Machine learning Homework- Optimisation

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Problem 1:

1. we know that sum of convex function is a convex function. $x^2, 2y, cos(sin(\sqrt{\pi}))$ are convex functions because linear and constant are convex.

   $-min\{-x^2, log(y)\} = max\{x^2, -log(y)\}$. The max is convex if both functions are convex which is true for our case. So, it is a convex function.

2. we know that the point-wise addition of two convex function is convex; but here $log(x)$ and $-x^3$ are concave; so overall is concave.

3. $-min\{log(3x+1), -x^4 - 3x^2 + 8x - 42\} = max\{-log(3x+1), x^4 + 3x^2 - 8x + 42\}$ The max is convex if both functions are convex which is true for our case. So, it is a convex function.

4. by plotting Fig. 1, we see that the function $-x^2y$ is not convex. So, it is not a convex function.

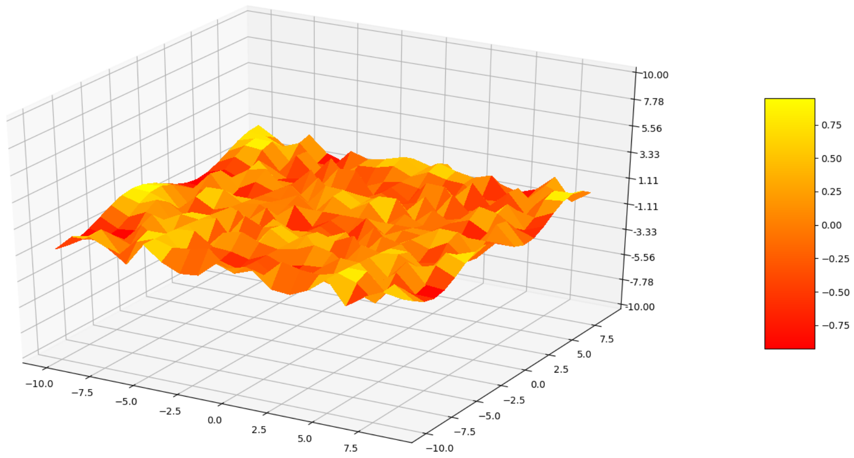

Figure 1:

Problem 2:

we know that the functions $f_1 and f_2$ are convex in $\mathcal{R}$.
so,

$$\lambda_1 f_1(x_1) + (1-\lambda_1) f_1(x_2) \geqslant f_1(\lambda_1 x_1 + (1-\lambda_1)x_2) \tag{1}$$

and

$$\lambda_1 f_2(x_1) + (1-\lambda_1) f_2(x_2) \geqslant f_2(\lambda_1 x_1 + (1-\lambda_1)x_2) \tag{2}$$

$h(x) = max(f_1(x), f_2(x))$ and given $f_1 \text{and} f_2$ are convex can be written as

$$\begin{cases} f_1(x) & ; x \geqslant C \\ f_2(x) & ; x \leqslant C \end{cases}$$

for $x_1, x_2 \geqslant C$; Eqn 1 is true and for $x_1, x_2 \leqslant C$; Eqn 2 is true.
for the case $x_2 \geqslant C$ and $x_1 \leqslant C$. We have to show two scenario to be true.

$$\lambda_1 f_1(x_1) + (1 - \lambda_1) f_2(x_2) \geqslant f_1(\lambda_1 x_1 + (1 - \lambda_1) x_2) \tag{3}$$

Eq. 3 is true because Eq. 1 is true and on the LHS $(1 - \lambda_1) f_2(x_2) > (1 - \lambda_1) f_1(x_2)$ from definition of $h(x)$.

$$\lambda_1 f_1(x_1) + (1 - \lambda_1) f_2(x_2) \geqslant f_2(\lambda_1 x_1 + (1 - \lambda_1) x_2) \tag{4}$$

Eq. 4 is true because Eq. 2 is true and on the LHS $\lambda_1 f_1(x_1) > \lambda_1 f_2(x_1)$ from definition of $h(x)$.

Problem 3:

we know that $f_1 \text{and} f_2$ are convex and their exist a minima $x_1 \text{and} x_2$ such that

$$f_1'(x_1) = 0 \; ; \; f_1''(x_1) > 0 \tag{5}$$

$$f_2'(x_2) = 0 \; ; \; f_2''(x_2) > 0 \tag{6}$$

for our function $f_1(f_2(x))$, the local minima can be found at

$$f_1'(f_2(x)).f_2'(x) = 0 \tag{7}$$

From Eq. 5 and Eq. 6 the possible candidates for minima of $f_1 f_2(x)$ is $x = x_2$ and $f_2(x) = x_1$.
For the critical point $x = x_2$ is minima if $(f_1 f_2(x))'' > 0$

$$f_2''(x).f_1'(f_2(x)) + f_2'(x).f_1''(f_2(x)).f_2'(x) \tag{8}$$

we know that $f_2'(x_2).f_1''(f_2(x_2)).f_2'(x_2) = 0$ from Eq. 6 and in the term $f_2''(x).f_1'(f_2(x))$,$f_2''(x)$ is +ve from Eq. 6. Therefore, the final sign of the Eq.8 depends on the sign of $f_1'(f_2(x))$,$f_2''(x)$. Since it is positive/negative we cannot say it is convex always.

For the critical point $f_2(x) = x_1$ is minima if $(f_1 f_2(x))'' > 0$.
we know that $f_2''(x).f_1'(f_2(x)) = 0$ from Eq. 7 and in the term $f_2'(x).f_1''(f_2(x)).f_2'(x)$,$f_2''(x)$ is always +ve from Eq. 6. Therefore, the statement is true in this case.
Overall, the statement is true for some f_1, f_2 and not true for other f_2, f_1

Problem 4:

Theorem: Suppose $f : x \mapsto \mathcal{R}$ is a differentiable function and $\mathcal{R}$ is convex. Then f is convex iff for $x, y \in \mathcal{X}$ then

$$f(y) \geqslant f(x) + (y - x)^T \nabla f(x) \tag{9}$$

for $\nabla f(x) = 0$, we get $f(y) \geqslant f(x)$ hence $f(x)$ is the lowest possible value when$\nabla f(x) = 0$