American Mathematics Competitions (AMC 8) Preparation

Volume 5

The American Mathematics Competitions 8 is a 25-question multiple-choice contest for students in the sixth through eighth grade. Accelerated fourth and fifth graders can also take part. The AMC 8 is administered in schools in November. The American Mathematics Competitions (AMC) publishes the Achievement Roll list recognizing students in 6th grade and below who scored 15 or above, and the Honor Roll list recognizing students who score in the top 5%, and the Distinguished Honor Roll list recognizing students who score in the top 1%.

This book can be used by 5^{th} to 8^{th} grade students preparing for AMC 8. Each chapter consists of (1) basic skill and knowledge section with plenty of examples, (2) about 30 exercise problems, and (3) detailed solutions to all problems.

We would like to thank the American Mathematics Competitions (AMC 8 and 10) for their mathematical ideas. Many problems (marked by ☆) in this book are inspired from these tests. We only cited very few problems directly from these tests for the purpose of comparison with our own solutions.

We wish to thank the following reviewers for their invaluable solutions, insightful comments, and suggestions for improvements to this book:
Alex Cheng (UT), Jin Cheng (CA), Felix Cui (NE), Albert Hao (CA), Sameer Khan (VA), Kathy Liu (VA), Priyo Majumdar (LA), Aadith Menon, Kalyanasundaram Seshadri (CA), Huili Shao (MA), Stephan Xie (TX), Cindy Ye (AR), Samuel Yoon (VA) and Sophia Zhang (CO).

ISBN-13: 978-1503019706
ISBN-10: 1503019705

Please contact mymathcounts@gmail.com for suggestions, corrections, or clarifications.

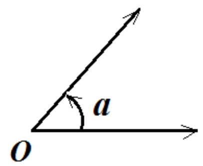
Table of Contents

This page is intentionally left blank.

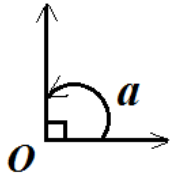
1.BASIC KNOWLEDGE

1.1. Terms

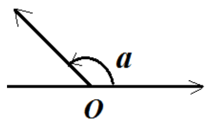
Acute angle: between 0 and 90°. $0° < a < 90°$.



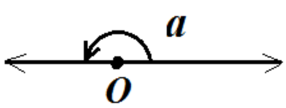
Right angle: $a = 90°$



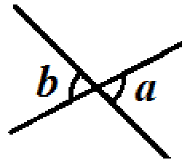
Obtuse angle: between 90 and 180°. $90° < a < 180°$.



Straight angle: $a = 180°$

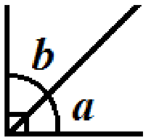


Vertical angles have equal measures. $a = b$.



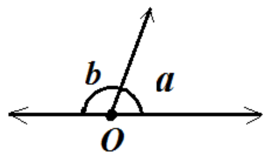
Complementary angles

If sum of the measures of two acute angles is 90°, the angles are said to be **complementary.** $a + b = 90°$.



Supplementary angles

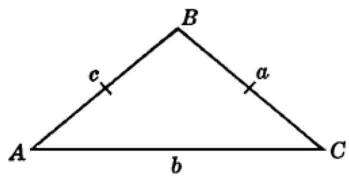
If sum of the measures of two angles is 180°, the angles are said to be **supplementary.** $a + b = 180°$.



Isosceles triangle

An isosceles triangle is a triangle with at least two congruent sides.

$AB = BC$. $\angle A = \angle C$

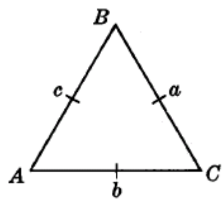


Isosceles Triangle

Equilateral triangle

An equilateral triangle is a triangle having three congruent sides.

$AB = BC = CA$. $\angle A = \angle B = \angle C = 60°$.



Equilateral Triangle

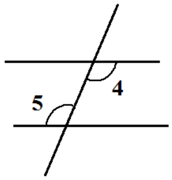
1.2. Relationship of Angles Formed by Parallel Lines

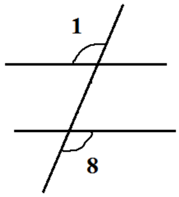
Alternate interior angles $\angle 4 = \angle 5$.

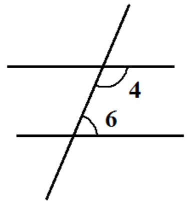
Alternate exterior angles $\angle 1 = \angle 8$.

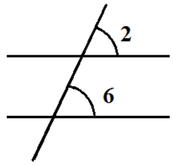
Interior angles on the same sides of transversal $\angle 4 + \angle 6 = 180$.

Corresponding angles $\angle 2 = \angle 6$.



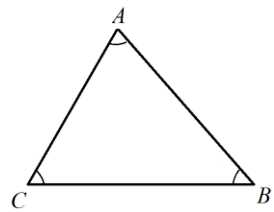






1.3. Angle – Measure – Sum Principles

Theorem 1. The sum of the measures of the angles of a triangle equals the measure of a straight angle, or 180°. $\angle A + \angle B + \angle C = 180°$

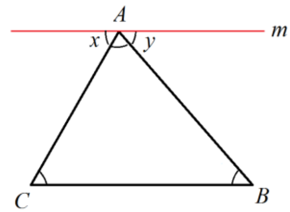


Proof:

We draw the line m parallel to BC, the base of the triangle.

So $\angle C = \angle x$, $\angle B = \angle y$.

Since $\angle A + \angle x + \angle y = 180°$, $\angle A + \angle B + \angle C = 180°$.

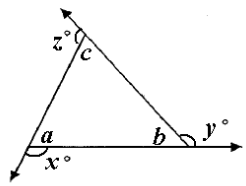


Theorem 2. The measure of each exterior angle of a triangle equals the sum of the measures of its two remote nonadjacent interior angles.

$z = a + b$

$x = b + c$

$y = c + a$



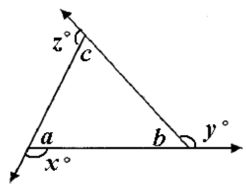
Proof:

$b + y = 180°$ (1)

$a + b + c = 180°$ (2)

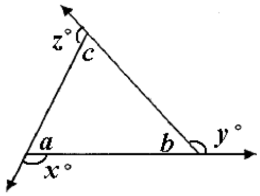
(1) – (2): $+ y - (c + a) = 0 \Rightarrow y = c + a$

Similarly we can prove $x = b + c$ and $z = a + b$.



<u>Theorem 3.</u> The sum of the measures of the exterior angles of a triangle equals 360°.

$x + y + z = 360°$



Proof:

$x = b + c$ (1)

$y = a + c$ (2)

$z = a + b$ (3)

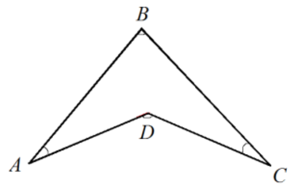
$a + b + c = 180°$ (4)

(1) + (2) + (3): $a + b + c + x + y + z = 540°$ (5)

Substituting (4) into (5): $x + y + z = 360°$.

<u>Theorem 4.</u> The four angles in the figure below have the following relationship:

$\angle D = \angle A + \angle B + \angle C$.

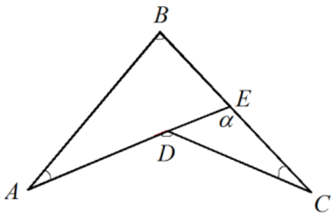


Proof:

Extend AD to meet BC at E.

By **<u>Theorem 2</u>**, $\angle D = \alpha + \angle C$ (1)

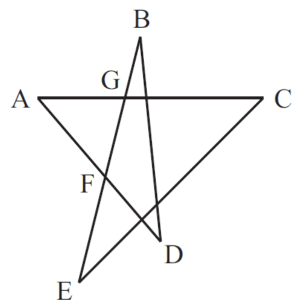
By **<u>Theorem 2,</u>** $\alpha = \angle A + \angle B$ (2)



Substituting (2) into (1): $\angle D = \angle A + \angle B + \angle C$.

2. EXAMPLES

☆**Example 1.** If $\angle A = 22°$ and $\angle AFG = \angle AGF$, Then $\angle B + \angle D =$
(A) 54° (B) 66° (C) 79° (D) 88° (E) 100°

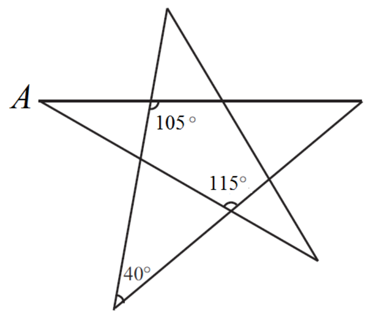


Solution: C.
$\angle A + \angle AFG + \angle AGF = 180°$ (1)

Since $\angle AFG = \angle AGF$, (1) can be written as
$\angle A + 2\angle AFG = 180°$ (2)
or $2\angle AFG = 180° - \angle A = 180° - 22° = 158°$.

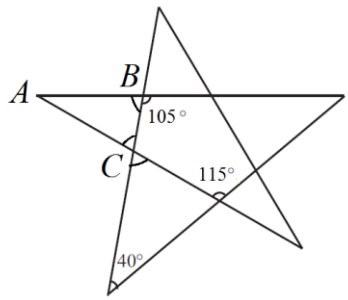
So $\angle AFG = 79°$.
By **Theorem 2,** $\angle AFG = \angle B + \angle D = 79°$.

☆**Example 2.** The degree measure of angle A is
(A) 20° (B) 30° (C) 35° (D) 40° (E) 45°.



Solution: B.
$\angle B = 180° - 105° = 75°$

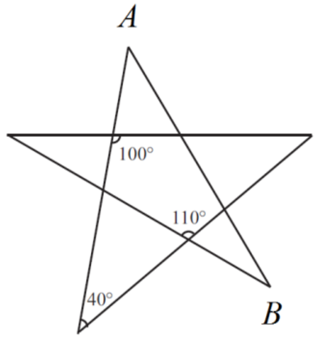
By **Theorem 2,** $115° = \angle C + 40°$
So $\angle C = 115° - 40° = 75°$.



By **Theorem 1,** $\angle A + \angle B + \angle C = 180°$
So $\angle A = 180° - (\angle B + \angle C) = 180° - 150° = 30°$.

☆**Example 3.** The degree measure of angles $A + B$ is

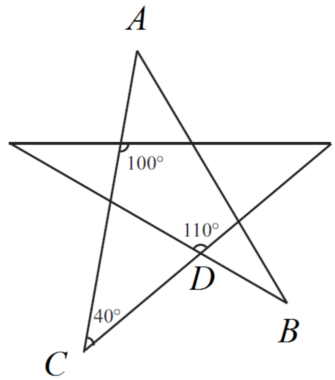
(A) 50° (B) 60° (C) 70° (D) 75° (E) 85°



Solution: C.
We label the angles as follows.

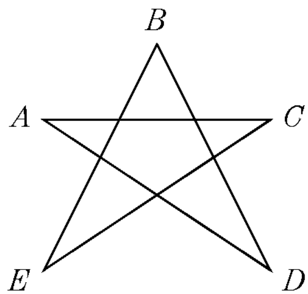
By the definition of vertical angles, we know that
$\angle D = 110°$.

By **Theorem 4,** $\angle D = \angle A + \angle B + \angle C$
$110° - (\angle A + \angle B + \angle C) = 110° - 40° - (\angle A + \angle B)$.
Thus $\angle A + \angle B = 110° - 40° = 70°$.



Example 4. The sum of the measures of angles A, B, C, D, and E in the accompanying figure is:

A. less than 180° B. 180° C. greater than 180° but less than 360°
D. 360° E . cannot be determined



Solution: B.
Draw line l that is parallel to the side AC.
So $\angle 1 = \angle 3$, $\angle 2 = \angle 4$.

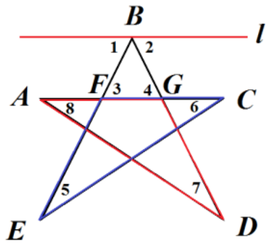
We also see that $\angle 1 + \angle B + \angle 2 = 180^{\circ}$ (1)
In triangle CEF, $\angle 3 = \angle 5 + \angle 6$.

In triangle ADG, $\angle 4 = \angle 7 + \angle 8$.

That is, $\angle 1 = \angle 5 + \angle 6$ (2)

$\angle 2 = \angle 7 + \angle 8$ (3)

Substituting (2) and (3) into (1): $\angle A + \angle B + \angle C + \angle D + \angle D = 180º$.

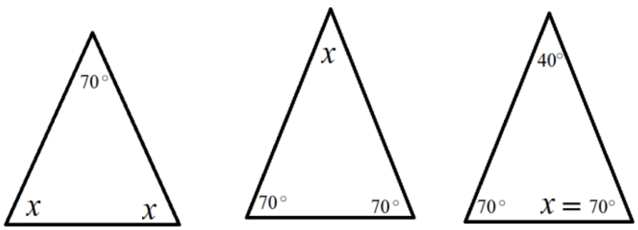


☆**Example 5.** 19. Two angles of an isosceles triangle measure 70° and x°. What is the sum of the all possible values of x ?
(A) 95° (B) 125° (C) 140° (D) 165° (E) 180°

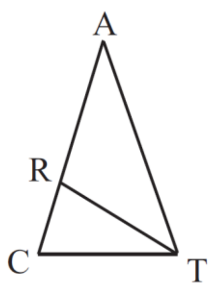
Solution: D.

We have the following three cases. So we get x = 55°, 40°, and 70°.

The answer is 55°+ 40°+ 70°= 165°.



☆**Example 6.** In triangle CAT, we have $\angle ACT = \angle ATC$ and $\angle CAT = 40°$. If TR bisects $\angle ATC$, then $\angle CRT =$
(A) 20° (B) 60° (C) 75° (D) 90° (E) 120°.



Solution: C.

Since $\angle ACT = \angle ATC$ and $\angle CAT = 40°$, $\angle ACT = \angle ATC = (180° - 40°)/2 = 70°$.

Since TR bisects $\angle ATC$, $\angle CTR = 70°/2 = 35°$.

By **<u>Theorem 1,</u>** $\angle CRT + \angle CTR + \angle C = 180°$

So $\angle CRT = 180° - (\angle CTR + \angle C) = 180° - (35° + 70°) = 75°$.

Example 7. If the complements of angle A and B are complementary, then the supplements of angles A and B:

A. are congruent B. are supplementary C. are complementary
D. differ by 90° E. add up to 270°

Solution: E.
The complements of angle A and B are $90-A$ and $90-B$, respectively.
Since $90-A$ and $90-B$ are complements, $90-A + 90-B = 90 \Rightarrow A+B=90$.
The supplements of angle A and B are $180-A$ and $180-B$, respectively.
$180-A + 180-B = 360 - (A+B) = 360-90 = 270^{\circ}$. The answer is E.

Example 8. The measure of an angle for which the measure of the supplement is four times the measure of the complement is:

A. 20° B. 45° C. 60° D. 75° E. none of these

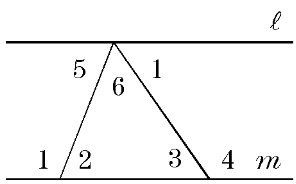
Solution: C.
Let the angle be x and its complement is y and its supplement is z.
We can write the following equations:

$x + y = 90°$ (1)
$x + z = 180°$ (2)
$z = 4y$ (3)
(2) − (1): $z - y = 90°$ (4)

Substituting (3) into (4):): $4y - y = 90° \Rightarrow y = 30°$.
From (1), $x = 60°$.

Example 9. In the figure, $\angle 1 = 7x + 10$. $\angle 5 = 3x$ and $l \parallel m$ .The measure of $\angle 2$ equals:

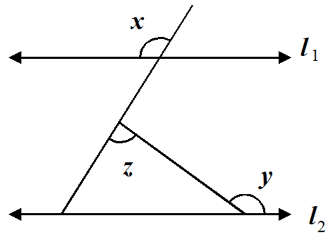
A. 17 B. 51 C. 87 D. 129 E. 139



Solution: B.
Since $l \parallel m$, $\angle 5 = \angle 2$.
We see that $\angle 1 + \angle 2 = 180^{\circ} \Rightarrow 7x + 10 + 3x = 180^{\circ} \Rightarrow x = 17^{\circ}$.

$\angle 2 = 3x = 3 \times 17^\circ = 51^\circ$.

Example 10. x, y, and z are the measures of the angles shown. $l_1 // l_2$. The measure of x is:

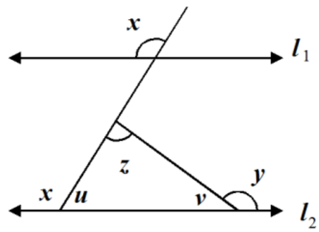


A. $180^\circ - y$ B. $180^\circ - z$ C. $180^\circ - z + y$

D. $180^\circ + z - y$ E. $z + y - 180^\circ$

Solution: D.

Method 1:



$x + u = 180^\circ$ (1)

$y + v = 180^\circ$ (2)

$180^\circ = xu + v + z$ (3)

(1) + (2) + (3):

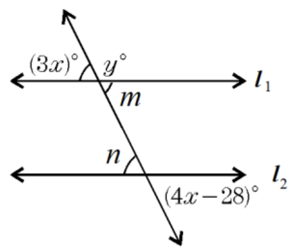
$x + u + y + v + 180^\circ = 180^\circ + u + v + z$

$\Rightarrow \quad x = 180^\circ + z - y$.

Method 2:

$y = 180^\circ - x + z \quad \Rightarrow \quad x = 180^\circ + z - y$.

Example 11. If lines l_1 and l_2 are parallel, then find the value of y.



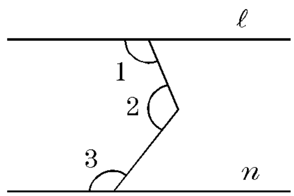
A. 84 B. 96 C. $90\frac{6}{7}$ D. $89\frac{1}{7}$ E. none of these

Solution: B.

$3x^\circ = 4x^\circ - 28 \quad \Rightarrow \quad x = 28^\circ$.

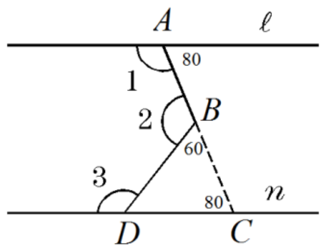
$y + 3x^\circ = 180^\circ \quad \Rightarrow \quad y = 180^\circ - 3x^\circ = 180^\circ - 3 \times 28^\circ = 96$.

Example 12. In the figure $l // n$, $\angle 1 = 100°$, and $\angle 2 = 120°$. Find $\angle 3$.



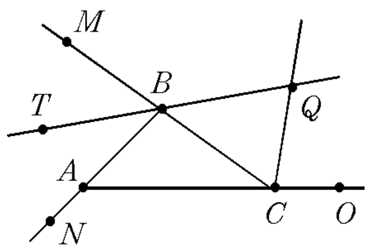
A. 0° B. 100° C. 120° D. 140° E. 150°

Solution: D.



As shown in the figure, extend AB to meet line n at C.
$\angle A = 180° - \angle 1 = 180° - 100° = 80°$.
$\angle BCD = \angle A = 80°$.
$\angle CBD = 180° - \angle 2 = 180° - 120° = 60°$.
By **Theorem 2**, $\angle 3 = \angle BCD + \angle CBD = 60° + 80° = 140°$.

Example 13. In the diagram, $\angle MBA$, $\angle NAC$, and $\angle OCB$ are exterior angles of triangle ABC. Lines TB and CQ intersect at point Q. Ray BT and ray CQ bisect $\angle MBA$ and $\angle OCB$ respectively. Then the measure of $\angle BQC$ is:



A. equal to the measure of $\angle CAN$.

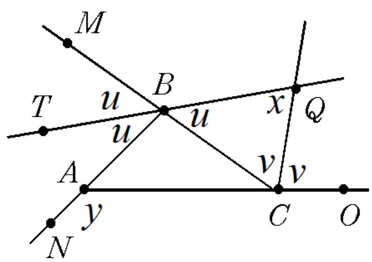
B. equal to the measure of $\frac{\angle CAN}{2}$.

C. equal to the measure of $\frac{\angle CAN}{3}$.

D. equal to the measure of $\frac{\angle CAN}{4}$.

E. none of these.

Solution: B.
We label some angles as in the figure.

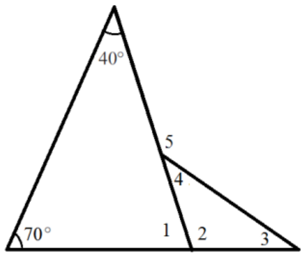


By **Theorem 3**, $y + 2u + 2v = 360°$ (1)
By **Theorem 1**, $u + v + x = 180°$ (2)
Multiplying (2) by 2: $2u + 2v + 2x = 360°$ (3)
(1) – (3): $y = 2x$
So the answer is B.

☆**Example 14.** $\angle 1 + \angle 2 = 180°$. $\angle 3 = \angle 4$. Find $\angle 5$.
A. 140° B. 145° C. 150° D. 160° E. 165°

Solution: B.

By **Theorem 1,** $\angle 1 = 180^\circ - (70^\circ + 40^\circ) = 70^\circ$.

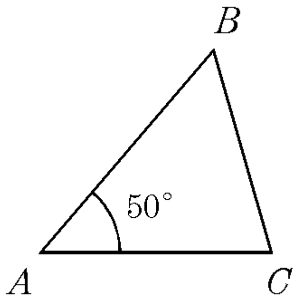


By **Theorem 2,** $\angle 1 = \angle 4 + \angle 3 = 70^\circ$.

Since $\angle 3 = \angle 4$, $\angle 4 = 70^\circ / 2 = 35^\circ$.
$\angle 5 = 180^\circ - \angle 4 = 180^\circ - 35^\circ = 145^\circ$.

Example 15. Given that ΔABC is a triangle such that $AB = AC$ and $\angle A = 50^\circ$. Then $\angle B$ is:

A. 50° B. 55° C. 60° D. 65° E. 70°

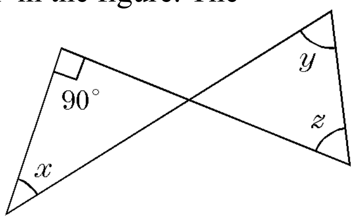


Solution: D.

Since $AB = AC$, $\angle B = \angle C = (180^\circ - 50^\circ)/2 = 65^\circ$.

Example 16. x, y, and z are the measures of the angles shown in the figure. The sum of y and z in terms of x is:

A. $2x$ B. $90^\circ + x$ C. $180^\circ - x$ D. $180^\circ - 2x$ E. $90^\circ - x$



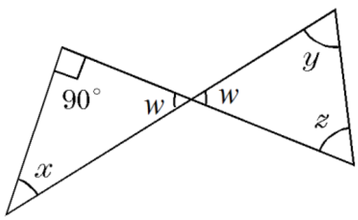
Solution: B.

$y + z + w = 180^\circ$ (1)

$x + w + 90^\circ = 180^\circ \Rightarrow w = 90^\circ - x$ (2)

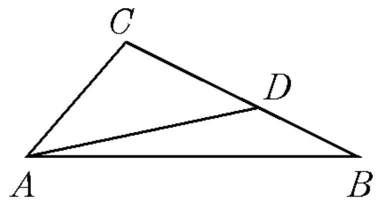
Substituting (2) into (1): $y + z + 90^\circ - x = 180^\circ$

$\Rightarrow y + z = 90^\circ + x$.



Example 17. In $\Delta\, ABC$, $\overline{AC} = \overline{CD}$ and $\angle CAB - \angle ABC = 40^\circ$. Then $\angle\, BAD$ equals:

A. 15° B. 20° C. 30° D. 35° E. 40°



Solution: B.

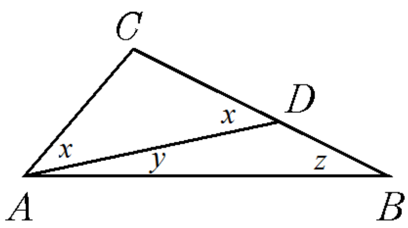
Since $\overline{AC} = \overline{CD}$, and $\angle CAD = \angle CDA = x$.

By **Theorem 2**, $x = y + z$ (1)

$\angle CAB - \angle ABC = 40^\circ$

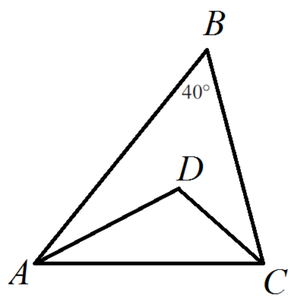
$\Rightarrow \quad x + y - z = 40^\circ$ (2)

Substituting (1) into (2): $2y = 40^\circ \quad \Rightarrow \quad y = 20^\circ$.



☆**Example 18.** The measure of angle ABC is 40°. AD bisects angle BAC, and DC bisects angle BCA. The measure of angle ADC is

A. 90° B. 105° C. 110° D. 125.5° E. 130°



Solution: C.

By **Theorem 1**, $\angle A + \angle B + \angle C = 180^\circ$

$\angle A + \angle C = 180^\circ - \angle B = 180^\circ - 40^\circ = 140^\circ$.

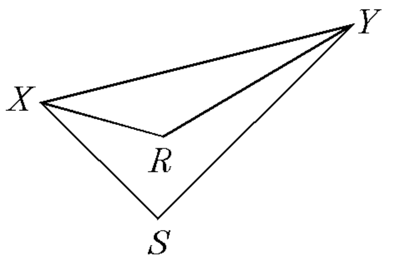
$\angle A/2 + \angle C/2 = 140^\circ/2 = 70^\circ$.

By **Theorem 4,** $\angle D = \angle A/2 + \angle C/2 + \angle B = 70^\circ + 40^\circ = 110^\circ$.

Example 19. In the given figure $\overrightarrow{XR}$ bisects $\angle YXS$, $\overrightarrow{YR}$ bisects $\angle XYS$, and $\angle\, S = a$. Express the measure of $\angle R$ in terms of a.

A. $90 + \dfrac{a}{2}$ B. $\dfrac{180 - a}{3}$ C. $\dfrac{2a + 90}{3}$

D. $180 + \dfrac{a}{2}$ E. $\dfrac{a - 90}{3}$

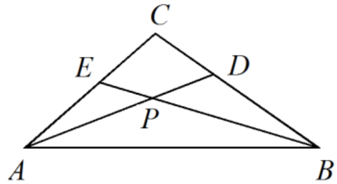


Solution: A.

By **Theorem 4,**, $\angle R = \angle YXS/2 + \angle XYS/2 + \angle S = (\angle YXS + \angle XYS)/2 + \angle S =$
$(180^\circ - \angle S)/2 + \angle S = (180^\circ - a)/2 + a = 90 + \dfrac{a}{2}$.

Example 21. In triangle ABC, $\overline{AD}$ and $\overline{BE}$ bisects angles A and B, respectively, and intersect in point P. The measure of angle ACB is 70°.
The measure of angle APE is:

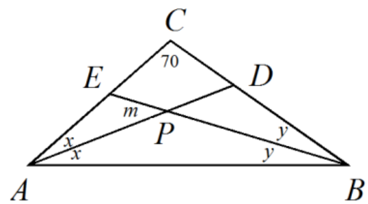
A. 50° B. 55° C. 60° D. 67°
E. cannot be determined.



Solution: B.

Method 1:

$2x + 2y = 180^\circ - 70^\circ = 110^\circ$ (1)
Dividing both sides of (1) by 2: $x + y = 55^\circ$ (2)
By **Theorem 2**, $\angle APE = m = x + y = 55^\circ$.



Method 2:

$2x + 2y = 180^\circ - 70^\circ = 110^\circ$ (1)
Dividing both sides of (1) by 2: $x + y = 55^\circ$ (2)
By **Theorem 4,** $\angle APB = 70^\circ + x + y = 70^\circ + 55^\circ = 125^\circ$.
$\angle APE = m = 180^\circ - 125^\circ = 55^\circ$.

Example 22. If the measures of the angles of a triangle are in the ratio 4 : 5 : 6, what is the measure of the smallest acute angle?

Solution: 48°.

The three angles will be $4x$, $5x$, and $6x$.
$4x + 5x + 6x = 180^\circ \Rightarrow 15x = 180^\circ. \Rightarrow x = 12^\circ$.
The measure of the smallest acute angle is $4x = 4 \times 12 = 48^\circ$.

3. PROBLEMS

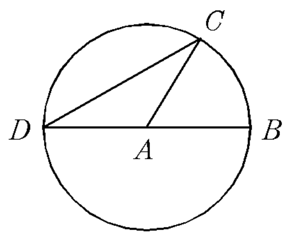
Problem 1. The measure of an angle whose supplement is $2\frac{1}{2}$ times that of its complement is:

A. 150^o B. 75^o C.$60°$ D. 45^o E. 30^o

Problem 2. The measure of an angle whose complement is one-third that of its supplement is:

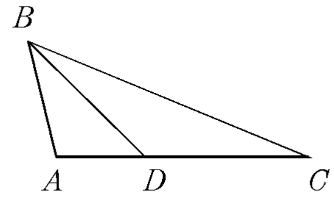
A. 75^o B. 45^o C. 60^o D. 135^o E. none of these

Problem 3. In the figure shown, $\overline{DB}$ is a diameter of the circle with center A. If $\angle CAB = 45°$, then $\angle CDB$ is:



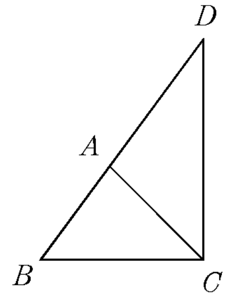
A. 15^o B. 20^o C. 22.5^o D. 25^o E. 27.5^o

Problem 4. In the given figure $\overline{AB} = \overline{AD}$, $\overline{BD} = \overline{CD}$, and $\angle C = 19°$, What is the measure of $\angle A$?



A. $104°$ B. $142°$ C. $76°$ D. $38°$ E. none of the above

Problem 5. In the figure shown, if $\angle ABC = 55^o$, $\angle ACB = 45^o$ and $CA = DA$, then $\angle DCB$ is:



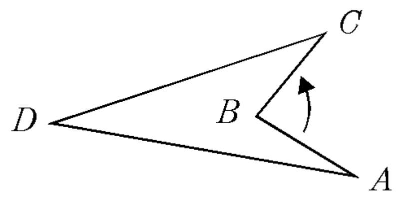
A.100^o B. 95^o C. 90^o D. 85^o E. 80^o

Problem 6. The tip on an arrow has the shape as shown. If the $\angle ABC$ marked by the curved arrow is an acute angle, then the sum of the interior angles of the quadrilateral $ABCD$:

A. is less than 180° B. is less than 360°

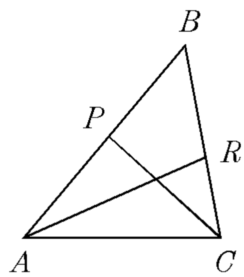
C. is exactly 360° D. is more than 360°

E. none of the above



Problem 7. In triangle ABC, $\angle A = 50^\circ$, $\angle C = 80^\circ$. $\overline{CP}$ bisects $\angle C$ and $\overline{AR}$ bisects $\angle A$. What is the measure of $\angle ARC$?

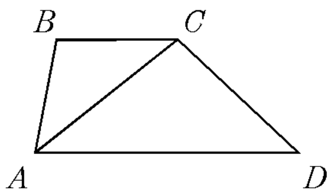
A. 105° B. 75° C. 65° D. 40° E. 25°



Problem 8. In the adjacent figure, sides $\overline{AD}$ and $\overline{BC}$ are parallel, segments $\overline{AC}$ and $\overline{CD}$ have equal length, $\angle ABC = 95^\circ$ and $\angle BAC = 35^\circ$. Then $\angle ACD$ is:

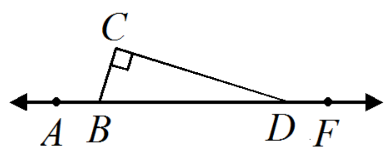
A. 80° B. 90° C. 100° D. 120°

E. none of the above



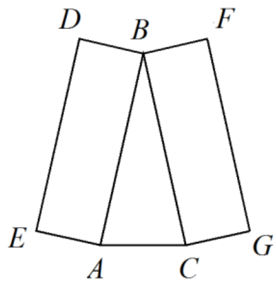
Problem 9. If in the figure, $\angle ABC = 110^\circ$, and $\angle C$ is a right angle, then $\angle CDF$ equals:

A. 110° B. 120° C. 140° D. 160° E. 170°



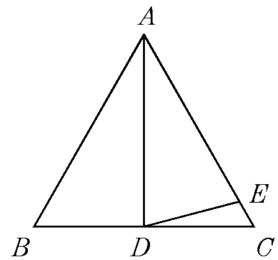
Problem 10. In the figure, $AB = CB$, quadrilaterals $ABDE$ and $CBFG$ are both rectangles, and $\angle BAC = 70°$. Find $\angle DBF$.

A.105^{o} B. 110^{o} C. 120^{o} D. 130^{o} E. 140^{o}



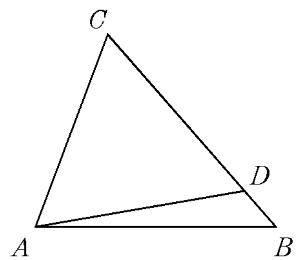
Problem 11. In the figure, ΔABC is equilateral, $\angle DAB = 30^{o}$, and $AE = AD$. Find $\angle EDC$.

A. 15^{o} B. 20^{o} C. $12\frac{1}{2}^{\circ}$ D. 30^{o} E. $7\frac{1}{2}^{\circ}$



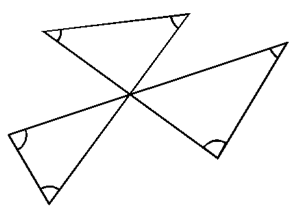
Problem 12. If in ΔABC, $AC = CD$ and $\angle CAB - \angle ABC = 30^{o}$, then $\angle BAD$ is:

A. 10^{o} B. 15^{o} C. 20^{o} D. $22\frac{1}{2}^{\circ}$ E. 30^{o}



Problem 13. In the given figure, the sum of the marked angles is:

A. 180^{o} B. 360^{o} C. 540^{o} D. 270^{o} E. cannot be determined.

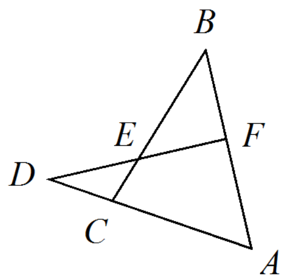


Problem 14. If the measures of two angles of a triangle are $(45 + x)$ degrees and $(45 - x)$ degrees, what is the measure of the third angle?

A. 45^o B. 60^o C. 90^o D. 180^o E. 135^o

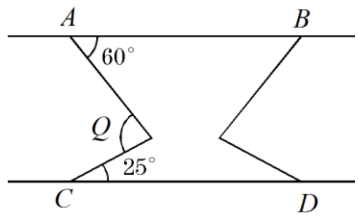
Problem 15. $\overline{DF}$ and $\overline{CB}$ intersect at E, $\overline{DA}$ and $\overline{CB}$ intersect at C, $\overline{AB}$ and $\overline{DF}$ intersect at F, $\overline{DF} \perp \overline{BA}$, $\angle FEC = 160^o$. $\angle A = \angle B$. $\angle ECA$ equals:

A. 20^o B. 40^o C. 60^o D. 70^o E. 140^o



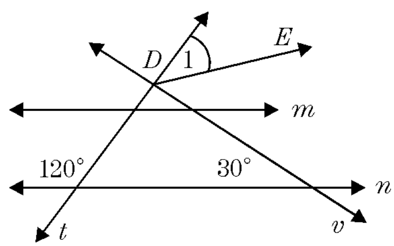
Problem 16. Find the measure of $\angle Q$ if $\overleftrightarrow{AB} // \overleftrightarrow{CD}$.

A.105^o B. 90^o C. 85^o D. 60^o E. 25^o



Problem 17. $m // n$, t and v are transversals intersecting at D; $\overline{DE}$ bisects the angle indicated. Determine measure of $\angle 1$.

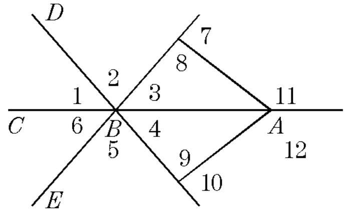
A. 30^o B. 45^o C. 60^o D. 90^o E. none of these



Problem 18. If $\overline{BC}$ bisects $\angle DBE$ and $\angle 7 = \angle 10$, then:

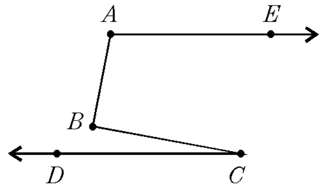
A. $\angle 3 \cong \angle 4$ B. $\angle 8 \cong \angle 9$ C. $\angle 11 \cong \angle 12$

D. *A* and *B* E. *A*, *B*, *C*



Problem 19. In the figure shown, $\overrightarrow{AE}$ is parallel to $\overrightarrow{CD}$ and *B* is a point between the rays. If the measure of $\angle BAE$ is 100° and the measure of $\angle ABC = 90^{\circ}$, find the measure of $\angle BCD$.

A. 5° B. 90° C. 100° D. 30° E .10°



Problem 20. What is the measure of an acute angle if twice the measure of its supplement is 27 more than five times the measure of its complement?

A. 17 B. 23 C. 31 D. 39 E. 47

Problem 21. The complement of an angle is $\frac{1}{7}$ of the supplement of that angle.

What is the complement?

A. 15° B. 60° C. 75° D.105° E. 120°

Problem 22. Find an angle whose supplement is 6 times the size of its complement.

A.30° B. 36° C.60° D.72° E. $\frac{180^{\circ}}{7}$

Problem 23. $\overrightarrow{OB}$ bisects $\angle AOC$. If $\angle AOB = 2x + 10$ and $\angle BOC = 8x - 14$, what is $\angle AOC$?

A. 22° B. 25° C. 36° D. 40° E. 44°

Problem 24. Twice the measure of the supplement of an angle is added to three times the measure of the complement of the same angle. The sum is the measure of an interior angle of a regular nine-sided polygon. What is the measure of the supplement of the angle?

A.82° B. 86° C. 90° D. 94° E. none of these

Problem 25. $\overrightarrow{OB}$ bisects $\angle AOC$. If $m\angle AOB = 3x + 16$ and $m\angle BOC = 8x - 14$, then $m\angle AOC =$

A. 6 B. 20 C. 34 D. 56 E. 68

4. SOLUTIONS

Problem 1. Solution: E.

Let the angle be x and its complement is y and its supplement is z.

We can write the following equations:

$x + y = 90°$ (1)

$x + z = 180°$ (2)

$z = 2\frac{1}{2}y$ (3)

(2) − (1): $z - y = 90°$ (4)

Substituting (3) into (4):): $2\frac{1}{2}y - y = 90° \Rightarrow \quad y = 60°$.

From (1), $x = 30°$.

Problem 2. Solution: B.

Let the angle be x, its complement be y and its supplement be z.

$y = \frac{1}{3}z \quad \Rightarrow \quad 3y = z$ (1)

$y = 90 - x$ (2)

$z = 180 - x$ (3)

(2) × 3: $3y = 3(90 - x) = 270 - 3x$ (4)

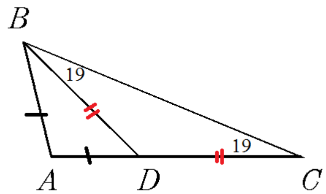
Substituting (3) and (4) into (1): $270 - 3x = 180 - x$. Solving for x: $x = 45^{\circ}$.

Problem 3. Solution: C.

Since $AD = AC = r$, $\angle CDB = \angle DCA$.

By **Theorem 2**, $\angle CAB = \angle CDB + \angle DCA = 2\angle CDB \Rightarrow \angle CDB = 45/2 = 22.5^{\circ}$.

Problem 4. Solution: A.



As shown in the figure, $\angle ADB = \angle ABD$

By **Theorem 2**, $\angle ADB = 19 \times 2 = 38^{\circ}$.

$\angle A = 180^{\circ} - 2\,\angle ADB = 180^{\circ} - 2 \times 38^{\circ} = 104^{\circ}$.

Problem 5. Solution: B.

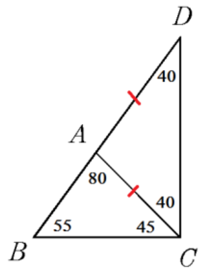
We know that $\angle ABC = 55^o$, $\angle ACB = 45^o$. So $\angle BAC = 180^o - (55^o + 45^o) = 80^o$.

By **Theorem 2**, $\angle BAC = \angle ADC + \angle ACD$.

We are given that and $CA = DA$. Thus $\angle ADC = \angle ACD$.

So $\angle ACD = 80^o/2 = 40^o$.

$\angle DCB = \angle ACB + \angle ACD = 45^o + 40^o = 90^o$.



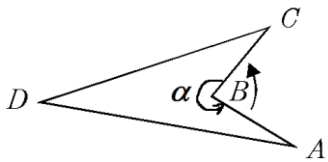
Problem 6. Solution: A.

By **Theorem 4**, $\angle ABC = \angle A + \angle D + \angle C$ (1)

$\angle ABC + \alpha = 360^o$ (2)

(2) – (1): $\alpha = 360^o - (\angle A + \angle D + \angle C)$

$\Rightarrow \quad \alpha + \angle A + \angle D + \angle C = 360^o$. The answer is C.

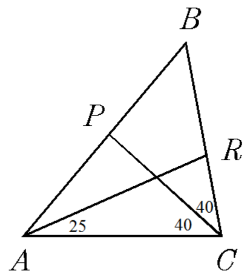


Problem 7. Solution: B.

Since $\overline{AR}$ bisects $\angle A$, $\angle DAC = 50°/2 = 25°$.

Since $\overline{CP}$ bisects $\angle C$, $\angle DCA = 80°/2 = 40°$.

In ΔARC, $\angle ARC = 180° - (25 + 40 + 40)° = 75°$.



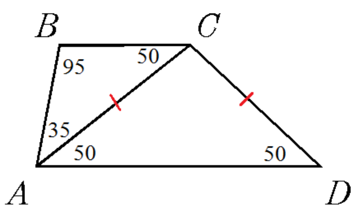
Problem 8. Solution: A.

In ΔABC, $\angle ACB = 180^o - (95^o + 35^o) = 50^o$.

Since $\overline{AD} // \overline{BC}$, $\angle CAD = \angle ACB = 50^o$.

Since $\overline{AC} = \overline{CD}$, $\angle CAD = \angle CDA = 50^o$.

In ΔACD, $\angle ACD = 180^o - (50^o + 50^o) = 80^o$.

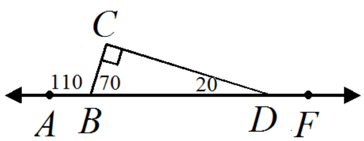


Problem 9. Solution: D.

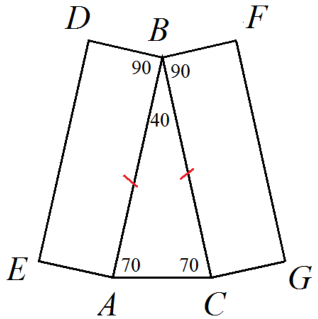
$\angle CBD = 180^o - 110^o = 110^o$.

$\angle CDB = 90^o - 70^o = 20^o$.

$\angle CDF = 180^o - 20^o = 160^o$.



Problem 10. Solution: E.



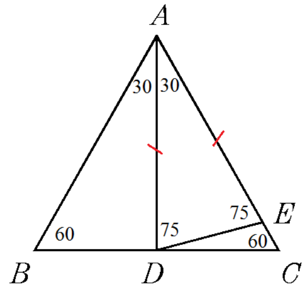
Since $AB = CB$, $\angle BCA = 70°$.

$\angle BCA = 70°$.

$\angle ABC = 180^{\circ} - 70^{\circ} - 70^{\circ} = 40°$.

$\angle DBF = 360^{\circ} - 90^{\circ} - 90^{\circ} - 40^{\circ} = 140°$.

Problem 11. Solution: A.



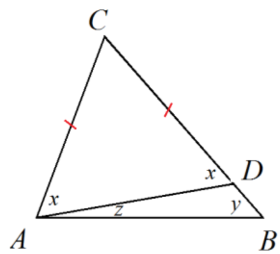
Since $\angle DAB = 30^{\circ}$, $\angle DAC = 30^{\circ}$.

Since $AE = AD$, $\angle ADE = \angle AED = (180 - 30)^{\circ}/ 2 = 75^{\circ}$.

