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SS 2018
Sheet 1P
13.04.2018

Scientific Computing II

Exercise 1: Smoothing Properties of (weighted) Jacobi Method

We want to numerically solve the Poisson problem

$$\begin{aligned} -\frac{d^2u}{dx^2} &= 0, \quad x \in (0,1), \\ u(0) = u(1) &= 0, \end{aligned}$$

with finite differences. Discretise the equation on a grid with $N+1$ points and mesh size $h := 1/N$.

(a) Implement an iteration of the Jacobi method in Matlab or in any other programming language.

(b) Carry out 10 relaxation steps with the initial condition $u_k(x) := \sin(\pi k x)$, $k = 1, 3, 7$, and mesh size $h := 1/8$. At which rate does the (discrete) error $e_k^{(n)} := \max_i |0 - u_{k,i}^{(n)}|$ decrease in each iteration step? ($u_{k,i}^{(n)}$ is the value of the i-th grid point at iteration n with initial condition $u_k(x)$.) Measure $r := e_k^{(n)}/e_k^{(n-1)}$ for this purpose. Compare your results with the analytical findings.

(c) What happens when decreasing the mesh size to $h := 1/16$ and keeping all other parameters unchanged?

(d) Carry out the same study for the weighted Jacobi method with $\omega = \frac{1}{3}, \frac{2}{3}, \frac{1}{2}$.

Note: Do not stick to the description given here. Try other values of h, N, and k. You can also try to introduce a right hand side $f(x)$.

Solution:

(a) The solution can be found in smoothers.m.

(b) In all simulations, we can observe a constant rate of error decrease. This rate exactly corresponds to the eigenvalue of our iterative matrix-vector operation, assuming a given input frequency. We obtain:

$$\begin{aligned} k=1: \quad r=0.9239 &= \cos(k\pi h) \\ k=3: \quad r=0.3827 &= \cos(k\pi h) \\ k=7: \quad r=0.9239 &= -\cos(k\pi h). \end{aligned} \tag{1}$$

The decay of the error is thus exactly as expected from our Fourier analysis in Exercise 3 where we predicted a decrease of $|\cos(k\pi h)|$.

(c) Changing the grid resolution implies a frequency modification with respect to the grid resolution. We know from Exercise 3, that frequencies at the very lower and upper end of the spectrum are removed very slowly. The frequency $k=3$ is—with respect to the fine grid—now quite at the "lower" end of the frequency spectrum. We thus expect a worse convergence compared to the coarser grid. The frequency $k=7$, however, is now in the middle of the spectrum. We therefore expect a very good damping of this error mode. We obtain the following rates in error reduction:

$$\begin{aligned} k=1: \quad r=0.9808 &= \cos(k\pi h) \\ k=3: \quad r=0.8315 &= \cos(k\pi h) \\ k=7: \quad r=0.1951 &= \cos(k\pi h). \end{aligned} \tag{2}$$

(d) For $\omega=1/3$, $h=1/8$, we obtain:

$$\begin{aligned} k=1: \quad r=0.9808 &= 1-\omega+\omega\cos(k\pi h) \\ k=3: \quad r=0.7942 &= 1-\omega+\omega\cos(k\pi h) \\ k=7: \quad r=0.3587 &= 1-\omega+\omega\cos(k\pi h). \end{aligned} \tag{3}$$

For $\omega=2/3$, $h=1/8$, we obtain:

$$\begin{aligned} k=1: \quad r=0.9493 &= 1-\omega+\omega\cos(k\pi h) \\ k=3: \quad r=0.5885 &= 1-\omega+\omega\cos(k\pi h) \\ k=7: \quad r=0.2826 &= -(1-\omega+\omega\cos(k\pi h)). \end{aligned} \tag{4}$$

If we consider the error plots of $(x_i, u_{k,i}^{(n)})$, we observe that for $\omega=1/3$, the error homogeneously decays without oscillating around the zero line. Since $\omega=1/3<1/2$, this is what we already expected after the Fourier analysis in Exercise 3 for the damped Jacobi method.
For $\omega=1/2$, $h=1/8$, we obtain:

$$\begin{aligned} k=1: \quad r=0.9619 &= 1-\omega+\omega\cos(k\pi h) \\ k=2: \quad r=0.8536 &= 1-\omega+\omega\cos(k\pi h) \\ k=3: \quad r=0.5913 &= 1-\omega+\omega\cos(k\pi h) \\ k=4: \quad r=0.5000 &= 1-\omega+\omega\cos(k\pi h) \\ k=5: \quad r=0.3087 &= 1-\omega+\omega\cos(k\pi h) \\ k=6: \quad r=0.1464 &= 1-\omega+\omega\cos(k\pi h) \\ k=7: \quad r=0.0381 &= 1-\omega+\omega\cos(k\pi h). \end{aligned} \tag{5}$$

Due to the monotony and the strict positivity for the weights $\omega = 1/3, 1/2$, we obtain an improvement of the error reduction rates for increasing frequencies k. We can thus obtain very good damping of all high frequencies by tuning the weighting parameter ω. Using $\omega = 1/2$ instead of $\omega = 1/3$ thus yields better reduction of the high frequencies in the Poisson problem.