COMP 3030 Fall 2021 Homework Assignment 4

due Wednesday, November 17, 2021, 11:59pm

Must be submitted on Crowdmark, **not** UM Learn

Page 1: Important definitions, notation, and hints!

- Notation: for any string W and any symbol x, we write $\#_x(W)$ to represent the number of occurrences of symbol x in W.
  Examples: $\#_0(\varepsilon) = 0, \#_1(\varepsilon) = 0, \#_0(0010) = 3, \#_1(0010) = 1, \#_0(111) = 0$

Difference between 2 consecutive

- Definition: For each integer $n \geq 1$, the *triangular number* T_n is equal to $\frac{n(n+1)}{2}$. Plugging in the possible values of n, this forms the sequence of triangular numbers: 1, 3, 6, 10, 15, ...
  - To show that a number t is a triangular number: find a value of n such that $t = \frac{n(n+1)}{2}$.
  - To show that a number t is not a triangular number: one approach is to demonstrate that t lies strictly between two consecutive triangular numbers. For example, the number 12 is not triangular, since it lies strictly between 10 and 15. (This is a useful technique to remember, it also works with many other sequences!)

- Notation: For any Turing machine M, denote by $\mathscr{L}(M)$ the set of strings accepted by M.
  More specifically, $\mathscr{L}(M)$ is exactly the set of strings W such that, when machine M is executed with string W as input, the machine M eventually enters the Accept state (i.e., within a finite number of steps).
  (I was sure I defined this in lecture, it should probably be in Lecture 35, but I can't find it.)

The assignment questions start on the next page.

2 player game

1. For each of the given languages L, prove that L is not regular using the Pumping Lemma. You should not use the Myhill-Nerode Theorem or closure facts. I strongly suggest you use the 'Two-Player Game' proof format that we saw in lectures.
   In each solution, you must write a proof that your choice of W is legal (according to the conditions of the Pumping Lemma), and write a proof that your choice of i is such that $XY^iZ \notin L$.

(6 marks) (a) $L = \{W \in \{0,1\}^* \mid W = 0^d 1^e 0^f$, where d, e, f are integers, $d, e, f > 0$, and $f > d + e\}$

(6 marks) (b) $L = \{W \in \{0,1\}^* \mid$ there exist an integer $k \geq 1$ and $U \in \{0,1\}^*$ such that $W = 1^k U$, and $\#_1(U) \leq k\}$

(6 marks) (c) $L = \{W \in \{1\}^* \mid W = 1^t$ where t is a triangular number$\}$

$1^{\frac{p(p+1)}{2}} 0$

$1^{tp} 0 = 1^a . 1^b 1^{tp-(a+b)} 0$

$X = 1^a$
$Y = 1^b$

(4 marks) 2. Let $L = \{W \in \{0,1,2\}^* \mid \#_1(W)$ is not a triangular number$\}$
   Prove that L is not regular.
   Hint: You can use any technique we've learned from this course (Pumping Lemma, Myhill-Nerode, closure facts), but one approach to this question is <u>*much*</u> *easier than the others!*

probable pumping lemma

for n

but I am comfortable with Myhill nerode

3. For each part of this question, you will be asked to draw a Turing machine diagram. Your diagrams should be clear and easily readable. Along with each diagram, you need to provide an English description of your machine (pretend that a classmate is having trouble understanding how your machine is accomplishing the given task, and you are writing a "walkthrough" to explain it). It could be helpful to label your states with letters, and you can refer to those letters when explaining how your machine works.

(6 marks) (a) Draw the transition diagram of a Turing machine that decides the following language:

$$L = \{W \in \{0,1\}^* \mid \#_1(W) = 1 + 3 \cdot \#_0(W)\}$$

what if $\#_1(W) = 1$ 1

(6 marks) (b) Draw the transition diagram of a Turing machine that satisfies the following specification.
   Input: Assume that the tape contents will initially be \$ followed by an integer $N > 1$ written in base-2 (binary), and assume that the first symbol of N is 1. The tape head is initially pointing to the \$.
   Output: Your machine should halt in the Accept state, and the tape contents should be a \$ followed by $N + 1$ written in binary. The tape head should be pointing at the \$ when the machine halts.
   For example, if the tape contents are initially \$111, then your machine should accept with tape contents \$1000 and the tape head pointing to the \$.

(3 marks) 4. This question is about the Turing machine that is on the final page of this assignment.

Assume that the tape of the machine initially contains a \$ on the leftmost square, followed by a non-empty binary string W, followed by a $\#$. The tape head is initially pointing to the \$.

Explain what is on the tape after the machine halts (stops running).
(In other words, what is the purpose of this machine? What does it accomplish?)

5. For each of the following, answer TRUE or FALSE. Getting it correct is worth 1 mark. For the remaining 2 marks, you must prove that your answer is correct.

