Md. Hasibul Haque Sir

Lecture : 01

Date : 04.02.2020

Fourier Analysis

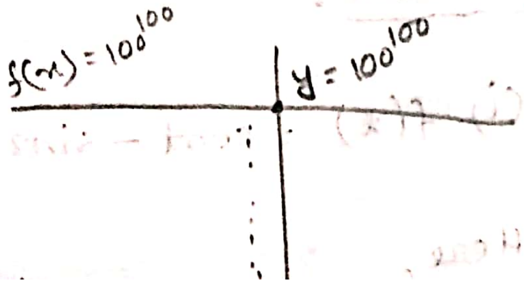
Fourier Analysis is a branch of Mathematics which consists of Fourier series, Fourier Transform and Fourier Integral. This branch of mathematics was named according to the name of the mathematician "Joseph Fourier".

Constant Function :

A function is said to be constant function if it can have only one value.

Example :

$y = f(x) = 100^{100}$



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Lecture: 02

Date: 09.02.2020

Periodic function:

A function is defined as periodic function if $f(t+T) = f(t)$.

Simply, if the ratios of the periods of individual periodic functions present in a ~~periodic~~ function can be converted into a ratio of natural numbers, then the function is called Periodic function.

$\square$ The lowest value of T for which $f(t+T) = f(t)$ is called the fundamental period of $f(t)$.

Example:

(i) $f(t) = \cos t - \sin 2t$

Here,

~~The~~ period of ~~cos~~ ~~is~~ $\cos t$ is $T_1 = 2\pi$ and $\sin 2t$ is $T_2 = \pi$.

$\therefore \frac{T_1}{T_2} = \frac{2\pi}{\pi} = 2$

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$\therefore f(t)$ is a periodic function.

(ii) $f(t) = \cos t + \sin 2t + \cos\sqrt{3}t$

Here,
The period of $\cos t$, $\sin 2t$ and $\cos\sqrt{3}t$ are 2π, π and $\frac{2\pi}{\sqrt{3}}$

$\therefore \frac{2\pi}{\pi} = 2$

Now,

$\frac{2}{\frac{2\pi}{\sqrt{3}}} = \cancel{2} \times \frac{\sqrt{3}}{\cancel{2}\pi} = \frac{\sqrt{3}}{\pi}$ is can't be transformed into a ratio of natural numbers.

$\therefore f(t)$ is an aperiodic function.

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Even function:

A function is said to be even if $f(-t) = f(t)$

$\cos t$ is an even function in it's natural domain, $t \in (-\infty$ to $\infty)$. But in the domain $t \in (0, \infty)$, $\cos t$ is a odd function.

Odd function:

A function is said to be odd if $f(-t) = -f(t)$

$\sin x$ is a odd function in it's natural domain $x \in (-\infty, \infty)$.

A ~~no~~ function may be even or odd or none of these.

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Lecture: 03

Date: 11.02.2020

Fourier series

Every periodic function can be represented as an infinite series of sine and cosine functions.

For example,

$$f(t) = \frac{1}{2}a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t + \dots$$

$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + \dots \dots$$

$$\boxed{\therefore f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin (n\omega t)\right)} \quad (1)$$

↑

DC Component

or, Average area/value of the function

(i) $\int_{-T/2}^{T/2} \cos n\omega t \, dt$

$$= \left[\frac{\sin(n\omega t)}{n\omega}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{n\omega}\left[\sin\left(n \times \frac{2\pi}{T} \times \frac{T}{2}\right) - \left\{\sin\left\{n \times \frac{2\pi}{T}\left(-\frac{T}{2}\right)\right\}\right\}\right]$$

$$= \frac{1}{n\omega}\left[\sin(n\pi) + \sin(n\pi)\right]$$

$$= \frac{2}{n\omega}\sin(n\pi)$$

$$= \frac{2}{n\omega} \times 0$$

$$= 0$$

Note that: The integration of a odd function over a complete period is zero.

