Proof:

The word lengths $l_1, l_2 \dots l_q$ are arranged in ascending order $l_1 \leq l_2 \leq l_3 \dots \leq l_q$.

If we choose "one-length" codewords for all q source symbols, satisfying prefix property, then

$$q \leq r.$$

If $q > r$, go for combinations of 'r' symbols to form instantaneous codewords.

Let 'n_i' represents the number of messages encoded into codewords of length 'i'.

for $i = 1$; $n_1 \leq r$.

for $i = 2$; for getting an instantaneous code, we must start encoding using $(r - n_1)$ symbols only as the first digits and the second digit can be any of 'r' symbols of the code alphabet.

$$n_2 \leq (r - n_1) r$$

$$n_2 \leq r^2 - n_1 r.$$

for $i = 3$, The number of msgs to be encoded to 3 digits which has to be different from code-words corresponding to n_1 and n_2 number of msgs must satisfy.

$$n_3 \leq \left[(r^2 - n_1 r) - n_2\right] . r$$

$$n_3 \leq r^3 - n_1 r^2 - n_2 r$$

Proceeding this way, we can write.

for ith codeword ;

$$n_i \leq r^i - n_1 r^{(i-1)} - n_2 r^{(i-2)} \dots - n_{(i-1)} r.$$

Then

$$n_i + n_1 r^{(i-1)} + n_2 r^{(i-2)} + \dots + n_{(i-1)} r \leq r^i$$

Multiply throughout by r^{-i};

$$n_i r^{-i} + n_1 r^{-1} + n_2 r^{-2} + \ldots + n_{(i-1)} r^{-(i-1)} \leq 1$$

Rearrange ;

$$n_i r^{-i} + n_{(i-1)} r^{-(i-1)} + n_{(i-2)} r^{-(i-2)} + \ldots + n_2 r^{-2} + n_1 r^{-1} \leq 1$$

Write as summation :

$$\sum_{m=1}^{i} n_m r^{-m} \leq 1$$

Rewrite the equation, since actual number of messages n_i has to be integer.

$$\sum_{m=1}^{i} n_m r^{-m} = n_1 r^{-1} + n_2 r^{-2} + \ldots + n_i r^{-i}.$$

$$= \underbrace{r^{-1} + r^{-1} + \ldots + r^{-1}}_{n_1 \text{ terms}} + \underbrace{r^{-2} + r^{-2} + \ldots + r^{-2}}_{n_2 \text{ terms}} + \ldots + \underbrace{r^{-i} + r^{-i} + \ldots + r^{-i}}_{n_i \text{ terms}} \leq 1$$

$$\sum_{m=1}^{i} n_m r^{-m} = \underbrace{\sum_{j=1}^{n_1} r^{-1}}_{\text{length } l_1} + \underbrace{\sum_{j=1}^{n_2} r^{-2}}_{\text{length } l_2} + \ldots + \underbrace{\sum_{j=1}^{n_i} r^{-i}}_{\text{length } l_i} \leq 1$$

Combining all groups; $n_1 + n_2 + \ldots + n_i = q$

$$\boxed{\sum_{i=1}^{q} r^{-l_i} \leq 1} \longrightarrow \text{Hence Proved.}$$