(3 marks) (a) For every Turing machine M, there exists a language L such that M decides L. False

(3 marks) (b) For every Turing machine M, there exists a language L such that M recognizes L. True

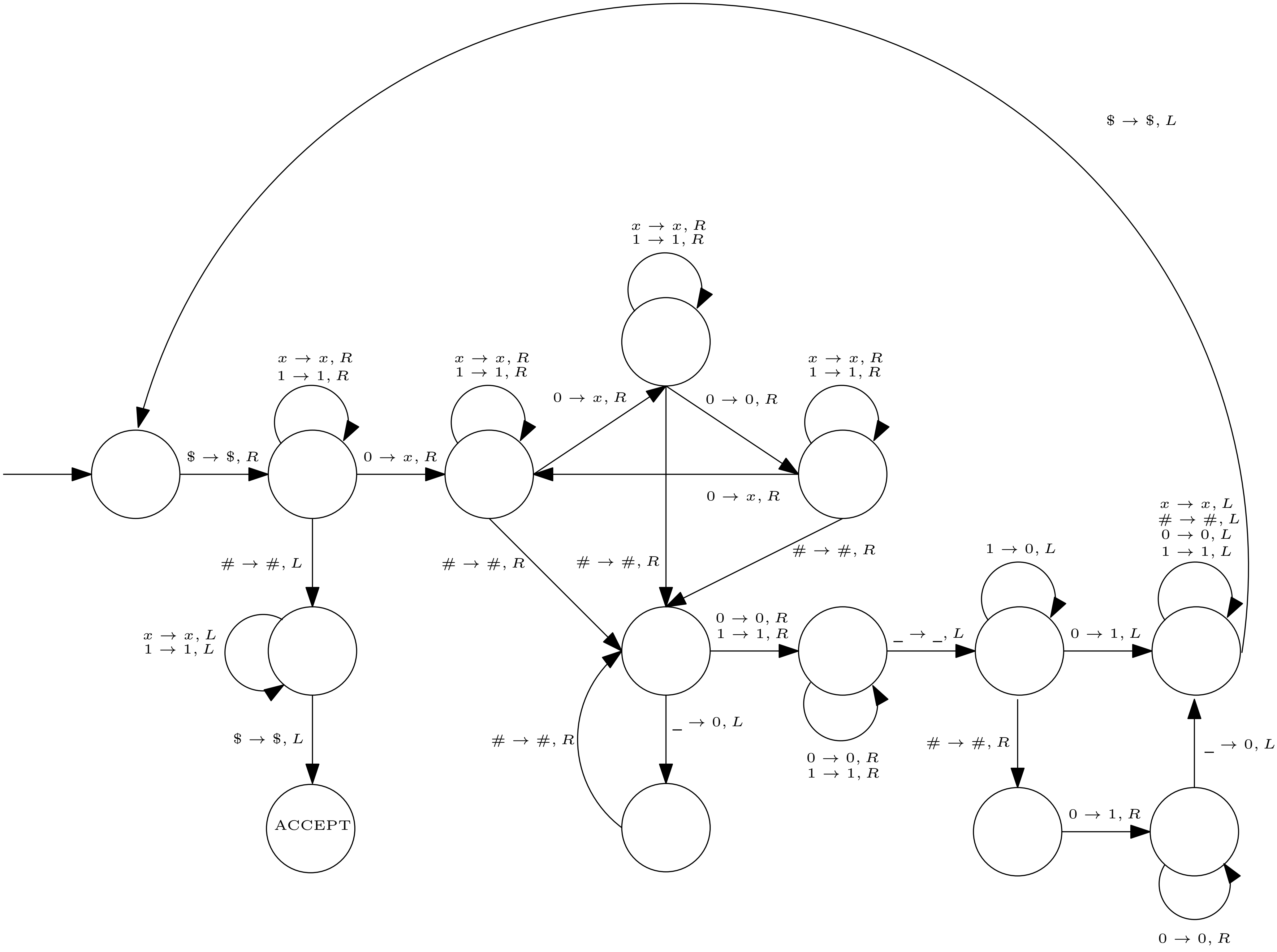
Use AcceptTM

AcceptTM

can we express TM's language using regular expression

DFA

for every DFA $\exists L$ M decides L.