In triangle ECD, by 3.2, $75^{\circ} = \angle EDC + 60^{\circ}$.

So $\angle EDC = 75^{\circ} - 60^{\circ} = 15^{\circ}$.

Problem 12. Solution: B.



Since $AC = CD$, and $\angle CAD = \angle CDA = x$.

By **<u>Theorem 2,</u>** $x = y + z$ (1)

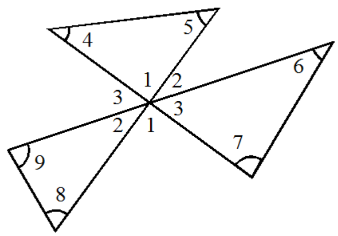
$\angle CAB - \angle ABC = 30^{\circ}$

$\Rightarrow \quad x + z - y = 30^{\circ}$ (2)

Substituting (1) into (2): $2y = 30^{\circ} \quad \Rightarrow \quad y = 15^{\circ}$.

Problem 13. Solution: B.

We label all angles as follows.



We have $\angle 1 + \angle 4 + \angle 5 = 180^{\circ}$ (1)

$\angle 3 + \angle 6 + \angle 7 = 180^{\circ}$ (2)

$\angle 2 + \angle 8 + \angle 9 = 180^{\circ}$ (3)

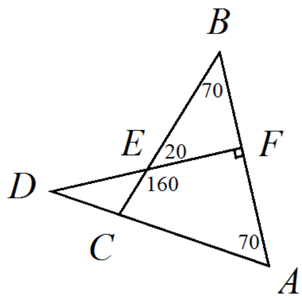
(1) + (2) + (3):

$\angle 4 + \angle 5 + \angle 6 + \angle 7 + \angle 8 + \angle 9 + \angle 1 + \angle 2 + \angle 3 = 540^\circ$ (4)

We see that $\angle 1 + \angle 2 + \angle 3 = 180^\circ$ (5)

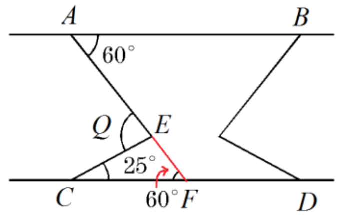
Substituting (5) into (4): $\angle 4 + \angle 5 + \angle 6 + \angle 7 + \angle 8 + \angle 9 = 540^\circ - 180^\circ = 360^\circ$.

Problem 14. Solution: C.
The third angle is $180^\circ - (45 + x + 45 - x) = 90^\circ$.



Problem 15. Solution: B.
Since $\angle FEC = 160^\circ$, $\angle BEF = 180^\circ - 160^\circ = 20^\circ$.
In ΔBEF, $\angle B = 90^\circ - 20^\circ = 70^\circ$.
We are given that $\angle A = \angle B$. So $\angle A = 70^\circ$.
In ΔABC, $\angle C = 180^\circ - 70^\circ - 70^\circ = 40^\circ$.

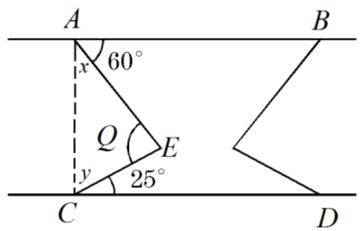
Problem 16. Solution: C.
Method 1:
Extend AE to meet CD at F. Since $\overline{AB} \,//\, \overline{CD}$, $\angle FAB = \angle BAF = 60^\circ$. By 3.2, $\angle Q = 25^\circ + 60^\circ = 85^\circ$.



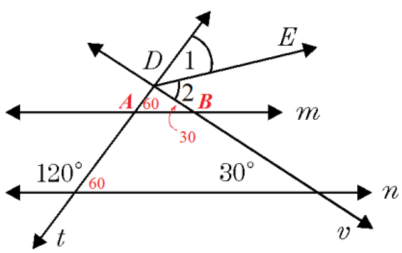
Method 2:
Connect AC. Since $\overline{AB} \,//\, \overline{CD}$, $x + 60 + y + 25^\circ = 180^\circ$

$\Rightarrow \quad x + 60 + y + 25^\circ = 95^\circ$ (1)

In triangle ACE, $\angle Q = 180^\circ - (x + y) = 180^\circ - 95^\circ = 85^\circ$.

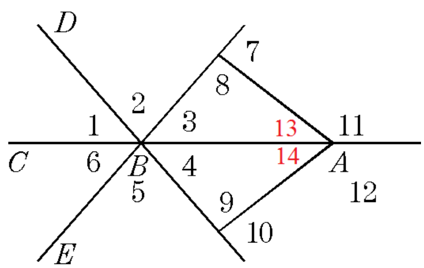


Problem 17. Solution: B.
We label some angles as shown in the figure.
In triangle ABD, by 3.2, $\angle 1 + \angle 2 = 60^\circ + 30^\circ = 90^\circ$.



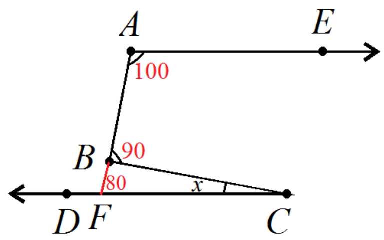
Since $\overline{DE}$ bisects the angle indicated, $\angle 1 = \angle 2$.
$\angle 1 = 90^\circ/2 = 45^\circ$.

Problem 18. Solution: E.



Since $\angle 7 = \angle 10$, $\angle 8 = \angle 9$ because they are the supplements of the two congruent angles.
Since $\overline{BC}$ bisects $\angle DBE$, $\angle 3 = \angle 4$.
Since $\angle 3 = \angle 4$ and $\angle 8 = \angle 9$, $\angle 13 = \angle 14$.
Since $\angle 13 = \angle 14$, $\angle 11 = \angle 12$.
So the answer is E.

Problem 19. Solution: E.



Extend AB to meet DC at F.
Since $\overrightarrow{AE}$ is parallel to $\overrightarrow{CD}$, $\angle BAE + \angle CFA = 180^\circ$.
So $\angle CFA = 180^\circ - \angle BAE = 180^\circ - 100^\circ = 80^\circ$.
In triangle DBC, by 3.2, $90^\circ = \angle BCD + 80^\circ$.
$\angle BCD = 90^\circ - 80^\circ = 10^\circ$.

Problem 20. Solution: D.
Let the angle be x and its complement is y and its supplement is z.
We can write the following equations:

$x + y = 90°$ (1)
$x + z = 180°$ (2)
$2z = 5y + 27$ (3)

$5 \times (1)$ and $2 \times (2)$:

$5x + 5y = 450°$ (4)
$2x + 2z = 360°$ (5)
$(4) - (5)$: $3x - 27 = 90° \Rightarrow x = 39°$.

Problem 21. Solution: A.
Let the angle be x and its complement is y and its supplement is z.

We can write the following equations:

$x + y = 90°$ (1)

$x + z = 180°$ (2)

$z = 7y$ (3)

(2) − (1): $z - y = 90°$ (4)

Substituting (3) into (4): $7y - y = 90°$ $\Rightarrow$ $y = 15°$.

Problem 22. Solution: D.

Let the angle be x and its complement is y and its supplement is z.

We can write the following equations:

$x + y = 90°$ (1)

$x + z = 180°$ (2)

$z = 6y$ (3)

(2) − (1): $z - y = 90°$ (4)

Substituting (3) into (4): $6y - y = 90°$ $\Rightarrow$ $y = 18°$.

$x + 18° = 90°$ $\Rightarrow$ $x = 72°$.

Problem 23. Solution: C.

Since $\overrightarrow{OB}$ bisects $\angle AOC$, $\angle AOB = \angle BOC$ $\Rightarrow$ $2x + 10 = 8x - 14$

$\Rightarrow$ $6x = 24$ $\Rightarrow$ $2x = 8$ $\Rightarrow$ $10x = 40$.

$\angle AOC = 2x + 10 + 8x - 14 = 10x - 4 = 40 - 4 = 36°$.

Problem 24. Solution: A.

Let the angle be x and its complement is y and its supplement is z.

We can write the following equations:

$x + y = 90°$ (1)

$x + z = 180°$ (2)

By 3.9, the measure of an interior angle of a regular nine-sided polygon is

$$\frac{(n-2)\times 180°}{n} = \frac{(9-2)\times 180°}{9} = 140°.$$

So we get $2z + 3y = 140°$ (3)

$3 \times (1)$ and $2 \times (2)$:

$3x + 3y = 270°$ (4)

$2x + 2z = 360°$ (5)

(4) + (5): $5x + 140° = 630°$ $\Rightarrow$ $x = 98°$.

$z = 180° - 98° = 82°$.

Problem 25. Solution: E.

Since $\overrightarrow{OB}$ bisects $\angle AOC$, $\angle AOB = \angle BOC$ $\Rightarrow$ $3x + 16 = 8x - 14$

$\Rightarrow$ $5x = 30$ $\Rightarrow$ $x = 6$.

$\angle AOC = 3x + 16 + 8x - 14 = 11x + 2 = 66 + 2 = 68°$.

1. BASIC KNOWLEDGE

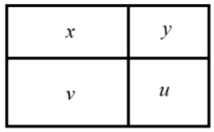
Rectangle: A quadrilateral with four right angles.

Properties of a rectangle:

The diagonals are congruent and bisect each other.
Opposite sides are congruent and parallel.

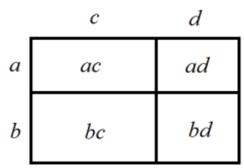
Square: A quadrilateral with four right angles and four congruent sides.

Property 1.
The rectangle is divided into four rectangles with areas as shown. The following relationship is true: $x \times u = y \times v$

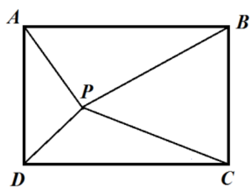


Proof:
We label the figure as follows. We see that

$x \times u = y \times v \quad \Rightarrow \qquad ac \times bd = ad \times bc \Rightarrow abcd = abcd$,
which is true.

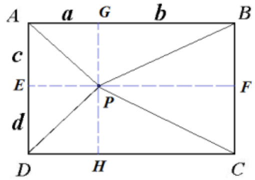


Property 2. P is any point inside rectangle $ABCD$, $AP^2 + PC^2 = BP^2 + PD^2$.



Proof:
Draw $EF \parallel AB$ through P.
Draw $GH \parallel AD$ through P.
Let $AG = DH = a$, $BG = CH = b$, $AE = BF = c$, $DE = CF = d$.

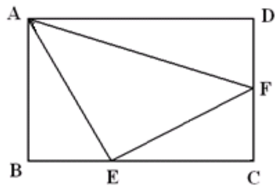
By Pythagorean Theorem,
$AP^2 = a^2 + c^2$



$CP^2 = b^2 + d^2$
$BP^2 = b^2 + c^2$
$DP^2 = d^2 + a^2$
Therefore, $AP^2 + PC^2 = BP^2 + PD^2$.

Note: The formula holds even if P is a point outside the rectangle $ABCD$.

Property 3. In rectangle $ABCD$, E and F are any two points on BC and CD, respectively. The following relationship is true:



$S_{ABCD} = 2S_{\Delta AEF} + BE \times DF$

Proof:
Let $AD = x$ and $AB = y$.
We know that $S_{ABCD} = S_{\Delta ABE} + S_{\Delta ADF} + S_{\Delta ECF} + S_{\Delta AEF}$.

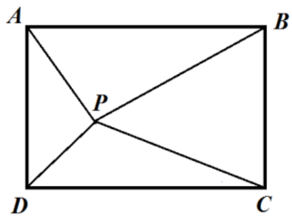
So $xy = \dfrac{y \times BE}{2} + \dfrac{x \times DF}{2} + \dfrac{(y - DF)(x - BE)}{2} + S_{\Delta AEF}$

$2xy = y \times BE + x \times DF + (xy - xDF - yBE + BE \times DF) + 2S_{\Delta AEF}$

$xy = BE \times DF + 2S_{\Delta AEF} \quad \Rightarrow \quad S_{ABCD} = 2S_{\Delta AEF} + BE \times DF$.

The conclusion is also true if $ABCD$ is a parallelogram.

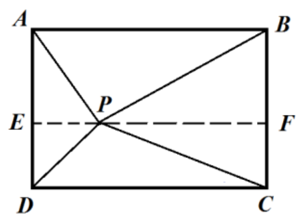
Property 4. P is any point inside rectangle $ABCD$,



$S_{\Delta APD} + S_{\Delta BPC} = S_{\Delta APB} + S_{DPC} = \dfrac{1}{2} S_{ABCD}$.

Proof:

Draw EF // AB through P.



$$S_{\Delta APD} = \frac{AD \times EP}{2} \qquad (1)$$

$$S_{\Delta BPC} = \frac{BC \times PF}{2} \qquad (2)$$

(1) + (2):

$$S_{\Delta APD}+S_{\Delta BPC}=\frac{AD\times EP}{2}+\frac{BC\times PF}{2}=\frac{1}{2}AD(EP+PF)=\frac{1}{2}AD\times DC=\frac{1}{2}S_{ABCD}$$

Similarly, $S_{\Delta APB}+S_{\Delta DPC}=+\frac{1}{2}S_{ABCD}$.

Therefore $S_{\Delta APD}+S_{\Delta BPC}=S_{\Delta APB}+S_{DBPC}=\frac{1}{2}S_{ABCD}$.

2. EXAMPLES

☆**Example 1.** A rectangle is divided into four rectangles with areas 45, 25, 15, and x. Find x.

A. 23 B. 27 C. 30 D. 32 E. none of these

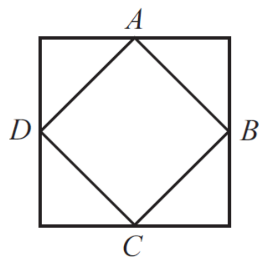
25	**45**
15	x

Solution: B.

$25x = 15 \times 45 \quad \Rightarrow \quad x = 27.$

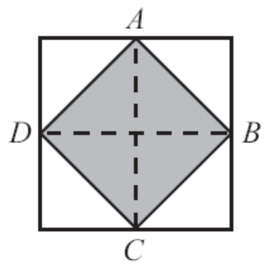
☆**Example 2.** Points A, B, C and D are midpoints of the sides of the larger square. If the larger square has area 2016, what is the area of the smaller square?

(A) 1005 (B) 1006 (C) 1008 (D) 504 (E) 540

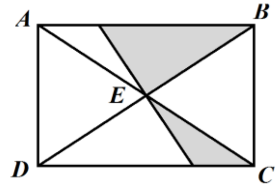


Solution: C.

Divide the larger square into 8 congruent triangles, as shown, 4 of which make up the smaller square. The area of the smaller square is 4/8 or 1/2 of the area of the larger square, so the area of the smaller square is equal to 2016/2 = 1008.



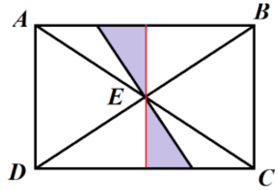
Example 3. *ABCD* is a rectangle and *E* is the intersection of the two diagonals. What percent of the total area is the same as the shaded area?



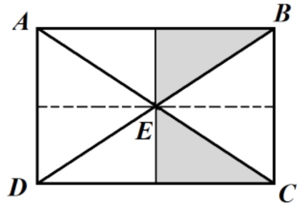
A. 15 B. 20 C. 25 D. 30 E. $33\frac{1}{3}$

Solution: C.

We see that the two shaded areas have the same area.



So the given figure can be converted in to the figure below.



The answer is then 2/8 = 1/4 = 25%.

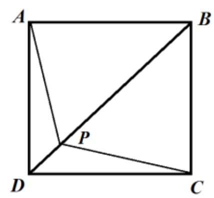
Example 4. A point inside a square is positioned so that the distances to the four vertices are 27, 21, 6 and *x* units. If *x* is a whole number, what is the value of *x*?

A. 15 B. 16 C. 17 D. 18 E. 19

Solution: D.

$a^2+c^2=b^2+d^2 \quad \Rightarrow \quad 27^2+6^2=21^2+x^2 \quad \Rightarrow \quad x=18$.

Example 5. *ABCD* is a square with the side length of 1. *BD* is the diagonal. *P* is a point on *BD*. Find $S_{\Delta PDC}$, the area of *ΔPDC* if $S_{\Delta APB}=\frac{2}{5}$.



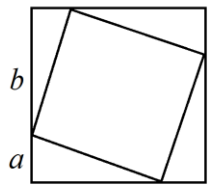
A. $\frac{\sqrt{2}}{4}$ B. $\frac{\sqrt{3}}{6}$ C. $\frac{3}{5}$ D. $\frac{1}{10}$ E. $\frac{3}{10}$

Solution: D.

By the **property 4**, we have $\frac{1}{2}=S_{\Delta APB}+S_{\Delta PDC}=\frac{2}{5}+S_{\Delta PDC}$.

Therefore $S_{\Delta PDC} = \frac{1}{2} - \frac{2}{5} = \frac{5-4}{10} = \frac{1}{10}$.

☆ **Example 6.** (2012 AMC problem 25) A square with area 4 is inscribed in a square with area 5, with one vertex of the smaller square on each side of the larger square. A vertex of the smaller square divides a side of the larger square into two segments, one of length *a* and the other of length *b*. What is the value of *ab*?

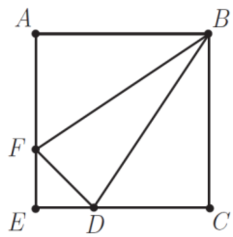


A. $\frac{1}{5}$ B. $\frac{2}{5}$ C. $\frac{1}{2}$ D. 1 E. 4

Solution: C.
Method 1 (official solution):
The area of the region inside the larger square and outside the smaller square has total area 5 − 4 = 1 and is equal to the area of four congruent right triangles, each with one side of length *a* and the other of length *b*. The area of each triangle is 1/4. If *ab*/2 = 1/4, then *ab* = 1/2.

Method 2 (our solution):
We know that $a^2 + b^2 = 4$ (1)
and $a + b = \sqrt{5}$ (2)
Squaring both side of (2): $a^2 + 2ab + b^2 = 5 \Rightarrow 4 + 2ab = 5 \Rightarrow 2ab = 5 - 4 = 1$.
Thus $\Rightarrow$ $ab = \frac{1}{2}$.

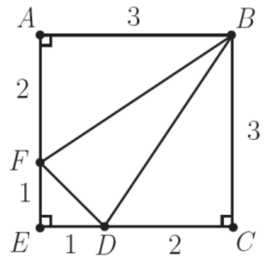
☆ **Example 7.** (2008 AMC problem 23) In square *ABCE*, *AF* = 2*FE* and *CD* = 2*DE*. What is the ratio of the area of *ΔBFD* to the area of square *ABCE* ?

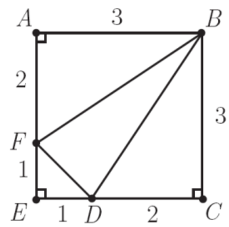


(A) 1/6 (B) 2/9 (C) 5/18 (D) 1/3 (E) 7/20

Solution: C.
Method 1 (official solution):

Because the answer is a ratio, it does not depend on the side length of the square. Let $AF = 2$ and $FE = 1$. That means square $ABCE$ has side length 3 and area $3^2 = 9$ square units. The area of ΔBAF is equal to the area of $\Delta BCD = \frac{1}{2} \times 3 \times 2 = 3$ square units. Triangle DEF is an isosceles right triangle with leg lengths $DE = FE = 1$. The area of ΔDEF is $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$ square units. The area of ΔBFD is equal to the area of the square minus the areas of the three right triangles: $9 - (3 + 3 + \frac{1}{2}) = 5/2$. So the ratio of the area of ΔBFD to the area of square $ABCE$ is $\frac{\frac{5}{2}}{9} = \frac{5}{18}$.



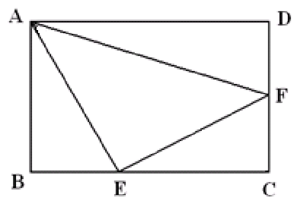


Method 2 (our solution):
Because the answer is a ratio, it does not depend on the side length of the square. Let $AF = 2$ and $FE = 1$. That means square $ABCE$ has side length 3 and area $3^2 = 9$ square units.

By **Property 3**, $S_{ABCD} = 2S_{\Delta BFD} + AF \times DC \Rightarrow 1 = 2 \times \frac{S_{\Delta BFD}}{S_{ABCD}} + \frac{AF \times DC}{S_{ABCD}} \Rightarrow$

$$1 = 2 \times \frac{S_{\Delta BFD}}{S_{ABCD}} + \frac{2 \times 2}{9} \Rightarrow 2 \times \frac{S_{\Delta BFD}}{S_{ABCD}} = 1 - \frac{4}{9} = \frac{5}{9} \Rightarrow \frac{S_{\Delta BFD}}{S_{ABCD}} = \frac{5}{18}.$$

☆ **Example 8.** $ABCD$ is a rectangle. E is a point on BC and F is a point on CD. The areas of the triangles ABE, ECF, and FDA are 4, 3, and 5, respectively. What is the area of the triangle AEF?
(A) 4 (B) 6 (C) 7 (D) 8 (E) 10



Solution: D.

Method 1: $S_{\Delta FDA}=5$, so $AD = x$, $DF=10/x$.

$S_{\Delta ADE}=4$, so $AB = y$, $BE = 8/y$

Thus $CE = x - 8/y$ and $CF = y - 10/x$.

Using $CE \times CF = 6$, we have $xy + 80/(xy) = 24$.

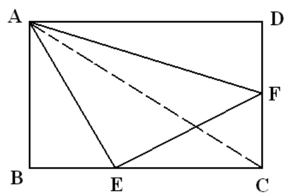
It follows that $xy = 20$ or 4, but 4 is clearly not feasible in this problem.

Method 2:

Connect AC. Let $S_{\Delta AEC}=x$ and $S_{\Delta CAF}=y$.

We have $x+4=y+5$ or

$x=y+1$ (1)



The ratio of the areas of 2 triangles with the same height is equal to the ratio of the bases.

In ΔABC and ΔAEC, $\dfrac{x+4}{x}=\dfrac{BC}{EC}$

In ΔACF and ΔECF, $\dfrac{y}{3}=\dfrac{AD}{EC}$

Since $AD = BC$, $\dfrac{x+4}{x}=\dfrac{y}{3}$ (2)

Solve x and y in (1) and (2): $x = 6$ and $y = 5$.

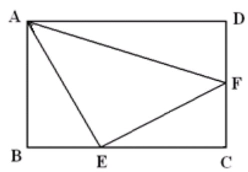
$S_{\Delta AEF}=x+y-3=8$

Method 3:

In rectangle $ABCD$, by **Property 3,** we have:

$S_{ABCD}=2S_{\Delta AEF}+BE\times DF$.

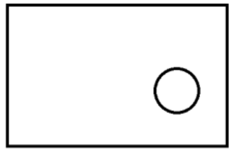
Then: $3+4+5+S=2S+BE\times DF$



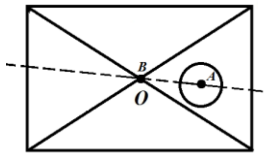
$$\Rightarrow \qquad 12 = S + \frac{BE \times AB}{AB} \times \frac{DF \times AD}{AD}$$

$$\Rightarrow \qquad 12 = S + \frac{2S_{\Delta ABE} \times 2S_{\Delta FDA}}{S_{ABCD}} \Rightarrow 12 = S + \frac{8 \times 10}{12 + S} \qquad \Rightarrow \qquad S = 8.$$

Example 9. There is a circle inside a rectangle. Please divide the circumference of the circle and the perimeter of the rectangle each into two equal parts with only one cut.

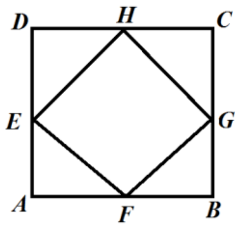


Solution:



The diameter cut the circle (center is 1 at point *B*) into two equal parts and the diagonals divide the rectangle into two equal parts (Property 3). By Property 4, any line contains points A and B will divides the circle and the rectangle into two congruent parts.

☆**Example 10.** The side length of square *ABCD* is 1. *EFGH* is a square inscribed in square *ABCD*. $AE = a$, $AF = b$. $S_{EFGH} = \frac{2}{3}$. Find $|b - a|$.



(A) $\frac{\sqrt{2}}{2}$. (B) $\frac{\sqrt{2}}{3}$. (C) $\frac{\sqrt{3}}{2}$. (D) $\frac{\sqrt{3}}{3}$. (E) $\frac{\sqrt{3}}{4}$.

Solution: D.

Since ΔAEF is a right triangle, $EF^2 = AE^2 + AF^2$.

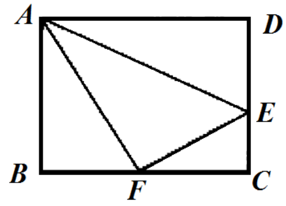
We also know that $\Delta AEF \cong \Delta DHE$. So $AF = DE$.

$$\begin{cases} a + b = 1 & (1) \\ a^2 + b^2 = \frac{2}{3} & (2) \end{cases}$$

$(1)^2 - (2)$: $2ab = \frac{1}{3}$ (3)

$(2) - (3)$: $(a - b)^2 = \frac{1}{3}$. Therefore $|a - b| = \frac{\sqrt{3}}{3}$.

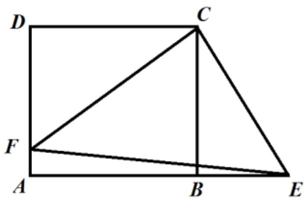
☆**Example 11.** As shown in the figure, rectangle $ABCD$ is folded along AE such that D is on F, which is a point on BC. Find CF if $AB = 12$ cm and $BC = 13$ cm.



(A) 4 (B) 6 (C) 7 (D) 8 (E) 10

Solution: D.
Since AE is the crease, $\Delta ADE \cong \Delta AFE$.
$AF = AD = 13$.
In right triangle ABF, $AF = 13$, $AB = 12$, so $BF = 5$.
$FC = BC - BF = 8$.

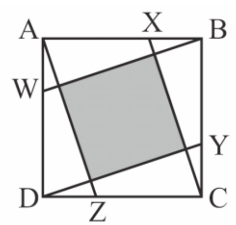
Example 12. The area of square $ABCD$ is 256. Point F is on AD. Point E is on the extension of AB. The area of right triangle CEF is 200. Find the value of BE.
(A) 10 (B) 11 (C) 12 (D) 15 (E) 18



Solution: C.
Since $\angle ECF = \angle BCD = 90°$, $\angle ECB = \angle FCD$.
We also know that $BC = CD$ and $\angle CBE = \angle CDF = 90°$.
Therefore $\Delta BCE \cong \Delta DCF$.

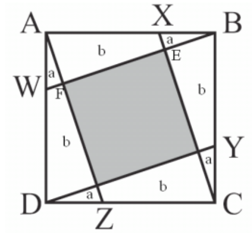
Then $CE = CF$ and ΔEFC is an isosceles right triangle.
$S_{\Delta EFC} = \frac{1}{2}CE^2 = 200$.
$CE^2 = 400$ and $BE = \sqrt{CE^2 - BC^2} = \sqrt{400-256} = 12$.

☆**Example 13.** The sides of unit square ABCD have trisection *points X, Y, Z* and *W*, as shown. If $AX:XB = BY:YC = CZ:ZD = DW:WA = 2:1$, what is the area of the shaded region? Express your answer as a common fraction.



A. $\frac{1}{4}$ B. $\frac{\sqrt{3}}{6}$ C. $\frac{3}{5}$ D. $\frac{2}{5}$ E. $\frac{3}{10}$

Solution: D.



$$\frac{S_{\Delta XBE}}{S_{\Delta ABF}} = \frac{a}{a+b} = \left(\frac{\frac{1}{3}}{1}\right)^2$$ So we have $b = 8a$.

$$S_{\Delta ADZ} = \frac{\frac{1}{3}\times 1}{2} = 2a+b = 10a$$

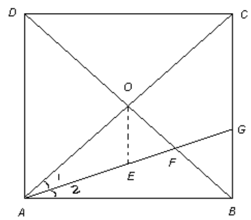
So we have 10 $a = \frac{1}{6}$ or $a = \frac{1}{60}$

We also know that the shaded area , $A = 1-4(a+b) = 1-36a$.

So $A = 1-36\times\frac{1}{60} = 1-\frac{3}{5} = \frac{2}{5}$.

Example 14. In square *ABCD*, *AC* and *BD* meet at *O*, angle bisector of $\angle CAB$ meets *BD* at *F*, and meets *BC* at *G*. Find the ratio of *OF* to *CG*.

A. $\frac{\sqrt{2}}{4}$ B. $\frac{3}{5}$ C. $\frac{3}{7}$ D. $\frac{1}{2}$ E. $\frac{3}{10}$

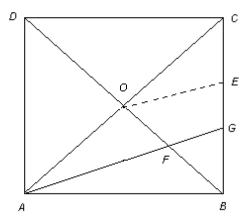


Solution: D.

Method 1:

Draw $OE // BC$, $OE = \frac{1}{2}CG$. Since $\angle 1 = \angle 2$, $\angle AOF = \angle ABG = 90°$ and $\angle AFO = \angle AGB$. Since $OE//BC$, $\angle OEG = \angle AGB$. Then $\angle AFO = \angle OEF$. It follows that $OE = OF \Rightarrow OF/CG = 1/2$.

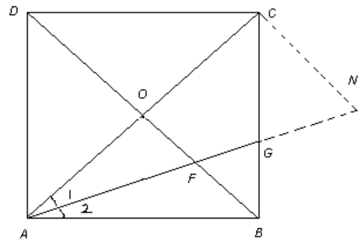
Method 2:



Draw $OE//AG$. $CE = EG$, $\angle GBA = 90°$, so $\angle GAB + \angle AGB = 90°$. $\angle AOB = 90°$, so $\angle OAF + \angle AFO = 90°$.

Since AG is the angle bisector of $\angle CAB$, $\angle CAG = \angle GAB \Rightarrow 90° - \angle CAG = \angle OFA$. $90° - \angle GAB = \angle AGB \Rightarrow \angle OFA = \angle AGB$.

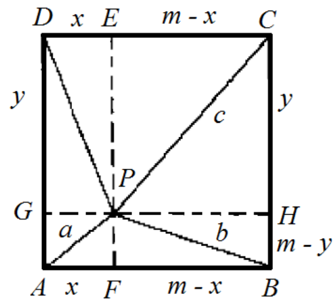
From parallel lines, $\angle GFB = \angle AFO$ and $\angle EOF = \angle GFB$. $\angle EOF = \angle OEG$. So $OFGE$ is an isosceles trapezoid. $OF = EG = \frac{1}{2} CG$.



Method 3:
Draw $CN \perp AC$. $\angle N = \angle CGN \Rightarrow \angle 1 = \angle 2$, $CG = CN$.
Since $OF \parallel CN$ and O is the midpoint of AC, $OF = \frac{1}{2} CN = \frac{1}{2} CG \Rightarrow$ $OF/CG = 1/2$.

Example 15. P is a point inside square $ABCD$. The distances from P to three vertices are 1, $\sqrt{2}$, and 3, respectively. Find the side length of the square.

A. $2\sqrt{2}$ B. $\sqrt{5}$ C. 5 D. 2 E. 4



Solution: B.
Let $DE = AF = x$, $DG = CH = y$.
$FB = EC = m - x$, $HB = GA = m - y$.
$PA = a$, $PB = b$, $PC = c$, $a \le b$, $a \le c$.
We know that $PD^2 = a^2 + c^2 - b^2$, or
$PD = \sqrt{a^2 + c^2 - b^2} = \sqrt{6}$.

By Pythagorean Theorem, we have $x^2 + (m - y)^2 = a^2$ (1)

and $x^2 + y^2 = a^2 + c^2 - b^2$ (2)

Therefore $m^2 - 2mx + a^2 + c^2 - b^2 = c^2$, or $2mx = m^2 + a^2 - b^2$ (3)

We also have $(m - x)^2 + y^2 = c^2$ (4)
Considering (2) and (4), we have $2my = m^2 + a^2 - b^2$ (5)

$(3)^2 + (5)^2$: $4m^2(a^2+c^2-b^2)=(m^2+a^2-b^2)^2+(m^2+c^2-b^2)^2 \Rightarrow$

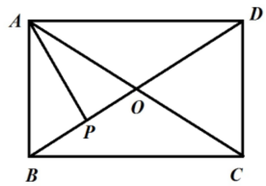
$m^4-(a^2+c^2)m^2+\frac{1}{2}[(a^2-b^2)^2+(c^2-b^2)^2]=0.$

Since $a = 1$, $b = \sqrt{2}$, $c = 3$, so $m^4-10m^2+25=0 \Rightarrow (m^2-5)^2=0$

Solving we get: $m=\sqrt{5}$.

Example 16. In rectangle $ABCD$, $AP \perp BD$ at P. $BP : PD = 1 : 3$. Find $\angle AOB$ if O is the intersection of two diagonals.

(A) 30° (B) 45° (C) 50° (D) 60° (E) 50°



Solution: D.

By the property 3, we have: $BO=\frac{1}{2}BD$.

We know that $BP : PD = 1 : 3$, so $BP=\frac{1}{4}BD=\frac{1}{2}BO=PO$.

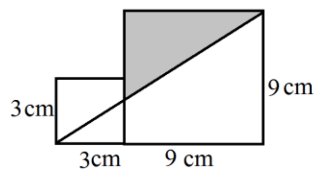
Since $AP \perp BD$ and $AP = AP$, Rt$\Delta APB \cong$ RtΔAPO.

Therefore $AB = AO \Rightarrow AO = BO$.

Triangle AOB is an equilateral triangle and $\angle AOB = 60°$.

Example 17. What is the area of the shaded region in the figure shown? Round your answer to the nearest square centimeter.

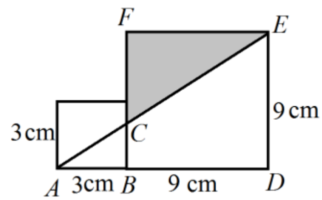
(A) 20 (B) 25 (C) 30 (D) 35 (E) 40



Solution: C.

The area of triangle ADE is $S_{\Delta ADE}=\frac{(3+9)\times 9}{2}=54$

Triangle CEF is similar to triangle ADE.



The area of triangle CEF can be obtained as following:

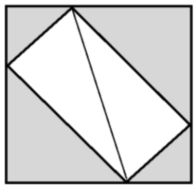
$$\frac{S_{\Delta CEF}}{S_{\Delta ADE}}=\left(\frac{9}{12}\right)^2=\left(\frac{3}{4}\right)^2=\frac{9}{16} \Rightarrow$$

$$S_{\Delta CEF}=\frac{9}{16}S_{\Delta ADE}=\frac{9}{16}\times 54=\frac{243}{8}=30.375 \approx 30.$$

Method 2: Triangle *CEF* is similar to triangle *ADE.*

$$\frac{ED}{CB}=\frac{AD}{AB} \quad \Rightarrow \quad CB=\frac{AB\times ED}{AD}=\frac{3\times 9}{12}=\frac{9}{4}$$

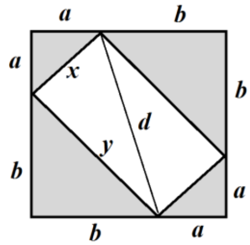
$$S_{\Delta CEF}=\frac{FE\times FC}{2}=\frac{9\times(9-\frac{9}{4})}{2}=\frac{243}{8}\approx 30$$



Example 18. A rectangle is inscribed in a square such that there is an isosceles triangle in each corner. The combined area of the four triangles is 200. Find the length of the rectangle's diagonal.

A. 20 B. 25 C. 30 D. 35 E. 40

Solution: A.



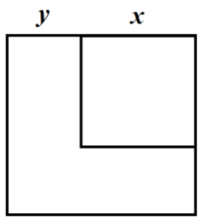
We label the line segments as shown in the figure.

We are giving that $\frac{a^2}{2}\times 2+\frac{b^2}{2}\times 2=200$

$\Rightarrow 2a^2+2b^2=400 \Rightarrow x^2+y^2=400$

So $d=\sqrt{x^2+y^2}=\sqrt{400}=20$

Example 19. A large square has a smaller square cut from its corner in such a way that the area of the square removed equals the area of the remaining region. If *x* represents the length of a side of the removed square, and *y* represents the remaining length, find the ratio $\frac{x}{y}$.



A. $\frac{5}{2}$ B. $\frac{12}{5}$ C. $\frac{1+\sqrt{2}}{1}$ D. $\frac{8}{3}$ E. $\frac{6(\sqrt{2}-1)}{1}$

Solution: C.

$(x+y)^2-x^2=x^2 \quad \Rightarrow \quad x^2+y^2+2xy-x^2=x^2$

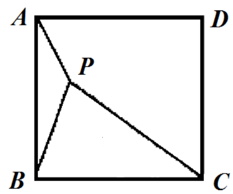
$\Rightarrow \quad y^2+2xy-x^2=0 \qquad (1)$

Dividing both sides of (1) by y^2 and letting $\frac{x}{y} = m$.

(1) becomes: $m^2 - 2m - 1 = 0 \Rightarrow \quad m_{1,2} = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$.

We ignore the negative value and get the answer: $\frac{x}{y} = 1 + \sqrt{2}$.

Example 20. P is a point inside square $ABCD$. Find $\angle APB$ if $PA:PB:PC = 1:2:3$.
(A) 105° (B) 110° (C) 120° (D) 125° (E) 135°

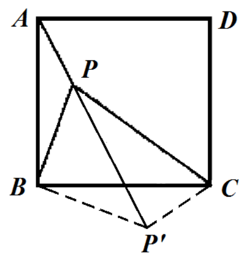


Solution: E.
Rotating ΔAPB 90° along point B to $\Delta CP'B$. Connect PP'.

Let $PA = k$.

$BP = BP' = 2k$, $PC = 3k$, $P'C = k$.

In $\Delta BPP'$, $\angle BPP' = 90°$.



So $pp' = 2\sqrt{2}k$.

$PP'^2 + P'C^2 = 8k^2 + k^2 = 9k^2$.

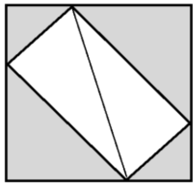
$PP'^2 + P'C^2 = PC^2 = 9k^2$

$\angle PP'C = 90°$.

In Rt$\Delta BPP'$, $\angle BP'P = \angle BPP' = 45°$
So $\angle BP'C = \angle BP'P + \angle PP'C = 135°$. $\angle APB = 135°$.

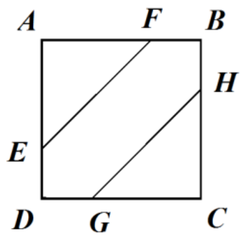
3. PROBLEMS

Problem 1. An isosceles right triangle is removed from each corner of a square piece of paper, so that a rectangle remains. The removed triangles are shown as gray in the picture below. Find the sum of the areas of the triangles cut off if the length of the diagonal d is 20 units.



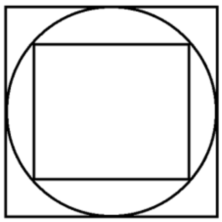
A. $120\sqrt{2}$ B. $140\sqrt{2}$ C. 180 D. 200 E. 240

Problem 2. $ABCD$ is a square of area 1. $\overline{EF}$ and $\overline{GH}$ area parallel to the diagonal and divide the square into three regions of equal area. Find the length of $\overline{EF}$.



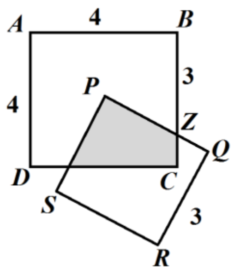
A. $\frac{1}{3}$ B. $\frac{1}{2}$ C. $\frac{\sqrt{3}}{3}$ D. $\frac{\sqrt{6}}{3}$ E. $\frac{2\sqrt{3}}{3}$

Problem 3. Find the ratio of the areas of the squares that circumscribe and inscribe a circle.



A. $\sqrt{2}$ B. $\frac{\pi}{\sqrt{2}}$ C. $\frac{\pi}{2}$ D. 2 E. $\frac{4}{\pi}$

Problem 4. What is the area of intersection of the two squares shown where P is the center of the square $ABCD$?

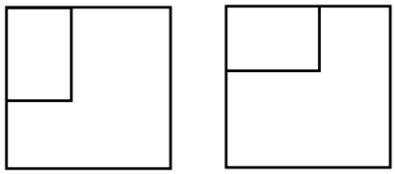


A. $\sqrt{2}$ cm^2 B. 2 cm^2 C. $2\sqrt{2}$ cm^2

D. 4 cm^2 E. $3\sqrt{2}$ cm^2

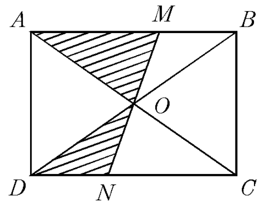
☆**Problem 5.** A 9' by 11' table sits in the corner of a square room, as shown. The owners desire to move the table to the position shown in the second figure. The side of the room is S feet. What is the smallest integer value of S for which the table can be moved as desired without tilting it or taking it apart.

A. 11 B. 12 C. 13 D. 14 E. 15



Problem 6. Given the rectangle $ABCD$ with segment MN as shown, find the area of the shaded triangles if the area of the rectangle is 100 square units.

A. 20 B. 25 C. 30 D. $33\frac{1}{3}$ E. none of these

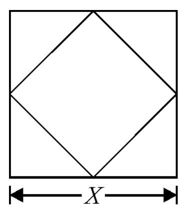


Problem 7. The ratio of the area of rectangle A to the area of rectangle B, when their respective altitudes are in the ratio 2 : 3 and their respective bases in the ratio 3 : 4, is:

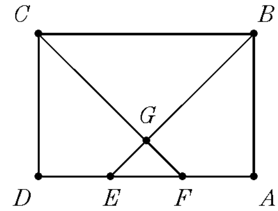
A. 1 : 2 B. 1 : 1 C. 2 : 3 D. 1 : 3 E. none of these

☆**Problem 8.** Find the ratio of the area of the inner square to the area of the outer square. The vertices of the inner square intersect the midpoints of the sides of the outer square.

A. $\frac{1}{2}$ B. $\frac{x\sqrt{2}}{2}$ C. $\frac{x^2}{2}$ D.x^2 E. none of these

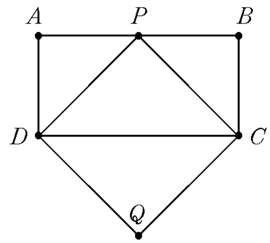


☆**Problem 9.** $ABCD$ is a rectangle of area 72 square units. Points E and F trisect side AD. Let G be the point of intersection of the segments CF and BE. The area of the quadrilateral $ABGF$ is:



A. cannot be determined uniquely from the given information
B. is 20 square units C. is 21 square units D. is 22 square units
E. is greater than 22 square units

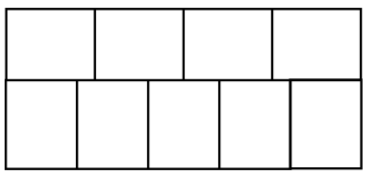
☆**Problem 10.** In the figure shown, P is the midpoint of side AB. If the area of rectangle $ABCD$ is 24, then the area of square $PCQD$ is:
A. 12 B. 18 C. 24 D. 25 E. 30



Problem 11. A point is selected inside a rectangle such that its distance from one vertex is 11 cm, its distance from the opposite vertex is 12 cm, and its distance from a third vertex is 3 cm. Its distance, in centimeters, from the fourth vertex is:
A. 20 B. 16 C. 18 D. 14 E. 13

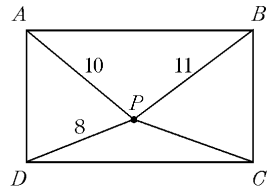
Problem 12. A farmer has sixty meters of fence with which to build a rectangular animal run as shown for her cows, horses, and pigs. She wants each type of animal to have the same area. What is the largest number of square meters that can be enclosed?
A. 112.5 B. 124 C. 128.5 D. 135 E. 136.5

Problem 13. Nine congruent rectangles are placed as shown to form a large rectangle whose area is 180. The perimeter of this figure is:
A. 54 B. 55 C. 56 D. 58 E. 59



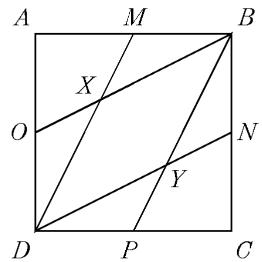
Problem 14. A rectangle *ABCD* contains a point *P* in its interior which is 10 units from *A*, 11 units from *B*, and 8 units from *D*. How far is *P* from *C* ?

