Mastering AMC 8

OmegaLearn.org

Contents

Preface

Motivation

This book was created to provide a comprehensive overview of the most important concepts on the AMC 8 math contest. The book includes video lectures for every chapter, formulas for every topic, and hundreds of examples and practice problems with detailed video solutions.

This book covers the following topics:

- Combinatorics
- Algebra
- Number Theory
- Geometry

How to Use This Book

To get the most out of this book, please give each problem your best effort before checking out the solution! Even if you don't solve it, the act of trying will help you gain useful insights about the problem and will help you develop the intuition needed for the topic.

Each chapter includes relevant formulas for the topic along with instructive **Example Problems** that show interesting applications of the concept.

Some of the examples are explained in the video, and some of the examples have detailed explanations. *The explanations provided aim to show not just the final solution, but the thought process behind finding the solution.*

Then, there is a **Practice Problems** section, which includes problems from AMC 8, AMC 10/12, MATHCOUNTS, BMT, EMCC, and many original problems from Omega Learn. These problems have detailed video solutions.

There is also an **Additional Problems** section with more problems (on the harder side) for extra practice.

Hopefully, the curated collection of examples and problems in this book will improve your problem solving skills and help you perform better on the AMC 8 contest. Good Luck!

Discord Server
Join our Discord Server to discuss problems from the book and for help with math contest preparation in general.

Feedback Form
If you have any feedback, find any errors, or think of any interesting problems that should be added here, please fill out this Feedback Form or email us at **info@omegalearn.org**.

Book Updates
We appreciate your feedback and will update the book regularly by adding new topics and problems. Please bookmark OmegaLearn.org to get the Latest Version of this book.

Information about math competitions

These videos provide some useful strategies for anyone just starting with math competitions or working towards specific competitions like the AMC 8.

1. All you need to know about Math Competitions from Elementary to High School
2. How to prepare for AMC 10/12 and qualify for AIME and USA(J)MO

Free Mastering AMC 8 Course

Mastering AMC 8 Course

This is the video course accompanying this book, and includes video lectures for every chapter. We explain all the important concepts from this book in detail, along with many useful examples showing the application of those concepts.

Free AMC 8 Fundamentals Classes

AMC 8 Fundamentals Classes

This is a crash course covering the most important fundamental concepts on the AMC 8.

Free AMC 8 Advanced/MATHCOUNTS Classes

AMC 8 Advanced/MATHCOUNTS Classes

This is a crash course covering the most important advanced concepts on the AMC 8.

Combinatorics

Chapter 1

Permutations

Video Lecture

1.1 Permutations Definition

Definition 1.1.1. A permutation is a possible arrangement of objects in a set where the order of objects matter.

Example 1.1

How many 3-digit passwords are possible where each digit is a number from 0 to 9?

Solution

This is a permutation since the order of the digits matters! Notice that each digit has 10 choices. From this, how do we find the total number of 3-digit passwords?

First, how many 1-digit passwords are there?

This is 10 because the password is simply one of the 10 1-digit numbers.

How can we try and find the number of 2-digit passwords?

Suppose that the first digit is a 3. Then, how many ways are there for the 2nd digit? This is just the number of 1-digit passwords, so it's just 10.

Now, what if the first digit is a 1?

Again, there are just 10 ways for the second digit. So for each of the 10 choices for the first digit, there are 10 choices for the 2nd digit. Doing this for all possible first digits, we see that there are $10 \times 10 = 100$ 2-digit passwords.

Now, how many 3-digit passwords are there?

Well, there are 10 choices for the first digit, and for each first digit, there are 100 2-digit passwords, so it's just

$$10 \times 100 = \boxed{1000}$$

Notice, we can also just see that there are 10 choices for the first digit. For each first digit, there are 10 choices for the 2nd digit. Finally, for each pair of first 2 digit, there are 10 choices for the 3rd digit, so the total number of ways is just

$$10 \times 10 \times 10 = 10^3 = \boxed{1000}$$

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1.2 Factorials

Definition 1.2.1 (Factorials)**.** A factorial is the product of all positive integers less than or equal to a given positive integer. In other words $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$.

Remark 1.2.2
For math competitions, it is recommended to memorize the following factorials as it will save you a lot of time:

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2 \\ 3! &= 6 \\ 4! &= 24 \\ 5! &= 120 \\ 6! &= 720 \end{aligned}$$

Definition 1.2.3. A permutation is a possible arrangement of objects in a set where the order of objects matter.

Example 1.2

Sohil has 1 wrapping paper in each of 4 different colors: red, orange, green, or blue. He needs to wrap all of them around a box in any order. How many ways are there for him to do this?

Solution

Notice that in this problem, we can't use the same wrapping paper twice so it's slightly more complicated. Let's begin this problem by approaching each layer at a time.

Starting with the innermost layer, how many ways are there to choose a color for this layer?

We can choose any of the 4 colors so there are just 4 ways for this to happen.

How many choices are there for the 2nd layer?

For the 2nd layer, since we only have 1 of each paper, only 3 of the colors are left since we already used one. Therefore, for the 2nd layer we have 3 choices for the color.

For the 3rd layer, we have already used 2 of the wrapping papers, so there are only 2 choices left. For the final layer, we have already used 3 of the papers, so we only have 1 choice for the final paper.

Now, how do we find the total number of ways?

We must multiply all of the numbers since all the choices are independent. For example, let's say the first wrapping paper chosen was red. Then, the number of ways to order the wrapping papers is simply the number of ways to arrange wrapping papers of 3 different colors. However, notice that this is also true for all the other color choices of the first wrapping paper (orange, green, and blue).

Therefore, in total, we must multiply by 4 to the number of ways of arranging 3 wrapping papers of different colors. Using this logic, we get that the total number of ways to arrange the wrapping papers is

$$4 \times 3 \times 2 \times 1 = \boxed{24}.$$

Notice that this is just 4!.

We will see later that we can find a general formula in terms of factorials to solve this type of problem.

Example 1.3

Find the number of ways to arrange 4 different books on a bookshelf.

Solution

This problem is very similar to the previous one. We must consider the number of ways for each book.

There are 4 choices for the first book on the bookshelf. After that there are 3 books left, so there are 3 choices for the 2nd book. Continuing on, there are 2 choices for the 3rd book, and 1 choice for the 4th book.

Therefore, the total number of ways is just

$$4 \times 3 \times 2 \times 1 = 4! = \boxed{24}$$

Example 1.4

There are 4 different AMC 8 videos. Every student in the class watches the videos in a different order. What is the maximum number of students in the class?

Video Solution

Theorem 1.2.4

The number of ways of arranging n distinct objects is just

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

1.3 Permutations Fundamentals

Example 1.5

Sam has 5 different stamps and is making a Christmas card where he puts 3 different stamps in a line. How many different Christmas card designs can he make?

Video Solution

Theorem 1.3.1 (Permutations Formula)

The number of ways to order k objects out of n total objects is

$${}^{n}P_{k} = \frac{n!}{(n-k)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-(k-1))$$

Example 1.6

Evaluate ${}^{7}P_{3}$.

Solution

If we compare the expression ${}^7P_3$ to the formula ${}^nP_k$, we have $n = 7$ and $k = 3$. Substituting these values in the formula for permutations,

$$\begin{aligned} {}^7P_3 &= \frac{7!}{(7-3)!} \\ &= \frac{7!}{4!} \\ &= 7 \times (7-1) \times \ldots \times (7-(3-1)) \\ &= 7 \times 6 \times 5 \\ &= \boxed{210} \end{aligned}$$

Remark 1.3.2
Notice how we can use this formula as a shortcut to the previous problem.

Example 1.7

Find the number of ways of selecting a president, vice president, and secretary from a group of 7 people.

Solution

This is very similar to the factorials problems, except now we only have to order positions.

How can we approach the problem?

Let's consider the number of ways for each position separately.

There are 7 choices for the president. Now, because there are only 6 people left, there are 6 choices for the vice president. Then, because 2 of the people are already chosen for the president and vice president, there are 5 choices for the secretary.

In total, the number of ways is

$$7 \times 6 \times 5 = \boxed{210}$$

1.4 Digit Permutations

Example 1.8

How many 4 digit numbers have all digits distinct?

Solution

Let's consider the number of ways to form this number digit by digit.

For the first digit, how many choices do we have? Remember, the first digit of a number cannot be 0 so there are 9 choices (any number from 1 to 9).

How do we handle the condition that the numbers must have distinct digits?

However, unlike the first digit, the 2nd digit can be 0. The 2nd digit can be any number from 0 to 9 except the digit chosen as the first digit, so there are

$$10 - 1 = 9$$

choices. For the 3rd digit, again, it can be anything from 0 to 9 except the 2 digits already chosen, so there are

$$10 - 2 = 8$$

choices. For the final digit, there are 10 - 3 = 7 choices.

So, the total number of possible numbers is

$$9 \times 9 \times 8 \times 7 = \boxed{4536}$$

Remark 1.4.1
Generally, in these types of problems, we want to consider the most restrictive conditions first. For example, in this problem, if we started at the units digit, we would have had issues in calculating the number of options for the thousands digit so we would have had to do casework, which is a more advanced technique that will be covered in a later chapter.

Example 1.9

How many 4 digit numbers exist such that the first digit is odd and the other 3 digits are even and all digits distinct?

Solution

The first digit has 5 choices (1, 3, 5, 7, or 9). The 2nd digit also has 5 choices (0, 2, 4, 6, 8).

Do we have to subtract any of the choices of the 2nd digit to make sure the numbers are distinct?

Keep in mind that although the digits have to be distinct, the 2nd digit is even and the first digit is odd, so there is no overlap.

The 3rd digit can be any even number except the one chosen for the 2nd digit, so it has 4 choices. The 4th digit can be any even number except those chosen for the 2nd and 3rd digit.

In total, the number of 4 digit numbers with these constraints is

$$5 \times 5 \times 4 \times 3 = \boxed{300}$$

Example 1.10

How many 5 digit palindromes are there? A palindrome is a number that reads the same forward and backward.

Video Solution

Example 1.11 (Omega Learn)

How many 6 digit numbers exist such that all the digits are distinct, and it's first 2 digits are 6 or more, the last 2 digits are 5 or less, and the 3rd digit is a nonzero multiple of 7?

Video Solution

1.5 Circular Arrangements

Theorem 1.5.1 (Circular Arrangements)

The number of ways of arranging n objects in a circle where rotations of the same arrangement aren't considered distinct is $(n-1)!$

The number of ways of arranging n objects in a circle where rotations of the same arrangement aren't considered distinct and reflections of the same arrangement aren't considered distinct is $\frac{(n-1)!}{2}$

Remark 1.5.2

The reason that this is true is because we can simply fix 1 person to be at the top and there are $(n-1)!$ ways to arrange the other people. This accounts for rotations since rotating an arrangement will result in someone else on top. We divide by 2 for reflections because of symmetry on both the left and right sides of the person chosen to be at the top.

Example 1.12

How many ways are there to arrange 5 people in a circle if rotations are not counted as distinct orientations?

Solution

When things are arranged in a circle, we can fix the position of one person, and look for arrangements of others.

How many other people are left to arrange?

We can simply apply the formula for 5 people where rotations don't matter. This is just

$$(5-1)! = 4! = \boxed{24}$$

Example 1.13 (Omega Learn)

How many ways are there to arrange 5 people around a circle such that 2 of the people, Alex and Bob, must sit next to each other. Note that rotations are not counted as distinct orientations.

Video Solution

Example 1.14 (Omega Learn)

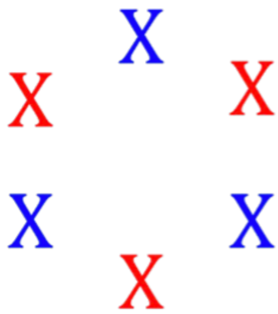
Gauss, Pauli, Einstein, Newton, Edison, and Faraday are sitting at a circular table. Einstein, Newton, and Faraday are enemies so they refuse to sit next to each other. With this condition, how many ways are there for the 6 physicists to sit at the table if rotations are not counted as distinct orientations?

Solution

Without the condition, the answer would just be $(6-1)! = 5! = 120$ as we can fix 1 of the 6 people at the top and permute the remaining 5 people.

Where must the enemies sit so that none of them are sitting next to each other?

Since there are only 6 total people, there must be exactly one other person between each of the enemies as seen in the diagram below where the red X's represent the enemies.



Should we again try to fix someone at the top to deal with rotations?

Yes! By fixing Gauss at the top (arbitrarily), we can now simply order the remaining 5 people without having to worry about the rotation condition.

How many ways to permute the enemies?

Note that the 3 possible locations of the enemies is fixed so there are 3! ways to permute them.

How many ways to permute the people with no enemies?

We already fixed Gauss (someone without enemies) so there are 2! ways to permute the other 2 people who don't have enemies.

In total, our answer is

$$3! \times 2! = \boxed{12}$$

1.6 Practice Problems

Problem 1.6.1

Evaluate ${}^6P_3$ and ${}^7P_2$

Video Solution

Problem 1.6.2

How many 3-digit numbers exist with all odd digits?

Video Solution

Problem 1.6.3

How many 3-digit numbers exist with all even digits?

Video Solution

Problem 1.6.4

How many 3-digit odd numbers are there with distinct digits?

Video Solution

Problem 1.6.5

If you have Alice, Betty, and Chase and you need to choose two of them to be the president and vice president of the math club, how many ways are there to do this?

Video Solution

Problem 1.6.6

Find the number of ways of selecting a president, vice president, and secretary from a group of 8 people.

Video Solution

Problem 1.6.7

Find the number of 3-digit numbers with all of its digits distinct.

Video Solution

Problem 1.6.8

Find the number of ways to arrange 5 different books on a bookshelf.

Video Solution

Problem 1.6.9

How many ways are there to pick your favorite and second favorite book amongst 5 books?

Video Solution

Problem 1.6.10

There are 10 students at a math contest. How many ways are there to select the first, second, and third place winners?

Video Solution

Problem 1.6.11

How many 4 digit numbers are not palindromes? A palindrome is a number that reads the same forward and backward.

Video Solution

Problem 1.6.12 (AMC 8)

A special type of license plate includes 3 distinct letters at the beginning and 4 single digit numbers after. How many such license plates exist?

Video Solution

Problem 1.6.13

There is a school race with 12 students. How many different ways to award the gold, silver, and bronze medals if no ties are possible?

Video Solution

Problem 1.6.14 (AMC 8)

How many integers between 1000 and 9999 have four distinct digits?

Video Solution

Problem 1.6.15 (AMC 8)

The Dragonvale Middle School chess team consists of two boys and three girls. A photographer wants to take a picture of the team to appear in the local newspaper. She decides to have them sit in a row with a boy at each end and the three girls in the middle. How many such arrangements are possible?

Video Solution

Problem 1.6.16 (AMC 8)

Professor Chang has nine different language books lined up on a bookshelf: two Arabic, three German, and four Spanish. How many ways are there to arrange the nine books on the shelf keeping the Arabic books together and keeping the Spanish books together?

Video Solution

Problem 1.6.17 (MATHCOUNTS)

How many permutations of the digits 1, 2, 3, 4 have 1 before 3 and 2 before 4? One such permutation to include is 2134.]

Video Solution

Problem 1.6.18 (AMC 10)

Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

Video Solution

Problem 1.6.19 (Omega Learn Math Competition)

Alice, Bob, Claire, Dave, Emma are sitting around a round table. Alice refuses to sit next to Bob. How many different ways can they sit around the table? (Rotations of the same arrangement are not considered different).

Video Solution

Additional Problems

Problem 1.6.20 (Omega Learn)

7 people are lining up for a photograph. If the 2 tallest people must be at 1st and 7th positions in any order and the shortest person must be in the center, how many ways are there for them to line up?

Problem 1.6.21 (EMCC 2014)

Five distinct boys and four distinct girls are going to have lunch together around a table. They decide to sit down one by one under the following conditions: no boy will sit down when more boys than girls are already seated, and no girl will sit down when more girls than boys are already seated. How many possible sequences of taking seats exist?

Hints

Chapter 2

Combinations

Video Lecture

2.1 Combinations Fundamentals

Definition 2.1.1. A combination is a collection of items where the order of the items does not matter.

Remark 2.1.2
Usually, the words permute, order does matter, etc. imply a permutation while the words choose, select, order doesn't matter, etc. imply a combination

Example 2.1

How many ways can I select 3 pencils from 6 different pencils?

Solution

Let's try to approach this similarly to permutations.

There are 6 choices for the first pencil, 5 choices for the 2nd pencil, and 4 choices for the 3rd pencil.

So the total number of ways is just

$$6 \times 5 \times 4 = 120$$

Is this the number of ways of choosing or arranging?

Remember that we are just selecting 3 pencils, so the order doesn't matter. Therefore, we are overcounting many cases. For example, choosing pencil A first and pencil B second is the same as choosing pencil B first and pencil A second.

So how do we account for this?

Well, to answer that question, we must consider how many ways there are to order the pencils. Let's say we have 3 pencils.

How many ways are there to order them?

This is just 3!because we are arranging 3 objects in order. Therefore, we are overcounting by a factor of 3!.

So the total number of ways to "choose" 3 pencils from 6 different pencils is

$$\frac{6 \times 5 \times 4}{3!} = 20$$

Example 2.2

How many ways are there to choose a committee of 3 people from 6 people?

Video Solution

Theorem 2.1.3 (Combinations Formula)

The number of ways to choose k objects out of a total of n objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdot\cdots\cdot(n-k+1)}{k!}$$

.

Example 2.3

Evaluate

$$\binom{6}{3}$$

Solution

We can see that $n = 6$ and $k = 3$. Substituting these values in the formula, we get

$$\frac{6 \times (6-1) \times \cdots \times (6-3+1)}{3!} = \frac{6 \times 5 \times 4}{6} = \boxed{20}$$

Remark 2.1.4
Notice that

$$\binom{n}{k} = \binom{n}{n-k}$$

This is true because choosing k objects that are part of your selection (left hand side) is equivalent to choosing the $n-k$ objects that are not be part of your selection (right hand side). For example,

$$\binom{6}{2} = \binom{6}{4}$$

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Example 2.4 (Omega Learn)

You are working on the Mastering AMC 8 Book, and you must choose 7 of the 10 combinatorics chapters to work on. How many ways are there to do this?

Video Solution

Example 2.5 (Omega Learn)

You are working on the Mastering AMC 8 Book, and you must choose 7 of the 10 combinatorics chapters to work on. However, the first 2 chapters are required to understand the remaining chapters so they must be chosen. How many ways are there to do this?

Video Solution

2.2 Binomial Identity

Theorem 2.2.1 (Binomial Identity)
The binomial identity states that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Remark 2.2.2
Both of the above methods can be used for calculating the number of ways of choosing any # of items from n different items.

Example 2.6

Nihir has an apple, orange, banana, pear, and raspberry. He wants to take a fruit basket to a picnic. How many different types of fruit baskets can he take using the fruits he has?

Solution

We can do this by listing the number of ways of choosing all possible combination of fruits.

Is there any easier way to solve this question?

Notice that for each fruit, we have 2 choices. You can either take it, or leave it. We then have to multiply this for all of the 5 fruits.

Therefore, the number of ways is simply

$$2^5 = \boxed{32}$$

Note that one of the fruit baskets will have none of the fruits (which is mathematically a possible type of fruit basket and the problem statement does not explicitly says that it is not allowed).