$\$ \to \$, L$
$x \to x, R$
$1 \to 1, R$
$\$ \to \$, R$
$0 \to x, R$
$0 \to x, R$
$0 \to 0, R$
$0 \to x, R$
$\# \to \#, L$
$\# \to \#, R$
$\# \to \#, R$
$\# \to \#, R$
$x \to x, L$
$1 \to 1, L$
$\$ \to \$, L$
ACCEPT
$\_ \to 0, L$
$\# \to \#, R$
$0 \to 0, R$
$1 \to 1, R$
$\_ \to \_, L$
$1 \to 0, L$
$0 \to 1, L$
$x \to x, L$
$\# \to \#, L$
$0 \to 0, L$
$1 \to 1, L$
$\# \to \#, R$
$0 \to 1, R$
$\_ \to 0, L$
$0 \to 0, R$

- The game goes back and forth as follows:

1. You pick the language L
2. Opponent picks a positive integer value for p
3. You pick a string $W \in L$ that has length at least p
4. Opponent splits $W = XYZ$ with $|Y| > 0$ and $|XY| \leq p$
5. You pick an $i \geq 0$ such that $XY^iZ \notin L$

- If you can make a legal pick in each of your turns, then you win the game! (and winning the game means you've proven that L is not regular)

$\#_1(W)$ is not triangle

P

$W = 0^P 1 \in L$

$a \leq P$

$XY = 0^a 0^{P-a} 1$

10^P

$XY = 10^a 0^{P-a}$

L is picked

opponent picks P

$$t_P = \frac{P(P+1)}{2}$$

$$t_{P+1} = \frac{(P+1)(P+2)}{2}$$

$$t_{P+1} - t_P = \frac{(P+1)(P+2) - P(P+1)}{2}$$

$$= \frac{(P+1)(P+2-P)}{2} = \frac{2(P+1)}{2}$$

$= P+1$

	1	2	3	4	5	
	1	3	6	10	15	21
P	1	2	3	4	5	
P+1	2	3	4	5	6	

tp

$tp+1$

$\frac{n(n+1)}{2} = P$

$1^{tp+1}0$

$XY2 = 1^{tp+1}0$

$XY = 1^{n}$

$XY =$

Also try anyhill narasla

$1^{tp+1}0$

$tp+1 = m$

$XY = 1^{q}$

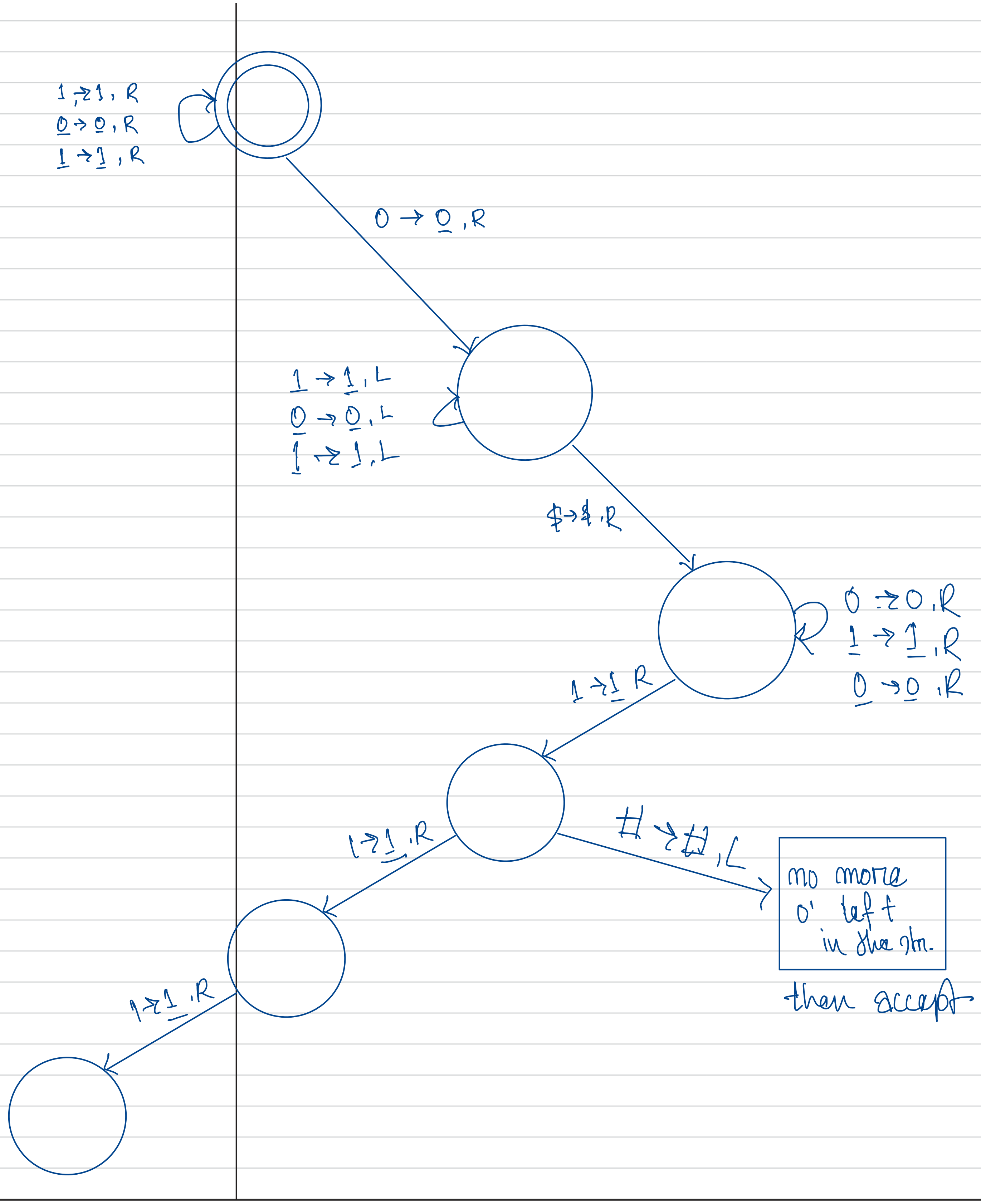
$XY2 = 1^{m}0$

$XY2 = 1^{q}1^{m-q}0$

$|Y| = n$

(a) Draw the transition diagram of a Turing machine that decides the following language:

$$L = \{W \in \{0,1\}^* \mid \#_1(W) = 1 + 3 \cdot \#_0(W)\}$$



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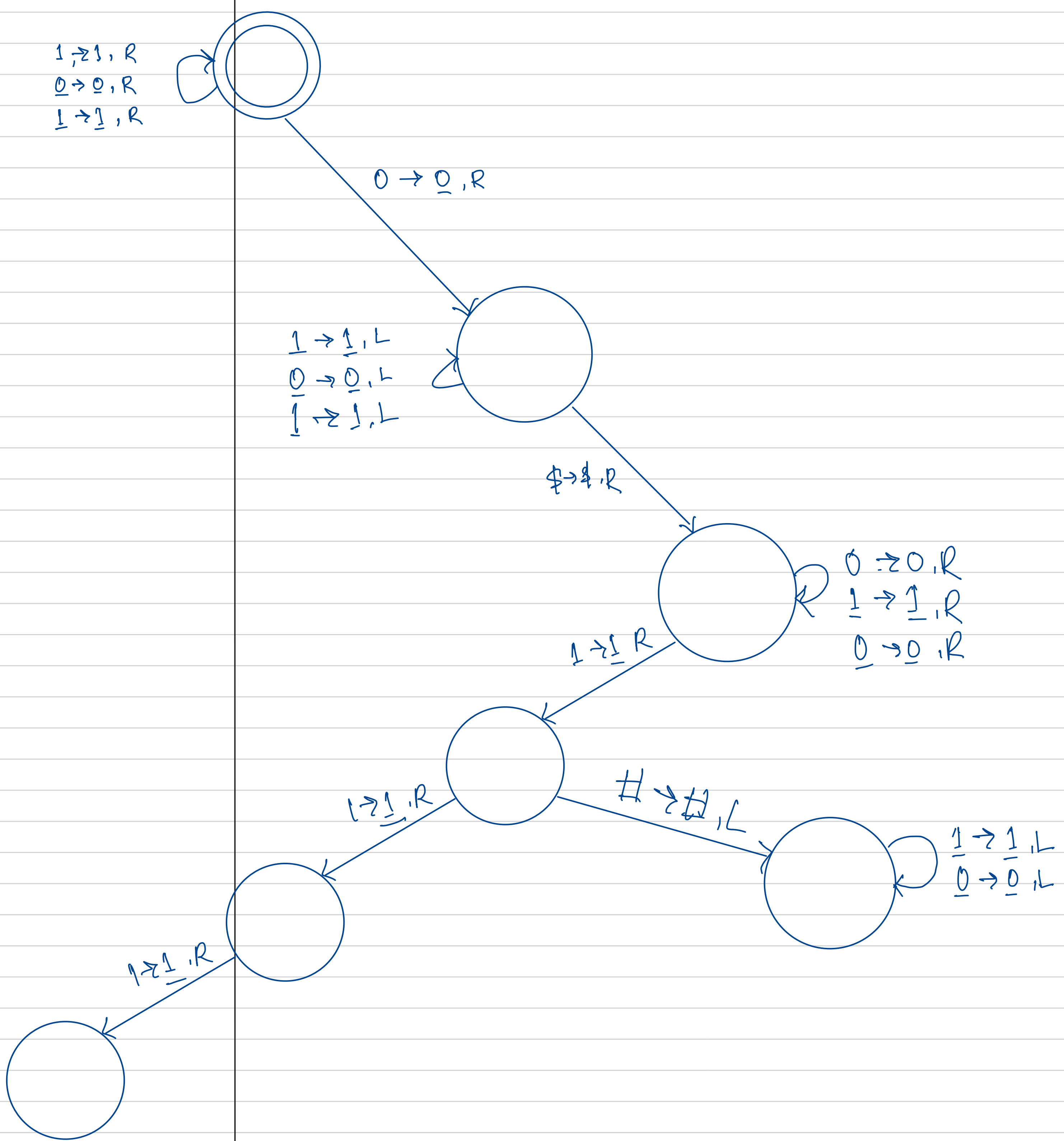