A. $\sqrt{81}$ B. $\sqrt{85}$ C. $\sqrt{90}$ D. $\sqrt{96}$ E. $\sqrt{100}$



Problem 15. Given a square *ABCD* with each edge 2 units long and having midpoints of each edge *M*, *N*, *O* and *P* as shown. $\overline{BO}$ and $\overline{DM}$ intersect at point *X*. $\overline{BP}$ and $\overline{DN}$ intersect at point *Y*. What is the area of quadrilateral *BXDY* ?

A. $\frac{1}{3}$ B. $\frac{2}{3}$ C. $\frac{3}{3}$ D. $\frac{4}{3}$ E. $\frac{5}{3}$



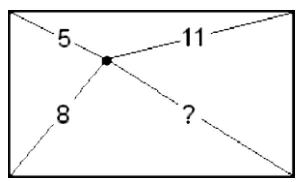
☆**Problem 16.** A rectangle is divided into four parcels as shown. If the areas of three of the parcels are 24.5, 42, and 56, what is the area of the fourth one?

A. 73.5 B. 42 C. 48 D.96 E. none of these

42	***x***
24.5	**56**

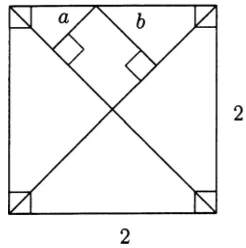
Problem 17. A rectangle has a point in its interior that is 5 units from one corner, 8 units from another and 11 units from the third. What is the distance of the point from the fourth corner?

A. 14 B. 10 C. $4\sqrt{10}$ D. $\sqrt{82}$ E. none of these



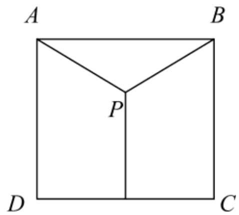
Problem 18. In the figure, the segments of length *a* and *b* lie on perpendiculars to the diagonals of a square of side length 2. Find *a* + *b*.

A. $\sqrt{2}-1$ B. $\sqrt{2}$ C. $\frac{\sqrt{2}}{2}$ D. $\sqrt{\sqrt{2}}$ E. $\sqrt{\sqrt{2}+1}$



Problem 19. *ABCD* is a square with $AB = s$. Point *P* is an interior point such that *AP*, *BP*, and the distance from *P* to $\overline{CD}$ are all equal. Find this distance.

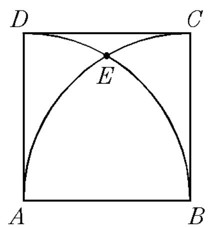
A. $\frac{5}{8}s$ B. $\frac{2}{3}s$ C. $\frac{\sqrt{2}}{2}s$ D. $\frac{3}{4}s$ E. $\frac{\sqrt{3}}{2}s$



☆**Problem 20.** Inside square *ABCD* with side of length *x*, quarter circle arcs with radius *x* are drawn using *A* and *B* as centers. These arcs intersect at a point *E* inside the square. The distance from *E* to side *CD* is:

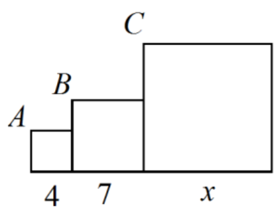
A. $\frac{x}{2}\sqrt{3}$ B. $\frac{x}{2}(1+\sqrt{3})$ C. $\frac{x}{2}(\sqrt{3}-1)$

D. $\frac{x}{2}(2-\sqrt{3})$ E. none of these



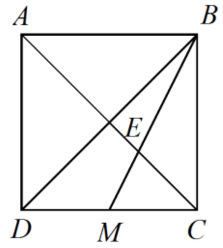
Problem 21. If the vertices *A, B*, and *C* in the 3 adjacent squares are collinear, then the value of *x* is:

A. 50 B. 49/4 C. 53/7 D. 11 E. 28

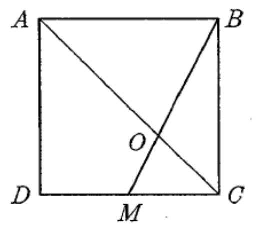


Problem 22. In the figure, quadrilateral *ABCD* is a square and *M* is the midpoint of *DC*. The ratio of the area of triangle *CEB* to the area of quadrilateral *AEMD* is:

A. 1 : 2 B. 2 : 5 C. 3 : 7 D. 3 : 8 E. 4 : 9



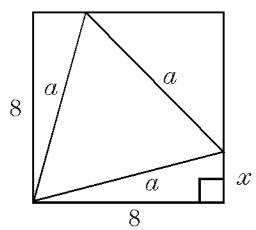
Problem 23. In square *ABCD*, *AD* is *s* centimeters, and *M* is the midpoint of $\overline{CD}$. What is the ratio of *OC* to *OA*? Express your answer as a common fraction.



☆**Problem 24.** An equilateral triangle is inscribed in a square as shown. Find *x*.

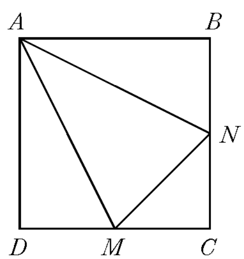
A. $16 + 8\sqrt{3}$ B. $8\sqrt{3} + 8$ C. $16 - 8\sqrt{3}$ D. $8\sqrt{3} - 8$

E. none of these



☆**Problem 25.** *ABCD* is a square with side s. *M* and *N* are midpoints of sides *DC* and *BC* respectively. Find the area of Δ*AMN*.

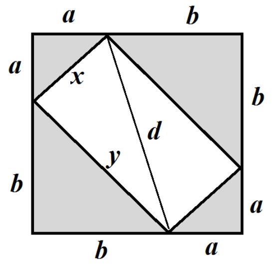
A. $\frac{s^2\sqrt{3}}{8}$ B. $\frac{s^2\sqrt{5}}{8}$ C. $\frac{s^2\sqrt{2}}{2}$ D. $\frac{5s^2}{16}$ E. $\frac{3s^2}{8}$



4. SOLUTION

Problem 1. Solution: D.

We label the line segments as shown in the figure.



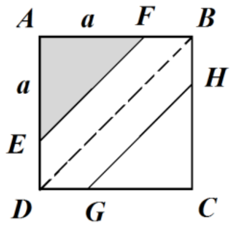
We are giving that $d = \sqrt{x^2 + y^2} = 20$. So $x^2 + y^2 = 400$.

We know by Pythagorean Theorem that $a^2 + a^2 = x^2$ and $b^2 + b^2 = y^2$. Thus $2a^2 + 2b^2 = 400$.

The sum of the areas of the triangles cut off is then

$$\frac{a^2}{2} \times 2 + \frac{b^2}{2} \times 2 = 200.$$

Problem 2. Solution: E.



Since *EF*//*DB*, we get *AE* = *AF* =*a*

We know that shaded area is 1/3. So we have $\frac{a \times a}{2} = \frac{1}{3}$

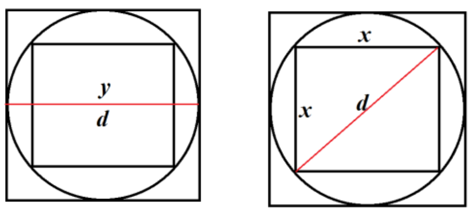
$$\Rightarrow \qquad a^2 = \frac{2}{3} \qquad \Rightarrow \qquad 2a^2 = \frac{4}{3}$$

Applying Pythagorean Theorem to triangle *AEF*:

$$EF^2 = AF^2 + AE^2 = 2a^2 = \frac{4}{3} \qquad \Rightarrow \qquad EF = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

Problem 3. Solution: D.

Let the diameter of the circle be *d*.



The area of the square that circumscribes the circle is d^2.

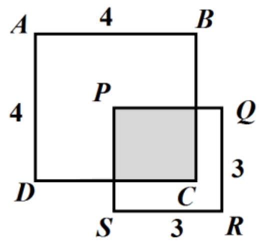
The area of the square that inscribes the circle is $x^2 = d^2/2$.

The ratio is $\dfrac{d^2}{\dfrac{d^2}{2}} = 2$.

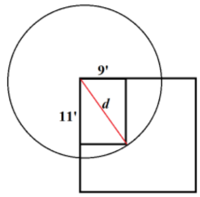
Problem 4. Solution: D.

The given problem is the same as the following problem as long as P is the center of the square $ABCD$.

The answer is then $2 \times 2 = 4$.



Problem 5. Solution: E.

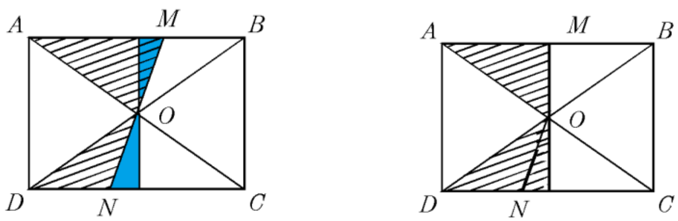


The smallest integer value of S is $\sqrt{11^2+9^2}=\sqrt{202}$.

We see that $14^2 = 196$ and $15^2 = 225$. So S must be greater than 14. 15 is the answer.

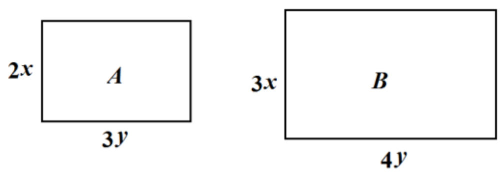
Problem 6. Solution: B.

The are of the shaded regions is $\frac{2}{8}\times 100 = 25$.



Problem 7. Solution: A.

The ratio of their areas is $\frac{2x\times 3y}{3x\times 4y}=\frac{1}{2}$.



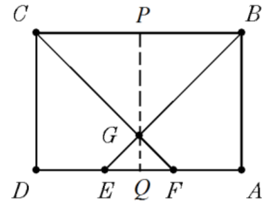
Problem 8. Solution: A.

Let the side length of the outer square be a.

The ratio of their areas is $\dfrac{a \times a}{\sqrt{2}a \times \sqrt{2}a} = \dfrac{1}{2}$.

Problem 9. Solution: C.

Method 1:



We know that $CD \times AD = CB \times AB = 72$.

We draw $PQ \perp BC$. $BGC \sim EGF$. $\dfrac{BC}{EF} = \dfrac{PG}{QG} = 3$.

The area of triangle CDF is $\dfrac{CD \times DF}{2} = \dfrac{CD \times \frac{2}{3}AD}{2} = 24$.

The area of triangle CBG is $\dfrac{CB \times PG}{2} = \dfrac{CD \times \frac{3}{4}AB}{2} = 27$.

The solution is $72 - 24 - 27 = 21$.

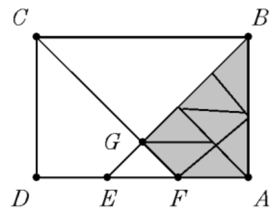
Method 2:

The area of triangle ABE is $\dfrac{AB \times AE}{2} = \dfrac{AB \times \frac{2}{3}AD}{2} = 24$.

The area of triangle EGF is $\dfrac{EF \times GQ}{2} = \dfrac{\frac{1}{3}AD \times \frac{1}{4}AB}{2} = 3$.

The solution is $24 - 3 = 21$.

Method 3:

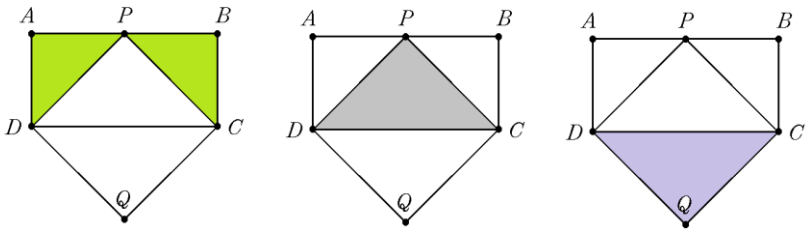


The area of triangle ABE is $\dfrac{AB \times AE}{2} = \dfrac{AB \times \frac{2}{3}AD}{2} = 24$.

We draw several parallel lines as shown in the figure below:

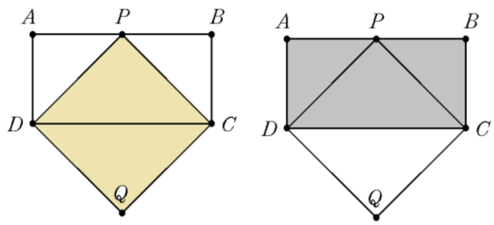
The shaded area is $\frac{7}{8}\times S_{\Delta ABE} = \frac{7}{8}\times 24 = 21$.

Problem 10. Solution: C.
As we can see, the shaded regions have the same areas (half of the area of rectangle *ABCD*).



So the area of square *PCQD* is the same as the area of rectangle *ABCD,* which is 24.

Problem 11. Solution:

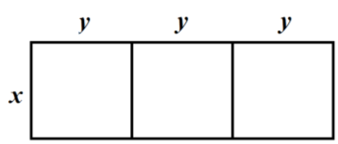


B.

$a^2 + c^2 = b^2 + d^2 \quad \Rightarrow$

$$11^2 + 12^2 = 3^2 + x^2 \quad \Rightarrow \quad x = 16.$$

Problem 12. Solution: A.

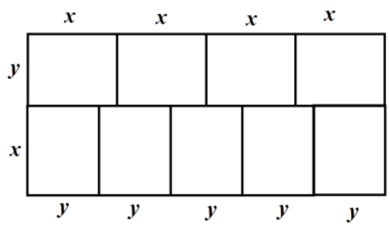


$4x + 6y = 60 \quad \Rightarrow \quad 2x + 3y = 30 \quad \Rightarrow \quad x = 15 - \frac{3}{2}y$.

$A = x \times 3y = (15 - \frac{3}{2}y)\times 3y = -\frac{9}{2}y^2 + 45y$

$$= -\frac{9}{2}(y^2 - 10y + 25 - 25) = -\frac{9}{2}(y-5)^2 + \frac{225}{2}.$$

The greatest area is $\frac{225}{2} = 112.5$.



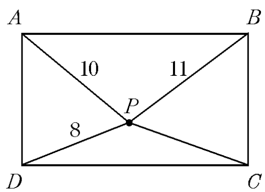
Problem 13. Solution: D.

We have $4x = 5y \quad \Rightarrow \quad y = \frac{4x}{5}$

$4x \times (x + y) = 180 \quad \Rightarrow \quad x \times (x + \frac{4x}{5}) = 45.$

So $x = 5$ and $y = 4$.

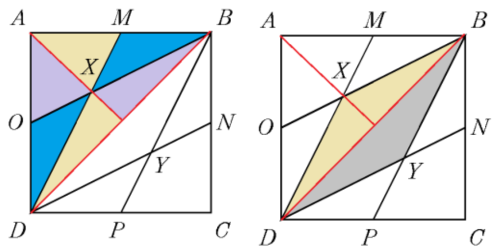
We want to find $10x + 2y = 12 \times 5 + 2 \times 4 = 58$.



Problem 14. Solution: B.

$a^2 + c^2 = b^2 + d^2 \quad \Rightarrow \quad 11^2 + 8^2 = 10^2 + x^2 \Rightarrow x = \sqrt{85}$.

Problem 15. Solution: D.



Each of the six colored small triangles has the same area, which is

$$\frac{1}{6} \times S_{\Delta ABD} = \frac{1}{6} \times \frac{AB \times AD}{2} = \frac{1}{3}.$$

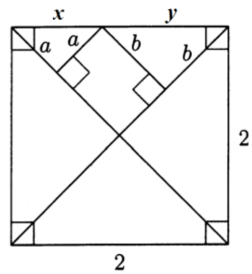
The area of quadrilateral $BXDY$ is $4 \times \frac{1}{3} = \frac{4}{3}$.

Problem 16. Solution: D.

$24.5x = 42 \times 56 \quad \Rightarrow \quad x = 96.$

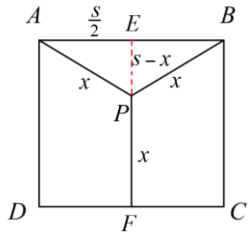
Problem 17. Solution: C.

$5^2 + x^2 = 8^2 + 11^2 \quad \Rightarrow \quad x = 4\sqrt{10}$.



Problem 18. Solution: B.
We label the line segments as x and y as shown in the figure.
We have $x = \sqrt{2}a$ and $x = \sqrt{2}b$.
We are given that $x + y = 2$, or $\sqrt{2}a + \sqrt{2}b = 2$.

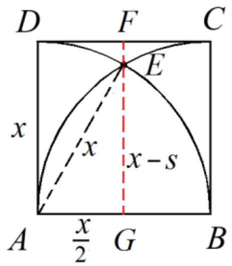
So $a + b = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.



Problem 19. Solution: A.
We extend *FP* to meet *AB* at *E*.
Let $AP = x$. $AE = s/2$. $EP = s - x$.
Applying Pythagorean Theorem to triangle *AEP*:

$$AE^2 + EP^2 = AP^2 \quad \Rightarrow \quad (\frac{s}{2})^2 + (s-x)^2 = x^2$$

$$\Rightarrow \quad \frac{s^2}{4} + s^2 - 2sx + x^2 = x^2 \quad \Rightarrow \quad \frac{5s}{4} = 2x \quad \Rightarrow \quad x = \frac{5s}{8}.$$



Problem 20. Solution: D.
Connect *AE*. Draw $FG \perp AB$ through *E*. Let $EF = s$.
Applying Pythagorean Theorem to triangle *AEG*:

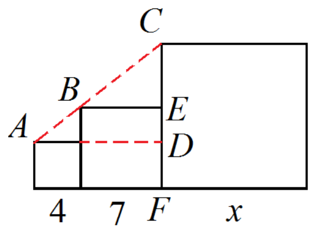
$$AE^2 - EG^2 = AG^2 \quad \Rightarrow \quad x^2 - (x-s)^2 = (\frac{x}{2})^2$$

$$\Rightarrow \quad x^2 - x^2 + 2xs - s^2 = \frac{x^2}{4} \Rightarrow \quad 4s^2 - 8xs + x^2 = 0$$

Using the quadratic formula we get: $s_{1,2} = \dfrac{8x \pm \sqrt{48x^2}}{2 \times 4} = \dfrac{8x \pm 4x\sqrt{3}}{2 \times 4} = \dfrac{2x \pm \sqrt{3}x}{2}$.

Since $s < x$, we get the answer: $s = \dfrac{2x - \sqrt{3}x}{2} = \dfrac{x}{2}(2 - \sqrt{3})$.

Problem 21. Solution: B.



$CE = x - 7$.

$\Delta ADC \sim \Delta BEC$. $\dfrac{AD}{BE} = \dfrac{CD}{CE} \quad \Rightarrow \quad \dfrac{11}{7} = \dfrac{x-4}{x-7}$

By the proportion property,

$$\Rightarrow \quad \frac{11}{7} = \frac{x-4}{x-7} = \frac{x+7}{x}.$$

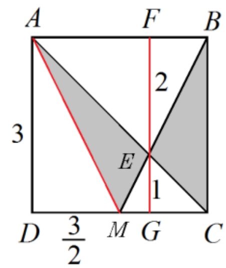
So we have $\dfrac{11}{7} = \dfrac{x+7}{x}$ or $11x = 7x + 49 \quad \Rightarrow \quad x = 49/4$.

Problem 22. Solution: B.

Method 1:

We assume that the side length of the square is 1.

Since $AB/MC = 1/2$, $FE/EG = 2/1$.

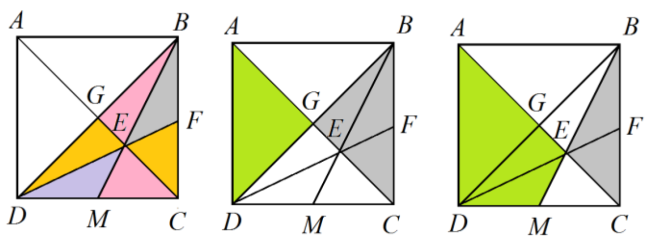


$$\frac{S_{\Delta BCE}}{S_{AEMD}} = \frac{S_{\Delta BCM} - S_{\Delta CEM}}{S_{\Delta ADM} + S_{\Delta AEM}} = \frac{S_{\Delta BCM} - S_{\Delta CEM}}{S_{\Delta ADM} + S_{\Delta BCM}}$$

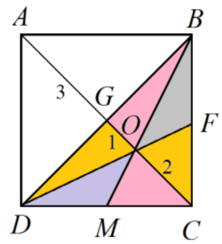
$$= \frac{\frac{1}{4} \times 9 - \frac{1}{2} \times \frac{3}{2} \times 1}{\frac{1}{4} \times 9 + \frac{1}{4} \times 9 - \frac{1}{2} \times \frac{3}{2} \times 1} = \frac{9-3}{9+9-3} = \frac{6}{15} = \frac{2}{5}.$$

Method 2:

We connect DF where F is the midpoint of BC. So E is the centroid of triangle BCD. Six smaller triangles have the same areas. We also see that the area of triangle AGD is the same as the sum of three smaller triangles. So the ratio is 2/5.

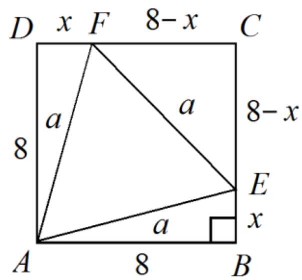


Problem 23. Solution: 1/2.



We connect DF where F is the midpoint of BC. So O is the centroid of triangle BCD. $CO/OG = 2/1$. Thus $OC/OA = 2/4 = 1/2$.

Problem 24. Solution: C.



Applying Pythagorean Theorem to triangles ECF and ABE:

$$8^2 + x^2 = a^2 \qquad (1)$$

$$(8-x)^2 + (8-x)^2 = a^2 \Rightarrow \quad a = \sqrt{2}(8-x) \qquad (2)$$

Substituting (2) into (1):

$$8^2 + x^2 = [\sqrt{2}(8-x)]^2 \Rightarrow \quad x^2 - 32x + 64 = 0 \qquad (3)$$

Solving equation (3) using the quadratic formula:

$$x_{1,2} = \frac{32 \pm \sqrt{32^2 - 4\times 64}}{2} = \frac{32 \pm 16\sqrt{3}}{2}.$$

We know that $x < 8$. So $x = \dfrac{32 - 16\sqrt{3}}{2} = 16 - 8\sqrt{3}$.

Problem 25. Solution: E.

$$S_{ABCD} = 2S_{\Delta AMN} + DM \times BN \quad \Rightarrow s^2 = 2S_{\Delta AMN} + \frac{s}{2}\times\frac{s}{2} \quad \Rightarrow \quad S_{\Delta AMN} = \frac{3s^2}{8}.$$

1. BASIC KNOWLEDGE

Similar triangles are triangles whose corresponding angles are congruent and whose corresponding sides are in proportion to each other. Similar triangles have the same shape but are not necessarily the same size.

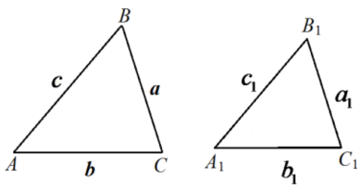
The symbol for "similar" is ~. The notation $\Delta ABC \sim \Delta A'B'C'$ is read "triangle *ABC* is similar to triangle *A*-prime *B*-prime *C*-prime."

1.1. Principles of Similar Triangles

Principle 1. (*SSS*) Corresponding sides (segments) of similar triangles are in proportion to each other.

If $\Delta ABC \sim \Delta A_1B_1C_1$, then $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$.

If $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$, then $\Delta ABC \sim \Delta A_1B_1C_1$.

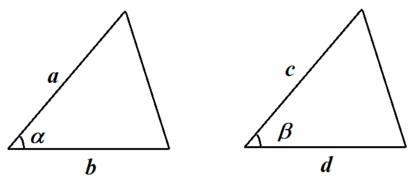


Principles 2. (*AA*) If two angles of one triangle are congruent respectively to two angles of the other triangle, the two triangles are similar by *AA* (angle, angle).

Corollary of Principle 2: Two right triangles are similar if they have one congruent cute angle.

Principles 3. (*SAS*) If two sides of one triangle are proportional to the corresponding parts of another triangle, and the ***included*** angles are congruent, the two triangles are similar by *SAS* (side, angle, side).

If $\frac{a}{c} = \frac{b}{d}$ and $\alpha = \beta$, then two triangles are similar.



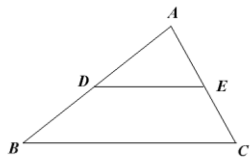
1.2. Important Theorems

Theorem 1. The ratio of the perimeters of two similar figures is:

$$\frac{P_{\Delta ABC}}{P_{\Delta A_1B_1C_1}} = \frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}.$$

The ratio of the areas of two similar figures is: $\frac{S_{\Delta ABC}}{S_{\Delta A_1B_1C_1}} = (\frac{a}{a_1})^2 = (\frac{b}{b_1})^2 = (\frac{c}{c_1})^2$.

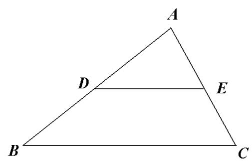
Theorem 2. A line parallel to a side of a triangle cuts off a triangle similar to the given triangle. If $DE//BC$, then $\Delta ABC \sim \Delta ADE$.



Proof:
Since $DE //BC$, $\angle ADE = \angle ABC$ and $\angle AED = \angle ACB$.
By the Principle 2 (AA), $\angle ADE = \angle ABC$.

If $\Delta ABC \sim \Delta AD$, then $\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}$; $\frac{AD}{DB} = \frac{AE}{EC}$; $\frac{AD}{AE} = \frac{DB}{EC}$.

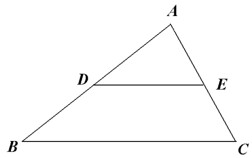
Theorem 3. In ΔABC, if D is the midpoint of AB, E is the midpoint of AC, then $DE//BC$ and $DE = \frac{1}{2}BC$.



Proof:
We see that $\frac{AD}{AB} = \frac{1}{2}$, $\frac{AE}{AC} = \frac{1}{2}$, and $\angle A = \angle A$.
By the **Principles 3.** (SAS), we know that $\Delta ABC \sim \Delta ADE$. So $\angle ADE = \angle ABC$ and $\angle AED = \angle ACB$. Thus $DE//BC$.

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{1}{2} \Rightarrow \qquad DE = \frac{1}{2}BC.$$

Theorem 4. In ΔABC, if D is the midpoint of AB, and $DE//BC$, then E is the midpoint of AC and $DE = \frac{1}{2}BC$.



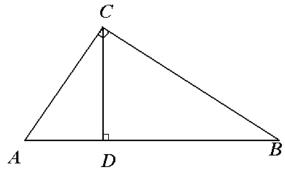
Theorem 5. If $\angle ACB = \angle ADC = 90°$, then $\Delta ABC \sim \Delta ACD \sim \Delta CBD$.

$AC^2 = AB \times AD$ (1)

$BC^2 = AB \times BD$ (2)

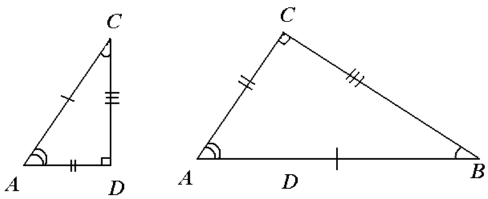
$CD^2 = AD \times BD$ (3)

$CD \times AB = AC \times BC$ (4)



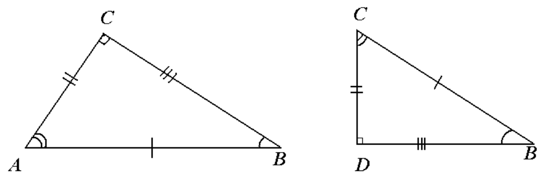
Proof:

(1). We separate two similar triangles as follows:



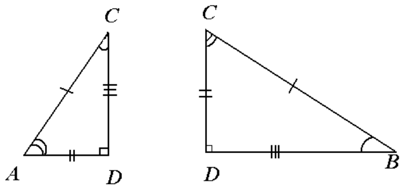
$$\frac{AC}{AB} = \frac{AD}{AC} \quad \Rightarrow \quad AC^2 = AB \times AD \qquad (1)$$

(2). We separate two similar triangles as follows:

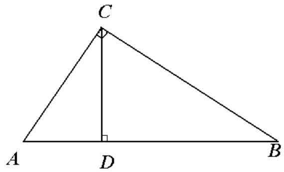


$$\frac{AB}{BC} = \frac{BC}{BD} \quad \Rightarrow \quad BC^2 = AB \times BD \qquad (2)$$

(3). We separate two similar triangles as follows:



$$\frac{AD}{CD}=\frac{CD}{BD} \quad \Rightarrow \quad CD^2 = AD \times BD \qquad (3)$$



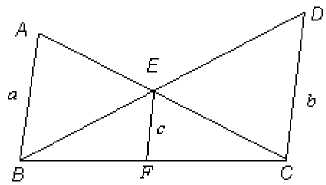
(4). The area of triangle ABC is $S_{\Delta ABC} = \frac{1}{2}AC \times BC$

$S_{\Delta ABC}$ can also be written as $S_{\Delta ABC} = \frac{1}{2}AB \times CD$

$$\frac{1}{2}AC \times BC = \frac{1}{2}AB \times CD \quad \Rightarrow \quad CD \times AB = AC \times BC$$

Theorem 6. Given $AB//EF//CD$. $AB = a$, $CD = b$, and $EF = c$. Then

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c} \quad \Rightarrow EF = c = \frac{ab}{a+b}.$$



Proof:

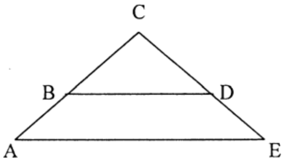
$\Delta ABC \sim \Delta EFC$. $\quad \frac{c}{a}=\frac{FC}{BC} \qquad (1)$

$\Delta DCB \sim \Delta EFB$. $\quad \frac{c}{b}=\frac{BF}{BC} \qquad (2)$

(1) + (2): $\frac{c}{a}+\frac{c}{b}=\frac{FC+BF}{BC}=1 \quad \Rightarrow \quad \frac{1}{a}+\frac{1}{b}=\frac{1}{c}$.

2. EXAMPLES

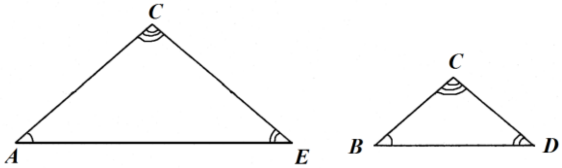
Example 1. Assume $\overline{BD}$ is parallel to $\overline{AE}$ in the figure shown. Which of the following segments corresponds to $\overline{CD}$ when we are considering the two similar triangles pictured?



A. $\overline{BD}$ B. $\overline{DE}$ C. $\overline{CE}$ D. $\overline{CA}$ E. $\overline{CE}$ and $\overline{CA}$

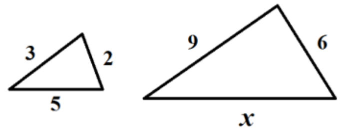
Solution: C.

CD faces $\angle CBD$. Since $\angle CBD = \angle CAE$. *CE* faces $\angle CAE$. So *CE* is the corresponding side of *CD*.



Example 2: These triangles are similar. Find x.

A. 9 B. 10 C. 15 D. 25 E. 12

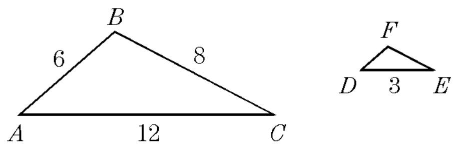


Solution: C.

We see that $\frac{3}{9} = \frac{2}{6}$. So we have $\frac{3}{9} = \frac{5}{x} \quad \Rightarrow \quad x = 15$.

Example 3. Triangle *ABC* is similar to triangle *DEF* as sketched. The perimeter of triangle *DEF* is:

A. 6.5 B. 9 C. 6 D. 7.5 E. none of these



Solution: A.

Since triangle *ABC* is similar to triangle *DEF*,

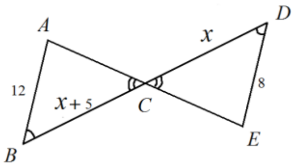
$$\frac{AB}{DF} = \frac{AC}{DE} \Rightarrow \frac{6}{DF} = \frac{12}{3} \Rightarrow DF = \frac{3}{2}.$$

$$\frac{BC}{FE} = \frac{AC}{DE} \Rightarrow \frac{8}{FE} = \frac{12}{3} \Rightarrow FE = 2.$$

The perimeter of triangle DEF is $3+2+\frac{3}{2}=6.5$.

Example 4. In the figure shown, segment AE intersects segment BD at point C. Find the length of line segment BD if $\angle ABC=\angle CDE$.

A. 9 B. 10 C. 15 D. 25 E. 5



Solution: D.

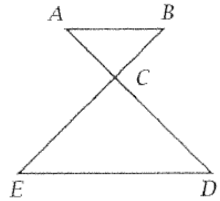
By AA, we know that triangle ABC is similar to triangle EDC.

$$\frac{12}{8}=\frac{x+5}{x} \quad \Rightarrow \quad x=10.$$

$BD=x+5+x=25$.

Example 5. If $AB \parallel DE$, $AB=5$, $CE=8$, and $DE=12.5$, find the measure of $\overline{BC}$.

A. 20 B. 7.8 C. 15 D. 3.2 E. 4



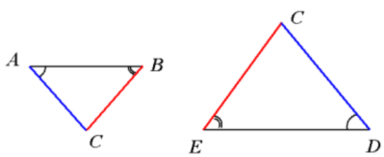
Solution: D.

Since $AB \parallel DE$, $\angle A=\angle D$ and $\angle B=\angle E$. $\Delta ABC \sim \Delta DEC$

AB and DE are corresponding sides. BC and CE are corresponding sides.

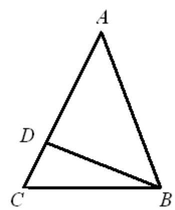
Since the corresponding sides of similar triangles are proportional to each other, we have:

$$\frac{AB}{DE}=\frac{BC}{CE} \quad \Rightarrow \quad \frac{5}{12.5}=\frac{BC}{8} \quad \Rightarrow \quad BC=3.2$$



Example 6. If $AB=AC$, $DB=CB$, $AB=12$ and $BC=5$, find the measure of $\overline{DC}$. Express your answer as a mixed number.

A. $2\frac{1}{12}$ B. $\frac{7}{3}$ C. 3 D. $\frac{29}{12}$ E. $\frac{4\sqrt{13}}{7}$

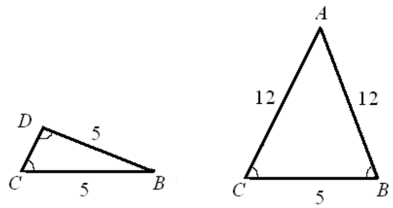


Solution: A.

As shown in the figure, ΔABC is similar to ΔBDC.

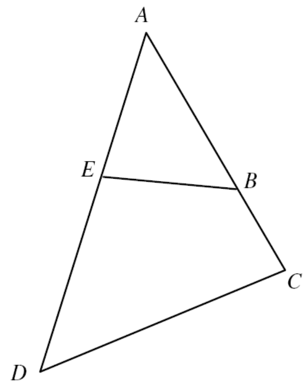
$$\frac{AB}{BC} = \frac{BC}{DC} \qquad \Rightarrow \qquad \frac{12}{5} = \frac{5}{DC}$$

$DC = 25/12 = 2\frac{1}{12}$.



Example 7. In the following diagram (not necessarily to scale), $\angle ABE = \angle ADC$, $AE = 6$, $BC = 2$, $BE = 3$, and $CD = 5$. $AB + DE$ is equal to

A. 46/3 B. 112/3 C. 13/2 D. 20 E. none of these



Solution: A.

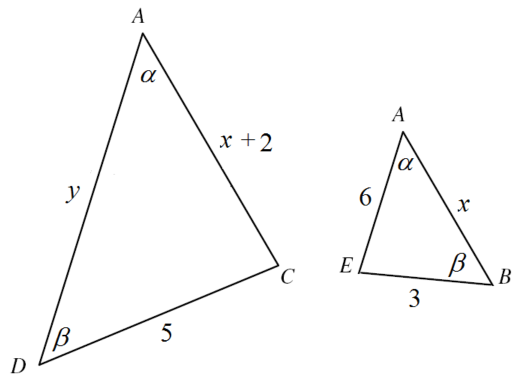
We separate two triangles and label each segments as shown in the figure.

We know that $\Delta ABE = \Delta ADC$.

So $\frac{5}{3} = \frac{x+2}{6} \quad \Rightarrow \quad x = 8$.

So $\frac{5}{3} = \frac{y}{x} = \frac{y}{8} \quad \Rightarrow \quad y = \frac{40}{3}$.

$AB + DE = 8 + y = \frac{40}{3} - 6 = \frac{46}{3}$.



Example 8. A triangle with sides 9, 12 and 15 is similar to another triangle which has longest side 25. The area of the larger triangle is:

A. 54 B. 96 C. 105 D. 210 E. 150.

Solution: E.

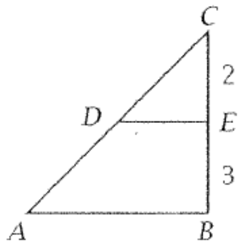
This is a right triangle with two legs of 9 and 12. The area is 9 × 12/2 = 54.

Let the area of another triangle be $S_{\Delta ABC}$. By the Theorem 1, we have

$\frac{S_{\Delta ABC}}{54} = (\frac{25}{15})^2$, or $S_{\Delta ABC} = (\frac{25}{15})^2 \times 54 = 150$.

Example 9. In right triangle ABC , $\overline{DE}$ is parallel to $\overline{AB}$, CE = 2 cm, and EB = 3 cm. If the area of ΔABC is 30 cm^2, what is the number of square centimeters in the area of ΔCDE? Express your answer as a decimal number.

A. 5.4 B. 9.6 C. 4.8 D. 3.3 E. 5.



Solution: C.

Since AB // DE, $\Delta DCE \sim \Delta ACB$

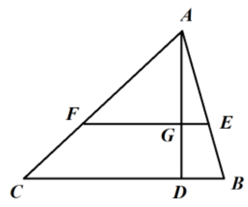
For similar figures, the ratio of their areas is the ratio of the square of the sides, that is,

$$\frac{S_{\Delta DEC}}{S_{\Delta ABC}} = (\frac{2}{3+2})^2 \quad \Rightarrow \quad S_{\Delta DEC} = (\frac{2}{3+2})^2 S_{\Delta ABC} = \frac{4}{25} \times 30 = 4.8$$

Example 10. A line parallel to the base of a triangle cuts the triangle into two regions of equal area. This line also cuts the altitude into two parts. Find the ratio of the two parts of the altitude.

A. 1 : 1 B. 1 : 2 C. 1 : $\sqrt{2}$ D. 1 : $\sqrt{2}$ +1 E. none of these

Solution: D.

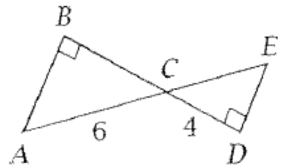


$$\frac{S_{\Delta ABC}}{S_{\Delta AEF}} = \left(\frac{AD}{AG}\right)^2 \quad \Rightarrow \quad 2 = \left(\frac{AG+GD}{AG}\right)^2$$

$$\Rightarrow \quad \frac{AG+GD}{AG} = \sqrt{2} \quad \Rightarrow \quad 1 + \frac{GD}{AG} = \sqrt{2}$$

$$\Rightarrow \quad \frac{GD}{AG} = \sqrt{2} - 1 \quad \Rightarrow \quad \frac{AG}{GD} = \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1 .$$

Example 11. In the figure, $AC = 6$ cm, $CD = 4$ cm, and $DE = 3$ cm. Find the number of square centimeters in the area of the triangle ABC .



A. $9\frac{1}{12}$ B. $7\frac{7}{3}$ C. 8 D. $8\frac{4}{5}$ E. $8\frac{16}{25}$

Solution: E.

Method 1: We know that $\Delta ABC \sim \Delta EDC$ ($\angle CAB = \angle CED$ and $\angle B = \angle D = 90°$).

We know that $CE = 5$ (ΔEDC is a $3 - 4 - 5$ right triangle).

$$\frac{S_{\Delta ABC}}{S_{\Delta EDC}} = (\frac{AC}{CE})^2 = (\frac{6}{5})^2 \Rightarrow \qquad S_{\Delta ABC} = (\frac{6}{5})^2 S_{\Delta EDC} = \frac{36}{25} \times 6 = \frac{216}{25} = 8\frac{16}{25}.$$

Method 2: We know that $CE = 5$ (ΔEDC is a $3 - 4 - 5$ right triangle).

Since both AB and DE are perpendicular to the same line segment BD, they are parallel. So $\Delta ABC \sim \Delta EDC$ ($\angle CAB = \angle CED$ and $\angle B = \angle D = 90°$).

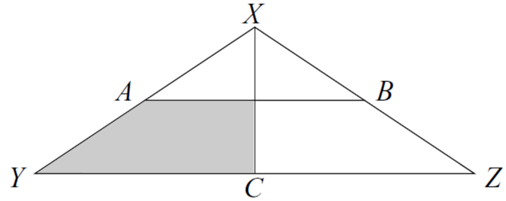
$$\frac{CE}{AC} = \frac{CD}{BC} = \frac{DE}{AB}$$

$$\frac{CE}{AC} = \frac{CD}{BC} \quad \Rightarrow \quad \frac{5}{6} = \frac{4}{BC} \quad \Rightarrow \quad BC = \frac{24}{5}$$

$$\frac{CE}{AC} = \frac{DE}{AB} \quad \Rightarrow \quad \frac{5}{6} = \frac{3}{AB} \quad \Rightarrow \quad AB = \frac{18}{5}$$

$$S_{\Delta ABC} = \frac{1}{2} AB \times BC = \frac{1}{2} \times \frac{24}{5} \times \frac{18}{5} = 8\frac{16}{25}.$$

☆**Example 12.** (2002 AMC 8 problem 20) The area of triangle XYZ is 8 square inches. Points A and B are midpoints of congruent segments XY and XZ. Altitude XC bisects YZ. The area (in square inches) of the shaded region is

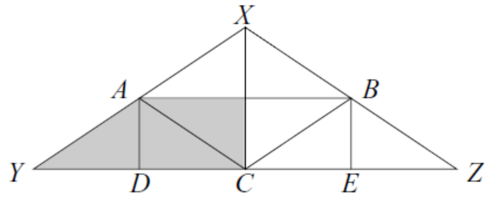


A. $1\frac{1}{2}$. B. 2 C. $2\frac{1}{2}$. D. 3. E. $3\frac{1}{2}$

Solution: D.

Method 1: (official solution)

Segments AD and BE are drawn perpendicular to YZ. Segments AB, AC and BC divide ΔXYZ into four congruent triangles. Vertical line segments AD, XC and BE divide each of these in half. Three of the eight small triangles are shaded, or 3/8 of ΔXYZ. The shaded area is (3/8) (8) = 3.



Method 2:

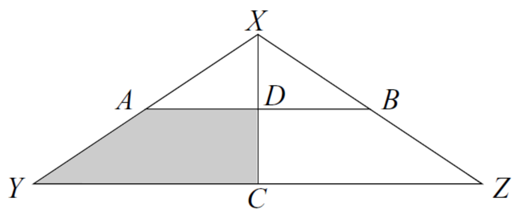
Segments AB, AC and BC divide ΔXYZ into four congruent triangles, so the area of ΔXAB is one-fourth the area of ΔXYZ. That makes the area of trapezoid $ABZY$ three-fourths the area of ΔXYZ. The shaded area is one-half the area of trapezoid $ABZY$, or three-eighths the area of ΔXYZ, and (3/8) (8) = 3.

Method 3 (our method):

ΔXYC is similar to ΔXAD. The ratio of their areas is $\frac{S_{\Delta XYC}}{S_{\Delta XAD}} = (\frac{XY}{XA})^2 = (\frac{2}{1})^2 = \frac{4}{1} \Rightarrow$

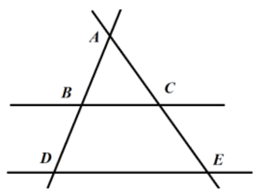
$\frac{S_{\Delta XYC}}{S_{\Delta XAD}} = \frac{4}{1} \Rightarrow \frac{4}{S_{\Delta XAD}} = \frac{4}{1} \Rightarrow S_{\Delta XAD} = 1 \Rightarrow$

$S_{AYCD} = S_{\Delta XYC} - S_{\Delta XAD} = 4 - 1 = 3$.



