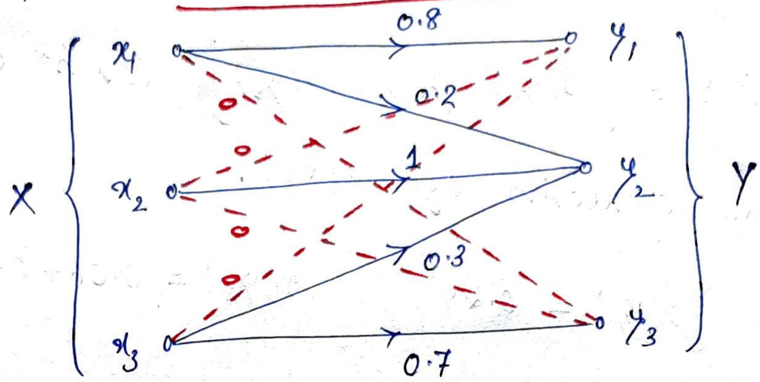
Q7. A discrete source transmits msgs x_1, x_2 and x_3 with probabilities 0.3, 0.4 and 0.3. The source is connected to channel as shown in figure. Calculate all the entropies. Also find the mutual information for the channel.



Sol. The given figure is known as channel diagram. using it we are able to write channel metrix, ie conditional probability matrix $P(Y/x)$.

$$P(Y/x) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \left[\begin{array}{ccc} 0.8 & 0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0.3 & 0.7 \end{array}\right] \end{array} \begin{array}{c} \\ =1 \\ =1 \\ =1 \end{array}$$

Sum of all the rows of $P(Y/x)$ is 1.

Also given $P(X) = \begin{array}{c} \begin{array}{ccc} p(x_1) & p(x_2) & p(x_3) \end{array} \\ \left[\begin{array}{ccc} 0.3 & 0.4 & 0.3 \end{array}\right] \end{array}$

The Joint Probability Matrix $P(XY)$ can be obtained by multiplying the rows of $P(Y/x)$ by $p(x_1)$, $p(x_2)$ and $p(x_3)$; (Not Matrix multiplication, it is elementary multiplication).

Then

$$P(XY) = \begin{bmatrix} 0.8 \times 0.3 & 0.2 \times 0.3 & 0 \times 0.3 \\ 0 \times 0.4 & 1 \times 0.4 & 0 \times 0.4 \\ 0 \times 0.3 & 0.3 \times 0.3 & 0.7 \times 0.3 \end{bmatrix}$$

$$P(XY) = \begin{array}{c|c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.24 & 0.06 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0.09 & 0.21 \end{bmatrix} \end{array}$$

Sum of all the entries of $P(XY)$ is 1

$P(Y) = [p(y_1) \;\; p(y_2) \;\; p(y_3)]$ can be obtained by adding column elements of $P(XY)$.

$$p(y_1) = 0.24 \qquad p(y_2) = 0.06 + 0.4 + 0.09 = 0.55$$

$$p(y_3) = 0.21$$

$$P(Y) = [0.24 \quad 0.55 \quad 0.21]$$

The conditional probability Matrix $P(X/Y)$ can be obtained by dividing the columns of $P(XY)$ by $p(y_1)$, $p(y_2)$ and $p(y_3)$ respectively (Elementary Division).

$$P(X/Y) = \begin{bmatrix} \frac{0.24}{0.24} & \frac{0.06}{0.55} & 0 \\ 0 & \frac{0.4}{0.55} & 0 \\ 0 & \frac{0.09}{0.55} & \frac{0.21}{0.21} \end{bmatrix}$$

$$P(X/Y) = \begin{array}{c|c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & 0.109 & 0 \\ 0 & 0.727 & 0 \\ 0 & 0.164 & 1 \end{bmatrix} \end{array}$$

Sum of all the columns of $P(X/Y)$ is 1

The entropies are:

$$H(X) = -\sum_{j=1}^{3} p(x_j) \log p(x_j)$$

$$= -\left[0.3 \log 0.3 + 0.4 \log 0.4 + 0.3 \log 0.3\right]$$

$$= \underline{1.571 \text{ bits/msg}} \quad \left(\frac{0.473}{\log_{10} 2}\right)$$

$$H(Y) = -\sum_{k=1}^{3} p(y_k) \log p(y_k)$$

$$= -\left[0.24 \log 0.24 + 0.55 \log 0.55 + 0.21 \log 0.21\right]$$

$$= \underline{1.441 \text{ bits/msg}}.$$

$$H(XY) = -\sum_{j=1}^{3}\sum_{k=1}^{3} p(x_j, y_k) \log p(x_j, y_k).$$

$$= -\left[0.24 \log 0.24 + 0.06 \log 0.06 + 0.4 \log 0.4 + 0.09 \log 0.09 + 0.21 \log 0.21\right]$$

$$= \underline{2.053 \text{ bits/msg}}$$

$$H(Y/X) = -\sum_{j=1}^{3}\sum_{k=1}^{3} p(x_j, y_k) \log p(y_k/x_j).$$

$$= -\left[0.24 \log 0.8 + 0.06 \log 0.2 + 0.4 \log 1 + 0.09 \log 0.3 + 0.21 \log 0.7\right]$$

$$= \underline{0.482 \text{ bit/msg}}.$$

$$H(X/Y) = -\sum_{j=1}^{3}\sum_{k=1}^{3} p(x_j, y_k) \log p(x_j/y_k)$$

$$= -\left[0.24 \log 1 + 0.06 \log 0.109 + 0.4 \log 0.727 + 0.21 \log 1\right]$$

$$= \underline{0.6124 \text{ bit/msg}}.$$

Mutual Information, $I(XY) = H(X) + H(Y) - H(XY)$

$= 1.571 + 1.441 - 2.053$

$= 0.959$ bit/msg.