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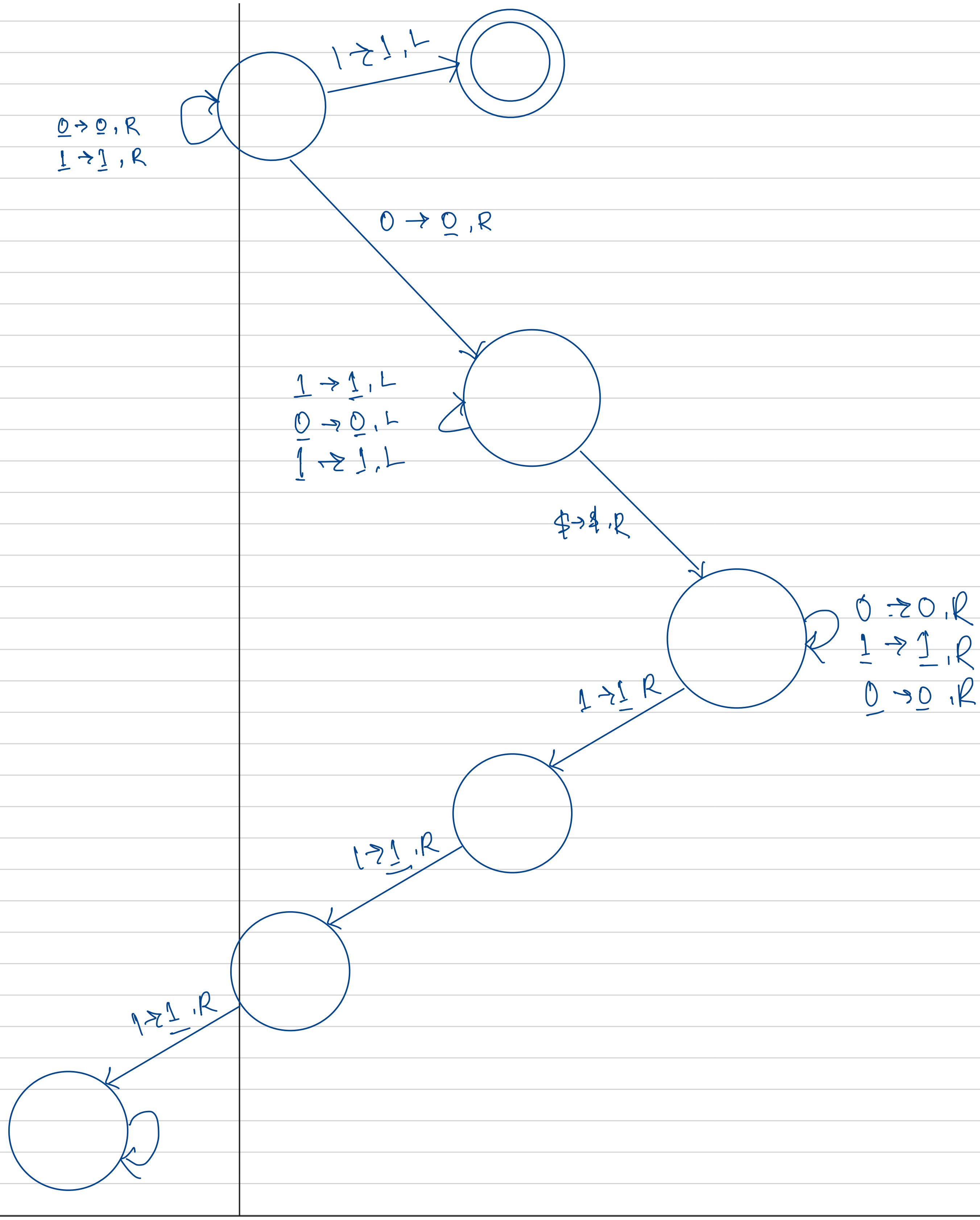


1 go through the if seen 1 mark it <u>1</u>

2 keep going R

2.1 if seen # we are done

2.2 if seen 1 count 3 more 1. (mark them)

2.2.1 then go to the start.