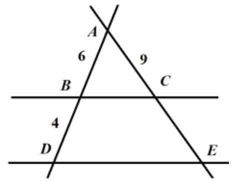
Example 13. In the figure, $\overline{BC}$ is parallel to $\overline{DE}$. The length of $\overline{AB}$ is 6, the length of $\overline{BD}$ is 4, and the length of $\overline{AC}$ is 9. What is the length of $\overline{CE}$?

A. $\frac{9}{2}$. B. 5 C. 6. D. $\frac{27}{2}$. E. Cannot be determined.



Solution: C.

By **Theorem 2,** $\frac{CE}{4} = \frac{9}{6}$ $\Rightarrow$ $CE = \frac{9}{6} \times 4 == 6$.

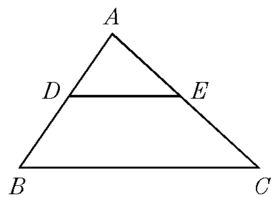


Example 14. In the sketch, *DE*//*BC* and *BD* is the square of *AD*. If *AC* = 21/4 and *EC* = 9/2, what is *BD* ?

A. 16 B. 36 C. 64 D. 144 E. none of these

Solution: B.

By **Theorem 2,** $\frac{AD}{AE} = \frac{BD}{EC}$ $\Rightarrow$ $\frac{\sqrt{BD}}{\frac{21}{4} - \frac{9}{2}} = \frac{BD}{\frac{9}{2}}$.



$\sqrt{BD} = \frac{\frac{9}{2}}{\frac{21}{4} - \frac{9}{2}} = \frac{\frac{9}{2}}{\frac{3}{4}} = 6$ $\Rightarrow$ $BD = 36$.

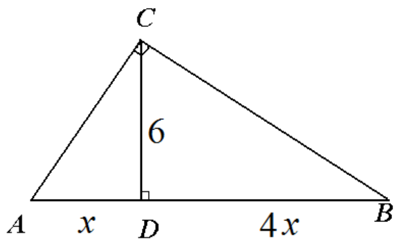
Example 15. In a right triangle, a perpendicular is dropped from the right angle to the hypotenuse and the segments of the hypotenuse have lengths of x inches and 4x inches. If the altitude is 6 inches in length, then x has length in inches, of:

A. 1/3 B. 2/3 C. 3/2 D. 3 E. cannot be determined

Solution: D.

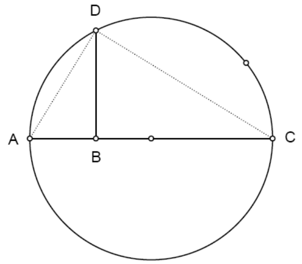
By **Theorem 5,** $CD^2 = AD \times BD$

$\Rightarrow 6^2 = x \times 4x$ $\Rightarrow$ $x = 3$



Example 16. *AC* is a diameter of a circle in which *AD* is a chord; *B* is a point on *AC* such that $DB \perp AC$. If $AB = 9$, and $BC = 16$, how long is *DB*?

A. 12 B. 13 C. 14 D. 15 E. 18.



Solution: A.

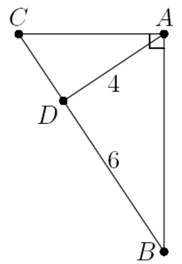
ΔADC is a right triangle since it is inscribed in a semicircle. If a perpendicular line is dropped from the right angle to the hypotenuse, then the square of its length is the product of the 2 segments it forms on the hypotenuse.

$(DB)^2 = AB \times BC \quad \Rightarrow \quad DB = \sqrt{9 \times 16} = 12$.

Example 17. What is length of *BC* in the right triangle ΔABC if $AD \perp BC$?

(A) 26/3 (B) 52 (C) $\frac{10\sqrt{13}}{3}$ (D) $\frac{8\sqrt{13}}{3}$ (E) 1/3

Solution: A.



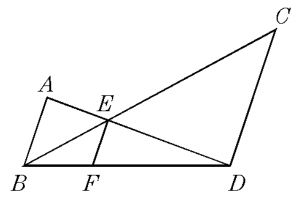
By the **Theorem 5** (3), we have $AD^2 = CD \times DB$

$\Rightarrow \quad 4^2 = CD \times 6 \quad \Rightarrow \quad CD = \frac{8}{3}$.

$BC = CD + DB = \frac{8}{3} + 6 = \frac{26}{3}$.

Example 18. In the diagram *AB//FE//DC,* and $AB = 2$ with $CD = 4$. Find the length of *EF*.

A. 4/3 B. 1 C. 3/4 D. 5/4 E. none of these



Solution: A.

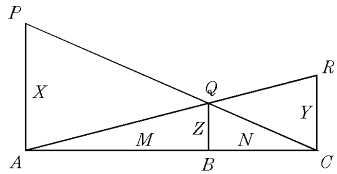
By **Theorem 6**, $EF = \frac{ab}{a+b} = \frac{2 \times 4}{2+4} = \frac{4}{3}$.

Example 19. In the figure, $\overline{PA}$, $\overline{QB}$ and $\overline{RC}$ are each perpendicular to $\overline{AC}$. Which of the following is correct?

(A) $\frac{Z}{X}=\frac{N}{M}$ (B) $\frac{Z}{X}=\frac{M+N}{N}$ (C) $\frac{Z}{Y}=\frac{M}{N}$ (D) $\frac{Z}{Y}=\frac{M+N}{M}$ (E) $\frac{Z}{Y}=\frac{M}{M+N}$

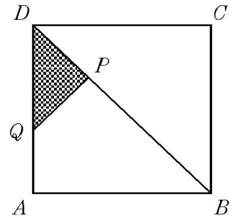
Solution: E

$\Delta CRA \sim \Delta BQA$. $\frac{Z}{Y}=\frac{AB}{AC} \quad \Rightarrow \quad \frac{Z}{Y}=\frac{M}{M+N}$.



Example 20. What fraction of the area of square $ABCD$ is represented by the area of ΔDPQ? $BP = BA$. $DP = QP$.

A. $\frac{1}{2}(\sqrt{2}-1)^2$ B. $\frac{1}{8}$ C. $\frac{\sqrt{2}}{2}$ D. $(\sqrt{2}-1)^2$ E. $\frac{\sqrt{2}}{6}$



Solution: A.

Since $\angle D = \angle D$, $\angle A = \angle DPQ = 90°$, ΔDPQ is similar to ΔABD.

$$\frac{S_{\Delta DPQ}}{S_{\Delta BAD}}=\left(\frac{DP}{AB}\right)^2 \qquad (1)$$

$$DP=\sqrt{2}AB-AB \qquad (2)$$

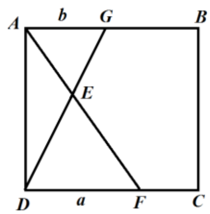
$$S_{\Delta BAD}=\frac{1}{2}S_{ABCD} \qquad (3)$$

Substituting (2) and (3) into (1):

$$\frac{S_{\Delta DPQ}}{\frac{1}{2}S_{ABCD}}=\left(\frac{\sqrt{2}AB-AB}{AB}\right)^2 \quad \Rightarrow \frac{S_{\Delta DPQ}}{S_{ABCD}}=\frac{1}{2}(\sqrt{2}-1)^2.$$

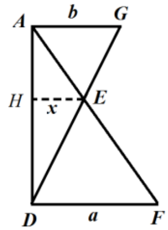
Example 21. In the unit square, find the distance from E to $\overline{AD}$ in terms of a and b, the lengths of $\overline{DF}$ and $\overline{AG}$, respectively.

A. $\frac{ab}{a+b}$ B. $\frac{b}{a+b}$ C. $\frac{a-b}{a+b}$ D. $\frac{a}{a+b}$ E. $\frac{2a-b}{a+b}$



Solution: A.

The given figure can be simplified into the figure below.

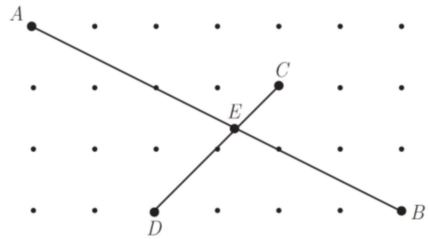


ΔAGD is similar to ΔHED. $\frac{x}{b}=\frac{HD}{AD}$ (1)

ΔDFA is similar to ΔHEA. $\frac{x}{a}=\frac{AH}{AD}$ (2)

(1) + (2): $\frac{x}{b}+\frac{x}{a}=\frac{HD+AH}{AD}=1 \Rightarrow x=\frac{1}{\frac{1}{a}+\frac{1}{b}}=\frac{ab}{a+b}$.

☆**Example 22.** (2000 AMC 10 problem 16) The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE.

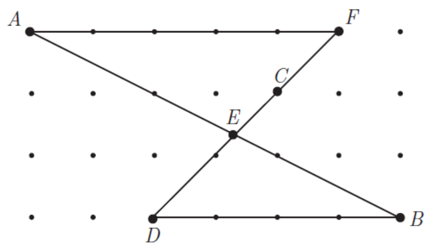


A. $\frac{4\sqrt{5}}{3}$ B. $\frac{5\sqrt{5}}{3}$ C. $\frac{12\sqrt{5}}{7}$ D. $2\sqrt{5}$

E. $\frac{5\sqrt{65}}{9}$

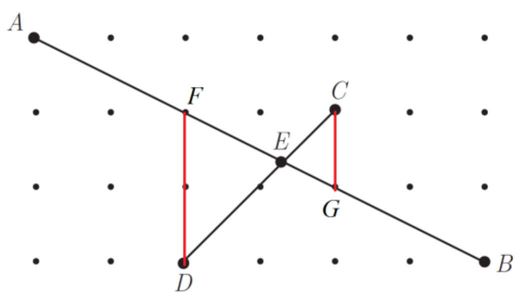
Solution: B.

Method 1 (official solution):



Extend DC to F. Triangle FAE and DBE are similar with ratio 5 : 4. Thus $AE = 5 \times AB/9$, $AB=\sqrt{3^2+6^2}=\sqrt{45}=3\sqrt{5}$, and AE = $5(3\sqrt{5})/9 = 5\sqrt{5}/3$

Method 2 (our solution):



Connect FD and CG as shown. Triangle FDE and GCE are similar with ratio 2 : 1. Thus $AF = FG = \sqrt{1^2+2^2}=\sqrt{5}$. Thus $FE=\frac{2}{3}\sqrt{5}$. $AE = AF + FE = \sqrt{5}+\frac{2}{3}\sqrt{5} = 5\sqrt{5}/3$

3. PROBLEMS

Problem 1. If two lines intersect at point P, and two triangles are formed by two parallels cutting the intersecting lines above and below their intersection P, the resulting triangles are necessarily:

A. congruent B. similar C. isosceles D. equilateral E. right triangles

Problem 2. A light pole is 30 feet tall. How long is the shadow cast by a woman 6 feet tall who is standing 8 feet from the pole?

A. 1.6 ft. B. 2 ft. C. 10 ft. D.16 ft. E.3ft.

Problem 3. Right triangles ABC and XYZ are similar, with A corresponding to X, B to Y, and C to Z. If $BC = 9$, $AC = 21$, and $YZ = 24$, then the length of $\overline{XZ}$ is:

A. 42 B.63 C. 49 D. 56 E. 72

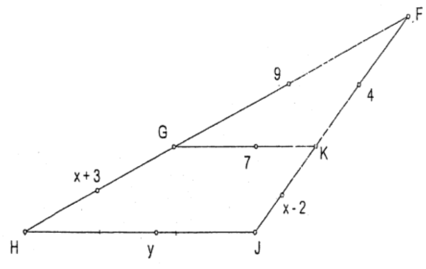
Problem 4. If $\Delta ABC - \Delta FED$, which of the following proportions is *not* true for this pair of similar triangles?

A. $\dfrac{AB}{FE} = \dfrac{AC}{FD}$. B. $\dfrac{AB}{FE} = \dfrac{BC}{ED}$. C. $\dfrac{CB}{DE} = \dfrac{CA}{DF}$. D. $\dfrac{DE}{CB} = \dfrac{FD}{AC}$.

E. $\dfrac{AB}{ED} = \dfrac{CB}{DE}$.

Problem 5. Given: $\overline{GK} \,//\, \overline{HJ}$, with lengths as shown. Find the perimeter of ΔHJF.

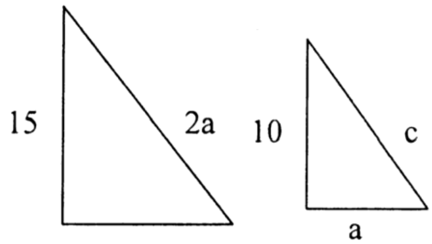
A. 40. B. 38. C. 35. D. 49. E. 50.



Problem 6. A vertical wall 20 feet high casts a shadow 8 feet wide on level ground. If Alex is 5 feet, 5 inches tall, how far away from the wall can he stand and still be entirely in the shade?

A. 2 feet, 4 inches B. 3feet, 9 inches C. 4 feet, 3inches D. 5 feet, 10 inches E. 6 feet, 2 inches

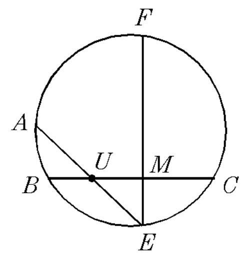
Problem 7. The right triangles in the figure below are similar. Find the value of *c*.



A. $5\sqrt{65}$ B. $\frac{4}{3}$ C. 20 D. $\frac{80}{3}$ E. $\frac{40\sqrt{7}}{7}$

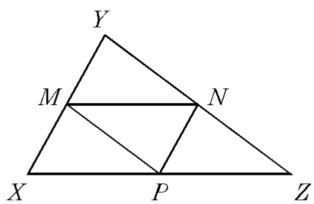
Problem 8. Chord *EF* is the perpendicular bisector of chord *BC*, intersecting it in *M*. Between *B* and *M*, point *U* is taken and $\overline{EU}$ extended meets the circle at *A*. Then for any selection of *U*, as described, ΔEUM is similar to triangle

A. *EFA* B. *EFC* C. *ABM* D. *ABU* E. *FMC*



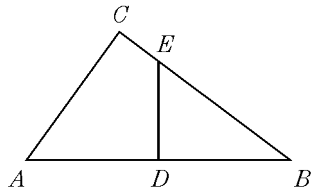
Problem 9. In ΔXYZ, points *M*, *N* , and *P* are midpoints. If $XY = 10$, $YZ = 15$ and $XZ = 17$, what is the perimeter of ΔMNP ?

A. 14 B. 16 C. $10\frac{2}{3}$ D. 21 E. cannot be determined



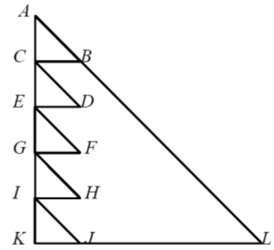
Problem 10. In the figure shown angle C is a right angle, line segments AD and DB are congruent, line segment AC has length 12, line segment AB has length 20 and DE is perpendicular to AB. Then the area of quadrilateral $ADEC$ is

A. 75 B. $58\frac{1}{2}$ C. 48 D. $37\frac{1}{2}$ E. none of these



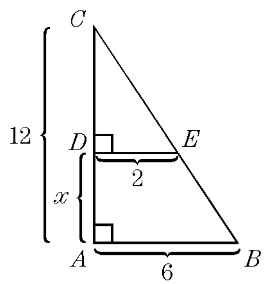
Problem 11. The triangles ΔABC, ΔCDE, ΔEFG, ΔGHI, ΔIJK in the figure above and to the right are congruent to each other and are similar to ΔAKL. If the area of ΔABC is 4, then the area of ΔAKL is:

A. 25 B. 60 C. 100 D. 120 E. none of these



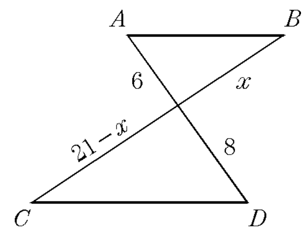
Problem 12. If ΔABC is a right triangle, and $DE \perp AC$. then x equals

A. 8 B. 4 C. 1 D. 3 E. 10



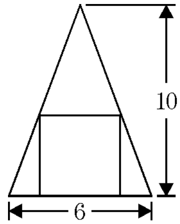
Problem 13. AB is parallel to CD. The value of x is:

A. 6 B. 7 C. 8 D. 9 E. 10



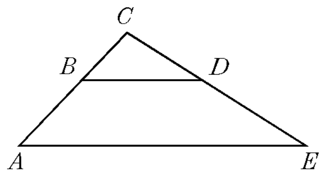
Problem 14. In a triangle with height 10 and base 6 a square is inscribed with a side along the base of the triangle as shown. The length of a side of the square is:

A. $3\frac{1}{2}$ B. $3\frac{3}{4}$ C. 4 D. $4\frac{1}{4}$ E. $4\frac{1}{2}$



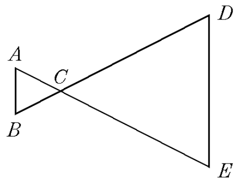
Problem 15. In the triangle, $BD \parallel AE$. $BD = \frac{3}{8} AE$. The ratio of the area of ΔBDC to ΔAEC is:

A. $\frac{9}{64}$ B. $\frac{3}{8}$ C. $\frac{\sqrt{6}}{4}4$ D. $\frac{3}{5}$ E. $\frac{9}{25}$



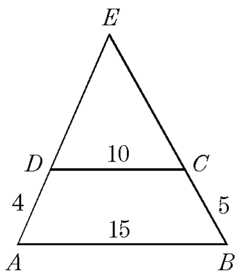
Problem 16. If AB is parallel to DE with $AC = a$, $CE = b$, and $AB = c$, then DE is:

A. $\frac{ac}{b}$ B. $\frac{bc}{a}$ C. $\frac{ab}{c}$ D. $\frac{a}{bc}$ E. $\frac{b}{ac}$



Problem 17. Sides AD and BC of trapezoid $ABCD$ are extended to point E. If $AB = 15$, $DC = 10$, $AD = 4$, and $BC = 5$, then DE is:

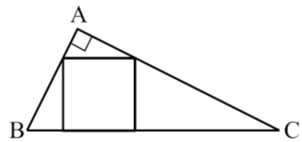
A. $2\frac{2}{3}$ B. 4 C. 8 D. 12 E. none of these



Problem 18. The base of a triangle is 24 inches. Two lines are drawn parallel to the base, terminating in the other two sides, and dividing the triangle into three equal areas. The length of the parallel closer to the base is:

A. $12\sqrt{3}$ inches B. $12\sqrt{6}$ inches C. 16 inches D. $8\sqrt{6}$ inches E. 12 inches

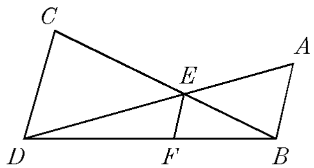
Problem 19. In the figure ΔABC is a right triangle with legs $AB = 6$ and $AC = 8$. A square is drawn as shown, with a side along AC and corners on AB and CB. Find the length of the side of the square.



A. 9/2 B. $\sqrt{19}$ C. $\dfrac{\sqrt{57}}{2}$ D. 120/27 E. 120/37

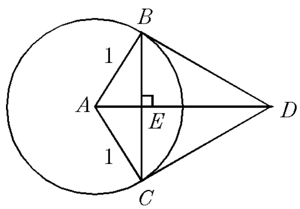
Problem 20. In the diagram $AB//FE//DC$, and $AB = 4$ with $CD = 8$. Find the length of EF.

A. 4/3 B. 2 C. 8/3 D. 3 E. none of these



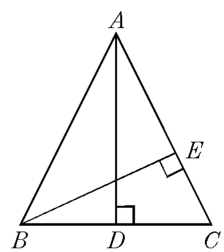
Problem 21. If A is the center of the circle through B and C, and DC and DB are tangents, suppose that $AB = 1$, $BC \perp AD$, and $AD = 2$. Then AE equals:

A. $\dfrac{\sqrt{3}}{2}$ B. $\dfrac{\sqrt{5}}{2}$ C. $\sqrt{3}$ D. $\dfrac{1}{\sqrt{3}}$ E. none of these



Problem 22. Suppose you are given isosceles triangle ABC, with perpendiculars AD and BE drawn to sides BC and AC. You can conclude that $\Delta ADC \sim \Delta BEC$ because:

A. $AB = AC$ B. $\angle ABC = \angle ACB$ C. The sides are proportional D. The triangles have two pairs of congruent angles E. The altitudes of a triangle are proportional to the sides opposite



Problem 23. The perimeters of two similar figures are 16 and 24 units, respectively. What is the ratio of their areas?

A. 2 : 3 B. 4 : 9 C. 4 : 6 D. 8 : 12 E. cannot be determined

Problem 24. A line intersects two sides of an equilateral triangle and is parallel to the third side. If this line divides the triangular region into a trapezoid and a smaller triangle having equal perimeters, then the ratio of the area of the smaller triangle to that of the trapezoid is:

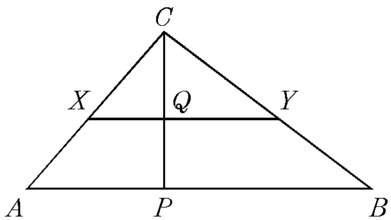
A. 9 : 7 B. $\sqrt{3}:2$ C. 7 : 4 D. 3 : 2 E. 16 : 9

Problem 25. Let ΔABC and ΔDEF be similar triangles such that $AB = 4$ and $DE = 10$. If the area of $\Delta ABC = 24$, what is the area of ΔDEF?

A. 60 B. 240 C. 150 D. 96 E. 120

Problem 26. In triangle ABC, $XY//AB$ such that the area of triangle CXY is equal to the area of trapezoid $ABYX$. CP is an altitude of triangle ABC. What is the ratio CQ/CP ?

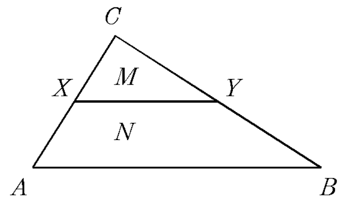
A. $1/\sqrt{2}$ B. 1/2 C. 1 D. $\sqrt{2}$ E. 2



Problem 27. Let ABC be a triangle with X and Y midpoints of the sides as shown. Let area of $\Delta CXY = M$, and area trapezoid $AXYB = N$. How are M and N related?

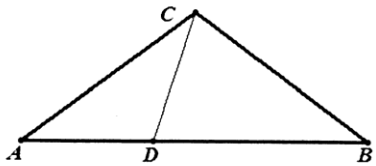
A. $N = M$ B.$N = \sqrt{2}\,M$ C. $N = 2M$

D. $N = 3M$ E. $N = 4M$



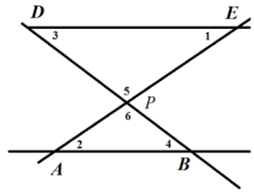
Problem 28. An isosceles triangle ABC with point D on AB, has $AC = BC = BD$ and $AD = DC$. If $AB = 2$, find the length of CD.

A. 1 B. $\sqrt{2}$ C. $\sqrt{5}-1$ D. $\sqrt{10}-1$ E. $3-\sqrt{5}$



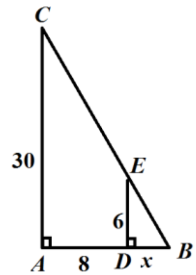
4. SOLUTIONS:

Problem 1. Solution: B.



Since $DE//AB$, we know that $\angle 1 = \angle 2$, $\angle 3 = \angle 4$. We also know that $\angle 5 = \angle 6$. We have no information about the sides. So we can only say that they similar.

Problem 2. Solution: B.



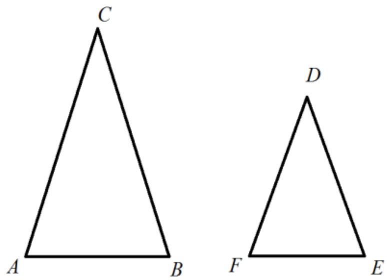
By Principle 2, $\Delta ABC \sim \Delta DBE$.

$$\frac{30}{6} = \frac{8+x}{x} \quad \Rightarrow \quad 5 = \frac{8}{x} + 1 \quad \Rightarrow \quad \frac{8}{x} = 4 \Rightarrow \; x = 2.$$

Problem 3. Solution: D

By **Principle 1**, $\dfrac{BC}{YZ} = \dfrac{AC}{XZ} \quad \Rightarrow \quad \dfrac{9}{24} = \dfrac{21}{XZ} \quad \Rightarrow \quad XZ = 56.$

Problem 4. Solution: E



Since $\Delta ABC - \Delta FED$, we have $\dfrac{AB}{FE} = \dfrac{AC}{FD} = \dfrac{BC}{ED}$.

We see that E is not correct.

$$\frac{AB}{CB} = \frac{FE}{DE} \quad \Rightarrow \quad \frac{AB}{FE} = \frac{CB}{DE}.$$

Problem 5. Solution: A.

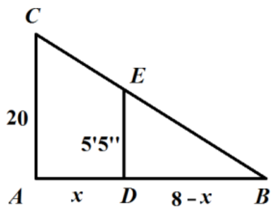
Since $\overline{GK} // \overline{HJ}$, $\Delta HJF \sim \Delta GKF$.

$$\frac{9}{4} = \frac{x+3}{x-2} \quad \Rightarrow \quad x = 6.$$

By **Theorem 1,** The ratio of the perimeters of two similar figures is:

$$\frac{P_{\Delta HJF}}{P_{\Delta GKF}} = \frac{9+x+3}{9} = \frac{18}{9} = 2 \quad \Rightarrow \quad P_{\Delta HJF} = 2 \times P_{\Delta GKF} = 2(9+7+4) = 40.$$

Problem 6. Solution: D.



By Principle 2, $\Delta ABC \sim \Delta DBE$.

$$\frac{20}{5\frac{5}{12}} = \frac{8}{8-x} \quad \Rightarrow \quad x = \frac{35}{6} = 5'\ 10''.$$

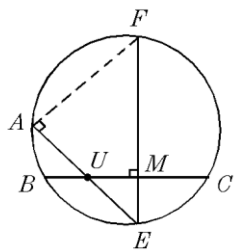
Problem 7. Solution: E.

For the first triangle, another leg is $\sqrt{(2a)^2 - 15^2}$. Since two triangles are similar,

$$\frac{15}{10} = \frac{\sqrt{(2a)^2 - 15^2}}{a} \quad \Rightarrow \quad \frac{9}{4} = \frac{(2a)^2 - 15^2}{a^2} \quad \Rightarrow \quad a = \frac{30\sqrt{7}}{7}.$$

By Pythagorean Theorem, $c = \sqrt{(a)^2 + 10^2} = \dfrac{40\sqrt{7}}{7}$

Problem 8. Solution: A.



Connect AF. $\angle UEM = \angle FEA$. Since EF is the perpendicular bisector of chord BC, EF is the diameter of the circle. So $\angle UME = \angle FAE = 90°$. Therefore ΔEUM is similar to ΔEFA.

Problem 9. Solution: D

Since MN are midpoints, $MN // XZ$. ΔXYZ is similar to ΔMYN. By **Theorem 1.** The ratio of the perimeters of two similar figures is: $\dfrac{P_{\Delta MNP}}{P_{\Delta XYZ}} = \dfrac{MY}{XY} = \dfrac{\frac{1}{2}XY}{XY} = \dfrac{1}{2}$

$$\Rightarrow \quad P_{\Delta MNP} = \frac{1}{2}P_{\Delta XYZ} = \frac{1}{2}(10+15+17) = 21.$$

Problem 10. Solution: B.

ΔABC is a 12-16-20 right triangle. So $CB = 16$. We also see that ΔABC is similar to ΔEBD.

By **Theorem 1.** the ratio of the areas of two similar figures is:

$\frac{S_{\Delta EBD}}{S_{\Delta ABC}} = (\frac{DB}{CB})^2 == (\frac{10}{16})^2 = \frac{25}{64} \Rightarrow \quad S_{\Delta EBD} = \frac{25}{64} S_{\Delta ABC} = \frac{25}{64}(\frac{1}{2} AC \times BC) = \frac{75}{2}$.

The area of quadrilateral *ADEC* is $S_{\Delta ABC} - S_{\Delta EBD} = \frac{1}{2} AC \times BC - \frac{75}{2} = 58\frac{1}{2}$.

Problem 11. Solution: C.

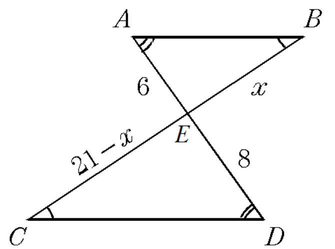
ΔABC is similar to ΔAKL. By **Theorem 1.** the ratio of the areas of two similar figures is: $\frac{S_{\Delta AKL}}{S_{\Delta ABC}} = (\frac{AK}{AC})^2 = (\frac{5}{1})^2 = 25 \quad \Rightarrow S_{\Delta AKL} = 25 S_{\Delta ABC} = 25 \times 4 = 100$.

Problem 12. Solution: A

ΔABC is similar to ΔDEC. $\frac{CA}{CD} = \frac{AB}{DE} \Rightarrow \quad \frac{12}{12 - x} = \frac{6}{2} \Rightarrow x = 8$.

Problem 13. Solution: D.

Since *AB* is parallel to *CD,* ΔABE is similar to ΔDCE.

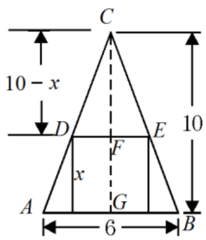


$\frac{x}{21 - x} = \frac{6}{8} = \frac{3}{4} \quad \Rightarrow \quad 4x = 63 - 3x$

$\Rightarrow \quad x = 9$.

Problem 14. Solution: B.

We draw *CG*, the height of the triangle to meet *DE* at *F* and *AB* at *G*. Since *AB* is parallel to *DE,* ΔABC is similar to ΔDEC.



$\frac{AB}{DE} = \frac{CG}{CF} \Rightarrow \frac{6}{x} = \frac{10}{10 - x} = \frac{16}{10} = \frac{8}{5} \Rightarrow x = \frac{30}{8} = \frac{15}{4} = 3\frac{3}{4}$.

Problem 15. Solution: A

Since $BD \parallel AE$, ΔBDC is similar to ΔAEC.

The ratio of the area of ΔBDC to ΔAEC is $\left(\frac{BD}{AE}\right)^2 = \left(\frac{3}{8}\right)^2 = \frac{9}{64}$.

Problem 16. Solution: B.
Since AB is parallel to DE, ΔABC is similar to ΔEDC.

$$\frac{AB}{DE} = \frac{AC}{CE} \quad \Rightarrow \quad \frac{c}{DE} = \frac{a}{b} \quad \Rightarrow \quad DE = \frac{bc}{a}.$$

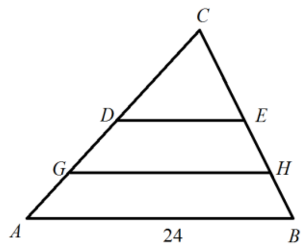
Problem 17. Solution: C.
Since $ABCD$ is a trapezoid, AB is parallel to DC.
Thus ΔABE is similar to ΔDCE.

$$\frac{AB}{DC} = \frac{AE}{DE} \quad \Rightarrow \quad \frac{15}{10} = \frac{4+DE}{DE} \quad \Rightarrow \quad \frac{3}{2} = \frac{4}{DE} + 1 \quad \Rightarrow \quad \frac{1}{2} = \frac{4}{DE} \quad \Rightarrow$$
$$DE = 8.$$

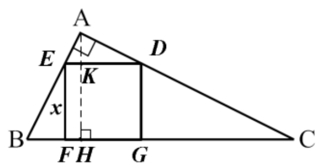
Problem 18. Solution: D.

$$\frac{S_{\Delta ABC}}{S_{\Delta GHC}} = \left(\frac{AB}{GH}\right)^2 \quad \Rightarrow \quad \frac{3}{2} = \left(\frac{24}{GH}\right)^2 \quad \Rightarrow$$

$$GH = 24\sqrt{\frac{2}{3}} = 24\sqrt{\frac{2\times 3}{3\times 3}} = 8\sqrt{6}.$$



Problem 19. Solution: E.
Method 1:
In ΔABC, $AB = 6$, $AC = 8$, and $BC = 10$.
Since $ED // FG$, $\Delta AED \sim \Delta ABC$



$$\frac{AE}{ED} = \frac{AB}{BC} \quad \Rightarrow \quad \frac{AE}{x} = \frac{6}{10} = \frac{3}{5} \quad \Rightarrow \quad AE = \frac{3}{5}x$$

$$\frac{AD}{ED} = \frac{AC}{BC} \quad \Rightarrow \quad \frac{AD}{x} = \frac{8}{10} = \frac{4}{5} \quad \Rightarrow \quad AD = \frac{4}{5}x$$

Draw $AH \perp BC$. AH meets ED at K.

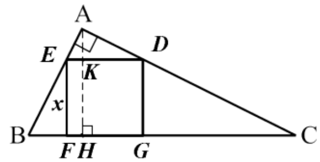
Since $S_{\Delta ABC} = \dfrac{AB \times AC}{2} = \dfrac{BC \times AH}{2}$, $AH = \dfrac{AB \times AC}{BC} = \dfrac{6 \times 8}{10} = \dfrac{24}{5}$

Since $S_{\Delta AED} = \dfrac{AE \times AD}{2} = \dfrac{ED \times AK}{2}$, we have $\dfrac{\frac{3}{5}x \times \frac{4}{5}x}{2} = \dfrac{x \times (\frac{24}{5} - x)}{2} \Rightarrow$

$\dfrac{3}{5} \times \dfrac{4}{5}x = \dfrac{24}{5} - x$. Solve for x: $x = \dfrac{120}{37}$.

Method 2:

In ΔABC, $AB = 6$, $AC = 8$, $BC = 10$.



Since $ED // BC$, $\Delta AED \sim \Delta ABC$

$\dfrac{AE}{ED} = \dfrac{AB}{BC} \quad \Rightarrow \quad \dfrac{AE}{x} = \dfrac{6}{10} = \dfrac{3}{5} \quad \Rightarrow \quad AE = \dfrac{3}{5}x$

Draw $AH \perp BC$. AH meets ED at K.

Since $S_{\Delta ABC} = \dfrac{AB \times AC}{2} = \dfrac{BC \times AH}{2}$, $AH = \dfrac{AB \times AC}{BC} = \dfrac{6 \times 8}{10} = \dfrac{24}{5}$

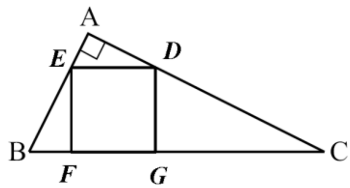
Since $EK // BH$, $\Delta AEK \sim \Delta ABH$, $\dfrac{AE}{AB} = \dfrac{AK}{AH}$.

$$\frac{\frac{3}{5}x}{6} = \frac{\frac{24}{5} - x}{\frac{24}{5}} \quad \Rightarrow \quad \frac{x}{10} \times \frac{24}{5} = \frac{24}{5} - x.$$

Solve for x: $x = \dfrac{120}{37}$.

Method 3:

Theorem: Square $DEFG$ inscribes in ΔABC, $\angle A = 90°$. If $AB = a$, $AC = b$, then $BF : FG : GC = a^2 : ab : b^2$



In our case, AB =6, AC =8, and BC =10.

$BF : FG : GC = AB^2 : AB \times AC : AC^2 = 6^2 : 6 \times 8 : 8^2$

$$FG = \frac{6 \times 8}{6^2 + 6 \times 8 + 8^2} \times BC = \frac{48}{148} \times 10 = \frac{120}{37}.$$

Problem 20. Solution: C.

By **Theorem 6**, $EF = \dfrac{ab}{a+b} = \dfrac{4 \times 8}{4+8} = \dfrac{8}{3}$.

Problem 21. Solution: E.

We know that B is the tangent point. So $BD \perp AB$.

By Theorem 5, $AB^2 = AD \times AE \quad \Rightarrow \quad 1^2 = 2 \times AE \qquad \Rightarrow \qquad AE = \dfrac{1}{2}$.

Problem 22. Solution: D.

We see that each triangle has the angle C and has a right angle. So D is the correct answer.

Problem 23. Solution: B.

By Theorem 1, we have

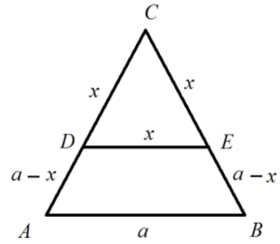
$$\frac{P_{\Delta ABC}}{P_{\Delta A_1B_1C_1}} = \frac{a}{a_1} \qquad (1)$$

$$\frac{S_{\Delta ABC}}{S_{\Delta A_1B_1C_1}} = \left(\frac{a}{a_1}\right)^2 \qquad (2)$$

Squaring both sides of (1): $\left(\dfrac{P_{\Delta ABC}}{P_{\Delta A_1B_1C_1}}\right)^2 = \left(\dfrac{a}{a_1}\right)^2 \qquad (3)$

So we get $\dfrac{S_{\Delta ABC}}{S_{\Delta A_1B_1C_1}} = \left(\dfrac{P_{\Delta ABC}}{P_{\Delta A_1B_1C_1}}\right)^2 = \left(\dfrac{16}{24}\right)^2 = \left(\dfrac{2}{3}\right)^2 = \dfrac{4}{9}$.

Problem 24. Solution: A.



We draw the figure. Since $DE//AB$ and ABC is an equilateral triangle, $CD = CE = DE$.

Since DEC and $ABED$ have the same perimeter, we get

$$3x = a - x + x + a - x + a \quad \Rightarrow \quad x = \frac{3a}{4}.$$

$$\frac{S_{\Delta DEC}}{S_{\Delta ABC}} = \left(\frac{x}{a}\right)^2 = \frac{9}{16} \quad \Rightarrow \quad \frac{S_{\Delta DEC}}{S_{\Delta ABC}} = \frac{S_{\Delta DEC}}{S_{\Delta DEC} + S_{ABDE}} = \frac{9}{16}$$

$$\frac{S_{\Delta DEC} + S_{ABDE}}{S_{\Delta DEC}} = \frac{16}{9} \quad \Rightarrow \quad \frac{S_{ABDE}}{S_{\Delta DEC}} + 1 = \frac{16}{9} \quad \Rightarrow \quad \frac{S_{ABDE}}{S_{\Delta DEC}} = \frac{7}{9}$$

$$\Rightarrow \quad \frac{S_{\Delta DEC}}{S_{ABDE}} = \frac{9}{7}.$$

Problem 25. Solution: C.

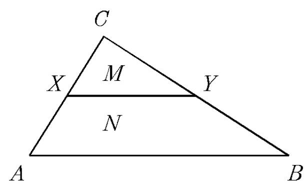
By Theorem 1, we have

$$\frac{S_{\Delta DEF}}{S_{\Delta ABC}} = \left(\frac{10}{4}\right)^2 = \frac{25}{4} \quad \Rightarrow \quad S_{\Delta DEF} = \frac{25}{4} \times S_{\Delta ABC} = 25 \times 6 = 150.$$

Problem 26. Solution: A.

$$\frac{S_{\Delta XYC}}{S_{\Delta ABC}} = \left(\frac{CQ}{CP}\right)^2 \quad \Rightarrow \quad \left(\frac{CQ}{CP}\right)^2 = \frac{1}{2} \quad \Rightarrow \quad \frac{CQ}{CP} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Problem 27. Solution: D.



$$\frac{S_{\Delta XYC}}{S_{\Delta ABC}} = \frac{S_{\Delta XYC}}{S_{\Delta XYCC} + S_{ABYX}} = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \quad \frac{S_{\Delta XYC}}{S_{\Delta XYC} + S_{ABYX}} = \frac{1}{4}$$

$$\Rightarrow \quad \frac{S_{\Delta XYC} + S_{ABYX}}{S_{\Delta XYC}} = 4.$$

$\frac{S_{ABYX}}{S_{\Delta XYC}}+1=4 \quad \Rightarrow \frac{S_{ABYX}}{S_{\Delta XYC}}=3 \quad \Rightarrow \quad N=3M$.

Problem 28. Solution: E

Let $x = CD$.

Since $\Delta ABC \sim \Delta ABC \quad \Rightarrow \quad CD : AC = AC : AB \quad \Rightarrow \quad 2x = (AC)^2$.

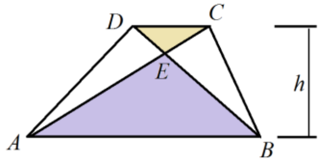
$AD + BD = AB \Rightarrow \quad x + AC = 2 \quad \Rightarrow \quad 2x = (2 - x)^2 \quad \Rightarrow \quad x^2 - 6x + 4 = 0$

$\Rightarrow \quad x = 3-\sqrt{5}$.

1.BASIC KNOWLEDGE

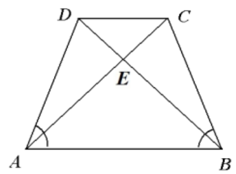
A trapezoid is a quadrilateral with one pair of parallel sides.

As shown in the figure, $AB//CD$, AB and DC are called the bases, h is called the height, and AC and BD are called the diagonals.



$\Delta ABE \sim \Delta CDE \Rightarrow \dfrac{AB}{DC} = \dfrac{AE}{CE} = \dfrac{BE}{DE}$

The base angles of an isosceles trapezoid are congruent. If the base angles of a trapezoid are congruent, the trapezoid is isosceles.



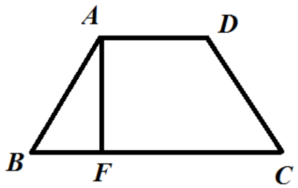
$AD = BC,\ AC = BD,\ \angle DAB = \angle CBA.\ \angle DCB = \angle CDA.$

$\angle DAB + \angle DCB = \angle DBA + \angle CDA = 180°.$

Properties

Property 1: In trapezoid $ABCD$, sides AD and BC are parallel to each other. If $AD = a$, $BC = b$, and $AF = h$, then S, the area of the trapezoid, is

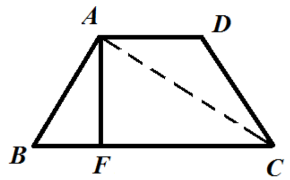
$$S = \frac{a+b}{2}h \qquad (1.1)$$



Proof:

Method 1:

Connect AC.

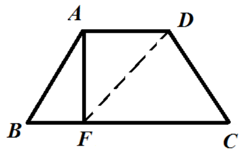


$$S_{\Delta ADC} = \frac{ah}{2}$$

$$S_{\Delta BCA} = \frac{bh}{2}$$

(1) + (2):

$$S = S_{\Delta ADC} + S_{\Delta BCA} = \frac{bh}{2} + \frac{ah}{2} = \frac{a+b}{2}h.$$

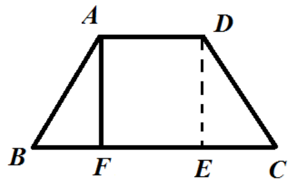


Method 2:
Connect *FD*.

$S_{\Delta ABF}=\frac{1}{2}BF\cdot AF$; $S_{\Delta FAD}=\frac{1}{2}AD\cdot AF$; $S_{\Delta FDC}=\frac{1}{2}FC\cdot AF$

$$S_{ABCD}=S_{\Delta ABF}+S_{\Delta FAD}+S_{\Delta FDC}=\frac{1}{2}BF\cdot AF+\frac{1}{2}AD\cdot AF+\frac{1}{2}FC\cdot AF$$

$$=\frac{1}{2}(BF+AD+FC)AF+\frac{1}{2}(AD+BC)AF=\frac{1}{2}(a+b)h$$



Method 3:
Draw $DE\perp BC$ at E. *AFED* is a rectangle.

$$S_{AFED}=FE\cdot AF$$

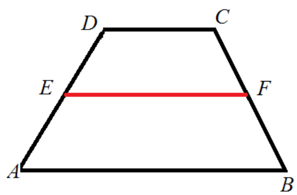
$$S_{\Delta ABF}=\frac{1}{2}BF\cdot AF$$

$$S_{\Delta DEC}=\frac{1}{2}EC\cdot DE$$

$$S_{ABCD}=S_{\Delta ABF}+S_{AFED}+S_{DECC}=\frac{1}{2}BF\cdot AF+FE\cdot AF+\frac{1}{2}EC\cdot AF$$

$$=\frac{1}{2}(BF+2FE+EC)AF=\frac{1}{2}(AD+BC)AF=\frac{1}{2}(a+b)h$$

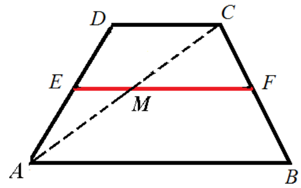
<u>Property 2:</u> The median of a trapezoid, the segment joining the midpoints of the non-parallel sides, is parallel to each of the parallel sides, and has a measure equal to one-half of the sum of their measures.



