standard diffusion

$$\frac{\partial c}{\partial t} = D\nabla^2 c - kc$$

Proportional metabolization

concentration gradient

Since it is 1D,

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} - kc \quad \ldots\ldots (1)$$

Since the boundary condition at $x=0$ is $c_0 \neq 0$, it's non-homogenous, so, splitting up the solution into

→ Transient solution $c_T(x,t)$

→ Steady state $c_\infty(x)$

So, $c(x,t) = c_T(x,t) + c_\infty(x) \quad \ldots\ldots (2)$

Solving for steady state, we set $\frac{\partial c}{\partial t} = 0$

$$\rightarrow 0 = D\frac{d^2 c_\infty}{dx^2} - kc_\infty$$

$$\rightarrow \frac{d^2 c_\infty}{dx^2} - \frac{k}{D}c_\infty = 0$$

Using general form $[f'' - k^2 f = 0 \rightarrow f(x) = A\cosh(kx) + B\sinh(kx)]$

$$\rightarrow c_\infty(x) = A\cosh\left(\sqrt{\frac{k}{D}}\,x\right) + B\sinh\left(\sqrt{\frac{k}{D}}\,x\right)$$

→ (S1)

Applying boundary conditions

$c_\infty(0) = c_0 \rightarrow A(1) + B(0) = c_0 \longrightarrow A = c_0$

$c_\infty(L) = 0 \rightarrow c_0 \cosh\left(\sqrt{\frac{K}{D}}L\right) + B\sinh\left(\sqrt{\frac{K}{D}}L\right) = 0$

$\hookrightarrow B = -c_0 \frac{\cosh(\sqrt{\cdots}L)}{\sinh(\sqrt{\cdots}L)} = -c_0 \coth\left(\sqrt{\frac{K}{D}}L\right)$

Plugging A and B to S1,

$c_\infty(x) = c_0 \cosh(\sqrt{\cdots}x) - c_0 \coth(\sqrt{\cdots}L)\sinh(\sqrt{\cdots}x)$

Using $[\sinh(A-B) = \sinh A \cosh B - \cosh A \sinh B]$

$$c_\infty(x) = c_0 \frac{\sinh\left(\sqrt{\frac{K}{D}}(L-x)\right)}{\sinh\left(\sqrt{\frac{K}{D}}L\right)} \quad \ldots\ldots \quad S2$$

Now for the transient solution,

Let's place $c = c_T + c_\infty$ into 1

$$\frac{\partial(c_T + c_\infty)}{\partial t} = D\frac{\partial^2(c_T + c_\infty)}{\partial x^2} - K(c_T + c_\infty)$$

Since c_∞ is not a function of t,

$$\rightarrow \frac{\partial c_T}{\partial t} = D\frac{\partial^2 c_T}{\partial x^2} - Kc_T \quad \ldots\ldots \quad T1$$

For new boundary conditions,

$c_T(0,t) = c(0,t) - c_\infty(0) = c_0 - c_0 = 0$

$c_T(L,t) = c(L,t) - c_\infty(L) = 0 - 0 = 0$

$c_T(x,0) = c(x,0) - c_\infty(x) = 0 - c_\infty(x) = -c_\infty(x)$

These are homogenous, so we can separate the variables as:

$$c_p(x,t) = X(x) \cdot T(t) \quad \ldots\ldots\ldots \quad T2$$

In T1,

$$XT' = DX''T - kXT$$

$$\rightarrow \frac{T'}{T} = D\frac{X''}{X} - k$$

$$\rightarrow \frac{T'}{T} + k = D\frac{X''}{X} = -\lambda D$$

Separation constant to ensure decaying solutions

So, we have,

Spatial: $X'' = -\lambda X$

Time: $\frac{T'}{T} + k = -\lambda D$

For spatial, using conditions $X(0)=0$, $X(L)=0$, it leads to standard sine eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1,2,3,\ldots$$

Time then becomes,

$$\frac{T'}{T} + k = -D\lambda_n$$

$$\rightarrow T' + (k + D\lambda_n)T = 0$$

$$\rightarrow$$

Using the standard form,

$$T_n(t) = e^{\left(-t\left(D\left(\frac{n\pi}{L}\right)^2 + K\right)\right)}$$

Thus, T2 becomes,

$$c_T(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-t\left(D\left(\frac{n\pi}{L}\right)^2 + K\right)} \quad \text{......} \quad T3$$

Now, we can apply $t = 0 \rightarrow c_T = -c_0(x)$,

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = -\frac{c_0 \sinh\left(\sqrt{\frac{K}{D}}(L-x)\right)}{\sinh\left(\sqrt{\frac{K}{D}} L\right)}$$

$\hookrightarrow$ Fourier sine series!

Using orthogonality,

$$b_n = \frac{2}{L} \int_0^L -c_\infty(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Solving,

$$b_n = -\frac{2 c_0 n\pi}{L^2\left(\frac{K}{D} + \left(\frac{n\pi}{L}\right)^2\right)} \quad \text{......} \quad T4$$

Finally, putting together everything,

$$c(x,t) = c_\infty(x) + c_T(x,t)$$

$$c(x,t) = c_0 \frac{\sinh\left(\sqrt{\frac{K}{D}}(L-x)\right)}{\sinh\left(\sqrt{\frac{K}{D}}L\right)} - \sum_{n=1}^{\infty} \frac{2\cos n\pi}{L^2\left(\frac{K}{D}+\left(\frac{n\pi}{L}\right)^2\right)} \sin\left(\frac{n\pi x}{L}\right) e^{-t\left(D\left(\frac{n\pi}{L}\right)^2+K\right)}$$

The final solution //