An array of identical elements all of identical magnitude and each with a progressive phase is referred to as a uniform array.

The array factor is given by,

$$AF = 1 + e^{j(kd\cos\theta+\beta)} + e^{2j(kd\cos\theta+\beta)} + \ldots + e^{j(n-1)(kd\cos\theta+\beta)}$$

$$AF = \sum_{n=1}^{N} e^{j(n-1)(kd\cos\theta+\beta)}$$

which can be written as, $AF = \sum_{n=1}^{N} e^{j(n-1)\Psi}$..............(4)

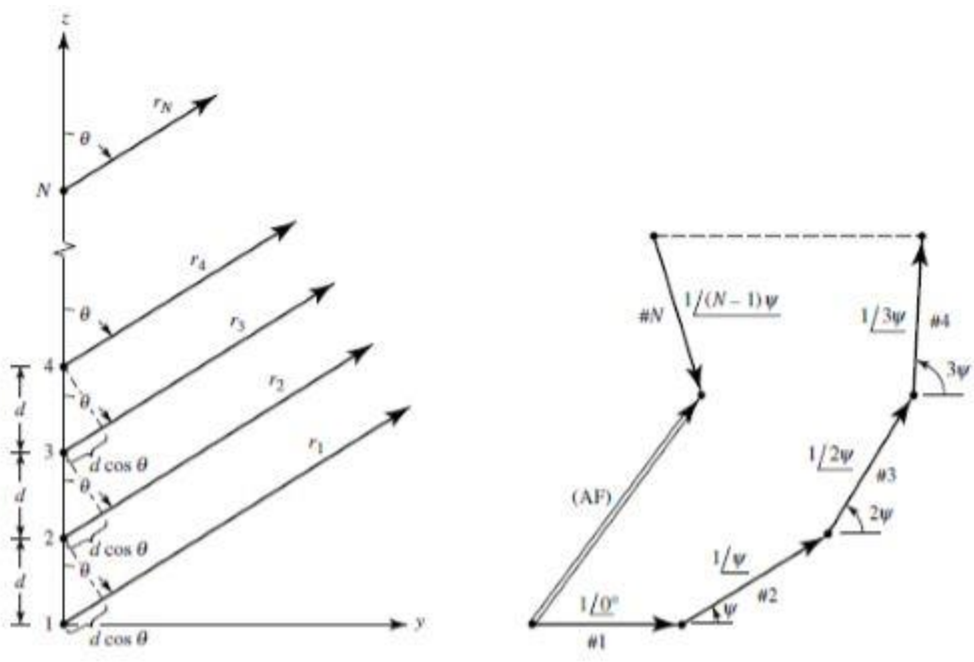
Where, $\Psi = kd\cos\theta + \beta$

Therefore, multiplying both sides of above eq (4) by $e^{j\Psi}$

, it can be written as

$$(AF)e^{j\Psi} = e^{j\Psi} + e^{2j\Psi} + e^{3j\Psi} + \ldots + e^{(N-1)j\Psi} + e^{Nj\Psi}$$

.... (5)



(a) Geometry (b) Phasor diagram

Fig.5 Far-field geometry and phasor diagram of N-element array of isotropic sources positioned along the z-axis

Subtracting eq(4) & eq(5),

$$AF(e^{j\Psi}-1)=(-1+e^{Nj\Psi})............(6)$$

which can also be written as,

$$AF=\left[\frac{e^{Nj\Psi}-1}{e^{j\Psi}-1}\right]=e^{j[(N-1)/2]\Psi}\left[\frac{e^{j(N/2)\Psi}-e^{-j(N/2)\Psi}}{e^{j(1/2)\Psi}-e^{-j(1/2)\Psi}}\right]$$
$$=e^{j[(N-1)/2]\Psi}\left[\frac{sin(\frac{N}{2}\Psi)}{sin(\frac{\Psi}{2})}\right]...............(7)$$

If the reference point is the physical centre of the array, the array factor of the above eq. reduces to,

$$AF=\left[\frac{sin(\frac{N}{2}\Psi)}{sin(\frac{\Psi}{2})}\right]...........(8)$$

For small values of Ψ, the above expression can be approximated by,

$$AF=\left[\frac{sin(\frac{N}{2}\Psi)}{(\frac{\Psi}{2})}\right]...........(9)$$
$$(AF)_n=\frac{N}{2}\left[\frac{sin(\frac{N}{2}\Psi)}{(\frac{\Psi}{2})}\right]...........(10)$$
$$(AF)_n=\left[\frac{sin(\frac{N}{2}\Psi)}{(\frac{N}{2})\Psi}\right]...........(11)$$

To find the nulls of array eq(10) & eq(11) are set equal to zero

$$\sin\left(\frac{N}{2}\psi\right)=0\Rightarrow\frac{N}{2}\psi|_{\theta=\theta_n}=\pm n\pi\Rightarrow\theta_n=\cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta\pm\frac{2n}{N}\pi\right)\right]$$
$$n=1,2,3,\ldots$$
$$n\neq N,2N,3N,\qquad\ldots..(12)\ldots\text{with eq }(9)$$

For n = N, 2N, 3N, . . ., (9) attains its maximum values because it reduces to a sin(0)/0 form. The values of n determine the order of the nulls (first, second, etc.). For a zero to exist, the argument of the arccosine cannot exceed unity. Thus the number of nulls that can exist will be a function of the element separation d and the phase excitation difference β.

The maximum values of (9) occur when

$$\frac{\psi}{2}=\frac{1}{2}(kd\cos\theta+\beta)|_{\theta=\theta_m}=\pm m\pi\Rightarrow\theta_m=\cos^{-1}\left[\frac{\lambda}{2\pi d}(-\beta\pm 2m\pi)\right]$$
$$m=0,1,2,\ldots\qquad\ldots\ldots(13)$$

The array factor of (11) has only one maximum and occurs when m = 0 in (13). That is,

$$\theta_m=cos^{-1}\left(\frac{\lambda\beta}{2\pi d}\right).........(14)$$

which is the observation angle that makes $\psi = 0$.

For the array factor of (11), there are secondary maxima (maxima of minor lobes) which occur approximately when the numerator of (11) attains its maximum value. That is,

$$\sin\left(\frac{N}{2}\psi\right) = \sin\left[\frac{N}{2}(kd\cos\theta + \beta)\right]\Big|_{\theta=\theta_s} \simeq \pm 1 \Rightarrow \frac{N}{2}(kd\cos\theta + \beta)\Big|_{\theta=\theta_s}$$

$$\simeq \pm\left(\frac{2s+1}{2}\right)\pi \Rightarrow \theta_s \simeq \cos^{-1}\left\{\frac{\lambda}{2\pi d}\left[-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right]\right\}.$$

$$s = 1, 2, 3, \ldots \qquad \ldots\ldots (15)$$

which can also be written as,

$$\theta_x \simeq \frac{\pi}{2} - \sin^{-1}\left\{\frac{\lambda}{2\pi d}\left[-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right]\right\}, \qquad s=1,2,3,\ldots. \qquad \ldots\ldots\ldots\ldots.(16)$$

For large values of d($d \gg \lambda$), it reduces to,

$$\theta_x \simeq \frac{\pi}{2} - \frac{\lambda}{2\pi d}\left[-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right], \qquad s=1,2,3,\ldots. \qquad \ldots\ldots\ldots\ldots\ldots\ldots$$