IInd Extension : Consider a binary source 'S' with symbols s_1 & s_2 with probabilities p_1 & p_2 where $p_1+p_2=1$.

Second extension of a binary source will have $2^2=4$ symbols given by.

$s_1 s_1$ with probabilities ; $p_1 p_1 = p_1^2$

$s_1 s_2$ " $p_1 p_2$.

$s_2 s_1$ " $p_2 p_1$ and

$s_2 s_2$ " $p_2 p_2 = p_2^2$.

Sum of all probabilities of the 2nd extended source.

$$p_1^2 + p_1 p_2 + p_2 p_1 + p_2^2 = p_1^2 + 2p_1 p_2 + p_2^2 = (p_1+p_2)^2 = \underline{\underline{1}}.$$

Entropy of basic binary source ;

$$H(S) = \sum_{k=1}^{2} p_k \log \frac{1}{p_k}$$

$$= p_1 \log \frac{1}{p_1} + p_2 \log \frac{1}{p_2}. \quad \text{——— (1)}$$

Entropy of the 2nd extended source ;

$$H(S^2) = \sum_{k=1}^{4} p_k \log \frac{1}{p_k}.$$

$$= p_1^2 \log \frac{1}{p_1^2} + p_1 p_2 \log \frac{1}{p_1 p_2} + p_1 p_2 \log \frac{1}{p_1 p_2} + p_2^2 \log \frac{1}{p_2^2}$$

$$= 2p_1^2 \log \frac{1}{p_1} + 2p_1 p_2 \log \frac{1}{p_1 p_2} + 2p_2^2 \log \frac{1}{p_2}.$$

$$= 2p_1^2 \log \frac{1}{p_1} + 2p_1 p_2 \log \frac{1}{p_1} + 2p_1 p_2 \log \frac{1}{p_2} + 2p_2^2 \log \frac{1}{p_2}.$$

$$= 2p_1 \underbrace{(p_1+p_2)}_{=1} \log \frac{1}{p_1} + 2p_2 \underbrace{(p_1+p_2)}_{=1} \log \frac{1}{p_2}$$

$$= 2\left[p_1 \log \frac{1}{p_1} + p_2 \log \frac{1}{p_2} \right]$$

$$\boxed{H(S^2) = 2\, H(S).} \quad \text{from eq (1)}.$$

Similarly the entropy of 3rd extended source can be

$$\boxed{H(S^3) = 3 \cdot H(S)}$$

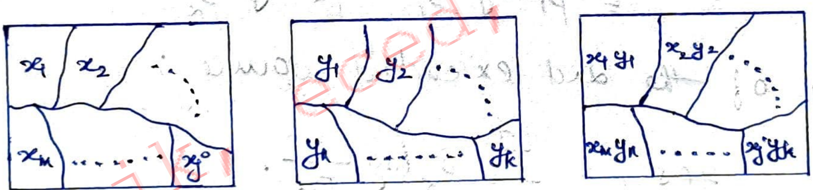
Generalizing, the nth extension of the basic binary source will have 2^n symbols and the entropy of the nth extended source is given by

$$\boxed{H(S^n) = n \cdot H(S)}$$

JOINT ENTROPY AND CONDITIONAL ENTROPY

So far we considered only a single probability scheme ie one dimensional probability scheme; probability belong either to the transmitter or to the receiver. That means we are able to study the behaviour of either transmitter or receiver.

But to study the behaviour of a communication system, we must simultaneously study the behaviour of the transmitter and receiver. This will give the concept of two dimensional probability scheme.



Sample space, S_1 $\qquad$ S_2 $\qquad$ $S = S_1 S_2$

There are two finite discrete sample spaces S_1 and S_2 and their product space will be $S = S_1 S_2$.

Let $[X] = [x_1, x_2, \ldots\ldots x_j \ldots\ldots x_m]$

and $[Y] = [y_1, y_2, \ldots\ldots y_k \ldots\ldots y_n]$

be the set of events in S_1 and S_2 respectively.

Each event x_j of S_1 occur in connection (relation) with any event y_k of S_2.

The complete set of events in $S = S_1 S_2$ is $[XY]$.

$$[XY] = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_k & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_k & \cdots & x_2y_n \\ \vdots & \vdots & & \vdots & & \vdots \\ x_jy_1 & x_jy_2 & \cdots & x_jy_k & \cdots & x_jy_n \\ \vdots & \vdots & & \vdots & & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_k & \cdots & x_my_n \end{bmatrix}$$

The three sets of complete probability schemes:

$$p(X) = [p(x_j)] = [p(x_1)\ p(x_2) \cdots p(x_j) \cdots p(x_m)]$$

$$p(Y) = [p(y_k)] = [p(y_1)\ p(y_2) \cdots p(y_k) \cdots p(y_n)]$$

$$p(XY) = [p(x_j, y_k)] =$$

$$\begin{array}{c|c} & \begin{array}{cccccc} y_1 & y_2 & \cdots & y_k & \cdots & y_n \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_m \end{array} & \begin{bmatrix} p(x_1, y_1) & p(x_1 y_2) & \cdots & p(x_1, y_k) & \cdots & p(x_1, y_n) \\ p(x_2 y_1) & p(x_2 y_2) & \cdots & p(x_2 y_k) & \cdots & p(x_2 y_n) \\ \vdots & \vdots & & \vdots & & \vdots \\ p(x_j y_1) & p(x_j y_2) & \cdots & p(x_j y_k) & \cdots & p(x_j y_n) \\ \vdots & \vdots & & \vdots & & \vdots \\ p(x_m y_1) & p(x_m y_2) & \cdots & p(x_m y_k) & \cdots & p(x_m y_n) \end{bmatrix} \end{array}$$

$p(XY)$ is called "Joint Probability Matrix". Properties of JPM are:

① By adding the elements of JPM columnwise, we can obtain the ~~probabilities~~ probability of output symbols.
$$\sum_{j=1}^{m} p(x_j, y_k) = p(y_k).$$

② By adding the elements of JPM rowwise, we can obtain the probability of input symbols.
$$\sum_{k=1}^{n} p(x_j, y_k) = p(x_j).$$

③ The sum of all the elements of JPM is equal to unity
$$\sum_{j=1}^{m} \sum_{k=1}^{n} p(x_j, y_k) = 1$$