Machine learning Homework- Deep Learning

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1 Activation Function

Problem 1:

The matrix operation $w^T + b$ is essentially a linear operation. When we stack linear operations over other linear operations we essentially get a linear function. It is impossible to approximate complex functions with just linear operations, therefore non-linearity is introduced to overcome this problem.

Problem 2:

The sigmoid activation function is

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

The tanh activation is

$$\begin{aligned}
tanh(x) &= \tfrac{e^{2x}-1}{e^{2x}+1} \\
tanh(\frac{x}{2}) &= \tfrac{e^{x}-1}{e^{x}+1} \\
tanh(\frac{x}{2}) &= \tfrac{1-e^{-x}}{1+e^{-x}} \\
tanh(\frac{x}{2}) + 1 &= \tfrac{1-e^{-x}}{1+e^{-x}} + 1 \\
tanh(\frac{x}{2}) + 1 &= \tfrac{2}{1+e^{-x}} \\
&= 2\sigma(x)
\end{aligned}$$

which implies that the $\sigma(x) = tanh(\frac{x}{2}) + \frac{1}{2}$ Now for each hidden layer we can take half the input weights which will make it $\frac{x}{2}$ and the output weights from hidden layer can be factored by $\frac{1}{2}$then add a constant of half to the bias term which gives us the sigmoid function.

Problem 3:

From the previous problem we know that,

$$\begin{aligned}
tanh(x) &= \tfrac{e^{2x}-1}{e^{2x}+1} \\
tanh(x) &= \tfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}
\end{aligned}$$

Taking derivative on both sides we get,

$$\begin{aligned}
\frac{d}{dx}(tanh(x)) &= \tfrac{d}{dx}(\tfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}) \\
&= (e^x + e^{-x})(e^x + e^{-x})^{-1} - (e^x + e^{-x})^{-2}(e^x - e^{-x})(e^x - e^{-x}) \\
&= 1 - (\tfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}})^2 \\
\frac{d}{dx}(tanh(x)) &= 1 - (tanh(x))^2
\end{aligned}$$

The advantage is that it is easy to compute the gradients during backpropagation.

2 Numerical Stability

Problem 4:

$$
\begin{aligned}
& a + log(\textstyle\sum_{i=1}^{N} e^{x_i - a}) \\
= \quad & a + log(\textstyle\sum_{i=1}^{N} e^{x_i} e^{-a}) \\
= \quad & a + log(e^{-a} \textstyle\sum_{i=1}^{N} e^{x_i}) \\
= \quad & a + log(e^{-a}) + log(\textstyle\sum_{i=1}^{N} e^{x_i}) \\
= \quad & a - a + log(\textstyle\sum_{i=1}^{N} e^{x_i}) \\
= \quad & log(\textstyle\sum_{i=1}^{N} e^{x_i})
\end{aligned}
$$

Hence proved the equivalence

Problem 5:

$$
\begin{aligned}
& \tfrac{e^{x_i - a}}{\sum_{i=1}^{N} e^{x_i - a}} \\
= \quad & \tfrac{e^{x_i} e^{-a}}{\sum_{i=1}^{N} (e^{x_i} e^{-a})} \\
= \quad & \tfrac{e^{x_i}}{\sum_{i=1}^{N} (e^{x_i})}
\end{aligned}
$$

Hence proved the equivalence

Problem 6:

We can write the equivalent formulation as

$$x - xy + log(1 + e^{-x}) \quad \forall x > 0 \tag{1}$$

$$xy + log(1 + e^{x}) \quad \forall x < 0 \tag{2}$$

$$
\begin{aligned}
& -(ylog(\sigma(x)) + (1 - y)log(1 - \sigma(x))) \\
&= -ylog(\sigma(x)) - log(1 - \sigma(x)) + ylog(1 - \sigma(x)) \\
&= -ylog(\tfrac{1}{1+e^{-x}}) - log(1 - \tfrac{1}{1+e^{-x}}) + ylog(1 - \tfrac{1}{1+e^{-x}}) \\
&= -ylog(\tfrac{1}{1+e^{-x}}) - log(\tfrac{e^{-x}}{1+e^{-x}}) + ylog(\tfrac{e^{-x}}{1+e^{-x}}) \\
&= -ylog(\tfrac{1}{1+e^{-x}}) + x + log(1 + e^{-x}) - xy - ylog(1 + e^{-x}) \quad = x - xy + log(1 + e^{-x})
\end{aligned}
$$

Since here x are logits and logits can never be negative so the end equation matches Eqn 1. hence it is proved.