Standard Diffusion

$$\frac{\partial c}{\partial t} = D\nabla^2_c - kc$$

Concentration gradient

Proportional metabolisation

Since it is 1D,

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} - kc \qquad — (1)$$

Since the boundary condition at $x=0$ is $c_0 \neq 0$, it's non-homogenous, so, splitting up the solution into

→ Transient solution $c_T(x,t)$
→ Steady state $c_\infty(x)$

So, $c(x,t) = c_T(x,t) + c_\infty(x)$ — (2)

Solving for stea state, we set $\frac{\partial c}{\partial t} = 0$

$$\rightarrow 0 = D\frac{d^2 c_\infty}{dx^2} - kc_\infty$$

$$\rightarrow \frac{d^2 c_\infty}{dx^2} - \frac{k}{D}c_\infty = 0$$

Using general form $[f'' - k^2 f = 0 \rightarrow f(x) = A\cosh(kx) + B\sinh(kx)]$

$$\rightarrow c_\infty\ ) = A\cosh\left(\sqrt{\frac{k}{D}}\,x\right) + B\sinh\left(\sqrt{\frac{k}{D}}\,x\right)$$

↳ S1

Applying boundary conditions

$C_\infty(0) = C_0 \rightarrow A(1) + B(0) = C_0 \rightarrow A = C_0$

$C_\infty(L) = 0 \rightarrow C_0 \cosh\left(\sqrt{\frac{k}{D}}\, L\right) + B \sinh\left(\sqrt{\frac{k}{D}}\, L\right) = 0$

$$\hookrightarrow B = -C_0 \frac{\cosh(\sqrt{\dots}\, L)}{\sinh(\sqrt{\dots}\, L)} = -C_0 \coth\left(\sqrt{\frac{k}{D}}\, L\right)$$

Plugging A and B to S1,

$$C_\infty(x) = C_0 \cosh(\sqrt{\dots}\, x) - C_0 \coth(\sqrt{\dots}\, L) \sinh(\sqrt{\dots}\, x)$$

Using $[\sinh(A - B) = \sinh A \cosh B - \cosh A \sinh B]$,

$$C_\infty(x) = C_0 \frac{\sinh\left(\sqrt{\frac{k}{D}}\,(L - x)\right)}{\sinh\left(\sqrt{\frac{k}{D}}\, L\right)} \quad \text{— S2}$$

Now for the transient solution,

Let's place $C = C_T + C_\infty$ into 1

$$\frac{\partial (C_T + C_\infty)}{\partial t} = D \frac{\partial^2 (C_T + C_\infty)}{\partial x^2} - k(C_T + C_\infty)$$

Since $C_\infty$ is not a function of $t$,

$$\rightarrow \frac{\partial C_T}{\partial t} = D \frac{\partial^2 C_T}{\partial x^2} - k C_T \quad \text{— T1}$$

For new boundary conditions,

$$c_T(0,t) = c(0,t) - c_\infty(0) = c_0 - c_0 = 0$$

$$c_T(L,t) = c(L,t) - c_\infty(L) = 0 - 0 = 0$$

$$c_T(x,0) = c(x,0) - c_\infty(x) = 0 - c_\infty(x) = -c_\infty(x)$$

These are homogenous, so we can separate the variables like

$$c_T(x,t) = X(x)\,T(t) \quad \text{— T2}$$

In T1,

$$XT' = DX''T - kXT$$

$$\rightarrow \frac{T'}{T} = D\frac{X''}{X} - k$$

$$\rightarrow \frac{T'}{T} + k = D\frac{X''}{X} = -\lambda D$$

separation constant to ensure decaying solutions

So we have

Spatial: $X'' = -\lambda X$

Time: $\frac{T'}{T} + k = -\lambda D$

For spatial, using conditions $X(0)=0$, $X(L)=0$, it leads to standard sine eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1,2,3,\ldots$$

Time then becomes

$$\frac{T'}{T} + k = -D\lambda_n$$

$$\rightarrow T' + (k + D\lambda_n) T = 0$$

Using the standard form ...

$$T_n(t) = e^{\left(-t\left(D\left(\frac{n\pi}{L}\right)^2 + k\right)\right)}$$

Thus, T2 Becomes,

$$c_T(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-t\left(D\left(\frac{n\pi}{L}\right)^2 + k\right)} \quad \text{— T3}$$

Now we can apply $t = 0 \rightarrow c_T = -c_0(x)$,

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = -c_0 \frac{\sinh\left(\sqrt{\frac{k}{D}}(L-x)\right)}{\sinh\left(\sqrt{\frac{k}{D}}\, L\right)}$$

↳ fourier sine series!

Using orthogonality

$$b_n = \frac{2}{L}\int_0^L -c_\infty(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Using wolfrom Alpha for this one

$$b_n = -\frac{2 c_0 n\pi}{L^2\left(\frac{k}{D} + \left(\frac{n\pi}{L}\right)^2\right)} \quad \text{— T4}$$

Finally putting together everything

$$c(x,t) = C_\infty(x) + C_T(x,t)$$

$$\boxed{c(x,t) = C_0 \frac{\sinh\left(\sqrt{\frac{k}{D}}(L-x)\right)}{\sinh\left(\sqrt{\frac{k}{D}}L\right)} - \sum_{n=1}^{\infty} \frac{2C_0 n\pi}{L^2\left(\frac{k}{D}+\left(\frac{n\pi}{L}\right)^2\right)} \sin\left(\frac{n\pi x}{L}\right) e^{-t\left(D\left(\frac{n\pi}{L}\right)^2+k\right)}}$$

The final solution