2.2.1.1 then keep going R until 0,

mark 0 . <u>0</u>

then go to the start

repeat 2

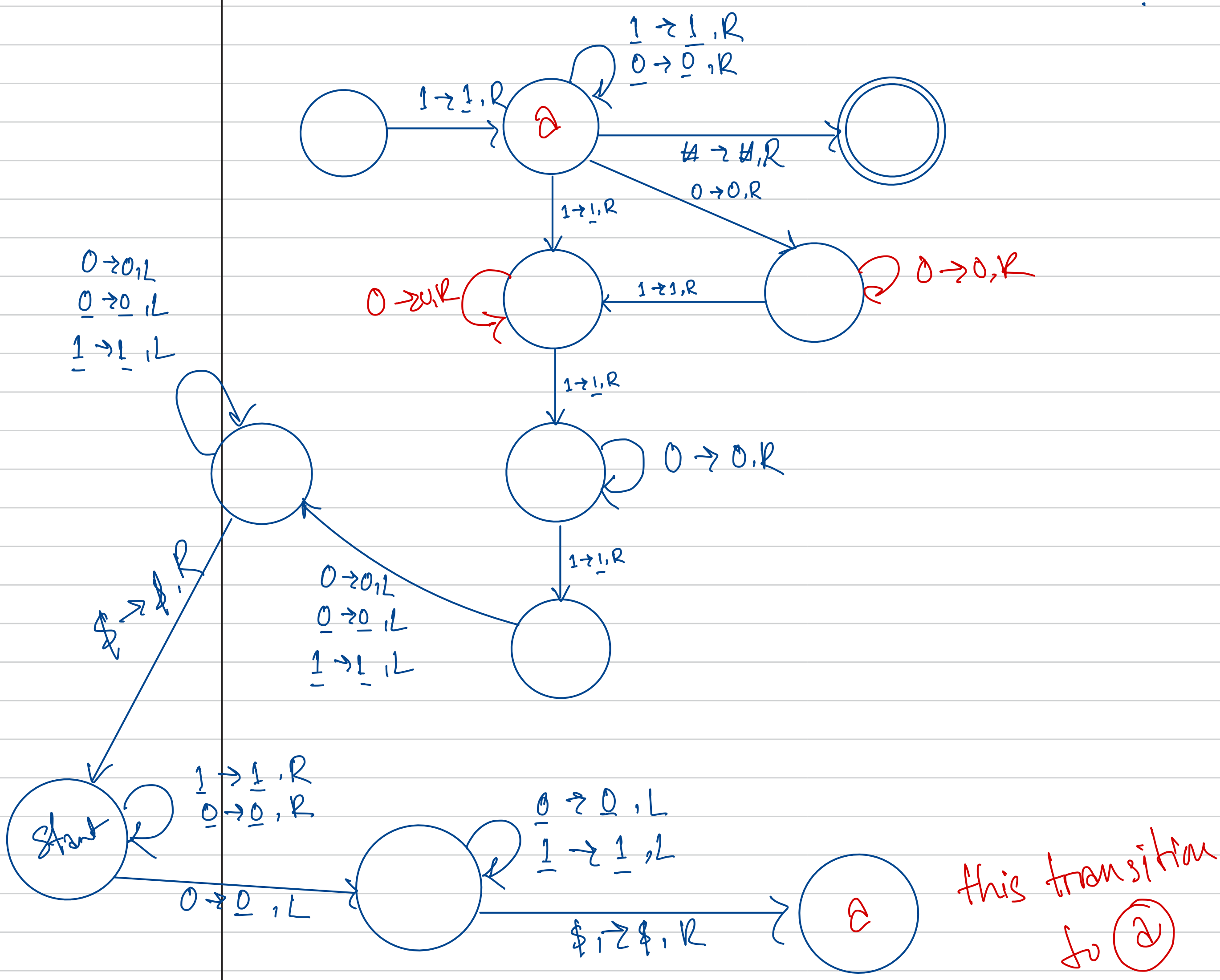
go thru the tape see 1.

case:1 no more string #

case:2 have seen 1 → check 3 1's

go to start check 1 0's

go to the sta.



myhill nerode:

let for $n \le 1$

$t_n = \frac{n(n+1)}{2}$ is a triangle number

$t_n + (n+1) = \frac{(n+1)(n+2)}{2}$ the next triangle number.