Example 2.7

You have 5 different pencils and 3 different erasers and must choose some of them to take to the AMC 8. How many ways are there to do this?

Video Solution

2.3 Tricky Combinations Examples

Example 2.8 (Omega Learn)

There are 6 experienced applicants and 7 inexperienced applicants applying for a job. Out of the experienced applicants, 2 managers are selected. Out of all of the other applicants who aren't selected for a manager (both experienced and inexperienced), 4 other employees are selected. How many ways are there to do this?

Solution

To solve this problem, we will find the number of ways of choosing managers and other employees separately.

First, how many ways are there to select 2 managers?

Since they must be from the 6 experienced applicants, there are

$$\binom{6}{2} = \frac{6 \times 5}{2!} = \frac{30}{2} = 15$$

Next, how many ways are there to select the 5 other employees?

Since 2 of the experienced applicants were already selected, there are 4 experienced applicants and 7 inexperienced applicants left.

Therefore, we must select 4 employees out of the 11 remaining people. We can do this in

$$\binom{11}{4} = \frac{11 \times 10 \times 9 \times 8}{4!} = \frac{11 \times 10 \times 72}{24} = 11 \times 10 \times \frac{72}{24} = 11 \times 10 \times 3 = 330$$

Therefore, in total, how many ways are there to select the 3 managers and 5 other employees?

Since there are 15 ways to select the managers and 330 ways to select the other employees, there will be $15 \times 330 = \boxed{4950}$ ways to select both the managers and the employees.

Example 2.9 (Omega Learn)

You have 12 different energy bars that you need to give to 3 people: Alice, Betty, and Chase. Alice needs 3 bars, Betty needs 4 bars, and Chase needs 5 bars. How many ways are there to distribute the 12 bars to satisfy their requirements?

Video Solution

Example 2.10 (EMCC)

How many ways are there to place 4 balls into a 4×6 grid such that no column or row has more than one ball in it? (Rotations and reflections are considered distinct.)

Video Solution

Example 2.11 (AIME)

Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of one face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.

Solution

First, let's deal with the orderings of the coins.

How many ways to arrange the 4 gold and 4 silver coins?

We can use the word rearrangement formula to get $\frac{8!}{4!\times 4!} = 70$. Now, we will the number of ways to put the coins face up or face down in the stack

If a coin is face down, what must the coin above it be?

It can be anything and the faces will never meet.

If a coin is face up, what must the coin above it be?

It must also be face up as if it's face down, then the faces will meet.

The moment the coin first becomes face up, all the remaining coins on top must be face up as well! If none of the coins are face up, then there is 1 orientation.

How many orientations exist if at least 1 of the coins is face up?

We simply have to pick a coin to be the first face up coin and every other coin's orientation is fixed. There are 8 ways to do this.

In total, we have 9 orientations for the faces of the coins and $\binom{8}{4} = 70$ ways to order the gold and silver coins giving us an answer of $70 \times 9 = \boxed{630}$.

2.4 Practice Problems

Problem 2.4.1

If you have Alice, Betty, Chase, and Dave and you need to choose two of them to be the leaders of the math club, how many ways are there to do this?

Video Solution

Problem 2.4.2

Evaluate $\binom{8}{4}$ and $\binom{9}{7}$

Video Solution

Problem 2.4.3

How many ways are there to select 3 fruits amongst 8 different fruits?

Video Solution

Problem 2.4.4

How many ways are there to choose 4 balls from a bag which contains 9 balls of different color?

Video Solution

Problem 2.4.5

Mrs. Jones is organizing a potluck and wants to serve 8 different dishes. Each family can bring 3 dishes. How many families can Mrs. Jones invite so that no two families bring the same combination of dishes?

Video Solution

Problem 2.4.6

There are 9 different flavors of ice cream at an ice cream shop. Joe wants to buy 2 or 3 scoops of different ice cream flavors. He doesn't like 2 of the flavors Mango and Chocolate together so won't buy them together . How many ways can he buy ice cream?

Video Solution

Problem 2.4.7 (AMC 10)

A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

Video Solution
Video Solution

Problem 2.4.8 (AMC 10)

Henry's Hamburger Haven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties and any collection of condiments. How many different kinds of hamburgers can be ordered?

Video Solution

Additional Problems

Problem 2.4.9 (AMC 8)

How many integers between 2020 and 2400 have four distinct digits arranged in increasing order? (For example, 2347 is one integer.)

Problem 2.4.10

How many 5 digit positive integers exist such that the third digit is 8, the first 3 digits ascending, and last 3 digits are descending?

Problem 2.4.11 (Omega Learn)

There are 12 astronauts who applied to go on a mission to explore Mars. 2 different rockets will be sent from Earth: one carrying 3 people and another carrying 4. Out of the 12 astronauts, 4 of them are very experienced so they must go on either rocket. How many ways are there to place the astronauts in the rockets?

Chapter 3

Word Rearrangements

Video Lecture

3.1 Word Rearrangements Fundamentals

Example 3.1

How many ways are there to arrange the letters in the word OMEGA?

Solution

This is just like arranging 5 distinct books on a shelf. Since all the letters are different, there are just

$$5! = \boxed{120}$$

ways to arrange the 5 letters.

Example 3.2

How many ways are there to arrange the letters in the word SUNNY?

Solution

This looks similar to the previous example.

Can we use the same factorial formula again?

If that were the case, then our answer would just be $5! = 120$. Unfortunately, this problem is more complicated because now there are 2 N's. Therefore, some arrangements will be overcounted. Let us call the two N's N_1 and N_2

For example, N_1USN_2Y and N_2USN_1Y are the same arrangement, but our factorial formula will count them as different arrangements.

How can we account for this overcouting?

The valid arrangement $NUSNY$ is counted twice. Notice that we count each arrangement twice where the position of two N's is swapped.

Is this the case for all rearrangements?

Yes, indeed. For any arrangement of the letters in the word SUNNY, we get 2 different arrangements of the word: N_1 on the left and N_2 on the right, N_2 on the left and N_1 on the right.

Therefore, the number of valid arrangements is just $\frac{1}{2}$ the original 5! arrangements, which gives

$$\frac{5!}{2} = \boxed{60}$$

Remark 3.1.1
Remember that the factorial formula only works if all the letters are different.

Example 3.3

How many ways are there to arrange the letters of the word APPLE?

Video Solution

Theorem 3.1.2 (Word Rearrangements)
The number of ways to order a word is

$$\frac{n!}{d_1! \times d_2! \times d_3! \times \dots}$$

where n is the number of letters and $d_1, d_2, d_3, \dots$ are the number of times each of the duplicate letters that appear in the word.

Remark 3.1.3
This may seem complicated, but it essentially means the number of rearrangements of an n-letter word is n!. However, since some letters may appear multiple times (let's say d times), we must divide by d! for all such letters because there are d! ways to arrange the duplicate letters that we are overcounting.

3.2 Word Rearrangements with Constraints

Example 3.4

How many ways are there to arrange the letters of the word OPOSSUM if the arrangement must end in an M?

Video Solution

Remark 3.2.1
This formula is not only true for words! This formula can also work for the number of ways of arranging objects where some objects are of same type. In fact, we will use this later in geometric counting.

Example 3.5

How many ways are there to rearrange the letters in COMPUTER such that the C, O, M, and P stay together (not necessarily in the same order)?

Solution

This problem is slightly different from the standard word rearrangement problems because we now have a constraint.

How should we deal with the condition that the COMP must stay together?

Let's treat COMP as 1 block. Since the letters anyways have to stay together, let call this block a $\boxed{COMP}$. Now, instead, we can find the number of arrangments of $\boxed{COMP}$UTER.

How many ways are there to arrange $\boxed{COMP}$UTER?

Notice that this is just 5! since all the letters (or symbols) are different.

Now, are we done, or is there something we forgot to account for?

Remember that letters in the word COMP can appear in any order inside the box, so we have to also account for rearrangements of these letters within the $\boxed{COMP}$.

For each unique arrangement of $\boxed{COMP}$UTER, how many ways are there to

arrange the letters in the word $\boxed{COMP}$?

There are 4! ways to arrange the 4 letters C O M P.

Therefore, in total, there are $5! \times 4! = 120 \times 24 = \boxed{2880}$ ways to rearrange COMPUTER such that the letters COMP remain together.

Example 3.6 (Omega Learn)

How many ways are there to arrange the letters in LOLLIPOP if the I must be next to both an L and O?

Video Solution

3.3 Practice Problems

Problem 3.3.1

How many ways are there to arrange the letters of the word MATH?

Video Solution

Problem 3.3.2

How many ways are there to arrange the letters of the word MISSISSIPPI?

Video Solution

Problem 3.3.3

Find the number of ways to order the letters in PINEAPPLE.

Video Solution

Problem 3.3.4

How many ways are there to misspell the word MISSPELLED?

Video Solution

Problem 3.3.5

How many arrangements of the word BOTTLE are there such that the 2 T's and O remain together?

Video Solution

Problem 3.3.6 (AMC 10)

A child builds towers using identically shaped cubes of different color. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

Video Solution

Additional Problems

Problem 3.3.7

You have 6 apples, 3 pears, and 2 oranges. Assuming the same type of fruits are indistinguishable, how many distinct ways are there to place the fruits in a line if the 2 oranges must be next to each other?

Problem 3.3.8 (AMC 10)

How many distinguishable arrangements are there of 1 brown tile, 1 purple tile, 2 green tiles, and 3 yellow tiles in a row from left to right? (Tiles of the same color are indistinguishable.)

Video Solution

Additional Problems

Problem 3.3.9 (MATHCOUNTS)

How many distinct four-letter permutations are possible using four of the ten letters in MATHCOUNTS?

Problem 3.3.10 (AMC 8)

In how many ways can the letters in **BEEKEEPER** be rearranged so that two or more **E**s do not appear together?

Problem 3.3.11 (AHSME)

How many distinguishable rearrangements of the letters in $CONTEST$ have both the vowels first? (For instance, $OETCNST$ is one such arrangement but $OTETSNC$ is not.)

Problem 3.3.12 (Omega Learn)

If the letters of the word SMILE are rearranged in all possible ways, and arranged in alphabetical order like in a dictionary, then what will be the rank of the word SLIME?

Answers

3.3 60

3.4 180

3.3.1 24

3.3.2 34650

3.3.3 30240

3.3.4 453599

3.3.5 72

3.3.6 1260

3.3.7 840

3.3.8 420

3.3.9 3360

3.3.10 24

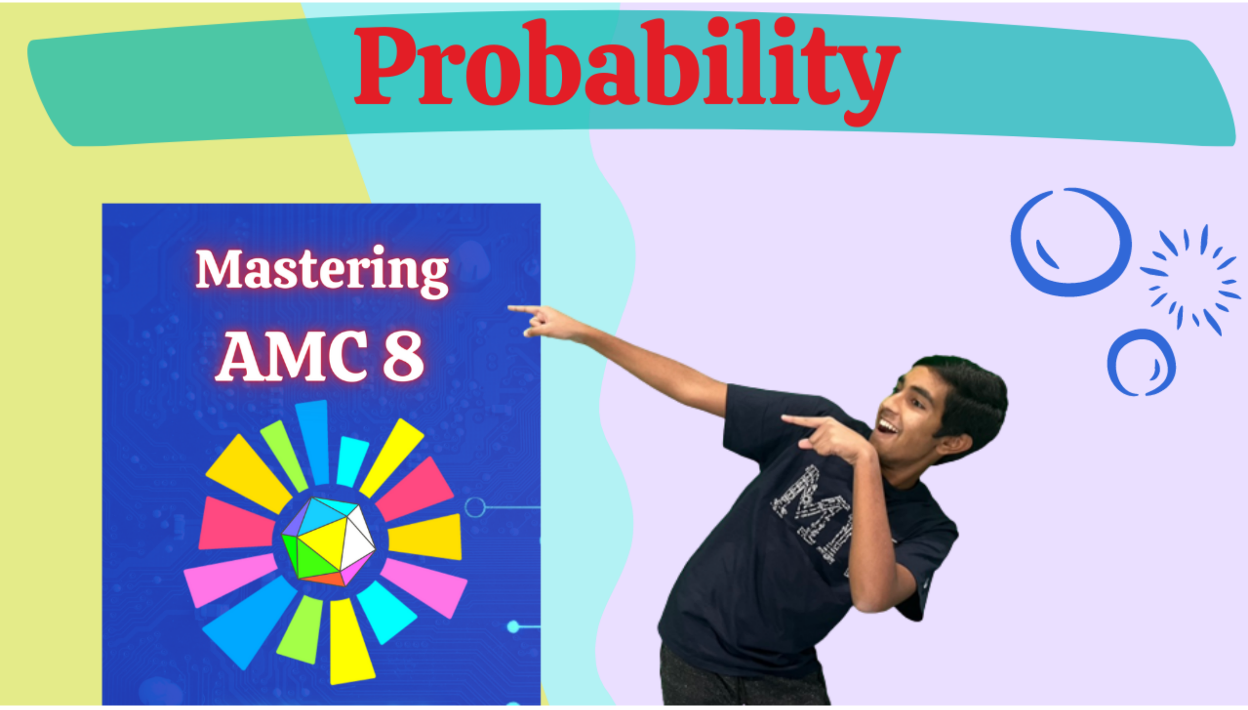
3.3.11 120

3.3.12 112

Chapter 4

Probability

Video Lecture



4.1 Probability Fundamentals

Definition 4.1.1. Probability is the likelihood of something happening. To calculate probability, you need to know how many possible options or outcomes there are and how many right combinations there are.

Theorem 4.1.2 (Probability)

$$\text{probability} = \frac{\text{Total number of desired outcomes}}{\text{Total number of possible outcomes}}$$

Example 4.1

What is the probability of rolling a prime number on a 6 sided dice?

Solution

To calculate probability, we need to find the number of successful and total outcomes.

How many successful outcomes are there?

We can see that there are 3 prime numbers from 1 to 6 (2, 3 and 5).

Next, how many total outcomes are there?

Well this is just 6 because it can be any number from 1 to 6.

Therefore, since the probability is just the number of successful outcomes divided by the total number of outcomes, the probability is

$$\frac{3}{6} = \boxed{\frac{1}{2}}$$

4.2 Distinguishability

Example 4.2

I have cards numbered from 1 to 10. What is the probability I pick a pair of 2 different cards that have an odd product?

Solution

This is a rather simple example however it demonstrates an important point about distinguishability in probability problems.

What must happen for the cards to have an odd product?

Both cards must be odd.

How many pairs of cards have both cards odd?

If order does not matter, then there are $\binom{5}{2} = 10$ combinations. However, if the order does matter, then there are $5 \times 4 = 20$ permutations.

How many total pairs of cards can be chosen?

If order does not matter, then there are $\binom{10}{2} = 45$ combinations. However, if the order does matter, then there are $10 \times 9 = 90$ permutations.

So does the order matter of the cards in the pair matter or not?

If the order of the cards in the pair does matter, then the probability is $\frac{20}{90} = \boxed{\frac{2}{9}}$. If the order of the cards in the pair does not matter, then the probability is $\frac{10}{45} = \boxed{\frac{2}{9}}$.

Why are the probabilities are the same?

When dealing with ordered pairs, the number of successful and total pairs were multiplied by 2! so they would cancel out when dividing!

Remark 4.2.1
In probability problems, whether you decide to multiply for order or not is up to you as both will give the same probability.

Example 4.3

The sum of 2 positive integers is 4. Find the probability that one of the integers is a 2.

Solution

There are 2 cases:

1. Both integers are 2
2. One integer is a 3 and the other is a 1

So is the probability just $\frac{1}{2}$?

No! While there are only 2 cases, they both have different chances of happening.

How many ways are there for each case to happen?

There is only 1 way where both integers can be 2 however there are 2 ways one of the integers is a 1 and the other is a 3 as we can flip them. The 2nd case is more likely.

What is the probability with this knowledge?

There are a total of 3 cases of which 1 satisfies the condition so the probability is $\boxed{\frac{1}{3}}$.

Remark 4.2.2
If you decide not to account for order when doing probability problems, make sure the number of orderings is consistent across all cases. In the first problem, every pair had 2 orderings so it was okay to neglect order. However, in the 2nd problem, the 2 cases had different number of orderings so this didn't work.

4.3 Casework in Probability

Concept 4.3.1
Often, when a problem asks for the probability of an event happening, we may have to do casework to find the number of successful outcomes and divide that by the total number of outcomes which is usually easy to find.

Example 4.4

What is the probability that the product of 4 dice rolls is a multiple of 648?

Video Solution

4.4 Probability of Independent Events

Example 4.5

Sohil randomly picks a number from 1 to 10. Sejal randomly picks a number from 1 to 25. What is the probability that the product of the numbers they choose is odd?

Solution

For this problem, although it is possible to find the number of successful and total outcomes amongst both picks of numbers, there is an easier way to solve the problem.

To start, what do we know about 2 numbers whose product is odd?

For the product of 2 numbers to be odd, both numbers must be odd because if any of the numbers are even, then the product will also have a factor of 2 in it.

Next, how can find the probability that Sohil and Sejal pick odd numbers?

We could find find the number of successful and total outcomes amongst both picks of numbers as mentioned earlier, but instead, we can simply find the probabilities of each of them picking an odd number.

The probability of Sohil picking an odd number is $\frac{5}{10} = \frac{1}{2}$.

The probability of Sejal picking an odd number is $\frac{13}{25}$ since there are 13 odd numbers from 1 to 25.

Now, how do we find the overall probability of both of them picking an odd number?

We must multiply the probabilities.

Therefore, the overall probability is

$$\frac{1}{2} \times \frac{13}{25} = \boxed{\frac{13}{50}}$$

.

Remark 4.4.1
Whenever we need to find the probabilities of 2 independent events happening (the results of the events don't depend on each other), we can simply find the probability of each event individually and multiply them together.

Example 4.6 (Omega Learn)

Person A rolls a dice with the numbers 1, 1, 2, 6, 15, 30 and Person B rolls a dice with the numbers 1, 2, 4, 18, 28, 44. What is the probability that the product of the 2 numbers is a multiple of 24?

Video Solution

4.5 Probability of Dependent Events

Example 4.7

Alex, Betty, Chase, Derek, Emma, Fiona, and George are racing in a marathon. If they finish in a random order, what is the probability that Chase is 1st and George is 6th?

Solution

Let's see how to solve this problem with dependent events.

Can we just use the technique above to solve this problem?

That won't work because this time, Chase' rank affects George's rank, such as in the case where Chase is 6th which would make it impossible for George to be 6th, so the events affect each other.

Then, how do we approach this question?

To do this, let's first consider the probability Chase is 1st and then find the probability

George is 6th given that Chase is 1st.

What is the probability that Chase is selected first?

Because there are 7 people who can be first and Chase is one of those persons, the probability is simply $\frac{1}{7}$ since they finish in a random order.

Next, what is the probability that George is selected 6th if Chase is first?

Notice that now out of the 7 original positions, only 6 are left. Therefore, since all positions are equally likely for George, the probability he is 6th is $\frac{1}{6}$.

Now, how do we find the overall probability?

Notice that we only need to find the probability that both Chase is first and George is 6th. Therefore, we can just find the probability of Chase being first and multiply that by the probability that George is 6th.

Therefore, our answer is just

$$\frac{1}{7} \times \frac{1}{6} = \boxed{\frac{1}{42}}$$

.