For any trapezoid *ABCD*, the following relationship is true: $EF=\frac{1}{2}(AB+CD)$ (1.2)

E and *F* are the midpoints of *AD* and *BC*, respectively.

Proof:



We know that E and F are midpoints of AD and BC, respectively.

It follows that $\frac{DE}{EA} = \frac{CF}{FB}$, which means that $EF//AB$.

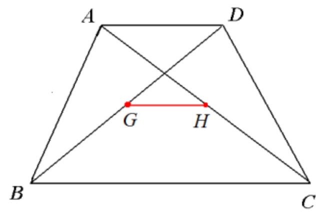
Connect AC. AC meets EF at M, where M is the midpoint of AC.

In triangle ABC, we have $MF = \frac{1}{2}AB$.

Similarly, we have $EM = \frac{1}{2}CD$. Therefore $EF = MF + EM = \frac{1}{2}(AB + CD)$.

Property 3: In trapezoid $ABCD$, the following relationship is true:

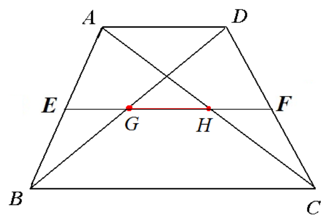
$$GH = \frac{1}{2}(BC - AD) \qquad (1.3)$$



G and H are the midpoints of the diagonals AC and BD, respectively.

Proof:

Since $ABCD$ is a trapezoid, $AD//BC$, and E and F are the midpoints of AB and DC, respectively.



So $EF = \frac{1}{2}(AD + BC) = \frac{1}{2}AD + \frac{1}{2}BC$ and $AD//EF//BC$.

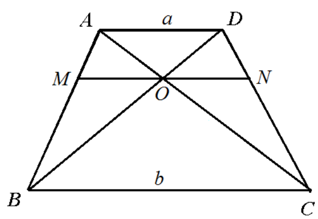
Therefore in triangle BAD, $BG = GD$.

In ΔCDA, $CH = HA$.

Therefore, $EG = \frac{1}{2}AD$, $HF = \frac{1}{2}AD$ and

$$GH = EF - EG - HF = \frac{1}{2}AD + \frac{1}{2}BC - \frac{1}{2}AD - \frac{1}{2}AD = \frac{1}{2}(BC - AD).$$

Property 4: The measure of the segment passing through the point of intersection of the diagonals of a trapezoid and parallel to the bases with its endpoints on the legs, is the harmonic mean between the measures of



the parallel sides. The harmonic mean of two numbers is defined as the reciprocal of the average of the reciprocals of two numbers.

$$MN = \frac{2ab}{a+b} \qquad (1.4)$$

Proof:

In trapezoid *ABCD*, *AD//BC* and *MN//BC*. Let *AD* = *a, BC* = *b*.

Since $\Delta ADO \sim \Delta CBO$, we have $\frac{a}{b} = \frac{OD}{OB}$.

Then we can have:

$$\frac{a}{b} = \frac{OD}{BD - OD} \quad \Rightarrow \quad \frac{b}{a} = \frac{BD - OD}{OD} = \frac{BD}{OD} - 1$$

$$\Rightarrow \quad \frac{b+a}{a} = \frac{BD}{OD} \quad \Rightarrow \quad \frac{OD}{BD} = \frac{a}{a+b} \qquad (1)$$

Similarly we have $\frac{OB}{BD} = \frac{b}{a+b}$ (2)

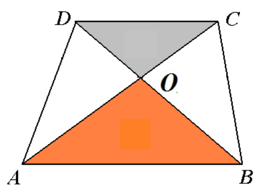
Since $\Delta BCD \sim \Delta OND$, it follows that $\frac{BC}{ON} = \frac{BD}{OD}$.

Then $ON = \frac{OD}{BD} \times BC = \frac{b}{a+b} \times a$ (3)

Since $\Delta ADB \sim \Delta BOM$, we have $\frac{BD}{BO} = \frac{AD}{MO}$.

Then $MO = \frac{BO}{BD} \times AD = \frac{a}{a+b} \times b$ (4)

(3) + (4): $MN = ON + MO = \frac{ab+ab}{a+b} = \frac{2ab}{a+b}$.



<u>Property 5:</u> In trapezoid *ABCD*, with *AB//CD*, let $S_{\Delta ABO} = m^2$, $S_{\Delta DOC} = n^2$. the following relationship of areas is true: $S_{ABCD} = (m+n)^2$ (1.5)

Proof:

Since $\Delta ABO \sim \Delta CDO$, $\dfrac{AO^2}{OC^2} = \dfrac{S_{\Delta AOB}}{S_{\Delta COD}} = \dfrac{m^2}{n^2}$

Simplifying yields $\dfrac{AO}{OC} = \dfrac{m}{n}$.

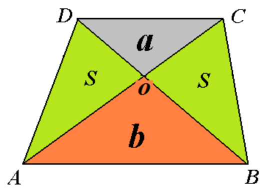
We also know that $\dfrac{AO}{OC} = \dfrac{S_{\Delta AOB}}{S_{\Delta COB}}$, so $S_{\Delta COB} = mn$.

Since $S_{\Delta AOD} = S_{\Delta COB}$, $S_{\Delta AOD} = mn$.

Therefore $S_{ABCD} = m^2 + mn + mn + n^2 = m^2 + 2mn + n^2 = (m+n)^2$.

Property 6: For any trapezoid *ABCD*, the following relationship of areas is true:

$$S_{\Delta AOD} = S = \sqrt{ab} \qquad (1.6)$$



Proof:

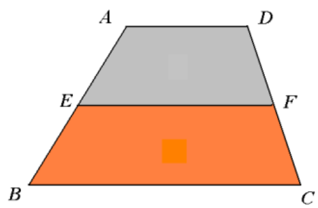
From (2.4), we have: $S_{ABCD} = (m+n)^2 = (\sqrt{a} + \sqrt{b})^2$

Or $2S + a + b = (\sqrt{a} + \sqrt{b})^2$

$\Rightarrow \quad 2S = (\sqrt{a} + \sqrt{b})^2 - a - b = a + b + 2\sqrt{ab} - a - b \quad \Rightarrow \quad S = 2\sqrt{ab}$.

Property 7: In trapezoid *ABCD*, the following relationship is true:

$$BC^2 - EF^2 = EF^2 - AD^2 \qquad (1.7)$$

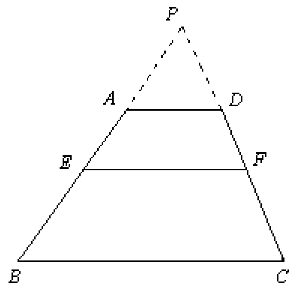


Proof:

Extend *BA* and *CD* to meet at *P*. Since *AD*//*EF*, $\Delta PAD \sim \Delta PEF$.

We have $\dfrac{S_{\Delta PAD}}{AD^2} = \dfrac{S_{\Delta PEF}}{EF^2}$.

Similarly, we have $\dfrac{S_{\Delta PEF}}{EF^2} = \dfrac{S_{\Delta PBC}}{BC^2} = \dfrac{S_{\Delta PAD}}{AD^2} = k$.



So $S_{\Delta PAD} = k \times AD^2$

$S_{\Delta PEF} = k \times EF^2$ $S_{\Delta ABC} = k \times BC^2$

Since $S_{ADFE} = S_{EFCB}$, $S_{\Delta PEF} - S_{\Delta PAD} = S_{\Delta PBC} - S_{\Delta PEF}$.

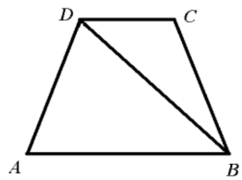
Or $k \times EF^2 - k \times AD^2 = k \times BC^2 - k \times EF^2$.

Dividing each term by k yields $AD^2 + BC^2 = 2EF^2$

or $BC^2 - EF^2 = EF^2 - AD^2$.

Property 8: In isosceles trapezoid $ABCD$, the following relationship is true:

$$BD^2 = AD^2 + AB \times CD \qquad (1.8)$$



Proof:

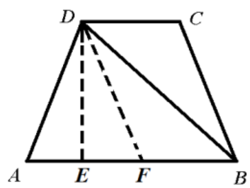
Draw $DF//BC$ to meet AB at F.

$BCDF$ is a parallelogram and AFD is an isosceles triangle, so

$BF = DC$, $BC = DF = AD$.

Draw $DE \perp AB$ at E.

By the Pythagorean Theorem,



$BD^2 = BE^2 + DE^2$ and $AD^2 = DE^2 + AE^2$.

Therefore $BD^2 = DE^2 + (AB - AE)^2$

$= DE^2 + AB^2 + AE^2 - 2AB \cdot AE$

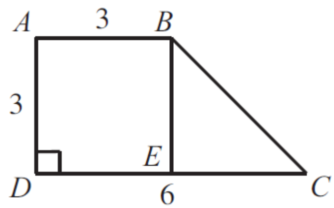
$= DE^2 + AE^2 + AB(AB - 2AE)$

$= AD^2 + AB(AB - AF) = AD^2 + AB \cdot FB = AD^2 + AB \cdot CD$.

2. EXAMPLES

☆ **Example 1.** (2007 AMC 8) In trapezoid *ABCD*, *AD* is perpendicular to *DC*, $AD = AB = 3$, and $DC = 6$. In addition, *E* is on *DC*, and *BE* is parallel to *AD*. Find the area of ΔBEC.

(A) 3 (B) 4.5 (C) 6 (D) 9 (E) 18



Solution: B.

Method 1:

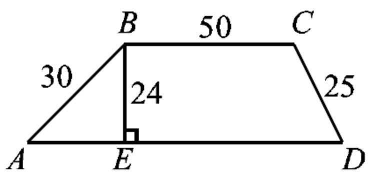
Note that *ABED* is a square with side 3. Subtract *DE* from *DC*, to find that *EC*, the base of ΔBEC, has length 3. The area of ΔBEC is $3 \times 3/2 = 4.5$.

Method 2:

The area of the ΔBEC is the area of the trapezoid *ABCD* minus the area of the square *ABED*. The area of ΔBEC is $(3 + 6)3/2 - 3^2 = 13.5 - 9 = 4.5$.

☆ **Example 2.** (2005 AMC 8) What is the perimeter of trapezoid *ABCD*?

(A) 180 (B) 188 (C) 196 (D) 200 (E) 204



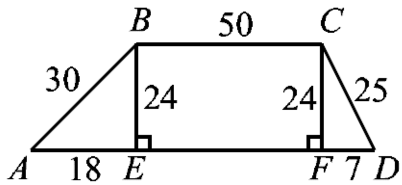
Solution: A.

Method 1 (official solution):

By the Pythagorean Theorem, $AE = \sqrt{30^2 - 24^2} = \sqrt{324} = 18$.

Also $CF = 24$ and $FD = \sqrt{25^2 - 24^2} = \sqrt{49} = 7$.

The perimeter of the trapezoid is $50 + 30 + 18 + 50 + 7 + 25 = 180$.



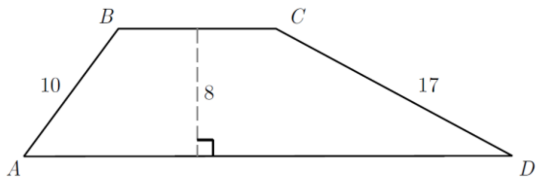
Method 2:

ΔAEB is a 3×6, 4×6, 5×6 right triangle.

ΔCDF is a 7 – 24 – 25 right triangle. $EF = BC = 50$. So the perimeter of the trapezoid is 50 + 30 + 18 + 50 + 7 + 25 = 180.

☆ **Example 3.** (2003 AMC 8 problem 21) The area of trapezoid $ABCD$ is 164 cm^2. The altitude is 8 cm, AB is 10 cm, and CD is 17 cm. What is BC, in centimeters?

(A) 9 (B) 10 (C) 12 (D) 15 (E) 20



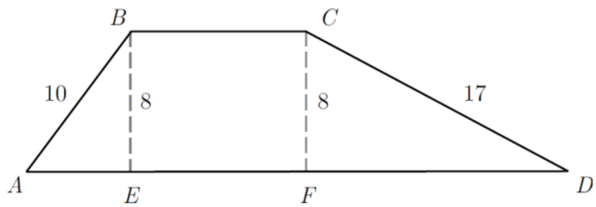
Solution: B.

Method 1:

Label the feet of the altitudes from B and C as E and F respectively.

Considering right triangles AEB and DFC, $AE = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$ cm,

and $FD = \sqrt{17^2 - 8^2} = \sqrt{225} = 15$ cm. So the area of ΔAEB is (6)(8)/2 = 24 cm2, and the area of ΔDFC is (15)(8)/2 = 60 cm^2. Rectangle $BCFE$ has area 164 − (24 + 60) = 80 cm^2. Because $BE = CF$ = 8 cm, it follows that BC = 10 cm.



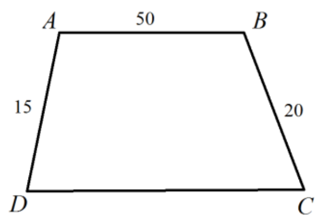
Method 2:

Let $BC = EF = x$. From the first solution we know that AE = 6 and FD = 15. Therefore, AD = x + 21, and the area of the trapezoid ABCD is

$$8 \times \frac{x+x+21}{2} = 164 \Rightarrow 4(2x+21) = 164 \quad \Rightarrow 2x + 21 = 41 \Rightarrow 2x = 20 \Rightarrow x = 10.$$

☆ **Example 4.** In trapezoid $ABCD$, AD =15, AB = 50, and BC=20. $DC - AB$ is a positive integer. What is the area of the trapezoid?

(A) 710 (B) 720 (C) 730 (D)740 (E) 750.

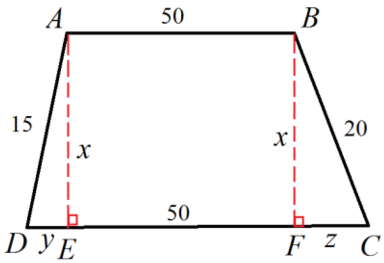


Solution: E.

Let E and F be the feet of the perpendiculars from A

and B to DC.

In right ΔAED, $15^2 - DE^2 = x^2 \Rightarrow \quad 15^2 - y^2 = x^2$ (1)
In right ΔBFC, $20^2 - FC^2 = x^2 \quad \Rightarrow \quad 20^2 - z^2 = x^2$ (2)
From (1) and (2) we get: $15^2 - y^2 = 20^2 - z^2$
$z^2 - y^2 = 20^2 - 15^2 \Rightarrow \; (z - y)(z + y) = 175 = 1 \times 175 = 5 \times 35 = 7 \times 25$ (3)
We see that $DC - AB = z + y$ is an integer. So $(z - y)$ must also be an integer.
From (3), we get



$$\left.\begin{aligned} z - y &= 1 \\ z + y &= 175 \end{aligned}\right\}$$

$$\left.\begin{aligned} z - y &= 5 \\ z + y &= 35 \end{aligned}\right\}$$

$$\left.\begin{aligned} z - y &= 7 \\ z + y &\; 25 \end{aligned}\right\}$$

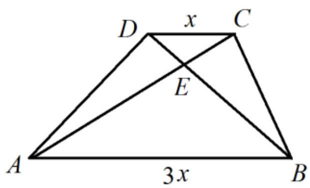
Only the last system of equations give the correct solutions with $y = 9$ and $z = 16$.
Substituting $y = 9$ into (1): $x = 12$.

$DC = DE + EF + FC = 9 + 50 + 16 = 75$. Then the area of trapezoid is $(AB + DC) \cdot AE/2 = (50 + 75) \cdot 12/2 = 125 \cdot 6 = 750$.

Example 5. Let $ABCD$ be a trapezoid with the measure of base AB three times that of base DC, and let E be the point of intersection of diagonals. If the measure of diagonal AC is 16, find the length of segment EC.

A. 3 B. 4 C. 5 D. 6 E. 8.

Solution: B.

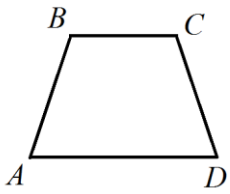


We know that $\Delta ABE \sim \Delta CDE \Rightarrow \quad \dfrac{AB}{DC} = \dfrac{AE}{CE}$

$\Rightarrow \qquad \dfrac{3x}{x} = \dfrac{16 - CE}{CE} \Rightarrow \qquad CE = 4$.

Example 6. If $ABCD$ is an isosceles trapezoid and $\angle BAD = 80°$, what is the measure of $\angle BCD$?

A. 100° B. 105° C. 110° D. 115° E. 120°.



Solution: A.

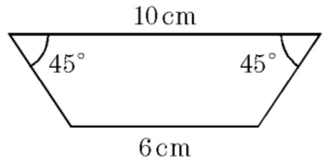
Since $ABCD$ is an isosceles trapezoid, $\angle BCD + \angle BAD = 180°$.

Thus $\angle BCD = 180° - \angle BAD = 180° - 80° = 100°$.

Example 7. Refer to the given trapezoid. Find the difference when the area is subtracted from the perimeter.

A. 32 B. $17 + \sqrt{218}$ C. $4\sqrt{2}$

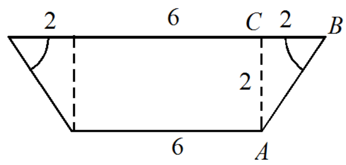
D. $16 + 2\sqrt{32}$ E. $52 + 2\sqrt{32}$



Solution: C.

We draw the height as shown in the figure. We see that triangle ABC is an isosceles right triangle with the sides of 2-2-$2\sqrt{2}$.

So the height is 2 and the area is $= \dfrac{6+10}{2} \times 2 = 16$.

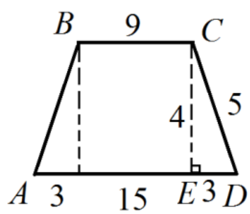


The perimeter is $10 + 6 + 4\sqrt{2} = 16 + 4\sqrt{2}$.

The difference is $16 + 4\sqrt{2} - 16 = 4\sqrt{2}$.

Example 8. Find the number of square centimeters in the area of the isosceles trapezoid whose parallel sides measure 9 cm and 15 cm and whose non-parallel sides measures 5 cm.

A. 27 B. 35 C. 48 D. 49 E. 50.



Solution: C.

We draw the perpendiculars to AD as shown in the figure.

We see that triangle CDE is a 3-4-5 right triangle. So $CE = 4$.

The area of the isosceles trapezoid is $= \dfrac{9+15}{2} \times 4 = 48$.

Example 9. The height of a trapezoid is 11 cm and one of its bases measures 32cm. If the area of the trapezoid is 649 cm^2, what is the measure in centimeters of the other base?

A. 70 B. 76 C. 80 D. 86 E. 69.

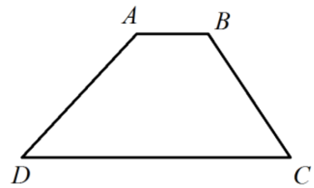
Solution: C.

Let b be the measure in centimeters of the other base. $S = \dfrac{1}{2}(a+b)h \Rightarrow$

$$649 = \frac{1}{2}(32+b) \times 11 \qquad \Rightarrow \qquad b = 86.$$

Example 10. Given trapezoid $ABCD$ with the measure of $\angle D$ (in degrees) equals 45, $AD = 8\sqrt{2}$, $AB = 4$, and $BC = 10$. Find the area of the trapezoid.

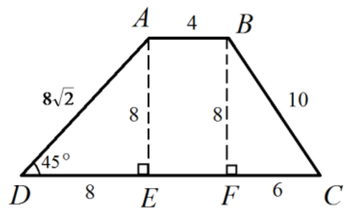
A. 80 B. 82 C. 84 D. 88 E. 90.



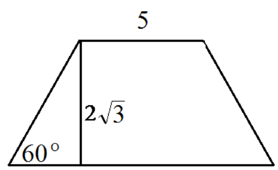
Solution: D.

We draw the heights of the trapezoid as shown. Triangle ADE is a 45-45-90° right triangle so $DE = AE = 8$.
Triangle BCF is a 6-8-10 right triangle so $FC = 6$.

$$S = \frac{1}{2}(a+b)h = \frac{1}{2}(4+18)8 = 88.$$



Example 11. The measure of one of the smaller base angles of an isosceles trapezoid is 60°. The shorter base is 5 inches long and the altitude is $2\sqrt{3}$ inches long. What is the number of inches in the perimeter of the trapezoid?

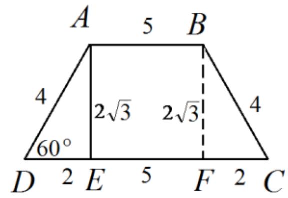


A. 20 B. 22 C. 26 D. 30 E. 35.

Solution: B.

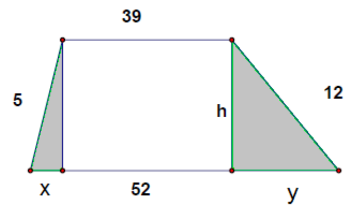
From the figure we know that triangle ADE is a 30-60-90° right triangle so DE = 2 and AD = 4.

So the perimeter is $5 + 4 \times 2 + 5 + 2 + 2 = 22$.



Example 12. In trapezoid $ABCD$ with bases AB and CD, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$. What is the area of $ABCD$?

A. 210 B. 220 C. 230 D. 240 E. 250.



Solution: A.

By the Pythagorean Theorem, we have:

$h^2 = 5^2 - x^2 = 12^2 - y^2$

$\Rightarrow \quad y^2 - x^2 = 12^2 - 5^2 = 119$ (1)

We know that $y + x = 52 - 39 = 13$. (2)

Therefore (1) becomes $(y-x)(y+x) = 119$

$\Rightarrow \ 13(y-x) = 119 \quad \Rightarrow \ y - x = \frac{119}{13}$ (3)

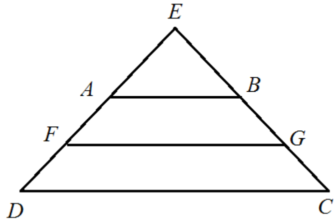
Solving the system of equations (2) and (3), we get $x = \frac{25}{13}$.

Therefore $h^2 = 5^2 - x^2 = 25 - \frac{5}{13} \quad \Rightarrow \quad h = \frac{60}{13}$.

The area of $ABCD$ is then $\frac{39+52}{2} \times \frac{60}{13} = 210$.

Example 13. The measures of the bases of trapezoid *ABCD* are 15 and 9, and the measure of the altitude is 4. Legs *DA* and *CB* are extended to meet at *E*. If *F* is the midpoint of *AD,* and *G* is the midpoint of *BC,* find the area of *ΔFGE.*

A. 40 B. 42 C. 43 D. 44 E. 48.



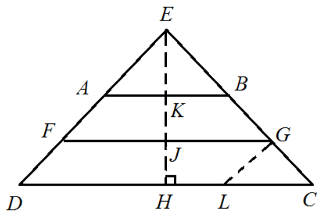
Solution: E.

Method 1: $\overline{FG}$ is the median of trapezoid *ABCD*, and

$FG = \dfrac{15+9}{2} = 12$ Since $\Delta EFG \sim \Delta EDC$, $\dfrac{EJ}{EH} = \dfrac{FG}{DC}$.

$KH = 4$ and $HJ = \dfrac{1}{2}KH = 2$. Therefore, $\dfrac{EJ}{EJ+2} = \dfrac{12}{15}$ and $EJ = 8$.

Hence, the area of $\Delta EFG = \dfrac{1}{2}(FG)(EJ) = \dfrac{1}{2}(12)(8) = 48$.



Method 2:

Since $\Delta EFG \sim \Delta EDC$,

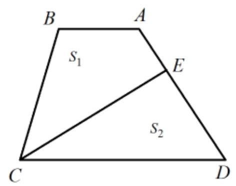
$$\frac{\text{Area of } \Delta EFG}{\text{Area of } \Delta EDC} = \frac{(FG)^2}{(DC)^2} = \frac{(12)^2}{(15)^2} = \frac{16}{25}.$$

Thus, $\dfrac{(\frac{1}{2})(FG)(EJ)}{(\frac{1}{2})(DC)(EH)} = \dfrac{(\frac{1}{2})(12)(EJ)}{(\frac{1}{2})(15)(EJ+2)} = \dfrac{16}{25}$.

Therefore, $EJ = 8$, and the area of $\Delta EFG = 48$.

Example 14. Given trapezoid *ABCD* with *AB*//*DC*. *CE* is the angle bisector of $\angle BCD$. $CE \perp AD$. $DE = 2AE$. *CE* cuts the trapezoid *ABCD* into two parts of areas S_1 and S_2. If $S_1 = 1$, find S_2.

A. 3/8 B. 5/8 C. 7/8 D. 1 E. 1/4.



Solution: D.

Extend *CB* and *DA* to meet at *F*. Since $CE \perp AD$ and *CE* is the angle bisector of $\angle BCD$, CE divides ΔCFD into two congruent parts. Thus ΔCFD is then an isosceles triangle and $CF = CD$. Since $CE \perp DF$, CE is the angle bisector of $\angle BCD$, and CE is also the median on DF,

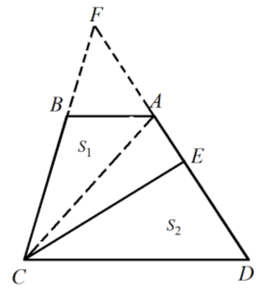
$S_{\Delta CEF} = S_{\Delta CDE}$.

Because $DE = EF$, $DE = 2AE$, so

$EA = AF = \frac{1}{4}FD$. $FB = \frac{1}{4}FC$.

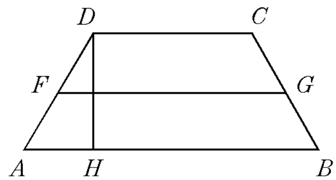
Thus $S_{\Delta FBA} = \frac{1}{4}S_{\Delta FAC} = \frac{1}{8}S_{\Delta FEC} = \frac{1}{8}S_1 = \frac{1}{8}$.

$S_2 = S_\Delta - S_{\Delta FBA} - S_1 = 2 - \frac{1}{8} - 1 = \frac{7}{8}$.



Example 15. In trapezoid $ABCD$, F is the midpoint of $\overline{AD}$ and G is the midpoint of $\overline{BC}$. If $FG = 9$ and $DH = 6$, then the area of the trapezoid is:

A. 78 B. 96 C. 72 D. 192 E. 54



Solution: E.

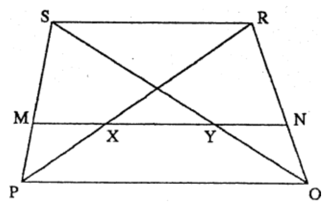
By **Property 2**, $FG = \frac{1}{2}(AB + CD) \Rightarrow 9 = \frac{1}{2}(AB + CD)$ (1)

By **Property 1**, $S = \frac{1}{2}(AB + CD)DH = \frac{1}{2}(AB + CD) \times 6$ (2)

Substituting (1) into (2): $S = \frac{1}{2}(AB + CD) \times 6 = 9 \times 6 = 54$.

Example 16. *PORS* is a trapezoid with *PO* a base. Median *MN* intersects the diagonals at *X* and *Y*. If *SR* = 12 and *XY* = 3, find *PO*.

A. 15 B. 16 C. 18 D. 21 E. 24



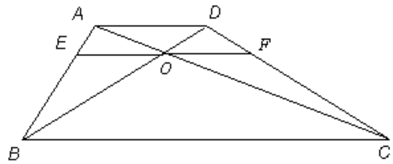
Solution: C.

By **Property 3**, $XY = \frac{1}{2}(PO - SR)$

$\Rightarrow \quad 3 = \frac{1}{2}(PO - 12) \quad \Rightarrow \quad PO = 18$.

Example 17. In trapezoid *ABCD, AD//BC. BD* and *AC* meet at *O*. *EF//BC*. *AD* = 12 and *BC* = 20. Find *EF*.

A. 10 B. 15 C. 20 D. 25 E. 30.

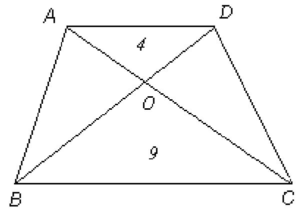


Solution: B.

By **Property 4**, $EF = \frac{2 \times 12 \times 20}{12 + 20} = 15$

Example 18. In trapezoid *ABCD*, *AD//BC*. $S_{\Delta BOC}$ = 9 cm^2. $S_{\Delta AOD}$ = 4 cm^2. Find the area of *ABCD*.

A. 10 B. 20 C. 25 D. 30 E. 40.



Solution: C.

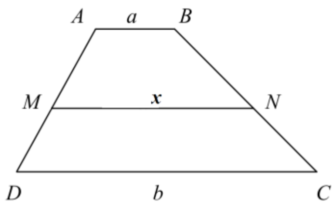
By **Property 5**,

$S_{ABCD} = (\sqrt{m} + \sqrt{n})^2 = (\sqrt{4} + \sqrt{9})^2 = 25\,\text{cm}^2$.

Example 19. Given trapezoid *ABCD* with *AB* // *CD*, *AB* = *a*, *CD* = *b*, and *MN* // *AB* such that trapezoid *ABNM* has area equal to trapezoid *MNCD*. If *MN* = *x*, solve for *x* in terms of *a* and *b*.

A. $\frac{a+b}{2}$ B. $\sqrt{ab}$ C. $\frac{2ab}{a+b}$

D. $\sqrt{\frac{a^2+b^2}{2}}$ E. $\frac{a^2+b^2}{2}$



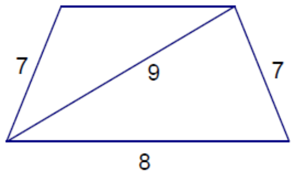
Solution: D.

By **Property 7**, $DC^2 - MN^2 = MN^2 - AB^2$

$\Rightarrow\ 2MN^2 = a^2 + b^2 \Rightarrow\ MN = \sqrt{\dfrac{a^2+b^2}{2}}$.

Example 20. The isosceles trapezoid pictured below has base of length 8, congruent sides with length 7, and a diagonal of length 9. Determine its area.

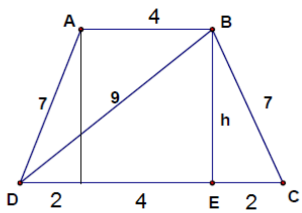
A. $18\sqrt{5}$ B. $3\sqrt{5}$ C. $10\sqrt{5}$ D. 40 E. 45.



Solution: A.

By the **property 8**, we have

$9^2 = 7^2 + AB \times 8 \qquad \Rightarrow \qquad AB = 4.$



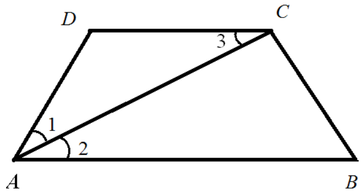
Let the height of the trapezoid be h. Applying Pythagorean Theorem to triangle BCE:

$h^2 = 7^2 - 2^2 \qquad \Rightarrow \qquad h = 3\sqrt{5}$.

The area is $\dfrac{8+4}{2} \times 3\sqrt{5} = 18\sqrt{5}$.

Example 21. As shown in the figure, isosceles trapezoid has the base angle 60°. The sum of two bases is 30 cm. The diagonal AC bisects the angle A. Find the perimeter of the trapezoid.

A. 10 B. 20 C. 30 D.4 E. 50.

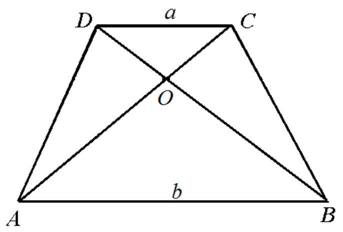


Solution: E.

Because $\angle B = 60°$ and $\angle BAC = 30°$, so triangle ABC is a right triangle where $\angle ACB = 90°$ and $AB = 2BC$.

Since $AB \parallel CD$, $\angle 3 = \angle 2$.
We also know that $\angle 1 = \angle 2$, so $\angle 3 = \angle 1$.
Thus, $DC = AD = BC$ and $AB = 2CD$.
Since $AB + CD = 30$, $2CD + CD = 30 \qquad \Rightarrow CD = 10$.
The perimeter of the trapezoid is $AB + BC + CD + DA = 5\ CD = 50$ (cm).

Example 22. Trapezoid *ABCD* has the area *S*. *AB* //*CD*, *AB* = *b*, *CD* = *a* ($a < b$). Diagonals *AC* and *BD* meet at *O*. If the area of ΔBOC is $\frac{2}{9}S$, find $\frac{a}{b}$.



A. 3/5 B. 5/8 C. 7/8 D. 1/2 E. 1/4.

Solution: D.

Let the area of ΔDOC be S_1 and the area of ΔAOB be S_2.

We have

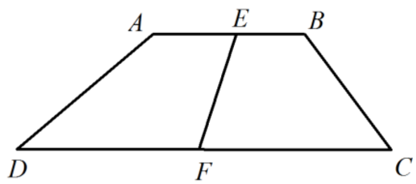
$$\begin{cases} S_1 + S_2 = \frac{5}{9}S \\ S_1 S_2 = (\frac{2}{9}S)^2 \end{cases}.$$

Solving for S_1 and S_2, we get $\begin{cases} S_1 = \frac{4}{9}S \\ S_2 = \frac{1}{9}S \end{cases}$ or $\begin{cases} S_1 = \frac{1}{9}S \\ S_2 = \frac{4}{9}S \end{cases}$.

Since $a < b$, then $S_1 < S_2$. Therefore $\frac{a}{b} = \sqrt{\frac{S_1}{S_2}} = \frac{1}{2}$.

Example 23. In trapezoid *ABCD*, *AB* = 5, *DC* = 11. *E* is the midpoint of *AB*. *F* is the midpoint of *DC*. $\angle D + \angle C = 90°$. What is the length of *EF*?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7



Solution: A.

Draw *AG* // *BC*. *ABCG* is a parallelogram. So *GC* = *AB* = 5. We also know that $\angle DAG = 90°$.

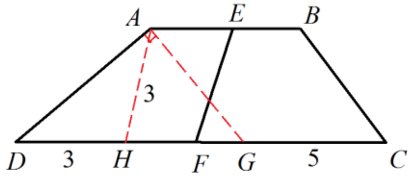
Connect *AH*, *H* is the midpoint of *DG*.

In right triangle ADG, $AH = DH = HG = \dfrac{1}{2}DG = \dfrac{1}{2}(11-5) = 3$.

$HF = DF - DH = \dfrac{1}{2}DC - 3 = 2.5$.

$EF = \dfrac{1}{2}AB = 2.5$.

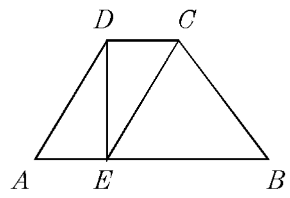
Thus $AEFH$ is a parallelogram and $EF = AH = 3$.



3. PROBLEMS

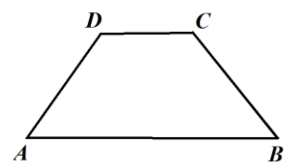
Problem 1. Find the area of the trapezoid *ABCD* given $AB = 14$, $DC = 6$, $\angle EDC$ is a right angle, and the area of ΔEDC is 30.

A. 75 B. 80 C. 90 D. 100 E. 120



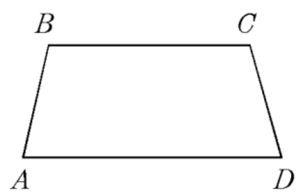
Problem 2. *ABCD* is a trapezoid. Given $\overline{AB} \parallel \overline{CD}$, $DC = 8$ m, $AC = 12$ m, $m\angle A = m\angle B = 45°$. The area of trapezoid *ABCD* is:

A. 20 m^2 B. 40 m^2 C. $20\sqrt{2}$ m^2
D. 32 m^2 E. 24 m^2



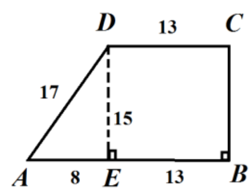
Problem 3. In quadrilateral *ABCD*, $\overline{CD} \perp \overline{BC}$, $\overline{BC} \parallel \overline{AD}$, $BC = 8$, $AD = 10$, and $CD = 3$. What is the area of *ABCD*?

A. 54 B. 30 C. 27 D. 240 E. 55



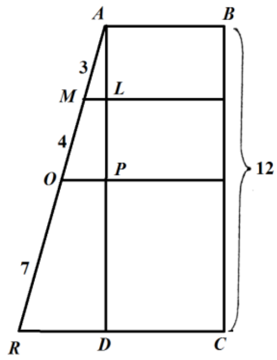
Problem 4. A trapezoid has parallel sides of lengths 13 inches and 21 inches. The longer of the two nonparallel sides is 17 inches and the shorter of the two nonparallel sides is perpendicular to a parallel side. What is the area in square inches of the trapezoid?

A. 221 B. 225 C. 247 D. 255 E. 289



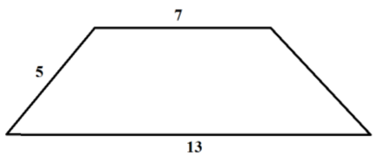
Problem 5. In the diagram, *ABCR* is a trapezoid with bases $\overline{AB}$ and $\overline{RC}$. *ABCD* is a rectangle. $AM = 3$, $MO = 4$, $OR = 7$, $BC = 12$. Find the perimeter of trapezoid *DROP*.

A. $4\sqrt{13}$ B. 26 C. $13+\sqrt{13}$ D. 24 E. $13+3\sqrt{13}$



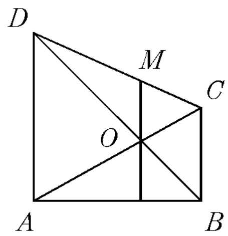
Problem 6. Find the area of this isosceles trapezoid.

A. 35 sq units B. 40 sq units C. 50 sq units D. 65 sq units E. 100 sq units

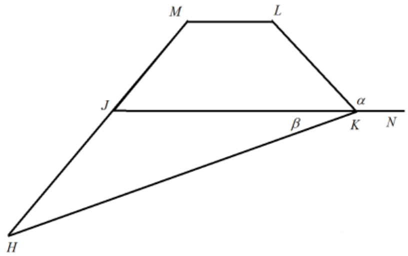


Problem 7. Given quadrilateral *ABCD* with $\overline{AD} \perp \overline{AB}$, $\overline{AB} \perp \overline{BC}$, $AD = 6$, and $BC = 3$. Let *O* be the point of intersection of the diagonals and let *M* be the point on $\overline{DC}$ such that $\overline{MO} \perp \overline{AB}$. Then *OM* is:

A. $\frac{3}{2}$ B. 2 C. $\frac{5}{2}$ D. 3 E. $\frac{7}{2}$



Problem 8. In the diagram, trapezoid *JKLM* is isosceles. Triangle *JHK* is isosceles with base $\overline{HK}$. Find the measure of β in terms of α.

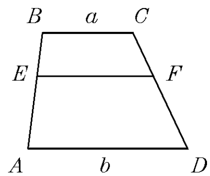


A. β = α. B. $\beta = \frac{\alpha}{2}$ C. $\beta = 180° - \frac{\alpha}{2}$.

D. β = 90° − α. E. $\beta = 90° - \frac{\alpha}{2}$.

Problem 9. Suppose quadrilateral *ABCD* is a trapezoid, $\overline{EF} // \overline{AD}$, trapezoid *AEFD* is similar to trapezoid *EBCF*, *BC* = *a*, and *AD* = *b*. Then *EF* is:

A. $\frac{2ab}{a+b}$ B. $\sqrt{ab}$ C. $\frac{a+b}{2}$ D. $\sqrt{\frac{a^2+b^2}{2}}$ E. $\frac{\sqrt{a}+\sqrt{b}}{2}$

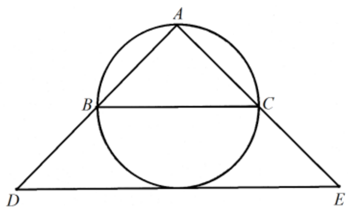


Problem 10. Trapezoid *ABCD*, with $\overline{AB}$ parallel to $\overline{CD}$, has median $\overline{XY}$. *AB* = 12 cm, and *XY* = 17 cm. Find *CD*.

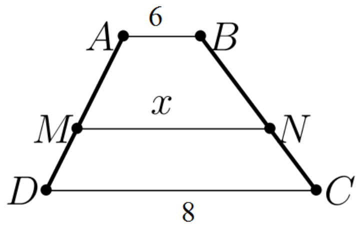
A. 14.5 cm B. 22 cm C. 24 cm D. 29 cm E. 34 cm

Problem 11. In the figure, $\overline{BC}$ is the base of isosceles triangle *ABC*. $\overline{BC}$ is a diameter of the circle, point *A* is on the circle, $\overline{BC} // \overline{DE}$, and $\overline{DE}$ is tangent to the circle. What is the ratio of the area of triangle *ABC* to the area of trapezoid *BCED*?

A. 1 : 4 B. 1 : 5 C. 2 : 5 D. 1 : 3 E. 3 : 4



Problem 12. Consider the trapezoid *ABCD* (see the diagram). Suppose $\overline{MN}$ is parallel to $\overline{DC}$, $AB = 6$, $DC = 8$, and $MN = x$. If the area of *ABNM* is half the area of *ABCD*, express *x* as a function of *a* and *b*.

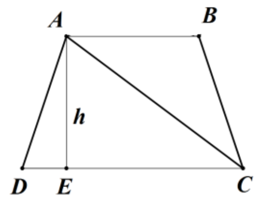


A. 10 B. 20 C. 30 D. 40 E. 50.

Problem 13. An isosceles trapezoid has parallel sides of lengths 10 and 28. The non-parallel sides are each of length 15. The height of the trapezoid is

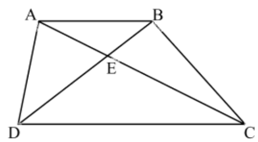
A. 12 B. 10 C. 8 D. 6 E. 4

Problem 14. *ABCD* is an isosceles trapezoid with $\overline{AB}$ parallel to $\overline{DC}$, $AC = DC$, and $AD = BC$. If the height h of the trapezoid is equal to *AB*, find the ratio $AB : DC$.

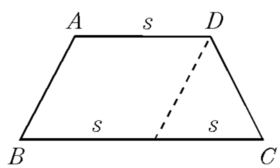


A. 2 : 3 B. 3 : 5 C. 4 : 5 D. 5 : 7 E. 5 : 9

Problem 15. In a trapezoid *ABCD* with *AB* parallel to *CD*, the diagonals intersect at point *E*. The area of triangle *ABE* is 32 and of triangle *CDE* is 50. Find the area of the trapezoid.



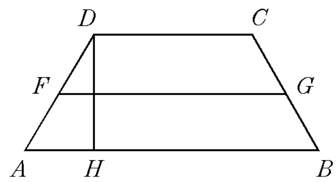
Problem 16. A trapezoid *ABCD* is separated into a parallelogram and a triangle by the dotted line through vertex *D* in the sketch shown. The ratio of area of triangle to area of trapezoid is:



A. $\frac{2}{3}$ B. $\frac{1}{3}$ C. $\frac{1}{2}$ D. $\frac{3}{4}$ E. none of these

Problem 17. In trapezoid *ABCD*, *F* is the midpoint of $\overline{AD}$ and *G* is the midpoint of $\overline{BC}$. If *FG* = 18 and *DH* = 12, then the area of the trapezoid is:

A. 54 B. $9\sqrt{6}$ C. 27 D. 18 E. 108

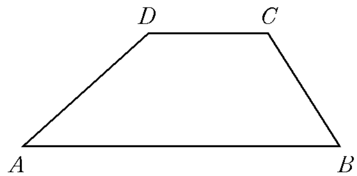


Problem 18. *ABCD* is a trapezoid with bases $\overline{AB}$ and $\overline{CD}$. The measure of $\angle A$ = 30° the measure $\angle B$ = 60°, *DC* = 8, and *AD* = 12. The perimeter of trapezoid *ABCD* is:

A. $34+6\sqrt{3}$ B. $28+9\sqrt{3}$

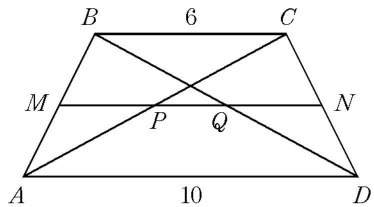
C. $28+24\sqrt{3}$ D. $28+13\sqrt{3}$

E. $28+12\sqrt{3}$



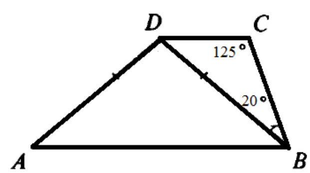
Problem 19. In trapezoid *ABCD*, *M* and *N* are the midpoints of $\overline{AB}$ and $\overline{CD}$, $\overline{BC} // \overline{AD}$, and diagonals $\overline{AC}$ and $\overline{BD}$ cut $\overline{MN}$ at *P* and *Q*. If *BC* = 6 and *AD* = 10, find *PQ*.