$\boxed{\square}$ The sum or product or subtraction of two or more even function is ~~zero~~ an even function.

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☐ The product of an even and a odd function is a odd function.

For example :

$$f(t) = \sin \omega t \cos 2\omega t$$

Here, $f(t)$ is a odd function.

☐ The sum of two or more odd function is an odd function.

☐ The product of two odd function is an even function.

(ii) $\int_{-T/2}^{T/2} \sin(n\omega t)\, dt$

$$= \left[-\frac{\cos(n\omega t)}{n\omega} \right]_{-T/2}^{T/2}$$

$$= -\frac{1}{n\omega} \left[\cos(n\omega t) \right]_{-T/2}^{T/2}$$

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$$= -\frac{1}{n\omega}\left[\cos\left(n\times\frac{2\pi}{T}\times\frac{T}{2}\right) - \cos\left\{n\times\frac{2\pi}{T}\left(-\frac{T}{2}\right)\right\}\right]$$

$$= -\frac{1}{n\omega}\left[\cos(n\pi) - \cos(n\pi)\right]$$

$$= -\frac{1}{n\omega}\times 0$$

$$= 0$$

Fourier Components

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left\{a_n\cos(n\omega t) + b_n\sin(n\omega t)\right\} \quad \text{——} ①$$

Now, integreating equation ① over the period $-T/2 < t < T/2$

$$\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{2}\int_{-T/2}^{T/2} a_0\,dt + \int_{-T/2}^{T/2} a_n\cos(n\omega t)\,dt + \int_{-T/2}^{T/2} b_n\sin(n\omega t)\,dt$$

Here,

$$\int_{-T/2}^{T/2}\cos(n\omega t)\,dt = 0$$

and, $\int_{-T/2}^{T/2}\sin(n\omega t)\,dt = 0$

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$$\therefore \int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{2}\int_{-T/2}^{T/2} a_0\,dt + a_n \times 0 + b_n \times 0$$

$$\Rightarrow \int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{2}a_0 \left[t\right]_{-T/2}^{T/2}$$

$$\Rightarrow \int_{0}^{T} f(t)\,dt = \frac{1}{2}a_0 \cdot T$$

$$\boxed{\therefore a_0 = \frac{2}{T}\int_0^T f(t)\,dt}$$

Multyplying equation ① by $\cos(m\omega t)$ and integrating both sides,

$$\therefore \int_{-T/2}^{T/2} f(t)\cos(m\omega t)\,dt = \frac{1}{2}a_0\int_{-T/2}^{T/2}\cos(m\omega t)\,dt + \int_{-T/2}^{T/2} a_n\cos(n\omega t)\cos(m\omega t)$$

$$+\int_{-T/2}^{T/2} b_n \sin(n\omega t)\cos(m\omega t)\,dt$$

Here,

$$\frac{a_0}{2}\int_{-T/2}^{T/2}\cos(m\omega t)\,dt = 0$$

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$$\int_{-T/2}^{T/2} b_n \sin(n\omega t)\cos(m\omega t)\,dt$$

$$= b_n \int_{-T/2}^{T/2} \frac{1}{2}\left\{\sin(m+n)\omega t - \sin(m-n)\omega t\right\} dt$$

$$= \frac{b_n}{2}\left[\left[\frac{-\cos(m+n)\omega t}{(m+n)\omega}\right]_{-T/2}^{T/2} - \left[-\frac{\cos(m-n)\omega t}{(m-n)\omega}\right]_{-T/2}^{T/2}\right]$$

$$= \frac{b_n}{2\omega}\left[\left[\frac{-1}{m+n}\left\{\cos(m+n).\frac{2\pi}{T}.\frac{T}{2} - \cos(m+n).\frac{2\pi}{T}.\left(-\frac{T}{2}\right)\right\}\right]\right.$$

$$\left. - \left[-\frac{1}{m-n}\left\{\cos(m-n).\frac{2\pi}{T}.\frac{T}{2} - \cos(m-n).\frac{2\pi}{T}\left(-\frac{T}{2}\right)\right\}\right]\right]$$

