Machine learning Homework- Constrained Optimisation & SVM

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Problem 1:

We construct the Lagrangian function as $\mathcal{L}(\theta, \alpha) = f_0(\theta) + \alpha f_1(\theta)$
forcing $\nabla_\theta \mathcal{L} = 0$. We get $\theta_1 = -\frac{1}{2\alpha}$ and $\theta_2 = \frac{\sqrt{3}}{2\alpha}$
substituting back the θ in the Lagrangian function and minimising with respect to α, we get $\alpha = \pm\frac{1}{2}$ we take only the positive values of $\alpha = 1/2$.
Therefore $\theta_1 = -1$
$\theta_2 = \sqrt{3}$

Problem 2:

Solved in a Jupyter Notebook attached at the end of problem of Problem 5.

Problem 3:

- Both use hyperplane for classification
- SVM have margins and perceptrons do not have them

Problem 4:

As we use KKT conditions for finding the minima of the contraint optimisation problem. Therefore, We staisfy the Slater theorem, which states if the constraint functions are affine, the duality gap is zero.

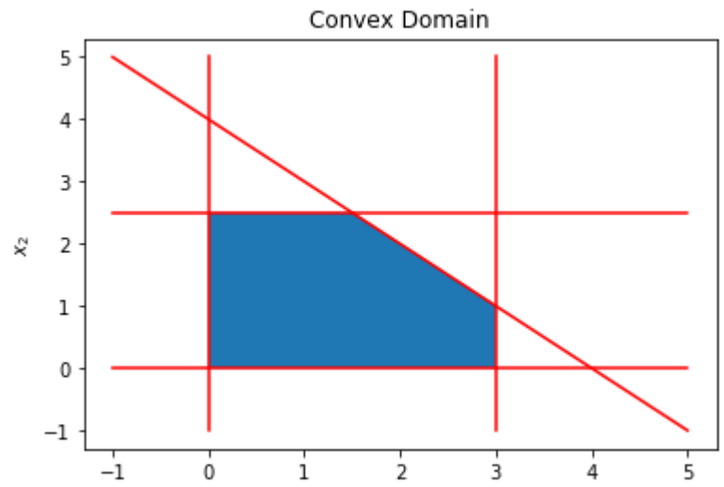
Problem 5:

1. from $g(\alpha)$ we know that $\alpha Q \alpha^T$ is equivalent to $-\sum_{i=1}^{N}\sum_{j=1}^{N} y_i y_j \alpha_i \alpha_j \mathbf{x}_i \mathbf{x}_j$. By rearranging the scalars we get,$-\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i y_i y_j \mathbf{x}_i \mathbf{x}_j \alpha_j$.
   Therefore $Q = (-yy^T(hadamard)XX^T)$
2. We know that $Q = -p^T p$ and also we know that $p^T p$ is positive semi definite due to its symmetric nature. So $a^t(p^T p)a \geq 0$ but we negative sign also. Therefore, Q is negative semi definite.
3. The negative semi definiteness allows the concave optimisation to be a maximisation problem.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

Visualisation

In [2]:
```
theta=np.linspace(-1,5,10)
f_theta=4*np.ones_like(theta)-theta
plt.plot(theta,f_theta,'-r')
plt.plot(theta,2.5*np.ones_like(theta),'-r')
plt.plot(theta,np.zeros_like(theta),'-r')
plt.plot(np.zeros_like(theta),theta,'-r')
plt.plot(3*np.ones_like(theta),theta,'-r')
plt.fill([0,3,3,1.5,0],[0,0,1,2.5,2.5])
plt.title('Convex Domain')
plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.show()
```



Closed Form Projection

In [3]:
```
def projectionPi(p):
    p[0]=max(min(p[0],3),0)
    p[1]=max(min(p[1],2.5),0)


    a=np.array([1.5,2.5])
    b=np.array([3,1])
    if 1.5<=p[0]<= 3 and 1<=p[1]<=2.5 and p[0]+p[1]>4:
        p=a+(np.dot(np.transpose(p-a),(b-a))/np.linalg.norm(b-a)**2)*(b-a)

    return p
```

Gradient Descent

In [4]:
```
def gradientDescent(x,lr):
    grad=np.array([2*(x[0]-2), 2*(2*x[1]-7)])
    x=projectionPi(x-lr*grad)

    return x
```

