COMP 3030 Fall 2021 Homework Assignment 5

due Tuesday, November 30, 2021, 11:59pm

Must be submitted on Crowdmark, **not** UM Learn

Page 1: Important definitions, notation, and hints!

- Notation: for any string W and any symbol x, we write $\#_x(W)$ to represent the number of occurrences of symbol x in W.
  Examples: $\#_0(\varepsilon) = 0, \#_1(\varepsilon) = 0, \#_0(0010) = 3, \#_1(0010) = 1, \#_0(111) = 0$

- In this assignment, if you need to prove something is recognizable or decidable, you should be using 'Turing machine pseudocode' (as in Lectures 49 and after) instead of trying to draw a transition diagram.

The assignment questions start on the next page.

1. Let function $D : \{0,1\}^* \to \mathbb{Z}$ be defined by $D(W) = \#_1(W) - \#_0(W)$.

(2 pts) (a) Is D one-to-one? Prove that your answer is correct.

(2 pts) (b) Is D onto? Prove that your answer is correct.

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2. Let function $C : \{0,1\}^* \to \{0,1\}^*$ be defined by $C(W) = 1 \cdot W$.

$c^{-1}(w) = w'$

(2 pts) (a) Is C one-to-one? Prove that your answer is correct.

(2 pts) (b) Is C onto? Prove that your answer is correct.

(3 pts) (c) Let $L = \{0,1\}^*$, that is, the set of all binary strings. Prove that L is a countable set.

$f(1) \to 1$

(5 pts) 3. Let $S = \{f \mid f : \{0,1\}^* \to \{0,1\}^*\}$. In other words, S is the set consisting of all functions whose domain and range are binary strings. Use the Diagonal Method to prove that S is uncountable. (Drawing a table can be a useful visual tool, but your submitted solution should not contain one.)

<u>Side note:</u> We can conclude from this result, that, even if we were able to write out every possible Java program that takes in a binary string and outputs a binary string, there are still infinitely many such functions out there that we cannot implement. This is because the number of different Java programs is countably infinite, yet this question proves that the number of functions is uncountably infinite!

p iff q

Is this a Tm?Can we say let's assume,Tr is a Tm that decides this regular expression?

(8 pts) 4. Prove that a language L is decidable if and only if $L \leq_m 000^*(11 + 111)$

Are all regular expression Turing decideable?

(7 pts) 5. Prove that the following language is decidable.

$\text{HighSteps}_{\text{TM}} = \{W \in \{0,1\}^* \mid W = \langle T \rangle$ and T is a TM that:

no matter which input $X \in \{0,1\}^*$ is given, machine T takes at least 420 steps$\}$

DFA / NFA

Highstep TM

Clarification: "*takes 1 step*" *means* "*follows 1 transition*"

(8 pts) 6. Prove that $\overline{\text{Equal}_{\text{TM}}}$ is unrecognizable by choosing an appropriate language L and proving that $L \leq_m \overline{\text{Equal}_{\text{TM}}}$.

(10 pts) 7. In this question, we'll consider a new kind of Turing machine, that I call an "American Turing Machine", or USTM. They are pretty much exactly the same as the Turing machines we've been using in this course (i.e., see Lectures 32-35), but with one difference: every state in the machine has a colour, Red or Blue.

Halt

Let $L = \{W \in \{0,1\}^* \mid W = \langle T, c \rangle$ where T is a USTM, $c \in \{\text{Red,Blue}\}$, and:
there exists at least one binary input string X such that T executed on input X enters a state coloured $c\}$

<u>Your task:</u> Prove that L is undecidable.

1. Let function $D : \{0,1\}^* \to \mathbb{Z}$ be defined by $D(W) = \#_1(W) - \#_0(W)$.

integer

(a) Is D one-to-one? Prove that your answer is correct.

(b) Is D onto? Prove that your answer is correct.

1

a

$x \neq y$

$D(x) \neq D(y)$

$D(x) = D(y)$

$\#_1(x) - \#_0(x) = \#_1(y) - \#_0(y)$

we can disprove by an example?

b

(c) Let $L = \{0,1\}^*$, that is, the set of all binary strings. Prove that L is a countable set.

Proof a bijection:

$2^3 = 8$

111100 → 111100

000111 → 1111

1

00

01

10

11

000

001

010

011

100

111

. .

count of 0 count of 1

$C(w) = 1 \cdot w$

one to one

L

x_t

$1 \cdot \{0,1\}^* \to \{0,1\}^*$

$f(x)$ = takes 1 the

Since C is a bijection from all natural numbers
$\mathbb{Z}$

Hence C is countable.

$f(\ ) \rightarrow$ even

if we can proof a bijection from a countable set to another set than that set is also countable

4. Prove that a language L is decidable if and only if $L \leq_m 000^*(11+111)$

is $000^*(11+111)$ a turing machine? can we say T_L decides $000^*(11+111)$

can we describe turing machine like,

keep going right until see a 1.

move two step left

accept.