$$= \frac{b_n}{2\omega}\left[\left[-\frac{1}{m+n}\left\{\cos(m+n)\pi - \cos(m+n)\pi\right\} + \frac{1}{m-n}\left\{\cos(m-n)\pi - \cos(m-n)\pi\right\}\right]\right]$$

$$= \frac{b_n}{2\omega}\left[-\frac{1}{m+n}\times 0 + \frac{1}{m-n}\times 0\right]$$

$$= \frac{b_n}{2\omega}\times 0 = 0$$

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Now,

$$\therefore I = \int_{-T/2}^{T/2} a_n \cos(n\omega t)\cos(m\omega t)\,dt$$

$$\Rightarrow I = \frac{a_n}{2}\int_{-T/2}^{T/2} \{\cos(m+n)\omega t + \cos(m-n)\omega t\}\,dt$$

If, $m = n$, then,

$$I = \frac{a_n}{2}\int_{-T/2}^{T/2} \{\cos(2m\omega t) + \cos 0^\circ\}\,dt$$

$$= \frac{a_n}{2}\left[\int_{-T/2}^{T/2} dt + \int_{-T/2}^{T/2} \cos(2m\omega t)\,dt\right]$$

$$= \frac{a_n}{2}\left[[t]_{-T/2}^{T/2} + 0\right]$$

$$= \frac{a_n}{2}\left[T/2 + T/2\right]$$

$$= \frac{a_n T}{2}$$

If, $m \neq n$,

$$I = \frac{a_n}{2}\int_{-T/2}^{T/2} \{\cos(m+n)\omega t + \cos(m-n)\omega t\}\,dt$$

$$= \frac{a_n}{2}(0+0)$$

$$= 0$$

$$\therefore \int_{-T/2}^{T/2} f(t)\cos(m\omega t)\,dt = 0 + \frac{a_n T}{2} + 0$$

[Taking the active ~~cond~~ cycle on, for $m = n$]

$$\boxed{\therefore a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(m\omega t)\,dt}$$

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Home Work

Multiplying eauation ① by $\sin(m\omega t)$ and integrating it,

$$\int_{-T/2}^{T/2} f(t)\sin(m\omega t)\,dt = \frac{1}{2}a_0\int_{-T/2}^{T/2}\sin(m\omega t)\,dt$$

$$+\int_{-T/2}^{T/2} a_n\cos(n\omega t)\sin(m\omega t)\,dt + \int_{-T/2}^{T/2} b_n\sin(m\omega t)\sin(n\omega t)\,dt$$

Here,

$$\frac{1}{2}a_0\int_{-T/2}^{T/2}\sin(m\omega t)\,dt = 0$$

$$\int_{-T/2}^{T/2} a_n\cos(n\omega t)\sin(m\omega t)\,dt = 0$$ [Because it's an odd function]

Now,

$$B = \int_{-T/2}^{T/2} b_n\sin(m\omega t)\sin(n\omega t)\,dt$$

$$= \frac{b_n}{2}\int_{-T/2}^{T/2}\left[\cos(m-n)\omega t + \cos(m+n)\omega t\right]dt$$

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$$= \frac{b_n}{2}\left[\frac{\sin(m-n)\cancel{\omega t}}{(m-n)\cancel{\omega}} + \frac{\sin(m+n)\omega t}{(m+n)\omega}\right]_{-T/2}^{T/2}$$

If $m \neq n$, then,

$$B = \frac{b_n}{2}\int_{-T/2}^{T/2}\{\cos(m-n)\omega t - \cos(m+n)\omega t\}\,dt$$

$$= \frac{b_n}{2}(0-0)$$

$$= 0$$

If $m = n$, then,

$$B = \frac{b_n}{2}\int_{-T/2}^{T/2}(\cos 0^\circ - \cos 2m\omega t)\,dt$$

$$= \frac{b_n}{2}\left[[t]_{-T/2}^{T/2} - 0\right]$$

$$= \frac{b_n}{2}\left[T/2 + T/2\right]$$

$$= \frac{b_n T}{2}$$

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$$\therefore \int_{-T/2}^{T/2} f(t) \sin(m\omega t)\, dt = 0 + 0 + \frac{b_n T}{2}$$

$$\boxed{\therefore b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(m\omega t)\, dt}$$

Lecture: 04+05

Date: 19.02.2020

and 25.02.2020.