A. $\frac{1}{2}(\sqrt{10}-\sqrt{6})$ B. $2(\sqrt{10}-\sqrt{6})$ C. 2 D. $\frac{8}{3}$ E. none of these



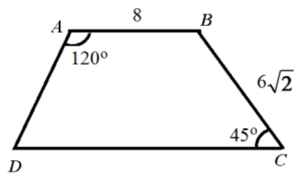
Problem 20: As shown in the figure, *ABCD* is a trapezoid. *AB* // *CD*. *BD* is the diagonal *BD* = *AD*. Find$\angle ADB$ if $\angle DCB$ = 125° and $\angle CBD$ = 20°.

A. 100° B. 110° C. 120° D. 125° E. 130°.



Problem 21: (2000 China Math Competition) Figure *ABCD* is a trapezoid with $AB \parallel DC$, $AB = 8$, $BC = 6\sqrt{2}$, $\angle BCD = 45°$ and $\angle BAD = 120°$

Find the area of *ABCD*.



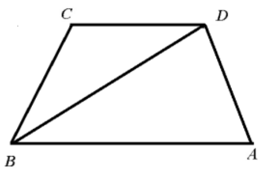
A. $66+6\sqrt{3}$ B. $66+\sqrt{3}$

C. $6+6\sqrt{3}$ D. $60+6\sqrt{3}$

E. $66+2\sqrt{3}$

Problem 22: In isosceles trapezoid, *AB//DC,* diagonal *BD* divides *ABCD* into two isosceles triangles as shown in the figure. Find each angle of trapezoid *ABCD.*

A. 100° B. 105° C. 108° D. 112° E. 120°.

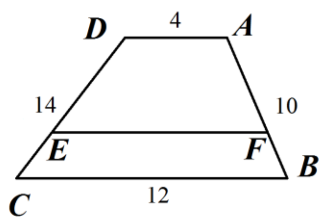


Problem 23: A rectangle is cut off two congruent isosceles right triangles as shown in the figure. The remainder of the rectangle is a trapezoid with two bases of lengths 20 and 30. Find the fractional parts of the rectangle that is cut off.

A. 1/6 B. 1/5 C. 1/4 D. 1/3 E. 1/2

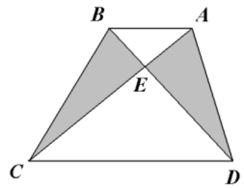
Problem 24: *ABCD* is a trapezoid with $AD = 4$, $BC = 12$, $AB = 10$, and $CD = 14$, as shown. Line *EF* parallel to the bases divides the trapezoid into two trapezoids of equal perimeters. Find the ratio *AF*/*FB*.

A. 2 : 1 B. 3 : 1 C. 4 : 5 D. 5 : 3 E. 5 : 2

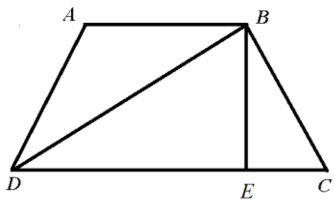


☆**Problem 25:** (AMC) The line joining the midpoints of the diagonals of a trapezoid has length 3. If the longer base is 97, then the shorter base is:
(A) 94 (B) 92 (C) 91 (D) 90 (E) 89

☆**Problem 26:** (AMC 12) Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and ΔAED and ΔBEC have equal areas. What is AE?
(A) 3 (B) 4 (C) 5 (D) 6 (E) 8



Problem 27: In isosceles trapezoid, $AB//DC$, $AD = BC$. $BE \perp DC$ at E. Find AB if $BE = AB$, $DB = DC = 10$.
(A) 9 (B) 8 (C) 7 (D) 6 (E) 5



4. SOLUTIONS

Problem 1. Solution: D.

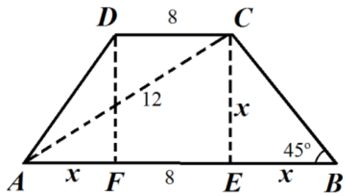
Since ΔEDC is a right triangle, its area is $S_{\Delta EDC} = \frac{1}{2} DC \times DE$

$\Rightarrow 30 = \frac{1}{2} \times 6 \times DE \Rightarrow DE = 10$.

So the area of the trapezoid is $S_{\Delta ABCD} = \frac{(DC + AB)}{2} \times DE = \frac{(6+14)}{2} \times 10 = 100$.

Problem 2. Solution: B.

We draw the height CE and DF as shown in the figure.



Connect AC. Let $BE = AF = CE = DF = x$.

Applying Pythagorean Theorem to triangle ACE:

$AC^2 = CE^2 + AE^2 \quad \Rightarrow \quad 12^2 = x^2 + (x+8)^2$

$\Rightarrow x = 2\sqrt{14} - 4$ and $x = -2\sqrt{14} + 4$ (ignored)

So the area of trapezoid $ABCD$ is:

$$S_{\Delta ABCD} = \frac{(DC + AB)}{2} \times CE = \frac{[8+8+2(2\sqrt{14}-4)]}{2} \times (2\sqrt{14}-4)$$

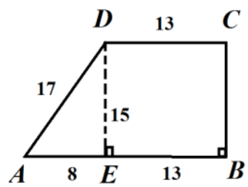
$$= (2\sqrt{14}+4)(2\sqrt{14}-4) = 40.$$

Problem 3. Solution: C

Since $\overline{CD} \perp \overline{BC}$, CD is the height of the trapezoid. So the area of trapezoid $ABCD$ is: $S_{\Delta ABCD} = \frac{(BC + AD)}{2} \times CD = \frac{8+10}{2} \times 3 = 27$.

Problem 4. Solution: D.

We draw the height of the trapezoid. We see that triangle AED is a 8-15-17 right triangle. So the area of trapezoid $ABCD$ is: $S_{\Delta ABCD} = \frac{(AB + DC)}{2} \times DE = \frac{13+21}{2} \times 15 = 255$.



Problem 5. Solution: E.

We see that $AO = OR = 7$. So $PD = 12/2 = 6$. We also know that $RD = 2OP$.

Applying Pythagorean Theorem to triangle ADR: $AR^2 = AD^2 + RD^2$

$\Rightarrow \quad 14^2 = RD^2 + 12^2 \quad \Rightarrow \quad RD = 2\sqrt{13}$. Then $OP = \dfrac{2\sqrt{13}}{2} = \sqrt{13}$.

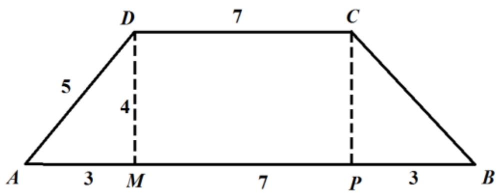
The perimeter of trapezoid $DROP$ is

$OR + RD + DP + OP = 7 + 2\sqrt{13} + 6 + \sqrt{13} = 13 + 3\sqrt{13}$.

Problem 6. Solution: B.

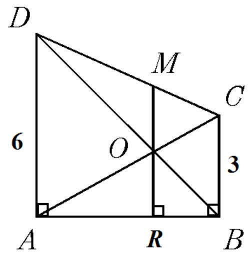
We draw the height of the trapezoid. We see that triangle ADM is a 3-4-5 right triangle. So the area of trapezoid $ABCD$ is:

$S_{\Delta ABCD} = \dfrac{(AB + DC)}{2} \times DM = \dfrac{13+7}{2} \times 4 = 40$.



Problem 7. Solution: B.

Since $\overline{AD} \perp \overline{AB}$ and $\overline{AB} \perp \overline{BC}$, $ABCD$ is a trapezoid.



By **<u>Property 4,</u>** $MR = \dfrac{2AD \times BC}{AD + BC} \Rightarrow MR = \dfrac{2 \times 6 \times 3}{6+3} = 4$

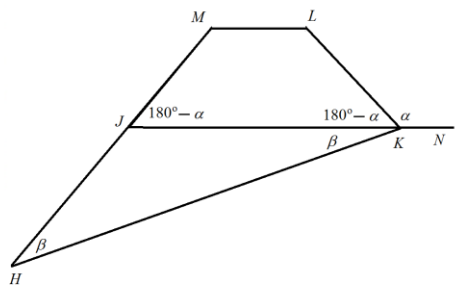
$$OR = \frac{2AD \times BC}{AD + BC} \quad \Rightarrow \quad MR = \frac{2 \times 6 \times 3}{6+3} = 4$$

By **<u>Theorem 6 (Chapter 27 Similar Triangles),</u>**.

$$OR = \frac{AD \times BC}{AD + BC} \quad \Rightarrow \quad OR = \frac{6 \times 3}{6+3} = 2$$

Therefore $OM = MR – OR = 4 – 2 = 2$.

Problem 8. Solution: E



Since trapezoid $JKLM$ is isosceles, $\angle LKJ = \angle MJK = 180° - \alpha$.

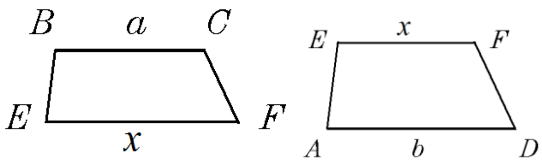
Triangle JHK is isosceles, $\angle JHK = \angle JKH = \beta$.

Notice that $\angle MJK$ is the exterior angle of triangle *JHK*, $\angle MJK = 2\beta$

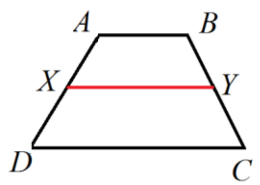
$\Rightarrow \quad 180° - \alpha = 2\beta \Rightarrow \beta = 90° - \frac{\alpha}{2}$.

Problem 9. Solution: B.

Trapezoid *AEFD* is similar to trapezoid *EBCF,* we have $\frac{b}{x} = \frac{x}{a} \Rightarrow \quad x = \sqrt{ab}$.



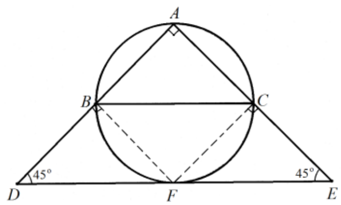
Problem 10. Solution: B.



By **Property 2**, $XY = \frac{1}{2}(AB + CD) \quad \Rightarrow 17 = \frac{1}{2}(12 + CD)$

$\Rightarrow \quad CD = 22$

Problem 11. Solution: D.

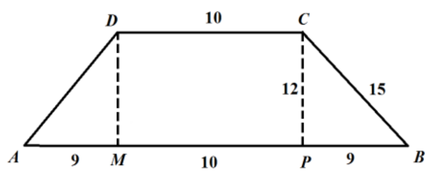


Since $\overline{BC}$ is the diameter, and *ABC* is the isosceles triangle, $\angle A = 90°$, $\angle D = \angle E = 45°$. When we connect BF and CF and we see that four triangles are congruent. So the ratio is 1:3.

Problem 12. Solution: E.

By **Property 7**, $DC^2 - MN^2 = MN^2 - AB^2 \quad \Rightarrow \quad 8^2 - x^2 = x^2 - 6^2 \Rightarrow x = 50$.

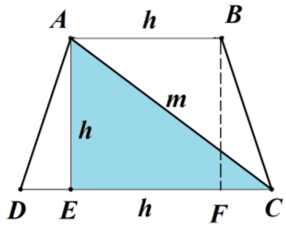
Problem 13. Solution: A.



We draw the height of the trapezoid. We see that triangle *CPB* is a 9-12-15 right triangle. So the height of the trapezoid *ABCD* is 12.

Problem 14. Solution: B.

Let $DC = AC = m$. Draw the height BF as shown in the figure. Since $ABCD$ is an isosceles trapezoid, $EF = h$, $FC = DE = \dfrac{m-h}{2}$.



Applying Pythagorean Theorem to triangle ACE:

$$AC^2 = AE^2 + EC^2 \quad \Rightarrow \quad m^2 = h^2 + (h + \frac{m-h}{2})^2 \quad \Rightarrow$$

$$m^2 = h^2 + \frac{1}{4}(m+h)^2 \quad \Rightarrow \quad 5(\frac{h}{m})^2 + 2\frac{h}{m} - 3 = 0$$

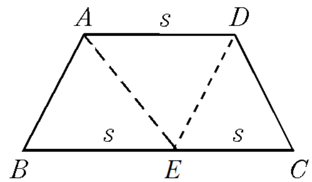
$$\Rightarrow \quad \frac{h}{m} = \frac{3}{5} \text{ and } \frac{h}{m} = -1 \text{ (ignored).}$$

Problem 15. Solution: 162.

By **Property 5**, $S_{ABCD} = (\sqrt{32} + \sqrt{50})^2 = 162$.

Problem 16. Solution: B.

We draw AE as shown in the figure. We see that three smaller triangles have the same area. So the answer is 2/3.



Problem 17. Solution: E.

By **Property 2**, $FG = \frac{1}{2}(AB + CD) \quad \Rightarrow 18 = \frac{1}{2}(AB + CD) \quad (1)$

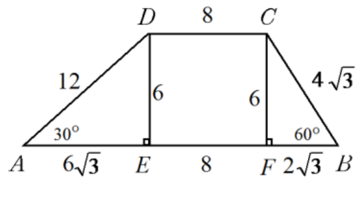
By **Property 1**, $S = \frac{1}{2}(AB + CD)DH = \frac{1}{2}(AB + CD) \times 12 \quad (2)$

Substituting (1) into (2): $S = \frac{1}{2}(AB + CD) \times 12 = 18 \times 6 = 108$.

Problem 18. Solution: E.

We draw the height CF and DE as shown in the figure.

We see that triangle ADE is a 6-$6\sqrt{3}$-12 right triangle.



We see that triangle BCF is a $2\sqrt{3}$-6-$4\sqrt{3}$ right triangle.

So the perimeter of trapezoid $ABCD$ is: $12+8+4\sqrt{3}+2\sqrt{3}+8+6\sqrt{3}=28+12\sqrt{3}$

Problem 19. Solution: C.

Method 1:

By **Property 2**, $MN=\frac{1}{2}(AB+CD)=8$. We know that $MP=QN=\frac{1}{2}BC=3$ So $PQ = 8 - 6 = 2$.

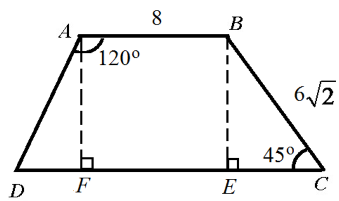
Method 2:

By **Property 3,** $PQ=\frac{1}{2}(AD-BC)=2$

Problem 20: Solution: B.

In ΔBDC, $\angle BDC = 35°$. Since DC is parallel to AB, $\angle DBA = 35°$. Since base angles of an isosceles triangle are equal, $\angle BAD = 35°$. Therefore $\angle ADB = 110°$.

Problem 21: Solution: A.



Draw $AF \perp DC$, and $BE \perp DC$. Since $BC = 6\sqrt{2}$, $\angle BCD = 45°$, thus $BE = EC = AF = 6$.

Since $\angle DAF = 30°$, $DF = 2\sqrt{3}$.

Therefore,

$$S_{ABCD}=\frac{1}{2}(AB+DF+FE+EC)\cdot AF=66+6\sqrt{3}$$

Problem 22: Solution: C.

We have $AD = BC = CD$, and $AB = BD$.

Let $\angle ABD = x$, $\angle A = y$. Then

$x + 2y = 180°$ (1)

$3x + y = 180°$ (2)

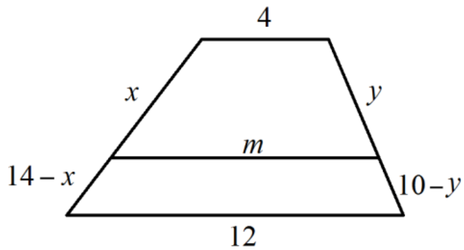
Solving for x and y, we get $x = 36°$ and $y = 72°$.

$\angle A = \angle B = 72°$, $\angle C = \angle D = 108°$.

Problem 23: Solution: A.

Each triangle has leg length 1/2 × (30 – 20) = 5 meters and area 1/2 × 5^2 = 25/2 square meters. Thus the flower beds have a total area of 25 square meters. The entire yard has length 30 and width 5, so its area is 150. The fraction of the yard occupied by the flower beds is 25/150 = 1/6.

Problem 24: Solution: A.



Let x and y be the two upper segments of the non-parallel sides. Then

$4 + y + x + m = 12 + 10 - y + 14 - x + m$

$\Rightarrow x + y = 16$.

Since $x : y = 14 : 10 = 7 : 5$, $\frac{7y}{5} + y = 8$

Solving for y, we have $y = \frac{20}{3}$, and $y : (10 - y) = \frac{20}{3} : \frac{10}{3} = 2 : 1$.

Problem 25: Solution: (C).

Method 1 (official solution):

The median of a trapezoid goes through the midpoints of the diagonals. Let x be the length of the shorter base.

So the length of median $= \frac{x}{2} + 3 + \frac{x}{2}$; $\frac{1}{2}(x + 97) = \frac{x}{2} + 3 + \frac{x}{2}$. So $x = 91$.

Method 2 (our solution):

Let the shorter base be x. By the Property 3 in (3.3), we have $3 = \frac{1}{2}(97 - x) \Rightarrow x$ = 91.

Problem 26: Solution: D.

Since ΔAED and ΔBEC have equal areas, $AB // CD$. $ABCD$ is a trapezoid. $\Delta ABE \sim \Delta CDE$.

$$\frac{AB}{CD} = \frac{AE}{CE} = \frac{AE}{AC - AE} \qquad \Rightarrow \qquad \frac{9}{12} = \frac{AE}{14 - AE} \qquad \Rightarrow \qquad AE = 6.$$

Problem 27: Solution: D.

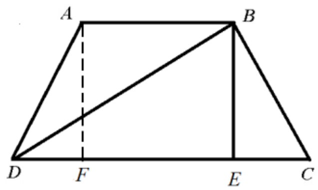
Method 1:

By **property 8**, we have $10^2 = BC^2 + AB \times 10$ (1)

Draw $AF \perp DC$ at E. We know that

$$BC^2 = BE^2 + EC^2 = AB^2 + \left(\frac{DC - AB}{2}\right)^2 = AB^2 + \left(\frac{10 - AB}{2}\right)^2 \quad (2)$$

Substituting (2) into (1), we get:



$$10^2 = AB^2 + \left(\frac{10 - AB}{2}\right)^2 + AB \times 10 \quad \Rightarrow$$

$$10^2 = AB^2 + \left(\frac{10 - AB}{2}\right)^2 + AB \times 10$$

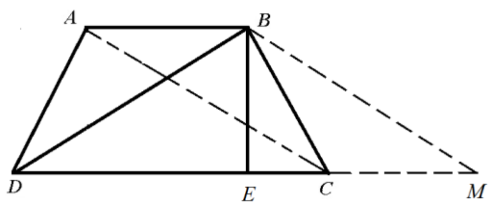
$\Rightarrow \; AB^2 + 4AB - 60 = 0$.

Solving the quadratic equation, we have $AB = 6$ or $AB = 10$ (extraneous, since $DB = DC = 10$).

Therefore $AB = 6$.

Method 2:

Connect AC and draw $BM//AC$ to meet DC at M.



Since $AB//DC$ and $BM//AC$, $AB = CM$, $AC = BM$.

Since $AD = BC$, $BD = AC$. Thus $BM = BD = 10$.

Since $BE \perp DC$, $DE = EM$.

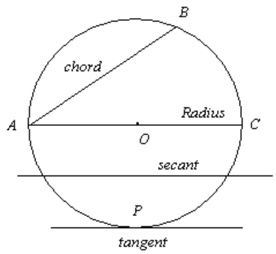
We know that $DC = 10$. Let $DE = x$.

It follows that $CE = 10 - x$ and $EM = x$, so $CM = EM - CE = 2x - 10$.

Since $BE = AB = CM$, $BE = 2x - 10$.
By the Pythagorean Theorem, we have $BE^2 + DE^2 = BD^2$, or
$(2x - 10)^2 + x^2 = 100$.
This can be simplified into $5x^2 - 40x = 0$.
Since $x > 0$, $x = 8$.
Therefore $AB = 2x - 10 = 6$.

1. BASIC KNOWLEDGE

Chord: a line segment joining two points on the circle.



Diameter: a chord through the center.

Radius: a line segment joining the center to a point on the circumference.

Tangent: a line that touches the circle at one and only one point.

Secant: a line that intersects the circle at one and only one point.

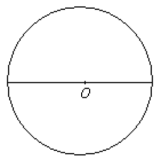
Minor arc: arc that is less than a semicircle.

Major arc: arc that is greater than a semicircle.

Concentric circles: circles that have the same center.

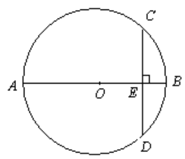
Diameter Principles

Principle 1. A diameter divides a circle into two equal parts.



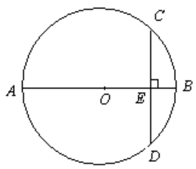
Principle 2. A diameter perpendicular to a chord bisects the chord and its arc.

If AB is the diameter of the circle and $AB \perp CD$, then $EC = ED$



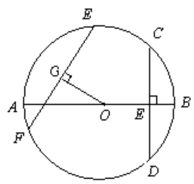
Principle 3. A perpendicular bisector of a chord passes through the center of the circle.

If $AB \perp CD$, and $EC = ED$, then AB is the diameter of the circle.



Principle 4. In the same or congruent circles, congruent chords are equally distanced from the center.

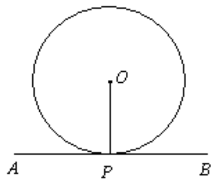
If $EF = CD$, then $OG = OE$



Tangent Principles

Principle 5. A tangent is perpendicular to the radius drawn to the point of contact.

AB tangent to the circle O at P, if we draw OP, then $OP \perp AB$.

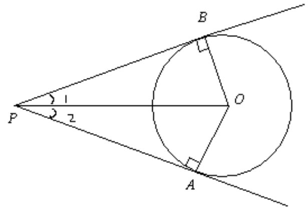


Principle 6. A line is tangent to a circle if it is perpendicular to a radius at its outer end.

Principle 7. A line passes through the center of a circle if it is perpendicular to a tangent at its point of contact.

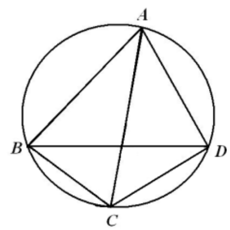
Principle 8. Tangents to a circle from an outside point are congruent.

$PA = PB$. $\angle 1 = \angle 2$.



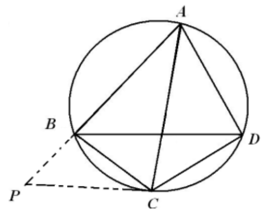
Length-measurement Principles

Principle 9. (Ptolemy's Theorem) If quadrilateral $ABCD$ is a cyclic quadrilateral, then $AC \times BD = AB \times CD + AD \times BC$.



Proof:

Method 1:



Extend AB to P such that $\angle PCA = \angle DCB$.

Then $\Delta ACP \sim \Delta DCB$. $\frac{AC}{CD} = \frac{AP}{BD}$.

So $AC \cdot BD = CD \cdot AP$ (1)

We also have $\angle CBP = \angle ADC$, $\angle BPC = \angle CBD = \angle CAD$.

Then $\Delta ACD \sim \Delta PCB$. We have $\frac{AD}{PB} = \frac{CD}{BC}$ or

$AD \cdot BC = CD \cdot PB$ (2)

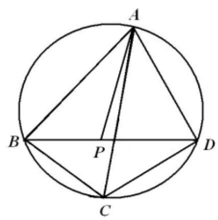
(1) – (2): $AC \cdot BD - AD \cdot BC = CD(AP - PB) = AB \cdot CD$.

That is, $AC \cdot BD = AB \cdot CD + BC \cdot AD$.

Method 2:

Draw the circumcircle of quadrilateral $ABCD$.

Take a point P at CD and connect AP such that $\angle PAB = \angle CAD$. Therefore $\Delta ABP \sim \Delta ACD$. So $\frac{AB}{AC} = \frac{BP}{CD}$ $\Rightarrow$



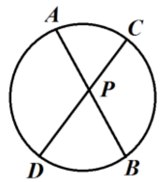
$AB \cdot CD = AC \cdot BP$ (1)

We also see that $\Delta ABC \sim \Delta APD$ ($\angle ACB = \angle ADP$, and $\angle BAC = \angle DAP$). So we have $BC \cdot AD = AC \cdot PD$ (2)

(1) + (2): $AB \cdot CD + BC \cdot AD = AC(BP + PD) = AC \cdot BD$.

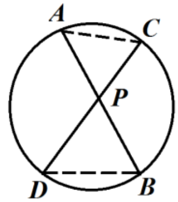
(Ptolemy is said to have provided the shortest proof of the Pythagorean Theorem).

Principle 10. Power of Points formula (1): If two chords intersect within a circle, the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other.
$PA \times PB = PC \times PD$.

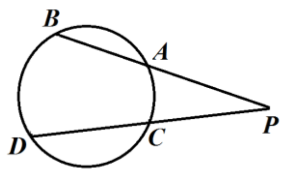


Proof:
Connect AC and BD. $\angle PAC = \angle PDB$.
We know that $\angle APC = \angle BPD$. Thus $\Delta PAC \sim \Delta PDB$.

$$\frac{PA}{PC} = \frac{PD}{PB} \quad \Rightarrow PA \cdot PB = PC \cdot PD.$$

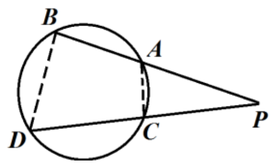


Principle 11. Power of Points formula (2):
$PA \times PB = PC \times PD$.

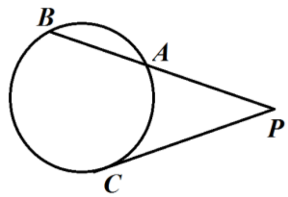


Proof:
Connect AC and BD. $\angle PAC = \angle PDB$.
We know that $\angle APC = \angle BPD$. Thus $\Delta PAC \sim \Delta PDB$.

$$\frac{PA}{PC} = \frac{PD}{PB} \quad \Rightarrow PA \cdot PB = PC \cdot PD.$$

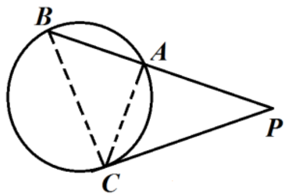


Principle 12. Power of Points formula (3):
$PC^2 = PB \times PA$



Proof:
Connect AC and BC. $\angle PAC = \angle PCB$.
We know that $\angle APC = \angle BPC$. Thus, $\Delta PAC \sim \Delta PCB$.

$$\frac{PA}{PC} = \frac{PC}{PB} \quad \Rightarrow PC^2 = PA \times PB$$



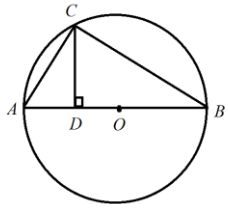
Principle 13. Triangle *ABC* is inscribed in the circle *O*. *AB* is the diameter. The height is draw from *C* to meet *AB* at *D*.

Then we have:

$AC^2 = AB \times AD$

$BC^2 = AB \times DB$

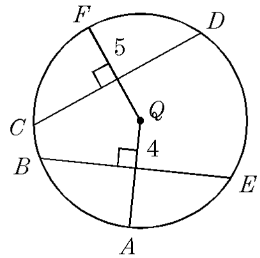
$CD^2 = AD \times BD$



2. EXAMPLES

Example 1. In circle *Q*, *BE* = *CD*. The length of *BE* is:

A. 9 B. $2\sqrt{41}$ C. $41\sqrt{2}$ D. 18 E. $2\sqrt{65}$

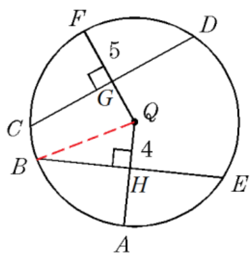


Solution: E.

Since *BE* = *CD*, GQ = QH = 4. Connect B*Q*. We see that *BQ* = *QF* = 5 + 4 = 9. Applying Pythagorean Theorem to triangle *BHQ*:

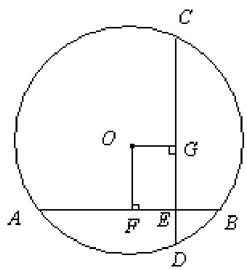
$BH^2 = BQ^2 - QH^2 \quad \Rightarrow \quad BH^2 = 9^2 - 4^2 = 65$

$\Rightarrow \quad BE = 2BH = 2\sqrt{65}$.



Example 2. If *AE* = 8, *EB* = 4, find *OG*.

A. 2 B. 3 C. 4 D. 5 E. 6



Solution: A.

AB = *AE* + *EB* = 8 + 4 = 12,

FB = 1/2 *AB* = 6, *FE* = *FB* – *EB* = 6 – 4 = 2 = *OG*

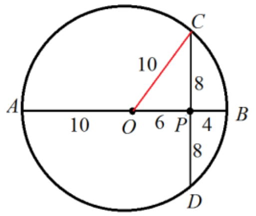
Example 3. 69. Point P is 6 units from the center of a circle of radius 10. Compute the number of chords with integral length that pass through P.

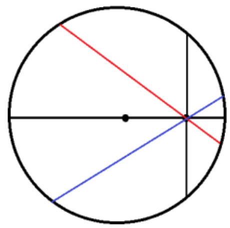
A. 2 B. 5 C. 7 D. 8 E. 10

Solution: D.

We draw the figure and we know that triangle OCE is a 6-8-10 right triangle. By **Principle** 2, we get $CD = 2 \times 8 = 16$. Thus the chord can be any integer value from 16 to 20 (the longest chord-the diameter).

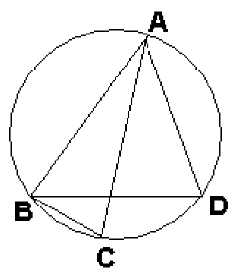
We can one chord for the length of 16 and the length of 20. For the lengths of 17, 18, and 19, we get two chords symmetrical to the diameter AB. Total we have $1 + 1 + 2 \times 3 = 8$ chords.





Example 4. Points A, B, C, and D lie on a circle with $AB = 4$ and $BC = 2$; AC is a diameter; and $\angle ABD = \angle CBD$. What is BD? (2003 NC Math Contest Geometry).

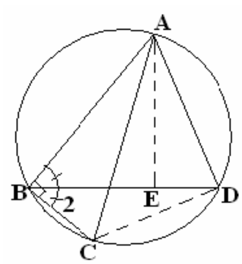
A. $2\sqrt{2}$. B. $3\sqrt{2}$. C. $\sqrt{10}$ D. 3 E. $2\sqrt{3}$.



Solution: B.

Method 1 (official Solution):

Draw segment CD. Since AC is a diameter, $\angle ABC$ and $\angle ADC$ are right angles. Because $\angle ABD \cong \angle CBD$ and $\angle ABC$ is a right angle, then $\angle ABC = \angle CBD = 45°$. Since $\angle CBD = 45°$ and $\angle CBD$ and $\angle CAD$ are inscribed angles intercepting CD, $\angle CAD = 45°$. ΔADC is an isosceles right triangle and $AD = CD = AC/\sqrt{2}$. Since ΔABC is a right triangle, $AC^2 = AB^2 + BC^2 = 20$, so $AC = 20$. Then $AD = CD = \sqrt{10}$. Draw a perpendicular from A to BD and let the foot of the



perpendicular be E. Then ΔADE is a right triangle so $DE^2 = AD^2 - AE^2 = 10 - 8 = 2$, so $DE = \sqrt{2}$. Then $DB = BE + ED = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$.

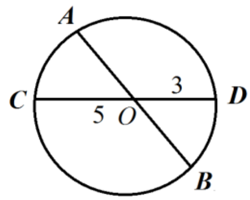
Method 2 (our solution):
Since $\angle ABD = \angle CBD$, $AD = DC$.
We know that AC is the diameter, so $AC = \sqrt{AB^2 + BC^2} = \sqrt{4^2 + 2^2} = 2\sqrt{5}$ and

$$AD = DC = \frac{AC}{\sqrt{2}} = \sqrt{10}$$

By Ptolemy's theorem, $AB \times CD + AD \times BC = AC \times BD$

$$4 \times \sqrt{10} + 2 \times \sqrt{10} = 2\sqrt{5} \times BD \qquad \Rightarrow BD = 3\sqrt{2}$$

Example 5: Given a circle with two intersecting chords, how long is chord AB if chord CD bisects chord AB at point O, and $CO = 5$ and $OD = 3$?
A. $\sqrt{15}$ B. $3\sqrt{2}$. C. $2\sqrt{15}$. D. 5 E. $15\sqrt{2}$

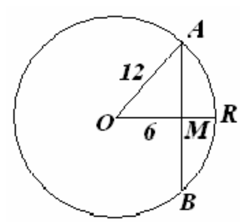


Solution: C.

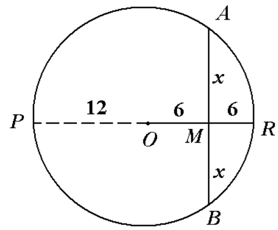
$$AO \times OB = CO \times OD \Rightarrow x^2 = 3 \times 5 = 15$$

$$\Rightarrow \quad x = \sqrt{15} \quad \Rightarrow \quad 2x = 2\sqrt{15}.$$

Example 6: A chord which is the perpendicular bisector of a radius of length 12 in a circle. What is the length of the chord? (2003 NC Math Contest Geometry).
A. $6\sqrt{3}$ B. $12\sqrt{3}$. C. $2\sqrt{13}$. D. 20 E. $13\sqrt{2}$



Solution: B.
Method 1 (o**<u>fficial solution):</u>**
Let O denote the center of the circle, and let OR and AB be the radius and the chord which are perpendicular bisectors of each other at M. Applying the Pythagorean theorem to the right triangle OMA yields $(AM)^2 = (OA)^2 - (OM)^2 = 12^2 - 6^2 = 108$, Thus $AM = 6\sqrt{3}$ and the required chord has length $12\sqrt{3}$.

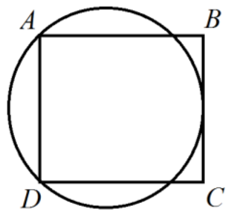


Method 2 (our solution): Extend RO to P and PR is the diameter.

$AM \times MB = RM \times MP \quad \Rightarrow x^2 = (12+6)\times 6 = 3\times 6^2$

$\Rightarrow \qquad x = 6\sqrt{3} \qquad \Rightarrow \qquad 2x = 12\sqrt{3}.$

Example 7: In the figure shown, a circle passes through two adjacent vertices of a square and is tangent to the opposite side of the square. If the side length of the square is 3, what is the area of the circle?

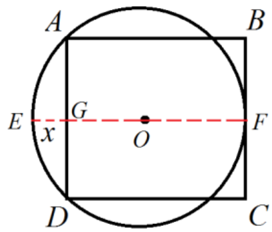


A. $\frac{9}{4}\pi$ B. $\frac{16}{4}\pi$ C. 6π D. $\frac{36}{25}\pi$ E. $\frac{225}{64}\pi$

Solution: E.

Draw EF, the diameter from the tangent point. Let $EG = x$.

We see that $GF = 3$.



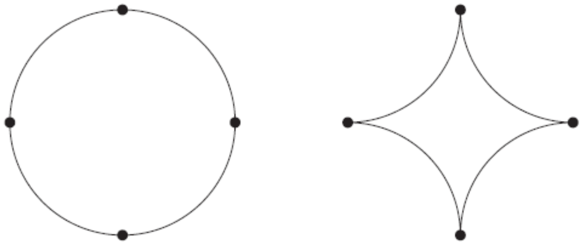
By **Principle 10**, $x\times 3 = \frac{3}{2}\times\frac{3}{2} \qquad \Rightarrow \qquad x = \frac{3}{4}$.

So $EF = EG + GF = \frac{3}{4} + 3 = \frac{15}{4}$.

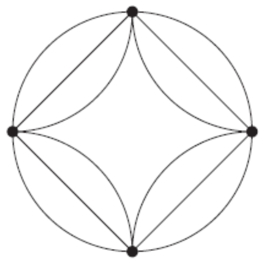
So the radius is $\frac{15}{8}$ and the area is $\frac{225}{64}\pi$.

☆**Example 8.** (2012 AMC 8 problem 24) A circle of radius 2 is cut in to four congruent arcs. The four arcs are joined to form the star figure shown. What is the ratio of the area of the star figure to the area of the original circle?

A. $\frac{4-\pi}{\pi}$ B. $\frac{1}{\pi}$ C. $\frac{\sqrt{2}}{\pi}$ D. $\frac{\pi-1}{\pi}$ E. $\frac{3}{\pi}$



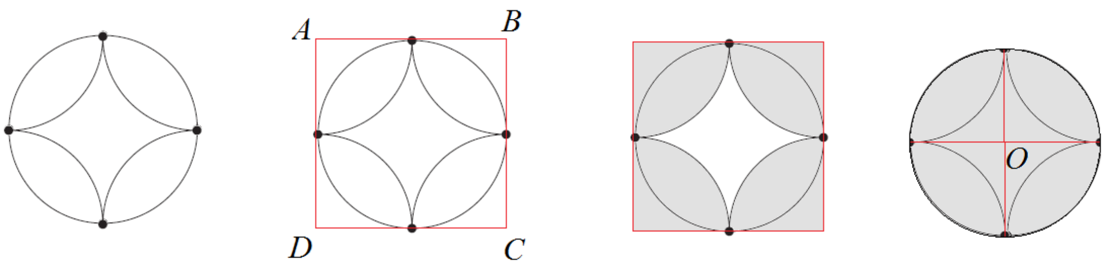
Solution: A.
Method 1 (official solution):
Translate the star into the circle so that the points of the star coincide with the points on the circle. Construct four segments connecting the consecutive points of the circle and the star, creating a square concentric to the circle.



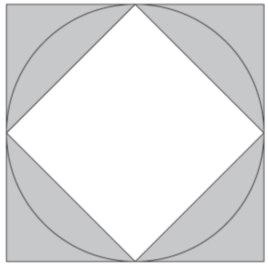
The area of the circle is $\pi(r)^2 = 4\pi$. The square is made up of four congruent right triangles with area $(2 \times 2)/2 = 2$, so the area of the square is $4 \times 2 = 8$.
The area inside the circle but outside the square is $4\pi - 8$.

This is also the area inside the square but outside the star. So, the area of the star is $8 - (4\pi - 8) = 16 - 4\pi$. The ratio of the area of the star figure to the area of the original circle is $(16-4\pi)/4\pi = (4-\pi)/\pi$.

Method 2 (our solution):
As shown in the figure below, the area of the star figure is the area of the square $ABCD$ – the area of the circle $O = 4 \times 4 - \pi \times 2^2 = 16 - 4\pi$.
The ratio of the area of the star figure to the area of the original circle is $(16-4\pi)/4\pi = (4-\pi)/(\pi \times 2^2) = (4-\pi)/\pi$.



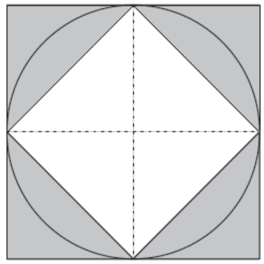
☆**Example 9.** (2011 AMC 8 problem 25) A circle with radius 1 is inscribed in a square and circumscribed about another square as shown. Which fraction is closest to the ratio of the circle's shaded area to the shaded area between the two squares?



(A) 1/2 (B) 1 (C) 3/2 (D) 2 (E) 5/2

Solution: A.

Method 1 (official solution):



The area of a circle of radius 1 is $\pi(1)^2 = \pi$. The side length of the big square is the diameter of the circle, which is 2, so its area is $2^2 = 4$. The big square can be divided into 8 congruent triangles, and the shaded area is made up of 4 of those triangles. The shaded area is half the area of the big square, which is 2. The requested ratio of the two shaded areas is $(\pi - 2)/2 \approx (3.14 - 2)/2 \approx 1/2$.

Method 2 (our solution):

We are asked to calculate the ratio of the two shaded areas as shown in the figure below.

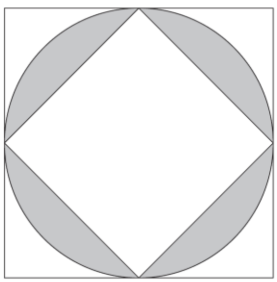


Figure *a*

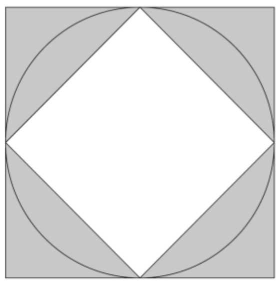
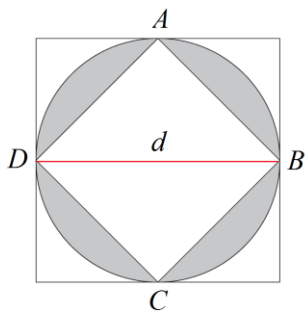


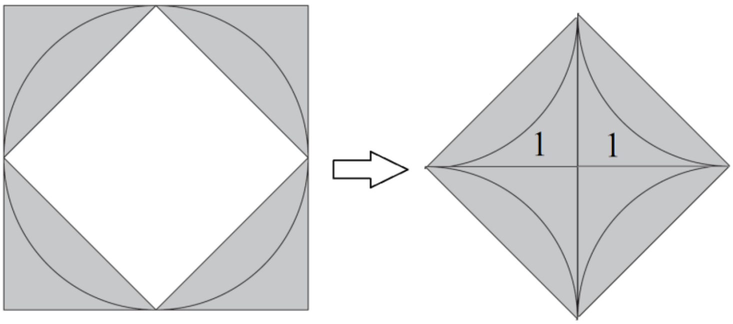
Figure *b*

The shaded area in figure *a* is the area of the circle – the area of the square *ABCD*

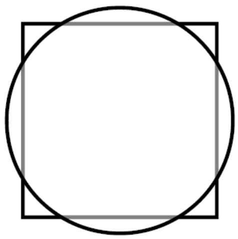


$$= \pi(r)^2 - \frac{1}{2}d^2 = \pi(1)^2 - \frac{1}{2} \times 2^2 = \pi - 2.$$

The shaded area of figure *b* can be converted into a rhombus and the area is
$\frac{1}{2} \times 2^2 = 2$.
The required ratio of the two shaded areas is then $(\pi - 2)/2 \approx (3.14 - 2)/2 \approx 1/2$.

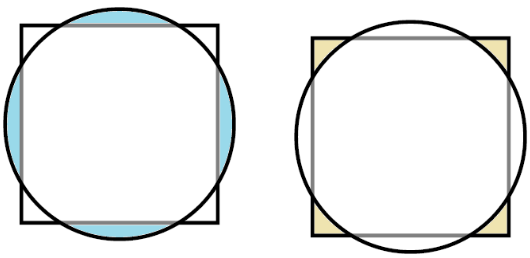


☆**Example 10.** A circle with radius $\frac{32}{\sqrt{\pi}}$ and a square share the same center. The total area of the regions that are inside the circle and outside the square is equal to the total area of the regions that are outside the circle and inside the square. What is the length of the side of the square?



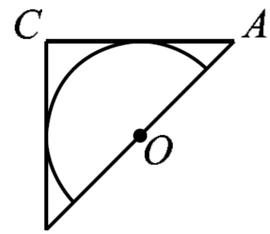
(A) 32 (B) 16 (C) 8 (D) 4 (E) 2

Solution: A.
Because the circle and square share the same interior region and the area of the two exterior regions indicated are equal, the square and the circle must have equal area.



The area of the circle is $\pi r^2 = \pi(\frac{32}{\sqrt{\pi}})^2 = 32^2$. Because the area of both the circle and the square is 32^2, the length of the side of the square is $\sqrt{32^2} = 32$.