Q10. A transmitter has an alphabet of 4 letters $[x_1, x_2, x_3, x_4]$ and the receiver has an alphabet of 3 letters $[y_1, y_2, y_3]$. The JPM is given

$$\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.3 & 0.05 & 0 \\ x_2 & 0 & 0.25 & 0 \\ x_3 & 0 & 0.15 & 0.05 \\ x_4 & 0 & 0.05 & 0.15 \end{array}$$

Calculate all the entropies and mutual information for the channel ?

Sol: We are able to find $P(X)$ and $P(Y)$ from $P(XY)$. Probabilities of the transmitter symbols are found by a summation of rows.

$$\begin{array}{c} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \end{array} \begin{bmatrix} 0.3 + 0.05 + 0 \\ 0 + 0.25 + 0 \\ 0 + 0.15 + 0.05 \\ 0 + 0.05 + 0.15 \end{bmatrix} \begin{array}{c} 0.35 \\ 0.25 \\ 0.2 \\ 0.2 \end{array}$$

$P(X) = [\ 0.35 \quad 0.25 \quad 0.2 \quad 0.2\ ]$

The probabilities of the receiver symbols are found by a summation of columns.

$$\begin{array}{cccc} & p(y_1) & p(y_2) & p(y_3) \\ & \left[\begin{array}{ccc} 0.3 & 0.05 & 0 \\ + & + & + \\ 0 & 0.25 & 0 \\ + & + & + \\ 0 & 0.15 & 0.05 \\ + & + & + \\ 0 & 0.05 & 0.15 \end{array}\right] \\ & 0.3 & 0.5 & 0.2 \end{array}$$

$$p(Y) = [\,0.3 \quad 0.5 \quad 0.2\,]$$

The Entropies are :

$$H(X) = -\sum_{j=1}^{4} p(x_j) \log p(x_j)$$

$$= -[0.35 \log 0.35 + 0.25 \log 0.25 + 2 \times 0.2 \log 0.2]$$

$$= 1.96 \text{ bits/msg}$$

$$H(Y) = -\sum_{k=1}^{3} p(y_k) \log p(y_k)$$

$$= -[0.3 \log 0.3 + 0.5 \log 0.5 + 0.2 \log 0.2]$$

$$= 1.49 \text{ bits/msg}$$

$$H(XY) = -\sum_{j=1}^{4} \sum_{k=1}^{3} p(x_j, y_k) \log p(x_j, y_k)$$

$$= -[0.3 \log 0.3 + 3 \times 0.05 \log 0.05 + 0.25 \log 0.25 + 2 \times 0.15 \log 0.15]$$

$$= 2.49 \text{ bits/msg.}$$

$$H(Y/X) = H(XY) - H(X)$$

$$= 2.49 - 1.96 = 0.53 \text{ bit/msg.}$$

$$H(X/Y) = H(XY) - H(Y)$$

$$= 2.49 - 1.49 = 1.00 \text{ bit/msg}$$

Mutual Information,

$$I(XY) = H(X) - H(X/Y)$$

$$= 1.96 - 1.00 = 0.96 \text{ bit/msg}$$

RELATIVE ENTROPY (Kullback- Leibler Distance)

It is a measure of the distance between two distributions. It is a log likelihood ratio. The relative entropy is a measure of the inefficiency of assuming that the distribution is $p_2(x)$ when the true distribution is $p_1(x)$.

Def: The relative entropy between two probability functions $p_1(x)$ and $p_2(x)$ is defined as

$$\boxed{D(p_1 \| p_2) = \sum_x p_1(x) \log \frac{p_1(x)}{p_2(x)}}$$

Properties:

① It is always a non negative value

② It is 'zero' only if $p_1 = p_2$.

CHANNEL CAPACITY

Claude E Shannon introduced the concept of channel capacity defined as the maximum of mutual information.

$$\boxed{C = \text{Max } I(XY) \text{ bits/msg.}}$$

$$= \text{Max}\left[H(X) - H(X/Y)\right]$$

$$= \text{Max}\left[H(Y) - H(Y/X)\right]$$

$$= \text{Max}\left[H(X) + H(Y) - H(XY)\right]$$

$$\boxed{C = r.\text{ Max } I(XY) \text{ bits/sec.}}$$

where r is the symbol rate in msg/sec.

Channel Efficiency And Redundancy:

The transmission efficiency or channel efficiency is defined as

$$\eta = \frac{\text{Actual transinformation}}{\text{Maximum transinformation}}$$

$$= \frac{I(XY)}{\max I(XY)}$$

$$\boxed{\eta = \frac{I(XY)}{C}}$$

The redundancy of the channel is defined as

$$R = 1 - \eta = 1 - \frac{I(XY)}{C}$$

$$\boxed{R = \frac{C - I(XY)}{C}}$$

Source Efficiency And Redundancy:

Source efficiency ; $\eta_s = \frac{H(X)}{\max H(X)}$

Source Redundancy, $R_s = 1 - \eta_s$.

Rate of information transmission over a channel:

Avg. Rate of information transmitted :

$$\boxed{R = r \cdot I(XY) \text{ Bits/sec.}}$$

$$= r \cdot \left[H(X) - H(X/Y)\right]$$

$$= r \cdot \left[H(Y) - H(Y/X)\right]$$