?

2 proofs.

T_L

if L is decidable then $L \leq_m 000^*(11+111)$

if $L \leq_m 000^*(11+111)$ then L is decidable

if L is decidable,

there is a TM T_L deciding L.

if L is decidable,

if $x \in L$ then $x \in 000^*(11+111)$

if $x \notin L$ then $x \notin 000^*(11+111)$

$f(x)$

if $x \in L$ then $f(x) = 0011$

if $x \notin L$ then $f(x) = 0110$

Hence

Since, all regular expression has DFA

if $w \in L$ then $f(w) \in \mathcal{L}(000^*(11+111))$

Given : L is decidable.

if $w \in L$ then $f(w) = 0011$

else if $w \notin L$ $f(w) = 0101$

As L is decidable

Prove $f(w)$ is the correct reduction:

First suppose that, $w \in L$

then since $f(w) = 0011$

and $0011 \in \mathcal{L}(000^*(11+111))$

Next suppose that, $w \notin L$

then since $f(w) = 0101$

and $0101 \notin \mathcal{L}(000^*(11+111))$

therefore $f(w)$ is the correct reduction.

Therefore $L \leq_m (000^*(11+111)$

Proof if: $L \leq_m 000^*(11+111)$ then L is decideable.

$\exists$ DFA such that it decides $000^*(11+111)$

therefore $000^*(11+111)$ is decideable,

following the definition of reduction,

if $000^*(11+111)$ is decideable then

L is also decideable.

(7 pts) 5. Prove that the following language is decidable.

$\text{HighSteps}_{\text{TM}} = \{W \in \{0,1\}^* \mid W = \langle T \rangle$ and T is a TM that:
no matter which input $X \in \{0,1\}^*$ is given, machine T takes at least 420 steps$\}$

Clarification: "takes 1 step" means "follows 1 transition"

$f(w) \Rightarrow$

$w \neq \langle T \rangle$ $\notin \text{HighSteps}_{\text{TM}}$

$w = \langle T \rangle$ takes less than 420 steps. $\notin \text{HighSteps}_{\text{TM}}$

$w = \langle T \rangle$ takes at least 420 steps $\in \text{HighSteps}_{\text{TM}}$

$f(w) \Rightarrow$

if $\rightarrow$

$w \neq \langle T \rangle$ $\notin \text{HighSteps}_{\text{TM}}$

$w = \langle T \rangle$ takes less than 420 steps. $\notin \text{HighSteps}_{\text{TM}}$

return $\langle\ \rangle$ $\ni$ Some L reject

$w = \langle T \rangle$ takes at least 420 steps $\in \text{HighSteps}_{\text{TM}}$

return $\langle\rangle$ $\ni$ some L accepts.

Halt

Prove both $\text{Highstep}_{\text{TM}}$ and It's complement recognizeable.

$w \neq \langle T, x \rangle$
$w = \langle T, x \rangle$

$\overline{\text{Highslop}_{TM}}$ is decideable.

we build a TM that simulates the $\langle T, x \rangle$ for

(7 pts) 5. Prove that the following language is decidable.

$A \leq_m B$

if B is decideable

$\swarrow$ A is also decide.

$\text{HighSteps}_{\text{TM}} = \{W \in \{0,1\}^* \mid W = \langle T \rangle$ and T is a TM that:

no matter which input $X \in \{0,1\}^*$ is given, machine T takes at least 420 steps$\}$

after 420 step does it decide

HTM

Clarification: "takes 1 step" means "follows 1 transition"

Testing Halt

$f(W) \Rightarrow$

$W \neq \langle T \rangle$ $\notin \text{HighSteps}_{\text{TM}}$

$W = \langle T \rangle$ takes less than 420 steps. $\notin \text{HighSteps}_{\text{TM}}$

$W = \langle T \rangle$ takes at least 420 steps $\in \text{HighSteps}_{\text{TM}}$

$f(W) \Rightarrow$

if $\rightarrow$

$W \neq \langle T \rangle$ $\notin \text{HighSteps}_{\text{TM}}$ return ε

$W = \langle T \rangle$ takes less than 420 steps. $\notin \text{HighSteps}_{\text{TM}}$

return $\langle M \rangle \ni$ Some Halt reject

$M \rightarrow$ simulate T if T takes less than 420 steps loop forever.

for $x \in \{0,1\}^{420}$

$W = \langle T \rangle$ takes at least 420 steps $\in \text{HighSteps}_{\text{TM}}$

return $\langle M \rangle \ni$ some L accepts.

return $\langle T \rangle$

$f(w)$

if $w \neq \langle T \rangle$ for any Turing machine T, then output ε.