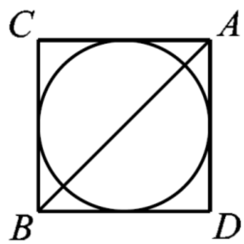
☆**Example 11.** (2005 AMC 8 problem 23) Isosceles right triangle *ABC* encloses a semicircle of area 2π. The circle has its center *O* on hypotenuse *AB* and is tangent to sides *AC* and *BC*. What is the area of triangle *ABC*?



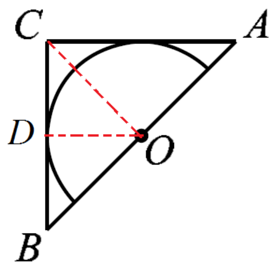
(A) 6 (B) 8 (C) 3π (D) 10 (E) 4π

Solution: B.

Method 1 (official solution):
Reflect the triangle and the semicircle across the hypotenuse *AB* to obtain
a circle inscribed in a square. The circle has area 4π.
The radius of a circle with area 4π is 2. The side length of the square is 4 and the area of the square is 16.
So the area of the triangle is 8.



Method 2 (our solution):
Connect *OC* and *OD*, where *D* is the tangent point.
We see that ΔOBC is an isosceles right triangle as well.



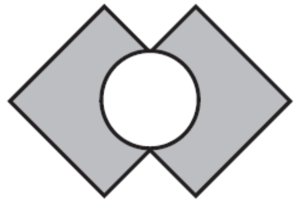
OD is the radius of the semicircle and $OD = \frac{1}{2}AC$.

$\frac{1}{2}\pi r^2 = 2\pi \quad \Rightarrow \quad OD^2 = 4 \quad \Rightarrow \quad OD = 2$

So $AC = 4$.

The area of triangle *ABC* = $\frac{1}{2}AC \times BC = \frac{1}{2}AC^2 = \frac{1}{2} \times 4^2 = 8$.

☆**Example 12.** (2004 AMC 8 problem 25) Two 4 × 4 squares intersect at right angles, bisecting their intersecting sides, as shown. The circle's diameter is the segment between the two points of intersection. What is the area of the shaded region created by removing the circle from the squares?

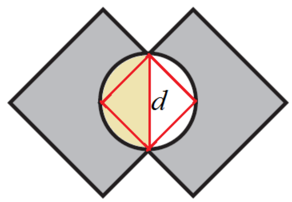


(A) $16-4\pi$ (B) $16-2\pi$ (C) $28-4\pi$
(D) $28-2\pi$ (E) $32-2\pi$

Solution: D.

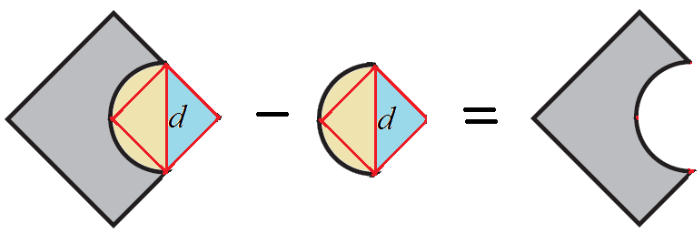
Method 1 (official solution):
The overlap of the two squares is a smaller square with side length 2, so the area of the region covered by the squares is 2(4 × 4) – (2 × 2) = 32 – 4 = 28.
The diameter of the circle has length $\sqrt{2^2+2^2} = \sqrt{8}$, the length of the diagonal of the smaller square. The shaded area created by removing the circle from the squares is $28 - \pi(\frac{\sqrt{8}}{2})^2 = 28-2\pi$.

Method 2 (our solution):
The overlap of the two squares is a smaller square with side length 2. The diameter of the circle has length $d = 2\sqrt{2}$ and the area of the semicircle is π.



The area of the triangle is $\frac{1}{2}\times\frac{d^2}{2}=\frac{8}{4}=2$.

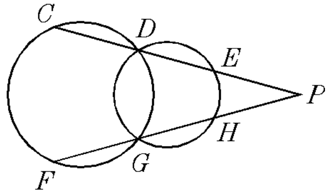
The shaded area shown in the figure below is $16-2-\pi-14-\pi$.



So the answer is $2\times(14-\pi)=28-2\pi$.

Example 13: In the given figure, $PE = 9$, $DE = 3$, $DC = 6$, and $PH = 8$. Find FG.

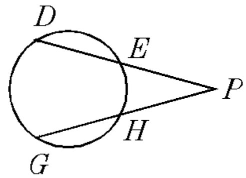
A. 16/3 B. 5/2 C. 11/2 D. 8/3 E. 5



Solution: B.

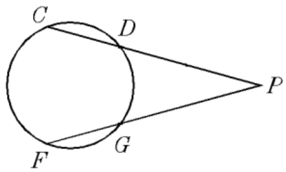
We separate the figure into two figures as shown.

By **Principle 11**, for the figure on the right:



$$PD \times PE = PG \times PH \Rightarrow (9+3) \times 9 = PG \times 8 \Rightarrow PG = \frac{27}{2}.$$

By **Principle 11**, for the figure on the right:

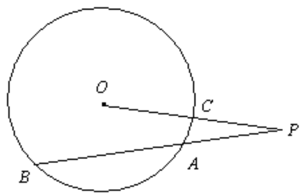


$$PC \times PD = PF \times PG \Rightarrow (6+12) \times 12 = (FG + \frac{27}{2}) \times \frac{27}{2}$$

$$\Rightarrow \quad FG = \frac{5}{2}.$$

Example 14: As shown in the figure, $PA = AB$, $PC = 2$, $PO = 5$, find PA.

A. $\sqrt{5}$ B. $2\sqrt{2}$. C. $2\sqrt{15}$. D. 3.5 E. $3\sqrt{2}$



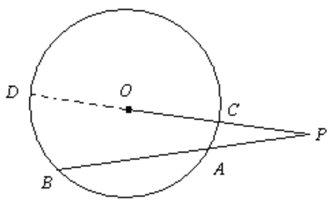
Solution: B.

Extend PO to D.

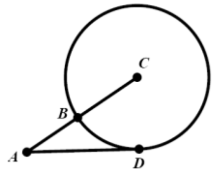
Since $PO = 5$, $PC = 2$, so $OC = 3$ and $OD = 3$, $PD = 8$.

$PD \times PC = PB \times PA$ or $8 \times 2 = (2PA) \times PA$, then

$PA^2 = 8 \Rightarrow PA = 2\sqrt{2}$.



Example 15: In the diagram, C is the center of the circle and AD is tangent to the circle at D. The line segment AC intersects the circle at B. If $AD = 10$ and $AB = 7$, find the radius of the circle. (2011 Georgia Southern University Math Contest).

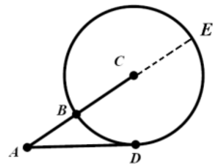


A. 51/14 B. 51/7 C. $2\sqrt{15}$. D. 7 E. 14

Solution: A.

Extend AC to meet the circle at E.

By the Power of Points formula:

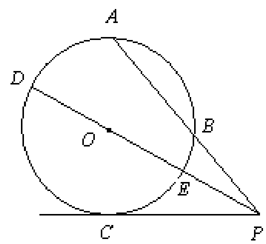


$$AD^2 = AE \times AB \qquad \Rightarrow 10^2 = (7 + BE) \times 7 \Rightarrow BE = \frac{51}{7}$$

The radius is 51/14, which is half of the diameter BE.

Example 16. PBA is the secant of the circle O. PC is tangent to O at C. PED passes through the center O. $\angle DPC = 45°$. $AB = BP = \sqrt{2}$.

Find the radius of the circle O.



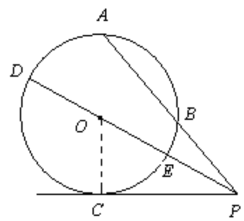
A. 2 B. 4 C. $\sqrt{3}$. D. π E. $\sqrt{2}$

Solution: A.

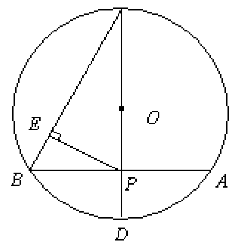
$$CP^2 = AP \times BP = (2\sqrt{2}) \times \sqrt{2} = 4$$

So $CP = 2$. Connecting OC. $OC \perp CP$, $\angle COP = \angle CPO = 45°$.

$CP = OC = 2$



Example 17. Point C is the middle point of arc ACB, CD is the diameter. $PE \perp CB$ at E. $BC = 10$, $CE : EB = 3 : 2$, find AB.



A. 10 B. 14 C. $7\sqrt{3}$. D. 12 E. $4\sqrt{10}$

Solution: E.

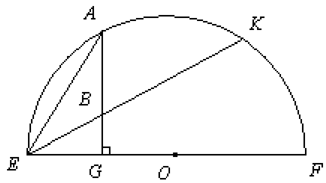
We are given that $\dfrac{CE}{EB}=\dfrac{3}{2}$. So $CE=\dfrac{3}{2}EB$.

$BC = CE+EB = \dfrac{3}{2}EB+EB = \dfrac{5}{2}EB$ and $\dfrac{5}{2}EB=10$, $EB = 4$; $CE = 6$.

$BP^2 = CB\times EB = 10\times 4$; $BP = 2\sqrt{10}$; $AB = 2\times BP = 4\sqrt{10}$.

Example 18. As shown in the figure is a semicircle and $AG\perp EF$. $EB = 4$ and $BK = 9$. Find AE.

A. 2 B. 4 C. 6 D. 8 E. $4\sqrt{2}$

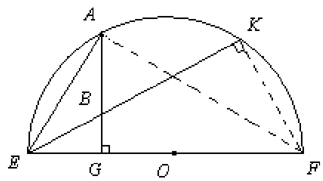


Solution: C.

Connect AF and KF.

We have $AE^2 = EF\times EG$

We also have $\Delta EBG \sim \Delta EFK$, $\dfrac{EF}{EB}=\dfrac{EK}{EG}$

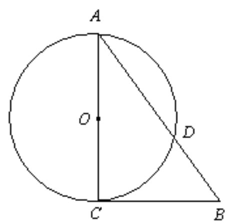


So $EF\times EG = BE\times EK$ or $AE^2 = EB\times EK = 4\times 9$.

$AE = 6$.

Example 19. ΔABC, $\angle C = 90°$, $AB = 10$, $BC = 2\sqrt{5}$. AC is the diameter of the circle O. AB meets the circle O at D. Find AD and CD.

A. 6 B. 4 C. $2\sqrt{3}$. D. 8 E. 14



Solution: B.

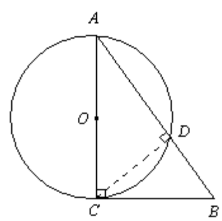
Connect CD. We get $\angle ADC = 90°$.

$\angle ACB = 90°$ (given).

Let $AD = x$, we get $BC^2 = AB\times DB$ or $(2\sqrt{5})^2 = 10\times DB$,

then $DB = 2$.

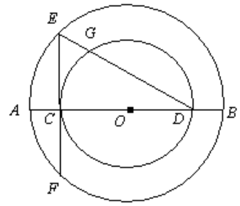
$AD = 8$, so $CD^2 = AD\times DB = 2\times 8 = 16$, then $CD = 4$.



Example 20. A small circle and a big circle are concentric. *EF* is tangent to the small circle at *C*. $AO = 6$, $CO = 4$. Find *EG*.

A. $\frac{10\sqrt{21}}{21}$ B. 2 C. $\sqrt{3}$. D. 4 E. $2\sqrt{2}$

Solution: A.



$AC = AO - CO = 6 - 4 = 2$.

$AC \times CB = EC \times CF = EC^2$.

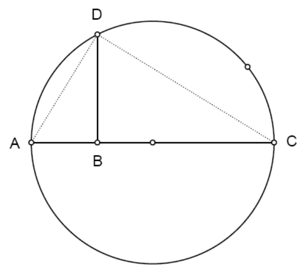
$EC^2 = 2 \times 10 = 20$. $CD = 8$. In the right triangle *ECD*:

$$ED = \sqrt{EC^2 + CD^2} = \sqrt{20+64} = 2\sqrt{2}$$

$$EC^2 = ED \times EG = EG \times 2\sqrt{21}\text{, so } EG = \frac{10\sqrt{21}}{21}.$$

Example 21. *AC* is a diameter of a circle in which *AD* is a chord; *B* is a point on *AC* such that $DB \perp AC$. If $AB = 9$, and $BC = 16$, how long is *DB*? (1999 NC Math Contest Geometry)

A. 10 B. 12 C. $10\sqrt{3}$. D. 14 E. $10\sqrt{2}$



Solution: B.

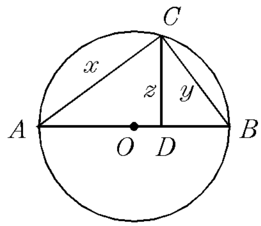
ΔADC is a right triangle since it is inscribed in a semicircle. If a perpendicular line is dropped from the right angle to the hypotenuse, then the square of its length is the product of the 2 segments it forms on the hypotenuse.

$$(DB)^2 = AB \times BC \qquad \Rightarrow \qquad DB = \sqrt{9 \times 16} = 12$$

PROBLEMS

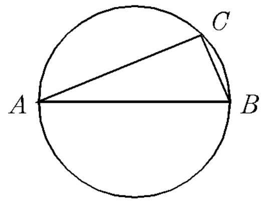
Problem 1. In the figure, AB is a diameter to circle O, $AO = 5$, $OD = 1$, and $CD \perp AB$. The length of $x + z$ is:

A. 5 B. $2(\sqrt{15}+\sqrt{6})$ C. 8 D. $\sqrt{72}$ E. $16\sqrt{2}$



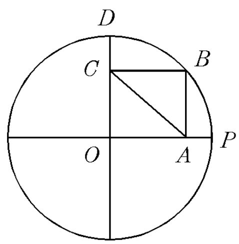
Problem 2. In the figure, AB is a diameter of the circle, $AB = 13$, and $AC = 12$ Then BC equals:

A. 3 B. 4 C. 5 D. $\sqrt{24}$ E. $\sqrt{26}$



Problem 3. In the figure, O is the center of the circle. Quadrilateral $OCBA$ is a rectangle, $OA = 5$, and $AP = 1$. Then CA equals:

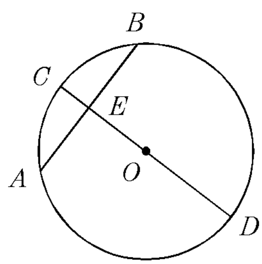
A. 6 B. $\sqrt{26}$ C. 6 5 D. 5.8 E. $\sqrt{30}$



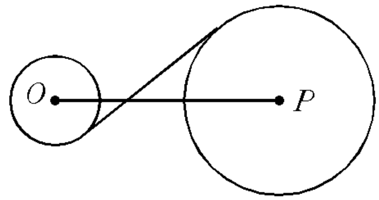
Problem 4. In circle O, chord AB is perpendicular to diameter CD at E. Which of the following is not necessarily true?

A. $OA = OB$ B. $AE = EB$ C. $AD = DB$

D. $CE = EO$ E. arc AC = arc CB

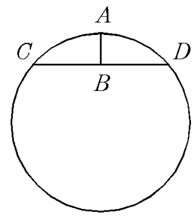


Problem 5. The centers of two circles, O and P, are 16 inches apart. The larger circle has a radius of 5 inches, and the smaller circle a radius of 3 inches. The length of the common internal tangent is:



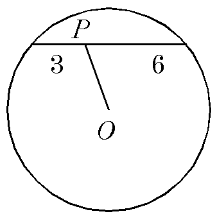
A. $10\sqrt{2}$ inches B. $6\sqrt{2}$ inches
C. $8\sqrt{5}$ D. $8\sqrt{3}$ inches E. none of the above

Problem 6. In the figure, AB is 4 and AB is perpendicular to CD at the midpoint of CD. If CD is 24, find the radius of the circle.



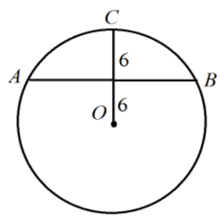
A. $4\sqrt{10}$ B. 20 C. 16 D. 24 E. $4\sqrt{3}$

Problem 7. In circle O, a point P divides a chord into segments 3 inches and 6 inches long. If the diameter of the circle is 11 inches, what is the distance from P to the center O ?



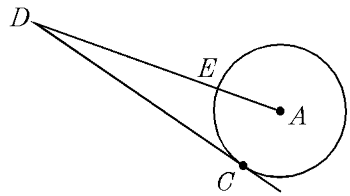
A. 3.5 inches B. 3.2 inches C. 3.8 inches D. 3.0 inches
E. 2.8 inches

Problem 8. A chord which is the perpendicular bisector of a radius of length 12 in a circle has length:

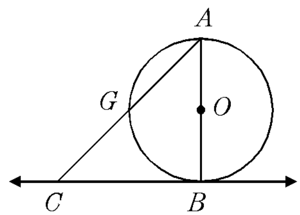


A. $3\sqrt{3}$ B. 27 C. $6\sqrt{3}$ D. $12\sqrt{3}$ E. none of the above

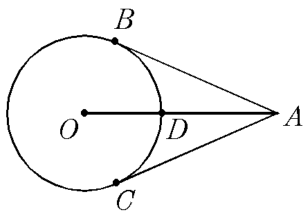
Problem 9. Given that $DE = 2$, $CD = 4$, A is the center of the circle, and CD is tangent to the circle, and the radius of the circle.
A. 2 B. 3 C. 4 D. 5 E. 6



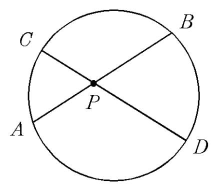
Problem 10. In the figure, AB is a diameter of circle O and CB is a tangent at B. If the radius of the circle is 6 and $AG = 8$, then CB is:
A. $4\sqrt{5}$ B. $6\sqrt{2}$ C. $6\sqrt{5}$ D. $2\sqrt{21}$ E. $8\sqrt{2}$.



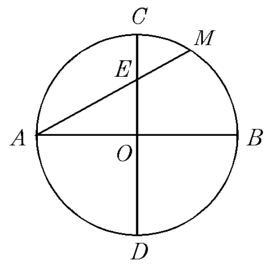
Problem 11. If AB and AC are tangents to the circle with center O, $\angle BAC = 60°$and $AB = 4$, then AD equals:
A. $\dfrac{\sqrt{3}}{3}$ B. $\sqrt{3}$ C. $\dfrac{2\sqrt{3}}{3}$ D. $\dfrac{4\sqrt{3}}{3}$ E. 3.



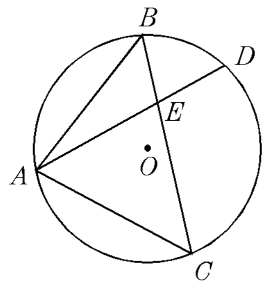
Problem 12. If AB and CD are any two chords of a circle which intersect at point P, which is the midpoint of AB, and $CP = 2$, and $PD = 18$, then AB equals:
A. 20 B. 18 C. $3\sqrt{5}$ D. 12 E. $5\sqrt{3}$.



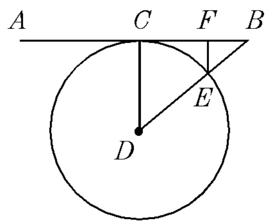
Problem 13. A circle with center O has perpendicular diameters, AB and CD. If AM is a chord intersecting CD at E, then $AE \times AM$ is:
A. $AO \times OB$ B. $AO \times AB$ C. $CE \times ED$ D. $CE \times ED$ E. $CO \times OE$



Problem 14. In a circle with center O, chord AB = chord AC. Chord AD cuts BC at E. If $AC = 12$ and $AE = 8$, then AD equals:
A. 14 B. 16 C. 18 D. 20 E. 21

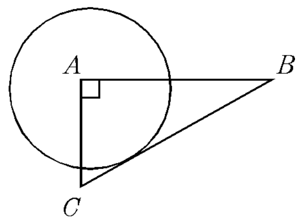


Problem 15. Line segment AB has length 16 inches and is tangent to the circle centered at D whose diameter is 12 inches. The point of tangency C is the midpoint of AB and FE is parallel to CD. Find the length of FB.
A. 16/3 B. 24/5 C. 12/5 D. 4 E. 16/5



Problem 16. In the figure shown, ABC is a right triangle and the circle centered at A is tangent to the hypotenuse BC. Find the radius of the circle if $AB = 2$ and $AC = 1$.

A. $\frac{2}{\sqrt{5}}$ B. $\sqrt{2}$ C. 1 D. $\frac{\sqrt{2}}{5}$ E. $\sqrt{5}$.

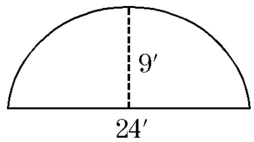


Problem 17. Two parallel chords on the same side of the center of a circle are 12 inches and 20 inches long and 2 inches apart. Find the radius of the circle

A. 8 in B. 15 in C. $2\sqrt{34}$ in D. $5\sqrt{13}$ 13 in E. $16\sqrt{2}$ in

Problem 18. A circular arch having width 24 feet and height 9 feet is to be constructed. What is the radius of the circle of which the arch is an arc?

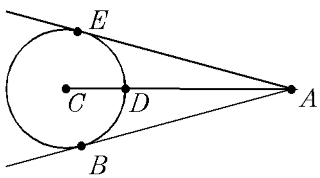
A. 10 B. 12.5 C. 13.5 D. 14 E. 16



Problem 19. In the figure the points A, D, and the center of the circle C are collinear, AB and AE are tangent to the circle. If $AB = b$ units and $AD = a$ units, what is the radius of the circle?

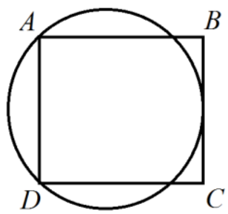
A. $\dfrac{b^2 - a^2}{2a}$ B. $\dfrac{b^2 + a^2}{a}$ C. $\dfrac{b^2 - a}{2}$

D. $a^2 - b^2$ E. $a^2 + b^2$



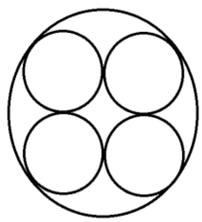
Problem 20. The length of each side of the square in the diagram is 8 ft. A circle is drawn through A and D tangent to BC. What is the radius of the circle?

A. 4 ft B. $4\sqrt{2}$ ft C. 5 ft D. $5\sqrt{2}$ ft E. 6 ft



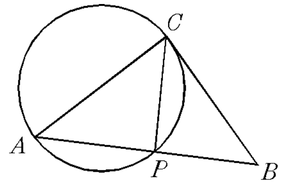
Problem 21. Four circles of radius r are mutually tangent inside a circle of radius one unit. The radius r is:

A. 1 B. 1/2 C. $\sqrt{2}$ - 1 D. 1/4 E. $\dfrac{\sqrt{5}}{5}$

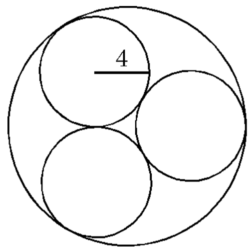


Problem 22. In the figure, AC is a diameter and BC is tangent at C. If $AB = 10$ and $AC = 8$, then the measure of PB equals:

A. 4.2 B. 6.4 C. 3 D. 3.6 E. none of these



Problem 23. Three circles each of radius 4 units are inscribed in a fourth circle in such a manner that the circles are mutually tangent to three other circles. The radius of the large circle is:

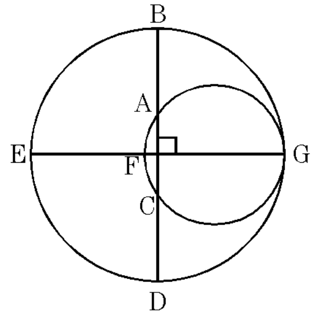


A. $\frac{2}{3}\sqrt{3}+4$ B. $\frac{4}{3}\sqrt{3}+4$ C. $\frac{32}{3}\sqrt{3}$

D. $\frac{16}{3}\sqrt{3}+8$ E. $\frac{8}{3}\sqrt{3}+4$

Problem 24. Two circles are internally tangent at point G as indicated in the diagram. If $AB = 6$, $EF = 8$, and $CD = 6$, find the sum of the radii of the two circles. BD passes through the center of the larger circle.

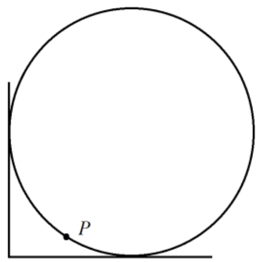
A. 11 B. 12 C. 13 D. 14 E. 15



Problem 25. The circular table in the diagram is pushed against two perpendicular walls. The point P on the edge of the table is a distance 2 dm from one wall and a distance of 9 dm from the other wall as shown in the figure. What is the radius of the table?

A. 11 dm B. $9\sqrt{2}$ dm C. $9\sqrt{3}$ dm

D. 17 dm E. 18 dm

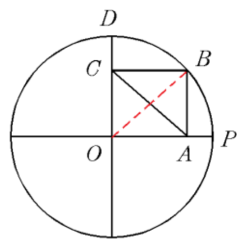


SOLUTIONS

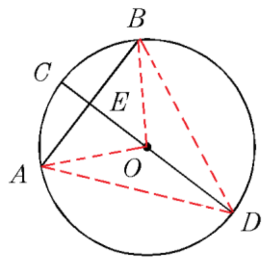
Problem 1. Solution: B.
$AO = 5$, $OD = 1$. So $DB = 4$.
By **Principle 13,** $CD^2 = AD \times DB = 6 \times 4 \quad \Rightarrow \quad CD = \sqrt{6 \times 4} = 2\sqrt{6}$.
Applying Pythagorean Theorem to triangle ACD:
$AC^2 = CD^2 + AD^2 = 6^2 + (2\sqrt{6})^2 = 60 \quad \Rightarrow \quad AC = 2\sqrt{15}$.
So the answer is $2(\sqrt{15} + \sqrt{6})$.

Problem 2. Solution: C.
Since AB is a diameter of the circle, triangle ABC is a 5-12-13 right triangle. So $BC = 5$.

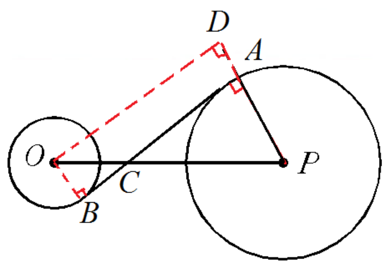
Problem 3. Solution: A.
$OP = OA + AP = 5 + 1 = 6$. Connect OB. We see that $CA = OB = OP = 6$.



Problem 4. Solution: D.
Since $AB \perp CD$, and CD is the diameter, CD bisects AB. So $AE = EB$, arc AC = arc CB, $OA = OB$, and $AD = DB$. So the answer is D.



Problem 5. Solution: D.
Connect OB, PA. Extend PA to D with $DA = OB$. Connect OD. So $OD \perp PD$. Applying Pythagorean Theorem to triangle ODP:

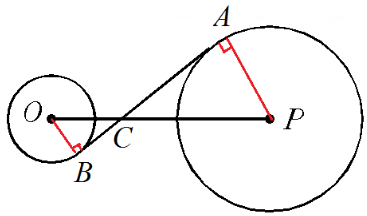


$OD^2 + DP^2 = OP^2 \quad \Rightarrow \quad OD^2 + (5+3)^2 = 16^2 \quad \Rightarrow OD = 8\sqrt{3}$.

OBAD is a rectangle so $AB = 8\sqrt{3}$.

Method 2:

Triangle *OBC* is similar to triangle *PAC*.



$\frac{OB}{AP} = \frac{OC}{CP} \quad \Rightarrow \quad \frac{3}{5} = \frac{16-CP}{CP} \Rightarrow$

$CP = 10$.

So $OC = 16 - 10 = 6$.

Applying Pythagorean Theorem to triangle *OBC*:

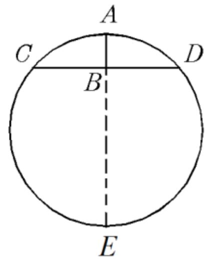
$OB^2 + BC^2 = OC^2 \quad \Rightarrow \quad 6^2 - 3^2 = BC^2 \Rightarrow BC = 3\sqrt{3}$.

Applying Pythagorean Theorem to triangle *ACP*:

$AP^2 + AC^2 = CP^2 \quad \Rightarrow \quad 5^2 + AC^2 = 10^2 \quad \Rightarrow AC = 5\sqrt{3}$.

$AB = 3\sqrt{3} + 5\sqrt{3} = 8\sqrt{3}$.

Problem 6. Solution: B.



Since *AB* is perpendicular to *CD* at the midpoint of *CD,*

$CB = BD = 12$.

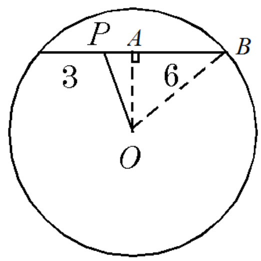
Extend *AB* to meet the circle at *E*.

By **Principle 10**, $AB \times BE = CB \times BD \quad \Rightarrow$

$4 \times BE = 12 \times 12 \Rightarrow \quad BE = 36$

The diameter is 36 + 4 = 40. So the radius is 20.

Problem 7. Solution: A.



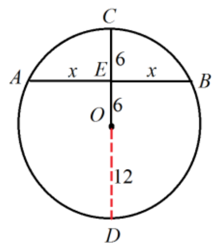
Connect *OB*. Draw $AO \perp CD$ and meets *PB* at *A*. By **Principle 2,** $AB = \frac{1}{2}(3 + 6) = 4.5$. $PA = 4.5 - 3 = 1.5$,

Applying Pythagorean Theorem to right triangles *ABO* and *APO*:

$BO^2 - AB^2 = PO^2 - PA^2$

$\Rightarrow \quad 5.5^2 - 4.5^2 = PO^2 - 1.5^2 \quad \Rightarrow PO = 3.5$.

Problem 8. Solution: D.



We draw the figure.

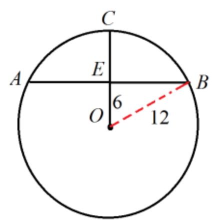
Since AB bisects CO, by **Principle 2,** CO is also bisects AB.

Method 1:

Extend CO to meet the circle at D.

By **Principle 10**, $x \times x = 6 \times 18 \quad \Rightarrow \quad x = 6\sqrt{3} \Rightarrow \quad AB = 12\sqrt{3}$.

Method 2:



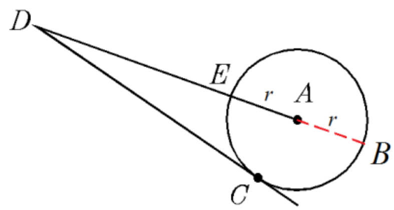
Connect OB.

Applying Pythagorean Theorem to right triangles EBO:

$EB^2 = OB^2 - EO^2 = 12^2 - 6^2 = 108 \quad \Rightarrow \quad EB = 6\sqrt{3}$

$AB = 12\sqrt{3}$.

Problem 9. Solution: B.



Let the radius be r. Extend DA to B as shown in the figure.

By **Principle 12**, $CD^2 = DB \times DE$

$\Rightarrow \quad 4^2 = (2 + 2r) \times 2 \Rightarrow \quad r = 3$.

Problem 10. Solution: C.

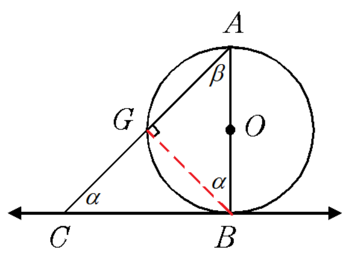
Connect BG. We know that $\angle AGB = 90°$.

Applying Pythagorean Theorem to right triangles AGB:

$GB^2 = AB^2 - AG^2 = 12^2 - 8^2 = 80 \quad \Rightarrow GB = 4\sqrt{5}$

We also see that $\Delta AGB \sim \Delta ABC$.

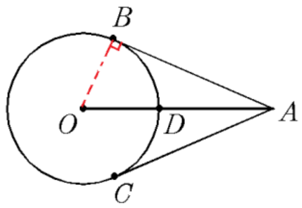
$\frac{AB}{AG} = \frac{CB}{GB} \quad \Rightarrow \quad \frac{12}{8} = \frac{CB}{4\sqrt{5}} \Rightarrow \quad CB = 6\sqrt{5}$



Problem 11. Solution: D.

Connect OB. By **Principle 5**, $OB \perp AB$. By **Principle 8**, $\angle OAB = \angle OAC = 30°$.

Triangle ABO is a 30-60-90 right triangle. So the ratio of the sides is 1: $\sqrt{3}$: 2.



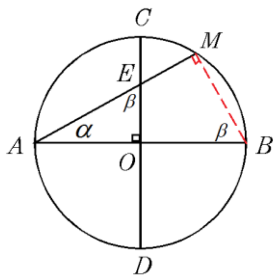
So $OB = \dfrac{4\sqrt{3}}{3}$, $OA = \dfrac{8\sqrt{3}}{3}$.

Thus $AD = OA - OD = OA - OB = \dfrac{8\sqrt{3}}{3} - \dfrac{4\sqrt{3}}{3} = \dfrac{4\sqrt{3}}{3}$.

Problem 12. Solution: D.

By **Principle 10**, $AP \times PD = CP \times PD \Rightarrow \dfrac{1}{2}AB \times \dfrac{1}{2}AB = 2 \times 18 \Rightarrow AB = 12$.

Problem 13. Solution: B.
Connect BM. $\angle AMB = 90°$.
We also see that $\Delta AMB \sim \Delta AOE$.

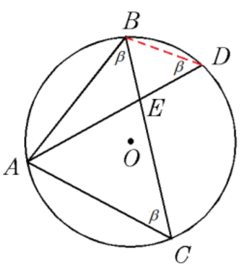


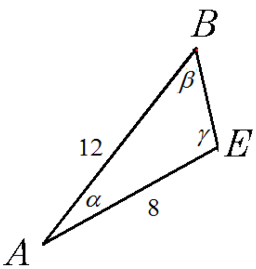
$$\frac{AM}{AO} = \frac{AB}{AE} \quad \Rightarrow \quad AM \times AE = AO \times AB .$$

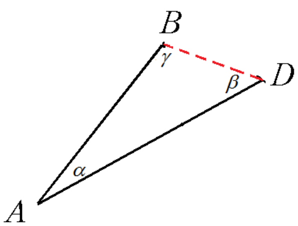
So the answer is B.

Problem 14. Solution: C.
Since $AB = AC$, $\angle ABC = \angle ACB = \beta$. Connect BD. Since both $\angle ADB$ and $\angle ACB$ face the same arc, $\angle ADB = \angle ACB = \beta$. We know that , $\angle BAD = \angle BAD$. Thus , $\Delta ADB \sim \Delta ABE$ (as shown in the figure). Therefore $\dfrac{AB}{AD} = \dfrac{AE}{AB} = \dfrac{BE}{BD}$

$$\Rightarrow \frac{12}{AD} = \frac{8}{12} \quad \Rightarrow AD = 18 .$$







Problem 15. Solution: E.

Method 1:

Since C is the midpoint of AB, $BC = 16/2 = 8$. We also know that $CD = 6$. So triangle BCD is a 6-8-10 right triangle. Then $BE = 10 – 6 = 4$.

We know that and FE is parallel to CD. Thus $\Delta BCD \sim \Delta BFE$.

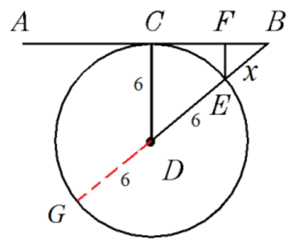
$$\frac{DE}{CF} = \frac{BE}{FB} \quad \Rightarrow \quad \frac{6}{8-FB} = \frac{4}{FB} = \frac{10}{8} = \frac{5}{4} \quad \Rightarrow \quad FB = \frac{16}{5}.$$

Method 2:

Since C is the midpoint of AB, $BC = 16/2 = 8$. Extend BD to meet the circle at G.

By **Principle 12**, $CB^2 = BG \times x \quad \Rightarrow \quad 8^2 = (12 + x) \times x \quad \Rightarrow$

$x = 4$.



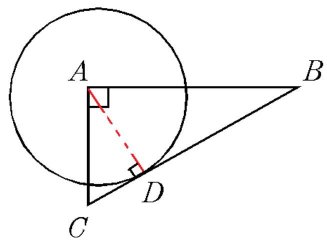
We know that and FE is parallel to CD. Thus $\Delta BCD \sim \Delta BFE$.

$$\frac{BC}{FB} = \frac{BD}{BE} \quad \Rightarrow \quad \frac{8}{FB} = \frac{10}{4} \quad \Rightarrow \quad FB = \frac{16}{5}.$$

Problem 16. Solution: A.

Applying the Pythagorean Theorem to triangle ABC:

$BC = \sqrt{AB^2 + AC^2} = \sqrt{12^2 + 1^2} = \sqrt{5}$.



Connect A to the tangent point D. Then $AD \perp BC$.

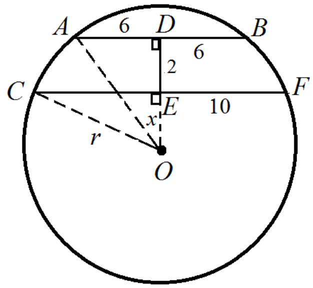
So $AC^2 = BC \times CD \quad \Rightarrow 1^2 = \sqrt{5} \times CD \Rightarrow CD = \frac{1}{\sqrt{5}}$.

Applying the Pythagorean Theorem to triangle ACD:

$$AD = \sqrt{AC^2 - CD^2} = \sqrt{1 - (\frac{1}{\sqrt{5}})^2} = \frac{2}{\sqrt{5}}.$$

This is the radius so the answer is A.

Problem 17. Solution: D.



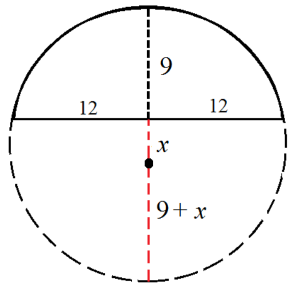
Let r be the radius of the circle. AB = 10. AD = 6. CF = 20. CE = 10. DE = 2. OE = x.
Applying Pythagorean Theorem to right triangles ADO and CEO:

$$AD^2 + DO^2 = CE^2 + OE^2 \quad \Rightarrow$$

$$6^2 + (2+x)^2 = 10^2 + x^2 \quad \Rightarrow x = 15.$$

$$r = \sqrt{CE^2 + OE^2} = \sqrt{10^2 + 15^2} = 5\sqrt{13}.$$

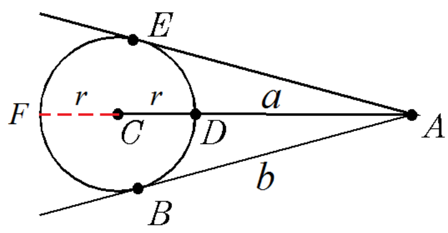
Problem 18. Solution: B.



By **Principle 10**, $9 \times (x + 9 + x) = 12 \times 12 \Rightarrow \quad x = \dfrac{7}{2}$.

The radius is $r = 9 + \dfrac{7}{2} = 12.5$.

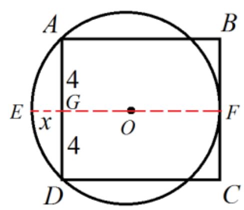
Problem 19. Solution: A.



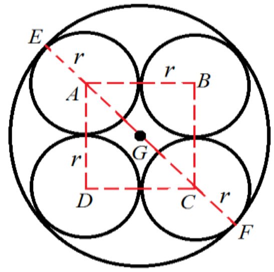
Extend AD to meet the circle at F. Let r be the radius of the circle.
By **Principle 12**, $AB^2 = AF \times AD \quad \Rightarrow$

$$b^2 = (a + 2r) \times a \Rightarrow \quad r = \frac{b^2 - a^2}{2a}.$$

Problem 20. Solution: C.



Draw EF, the diameter from the tangent point. Let $EG = x$.
We see that GF = 8.
By **Principle 10**, $x \times 6 = 4 \times 4 \quad \Rightarrow \quad x = 2$.
So EF = EG + GF = 2 + 8 = 10. So the radius is 10/2 = 5.

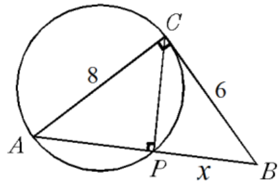


Problem 21. Solution: C.

We draw the diameter EF as shown in the figure.

Applying Pythagorean Theorem to right triangles *ADC*:

$(2r)^2 + (2r)^2 = (2 - 2r)^2 \qquad \Rightarrow \qquad r = \sqrt{2} - 1$.



Problem 22. Solution: D.

We see that $\angle ACB = \angle APC = 90°$.

ΔABC is 6-8-10 right triangle. So $AB = 10$.

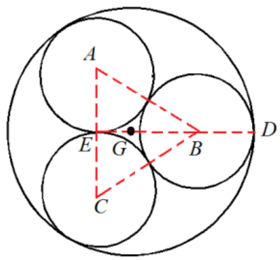
By **Principle 12**, $BC^2 = AB \times PB \quad \Rightarrow 6^2 = 10 \times x$

$x = 3.6$.

Problem 23. Solution: E.

We connect the centers of three small circles and we know that triangle *ABC* is an equilateral triangle with the side of 8. So the height *BE* is

$\frac{8}{2} \times \sqrt{3} = 4\sqrt{3}$.

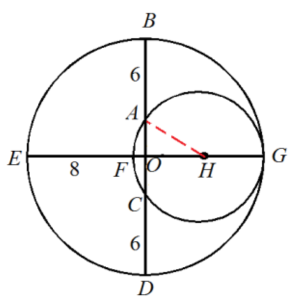


The centriod is *G* and $GB = \frac{2}{3} \times BE = \frac{2}{3} \times 4\sqrt{3} = \frac{8}{3}\sqrt{3}$. The radius is $GB + BD = \frac{8}{3}\sqrt{3} + 4$

Problem 24. Solution: D.

Let *O* be the center of the large circle and *H* be the center of the small circle.

Let r be the radius of the small circle and *r* be the radius of the large circle. Let $AO = m = CO$.



We see that $R = \frac{8 + 2r}{2} = r + 4$.

$OH = OG - GH = R - r = 4$.

$AO + OC = 2m = 2R - 12 = 2(r + 4) - 12 = 2r - 4$. So $m = r - 2$.

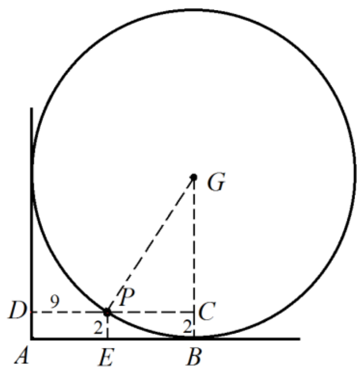
Applying Pythagorean Theorem to triangle AOH: $AH^2 = AO^2 + OH^2$

$\Rightarrow r^2 = (r-2)^2 + 4^2 \quad \Rightarrow \quad r = 5$ and $R = 5 + 4 = 9$.

So the answer is 5 + 9 = 14.

Problem 25. Solution: D.

Let the center of the circle be G, radius be r. Let the tangent point be B. Connect GB and GP. Draw DC // AB and meets GB at C. We label each line segments as shown.



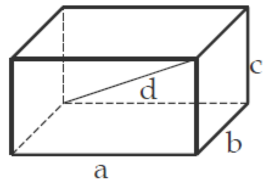
Applying Pythagorean Theorem to triangle PCG:

$PG^2 = PC^2 + GC^2 \quad \Rightarrow$

$r^2 = (r-9)^2 + (r-2)^2 \quad \Rightarrow \quad r = 17$ or 5 (ignored).

1. BASIC KNOWLEDGE

1.1. Rectangular solid



Volume, *V*

$V = LWH$

L: length of the base of the solid
W: width of the base of the solid
H: height of the solid
d: space diagonal of the solid

Surface area, *S*

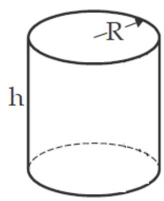
$S = 2(LW + WH + HL)$

Note: For a cube, $L = W = H$.

1.2. Cylinder

Volume, *V*

$V = \pi R^2 h = \frac{1}{4}\pi D^2 h$

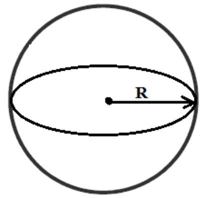


R: the radius of the circular base of the cylinder
D: the diameter of the circular base of the cylinder
h = the height of the cylinder

Surface area, *S*

$S = 2\pi R^2 + 2\pi Rh$

1.3. Sphere



Volume, *V*

$$V = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi \times D^3$$

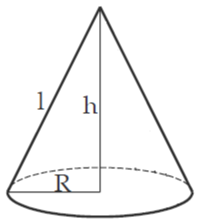
R: the radius of the circular base of the sphere
D: the diameter of the circular base of the sphere

Surface area, *S*

$$S = 4\pi R^2 = \pi D^2$$

Note: S is also the area of the biggest circle achieved when a sphere is sliced into two pieces.