Q-01: Suppose, we have a given function, $f(t) = t\sin t$ which is defined over a single period $[-\pi, \pi]$. Find the fourier components for the given $f(t)$.

Solution:

If a fourier series is given by,

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega t) + b_n \sin(n\omega t)\right)$$

Then,

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt$$

Here,

$$f(t) = t\sin t$$

$$T = 2\pi$$

$$\therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

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$$\therefore a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} t \sin t \, dt$$

$$= \frac{1}{\pi}\left[t.(-\cos t) - \int -\cos t \, dt\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-t\cos t + \sin t\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\left(\pi\cos\pi - \{(-\pi)\cos(-\pi)\}\right) + \left(\sin\pi - \sin(-\pi)\right)\right]$$

$$= \frac{1}{\pi}\left[-(-\pi - \pi) + (0 - 0)\right]$$

$$= \frac{1}{\pi}\left(-(-2\pi)\right)$$

$$= \frac{1}{\pi} \times 2\pi$$

$$= 2$$

Now,

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} t \sin t \cos nt \, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{2}\left\{\sin(n+1)t - \sin(n-1)t\right\} dt$$

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$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\{t \sin(n+1)t\}\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi}\{t \sin(n-1)t\}\,dt$$

$$= \frac{1}{2\pi}\left[t \cdot \frac{-\cos(n+1)t}{n+1} - \int -\frac{\cos(n+1)t}{n+1}\,dt\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\left[t \cdot \frac{-\cos(n-1)t}{n-1} - \int \frac{-\cos(n-1)t}{n-1}\,dt\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(n+1)}\left[-t\cos(n+1)t + \frac{\sin(n+1)t}{n+1}\right]_{-\pi}^{\pi} - \frac{1}{2\pi(n-1)}\left[-t\cos(n-1)t + \frac{\sin(n-1)t}{n-1}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(n+1)}\left[-\left(\pi\cos(n+1)\pi - (-\pi)\cos(n+1)(-\pi)\right) + 0\right] - \frac{1}{2\pi(n-1)}\left[-\{\pi\cos(n-1)\pi - (-\pi)\cos(n-1)(-\pi)\} + 0\right]$$

$$= \frac{1}{2\pi(n+1)}\left[-\left(2\pi\cos(n+1)\pi\right)\right] - \frac{1}{2\pi(n-1)}\left[-2\pi\cos(n-1)\pi\right]$$

$$= \left(\frac{1}{2\pi(n+1)} \times 2\pi\right) - \left(\frac{1}{2\pi(n-1)} \times 2\pi\right) \quad [\text{If } n \neq 1]$$

$$= \frac{1}{n+1} - \frac{1}{n-1}$$

$$= \frac{n-1-n-1}{n^2-1}$$

$$= \frac{-2}{n^2-1} = \frac{2}{1-n^2}$$

Again,

If $n=1$,

$$a_1 = \frac{2}{2\pi}\int_{-\pi}^{\pi} t\sin t\cos t\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{2}\sin 2t\,dt$$

$$= \frac{1}{2\pi}\left[t\cdot\frac{-\cos 2t}{2} - \int\frac{-\cos 2t}{2}\,dt\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{t\cos 2t}{2} + \frac{\sin 2t}{4}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left[-(\pi\cos 2\pi - (-\pi)\cos(-2\pi)) + 0\right]$$