In [5]:
```
x=np.array([2.5,1])
for i in range(2):
    x=gradientDescent(x,0.05)
    print(x)
```

```
[2.45 1.5 ]
[2.2525 1.7475]
```

Programming assignment 6: SVM

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

from sklearn.datasets import make_blobs

from cvxopt import matrix, solvers
```

Exporting the results to PDF

Once you complete the assignments, export the entire notebook as PDF and attach it to your homework solutions. The best way of doing that is

1. Run all the cells of the notebook.
2. Download the notebook in HTML (click File > Download as > .html)
3. Convert the HTML to PDF using e.g. https://www.sejda.com/html-to-pdf or `wkhtmltopdf` for Linux (tutorial)
4. Concatenate your solutions for other tasks with the output of Step 3. On a Linux machine you can simply use `pdfunite`, there are similar tools for other platforms too. You can only upload a single PDF file to Moodle.

This way is preferred to using `nbconvert`, since `nbconvert` clips lines that exceed page width and makes your code harder to grade.

Your task

In this sheet we will implement a simple binary SVM classifier.

We will use **CVXOPT** http://cvxopt.org/ - a Python library for convex optimization. If you use `Anaconda`, you can install it using

```
conda install cvxopt
```

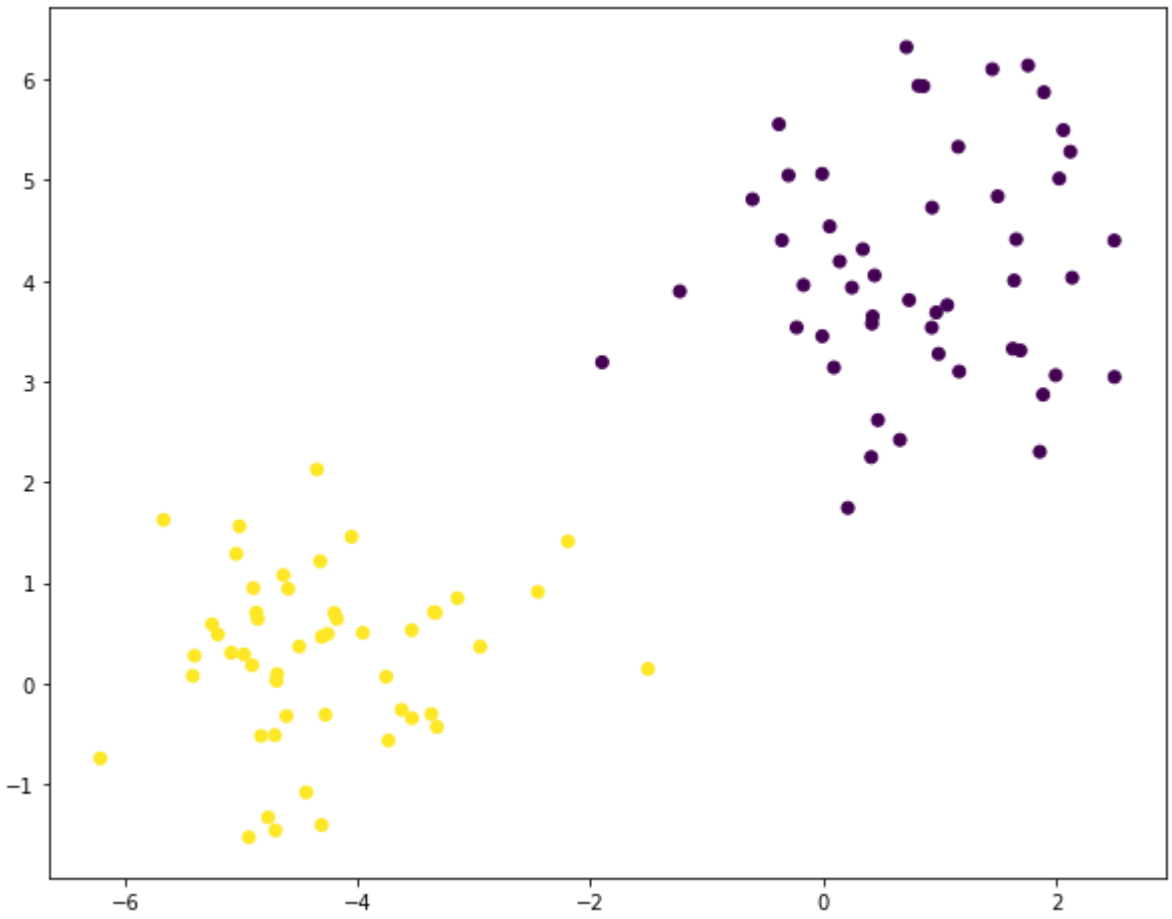
As usual, your task is to fill out the missing code, run the notebook, convert it to PDF and attach it you your HW solution.

Generate and visualize the data

In [2]:
```
N = 100  # number of samples
D = 2  # number of dimensions
C = 2  # number of classes
seed = 3  # for reproducible experiments

