

Machine learning Homework- Deep Learning

Abinav Ravi Venkatakrishnan - 03694216 and Abhijeet Parida - 03679676

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1 Activation Function

Problem 1:

The matrix operation $w^T + b$ is essentially a linear operation. When we stack linear operations over other linear operations we essentially get a linear function. It is impossible to approximate complex functions with just linear operations, therefore non-linearity is introduced to overcome this problem.

Problem 2:

The sigmoid activation function is

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

The tanh activation is

$$\begin{aligned} \tanh(x) &= \frac{e^{2x} - 1}{e^{2x} + 1} \\ \tanh\left(\frac{x}{2}\right) &= \frac{e^x - 1}{e^x + 1} \\ \tanh\left(\frac{x}{2}\right) &= \frac{1 - e^{-x}}{1 + e^{-x}} \\ \tanh\left(\frac{x}{2}\right) &= (1 - e^{-x})\sigma(x) \end{aligned}$$

Problem 3:

From the previous problem we know that,

$$\begin{aligned} \tanh(x) &= \frac{e^{2x} - 1}{e^{2x} + 1} \\ \tanh(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \end{aligned}$$

Taking derivative on both sides we get,

$$\begin{aligned} \frac{d}{dx}(\tanh(x)) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) \\ &= (e^x + e^{-x})(e^x + e^{-x})^{-1} - (e^x + e^{-x})^{-2}(e^x - e^{-x})(e^x - e^{-x}) \\ &= 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 \\ \frac{d}{dx}(\tanh(x)) &= 1 - (\tanh(x))^2 \end{aligned}$$

The advantage is that it is easy to compute the gradients during backpropagation.

2 Numerical Stability

Problem 4:

$$\begin{aligned}
& a + \log(\sum_{i=1}^N e^{x_i - a}) \\
= & a + \log(\sum_{i=1}^N e^{x_i} e^{-a}) \\
= & a + \log(e^{-a} \sum_{i=1}^N e^{x_i}) \\
= & a + \log(e^{-a}) + \log(\sum_{i=1}^N e^{x_i}) \\
= & a - a + \log(\sum_{i=1}^N e^{x_i}) \\
= & \log(\sum_{i=1}^N e^{x_i})
\end{aligned}$$

Hence proved the equivalence

Problem 5:

$$\begin{aligned}
& \frac{e^{x_i - a}}{\sum_{i=1}^N e^{x_i - a}} \\
= & \frac{e^{x_i} e^{-a}}{\sum_{i=1}^N (e^{x_i} e^{-a})} \\
= & \frac{e^{x_i}}{\sum_{i=1}^N (e^{x_i})}
\end{aligned}$$

Hence proved the equivalence

Problem 6:

We can write the equivalent formulation as

$$x - xy + \log(1 + e^{-x}) \quad \forall x > 0 \quad (1)$$

$$xy + \log(1 + e^x) \quad \forall x < 0 \quad (2)$$

$$\begin{aligned}
& -(y \log(\sigma(x)) + (1 - y) \log(1 - \sigma(x))) \\
= & -y \log(\sigma(x)) - \log(1 - \sigma(x)) + y \log(1 - \sigma(x)) \\
= & -y \log\left(\frac{1}{1 + e^{-x}}\right) - \log\left(1 - \frac{1}{1 + e^{-x}}\right) + y \log\left(1 - \frac{1}{1 + e^{-x}}\right) \\
= & -y \log\left(\frac{1}{1 + e^{-x}}\right) - \log\left(\frac{e^{-x}}{1 + e^{-x}}\right) + y \log\left(\frac{e^{-x}}{1 + e^{-x}}\right) \\
= & -y \log\left(\frac{1}{1 + e^{-x}}\right) + x + \log(1 + e^{-x}) - xy - y \log(1 + e^{-x}) = x - xy + \log(1 + e^{-x})
\end{aligned}$$

Since here x are logits and logits can never be negative so the end equation matches Eqn 1. hence it is proved.