Q6. In a communication system, a transmitter has three input symbols $A = \{a_1, a_2, a_3\}$ and receiver also has three ouput symbols $B = \{b_1, b_2, b_3\}$. The matrix given below shown JPM with some marginal probabilities: Find the missing probabilities (*) in the table.

$$p(AB) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}\begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 1/12 & * & 5/36 \\ 5/36 & 1/9 & 5/36 \\ * & 1/6 & * \end{bmatrix} \end{array}$$

$$p(B) = \begin{bmatrix} 1/3 & 14/36 & * \end{bmatrix}$$

Sol. use property ① of JPM.

$$p(b_1) = p(a_1 b_1) + p(a_2 b_1) + p(a_3 b_1) \leftarrow \text{1st column.}$$

$$1/3 = 1/12 + 5/36 + p(a_3, b_1).$$

$$p(a_3, b_1) = \frac{1}{3} - \left(\frac{1}{12} + \frac{5}{36}\right) = \underline{\underline{\frac{1}{9}}}.$$

Again property ①;

$$p(b_2) = p(a_1 b_2) + p(a_2 b_2) + p(a_3 b_2) \leftarrow \text{2nd column}$$

$$14/36 = p(a_1, b_2) + 1/9 + 1/6$$

$$p(a_1 b_2) = \underline{\underline{1/9}}$$

Property ③ of JPM;

$$\sum_{j=1}^{3} \sum_{k=1}^{3} p(a_j, b_k) = p(a_1 b_1) + p(a_1 b_2) + p(a_1 b_3) + p(a_2 b_1) + p(a_2 b_2) + p(a_2 b_3) + p(a_3 b_1) + p(a_3 b_2) + p(a_3 b_3) = 1$$

$$= \frac{1}{12} + \frac{1}{9} + \frac{5}{36} + \frac{5}{36} + \frac{1}{9} + \frac{5}{36} + \frac{1}{9} + \frac{1}{6} + p(a_3 b_3) = 1$$

$$p(a_3 b_3) = \underline{\underline{0}}.$$

A probability scheme $x_j$ is said to be complete when
$\sum_{j=1}^{m} p(x_j) = 1$.

Similarly $\sum_{k=1}^{3} p(b_k) = 1$

$$p(b_1) + p(b_2) + p(b_3) = 1$$

$$1/3 + 14/36 + p(b_3) = 1$$

$$p(b_3) = 5/18$$

The associated entropies of $P(X)$, $P(Y)$ and $P(XY)$ are:

Marginal Entropy of X ;

$$\boxed{H(X) = -\sum_{j=1}^{m} p(x_j) \log p(x_j)}$$

where $p(x_j) = \sum_{k=1}^{n} p(x_j, y_k)$.

Marginal Entropy of Y ;

$$\boxed{H(Y) = -\sum_{k=1}^{n} p(y_k) \log p(y_k)}$$

where $p(y_k) = \sum_{j=1}^{m} p(x_j, y_k)$.

Joint Entropy of X and Y ;

$$\boxed{H(XY) = -\sum_{j=1}^{m} \sum_{k=1}^{n} p(x_j, y_k) \log p(x_j, y_k).}$$

Then the conditional probability $P(X/Y)$ is given by

$$P(X/Y) = \frac{P(XY)}{P(Y)}$$

Consider an event "$y_k$", may occur in relation with $x_1, x_2 \ldots x_m$.

ie $[X/y_k] = \left[ \frac{x_1}{y_k} \quad \frac{x_2}{y_k} \quad \ldots \quad \frac{x_j}{y_k} \quad \ldots \quad \frac{x_m}{y_k} \right]$.

Then $p(X/y_k) = \left[p(x_1/y_k) \;\; p(x_2/y_k) \cdots\cdots\cdots p(x_m/y_k)\right]$

$$= \left[\frac{p(x_1 y_k)}{p(y_k)} \;\; \frac{p(x_2 y_k)}{p(y_k)} \;\cdots\cdots\; \frac{p(x_m y_k)}{p(y_k)}\right] \quad \text{--- (1)}$$

$$\sum_{j=1}^{m} p(x_j, y_k) = p(x_1 y_k) + p(x_2 y_k) + \cdots + p(x_m y_k) = p(y_k) \quad \text{--- (2)}$$

Comparing eqn (1) & (2), we can said that the sum of elements in matrix (1) is unity. So the probability scheme is complete.

The Conditional Entropy associated with an event '$y_k$'

$$H(X/y_k) = -\sum_{j=1}^{m} \frac{p(x_j, y_k)}{p(y_k)} \log \frac{p(x_j, y_k)}{p(y_k)}$$

$$= -\sum_{j=1}^{m} p(x_j/y_k) \log p(x_j/y_k).$$

Then for the whole event set 'Y': Conditional entropy of the system:

$$H(X/Y) = \sum_{k=1}^{n} p(y_k) \cdot H(X/y_k).$$

$$= -\sum_{k=1}^{n} p(y_k) \cdot \sum_{j=1}^{m} p(x_j/y_k) \log p(x_j/y_k).$$

$$= -\sum_{j=1}^{m}\sum_{k=1}^{n} p(x_j/y_k)\, p(y_k) \log p(x_j/y_k).$$

$$= -\sum_{j=1}^{m}\sum_{k=1}^{n} p(x_j, y_k) \log p(x_j/y_k).$$

Average Conditional Entropy or Conditional Entropy:

$$\boxed{H(X/Y) = -\sum_{j=1}^{m}\sum_{k=1}^{n} p(x_j, y_k) \log p(x_j/y_k)}$$