Remark 4.5.1
When the events are dependent like in the example above, the probability of both events happening (say A and B) is the probability of event A happening times the probability that event B happens given that event A already happened.

Example 4.8

On Saturday, there is a 20% chance of rain. On Sunday, there is a 30% chance of rain if it rained on Saturday, but only a 10% chance of rain if it didn't rain on Saturday. What is the probability that it will rain on both days or neither day?

Video Solution

Example 4.9 (AIME)

There is a 40% chance of rain on Saturday and a 30% chance of rain on Sunday. However, it is twice as likely to rain on Sunday if it rains on Saturday than if it does not rain on Saturday. The probability that it rains at least one day this weekend is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

Video Solution

4.6 Dependent or Independent?

Example 4.10 (Omega Learn)

Mark must choose a seat from 5 x 5 grid of chairs in the classroom. His enemy Steve does not like Mark so wants to choose a seat that is not in the same row or column as Mark. However, to his dismay, the teacher randomly assigns seats. What is the probability that Steve is not sitting in the same row or column as Mark?

Video Solution

4.7 Practice Problems

Problem 4.7.1

Two 6-sided dice are rolled. What is the probability that the sum of the numbers rolled is 2?

Video Solution

Problem 4.7.2

Two 6-sided dice are rolled. What is the probability that the sum of the numbers rolled is 4?

Video Solution

Problem 4.7.3

Two dice are rolled. What is the probability that the sum of the numbers rolled is 6?

Video Solution

Problem 4.7.4 (AMC 8)

A fair coin is tossed 3 times. What is the probability of at least two consecutive heads?

Video Solution

Problem 4.7.5 (AMC 8 Modified)

A fair 6 sided die is rolled twice. What is the probability that the first number that comes up is greater than the second number?

Video Solution

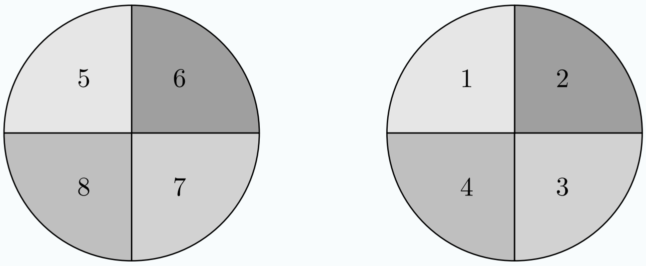
Problem 4.7.6 (AMC 8)

A box contains five cards, numbered 1, 2, 3, 4, and 5. Three cards are selected randomly without replacement from the box. What is the probability that 4 is the largest value selected?

Video Solution

Problem 4.7.7 (AMC 8)

The arrows on the two spinners shown below are spun. Let the number N equal 10 times the number on Spinner A, added to the number on Spinner B. What is the probability that N is a perfect square number?



Spinner A Spinner B

Problem 4.7.8 (AMC 8)

A top hat contains 3 red chips and 2 green chips. Chips are drawn randomly, one at a time without replacement, until all 3 of the reds are drawn or until both green chips are drawn. What is the probability that the 3 reds are drawn?

Video Solution

Problem 4.7.9 (AMC 8)

Two cards are dealt from a deck of four red cards labeled A, B, C, D and four green cards labeled A, B, C, D. A winning pair is two of the same color or two of the same letter. What is the probability of drawing a winning pair?

Video Solution

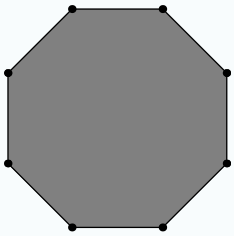
Problem 4.7.10 (AMC 10)

Three fair six-sided dice are rolled. What is the probability that the values shown on two of the dice sum to the value shown on the remaining die?

Video Solution

Problem 4.7.11 (AMC 8)

From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?



Video Solution

Problem 4.7.12 (AMC 10)

When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n?

Video Solution

Problem 4.7.13 (AMC 10)

Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same?

Video Solution

Problem 4.7.14 (AMC 8)

On a beach 50 people are wearing sunglasses and 35 people are wearing caps. Some people are wearing both sunglasses and caps. If one of the people wearing a cap is selected at random, the probability that this person is is also wearing sunglasses is $\frac{2}{5}$. If instead, someone wearing sunglasses is selected at random, what is the probability that this person is also wearing a cap?

Video Solution

Problem 4.7.15 (Omega Learn Math Contest)

John has 9 red apples and 3 green apples. He randomly selects and eats an apple. Then, 3 of the red apples get rotten and are thrown away. From the remaining apples he randomly selects and eats another apple. What the probability that both of the selected apples were red is $\frac{a}{b}$, what is the value of $a+b$?

Video Solution

Problem 4.7.16 (AMC 10)

Two different numbers are selected at random from $\{1,2,3,4,5\}$ and multiplied together. What is the probability that the product is even?

Video Solution

Problem 4.7.17 (MATHCOUNTS)

A fair tetrahedral die, whose faces are numbered 1, 2, 3, and 4 is rolled three times. What is the probability that the sum of the numbers rolled is 7? Express your answer as a common fraction.

Video Solution

Problem 4.7.18 (AMC 8)

Two different numbers are randomly selected from the set $\{-2,-1,0,3,4,5\}$ and multiplied together. What is the probability that the product is 0?

Video Solution

Problem 4.7.19 (Omega Learn Math Competition)

Joe is standing at the point (0,0) in the infinite coordinate plane. On any given move, there is a $\frac{1}{3}$ chance of him moving up, $\frac{1}{3}$ chance of him moving right, $\frac{1}{6}$ chance of him moving down, and $\frac{1}{6}$ chance of him moving left. If the probability he will reach the point (3,3) in less than or equal to 8 moves can be described as $\frac{a}{3^b}$, then what is the value of $a+b$?

Video Solution

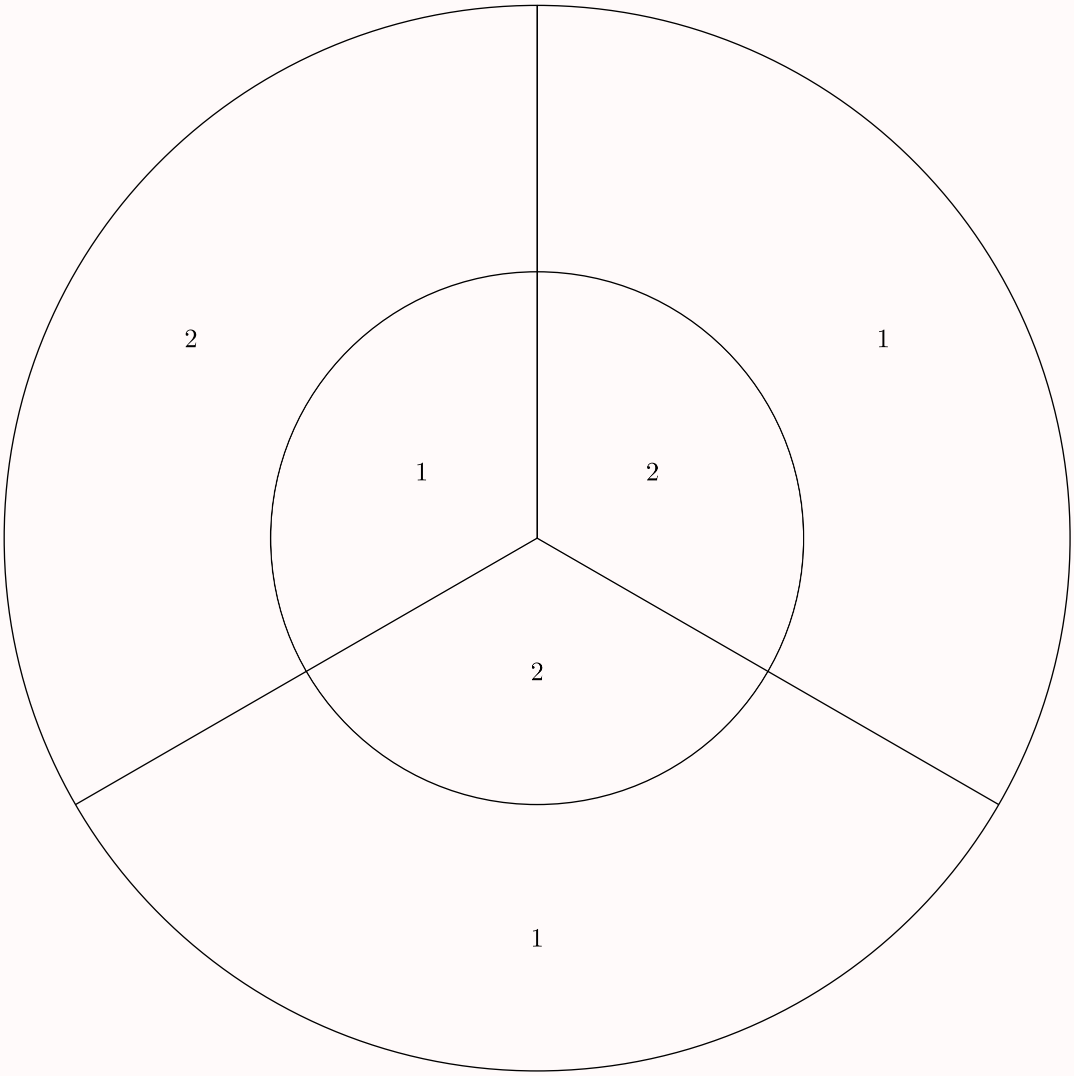
Problem 4.7.20 (Omega Learn Math Competition)

The Blazers and Cortzans are playing a series of 7 games. A team wins the series when they have won 4 games. Both teams have an equal likelihood of winning a game. If the probability the Cortzans win the series and lose less than 3 games can be expressed as $\frac{m}{n}$, find the value of $m+n$?

Video Solution

Problem 4.7.21 (AMC 8)

On the dart board shown in the figure below, the outer circle has radius 6 and the inner circle has radius 3. Three radii divide each circle into three congruent regions, with point values shown. The probability that a dart will hit a given region is proportional to the area of the region. When two darts hit this board, the score is the sum of the point values of the regions hit. What is the probability that the score is odd?



Video Solution

Problem 4.7.22 (Omega Learn Math Competition)

Eric is stacking twenty six Jenga Blocks. For n $\geq$ 1, the probability that placing the nth block causes the whole tower to topple is $\frac{1}{26-n}$ assuming the tower still stands after placing $n-1$ blocks. What is the expected number of blocks placed before the block that causes the tower to topple?

Video Solution

Additional Problems

Problem 4.7.23 (AMC 8)

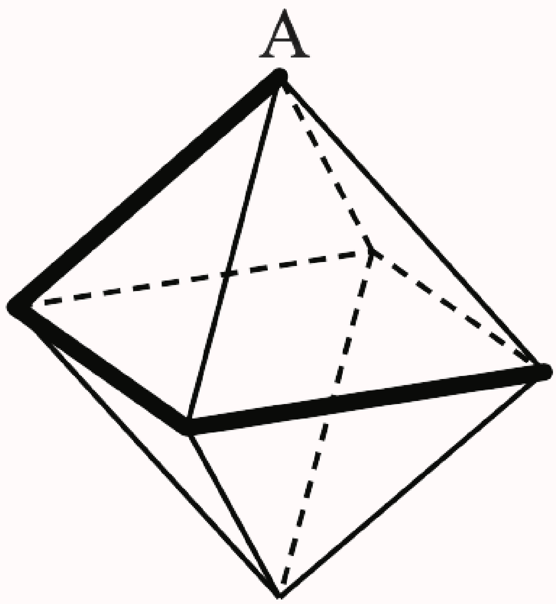
Jamal has a drawer containing 6 green socks, 18 purple socks, and 12 orange socks. After adding more purple socks, Jamal noticed that there is now a 60% chance that a sock randomly selected from the drawer is purple. How many purple socks did Jamal add?

Problem 4.7.24 (MATHCOUNTS)

A positive integer divisor of 11! is chosen at random. What is the probability that this divisor is prime? Express your answer as a common fraction.

Problem 4.7.25 (MATHCOUNTS)

Consider the regular octahedron as shown. Each edge of the octahedron has a length of 1. An ant starts at the vertex A and crawls a total distance of 3 units along the edges of the octahedron. Any time the ant reaches a vertex of the octahedron, it randomly chooses an edge to next crawl on that is different from the edge it just left. One such path the ant may take is shown. What is the probability that the ant will end up a distance greater than 1 from its starting point A ? Express your answer as a common fraction.



Answers

4.4 $\frac{5}{1296}$

4.6 $\frac{1}{6}$

4.8 78%

4.9 107

4.10 $\frac{2}{3}$

4.7.1 $\frac{1}{36}$

4.7.2 $\frac{1}{12}$

4.7.3 $\frac{5}{36}$

4.7.4 $\frac{3}{8}$

4.7.5 $\frac{5}{12}$

4.7.6 $\frac{3}{10}$

4.7.7 $\frac{1}{8}$

4.7.8 $\frac{2}{5}$

4.7.9 $\frac{4}{7}$

4.7.10 $\frac{5}{24}$

4.7.11 $\frac{5}{7}$

4.7.12 84

4.7.13 $\frac{1}{3}$

4.7.14 $\frac{7}{25}$

4.7.15 47

4.7.16 0.7

4.7.17 $\frac{3}{16}$

4.7.18 $\frac{1}{3}$

4.7.19 468

4.7.20 43

4.7.21 $\frac{35}{72}$

4.7.22 12

4.7.23 9

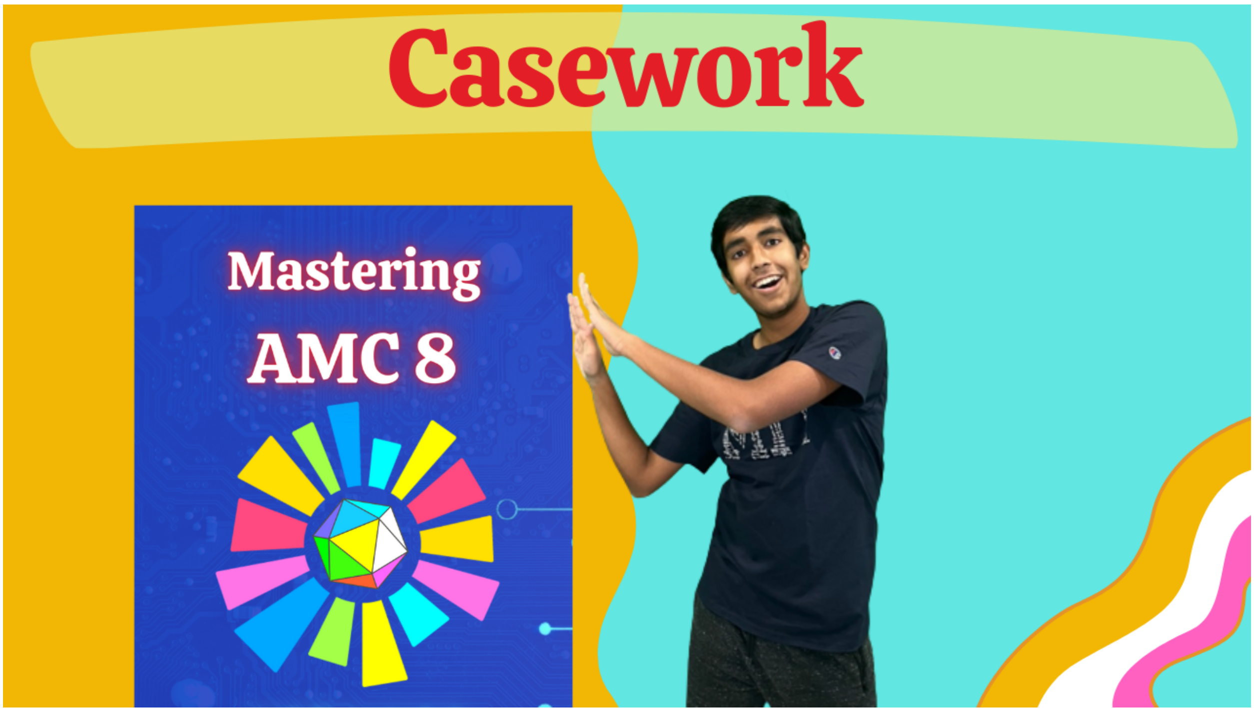
4.7.24 $\frac{1}{108}$

4.7.25 $\frac{2}{9}$

Chapter 5

Casework

Video Lecture



5.1 Casework Fundamentals

Concept 5.1.1

Casework is solving counting or probability problems by considering the different cases and adding them together. Many counting or probability problems can be solved by dividing a problem into several cases and calculating arrangements and probabilities for each case before summing them together.
You should try casework when there are no other obvious approaches as it is one of the most common techniques on the AMC 8.

Example 5.1

Two dice are rolled. What is the number of the ways that the sum of the numbers rolled is 3?

Solution

To do this, let's take cases based on the first roll.

How many ways are there for two dice to sum to 3?

Case 1: First roll is a 1

If the first roll is a 1, then the second roll must be 2.

Case 2: First roll is a 2

If the first roll is a 2, then the second roll must be 1.

Case 3: First roll is a 3 or more

If the first roll is anything 3 or more, then the sum will always be at least 4 (since the 2nd roll is at least 1), so this is not possible.

Therefore, in total, there are $\boxed{2}$ ways.

Example 5.2

Two cards are dealt from a deck of four red cards labeled A, B, C, D and four green cards labeled A, B, C, D. Winning pair is two of the same color or two of the same letter. How many ways are there to draw a winning pair?

Solution

This is a another casework problems. The first step is to identify the different cases.

What are the different ways to draw a winning pair?

We can either have the same color or the same letter. Note that having the same color and having the same letter are not possible as that would mean drawing the same card twice which is not possible. We consider the cases separately.

Case 1: The 2 cards are the same color.

For this case, we have 2 choices for which color both cards will be.

How many ways are there to select 2 cards from 4 cards of a given color?

Since there are 4 cards of each color, and we must select 2 of them, there are just $\binom{4}{2}$ ways to select 2 cards.

Therefore, the number of ways to pick 2 cards of the same color will just be the number of colors times the number of ways to select 2 cards from that color, which is

$$2 \times \binom{4}{2} = 2 \times \frac{4 \times 3}{2} = 12$$

Case 2: The 2 cards are the same letter.

For this case, we have 4 choices for which letter both cards will have.

Now, how many color choices are possible for the 2 cards?

Remember that order does not matter in our winning pair. For each letter, there are only 2 cards available to draw (red or green). Because we need to draw 2 cards for a winning pair, we need to draw both of the 2 cards available to form a winning pair, so there is only 1 way to do this. Another way to think about this is that there are $\binom{2}{2} = 1$ way to pick the colors.

Therefore, the number of ways to pick 2 cards of the same letter is just $4 \times 1 = 4$.

In total, the total number of winning pairs is the number of pairs of cards with the same color plus the number of pairs of cards with the same letter which is

$$12 + 4 = \boxed{16}$$

Example 5.3 (Omega Learn)

Sohil has 10 boxes. Each box has a green, orange, red, and blue ball. He randomly chooses one of the boxes and then uniformly at random picks a ball from that box. He repeats this process of randomly choosing any box (allowed to choose same box again) and randomly choosing a ball in that box. What is the probability that the balls are the same color?

Video Solution

5.2 Harder Casework Examples

Example 5.4 (Omega Learn)

There are 3 identical blue boxes, 6 identical green boxes, and 5 distinct items. Each of the items must be placed into either a green or blue box, and each box can contain a maximum of 1 item. Assuming all of the boxes are different, how many ways are there to do this?