$$= \frac{1}{4\pi}\left[-(\pi+\pi)\right]$$

$$= -\frac{1}{2}$$

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Now,

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} t \sin t \sin(nt)\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t}{2}\left\{\cos(n-1)t - \cos(n+1)t\right\} dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} t\cos(n-1)t\, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} t\cos(n+1)t\, dt$$

$$= \frac{1}{2\pi}\left[t.\frac{\sin(n-1)t}{n-1} - \int \frac{\sin(n-1)t}{n-1}\, dt\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\left[t.\frac{\sin(n+1)t}{n+1} - \int \frac{\sin(n+1)t}{n+1}\, dt\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(n-1)}\left[t\sin(n-1)t + \frac{\cos(n-1)t}{n-1}\right]_{-\pi}^{\pi} - \frac{1}{2\pi(n+1)}\left[t\sin(n+1)t + \frac{\cos(n+1)t}{n+1}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(n-1)}\left[\left\{\pi\sin(n-1)\pi - \left[(-\pi)\sin(n-1)(-\pi)\right]\right\} + \frac{1}{n-1}\left\{\cos(n-1)\pi - \cos(n-1)(-\pi)\right\}\right]$$

$$- \frac{1}{2\pi(n+1)}\left[\left\{\pi\sin(n+1)\pi - \left[(-\pi)\sin(n+1)(-\pi)\right]\right\} + \frac{1}{n+1}\left\{\cos(n+1)\pi - \cos(n+1)(-\pi)\right\}\right]$$

$$= \frac{1}{2\pi(n-1)}\left[\{0-0\} + \frac{1}{n+1}\{-1-(-1)\}\right] - \frac{1}{2\pi(n+1)}\left[\{0-0\} + \frac{1}{n+1}\{-1-(-1)\}\right]$$

$$= \frac{1}{2\pi(n-1)}\left[\frac{1}{n-1}(-1+1)\right] - \frac{1}{2\pi(n+1)}\left[\frac{1}{n+1}(-1+1)\right]$$

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$$= \left(\frac{1}{2\pi (n-1)} \times \frac{1}{n-1} \times 0 \right) - \left(\frac{1}{2\pi (n+1)} \times \frac{1}{n+1} \times 0 \right)$$

$$= (0 - 0)$$

$$= 0$$

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Lecture : 06

Date : 01.03.2020

Q-02: A function $f(t)$ is defined as,

$$f(t) = \begin{cases} -1 & ; -\pi \leq t < 0 \\ 1 & ; 0 \leq t \leq \pi \end{cases}$$

Find it's fourier co-efficients as well as it's Fourier Series representation. Hence, show that,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \ldots$$

Solution :

We assume that the function $f(t)$ is defined over it's single period.

Where,

Period $= 2\pi$

So, it's fourier series representation is,

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega t) + b_n \sin(n\omega t) \right)$$

$$= \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right) \quad \text{--- (1)} \quad \left[\because T = 2\pi \; \therefore \omega = 1 \right]$$

Where,

Fourier Co-efficients are,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \quad \text{——— (2)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \quad \text{——— (3)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt \quad \text{——— (4)}$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -1\,dt + \int_{0}^{\pi} dt\right]$$

$$= \frac{1}{\pi}\left\{\left[-t\right]_{-\pi}^{0} + \left[t\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{-(0-(-\pi))\right\} + (\pi - 0)\right]$$

$$= \frac{1}{\pi}\left[-\pi + \pi\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -1\cos(nt)\,dt + \int_{0}^{\pi}\cos(nt)\,dt\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{\sin(nt)}{n}\right]_{-\pi}^{0} + \left[\frac{\sin(nt)}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[-\{\sin 0 - \sin(-n\pi)\} + \{\sin(n\pi) - \sin 0\}\right]$$

$$= \frac{1}{n\pi}\left[-\{0 + \sin(n\pi)\} + \{0 - 0\}\right]$$

$$= \frac{1}{n\pi}\left[-\{0+0\} + 0\right]$$

$$= 0$$

Again,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -1\sin(nt)\,dt + \int_{0}^{\pi}\sin(nt)\,dt\right]$$

$$= \frac{1}{\pi}\left[-\left[\frac{-\cos(nt)}{n}\right]_{-\pi}^{0} + \left[\frac{-\cos(nt)}{n}\right]_{0}^{\pi}\right]$$