X, y = make_blobs(n_samples=N, n_features=D, centers=2, random_state=seed)
y[y == 0] = -1  # it is more convenient to have {-1, 1} as class labels (instea
d of {0, 1})
y = y.astype(np.float)
plt.figure(figsize=[10, 8])
```

```
plt.scatter(X[:, 0], X[:, 1], c=y)
plt.show()
```



Task 1: Solving the SVM dual problem

Remember, that the SVM dual problem can be formulated as a Quadratic programming (QP) problem. We will solve it using a QP solver from the `CVXOPT` library.

The general form of a QP is

$$\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x} - \mathbf{q}^T\mathbf{x}$$

$$\text{subject to} \quad \mathbf{G}\mathbf{x} \preceq \mathbf{h}$$

$$\text{and} \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\preceq$ denotes "elementwise less than or equal to".

Your task is to formulate the SVM dual problems as a QP and solve it using `CVXOPT`, i.e. specify the matrices $\mathbf{P}, \mathbf{G}, \mathbf{A}$ and vectors $\mathbf{q}, \mathbf{h}, \mathbf{b}$.

In [3]:
```
def solve_dual_svm(X, y):
    """Solve the dual formulation of the SVM problem.

    Parameters
    ----------
    X : array, shape [N, D]
        Input features.
    y : array, shape [N]
        Binary class labels (in {-1, 1} format).

    Returns
    -------
    alphas : array, shape [N]
```

```
        Solution of the dual problem.
    """
    # TODO
    # These variables have to be of type cvxopt.matrix
    N,D = X.shape
    M = y[:,None]*X
    P = matrix(M.dot(M.T))
    q = matrix(-np.ones([N,1]))
    G = matrix(-np.eye(N))
    h = matrix(np.zeros(N))
    A = matrix(y.reshape(1, -1))
    b = matrix(np.zeros(1))
    solvers.options['show_progress'] = False
    solution = solvers.qp(P, q, G, h, A, b)
    alphas = np.array(solution['x'])
    return alphas
```

Task 2: Recovering the weights and the bias

In [4]:
```
def compute_weights_and_bias(alpha, X, y):
    """Recover the weights w and the bias b using the dual solution alpha.

    Parameters
    ----------
    alpha : array, shape [N]
        Solution of the dual problem.
    X : array, shape [N, D]
        Input features.
    y : array, shape [N]
        Binary class labels (in {-1, 1} format).

    Returns
    -------
    w : array, shape [D]
        Weight vector.
    b : float
        Bias term.
    """
    w = np.dot(np.transpose(X), alpha * y[:, None])
    sv = (alpha > 1e-4).reshape(-1)
    bias = y[sv] - np.dot(X[sv, :], w).reshape(-1)
    b = np.mean(bias)

    return w, b
```

Visualize the result (nothing to do here)

In [5]:
```
def plot_data_with_hyperplane_and_support_vectors(X, y, alpha, w, b):
    """Plot the data as a scatter plot together with the separating hyperplane.

    Parameters
    ----------
    X : array, shape [N, D]
        Input features.
    y : array, shape [N]
        Binary class labels (in {-1, 1} format).
    alpha : array, shape [N]
        Solution of the dual problem.
    w : array, shape [D]
        Weight vector.
    b : float
```