Solution

There are 5 items to be put in some of the 9 green and blue boxes, so we must choose 5 of these 9 boxes to put the items into.

How can we divide this problem into cases?

We can choose our cases based on the number of green and blue boxes we are selecting. The cases are

1. 3 blue boxes, 2 green boxes
2. 2 blue boxes, 3 green boxes

3. 1 blue box, 4 green boxes

4. 0 blue boxes, 5 green boxes

Notice that this covers all the cases because it covers all possible number of blue boxes. Clearly, 6 green boxes is impossible since we only have to select 5 of them to put items into. Also, make sure to keep in mind that the boxes of the same color are identical.

Case 1: 3 blue boxes, 2 green boxes

For this case, how many ways are there to put 5 items into 3 blue and 2 green boxes?

We can count this be seeing that we simply have to choose 3 of the items to place into the blue boxes, and the other items will be put in green boxes. We don't have to worry about order of the items in the blue boxes (or the green boxes) because the boxes of the same color are identical. We can do this in $\binom{5}{3}$ ways.

Case 2: 2 blue boxes, 3 green boxes Again, we do this by choosing 2 of the items to put into blue boxes, and the remaining 3 items will be put into green boxes. We can do this in $\binom{5}{2}$ ways.

Case 3: 1 blue boxes, 4 green boxes Out of the 5 items, we must choose 1 of them for a blue box and the remaining 4 will go in green boxes. We can do this in $\binom{5}{1}$ ways.

Case 4: 0 blue boxes, 5 green boxes Because all of the items must go in green boxes, there is only 1 way to do this.

In total among all the cases, there are

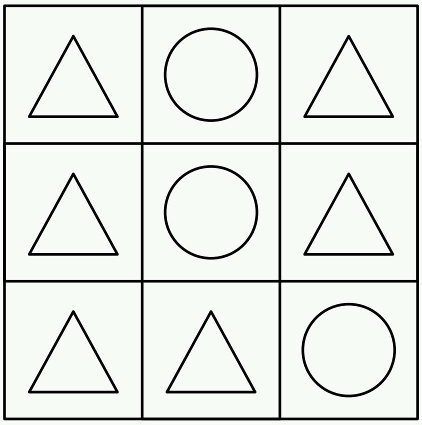
$$\binom{5}{3}+\binom{5}{2}+\binom{5}{1}+1=\frac{5\times4\times3}{3!}+\frac{5\times4}{2!}+\frac{5}{1!}+1=\frac{60}{6}+\frac{20}{2}+\frac{5}{1}+1=10+10+5+1=\boxed{26}$$

Remark 5.2.1

For casework problems, pick one attribute to change and find the number of ways in each of the cases. In the previous problem, this was the number of blue boxes. Sometimes, the problem may require subcases.

Example 5.5 (AMC 8)

A $\triangle$ or $\bigcirc$ is placed in each of the nine squares in a 3-by-3 grid. Shown below is a sample configuration with three $\triangle$s in a line. How many configurations will have three $\triangle$s in a line and three $\bigcirc$s in a line?



Video Solution

5.3 Practice Problems

Problem 5.3.1 (AMC 8)

How many different combinations of 5 dollar bills and 2 dollar bills can be used to make a total of 17 dollars if the order of the bills does not matter?

Video Solution

Problem 5.3.2 (AMC 8)

The "Middle School Eight" basketball conference has 8 teams. Every season, each team plays every other conference team twice (home and away), and each team also plays 4 games against non-conference opponents. What is the total number of games in a season involving the "Middle School Eight" teams?

Video Solution

Problem 5.3.3

How many 3-digit even numbers are there with distinct digits?

Video Solution

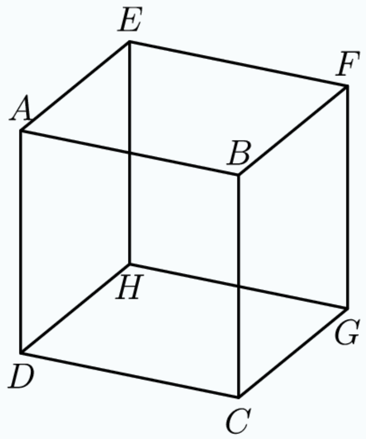
Problem 5.3.4 (AMC 8)

Paul owes Paula 35 cents and has a pocket full of 5-cent coins, 10-cent coins, and 25-cent coins that he can use to pay her. What is the difference between the largest and the smallest number of coins he can use to pay her?

Video Solution

Problem 5.3.5 (AMC 8)

How many pairs of parallel edges does a cube have?



Video Solution

Problem 5.3.6 (AMC 8)

A fair coin is tossed 3 times. What is the probability there will be at least 2 consecutive heads?

Video Solution

Problem 5.3.7 (AMC 8)

How many 4-digit positive integers have four different digits, where the leading digit is not zero, the integer is a multiple of 5, and 5 is the largest digit?

Video Solution

Problem 5.3.8 (AMC 10)

Three fair six-sided dice are rolled. What is the probability that the values shown on two of the dice sum to the value shown on the remaining die?

Video Solution

Problem 5.3.9 (AMC 10)

How many rearrangements of *abcd* are there in which no two adjacent letters are also adjacent letters in the alphabet? For example, no such rearrangements could include either *ab* or *ba*.

Video Solution

Problem 5.3.10 (AMC 10)

A license plate in a certain state consists of 4 digits, not necessarily distinct, and 2 letters, also not necessarily distinct. These six characters may appear in any order, except that the two letters must appear next to each other. How many distinct license plates are possible?

Video Solution

Problem 5.3.11 (Omega Learn Math Contest)

Given the set 1, 2, 3, 4, 5, 6, 7, how many non-empty subsets of this set are there such that the sum of the elements are even?

Video Solution

Problem 5.3.12 (MATHCOUNTS)

The Beavers, Ducks, Platypuses, and Narwhals are the only four basketball teams remaining in a single-elimination tournament. Each round consists of the teams playing in pairs with the winner of each game continuing to the next round. If the teams are randomly paired and each has an equal probability of winning any game, what is the probability that the Ducks and the Beavers will play each other in one of the two rounds? Express your answer as a common fraction.

Video Solution

Problem 5.3.13 (AMC 10)

A farmer's rectangular field is partitioned into 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat, soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field?

Video Solution

Problem 5.3.14 (AMC 10)

How many ways are there to place 3 indistinguishable red chips, 3 indistinguishable blue chips, and 3 indistinguishable green chips in the squares of a 3×3 grid so that no two chips of the same color are directly adjacent to each other, either vertically or horizontally?

Video Solution

Additional Problems

Problem 5.3.15 (AMC 8)

Abby, Bridget, and four of their classmates will be seated in two rows of three for a group picture, as shown.

X X X
X X X

If the seating positions are assigned randomly, what is the probability that Abby and Bridget are adjacent to each other in the same row or the same column?

Problem 5.3.16 (AMC 8)

When three positive integers a, b, and c are multiplied together, their product is 100. Suppose $a < b < c$. In how many ways can the numbers be chosen?

Problem 5.3.17 (AMC 8)

A palindrome is a number that has the same value when read from left to right or from right to left. (For example, 12321 is a palindrome.) Let N be the least three-digit integer which is not a palindrome but which is the sum of three distinct two-digit palindromes. What is the sum of the digits of N?

Problem 5.3.18 (EMCC)

Fred and George have a fair 8-sided die with the numbers 0,1,2,9,2,0,1,1 written on the sides. If Fred and George each roll the die once, what is the probability that Fred rolls a larger number than George?

Problem 5.3.19 (AMC 12)

On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of dots?

Problem 5.3.20

Three unique fair tetrahedral die, whose faces are numbered 1, 2, 3, and 4 are rolled. How many ways can the sum of the numbers rolled be 7?

Problem 5.3.21 (EMCC)

Chad and Jordan independently choose two-digit positive integers. The two numbers are then multiplied together. What is the probability that the result has a units digit of zero?

Problem 5.3.22 (BMMT)

At the Berkeley Sandwich Parlor, the famous BMT sandwich consists of up to five ingredients between the bread slices. These ingredients can be either bacon, mayo, or tomato, and ingredients of the same type are indistiguishable. If there must be at least one of each ingredient in the sandwich, and the order in which the ingredients are placed in the sandwich matters, how many possible ways are there to prepare a BMT sandwich?

Problem 5.3.23 (Omega Learn)

How many 4-digit numbers exist such that the 2nd digit is even, the 3rd digit is a multiple of 3 that is not a multiple of 2, and the 4th digit is a multiple of 4 if all of the digits are greater than 0 and different?

Problem 5.3.24 (MATHCOUNTS)

The number 40,231 is a five-digit positive integer that uses five consecutive digits, although not necessarily in order. How many such five-digit numbers are there?

Problem 5.3.25 (MATHCOUNTS)

In the grid shown, how many ways are there to spell the word "QUEUE" by moving one square at a time either horizontally or vertically, and provided squares may be revisited?

E	U	E	U	E
U	E	U	E	U
E	U	Q	U	E
U	E	U	E	U
E	U	E	U	E

Problem 5.3.26 (EMCC)

Jasmine rolls a fair 6-sided die, with faces labeled from 1 to 6, and a fair 20-sided die, with faces labeled from 1 to 20. What is the probability that the product of these two rolls, added to the sum of these two rolls, is a multiple of 3?

Answers

5.3 $\frac{9}{40}$

5.5 84

5.3.1 2

5.3.2 88

5.3.3 328

5.3.4 5

5.3.5 18

5.3.6 $\frac{3}{8}$

5.3.7 84

5.3.8 $\frac{5}{24}$

5.3.9 2

5.3.10 $5 \times 10^4 \times 26^2$

5.3.11 63

5.3.12 $\frac{1}{2}$

5.3.13 84

5.3.14 36

5.3.15 $\frac{7}{15}$

5.3.16 4

5.3.17 2

5.3.18 $\frac{23}{64}$

5.3.19 $\frac{11}{21}$

5.3.20 12

5.3.21 $\frac{27}{100}$

5.3.22 192

5.3.23 72

5.3.24 696

5.3.25 132

5.3.26 $\frac{13}{60}$

Chapter 6

Complementary Counting

Video Lecture

6.1 Complementary Counting Fundamentals

Complementary counting is the problem solving technique of counting the opposite of the problem constraint and subtracting that from the total number of cases. In combinatorics problems, the keyword "at least" generally indicates that complementary counting may be helpful.

Example 6.1

I roll 2 fair 6-sided dice. How many ways are there for the sum of the numbers to be 11 or less?

Solution

We could do this problem by casework by considering the number of ways to roll all of the numbers from 1 to 11. However, this will take a lot of time and is very tedious. Instead, let's try finding the opposite of what we are trying to count.

What is the opposite of rolling a sum of numbers 11 or less?

Notice that the sum of 2 numbers on 2 dice is at most 12 because the maximum roll on each dice is 6. Therefore, the only possibility for the sum of numbers on 2 dice that results in it not being 11 or less is for the sum to be 12.

How many ways are there to have a sum of 12 on 2 dice?

The only possibility for this is if both dice roll a 6, so there is only 1 way.

Next, we must find the total number of ways to roll 2 dice. This is just $6 \times 6 = 36$.

Then, how many rolls of 2 dice result in a sum that is 11 or less?

This is just the number of total possible rolls minus the number of rolls that sum to a number more than 11, which is

$$36 - 1 = \boxed{35}$$

Example 6.2 (Omega Learn)

You have 7 slips of paper numbered 1 to 7. How many ways are there to choose any subset of them so that you have at least 2 odd numbers and 1 even number?

Video Solution

Example 6.3 (AMC 10)

When a certain unfair die is rolled, an even number is 3 times as likely to appear as an odd number. The die is rolled twice. What is the probability that the sum of the numbers rolled is even?

Video Solution

Remark 6.1.1
Often we might also have to use the fact that the number of ways to choose any number of items from n items is 2^n in conjunction with complementary counting.

Example 6.4 (AIME)

A positive integer is called ascending if, in its decimal representation, there are at least two digits and each digit is less than any digit to its right. How many ascending positive integers are there?

Video Solution

6.2 Complementary Counting with Casework

Example 6.5

I roll 2 fair 6-sided dice. What is the probability that the sum of the numbers is 5 or more?

Solution

In this problem, we will first find the number of ways to have a sum of 5 or more. We could do casework, but there is a much faster solution with complementary counting.

What is the opposite of rolling a sum of numbers 5 or more?

Because the minimum sum of 2 dice is 2 (since each dice must roll at least a 1), the dice rolls must sum to 2, 3, or 4 for the sum to not be 5 or more. We divide this into cases.

Case 1: The 2 dice sum to 2

There is only 1 way for this to happen (both dice are 1).

Case 2: The 2 dice sum to 3

If the first dice roll is a 1, then the 2nd dice roll must be a 2. If the first dice roll is a 2, then the 2nd dice roll must be a 1. If the first dice roll is 3 or more, then the sum will be at least 4 or more, so this case doesn't count.

So, in total there are 2 ways for the sum to be 3.

Case 3: The 2 dice sum to 4

If the first dice roll is a 1, then the 2nd dice roll must be a 3. If the first dice roll is a 2, then the 2nd dice roll must be a 2. If the first dice roll is a 3, then the 2nd dice roll must be a 1. If the first dice roll is 4 or more, then the sum will be at least 5 or more, so this case doesn't count.

So in total there are 3 ways for this case.

Therefore, amongst all 3 cases, there are a total of $1+2+3=6$ ways to not have a sum of 5 or more.

Next, the total number of dice rolls is $6\times 6=36$. Therefore, the number of rolls resulting in a sum of 5 or more is just $36-6=30$.

Therefore, the probability of this happening is

$$\frac{30}{36}=\boxed{\frac{5}{6}}$$

Remark 6.2.1
Often times, we have to use complementary counting in conjunction with other concepts like casework and probability.

Example 6.6

How many subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8 , 9, 10\}$ have at least 1 even number and do not contain all of the elements?

Solution

How can we use complementary counting?

We can find the total number of subsets and subtract the subsets which have no even numbers and contain all of the elements. Note that both having no even numbers and containing all of the elements cannot happen, so we don't have to worry about overlap between the subsets that don't work.

Case 1: Subset contains all of the elements

There is only 1 subset that contains all of the elements (the original set).

Case 2: Subset contains no even elements

If there are no even elements, then we simply have to pick some odd numbers from $\{1, 3, 5, 7, 9\}$ for our subset.

How many ways are there to do this?

Remember that each element has 2 choices (to be in subset or not in subset), so the number of ways to form our subset from just odd numbers is $2^5 = 32$.
Therefore in total, $1+32 = 33$ ways don't work.

Next, how many total subsets are there?

Since there are 10 elements and each element has 2 choices, there are $2^{10} = 1024$ subsets.

Therefore the number of valid subsets is $1024-33 = \boxed{991}$

Example 6.7 (Omega Learn)

There are 3 sections of a dart board. The outermost section is worth 2 point if hit. The middle section is worth 5 points if hit. The innermost section is worth 10 points if hit. The probability you hit inner section is $\frac{1}{4}$, the probability you hit the middle section is $\frac{3}{8}$, the probability you hit the outermost section is $\frac{2}{8}$, and the probability you miss the dartboard entirely is $\frac{1}{8}$. You throw 3 darts at the dartboard. What is the probability you get less than 20 points?

Video Solution

6.3 Practice Problems

Problem 6.3.1

If you toss four coins, in how many ways will there be at least 1 head?

Video Solution

Problem 6.3.2

If you toss four coins, in how many ways will there be at least 2 heads?

Video Solution

Problem 6.3.3

How many three digit numbers contain the digit 5 at least once?

Video Solution

Problem 6.3.4

How many 3 digit numbers have at least 2 digits that are the same?

Video Solution

Problem 6.3.5

How many ways are there to select a group of 3 people from 5 women and 3 men if there must be at least 1 man in the group?

Video Solution

Problem 6.3.6 (AMC 8)

A box contains five cards, numbered 1, 2, 3, 4, and 5. Three cards are selected randomly without replacement from the box. What is the probability that 4 is the largest value selected?

Video Solution

Problem 6.3.7

How many 4 digit numbers are not palindromes? A palindrome is a number that reads the same forward and backward.

Video Solution

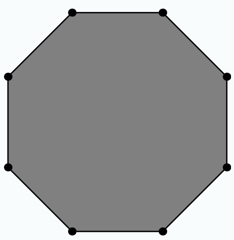
Problem 6.3.8 (Omega Learn Math Contest)

Alice, Bob, Claire, Dave, Emma are sitting around a round table. Alice refuses to sit next to Bob. How many different ways can they sit around the table? (Rotations of the same arrangement are not considered different).

Video Solution

Problem 6.3.9 (AMC 8)

From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?



Video Solution

Problem 6.3.10 (AMC 12)

How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

Video Solution

Problem 6.3.11

There are 9 different flavors of ice cream at an ice cream shop. Joe wants to buy 2 or 3 scoops of different ice cream flavors. He doesn't like 2 of the flavors Mango and Chocolate together so won't buy them together. How many ways can he buy ice cream?

Video Solution

Additional Problems

Problem 6.3.12 (AMC 10)

Una rolls 6 standard 6-sided dice simultaneously and calculates the product of the 6 numbers obtained. What is the probability that the product is divisible by 4?

Problem 6.3.13 (AMC 8)

Zara has a collection of 4 marbles: an Aggie, a Bumblebee, a Steelie, and a Tiger. She wants to display them in a row on a shelf, but does not want to put the Steelie and the Tiger next to one another. In how many ways can she do this?

Problem 6.3.14

There are 10 slips of paper numbered 1 to 5 in a bag, each equally likely to be drawn.

You draw a slip, and then immediately put it back in the bag. What is the least number of slips you must draw in order to have at least a $\frac{1}{2}$ chance of drawing the number 3?

Problem 6.3.15

You have 6 identical pears and 5 identical apples. You choose 4 fruits at random. What is the probability of picking at least 1 pear?

Answers

6.2 77

6.3 $\frac{5}{8}$

6.4 902

6.7 189/256

6.3.1 15

6.3.2 11

6.3.3 252

6.3.4 252

6.3.5 46

6.3.6 $\frac{3}{10}$

6.3.7 8910

6.3.8 12

6.3.9 $\frac{5}{7}$

6.3.10 240

6.3.11 112

6.3.12 $\frac{59}{64}$

6.3.13 12

6.3.14 4

6.3.15 $\frac{65}{66}$

Chapter 7

Principle of Inclusion Exclusion (PIE)

Video Lecture

7.1 PIE Strategies

Definition 7.1.1. Overcounting is the process of counting more than what you need and then systematically subtracting the parts which do not belong.

Definition 7.1.2. The principle of inclusion and exclusion (PIE) is a counting technique that uses overcounting to compute the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

7.2 PIE for 2 Events

Example 7.1

Find how many numbers from 1 to 100 (inclusive) that are divisible by 2 or 7.

Solution

To solve this problem, let's first find how many numbers are divisible by 2 and how many numbers are divisible by 7

How many numbers from 1 to 100 are divisible by 2?

Every 2nd number is divisible by 2, so all the numbers from 2×1 to 2×50 work. Therefore, from 1 to 100 there are 50 numbers divisible by 2.

How many numbers from 1 to 100 are divisible by 7?