1.4. Cone



Volume of a cone, *V*

$$V = \frac{1}{3}\pi \times R^2 h = \frac{1}{12}\pi \times D^2 h$$

R: the radius of the right cone
D: the diameter of the cone
h: the perpendicular height of the cone
l: the slant height of a right cone

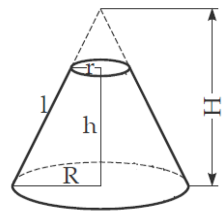
$$l = \sqrt{R^2 + h^2}$$

Note: The volume of the cone is $\frac{1}{3}$ of the volume of the cylinder of the same radius and height.

Surface Area of a cone, *S*

$$S = \pi Rl + \pi R^2$$

Volume of the frustum of a right circular cone, *V*



$$V = \frac{1}{3}\pi\, h(R^2 + r^2 + Rr)$$

R and r represent the radius of the bases.

1.5. Pyramid

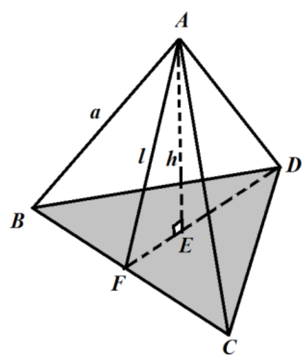
Volume of a Rectangular Pyramid, *V*

A pyramid is a solid figure with a polygonal base (in our case a rectangle) and triangular faces that meet at a common point (the apex).

$$V = \frac{1}{3}abc\,.$$

a: the length of the base of the pyramid.
b: the width of the base of the pyramid.
c: the perpendicular height of the pyramid.

Note: The volume of the pyramid is $\frac{1}{3}$ of the volume of the rectangular solid of the same length, width, and height.



Volume of a Triangular Pyramid, *V*

$$V = \frac{1}{12}\sqrt{2}a^3$$

a: the edge length.

l: the slant height (the height of the equilateral triangle ABC)

$l = \frac{1}{2}a\sqrt{3}$

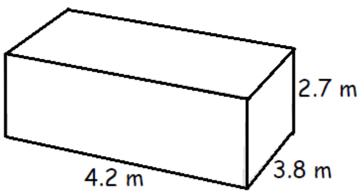
h: the height of the regular tetrahedron. $h = \frac{1}{3}a\sqrt{6}$

The surface area of the tetrahedron is simply four times the area of a single equilateral triangle face: $S = \sqrt{3}a^2$

2. EXAMPLES

Example 1. Find the surface area of the figure below. Express your answer to the nearest integer.

A. 94. B. 86. C. 75. D. 70. E. 64.



Solution: C.

This figure is a box (officially called a rectangular prism). We are given the lengths of each of the length, width, and height of the box, thus we only need to plug the values into the formula.

$S = 2(LW + WH + HL) = 2(4.2 \times 3.8 + 4.2 \times 2.7 + 3.8 \times 2.7) = 75.12 = 75$ square meters.

Example 2. The number of centimeters in the length, width and height of a rectangular carton are consecutive integers. Find the smallest 6-digit number that could represent the number of cubic centimeters in the volume)

A. 103,776. B. 103,786. C. 130,776. D. 103,677. E. 103,777.

Solution: A.

Let the smallest value of the length, width and height be $a - 1$. Since the numbers are consecutive integers, then the other two dimensions are a and $a + 1$.

$V = (a - 1) \times a \times (a + 1) = a^3 - a$

We are seeking for a 6-digit number (a^3) that is just over 100000.
$46^3 = 97336$ and $47^3 = 103,823$.

So $a = 47$. $V = a^3 - a = 103823 - 47 = 103,776$.

Example 3. The sum of the measures of the edges of a cube is 48 cm. What is the volume of the cube in cubic centimeters?

A. 125. B. 216. C. 343. D. 27. E. 64.

Solution: E.
Let the measure of the edges of the cube be a. There are 12 edges for a cube.
$12a = 48 \Rightarrow \quad a = 4$.
$V = a \times a \times a = a^3 = 4^3 = 64$ (cm^3).

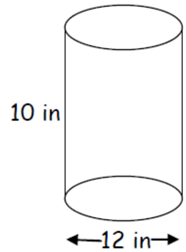
Example 4. The side, front, and bottom faces of a rectangular solid have areas of 32, 24, and 48 square units respectively. What is the number of cubic units in the volume of the solid?

A. 194. B. 192. C. 175. D. 170. E. 164.

Solution: B.
Let the dimensions of the solid be a, b, and c.
$a \times b = 32$ (1)
$b \times c = 24$ (2)
$c \times a = 48$ (3)
(1) × (2) × (3):
$(a \times b \times c)^2 = 32 \times 24 \times 48 \quad \Rightarrow \quad a \times b \times c = \sqrt{32 \times 24 \times 48} = 192$.

Example 5. Find the volume and surface area of the figure below. Express the sum in terms of π.

A. 360π B. 380π C. 300π D. 340π E. 552π



Solution: E.

The figure is a cylinder, with the diameter of its circular base as 12 inches, and height of the cylinder as 10 inches.

To calculate the volume and surface area, we simply need to plug the values into the formulas.

Surface Area: $S = 2(\pi r^2) + 2\pi rh = 2(\pi \cdot 6^2) + 2\pi(6)(10) = 72\pi + 120\pi = 192\pi$ square inches.

Volume: $V = \frac{1}{4}\pi D^2 h = \frac{\pi}{4} \times 12^2 \times 10 = 360\pi$. $360\pi + 192\pi = 552\pi$

Example 6. If the radius of a cylinder is doubled and its altitude is cut in half, what is the ratio of the volume of the original cylinder to the volume of the altered cylinder? Express your answer in the form *a* / *b*.

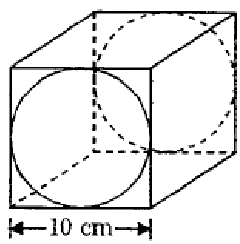
A. 1/4. B. 1/2. C. 1/5. D. 1/7. E. 1/6.

Solution: B.

$$\frac{a}{b} = \frac{V_1}{V_2} = \frac{\pi R_1^2 h_1}{\pi R_2^2 h_2} = \frac{R_1^2 h_1}{R_2^2 h_2} = \left(\frac{R_1}{R_2}\right)^2 \times \left(\frac{h_1}{h_2}\right) = \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{\frac{1}{2}}\right) = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2}.$$

Example 7. What is the remaining volume in cubic centimeters if a cylinder with a radius of 5 cm is removed from a cube whose sides measure 10 cm? Express your answer in terms of π.

A. 1000 – 250π. B. 1000 – 230π. C. 1000 – 300π.
D. 1000. E. 550π.



Solution: A.

The volume of the cube: $a^3 = 10^3 = 1000$.

The volume of the cylinder: $\frac{1}{4}\pi D^2 h = \frac{1}{4}\pi \times 10^2 \times 10 = 250\pi$.

The answer: 1000 – 250π.

Example 8. The radius of a right circular cylinder is decreased by 20% and its height is increased by 25%. What is the absolute value of the percent change in the volume of the cylinder?

A. 19%. B. 12%. C. 17%5. D. 20%. E. 16%.

Solution: D.

The volume of the original cylinder: $V_1 = \pi R_1^{\ 2} h_1$

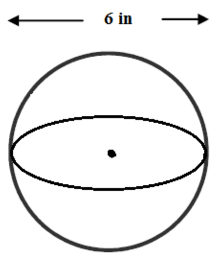
The volume of the new cylinder: $V_2 = \pi R_2^{\ 2} h_2 = \pi (0.8R)^2 \times (1.25h) = 0.8\pi R^2 h$

The absolute value of the percent change in the volume of the cylinder:

$$\left|\frac{V_1 - V_2}{V_1}\right| = \left|\frac{\pi R_1^{\ 2} h_1 - 0.8\pi R^2 h}{\pi R_1^{\ 2} h_{12}}\right| = \frac{0.2\pi R_1^{\ 2} h_1}{\pi R_1^{\ 2} h_1} = 0.2 = 20\%$$

Example 9. Find the sum of the numerical values of the volume and surface area of the figure below.

A. 74π B. 76π C. 60π D. 68π E. 72π



Solution: E.

The figure is a sphere. We are given that the diameter of the sphere is
6 inches, so we can just plug this number into the formulas to calculate the volume and surface area.

Volume: $V = \frac{1}{6}\pi \times D^3 = \frac{1}{6}\pi \times 6^3 = 36\pi$ cubic inches.

Surface Area: $S = \pi D^2 = 36\pi$ square inches.

The answer is 36π + 36π = 72π.

Example 10. Before cooking, how many spherical meatballs of radius 1 can you make from 1 spherical meatball of radius 3?

A. 37 B. 38 C. 30 D. 27 E. 20

Solution: D.

Let V_1 be the volume of the big sphere and V_2 be the volume of the smaller sphere.

$$V_1 = \frac{4}{3}\pi R_1^3$$

$$V_2 = \frac{4}{3}\pi R_2^3$$

$$\frac{V_1}{V_2} = \frac{\frac{4}{3}\pi R_1^3}{\frac{4}{3}\pi R_2^3} = (\frac{R_1}{R_2})^3 = (\frac{3}{1})^3 = 27$$

We can make 27 spherical meatballs of radius 1 from 1 spherical meatball of radius 3.

Example 11. A spherical drop of oil with a radius of 2 mm is dropped onto the surface of a pool of water. If the oil spreads out uniformly into a circular region of radius 8 mm, find the number of millimeters in the depth of the oil dispersion.

A. $\frac{1}{6}$. B. $\frac{3}{32}$. C. $\frac{1}{5}$. D. $\frac{1}{7}$. E. $\frac{1}{9}$.

Solution: A.

Let V_1 be the volume of the spherical drop and V_2 be the volume of the circular region containing the oil.

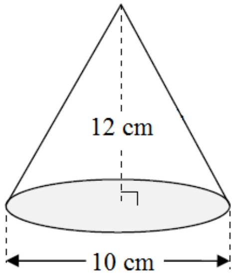
$$V_1 = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi \times 2^3 = \frac{32}{3}\pi$$

$$V_2 = \pi R_2^2 h = 64\pi h$$

Since $V_1 = V_2$, $64\pi h = \frac{32}{3}\pi \quad \Rightarrow \qquad h = \frac{1}{6}$

The number of millimeters in the depth of the oil dispersion is 1/6.

Example 12. Find the volume of the figure below.
A. 106π B. 107π C. 108π D. 120π E. 100π



Solution: E.
The height is given as 12 centimeters and the diameter is 10 centimeters.

We can plug these values into the volume of a cone formula.
$V = \frac{1}{12}\pi \times D^2 h = \frac{1}{12}\pi \times 10^2 \times 12 = 100\pi$ cubic centimeters.

Example 13. The volume of a cylinder is 54π cm^3. Find the number of cubic centimeters in the volume of a cone with the same radius and the same height.
A. 16π B. 17π C. 18π D. 20π E. 22π

Solution: C.
Let V be the volume of the cone.
Since the volume of a cylinder is 54π cm^3, we have: $54\pi = \pi R^2 h \Rightarrow$
$R^2 h = 54$
The volume of the cone: $V = \frac{1}{3}\pi \times R^2 h = \frac{1}{3}\pi \times 54 = 18\pi$.

Example 14. What is the surface area, in square centimeters, of a circular cone whose base has a circumference of 24π cm and whose height is 5 cm?
A. 500π B. 400π C. 300π D. 200π E. 250π

Solution: C.
The radius of the base is $24\pi = 2\pi R \qquad \Rightarrow \qquad R = 12$

$l = \sqrt{R^2 + h^2} = \sqrt{12^2 + 5^2} = 13$

$S = \pi Rl + \pi R^2 = 12 \times 13\pi + \pi \times 12^2 = 300\pi$.

Example 15. What is the surface area, in square centimeters, of a circular cone whose base has a radius of 9 cm and whose height is 12 cm?

A. 274π B. 246π C. 216π D. 268π E. 272π

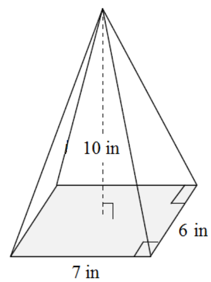
Solution: C.

$l = \sqrt{R^2 + h^2} = \sqrt{12^2 + 9^2} = 15$

$S = \pi Rl + \pi R^2 = 9 \times 15\pi + \pi \times 9^2 = 216\pi$.

Example 16. Find the volume of the figure below.

A. 174 B. 176 C. 160 D. 150. E. 140.



Solution: E.

The base of this figure is a rectangle and the sides of the figure are triangles, thus this figure is a rectangular pyramid. The height (perpendicular height) is 10 inches. The length of the base is 7inches, and the width of the base is 5 inches.

$V = \frac{1}{3}abc = \frac{1}{3} \times 7 \times 6 \times 10 = 140$ cubic inches.

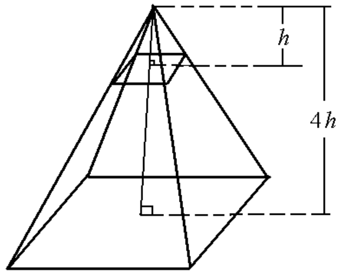
Example 17. What is the volume, in cubic centimeters, of a pyramid with a square base whose edge measures 6 cm and whose height is 8 cm?

A. 74 B. 76 C. 86 D. 96. E. 98.

Solution: D.

$V = \frac{1}{3}abc = \frac{1}{3} \times 6 \times 6 \times 8 = 96$ cubic inches.

Example 18. A square pyramid has a base edge of 32 inches and an altitude of 1 foot. A square pyramid whose altitude is one-fourth of the original altitude is cut away at the vertex. The volume of the remaining frustum is what fractional part of the volume of the original pyramid? Express your answer as a common fraction.



A. $\frac{63}{64}$. B. $\frac{31}{32}$. C. $\frac{15}{16}$. D. $\frac{3}{4}$. E. $\frac{1}{64}$.

Solution: A.

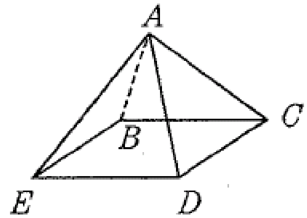
Let V_1 be the volume of the original pyramid and V_2 be the volume of the smaller pyramid that was sliced off.

Since the original pyramid is similar to the smaller pyramid that was sliced off, the ratio of their volumes is equal to the cube of the ratio of their heights**.**

We have $\frac{V_2}{V_1}=\left(\frac{h}{4h}\right)^3=\frac{1}{64}$.

The answer will be: $\frac{V_1-V_2}{V_1}=1-\frac{V_2}{V_1}=1-\frac{1}{64}=\frac{63}{64}$.

Example 19. Given right square pyramid *ABCDE* with square base *BCDE*. Perimeter of square *BCDE* is $32\sqrt{2}$. If $AB = AC = AD = AE = 10$, find the number of cubic units in the volume of the pyramid.



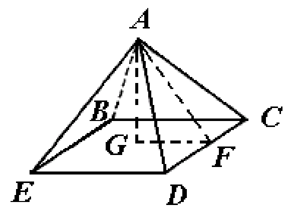
A. 274 B. 276 C. 286 D. 296. E. 256.

Solution: E.

Since the perimeter of square *BCDE* is $32\sqrt{2}$, the side length is then $8\sqrt{2}$ and $DF = 4\sqrt{2}$.

By Pythagorean Theorem, $AF=\sqrt{AD^2-DF^2}=\sqrt{100-32}=\sqrt{68}=2\sqrt{17}$, and

$AG = \sqrt{AF^2 - GF^2} = \sqrt{68-(4\sqrt{2})^2} = \sqrt{68-32} = \sqrt{36} = 6$



The volume of the pyramid

$V = \frac{1}{3}abc = \frac{1}{3} \times 8\sqrt{2} \times 8\sqrt{2} \times 6 = 256$

Example 20. If the volume of a tetrahedron is doubled without changing its shape, by what factor is the surface area increased?

A. $\sqrt[3]{2}$ B. $\sqrt[3]{4}$ C. 2 D. $\sqrt{3}$ E. 4

Solution: B.

Let V_1 be the volume of the new tetrahedron and V_2 be the volume of the original tetrahedron.

$$V_2 = \frac{1}{12}\sqrt{2}a_2^{\ 3} \qquad (1)$$

$$S_2 = \sqrt{3}a_2^{\ 2} \qquad (2)$$

$$V_1 = \frac{1}{12}\sqrt{2}a_1^{\ 3} \qquad (3)$$

$$S_1 = \sqrt{3}a_1^{\ 2} \qquad (4)$$

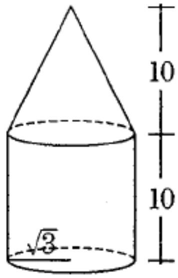
(1) ÷ (3): $\frac{V_1}{V_2} = \frac{\frac{1}{12}\sqrt{2}a_1^{\ 3}}{\frac{1}{12}\sqrt{2}a_2^{\ 3}} = (\frac{a_1}{a_2})^3 = 2 \quad \Rightarrow \quad \frac{a_1}{a_2} = 2^{\frac{1}{3}} \Rightarrow \quad (\frac{a_1}{a_2})^2 = 2^{\frac{2}{3}}$ (5)

(2) ÷ (4): $\frac{S_1}{S_2} = \frac{\sqrt{3}a_1^{\ 2}}{\sqrt{3}a_2^{\ 2}} = (\frac{a_1}{a_2})^2$ (6)

Substituting (5) into (6): $\frac{S_1}{S_2} = (\frac{a_1}{a_2})^2 = \sqrt[3]{4}$.

Example 21. Find the volume of this solid in terms of π.

A. 44π B. 46π C. 42π D. 40π E. 22π



Solution: D.
The volume of the solid consists of two parts: V_1, the volume of the cone and V_2, the volume of the cylinder.

$$V_1 = \frac{1}{3}\pi \times R^2 h = \frac{1}{3}\pi \times (\sqrt{3})^2 \times 10 = 10\pi$$

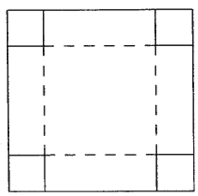
$$V_2 = \pi R^2 h = \pi (\sqrt{3})^2 \times 10 = 30\pi$$

The answer is $30\pi + 10\pi = 40\pi$.

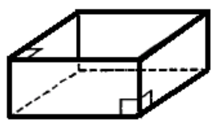
Example 22. Six congruent spherical solids, each of radius 1 centimeter, are packed tightly in a box with dimensions 2 centimeters by 4 centimeters by 6 centimeters. To the nearest cubic centimeter, what is the volume of the region inside the box not occupied by the solids? Express your answer in term of π.
A. $48 - 8\pi$ B. $50 - 8\pi$. C. $96 - 16\pi$. D. 4 0. E. 8π.

Solution: A.
The volume of the rectangular box: $V_1 = abc = 2 \times 4 \times 6 = 48$.
The volume of the 6 spheres: $V_2 = 6 \times \frac{4}{3}\pi R^3 = 8\pi$.
The volume of the region inside the box not occupied by the spheres:
$V_1 - V_2 = 48 - 8\pi$.

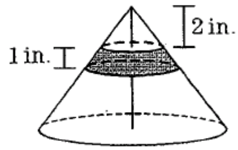
Example 23. A 10-inch by 10-inch piece of cardboard has a 2-inch by 2-inch square cut out of each corner. The sides are then folded up to form a box. What is the volume, in cubic inches, of the box?
A. 74 B. 76 C. 86 D. 96. E. 72.



Solution: E.
The following box can be formed with dimensions 6 inches ($10 - 2 - 2 = 6$), 6 inches ($10 - 2 - 2 = 6$), and 2 inches.
The volume is $6 \times 6 \times 2 = 72$.



Example 24. The right circular cone shown has a height of 8 inches and the radius of its base is 6 inches. A one-inch thick slice, parallel to the base and two inches from the vertex, as shown, is removed. Find the volume, in cubic inches, of the slice.

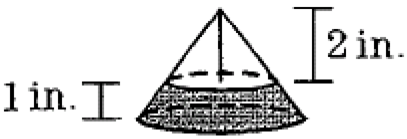


A. $\frac{57}{16}\pi$. B. $\frac{512}{19}\pi$. C. $\frac{512}{27}\pi$. D. $\frac{96}{7}\pi$. E. $\frac{17}{9}\pi$.

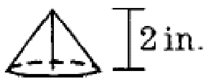
Solution: A.

Method 1:

Let the volume of the original cone be V, and the volumes of the following two cones be V_1 and V_2:



V_1



V_2

$$V = \frac{1}{3}\pi \times R^2 h = \frac{1}{3}\pi \times 6^2 \times 8 = 96\pi$$

Since the original cylinder is similar to the smaller cylinder that was sliced off, the ratio of their volumes is equal to the cube of the ratio of their heights.

$$\frac{V_1}{V} = \left(\frac{3}{8}\right)^3 = \frac{27}{512} \qquad \Rightarrow \qquad V_1 = \frac{27}{512}V$$

$$\frac{V_2}{V} = \left(\frac{2}{8}\right)^3 = \frac{1}{64} \qquad \Rightarrow \qquad V_2 = \frac{1}{64}V$$

The answer is $V_1 - V_2 = \frac{27}{512}V - \frac{1}{64}V = \frac{19}{512}V = \frac{19}{512} \times 96\pi = \frac{57}{16}\pi$.

Method 2:

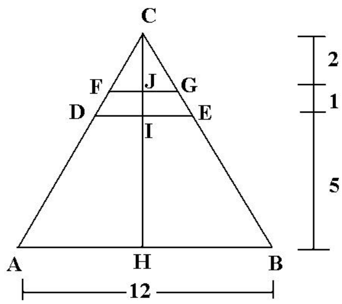
In order to calculate the volume of the slice, we must figure out the lengths of DE and FG (see figure below).

Using similar triangle ratios, we have:

$$\frac{AB}{DE} = \frac{HC}{IC} \quad \Rightarrow \quad \frac{12}{DE} = \frac{8}{3} \quad \Rightarrow \quad DE = \frac{9}{2} \Rightarrow R = \frac{9}{4}$$

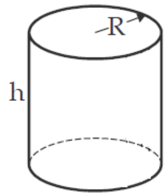
$$\frac{AB}{FG} = \frac{HC}{CJ} \quad \Rightarrow \quad \frac{12}{FG} = \frac{8}{2} \quad \Rightarrow \quad FG = 3 \quad \Rightarrow r = \frac{3}{2}$$

$$V = \frac{1}{3}\pi\, h(R^2 + r^2 + Rr) = \frac{1}{3}\pi \times 1 \times [(\frac{9}{4})^2 + (\frac{3}{2})^2 + \frac{9}{4} \times \frac{3}{2}] = \frac{57}{16}\pi\ .$$



3. PROBLEMS

Problem 1. A sealed can of soup has the shape of a right circular cylinder which has a radius of 3 cm and a height of 10 cm. The total area of the exterior surface is:

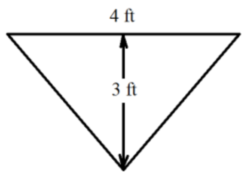


A. 60π cm^2 B. 78π cm^2 C. 69π cm^2 D. 129π cm^2
E. none of the above

Problem 2. The total surface area of the rectangular solid 3 ft by 4 ft by 5 ft is:
A. 60 square feet B. 12 square feet C. 300 square feet
D. 94 square feet E. 47 square feet

Problem 3. A rectangular box having no top is three times as long as it is wide and two times as tall as it is wide. If its volume is 750 cm^3, then its surface area in square centimeters is:
A. 195 B. 475 C. 550 D. 270 E. none of the above

Problem 4. A trough is 12 ft long and its ends are in the form of inverted isosceles triangles (see figure), having an altitude of 3 ft and a base of 4 ft What is the depth of the water when the trough is filled to one-half of its capacity?



A. $\frac{2}{3}$ ft B. $\frac{3}{2}$ ft C. $\frac{3}{\sqrt{2}}$ ft D. $\frac{\sqrt{2}}{3}$ ft E. $\frac{2}{\sqrt{3}}$ ft

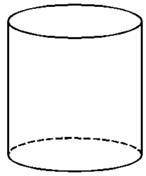
Problem 5. The ratio of the surface areas of two spheres is 4 : 9. The ratio of their volumes is:
A. 2 : 3 B. 4 : 9 C. 8 : 27 D. 16 : 81 E. 64 : 729

Problem 6. A sphere has a radius of k units. A cylinder inscribed in the sphere has a height of $(\frac{6}{5})\, k$ units. The ratio of the volume of the sphere to the volume of the cylinder is:

A. 125 : 72 B. 50 : 27 C. 25π : 48 D. 6 : 5 E. 250 : 81

Problem 7. A right circular cylinder has altitude 8 inches and radius 4 inches. Find the total surface area.

A. 80π in^2 B. 160π in^2 C. 96π in^2
D. 48π in^2 E. $(32 + 32\pi)$ in^2



Problem 8. The total surface area of a regular triangular pyramid all of whose edges are 6 is:

A. $18\sqrt{3}$ B. $27\sqrt{3}$ C. $36\sqrt{3}$ D. $45\sqrt{3}$ E. $27\sqrt{3} + 36$

Problem 9. Suppose two pyramids are similar and the surface area of the larger is 16 times the surface area of the smaller. How many times the volume of the smaller pyramid is the volume of the larger pyramid?

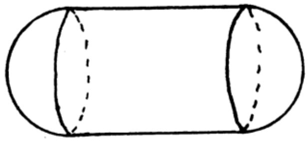
A. 4. B. 8. C. 16. D. 32. E. 64.

Problem 10. Let A be the vertex of a right circular cone with base centered at O and let B be a point on the circumference of its circular base. If the length of $\overline{AO}$ is 10 and length of $\overline{AB}$ is 14, what is the lateral surface area of the cone?

A. 70π B. $80\pi\sqrt{3}$ C. $100\pi\sqrt{2}$ D. 140π E. $56\pi\sqrt{6}$

Problem 11. A propane tank is in the shape of a cylinder with hemispherical end caps. The cylinder has diameter 4 feet and the total length of the tank is 12 feet. What is the volume of the tank in cubic feet?

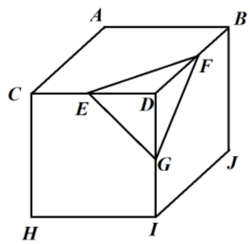
A. $\frac{32\pi}{3}$ B. $\frac{128\pi}{3}$ C. $\frac{176\pi}{3}$ D. 48π E. $\frac{1024\pi}{3}$



Problem 12. A frustum of a right circular cone has radii 4 cm and 6 cm with height 7 cm, as pictured. What is its volume?

A. 364π cm^3. B. $\frac{532}{3}\pi$ cm^3. C. 175π cm^3. D. 84π cm^3. E. $\frac{140}{3}$ cm^3.

Problem 13. In the diagram, each edge of the cube has length 24. E, F, and G are the midpoints of $\overline{CD}$, $\overline{BD}$, and $\overline{DI}$. Find the volume of the pyramid $DEFG$.



A. $72\sqrt{3}$ units^3. B. 144 units^3. C. $144\sqrt{3}$ units^3.
D. 288 units^3. E. 432 units^3.

Problem 14. When the radius of the base of a cylinder was doubled and its height remained the same, the total surface area of the new cylinder was three times as large as the original. What was the ratio of the radius to the height of the original cylinder?

A. 1 : 3 B. 1 : 2 C. 1 : 1 D. 2 : 1 E. 3 : 1

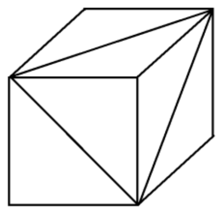
Problem 15. A solid sphere with radius 3 cm is melted down and recast as a cylinder with the same radius. What is the height of the cylinder?

A. $\frac{4}{3}$ cm B. 3 cm C. 4 cm D. 9 cm E. 12 cm

Problem 16. Suppose the side, front, and bottom faces of a right rectangular solid have areas of 12 cm^2, 8 cm^2, and 6 cm^2 respectively, then the volume is:

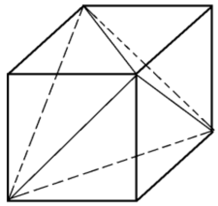
A. 576 cm^3 B. 24 cm^3 C. 9 cm^3 D. 104 cm^3 E. none of these

Problem 17. A cube with sides of length 4 cm has three face diagonals drawn as shown to form the edges of a new (6-edged) solid. What is the number of cubic centimeters in the volume of the new solid?



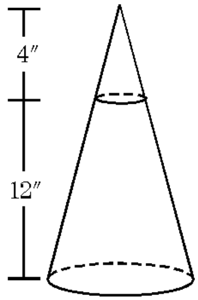
A. $10\frac{2}{3}$. B. $10\frac{1}{3}$. C. $10\frac{1}{5}$. D. $10\frac{1}{7}$. E. $10\frac{1}{9}$.

Problem 18. A cube with sides of length 4 cm has three face diagonals drawn from a vertex. These three diagonals, along with three other face diagonals, form the edges of a new (6-edged) solid. What is the number of cubic centimeters in the volume of the new solid?



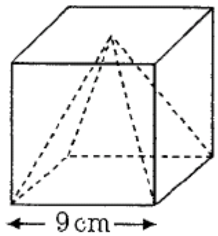
A. $21\frac{1}{3}$. B. $20\frac{1}{3}$. C. $21\frac{3}{5}$. D. $21\frac{3}{7}$. E. $20\frac{3}{5}$.

Problem 19. A plane parallel to the base of a cone divided the cone into two pieces as indicated in the diagram. The radius of the cone is 8″. What is the ratio of the volume of the top piece to the volume of the bottom piece?



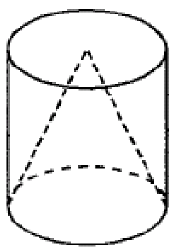
A. 1 : 63 B. 1 : 64 C. 1 : 4 D. 1 : 3 E. 1 : 16

Problem 20. The center of the top of the cube is the apex of a pyramid whose base is the bottom of the cube. Find the volume in cubic centimeters of the space enclosed by the cube but not the pyramid.



A. 486. B. 488. C. 516. D. 432. E. 464.

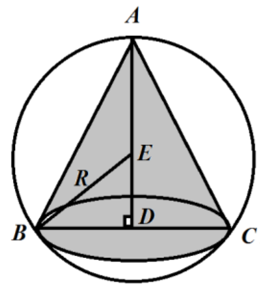
Problem 21. A right circular cone is inscribed in a right circular cylinder. The volume of the cylinder is 72π cubic centimeters. What is the number of cubic centimeters in the space inside the cylinder but outside the cone?



A. 44π B. 46π C. 48π D. 40π E. 42π

Problem 22. A right circular cone is inscribed in a sphere with the surface area of 400π as shown. The diameter of the base is equal to the altitude of the cone. If $ED = 6$, what is the volume of the cone ?

A. $\frac{1032\pi}{3}$ B. $\frac{1028\pi}{3}$ C. $\frac{1076\pi}{3}$ D. 341π E. $\frac{1024\pi}{3}$

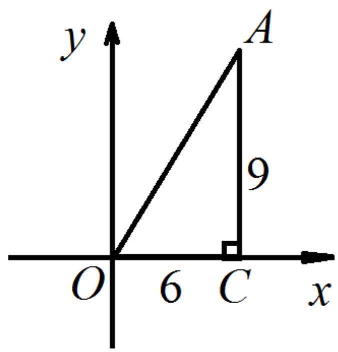


Problem 23. The volume of a cone, in cubic centimeters, made from a circular sector of radius 3 cm and central angle 40° is:

A. $\frac{4\sqrt{5}}{81}\pi$ B. $\frac{5\sqrt{4}}{81}\pi$ C. $\frac{20}{81}\pi$ D. $\frac{4\sqrt{3}}{81}\pi$ E. $\frac{4\sqrt{2}}{81}\pi$.

Problem 24. Triangle AOC is rotated about the y-axis. What is the volume of the figure generated by the rotation?

A. 244π B. 246π C. 228π D. 220π E. 216π



4. SOLUTIONS:

Problem 1. Solution: B.

$V = \pi R^2 h = \frac{1}{4}\pi D^2 h \quad \Rightarrow \qquad V = \pi R^2 h = \pi \times 3^2 \times 10 = 60\pi$.

Problem 2. Solution: D.

$S = 2(ab + bc + ca) = 2(3\times 4 + 4\times 5 + 5\times 3) = 94$.

Problem 3. Solution: B.

We know that $L = 3W$ and $H = 2W$.

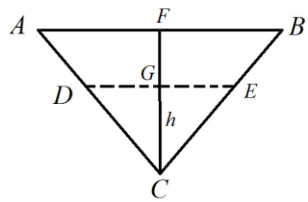
$V = LWH,\ 750 = 3W \times W \times 2W \quad \Rightarrow \quad W = 5$.

So $L = 3W = 15$ and $H = 2W = 10$.

$S = 2(LW + WH + HL) \qquad \Rightarrow \qquad S = 2(15\times 5 + 5\times 10 + 10\times 15) = 550$

Since the rectangular box having no top, the answer is 550 – 75 = 475.

Problem 4. Solution: C.



We draw $DE // AB$ such that $GC = h$, where h is the depth of the water.

As long as the area of triangle DEC is the same as the area of the trapezoid $ABED$, their corresponding volumes will be the same.

We know that triangle DEC is similar to triangle ABC.

So we have $\frac{S_{\Delta DEC}}{S_{\Delta ABC}} = (\frac{FC}{GC})^2 \quad \Rightarrow \qquad \frac{1}{2} = (\frac{3}{h})^2 \qquad \Rightarrow \qquad h = \frac{3}{\sqrt{2}}$.

Problem 5. Solution: C.

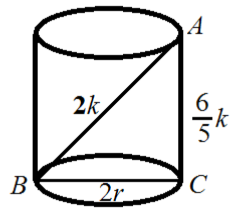
$S = 4\pi R^2 = \pi D^2$.

$$\frac{S_1}{S_2}=\frac{4\pi R_1^{\ 2}}{4\pi R_2^{\ 2}}=(\frac{R_1}{R_2})^2 \quad\Rightarrow\quad \frac{4}{9}=(\frac{R_1}{R_2})^2 \quad\Rightarrow\quad \frac{2}{3}=\frac{R_1}{R_2}$$

$$\Rightarrow\quad \frac{8}{27}=(\frac{R_1}{R_2})^3 \quad\Rightarrow\quad \frac{8}{27}=\frac{\frac{4}{3}\pi R_1^{\ 3}}{\frac{4}{3}\pi R_2^{\ 3}}=\frac{V_1}{V_2}.$$

Problem 6. Solution: A.

We draw the figure as shown and we know that AB is the diameter of the sphere. Let r be the radius of the cylinder.



Applying Pythagorean Theorem to right triangle ABC:

$$(2k)^2=(2r)^2+(\frac{6}{5}k)^2$$

$$\frac{V_S}{V_C}=\frac{\frac{4}{3}\pi R^3}{\pi r^2 h}=\frac{4}{3}\frac{k^3}{r^2\times\frac{6}{5}k}=\frac{10}{9}(\frac{k}{r})^2 \qquad (1)$$

Applying Pythagorean Theorem to right triangle ABC:

$$(2k)^2=(2r)^2+(\frac{5}{6}k)^2 \quad\Rightarrow\quad (\frac{k}{r})^2=\frac{25}{16} \qquad (2)$$

Substituting (2) into (1) $\frac{V_S}{V_C}=\frac{10}{9}\times\frac{25}{16}=\frac{125}{72}$.

Problem 7. Solution: C.

$S=2\pi R^2+2\pi Rh=2\pi\times 4^2+2\pi\times 4\times 8=96\pi$.

Problem 8. Solution: C.

$S=\sqrt{3}a^2= 36\sqrt{3}$.

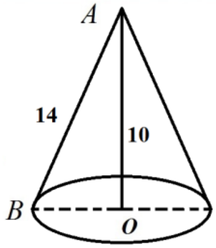
Problem 9. Solution: E.

Since two pyramids are similar, we have $\frac{S_1}{S_2}=\frac{16}{1}=(\frac{a_1}{a_2})^2 \quad\Rightarrow\frac{4}{1}=\frac{a_1}{a_2}$ (1)

Since two pyramids are similar, we have $\frac{V_1}{V_2} = (\frac{a_1}{a_2})^3$ (2)

Substituting (1) into (2): $\frac{V_1}{V_2} = (\frac{4}{1})^3 = 64$.

Problem 10. Solution: E.



$l = \sqrt{R^2 + h^2} \quad \Rightarrow \quad 14 = \sqrt{R^2 + 10^2} \Rightarrow \quad R = 4\sqrt{6}$

$S = \pi Rl + \pi R^2$.

The lateral area is $S_l = \pi Rl = \pi \times 4\sqrt{6} \times 14 = 56\pi\sqrt{6}$.

Problem 11. Solution: B.

The solid is one sphere and one cylinder.

The volume of the cylinder will be: $V_C = \frac{1}{4}\pi \times 4^2 \times (12-4) = 32\pi$

The volume of the sphere will be: $V_S = \frac{1}{6}\pi \times 4^3 = \frac{32}{3}\pi$

The answer will be $\frac{32}{3}\pi + 48\pi = \frac{128\pi}{3}$.

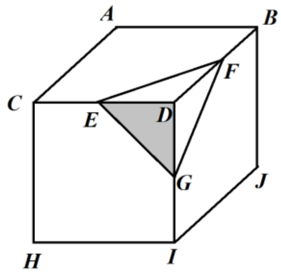
Problem 12. Solution: B.

$V = \frac{1}{3}\pi h(R^2 + r^2 + Rr) \quad \Rightarrow \quad V = \frac{1}{3}\pi \times 7 \times (4^2 + 6^2 + 4 \times 6) = \frac{532}{3}\pi$.

Problem 13. Solution: D.

The pyramid *DEFG* is a triangular pyramid with the base *EDG* and the height *DF*. So the volume will be

$V = \frac{1}{3}\frac{ED \times DG}{2} \times DF = \frac{12 \times 12}{6} \times 12 = 288$.



Problem 14. Solution: C.

By $S = 2\pi R^2 + 2\pi Rh$, we have $S_1 = 2\pi R_1^2 + 2\pi R_1 h_1$ and

$S_2 = 2\pi (2R_1)^2 + 2\pi (2R_1)h_1$.

We are given that $S_2 = 3S_1$. So $6\pi R_1^2 + 6\pi R_1 h_1 = 2\pi (2R_1)^2 + 2\pi (2R_1)h_1 \quad \Rightarrow$

$2\pi R_1 h_1 = 2\pi R_1^2 \qquad \Rightarrow \qquad \dfrac{R_1}{h} = 1:1$.

Problem 15. Solution: C.

The volume of the sphere is $V_S = \dfrac{4}{3}\pi R^3$.

The volume of the cylinder is $V_C = \pi R^2 h$.

We know that $V_S = V_C$. So: $\pi R^2 h = \dfrac{4}{3}\pi R^3 \qquad \Rightarrow \qquad h = \dfrac{4}{3}R = 4$.

Problem 16. Solution B.

Let the sides be *a, b*, and *c*.

$ab = 12$ (1)

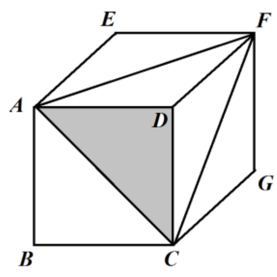
$bc = 8$ (2)

$ca = 6$ (3)

(1) × (2) × (3): $(abc)^2 = 24^2 \qquad \Rightarrow \qquad abc = 24$.

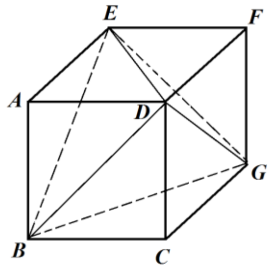
Problem 17. Solution: A.

The pyramid *DAFC* is a triangular pyramid with the base *ADC* and the height *DF*.



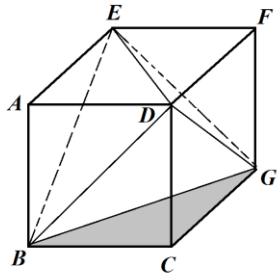
So the volume will be

$$V = \frac{1}{3}\frac{AD \times DC}{2} \times DF = \frac{4 \times 4}{6} \times 4 = 10\frac{2}{3}.$$



Problem 18. Solution: A.

The pyramid *DBEG* is a triangular pyramid.
We use the indirect way to get its volume.
The volume of *DBEG* = the volume of the cube – 4 × the volume of the triangular pyramid D*BCG* (with the base *BCG* and the height *CD*).



$$V = 4^3 - 4\times\frac{1}{3}\times\frac{BC\times CG}{2}\times DC$$

$$= 64 - \frac{4\times4\times4\times4}{6} = \frac{64}{3} = 21\frac{1}{3}.$$

Problem 19. Solution: A.

Let the volume of the original cone be V_1, and the volume of the small cone be V_2:

$$V_1 = \frac{1}{3}\pi\times R^2 h = \frac{1}{3}\pi\times 8^2\times 16 = \frac{1024\pi}{3}$$

Since the original cylinder is similar to the smaller cylinder that was sliced off, the ratio of their volumes is equal to the cube of the ratio of their heights.

$$\frac{V_1}{V_2} = \left(\frac{16}{4}\right)^3 = 64 \qquad \Rightarrow \qquad V_2 = \frac{1}{64}V_1 = \frac{16}{3}\pi .$$

$$\frac{V_2}{V_1 - V_2} = \frac{\frac{16}{3}\pi}{\frac{1024}{3}\pi - \frac{16}{3}\pi} = \frac{1}{63}.$$

Problem 20. Solution: A.

The height of the pyramid is 9.

Its volume is $V = \frac{1}{3}\times 9^3$.

The answer is $9^3 - \frac{1}{3}\times 9^3 = 486$.

Problem 21. Solution: C.

The volume of the cone is $\frac{1}{3}$ of the volume of the cylinder.

So answer is $72\pi - \frac{1}{3} \times 72\pi = 48\pi$.

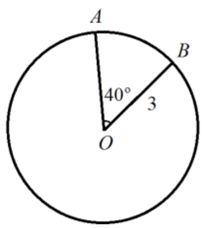
Problem 22. Solution: E.

$S = 4\pi R^2 \quad \Rightarrow \quad 400\pi = 4\pi R^2 \quad \Rightarrow \quad R = 10$.

Since $ED = 6$, $BD = 8$. then $AD = 16$.

The volume of the cone is $V = \frac{1}{3}\pi \times 8^2 \times 16 = \frac{1024}{3}\pi$

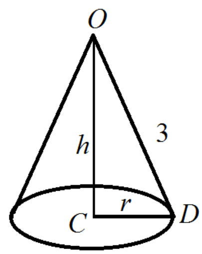
Problem 23. Solution: A.



As shown in the figures, the arc AB is the circumference of the circle C. Let the length of arc AB be x. Then $\frac{360°}{2\pi \times 3} = \frac{40°}{x} \Rightarrow$

$$\frac{3}{2\pi} = \frac{1}{x} \quad \Rightarrow \quad x = \frac{2\pi}{3}.$$

Then $2\pi r = \frac{2\pi}{3} \quad \Rightarrow \quad r = \frac{1}{3}$.

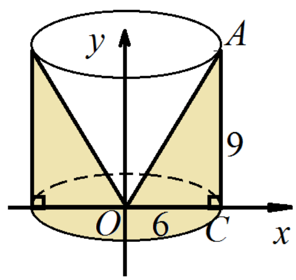


Applying Pythagorean Theorem to right triangle OCD:

$$h^2 = 3^2 - (\frac{1}{3})^2 = \frac{80}{9} \Rightarrow h = \sqrt{\frac{80}{9}} = \frac{4\sqrt{5}}{3}$$

$$V = \frac{1}{3}\pi \times (\frac{1}{3})^2 \times \frac{4\sqrt{5}}{3} = \frac{4\sqrt{5}}{81}\pi .$$

Problem 24. Solution: E.



As shown below in the figure below,

Volume of the figure generated = Volume of the cylinder – Volume of the cone

$= \pi r^2 h - \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^2 h = \frac{2}{3}\pi 6^2 \times 9 = 216\pi$.

A

B

C

D

E

F

S

T

V

W

Y