```
        Bias term.
    """
    plt.figure(figsize=[10, 8])
    # Plot the hyperplane
    slope = -w[0] / w[1]
    intercept = -b / w[1]
    x = np.linspace(X[:, 0].min(), X[:, 0].max())
    plt.plot(x, x * slope + intercept, 'k-', label='decision boundary')
    # Plot all the datapoints
    plt.scatter(X[:, 0], X[:, 1], c=y)
    # Mark the support vectors
    support_vecs = (alpha > 1e-4).reshape(-1)
    plt.scatter(X[support_vecs, 0], X[support_vecs, 1], c=y[support_vecs], s=25
0, marker='*', label='support vectors')
    plt.xlabel('$x_1$')
    plt.ylabel('$x_2$')
    plt.legend(loc='upper left')
```

The reference solution is

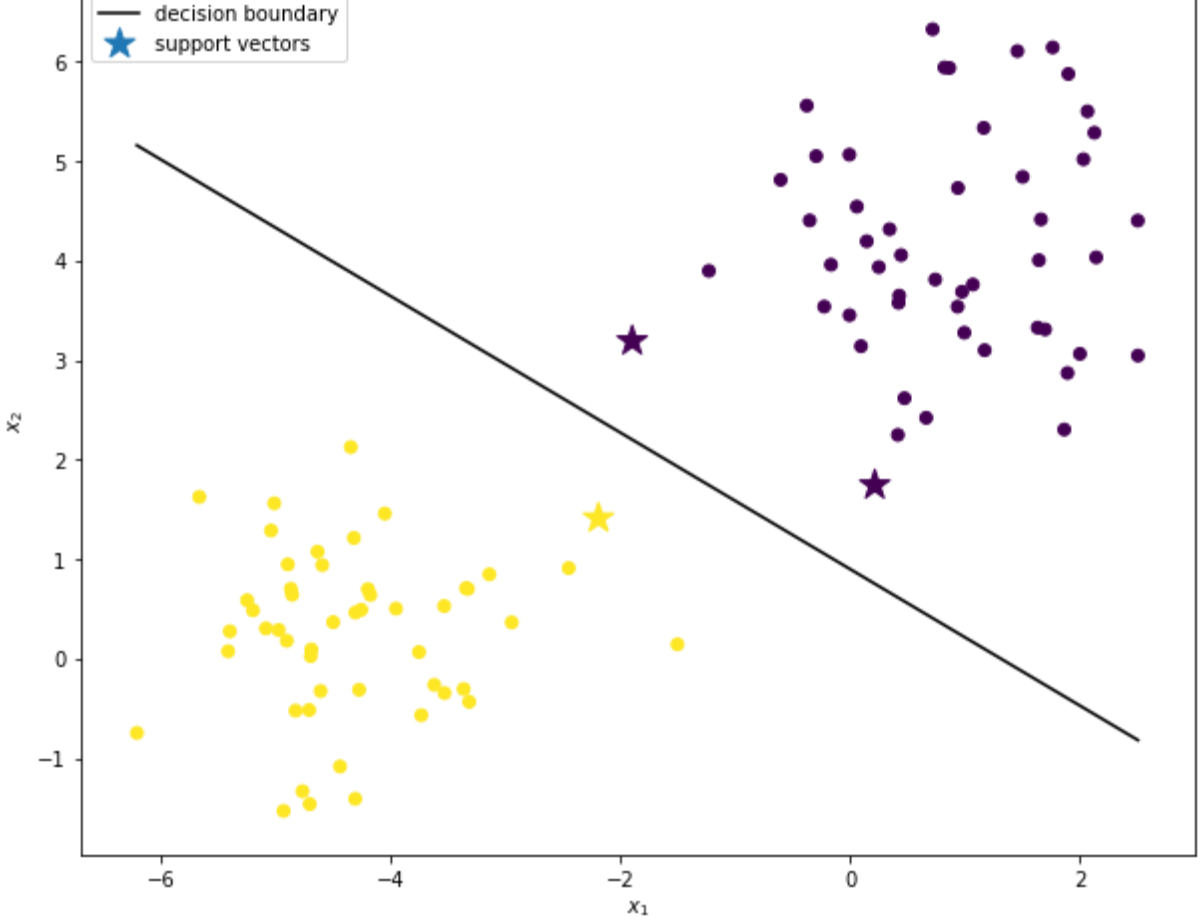
```
w = array([[-0.69192638],
           [-1.00973312]])


b = 0.907667782
```

Indices of the support vectors are

```
[38, 47, 92]
```

In [6]:
```
alpha = solve_dual_svm(X, y)
w, b = compute_weights_and_bias(alpha, X, y)
plot_data_with_hyperplane_and_support_vectors(X, y, alpha, w, b)
plt.show()
```


decision boundary
support vectors
6
5
4
3
2
1
0
−1
x_2
−6
−4
−2
0
2
x_1