Every 7th number is divisible by 7, so all of the numbers from 7×1 to 7×14 work. We calculate this by seeing that $\frac{100}{7}$ is a little more than 14 so 7×14 is the largest multiple of 7 less than 100. Therefore, there from 1 to 100 are 14 numbers divisible by 7.

Are there any numbers overcounted between the multiples of 2 and the multiples of 7?

For a number to be both a multiple of 2 and a multiple of 7, it must be a multiple of $7 \times 2 = 14$. Therefore, we are overcounting these numbers since we are counting them twice as part of the multiples of 2 and multiples of 7.

How many numbers from 1 to 100 are divisible by 14?

Every 14th number is divisible by 14, so all of the numbers from 14×1 to 14×7 work. Therefore, from 1 to 100 there are 7 numbers divisible by 14.

Therefore, in total, the number of numbers that are multiple of 2 or 7 are the number of multiples of 2 plus the number of multiples of 7 minus the multiples of 14.

This gives us an answer of $50+14-7=\boxed{57}$.

Definition 7.2.1 (Union Symbol)**.** $|A\cup B|$ is the union of elements in both A and B (i.e. all elements that are in set A or set B, but duplicates are only written once)

Definition 7.2.2 (Intersection Symbol)**.** $|A\cap B|$ is the intersection of elements in both A and B (i.e. only those elements which are in both sets)

Theorem 7.2.3 (Principle of Inclusion Exclusion for 2 Sets)
Given two sets, $|A_1|$ and $|A_2|$

$$|A_1\cup A_2| = |A_1|+|A_2|-|A_1\cap A_2|$$

Basically, we count the number of possibilities in 2 "things" and subtract the duplicates.

Example 7.2 (Omega Learn)

How many subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ have at most 4 even numbers and at most 2 elements that are multiples of 3?

This is another problem we will solve using the principle of inclusion and exclusion and the subsets formula we learned earlier.

Solution

How we can apply the Principle of Inclusion and Exclusion to this problem?

Notice how there are 5 even numbers, and we have to find how many subsets have at most 4 even numbers. Similarly, there are 3 multiples of 3, and we have to find how many elements have at most 2 multiples of 3.
We can see that there are fewer cases that don't satisfy the condition, so it is better to use complementary counting. We will first count the number of subsets that have more than 4

even numbers and have more than 2 multiple of 3.

First, how many subsets have more than 4 even numbers?

Since there are 5 multiples of 2 from 1 to 10 (2, 4, 6, 8, 10), to have more than 4 multiples of 2 means having all of the multiples of 2 (only 1 way to choose this). From the remaining 5 numbers, there are 2^5 possible subsets. Therefore, in total, there are $1 \times 2^5 = 2^5$ subsets that have more than 4 multiples of 2.

Next, how many subsets have more than 2 multiples of 3?

Since there are 3 multiples of 3 from 1 to 10 (3, 6, 9), to have more than 2 multiples of 3 means having all of the multiples of 3 (only 1 ways to choose this). From the remaining 7 numbers, there are 2^7 possible subsets. By similar logic, in total, there will be $1 \times 2^7 = 2^7$ possible subsets.

How many subsets are we overcounting that have all of the even numbers and multiples of 3 from 1 to 10?

For a subset to have all the multiples of 2 and 3, it must contain all the elements from 2, 3, 4, 6, 8, 9, 10 (only 1 way to choose this). From the remaining 3 numbers, there are $1 \times 2^3 = 2^3$ possible subsets.

Therefore, in total, the number of subsets that don't satisfy our original condition is $2^7 + 2^5 - 2^3 = 128 + 32 - 8 = 152$.
The total number of possible subsets of the original set of 10 numbers is $2^{10} = 1024$, so the number of valid subsets is $1024 - 152 = \boxed{872}$

Remark 7.2.4
When the number of cases that don't work is much less than the number of cases that do work, it can often be helpful to use complementary counting along with PIE.

Example 7.3 (Omega Learn)

Mark and George are playing a guessing game. George randomly chooses 2 numbers between 1 and 25. Mark then asks George if exactly 1 of the 2 numbers is divisible by 7. George, truthfully, answers yes. Mark then proceeds to guess 1 multiple of 7 and 1 number that is not a multiple of 7. What is the probability that Mark guesses at least 1 number correctly?

Video Solution

Example 7.4 (Omega Learn)

How many different ways are there to arrange the letters in the word LOLLIPOP if each of the L's must be next to an O?

Video Solution

7.3 PIE for 3 Events

Theorem 7.3.1 (Principle of Inclusion Exclusion for 3 Sets)
Given three sets, $|A_1|$, $|A_2|$, $|A_3|$,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

In this formula, we count the number of possibilities in 3 ”things”, subtract the possibilities that are duplicates in all 3 pairs of sets, and add back the number of duplicates that are in all 3 sets.

Example 7.5

How many numbers less than or equal to 193 are multiples of 2, 3, or 5?

Solution

First, let's find how many numbers are divisible by 2, how many numbers are divisible by 3, and how many numbers are divisible by 5.
Every 2nd number is divisible by 2, so all the numbers from 2×1 to 2×96 work. Therefore, there are 96 numbers divisible by 2 less than or equal to 193

Every 3rd number is divisible by 3, so all the numbers from 3×1 to 3×64 work. Therefore, there are 64 numbers divisible by 3 less than or equal to 193

Every 5th number is divisible by 5, so all the numbers from 5×1 to 5×38 work. Therefore, there are 38 numbers divisible by 5 less than or equal to 193.

Next, we must subtract numbers divisible by two of the numbers 2, 3, or 5.

For this to happen, what values must the numbers be divisible by?

Numbers divisible by 2 and 3 or numbers divisible by $2 \times 3 = 6$ are counted twice. Similarly, numbers divisible by 2 and 5 or $2 \times 5 = 10$ are counted twice and numbers divisible by 3 and 5 or $3 \times 5 = 15$ are counted twice. Therefore, we are overcounting multiples of each of 6, 10, and 15, so we must subtract them.

Every 6th number is divisible by 6, so all the numbers from 6×1 to 6×32 work. Therefore, there are 32 numbers divisible by 6 less than or equal to 193
Every 10th number is divisible by 10, so all the numbers from 10×1 to 10×19 work. Therefore, there are 19 numbers divisible by 10 less than or equal to 193
Every 15th number is divisible by 5, so all the numbers from 15×1 to 15×12 work. Therefore, there are 12 numbers divisible by 15 less than or equal to 193.

If we subtract the overcounted numbers above, did we remove too much?

Next, we look at how many times multiples of 2, 3, and 5, or multiples of $2 \times 3 \times 5 = 30$ were counted. Well, they were originally counted 3 times as part of the multiples of 2, 3, and 5. Then, they were subtracted 3 times as part of the multiples of 6, 10, and 15. Therefore, they have been counted 0 times! But, we need to count them exactly once, so we must add back the number of multiples of $2 \times 3 \times 5 = 30$.

Every 30th number is divisible by 30, so all the numbers from 30×1 to 30×6 work. Therefore, there are 6 numbers divisible by 30 less than or equal to 193.

In total, to find the number of multiples of 2, 3, or 5 we have to add the number of multiples of 2, the number of multiples of 3, the number of multiples of 5. Then, we have to subtract the number of multiples of 6, 10, and 15. And finally, we have to add back the number of multiples of 30.

This gives us an answer of $96+64+38-32-19-12+6=\boxed{141}$.

Example 7.6 (AMC 8)

Five different awards are to be given to three students. Each student will receive at least one award. In how many different ways can the awards be distributed?

Video Solution

7.4 PIE for any Number of Events (Optional)

Concept 7.4.1 (PIE Generalization)

The same principle to applies to more than when counting more than 3 events. For any n events, to find the total number of ways such that any of the events occur

1. Find number of ways each individual event occurs in.
2. For every pair of 2 events, subtract number of ways where both of them occur
3. For every triplet of 3 events, add back number of ways where all 3 of them occur
4. For every quadruplet of 4 events, subtract number of ways where all 4 occur $\vdots$
5. For all n events, add (if n is odd) or subtract (if n is even) number of ways where all of them occur

Alternate between adding the number of ways and subtracting the number of ways until you reach the case where all n events occur. Whether you add or subtract the number of ways where all n events occur depends on whether n is odd since we alternate between adding and subtracting.

Theorem 7.4.2 (Principle of Inclusion Exclusion Generalized Formula)
Stated more formally, if $(A_i)_{1\le i\le n}$ are finite sets, then:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}|A_i| - \sum_{i<j}|A_i\cap A_j| + \sum_{i<j<k}|A_i\cap A_j\cap A_k| - \cdots + (-1)^{n-1}|A_1\cap\cdots\cap A_n|$$

.

Remark 7.4.3
Don't be afraid of the complex notation. There is no need to memorize or understand this formula if you understand the concept above

7.5 Practice Problems

Problem 7.5.1

How many positive numbers less than or equal to 100 are multiples of 2 or 3?

Video Solution

Problem 7.5.2

How many positive numbers less than or equal to 100 are NOT multiples of 2 or 3?

Video Solution

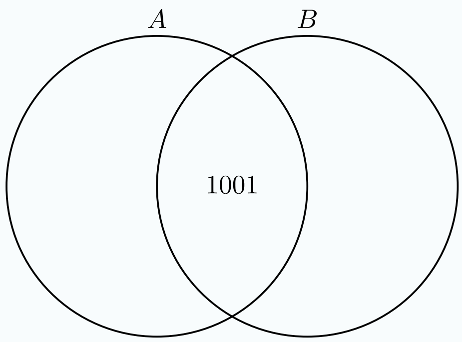
Problem 7.5.3

Out of 200 students, there are 100 taking Geometry, 70 taking Algebra, and 30 taking both. How many students are taking neither?

Video Solution

Problem 7.5.4 (AMC 8)

Sets A and B, shown in the Venn diagram, have the same number of elements. Their union has 2007 elements and their intersection has 1001 elements. Find the number of elements in A.



Problem 7.5.5

In a town of 351 adults, every adult owns a car, motorcycle, or both. If 331 adults own cars and 45 adults own motorcycles, how many of the car owners do not own a motorcycle?

Problem 7.5.6 (AMC 8)

At Euler Middle School, 198 students voted on two issues in a school referendum with the following results: 149 voted in favor of the first issue and 119 voted in favor of the second issue. If there were exactly 29 students who voted against both issues, how many students voted in favor of both issues?

Video Solution

Problem 7.5.7

How many positive numbers less than or equal to 200 are multiples of 2 and 3 but not a multiple of 5?

Video Solution

Problem 7.5.8 (AMC 10)

There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

Problem 7.5.9 (AMC 8)

Mrs. Sanders has three grandchildren, who call her regularly. One calls her every three days, one calls her every four days, and one calls her every five days. All three called her on December 31, 2016. On how many days during the next year did she not receive a phone call from any of her grandchildren?

Video Solution

Problem 7.5.10 (AMC 10)

Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

Video Solution

Problem 7.5.11 (AIME)

Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left to right as it does right to left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Video Solution

Additional Problems

Problem 7.5.12 (BmMT)

Students are being assigned to faculty mentors in the Berkeley Math Department. If there are 7 distinct students and 3 distinct mentors, and each student has exactly one mentor, in how many ways can students be assigned to mentors such that each mentor has at least one student?

Problem 7.5.13 (AMC 12)

Call a number prime-looking if it is composite but not divisible by $2, 3, or 5$. The three smallest prime-looking numbers are $49, 77$, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

Problem 7.5.14 (MATHCOUNTS)

How many different subsets of {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} contain at least one element in common with each of the sets {2, 4, 6, 8, 10, 12}, {3, 6, 9, 12} and {2, 3, 5, 7, 11}?

Answers

7.3 $\frac{4}{11}$

7.4 108

7.6 150

7.5.1 67

7.5.2 33

7.5.3 60

7.5.4 1504

7.5.5 306

7.5.6 99

7.5.7 27

7.5.9 146

7.5.10 28

7.5.11 059

7.5.12 1806

7.5.13 100

7.5.14 3682

Chapter 8

Stars and Bars

Video Lecture

8.1 Stars and Bars Fundamentals

Example 8.1

How many ways are there to distribute 8 identical computers amongst 3 students?

Solution

Make sure to keep in mind that the computers are identical. If the computers were not identical, then we would just have 3 choices for each computer, so our answer would just be 3^8.

However, when the computers are identical, only the number of computers each student has matters. To start, let's say we have 8 computers in a line: CCCCCCCC. Therefore, we must divide these computers amongst 3 students.

How can we represent the 8 computers being divided amongst the 3 students?

We can think about having 3 groups. We can simply let the leftmost group be for student one, the middle group for student 2, and the rightmost group for student 3 because only the number of computers each student has matters, so ordering is irrelevant.

How can we divide the 8 C's amongst 3 groups?

We can do this by just placing 2 bars somewhere in between the 8 C's! Everything left of the first bar goes to student 1, everything in the middle goes to student 2, and everything to the right of the 2nd bar goes to student 3. Therefore, the number of ways to distribute the computers is simply the number of ways to insert 2 bars in between 8 C's. Examples:

$CC|CCCCC|C$ - 2 for student 1, 5 for student 2, 1 for student 3
$|CCCC|CCCC$ - 0 for student 1, 4 for student 2, 4 for student 3
$C|CCCCCCC|$ - 1 for student 1, 7 for student 2, 0 for student 3
$CCCCCCCC||$ - 8 for student 1, 0 for student 2, 0 for student 3

How do we find the number of ways to insert 2 bars in between 8 C's?

We can think of having 10 slots (each slot for either a bar or a C). Then, we can choose 2 of these slots for bars.

Therefore, the numbers of ways to select two slots out of the 10 slots (i.e. choose two slots to put the two bars out of the 10 slots) is

$$\binom{10}{2} = \frac{10 \times 9}{2} = \boxed{45}$$

Example 8.2

How many ways are there to distribute 8 identical pencils amongst 3 people?

Video Solution

Theorem 8.1.1 (Stars and Bars)
The number of ways to place n indistinguishable objects into k distinguishable bins is

$$\binom{n+k-1}{n}$$

Remark 8.1.2
The reason this is true is because you can consider placing $k-1$ bars in between n objects which would be equivalent to having $n+(k-1)$ slots and choosing $k-1$ slots from those. This can be written as:

$$\binom{n+k-1}{k-1}$$

which is equivalent to

$$\binom{n+k-1}{n}$$

This general idea is important to understand because tricky problems will often add additional conditions.

Remark 8.1.3
Make sure to remember that Stars and Bars only works if the objects that are being distributed are identical!

8.2 Stars and Bars with Constraints

Concept 8.2.1
Stars and bars is extremely useful, and can often be adapted based on situations. For example, if each bin has to have at least 1 object in it we assign each bin 1 object to start off with and apply our formula with $n-k$ objects and k distinguishable bins.

Example 8.3

7 astronauts are stranded in space with only 10 identical meals left. Every astronaut must receive at least 1 meal or else, they will starve. How many ways are there to distribute meals under this constraint?

Solution

This problem is very similar to the previous problem, except we now have an additional constraint, so the formula won't work. Let's focus on the constraint, since it's the most complicated part of the problem.

How can we deal with the condition that every astronaut must get at least 1 meal?

If every astronaut must receive at least 1 meal and all the meals are identical, we can simply pre-distribute 7 of the meals and give 1 to each astronaut. Notice that now, no matter where the remaining meals are distributed, every astronaut will get 1 meal or more.

Now, how can we use stars on bars on the remaining problem?

After 7 meals are distributed, only 3 remain. There is no constraint on who these 3 meals can go to, so it can go to any of the 7 astronauts. To divide these 3 meals into 7 groups, we can place 6 bars in between the 3 meals. Then, out of the 3+6=9 slots for meals and bars, we must choose 6 of them for bars.

We can do this in $\binom{9}{6} = \binom{9}{3} = \frac{9\times8\times7}{3!} = \frac{9\times8\times7}{6} = \boxed{84}$.

Example 8.4

What's the probability that the sum of the top faces on five regular 6-sided dice is 24?

Solution

Before we begin, note that each dice must be a value from 1 to 6. There is no way to use stars and bars directly because it will not account for the constraint that the maximum of each dice roll is a 6.

How can we use stars and bars for this problem?

Let the default for each dice be a 6. Then the maximum score is 30, we must essentially distribute 6 negatives to the 5 dice (-1, -1, -1, -1, -1, -1).
We can see that to distribute 6 negatives to 5 dice, we must have 4 bars to separate the negatives into 5 groups for the dice. We then have 10 slots and must choose 4 of them for bars. We can do this in $\binom{10}{4} = 210$ ways.

Are there any cases we are overcounting?

Because the minimum value of a dice roll is 1, each dice can only take at most 5 negatives. Therefore, we are overcounting the possibilities where 1 dice receives 6 negatives.

How many ways are there for 1 dice to have 6 negatives?

There are 5 dice, so there are 5 choices for which dice will have all 6 negatives. From here, we can calculate our final answer by subtracting the 5 ways that result in 1 dice receiving all 6 negatives from $\binom{10}{4}$. This gives us our answer, the number of ways to distribute 6 negatives to 5 dice so that no dice gets more than 5 negatives.

The answer is $\binom{10}{4} - 5 = \boxed{205}$

Example 8.5

How many ways are there to choose positive integers a, b, c, and d such that $a+b+c+d < 13$?

Video Solution

Example 8.6

How many ways are there for 7 people to sit in a line if 3 of them are enemies and refuse to sit next to each other?

We will worry about the ordering about the 3 enemies and remaining 4 people later. For now, let's find the number of possible ways to choose 3 locations for the 7 enemies.

Without the condition, how many ways are there to do this?

There are 7 possible locations and the enemies must sit at 3 of them so $\binom{7}{3}$.

What must happen so that the enemies don't sit next to each other?

There must be at least 1 other person between each of the enemies. Let w be the number of people left of the 1st enemy, let x be the number of people between the 1st and 2nd enemies, let y be the number of people between the 2nd and 3rd enemies, and let z be the number of people right of the 3rd enemy.

What conditions must w, x, y, z satisfy?

Firstly, since there are 4 people who are not enemies, we must have $w + x + y + z = 4$. However, by our observation earlier, x and y must be at least 1.

How can we use stars and bars to find the number of solutions to this equation?

We start by assigning 1 star to x and y. Then, there are $\binom{5}{3}$ solutions by stars and bars.

How many orderings of the enemies and non-enemies are possible?

There are 3! ways to order the enemies and 4! ways to order the other people so our answer is $\binom{5}{3} \times 3! \times 4! = \boxed{1440}$

Example 8.7 (Omega Learn)

You have 11 apples and 5 bananas. You place them into 3 baskets. How many ways are there to do this if each basket must have more apples than bananas and at least 1 of any fruit?

Video Solution

8.3 Practice Problems

Problem 8.3.1

How many ways are there to distribute 5 identical candies to 3 children?

Video Solution

Problem 8.3.2 (AMC 10)

When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n?

Video Solution

Problem 8.3.3 (AMC 10)

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

Video Solution

Problem 8.3.4 (AMC 8)

Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

Video Solution

Problem 8.3.5 (Omega Learn Math Contest)

Three squirrels have to distribute 19 identical acorns amongst themselves.

- The first squirrel demands to have a positive odd number of acorns.
- The second squirrel is fine with any non-negative even number of acorns.
- The third squirrel insists he must get an even number of acorns greater than or equal to 4.

How many different ways can they distribute the acorns?

