Then the average information per msg ie.

$$\text{Entropy} = \frac{\text{Total Information}}{\text{No of Msgs.}}$$

$$H = \frac{I_t}{L} = p_1 \log \frac{1}{p_1} + p_2 \log \frac{1}{p_2} + \dots + p_M \log \frac{1}{p_M}$$

$$= \sum_{k=1}^{M} p_k \log \frac{1}{p_k}$$

$$* \boxed{H = -\sum_{k=1}^{M} p_k \log p_k \text{ bits/msg.}} *$$

Q.2. Let $X = \begin{cases} a & \text{with prob } 1/2 \\ b & \text{with prob } 1/4 \\ c & \text{with prob } 1/8 \\ d & \text{with prob } 1/8 \end{cases}$; find $H(x) = ?$

Sol:

$$H(x) = -\sum_{k=1}^{M} p_k \log p_k = -\sum_{k=1}^{4} p_k \log p_k$$

$$= -[p_1 \log p_1 + p_2 \log p_2 + p_3 \log p_3 + p_4 \log p_4]$$

$$= -[1/2 \log 1/2 + 1/4 \log 1/4 + 1/8 \log 1/8 + 1/8 \log 1/8]$$

$$= 7/4 \text{ bits/msg} = 1.75 \text{ bits/msg.}$$

Q3. A source emits one of 4 possible symbols x_0 to x_3 during each signalling interval. The symbols occurs with probabilities

$x_0 \longrightarrow p_0 = 0.4$

$x_1 \longrightarrow p_1 = 0.3$

$x_2 \longrightarrow p_2 = 0.2$

$x_3 \longrightarrow p_3 = 0.1$

Find the amount of information gained by observing the source emitting each of these symbols and also the entropy of the source?

Sol: Self-information, $I_k = \log_2 \frac{1}{p_k}$ bits

when $k=0$; $I_0 = \log \frac{1}{p_0} = \log \frac{1}{0.4} = 1.322$ bits

$k=1$; $I_1 = \log \frac{1}{p_1} = \log \frac{1}{0.3} = 1.737$ bits

$k=2$, $I_2 = \log \frac{1}{p_2} = \log \frac{1}{0.2} = 2.322$ bits

$k=3$, $I_3 = \log \frac{1}{p_3} = \log \frac{1}{0.1} = 3.322$ bits

Entropy of the source;

$$H(x) = \sum_{k=0}^{3} p_k \log \frac{1}{p_k} \text{ bits / msg.}$$

$$= \sum_{k=0}^{3} p_k \cdot I_k$$

$$= p_0 I_0 + p_1 I_1 + p_2 I_2 + p_3 I_3.$$

$$= (0.4)(1.322) + 0.3 \times 1.737 + 0.2 \times 2.322 + 0.1 \times 3.322$$

$$H(x) = 1.8465 \text{ bits/msg}$$

Notes:

① → $H(x) \geq 0$; $0 \leq p(x) \leq 1 \Rightarrow \log \frac{1}{p(x)} \geq 0$

② → If there is only one msg is possible, then

$M = 1$ and $p_k = p_1 = 1$

$$H = p_1 \log \frac{1}{p_1} = \log 1 = 0$$

Here the reception of msg conveys no information.

③ → There will be only one msg out of M msgs having prob '1' and all others are '0'.

M msgs → $p_1 = 1$ and $p_2 = p_3 = \cdots = p_M = 0.$

$$H = \sum_{k=1}^{M} p_k \log \frac{1}{p_k}$$

$$= p_1 \log \frac{1}{p_1} + \sum_{k=2}^{M} p_k \log \frac{1}{p_k}$$

$$= p_1 \log \frac{1}{p_1} + \lim_{p \to 0} \left\{ p \log \frac{1}{p} + p \log \frac{1}{p} + \cdots \right\}.$$

$$= \log 1 + 0 = 0.$$

If all the probabilities are '0' except for one, then the entropy will be 'zero'.

④ → All the other cases entropy will be > 0.

For a binary system ($M=2$):

$$H = \sum_{k=1}^{2} p_k \log \frac{1}{p_k} = p_1 \log \frac{1}{p_1} + p_2 \log \frac{1}{p_2}$$

Let $p_1 = p$ and $p_2 = 1-p$

$$H = p \log \frac{1}{p} + (1-p) \log \frac{1}{(1-p)} = H(p)$$

Let $p_1 = p = 1-q$ and $p_2 = 1-p = q$

$$H = (1-q) \log \frac{1}{1-q} + q \log \frac{1}{q} = H(q)$$

ie $H = H(p) = H(q)$

$$= p \log \frac{1}{p} + q \log \frac{1}{q}.$$

⑤ → Condition of max. entropy:

$$H = p \log \frac{1}{p} + (1-p) \log \frac{1}{(1-p)} \quad \text{——— ①}$$

Differentiate ① wrt p and equate to zero.

$$\frac{dH}{dp} = p \cdot p \cdot x \cdot \frac{-1}{p^2} + \log \frac{1}{p} + (1-p) \cdot (1-p) \cdot x \, \frac{-1}{(1-p)^2} \times -1 + \log \frac{1}{(1-p)} \times -1$$

$$= -x - \log p + x + \log (1-p)$$

$$= -\log p + \log (1-p)$$

$$\frac{dH}{dp} = 0 \Rightarrow -\log p + \log (1-p) = 0$$

$$\log p = \log (1-p)$$

$$\Rightarrow p = 1-p$$

$$2p = 1$$

$$\underline{p = 1/2}$$

Also $\dfrac{d^2H}{dp^2} = -\dfrac{1}{p} - \dfrac{1}{1-p} < 0.$

Hence H has a max at $p = 1/2$.