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$$= \frac{1}{n\pi}\left[\left[\cos(nt)\right]_{-\pi}^{0} - \left[\cos(nt)\right]_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[\left\{\cos 0^\circ - \cos(-n\pi)\right\} - \left\{\cos(n\pi) - \cos(0^\circ)\right\}\right]$$

$$= \frac{1}{n\pi}\left[\left(\cos 0^\circ - \cos(n\pi)\right) - \cos(n\pi) + \cos 0^\circ\right]$$

$$= \frac{1}{n\pi}\left[2\cos 0^\circ - 2\cos(n\pi)\right]$$

$$= \begin{cases} 0 & ; n = 2, 4, 6, \ldots \ldots \\ \frac{4}{n\pi} & ; n = 1, 3, 5, 7, \ldots \ldots \end{cases}$$

$$= \frac{2}{n\pi}\left\{1 - (-1)^n\right\} \quad ; n = 1, 2, 3, \ldots \ldots$$

From equation ①,

$$f(t) = 0 + 0 + \sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}\left\{1 - (-1)^n\right\}\sin(nt)\right\}$$

$$= \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin(3t) + \frac{4}{5\pi}\sin(5t) + \ldots \ldots$$

This is the required Fourier series.

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2nd Part :

When, $x = \frac{\pi}{2}$

$f(\frac{\pi}{2}) = 1$

Again,

$$f(\frac{\pi}{2}) = \frac{4}{\pi} \sin(\frac{\pi}{2}) + \frac{4}{3\pi} \sin(\frac{3\pi}{2}) + \frac{4}{5\pi} \sin(\frac{5\pi}{2}) + \dots$$

$$\Rightarrow 1 = \frac{4}{\pi} - \frac{4}{3\pi} + \frac{4}{5\pi} - \dots \dots$$

$$\Rightarrow 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \dots \dots\right)$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots \dots$$

(Showed)

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Q-03: Find fourier series expansion of

$$g(t) = \begin{cases} t & ; 0 < t < \pi \\ \pi & ; \pi < t < 2\pi \end{cases}$$

which is defined over it's single period and show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \dots$$

and $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \dots$$

Solution:

We assume that, the function $g(t)$ is defined over it's single period where,

Period, $T = 2\pi$

$$\therefore \omega = \frac{2\pi}{T} = 1$$

$\therefore$ The Fourier series representation/expansion of $g(t)$

$$g(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nt) + b_n \sin(nt)\right) \quad [\text{As}, \omega = 1]$$

Where Fourier co-efficients are,

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} f(t)\, dt$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} t\, dt + \int_{\pi}^{2\pi} \pi\, dt\right]$$

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$$= \frac{1}{\pi}\left[\left[\frac{t^2}{2}\right]_0^{\pi} + \left[\pi t\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \left\{(\pi \times 2\pi) - (\pi^2)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi^2 - \pi^2\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2 + 2\pi^2}{2}\right]$$

$$= \frac{1}{\pi} \times \frac{3\pi^2}{2}$$

$$= \frac{3\pi}{2}$$

Again,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} t\cos nt\,dt + \int_{\pi}^{2\pi} \pi\cos nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\left[t\cdot\frac{\sin(nt)}{n} - \int\frac{\sin(nt)}{n}\,dt\right]_0^{\pi} + \pi\left[\frac{\sin(nt)}{n}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{n\pi}\left[\left[t\sin(nt) + \frac{1}{n}\cos(nt)\right]_0^{\pi} + \pi\left[\sin(2n\pi) - \sin(n\pi)\right]\right]$$

$$= \frac{1}{n\pi}\left[\left[\pi \sin(n\pi) + 0 + \frac{1}{n}\left\{\cos(n\pi) - \cos 0^{\circ}\right\}\right] + \pi \times 0\right]$$