$t_n + 1 + (n-1)$

$t_n + n$

$t_n + 2 + (n-2)$

let w is a string that contain $t_n + 1$ 1's.

w' " " " $t_n + 2$ 1's

let, x has $n-1$ 1's

then $w \cdot x \in L$ as $\#_1(w \cdot x)$ is $t_n + 1 + (n-1)$
$= t_n + n$

then $w' \cdot x \notin L$ as $\#_1(w' \cdot x)$ is $t_n + 2 + (n-1)$

$t_n + 2 + n - 1$

$t_n + n + 1$

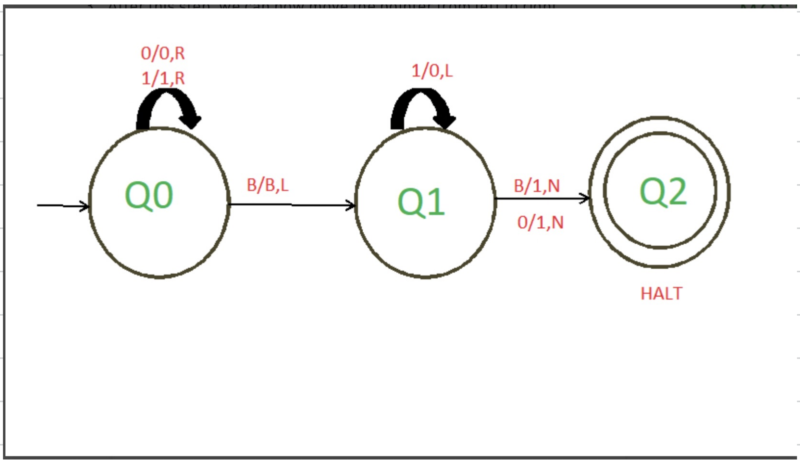
is the next triang

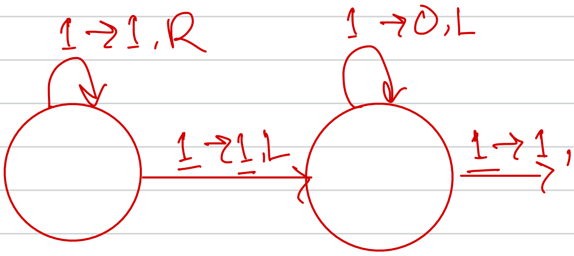
Hence we have n such num...

therefore we n equivalence

cls

Binary addition





1. c

$1^{t_p}0 = 1^a . 1^b 1^{t_p - (a+b)} 0$

$x = 1^a$

$y = 1^b$

Since $|Y| \leq P$

$|Y| < P+1$

$1^a . 1^b . \boxed{1^b}\ 1^{t_p - (a+b)} 0$

is not triangular number.

1. b

(b) $L = \{W \in \{0,1\}^* \mid$ there exist an integer $k \geq 1$ and $U \in \{0,1\}^*$ such that $W = 1^k U$, and $\#_1(U) \leq k\}$

$1^P 0 1^P \in W$

$XY = 1^a . 1^b$

$XYZ = 1^a . 1^b . 1^{P-(a+b)} 0 1^P$

$XY^0Z = 1^a . 1^{P-(a+b)} 0 1^P$

$= 1^{a+P-a-b} 0 1^P$

$= 1^{P-b} 0 1^P \notin L$

$k = P-b < P$

since $b = |Y|$

and $b > 0$.

1. (a)

(a) $L = \{W \in \{0,1\}^* \mid W = 0^d 1^e 0^f,$ where d, e, f are integers, $d, e, f > 0$, and $f > d + e\}$

let,

$\omega = 0^P 1 0^{P+2}$

$X \cdot Y \cdot Z = \underbrace{0^a}_{X} \cdot \underbrace{0^b}_{Y} \cdot \underbrace{0^{P-(a+b)} 1 0^{P+2}}_{Z}$

$\boxed{(Y)^{P+3} \quad 0^{b(P+3)} = 0^{bP+3b}}$ Later.

let, $(Y)^P = (0^b)^P$

$XY^PZ = 0^a 0^{Pb} 0^{P-(a+b)}$

$= 0^{\cancel{a}+Pb+P-\cancel{a}-b}$

$= 0^{Pb+P-b}$

$P+P-1$

$0^a \underbrace{0^b 0^b}_{Y \cdot Y} 0^{P-(a+b)} 1 0^{P+2}$

$0^{\cancel{a}+b+b+P-\cancel{a}-\cancel{b}}$

$0^a (0^b)^P 0^{P-(a+b)} 1 0^{P+2}$

$0^a 0^{Pb} 0^{P-(a+b)} 1 0^{P+2}$

$0^{\cancel{a}+Pb+P-\cancel{a}-b}$

$0^{Pb+P-b} 1 0^{P+2}$

$0^{P+b(P-1)}$

0

$P+b(P-1)+1$

$P+1+1(P-1)$

$P+1+P-1$

$2P$

$b=1$ $2P$ $P=1$

$1+2$

2

1.(a)

(a) $L = \{W \in \{0,1\}^* \mid W = 0^d 1^e 0^f$, where d, e, f are integers, $d, e, f > 0$, and $f > d + e\}$