Video Solution

Additional Problems

Problem 8.3.6

How many positive 5 digit integers exist such that each digit is greater than or equal to the previous digit?

Problem 8.3.7 (Omega Learn)

Find the number of solutions to $a+b+c+d<19$ where a, b, c, d are positive integers, a and c must be odd, and b and d must be even.

Answers

8.2 45

8.5 495

8.7 210

8.3.1 21

8.3.2 84

8.3.3 28

8.3.4 190

8.3.5 36

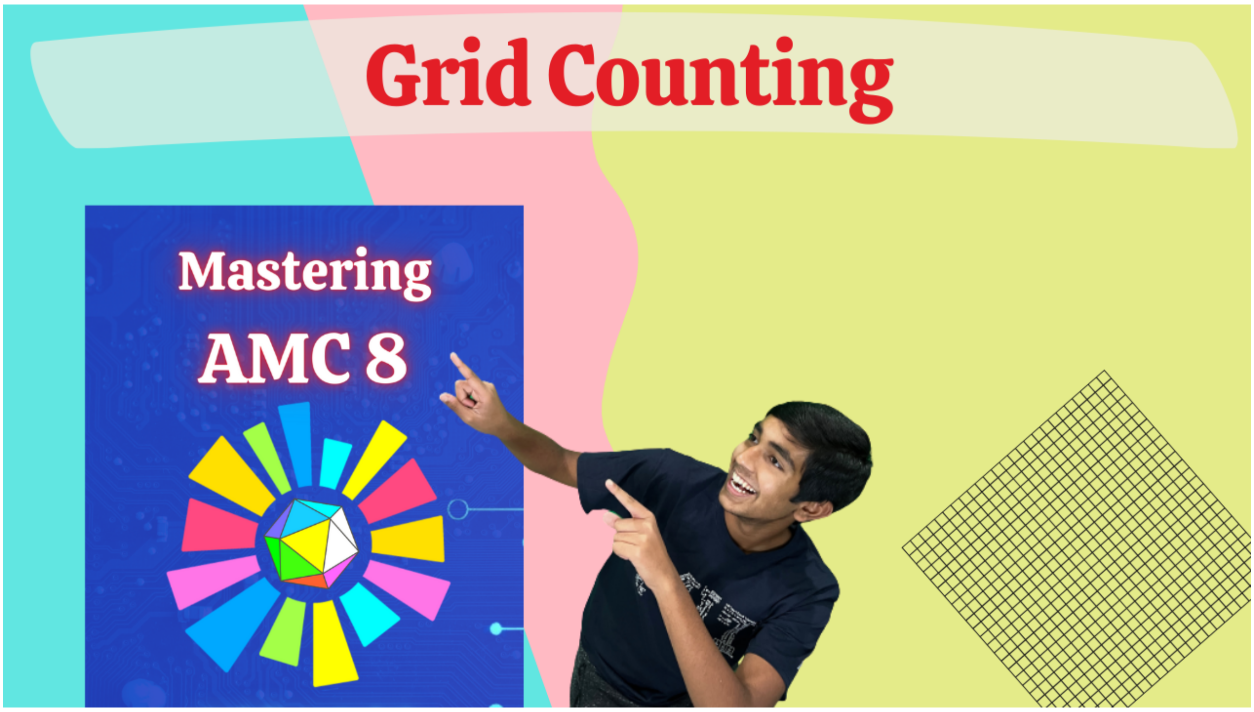
8.3.6 1287

8.3.7 210

Chapter 9

Geometric Counting

Video Lecture

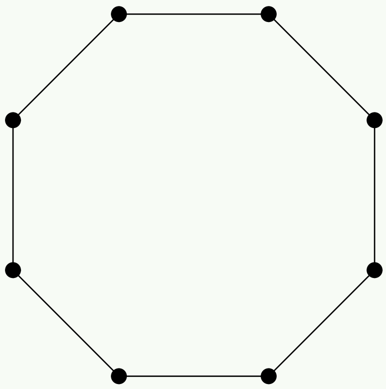



Great Pathfinding Adventure
Mastering
AMC 8

9.1 Geometric Counting Fundamentals

Example 9.1

How many triangles can be formed by connecting vertices in a regular octagon?



Solution

You could solve this by casework on what type of triangle you will select. However, there is a very smart and clever solution.

How many vertices defines a triangle?

This seems like a silly question, obviously 3.

How many ways are there to select 3 vertices from an octagon?

Since an octagon has 8 vertices, this will just be $\binom{8}{3}$.

Do all combinations of 3 vertices form a triangle?

This is a very important question to ask because it's not always true. For example, if 3 points are collinear (you can draw a line through all 3 of them) then the answer would actually be no, and we would have to subtract the extra cases. Luckily, in an octagon, no 3 points are collinear, so all combinations of 3 vertices do indeed form a triangle.

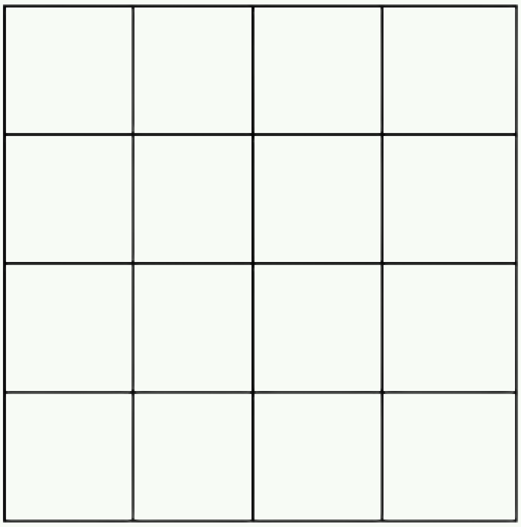
Therefore, our answer is just

$$\binom{8}{3} = \frac{8 \times 7 \times 6}{3!} = \frac{8 \times 7 \times 6}{6} = \boxed{56}$$

9.2 Number of Squares in Grid

Example 9.2

How many squares of all sizes can be formed from a 4x4 grid of unit squares?



Solution

Keep in mind that the answer is not 16 because we can't just count unit squares!

What's the easiest and most systematic way to count all possible squares?

We can break the problem down into 1x1, 2x2, 3x3, and 4x4 squares.

Case 1: 1x1 squares

For this case, there are just $4 \times 4 = 16$ squares.

Case 2: 2x2 squares

Notice how there are 3 possible choices of rows and 3 possible choices of columns to form a 2x2 square. In total, there are $3 \times 3 = 9$ possible 2x2 squares since each choice of two rows and two columns produces a 2x2 square.

Case 3: 3x3 squares

Notice how there are 2 possible choices of rows and 2 possible choices of columns to form a 3x3 square. In total, there are $2 \times 2 = 4$ possible 3x3 squares since each choice of two rows and two columns produces a 3x3 square.

Case 4: 4x4 squares

There is just one 4x4 square (the entire square).

In total, the number of squares of all sizes is

$$16+9+4+1 = \boxed{30}$$

Theorem 9.2.1

The number of squares in a $n \times n$ grid of squares is

$$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

.

Remark 9.2.2

This comes from the fact that there are n^2 1x1 squares, $(n-1)^2$ 2x2 squares, ..., 2^2 (n - 1)x(n - 1) squares, and 1^2 nxn square.

9.3 Number of Rectangles in Grid

Example 9.3

How many rectangles are in a 3x5 grid of rectangles?

Solution

This is another problem that can be solved by tedious casework but has a very slick solution.

What defines a rectangle in a grid?

4 points? Not quite because 4 points don't necessarily make a rectangle. 2 vertical lines and 2 horizontal lines define a rectangle as shown in the picture below. In other words, any 2 vertical lines and horizontal lines in this grid will form a rectangle.

How can we count the number of rectangles in a grid based on this?

We simply have to choose 2 vertical lines and 2 horizontal lines!

There are 4 horizontal lines, so there are $\binom{4}{2}$ ways to select 2 horizontal lines that define a rectangle. Similarly, there are 6 vertical lines, so there are $\binom{6}{2}$ ways to select 2 vertical lines that define a rectangle.

Therefore, the number of rectangles is just

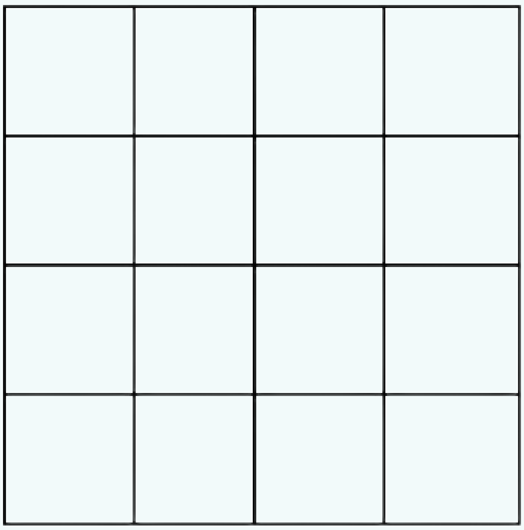
$$\binom{4}{2} \times \binom{6}{2} = \frac{4 \times 3}{2} \times \frac{6 \times 5}{2} = 6 \times 15 = \boxed{90}$$

Theorem 9.3.1

The general formula for the number of rectangles of all sizes in a rectangular grid of size $m \times n$ is

$$\binom{m+1}{2} \times \binom{n+1}{2}$$

.



Remark 9.3.2

Each combination of two horizontal lines and two vertical lines creates a unique rectangle.

We have
$$\binom{m+1}{2}$$
ways to choose two horizontal lines and
$$\binom{n+1}{2}$$
ways to choose two vertical lines.

Example 9.4 (Omega Learn)

How many rectangles, which are not squares, are in a 5×4 grid of squares?

Video Solution

Example 9.5 (Omega Learn)

How many squares with side length $\sqrt{5}$ can be found in a 5 by 5 grid of points?

Video Solution

9.4 Path Counting

Example 9.6

A rabbit is walking in a coordinate plane and has to get from (0,0) to the carrot at the point (5,5). If the rabbit can only move 1 unit up or 1 unit right on any given move, how many ways are there for the rabbit to get to the carrot?

Solution

This is yet another problem that is tedious to solve with casework. Instead, let's find a simpler way to solve this problem. Lets denote a right move by R and an up move by U.

How many right and up moves does the rabbit need to get to the carrot?

Because it's going from (0,0) to (5,5), it needs 5 right moves and 5 up moves. Therefore, we just need to count the number of ways to make 5 right moves and 5 up moves in any order.

How can we represent the number of ways to make 5 right moves and 5 up moves (in any order)?

This can be thought of as rearrangements of 5 R and 5 U: RRRRRUUUUU. We just need to find all possible rearrangements of this word. Given the total number of characters is 10, and counting the duplicates (5 R's and 5 U's), we can use the word rearrangement formula:

$$\frac{10!}{5! \times 5!} = 252$$

Theorem 9.4.1

The number of ways to get from (0,0) to a point (m,n) moving only up and right is

$$\binom{m+n}{m}$$

Remark 9.4.2

Imagine any arrangement string of m R's and n U's. Each string would correspond to a unique path, so to count this, we can use the word rearrangement formula.

Example 9.7

How many ways to go from (0, 0) to (5, 6) moving only right or up if you cannot visit all of (1, 2), (3, 4), and (5, 5) because there is a monster that will eat you if you visit all 3 locations?

[Video Solution]

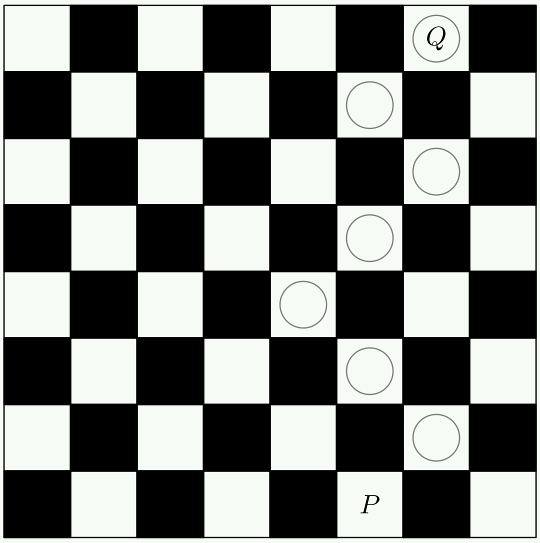
Example 9.8 (BMMT)

Sally is inside a pen consisting of points (a,b) such that $0 \le a \le b \le 4$. If she is currently on the point (x,y), she can move to either $(x,y+1)$, $(x,y-1)$, or $(x+1,y)$. Given that she cannot revisit any point she has visited before, find the number of ways she can reach (4, 4) from (0, 0).

Video Solution

Example 9.9 (AMC 8)

A game board consists of 64 squares that alternate in color between black and white. The figure below shows square P in the bottom row and square Q in the top row. A marker is placed at P. A step consists of moving the marker onto one of the adjoining white squares in the row above. How many 7-step paths are there from P to Q? (The figure shows a sample path.)

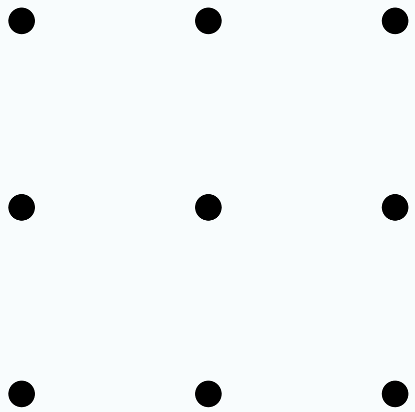


Video Solution

9.5 Practice Problems

Problem 9.5.1

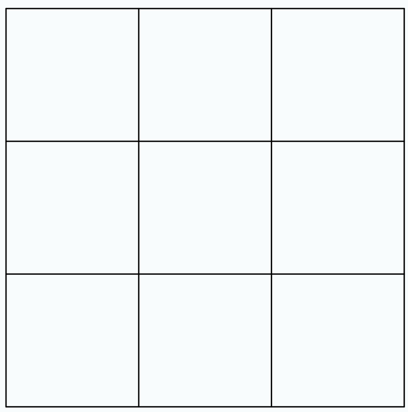
How many triangles can be formed by connecting 3 points in the figure below?



Video Solution

Problem 9.5.2

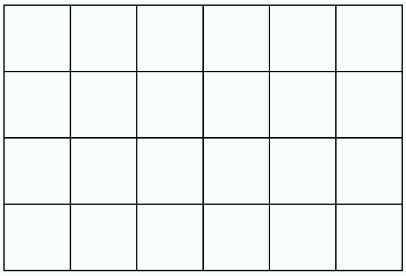
How many squares of all sizes can be formed from a 3x3 grid of unit squares?



Video Solution

Problem 9.5.3

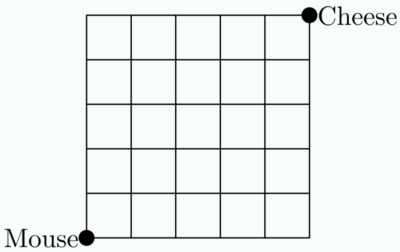
How many rectangles of any size are in a 4×6 grid of squares?



Video Solution

Problem 9.5.4

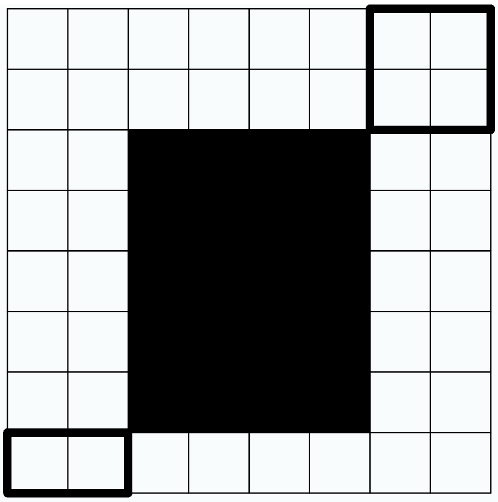
A mouse is standing on a grid at location (0,0). There is cheese on the grid at (5,5). He can only move right or up. How many different paths can he take to the cheese?



Video Solution

Problem 9.5.5 (AMC 8)

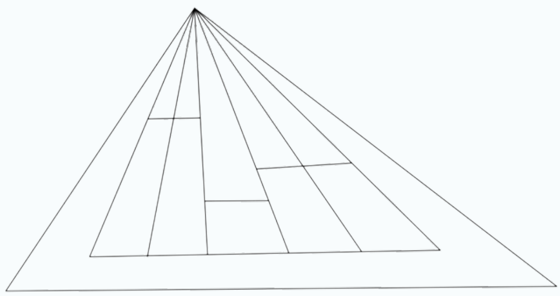
Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?



Video Solution

Problem 9.5.6 (Omega Learn Math Contest)

How many triangles are in this figure?



Video Solution

Problem 9.5.7 (Omega Learn Math Contest)

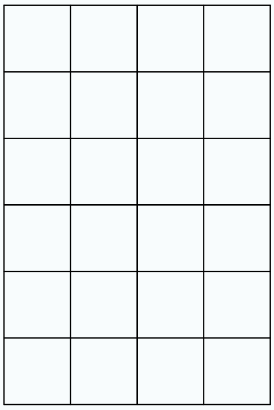
Joe is standing at the point (0,0) in the infinite coordinate plane. On any given move, there is a $\frac{1}{3}$ chance of him moving up, $\frac{1}{3}$ chance of him moving right, $\frac{1}{6}$ chance of him moving down, and $\frac{1}{6}$ chance of him moving left. If the probability he will reach the point (3,3) in less than or equal to 8 moves can be described as $\frac{a}{3^b}$, then what is the value of a + b?

Video Solution

Additional Problems

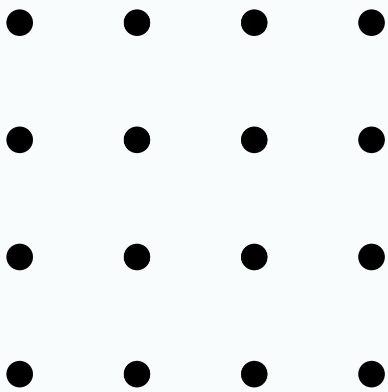
Problem 9.5.8

How many rectangles of any size are in a 4×6 grid of squares?



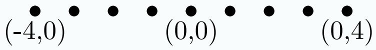
Problem 9.5.9

How many triangles can be formed by connecting 3 points in the figure below?



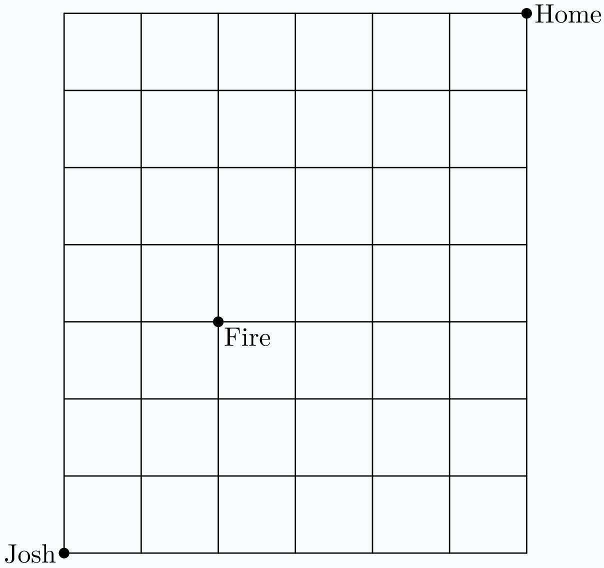
Problem 9.5.10 (EMCC)

A frog is located at the origin. It makes four hops, each of which moves it either 1 unit to the right or 1 unit to the left. If it also ends at the origin, how many 4-hop paths can it take?



