regression model: $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$

The least squares estimator can be found by minimising the squared Euclidean distance function:

$$Q = d(b_0, b_1) = \sum_{i=1}^{n} \left(Y_i - b_0 - b_1 x_i\right)^2$$

The estimates for β_0 and β_1 that minimise the above distance function can be found by solving the following normal equations:

$$\frac{\partial d}{\partial b_0} = -2\sum_{i=1}^{n}\left(Y_i - b_0 - b_1 x_i\right) = 0$$
$$\frac{\partial d}{\partial b_1} = -2\sum_{i=1}^{n} x_i\left(Y_i - b_0 - b_1 x_i\right) = 0$$

A bit of algebra simplifies the two equations:

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n}\left(b_0 + b_1 x_i\right) = nb_0 + b_1\sum_{i=1}^{n} x_i = n\left(b_0 + b_1\bar{x}\right)$$
$$\sum_{i=1}^{n} x_i Y_i = b_0\sum_{i=1}^{n} x_i + b_1\sum_{i=1}^{n} x_i^2 = n\left(b_0\bar{x} + b_1\bar{x}^2\right) + b_1 S_{xx}$$

Where,

$$\begin{aligned}
S_{xx} &= \sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2 = \sum_{i=1}^{n}\left(x_i^{\,2} - 2x_i\bar{x} + \bar{x}^2\right) \\
&= \sum_{i=1}^{n} x_i^{\,2} - \sum_{i=1}^{n} x_i\bar{x} - \sum_{i=1}^{n} x_i\bar{x} + \sum_{i=1}^{n}\bar{x}^2 \\
&= \sum_{i=1}^{n}\left(x_i - \bar{x}\right)x_i - \bar{x}\sum_{i=1}^{n} x_i + n\bar{x}^2 \\
&= \sum_{i=1}^{n}\left(x_i - \bar{x}\right)x_i - n\bar{x}^2 + n\bar{x}^2 \\
&= \sum_{i=1}^{n}\left(x_i - \bar{x}\right)x_i
\end{aligned}$$

Solving the first equations gives us: $b_0 = \bar{Y} - b_1\bar{x}$

Substituting this in the second equation gives us:

$$\sum_{i=1}^{n} x_i Y_i = n\left(\left(\overline{Y} - b_1\overline{x}\right)\overline{x} + b_1\overline{x}^2\right) + b_1 S_{xx} = n\overline{x}\overline{Y} + b_1 S_{xx}$$

Thus,

$$b_1 = \frac{\sum_{i=1}^{n} x_i Y_i - n\overline{x}\overline{Y}}{S_{xx}} = \frac{\sum_{i=1}^{n} x_i Y_i - \sum_{i=1}^{n} \overline{x} Y_i}{S_{xx}} = \frac{\sum_{i=1}^{n} \left(x_i - \overline{x}\right) Y_i}{S_{xx}} = \frac{S_{xy}}{S_{xx}}$$

Where,

$$\begin{aligned} S_{xy} &= \sum_{i=1}^{n} \left(x_i - \overline{x}\right)\left(Y_i - \overline{Y}\right) = \sum_{i=1}^{n} \left[\left(x_i - \overline{x}\right) Y_i - x_i \overline{Y} + \overline{x}\overline{Y}\right] \\ &= \sum_{i=1}^{n} \left(x_i - \overline{x}\right) Y_i - \sum_{i=1}^{n} \left(x_i \overline{Y} - \overline{x}\overline{Y}\right) = \sum_{i=1}^{n} \left(x_i - \overline{x}\right) Y_i \\ &= \sum_{i=1}^{n} x_i Y_i - n\overline{x}\overline{Y} \end{aligned}$$

Thus, our estimates are:

$$b_1 = \frac{S_{xy}}{S_{xx}} \qquad \text{and} \qquad b_0 = \overline{Y} - \frac{S_{xy}}{S_{xx}}\overline{x}$$