if $w = \langle T \rangle$ for some Turing machine T, then output $\langle M \rangle$

1. Simulate T on input x
2. if T stops before 420 steps, then enter an infinite loop

$\underline{Accept_{DFA}}$

to $accept_{DFA}$, $accept_{NFA}$ or $accept_{RE}$

reduction function from TM to $accept_{DFA}$

$HighSteps_{TM} \leq_m Accept_{TM}$

$w \in HS_{TM}$

$f(w) = \langle T, \varepsilon \rangle \in Accept_{TM}$

$w \notin HS_{TM}$

case 1: $w \neq \langle T \rangle$

$f(w) = \langle T, \varepsilon \rangle \notin Accept_{TM}$

$w = \langle T \rangle$

M_x simulates T, with input x.
if accepts the infinite loop
if rejects then rejects.

case 2: $w = \langle T \rangle$

return M_x

M_x simulates T

$M_x \rightarrow$

can we simulate multiple input x?

1. Simulates T with input x
2. After simulating 420 transition it halts and accept.
3. if it stops before 420 transition then, reject

Now, $w \in HS_{TM}$

$f(w) = \langle M_x, x \rangle \in Accept_{TM}$

Next, $w \notin HS_{TM}$

can we use something that is true about T

1. $w \neq \langle T \rangle$

$f(w) = \langle M_x, x \rangle \notin Accept_{TM}$

2. $w = \langle T \rangle$ but $\exists x \ni \langle T, x \rangle$ T stops before 420 steps

$f(w) = \langle M_x, x \rangle \in Accept_{TM}$

$HighSteps_{TM} \leq_m Accept_{TM}$

$\omega \in HS_{TM}$

$f(\omega) = \langle T, \varepsilon \rangle \in Accept_{TM}$

$\omega \notin HS_{TM}$

case 1: $\omega \neq \langle T \rangle$

$f(\omega) = \langle T, \varepsilon \rangle \notin Accept_{TM}$

$\omega = \langle T \rangle$

M_x simulates T, with input x.
if accepts the infinite loop
if rejects then rejects.

case 2: $\omega = \langle T \rangle$

return M_x

for x in $\{0,1\}^{420}$

Simulate $T \leftarrow x$

M_x simulates T

$M_x \rightarrow$

can we simulate multiple input x?

1. Simulates T with input x $\forall x \in \{0,1\}^{420}$
2. After simulating 420 transition it moves to step 1
3. if it stops before 420 transition then, reject

Homeland

Accept.

Now, $\omega \in HS_{TM}$

$f(\omega) = \langle M_x, x \rangle \in Accept_{TM}$

Next, $\omega \notin HS_{TM}$

can we use something that is true about T

1. $\omega \neq \langle T \rangle$

$f(\omega) = \langle M_x, x \rangle \notin Accept_{TM}$

2. $\omega = \langle T \rangle$ but $\exists x \ni \langle T, x \rangle$ T stops before 420 steps

$f(\omega) = \langle M_x, x \rangle \in Accept_{TM}$

$HighSteps_{TM} \leq_m Accept_{TM}$

$M_c \to$

① Simulate T with input x for all $x \in \{0,1\}^{420}$

② for any simulation if T halts before 420 transitions the reject.

③ Else accept.

for x in $\{0,1\}^{100}$

Simulate T with input x. upto 100 steps

if T halts before 100 steps reject

accept.

6. Prove that $\overline{\text{Equal}_{\text{TM}}}$ is unrecognizable by choosing an appropriate language L and proving that $L \leq_m \overline{\text{Equal}_{\text{TM}}}$.

$\overline{\text{Equal}_{TM}}$ $\qquad$ Accept_{TM}

$\overline{\text{Accept}_{TM}} \leq_m \overline{\text{Equal}_{TM}}$

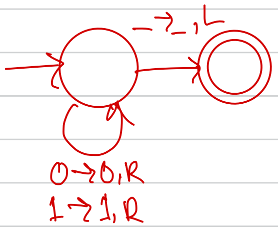
can we construct a TM that accepts one string?

$\overline{\text{Accept}_{TM}}$, Equal_{TM} are unrecognizable.

We need to find a L that can be reduced to $\overline{\text{Equal}_{TM}}$

can we buil a turing machine that accepts all binary string?

let, M_1 = accepts exactly one string.



$f(w)$

if $w \neq \langle T, x \rangle$ then $w \in \overline{\text{Accept}_{TM}}$,

$\langle M_\emptyset, M_U \rangle$

if $w = \langle T, x \rangle$ and T doesn't Accept x then, $\langle M_\emptyset, M_U \rangle$

M_U

case 1: T rejects x then,

is $M_\emptyset$ decideable

if $M_{\emptyset}$ decideable then M_U is also decideable.