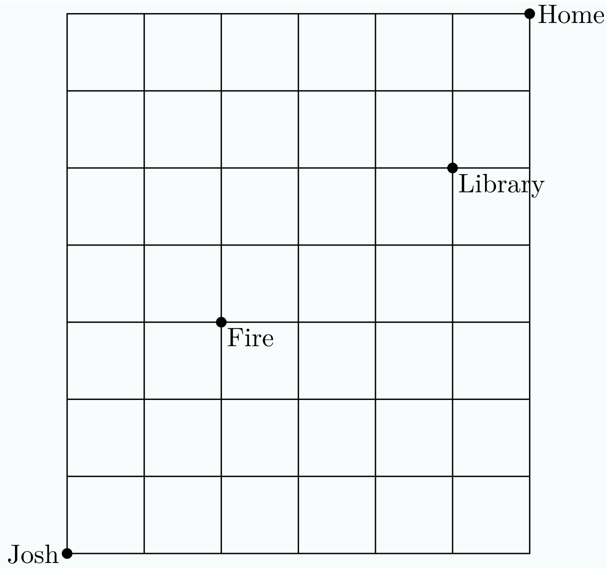
Problem 9.5.11

Josh starts at the point (0,0) and needs to travel to his home at the point (6,7). There is a fire at the point (2,3) so he cannot travel there. How many ways are there for Josh to reach his home if he can only move up or right?



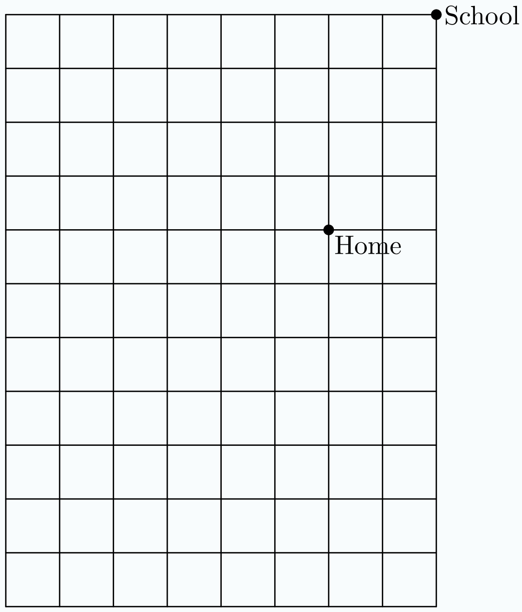
Problem 9.5.12

Josh starts at the point (0,0) and needs to travel to his home at the point (6,7). There is a fire at the point (2,3) so he cannot travel there. Also, he needs to pick up a book from the library at the point (5,5). How many ways are there for Josh to pick up the book and reach his home if he can only move up or right?



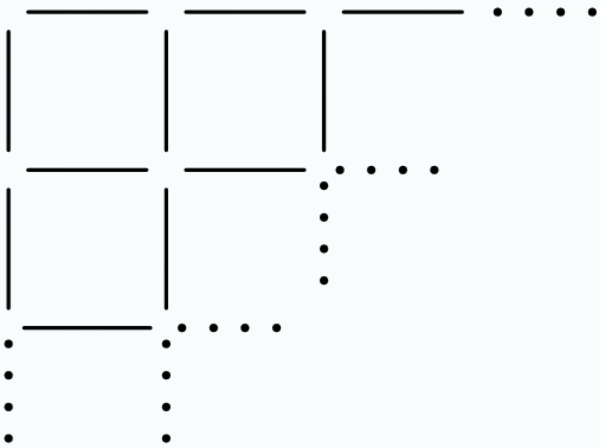
Problem 9.5.13

Josh is now home at the point (6,7) but realizes he forgot something at school which is at the point (8,11). Because he is in a rush to get there before it closes, he is now willing to move diagonally (up-right) in addition to being able to go up and right. How many ways can he reach his school?



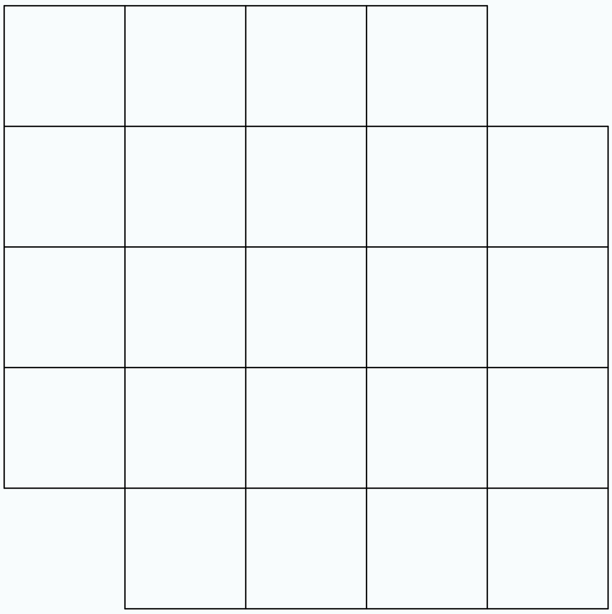
Problem 9.5.14 (AMC 8)

Toothpicks are used to make a grid that is 60 toothpicks long and 32 toothpicks wide. How many toothpicks are used altogether?



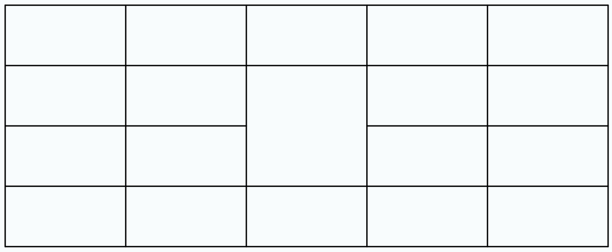
Problem 9.5.15

How many squares are in a 5x5 grid of squares with 2 diagonally opposite corners missing?



Problem 9.5.16

How many rectangles are in the figure below?



Problem 9.5.17 (BmMT)

Leanne and Jing Jing are walking around the xy plane. In one step, Leanne can move from any point (x,y) to $(x+1,y)$ or $(x,y+1)$ and Jing Jing can move from (x,y) to $(x-2,y+5)$ or $(x+3,y-1)$. The number of ways that Leanne can move from $(0,0)$ to $(20,20)$ is equal to the number of ways that Jing Jing can move from $(0,0)$ to (a,b), where a and b are positive integers. Compute the minimum possible value of $a+b$.

Answers

9.4 110

9.5 8

9.7 408

9.8 625

9.9 28

9.5.1 76

9.5.2 14

9.5.3 210

9.5.4 252

9.5.5 18

9.5.6 23

9.5.7 468

9.5.8 210

9.5.9 516

9.5.10 6

9.5.11 1016

9.5.12 456

9.5.13 41

9.5.14 3932

9.5.15 46

9.5.16 114

9.5.17 100

Chapter 10

Recursion

Video Lecture

10.1 Recursion Fundamentals

Definition 10.1.1. Recursion is the process of finding smaller values and using them to calculate larger values.

Example 10.1

Jeff is climbing a 6 stair staircase. If he can climb 1 or 2 stairs at a time, in how many distinct ways can he get to the top?

Solution

We could do casework, but that would take a long time. Instead, let's solve this problem for small values first.

How many ways are there for Jeff to reach the 0th stair (ground)?

Jeff is already there at the start, so there is just 1 way to do that.

How many ways are there for Jeff to reach the 1st stair?

There is only 1 way because he must climb 1 step to reach there from the ground (0th stair).

How many ways are there for Jeff to reach the 2nd stair?

There are 2 ways: 2 single steps or 1 double step from ground (0th stair)

How can Jeff reach the 3rd stair?

Jeff either has to take a single step from the 2nd stair, or a double step from the 1st stair. Therefore, the number of ways is simply the number of ways to reach the first step plus the number of ways to reach the 2nd step, which is $1+2=3$.

How can use the same logic to find the number of ways to reach the nth stair?

To reach the nth stair, Jeff must take a 1 step from the $(n-1)$th stair or a double step from the $(n-2)$th stair. Therefore, the number of ways of reaching the nth step is the number of ways of reaching the $(n-1)$th plus the number of ways of reaching the $(n-2)$th stair.

Let's define a function, f(n), such that f(n) represents the number of ways to reach the nth step. Then, we can write the recursion

$$f(n) = f(n-1) + f(n-2)$$

.
The next step is to just iteratively calculate the values of $f(n)$ for all values of n until we find $f(6)$, since the staircase has 6 steps. We already calculated $f(1)$ and $f(2)$ so we can use them as base cases.

$f(1) = 1$
$f(2) = 2$
$f(3) = f(3-1) + f(3-2) = f(2) + f(1) = 2+1 = 3$
$f(4) = f(4-1) + f(4-2) = f(3) + f(2) = 3+2 = 5$
$f(5) = f(5-1) + f(5-2) = f(4) + f(3) = 5+3 = 8$
$f(6) = f(6-1) + f(6-2) = f(5) + f(4) = 8+5 = \boxed{13}$

Remark 10.1.2
Notice how the numbers follow the Fibonacci sequence. This is because the recursions for our sequence and the Fibonacci sequence are the same. However, this is not always the case for all recursion problems.

10.2 Recursion with Constraints

Example 10.2

Jeff is climbing a 7 stair staircase. Jeff can climb 1 or 2 stairs at a time on every step, but on the last step, he is allowed to climb 3 stairs. In how many distinct ways can he get to the top?

Video Solution

Concept 10.2.1
To solve a problem with recursion requires a multi-step process:

1. Solve the problem for a few smaller base cases manually
2. In general, you will need as many base cases as the number of terms on the right hand side of your recursive function
3. For a given $f(n)$, figure out a recursive equation in terms of previous values of the function by considering different ways to get to that point

4. Iteratively calculate values of $f(n)$ until you reach the desired number

Remark 10.2.2
You can also use f(0) = 1 as a base case because to get to the 0th stair, you have to take no steps and there is only 1 way to do so. This can be a little faster, however it could be a little confusing, so it won't be used in the examples. Nevertheless, feel free to use it in the problems section.

Example 10.3

Mike is climbing a staircase with 10 stairs. He is in a rush, so he will climb 2 or 3 steps at a time for most of the journey. If he reaches the 9th stair, he is allowed to climb 1 stair to reach the 10th stair. How many ways are there for Mike to reach the top of the staircase?

Solution

We can do a similar approach to the previous problem. We first begin by finding the base cases.
There is no way to reach the 1st stair since he cannot take a 1 step, so $f(1) = 0$.
There is 1 way to reach the 2nd stair by taking a double step, so $f(2) = 1$.
There is 1 way to reach the 3rd stair by taking a triple step. If the first step is a double step, then it's impossible to reach the 3rd stair since there are no single steps. Therefore, $f(3) = 1$.

Next, what is the recursive function for this problem?

To get the the nth stair, we must either take a 2 step from the $(n-2)$th stair or a 3 step from the $(n-3)$th stair. So, the recursion is

$$f(n) = f(n-2) + f(n-3)$$

Let's now evaluate the values of f(n)

$f(1) = 0$
$f(2) = 1$
$f(3) = 1$
$f(4) = f(4-2) + f(4-3) = f(2) + f(1) = 1$
$f(5) = f(5-2) + f(5-3) = f(3) + f(2) = 2$
$f(6) = f(6-2) + f(6-3) = f(4) + f(3) = 2$
$f(7) = f(7-2) + f(7-3) = f(5) + f(4) = 3$
$f(8) = f(8-2) + f(8-3) = f(6) + f(5) = 4$
$f(9) = f(9-2) + f(9-3) = f(7) + f(6) = 5$

$f(10) = f(10-2) + f(10-3) = f(8) + f(7) = 7$

So, is our answer just $f(10) = 7$?

No, because once he reaches stair 9, he is allowed to take a single step. So, we must add the number of ways to reach stair 9 to the number of ways to reach stair 8 and the number of ways to reach stair 7. Therefore, we must find:
$f(10) = f(8) + f(7) + f(9) = \boxed{12}$

Remark 10.2.3
In general, it never hurts to find too many base cases. In your recursion, if you run into any values you are not sure how to calculate (like f(2), f(3)), you can always go back and evaluate more base cases.

10.3 Probability Recursions

Example 10.4 (AMC 8)

A cricket randomly hops between 4 leaves, on each turn hopping to one of the other 3 leaves with equal probability. After 4 hops what is the probability that the cricket has returned to the leaf where it started?

Video Solution

Concept 10.3.1
For probability recursions, it can be a good idea to simply make a table and find the probability of being at each location every time interval. Remember to use the fact that the total probability at each hop must always be 1.

10.4 Practice Problems

Problem 10.4.1

Jeff is climbing a 7 step staircase. If he can climb 1 or 2 steps at a time, in how many ways can he get to the top?

Video Solution

Problem 10.4.2 (AMC 8)

Everyday at school, Jo climbs a flight of 6 stairs. Jo can take the stairs 1, 2, or 3 at a time. For example, Jo could climb 3, then 1, then 2. In how many ways can Jo climb the stairs?

Video Solution

Problem 10.4.3 (Omega Learn Math Contest)

Brandon is trying to reach the top of a 7 step staircase. For each jump except the last jump, he can either jump 1 or 3 steps forward. On the last jump, he can jump 1, 2, or 3 steps forward. How many different ways can he reach the top of the staircase?

Video Solution

Problem 10.4.4 (AMC 12)

Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of 1, 2, 3, . . . , 12, including the empty set, are spacy?

Video Solution

Problem 10.4.5 (AIME)

A collection of 8 cubes consists of one cube with edge-length k for each integer k, $1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the following rules: Any cube may be the bottom cube in the tower. The cube immediately on top of a cube with edge-length k must have edge-length at most k + 2. How many towers can be constructed?

Video Solution

Additional Problems

Problem 10.4.6

Jeff is climbing a 10 step staircase. However, there are spiders on the 3rd and 9th steps, and he refuses to step on those stairs. If he can climb 1 or 2 steps at a time, in how many different ways can he get to the top?

Answers

10.2 26

10.4 $\frac{7}{27}$

10.4.1 21

10.4.2 24

10.4.3 13

10.4.4 129

10.4.5 458

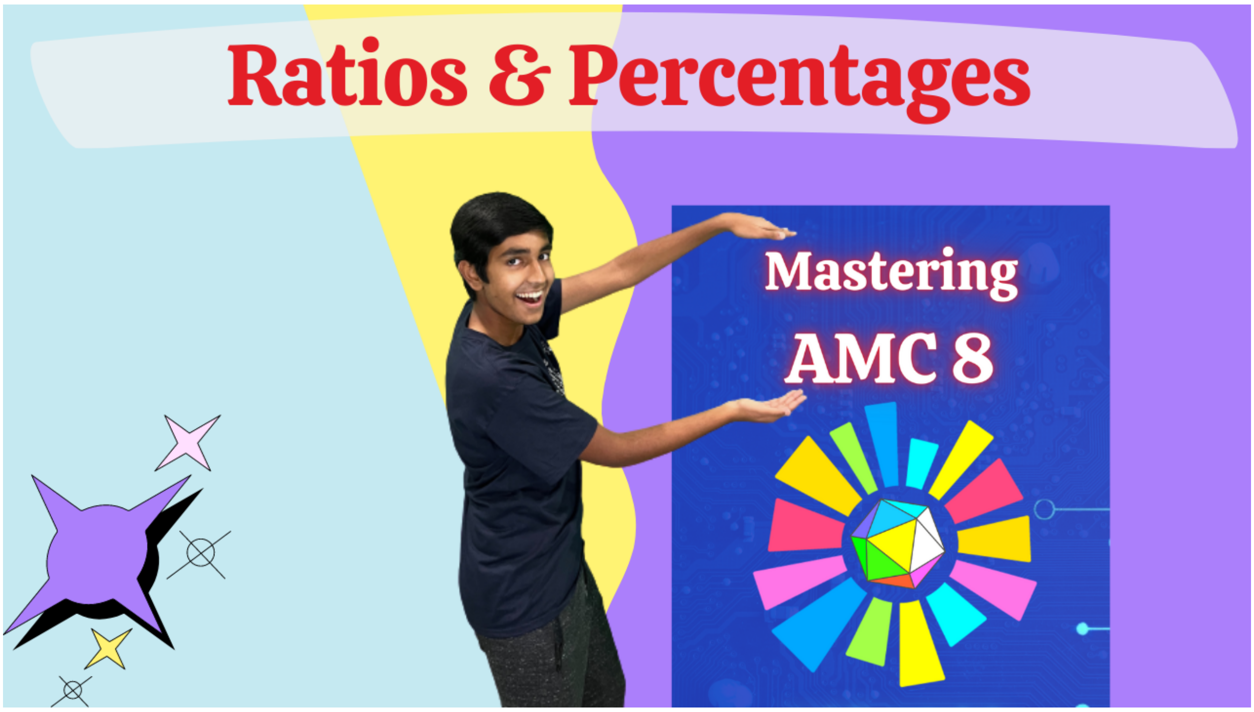
10.4.6 10

Algebra

Chapter 11

Ratios and Percentages

Video Lecture




Work & Rate
Mastering
AMC 8
MIT

11.1 Ratios Fundamentals

Example 11.1 (MATHCOUNTS)

The Ten Finger calculator company periodically checks random calculators before shipping crates out to customers. On Wednesday, 12 calculators from each of 64 crates of 144 calculators were tested. Two of the tested calculators were found to be defective. Based on this rate of defect, how many total calculators are expected to be defective?

Solution

Let's approach this problem one step at a time.

What fraction of the total calculators were tested?

For each crate, 12 out of 144 calculators were tested, so $\frac{1}{12}$ of them were tested.

From this, how many total calculators are expected to be defective?

We can think of dividing the calculators into 12 groups. Each group will make up $\frac{1}{12}$ of the total calculators. As found above, when $\frac{1}{12}$ of the calculators, or 1 group of calculators, were tested, 2 were defective.

Therefore, amongst all 12 groups, there will be $12 \times 2 = \boxed{24}$ defective calculators.

Example 11.2 (Omega Learn)

Sohil has a jar with many marbles. 20% of them are small, 40% are medium, and 40% are large. For each size, 30% of the marbles are red, 20% blue, 40% green, and 10% yellow. Half of large yellow and small green marbles have a special design on them. What percentage of the total marbles have a special design?

Solution

We can approach this problem by finding the percentage of "large yellow" and "small green" marbles.

What percent of marbles are large and yellow?

Since 40% of the marbles are large and 10% of the large marbles are yellow, $10\% \times 40\% =$

$\frac{1}{10} \times 40\% = 4\%$ of the marbles are large and yellow.

What percent of marbles are small and green?

Since 20% of the marbles are small and 40% of the small marbles are green, $20\% \times 40\% = \frac{1}{5} \times 40\% = 8\%$ of the marbles are small and green.

From this, what percent of marbles have a special design?

A total of $4\% + 8\% = 12\%$ of marbles are large and yellow or small and green. Therefore, since half of them have a special design, $\frac{1}{2} \times 12\% = \boxed{6\%}$ have a special design.