$0^P 1^P 0^{2P+1}$

$X = 0^a$; $Y = 0^b$

$XYZ = 0^a 0^b 0^{P-a-b} 1^P 0^{2P+1}$

$b = 2$

$2b = 4$

00

0000

$(00)^2 = 0000$

$X(Y)^2 Z = 0^a 0^{2b} 0^{P-a-b} 1^P 0^{2P+1}$

$= 0^{P+b} 1^P 0^{2P+1}$

$d = P+b$

$e = P$

$f = 2P+1$

Since $|Y| > 0$

$b > 0$

$2P + b > 2P$

$2P + b$

$X(Y)^3 Z = 0^a 0^{3b} 0^{P-a-b} 1^P 0^{2P+1}$

$= 0^{P+2b} 1^P 0^{2P+1}$

$d = P + 2b$

$e = P$

$f = 2P + 1$

$P + b + P \geq 2P + 1$

$2P + b \geq 2P + 1$

$b > 0$

$b \geq 1$

$b > 0 \Rightarrow 2b > 1$ if b is an integer.

$P = P$

$2P = 2P$

since b is an integer, $b > 0$

$b \geq 1$

$2p + b \geq 2p+1$

$d+e \geq f$

$\therefore 0^a 0^{2b} 0^{p-a-b} 1^p 0^{2p+1}$ not in L.

5. For each of the following, answer TRUE or FALSE. Getting it correct is worth 1 mark. For the remaining 2 marks, you must prove that your answer is correct.

arks) (a) For every Turing machine M, there exists a language L such that M decides L.

arks) (b) For every Turing machine M, there exists a language L such that M recognizes L.

acct M

Does every turing machine has a reject state

Since $accept_{TM}$ is Turing undecidable.

that means there exists.

$W = \langle T, x \rangle$

where T a turing machine doesn't halt

on x, hence

the statement is false

$\forall M \; \exists L \ni M$ decides L

number of turing machine is finite.

turing solveable

Any thing DFA can do TM can do.

$\exists M \; \forall L \ni M$ doesn't halt for $x \in L$

$\exists M \; \forall L \ni M$

Since $Accept_{TM}$ is turing recognizeable:

for all $w = \langle T, x \rangle$

T there exist x.

such that, T accepts x

$Accept_{TM}$ = {Set all TM for which there is atleast 1 X ∃ it accepts x, }

{

Can we represent $\forall x \in L$ where L is an arbitrary language with arbitrary alphabets, using binary representation ?

$\forall$ Tm $\in Accept_{TM}$

L = { }

Tm for which it never halts. for any input x for any L

Hence.

There is at least one TM for which there exis $x \in L$ such that <Tm, X> doesn't halt.

hence. that Tm doesn't

Suppose we have a TM that decides $L = \{ \}$

Since $Accept_{TM}$ is not decidable

there exist $\langle T, x \rangle$ for which the TM deciding $Accept_{TM}$ never halts.

because $T \leftarrow x$ never halts.

now if T decides L the $x \notin L$

Recognizers

can a TM accept empty language.

winnipeg

work

$Accept_{TM}$ is turing recognizeable or recognizeable

Hence,

1. if the format of $w \neq \langle T, x \rangle$ then T_A rejects
2. if the format of $w = \langle T, x \rangle$ then,

if T accepts x then T_A accepts

else if T doesnot accept then T_A doesn't accept

Now $\forall \langle T, x_n \rangle$ where $n \geq 0$ and T_A accepts $w = \langle T, x_n \rangle$

$\mathcal{L}(T) = \{x_0 \cdots x_n\}$

Now, if $x \notin \mathcal{L}(T)$ the T_A doesn't accept

x, Hence. T_A recognizes

Therefor, $\forall T$ $\exists \mathcal{L}(T)$ or L such that, T recognizes L.

Decides

can a TM accept empty language.

winnipeg

work

$Accept_{TM}$ is not turing decidable

Hence,

1. if the format of $\omega \neq \langle T, x \rangle$ than T_A rejects
2. if the format of $\omega = \langle T, x \rangle$ than,

if T accepts x than T_A accepts

else if T doesnot accept than T_A doesn't accept

i.e. T reject or doesn't halt than T_A reject or doesn't halt

Now $\forall \langle T, x_n \rangle$ where $n \geq 0$ and T_A accepts $\omega = \langle T, x_n \rangle$

$\boxed{\cancel{\mathcal{L}(T) = \{x_0 .. x_n\}}}$ $\mathcal{L}(T) \in P(\{x_0 .. x_n\})$

Now, if $x \notin \mathcal{L}(T)$ .the T_A reject or doesn't halt

Therefore, $\forall T$ $\exists \mathcal{L}(T)$ or L such that, T decides L

Now for $\forall L \in P(\{x_0 .. x_n\})$ if $x \notin L$, the T_A rejects

or doesn't halt which implies T also doesn't reject

$P(\{0,1\}^*)$

for $L \in P(\{0,1\}^*)$, if T accepts x, $\forall x \in L$

then