$$= \frac{1}{n\pi}\left[0 + \frac{1}{n}\left(\cos(n\pi) - 1\right)\right]$$

$$= \frac{1}{n\pi} \times \frac{1}{n}\left\{\cos(n\pi) - 1\right\}$$

$$= \begin{cases} -\frac{2}{n^2\pi} & ; n = 1, 3, 5, \ldots \ldots \\ 0 & ; n = 2, 4, 6, \ldots \ldots \end{cases}$$

$$= \frac{1}{n^2\pi}\left\{(-1)^n - 1\right\} \quad ; n = 1, 2, 3, \ldots$$

Again,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} t\sin(nt)\,dt + \int_{\pi}^{2\pi} \pi\sin(nt)\,dt\right]$$

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$$= \frac{1}{\pi}\left[t \cdot \frac{-\cos(nt)}{n} - \int \frac{-\cos(nt)}{n}\,dt\right]_0^{\pi} + \frac{1}{\pi}\cdot\pi\left[\frac{-\cos(nt)}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{n\pi}\left[-t\cos(nt) + \frac{1}{n}\cdot\sin(nt)\right]_0^{\pi} + \frac{1}{n}\left[-\{\cos(2n\pi) - \cos(n\pi)\}\right]$$

$$= \frac{1}{n\pi}\left[-\pi\cos(n\pi) + 0 + \frac{1}{n}\{\sin(n\pi) - \sin 0^{\circ}\}\right] - \frac{1}{n}\left(1 - \cos(n\pi)\right)$$

$$= \frac{1}{n\pi}\left[\left(-\pi\cos(n\pi) + 0 + 0\right]\right. - \frac{1}{n}\left(1 - \cos(n\pi)\right)$$

$$= \frac{1}{n\pi}\left(-\pi\cos(n\pi)\right)$$

$$= -\frac{1}{n}\cos(n\pi) - \frac{1}{n} + \frac{1}{n}\cos(n\pi)$$

$$= -\frac{1}{n}$$

$\therefore$ ~~The~~ From equation (1),

$$g(t) = \frac{1}{2}\times\frac{3\pi}{2} + \frac{1}{n^2\pi}\{(-1)^n - 1\}\cos(nt) - \frac{1}{n}\sin(nt)$$

$$= \frac{3\pi}{4} + \frac{1}{\pi}(-2)\cos t + \frac{1}{9\pi}(-2)\cos(3t) + \frac{1}{25\pi}(-2)\cos(5t) + \ldots$$

$$\ldots - \frac{1}{1}\sin t - \frac{1}{2}\sin(2t) - \frac{1}{3}\sin(3t) - \frac{1}{4}\sin(4t) - \ldots$$

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$$= \frac{3\pi}{4} - \frac{2}{\pi}\cos t - \frac{2}{9\pi}\cos 3t - \frac{2}{25\pi}\cos 5t - \dots - \sin t$$

$$- \frac{1}{2}\sin(2t) - \frac{1}{3}\sin(3t) - \frac{1}{4}\sin(4t) - \dots \dots$$

This is the required Fourier series.

2nd - Part :

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

From fourier series,

$$f\left(\frac{\pi}{2}\right) = \frac{3\pi}{4} - 0 - 0 - 0 - 0 - \dots \dots - \frac{1}{2} - 0 + \frac{1}{3} - 0 - \frac{1}{5}$$

$$- 0 + \frac{1}{7} - 0 - \dots \dots$$

$$\Rightarrow \frac{\pi}{2} = \frac{3\pi}{4} - \frac{1}{2} + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots$$

$$\boxed{\Rightarrow \pi = \frac{3\pi}{2}}$$

$$\Rightarrow \frac{\pi}{2} - \frac{3\pi}{4} = -\frac{1}{2} + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots$$

$$\Rightarrow \frac{-\pi}{4} = -\frac{1}{2} + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} -$$

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