Example 11.3 (AMC 10)

Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is $9:1$, and the ratio of blue to green marbles in Jar 2 is $8:1$. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

Video Solution

Example 11.4 (Omega Learn)

There are 994 students in the Omega Middle School. $\frac{2}{7}$ of them are in 6th grade, $\frac{2}{7}$ of them are in 7th grade, and $\frac{3}{7}$ of them are in 8th grade. $\frac{1}{4}$ of the 6th graders are interested in math, $\frac{3}{4}$ of the 7th graders are interested in math, and $\frac{1}{2}$ of the 8th graders are interested in math. Of the students who are interested in math, $\frac{5}{7}$ will take the AMC 8 this year. How many students will take the AMC 8 from Omega Middle School?

Video Solution

Example 11.5 (AMC 8)

Steph scored 15 baskets out of 20 attempts in the first half of a game, and 10 baskets out of 10 attempts in the second half. Candace took 12 attempts in the first half and 18 attempts in the second. In each half, Steph scored a higher percentage of baskets than Candace. Surprisingly they ended with the same overall percentage of baskets scored. How many more baskets did Candace score in the second half than in the first?

	First Half	Second Half
Steph	$\frac{15}{20}$	$\frac{10}{10}$
Candace	$\frac{\square}{12}$	$\frac{\square}{18}$

Video Solution

11.2 Rate and Work

Theorem 11.2.1

$$\text{Work} = \text{Rate} \times \text{Time}$$

Equivalently,

$$\text{Rate} = \frac{\text{Work}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Work}}{\text{Rate}}$$

Example 11.6 (Omega Learn)

10 workers from a company each working at a constant rate can build 10 houses in 12 years. However, after 6 years, 5 of them retire. The other workers continue working at the same original rate. The project is falling behind, so 2 years later, the company hires 10 more workers who each work twice as fast as the original workers. After how many total years will the 10 houses be complete?

Solution

To solve this problem, we will have to use ratios to compare the amount of work being done.

After 6 years, what fraction of the 10 houses will be built?

Since 6 years is half of the time needed to build 10 houses, $\frac{1}{2}$ of the houses will be completed by 6 years.

Next, we can see that for 2 years, only 5 of the original workers are working.

In these 2 years, what fraction of the houses will be built?

2 years is $\frac{1}{6}$ of the time needed to build all of the houses. However, only $\frac{1}{2}$ of the original workers are working. Therefore, in total, $\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$ of the houses will be built in these 2 years.

Next, we must find the rate of building houses when the 10 workers are added.

Compared to the original rate, how fast do the 5 original and 10 new workers work?

Since each of the 10 new workers work 2 times faster than one of the original workers, the 5 original workers that remain and the 10 new workers can do as much work as $5 + 10 \times 2 = 25$ original workers. Therefore, with the addition of the new workers, they can work $\frac{25}{10} = \frac{5}{2}$ times faster.

What fraction of the houses are left to be built?

$\frac{1}{2}$ of the houses were built in the first 6 years, and $\frac{1}{12}$ of the houses were built in the next 2 years, so

$$1 - \frac{1}{2} - \frac{1}{12} = \frac{12}{12} - \frac{6}{12} - \frac{1}{12} = \frac{5}{12}$$

of the houses are left to be built.

How long would the 10 original workers have taken to build $\frac{5}{12}$ of the houses?

Since they could build all of the houses in 12 years, they could build $\frac{5}{12}$ of the houses in $\frac{5}{12} \times 12 = 5$ years.

How long will it take the new team of workers to build $\frac{5}{12}$ of the houses?

The new team of workers can work $\frac{5}{2}$ faster than the original team, so they will take $\frac{2}{5}$ the time. Therefore, they can build $\frac{5}{12}$ of the houses in $\frac{2}{5} \times 5 = 2$ years.

The 10 original workers built $\frac{1}{2}$ of the houses in 6 years, the remaining 5 original workers built $\frac{1}{12}$ of the houses in 2 years, and the new team of workers built $\frac{5}{12}$ of the houses in 2 years. Therefore, the total number of years it took to build the houses is $6+2+2=\boxed{10}$.

Theorem 11.2.2

The if one person can do something is a amount of time, and someone else can do it in b amount of time, together they can do it in

$$\frac{ab}{a+b}$$

time.

Remark 11.2.3

This not only applies to work. For example, if a problem says 2 faucets take a and b hours to fill a tub, together they can fill a tub in

$$\frac{ab}{a+b}$$

hours.

Example 11.7 (Omega Learn)

Alex, using substitution or elimination, can solve 20 two variable equations in 10 minutes. Bob, who uses the diagonal product method, can solve 20 two variable equations in 2 minutes. How much time will it take both of them working together to solve 60 two variable equations?

Video Solution

Example 11.8 (Omega Learn)

All experienced workers work at a constant rate and all new workers work at a different constant rate. 12 experienced workers and 6 new workers can build a house in 6 months. 6 experienced workers and 12 new workers can build a house in 9 months. How many months will it take 9 experienced workers and 9 new workers to build a house?

Video Solution

Example 11.9 (Omega Learn)

Two rocket scientists Bill Einstein and Albert Nye are building a spaceship, and they will work 6 hours a day for standard pay. It would take Albert a total of 48 hours to build the rocket individually, and it would take Bill a total of 36 hours. The manager can motivate any of them to work extra hours every day by giving them an extra \$100 for each hour. The manager only has \$1000 to give to Albert and Bill for extra pay. What is the minimum number of whole days that the rocket can be completed?

Video Solution

11.3 Practice Problems

Problem 11.3.1 (AMC 8)

If the degree measures of the angles of a triangle are in the ratio $3 : 3 : 4$, what is the degree measure of the largest angle of the triangle?

Video Solution

Problem 11.3.2 (AMC 8)

There are 270 students at Colfax Middle School, where the ratio of boys to girls is $5 : 4$. There are 180 students at Winthrop Middle School, where the ratio of boys to girls is $4 : 5$. The two schools hold a dance and all students from both schools attend. What fraction of the students at the dance are girls?

Video Solution

Problem 11.3.3 (AMC 8)

Gilda has a bag of marbles. She gives 20% of them to her friend Pedro. Then Gilda gives 10% of what is left to another friend, Ebony. Finally, Gilda gives 25% of what is now left in the bag to her brother Jimmy. What percentage of her original bag of marbles does Gilda have left for herself?

Video Solution

Problem 11.3.4 (AMC 8)

All of Marcy's marbles are blue, red, green, or yellow. One third of her marbles are blue, one fourth of them are red, and six of them are green. What is the smallest number of yellow marbles Marcy can have?

Video Solution

Problem 11.3.5 (AMC 8)

A number of students from Fibonacci Middle School are taking part in a community service project. The ratio of 8^{th}-graders to 6^{th}-graders is $5:3$, and the the ratio of 8^{th}-graders to 7^{th}-graders is $8:5$. What is the smallest number of students that could be participating in the project?

Video Solution

Problem 11.3.6 (AMC 8)

Chloe and Zoe are both students in Ms. Demeanor's math class. Last night they each solved half of the problems in their homework assignment alone and then solved the other half together. Chloe had correct answers to only 80% of the problems she solved alone, but overall 88% of her answers were correct. Zoe had correct answers to 90% of the problems she solved alone. What was Zoe's overall percentage of correct answers?

Video Solution

Additional Problems

Problem 11.3.7 (AMC 8)

Suppose 15% of x equals 20% of y. What percentage of x is y?

Problem 11.3.8 (AMC 10)

Suppose $A > B > 0$ and A is $x\%$ greater than B. What is x?

(A) $100\left(\frac{A-B}{B}\right)$ **(B)** $100\left(\frac{A+B}{B}\right)$ **(C)** $100\left(\frac{A+B}{A}\right)$ **(D)** $100\left(\frac{A-B}{A}\right)$ **(E)** $100\left(\frac{A}{B}\right)$

Problem 11.3.9 (MATHCOUNTS)

On Wednesday, $\frac{1}{3}$ of the students in Mr. Short's homeroom had drama practice, $\frac{1}{4}$ of his other homeroom students had band practice. If 6 students had band practice, how many students are in Mr. Short's homeroom?

Problem 11.3.10 (AMC 10)

The ratio of w to x is $4:3$, the ratio of y to z is $3:2$, and the ratio of z to x is $1:6$. What is the ratio of w to y?

Problem 11.3.11 (AMC 8)

A store increased the original price of a shirt by a certain percent and then decreased the new price by the same amount. Given that the resulting price was 84% of the original price, by what percent was the price increased and decreased?

Problem 11.3.12 (MATHCOUNTS)

Andre can complete $\frac{5}{6}$ of a job in $\frac{3}{4}$ of the time that it takes Michael to do the whole job. What is the ratio of the rate at which Andre works to the rate at which Michael works? Express your answer as a common fraction.

Problem 11.3.13 (AMC 8)

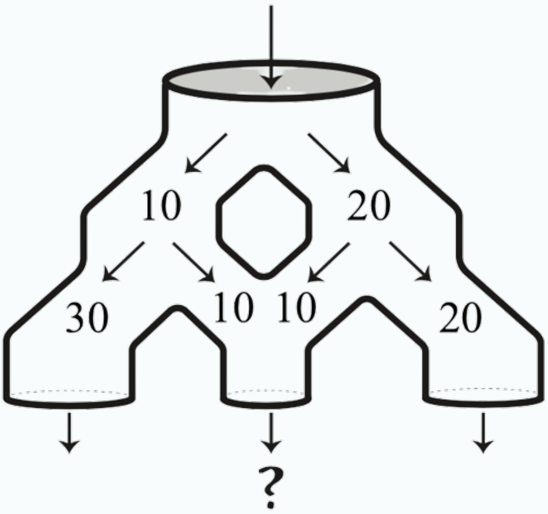
When the World Wide Web first became popular in the 1990s, download speeds reached a maximum of about 56 kilobits per second. Approximately how many minutes would the download of a 4.2-megabyte song have taken at that speed? (Note that there are 8000 kilobits in a megabyte.)

Problem 11.3.14 (MATHCOUNTS)

Grapes are 80% water by weight. When a bushel of grapes dries in the sun for two weeks, it loses 50% of its total weight. All of the weight loss is due to the loss of water. After drying for two weeks, what percentage of the grapes is water by weight? Express your answer to the nearest percent.

Problem 11.3.15 (MATHCOUNTS)

Ten thousand marbles are released into the top pipe as shown and roll down the pipe system. Anytime the pipe forks, the marbles split in proportion to the cross-sectional areas of the pipes. All pipes have circular cross-sections with diameters as indicated in the figure. How many marbles exit through the bottom, middle pipe?



Problem 11.3.16 (MATHCOUNTS)

Brian has earned 65%, 80% and 92% on his three pre-final exams. These exams are not weighed equally: the lowest counts for only 20% of his overall grade, while the other two count for 25% each. If the final exam is the remainder of the overall grade and there are no opportunities for extra credit, what is the highest grade Brian can earn in the class? Express your answer to the nearest whole percent.

Answers

11.3 5

11.4 355

11.5 9

11.7 5

11.8 $\frac{36}{5}$

11.9 3

11.3.1 72

11.3.2 89

11.3.3 54

11.3.4 4

11.3.5 89

11.3.6 93

11.3.7 75

11.3.8 $100\left(\frac{A-B}{B}\right)$

11.3.9 36

11.3.10 $16:3$

11.3.11 40

11.3.12 $\frac{10}{9}$

11.3.13 10

11.3.14 60%

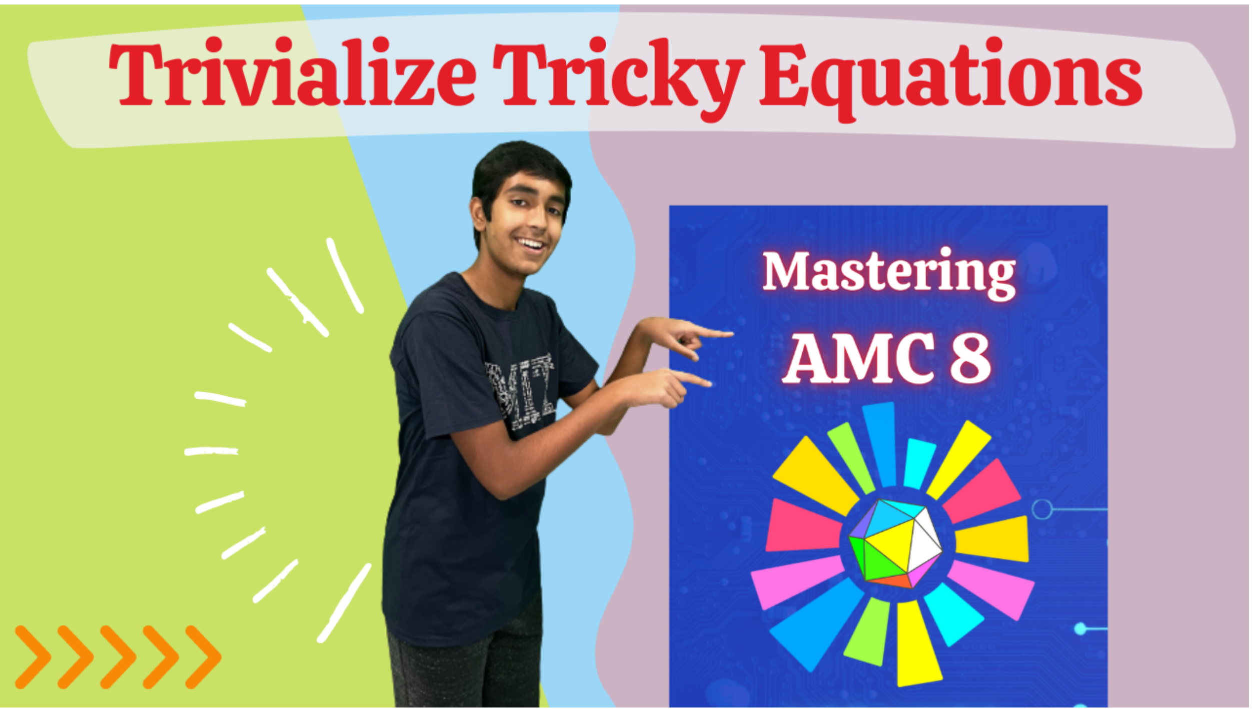
11.3.15 1800

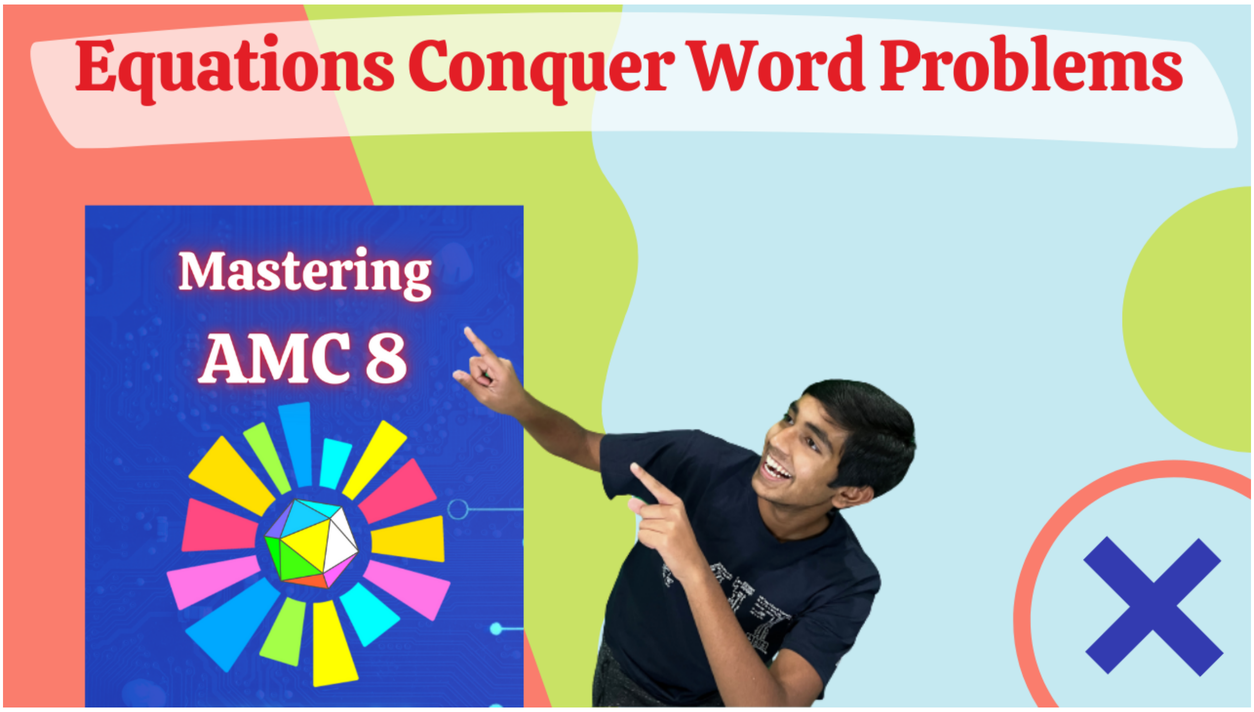
11.3.16 86%

Chapter 12

Algebraic Manipulations and Equations

Video Lecture




Equations Conquer Word Problems
Mastering
AMC 8

12.1 System of Equations Basics

Concept 12.1.1 (Factoring a Quadratic)
If we have a quadratic of the form x^2+bx+c, it can be factored into $(x+p)(x+q)$ such that $p+q=b$ and $pq=c$. We can generally do this by guess and check.

Then, we just solve the linear equations of the form $x+r=0$.

Concept 12.1.2 (Vietas Formulas)
If we have a quadratic of the form x^2+bx+c with 2 roots r and s, then $r+s=-b$ and $rs=c$. Sometimes, you may not need to calculate the roots and just find expression in terms of the roots.

Theorem 12.1.3 (Quadratic Formula)
The solutions to the quadratic equation

$$ax^2+bx+c=0$$

are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Remark 12.1.4
If you are new to quadratics, I recommend checking out the following links:

Basics of factoring Quadratics

Basics of Quadratic Formula

Remark 12.1.5 (Solving 2 variable equations)
The most common ways of solving equations are substitution and elimination.
Linear Equations Basics

There is an easier way to solve 2 variable equations using the diagonal product method
Diagonal Product Method to Solve 2 Variable Equations

12.2 Advanced Equation Solving Techniques

Concept 12.2.1 (Solving General Systems of Equations)

Once you've found the system of equations in your word problem and/or the problem itself gives you a system of equation, you should consider one or more of the following methods for solving them.

1. Using Symmetry
2. Substituting Variables
3. Elimination
4. Adding/Subtracting/Multiplying Equations
5. Difference of Squares: $x^2 - y^2 = (x-y)(x+y)$
6. Binomial Expansions:
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x-y)^2 = x^2 - 2xy + y^2$$
7. Other common factorization tricks (see algebraic number theory chapter for more)
8. Squaring equations if you have square roots
9. Introducing new variables and substituting
10. Graphing equation (Recommend graph paper)

Example 12.1

If

$$2x + y + z = 36$$
$$x + 2y + z = 40$$
$$x + y + 2z = 32$$

Find $x + y + z$.

Solution

This is a classic equation problem that uses a very powerful technique.

Do you notice anything similar about the equations?

Indeed! All 3 equations have 2 of one of the variables x, y, z and 1 of the other 2 variables. So, the equations are symmetric!

How can we use the symmetry?

Let's rewrite the equations as follows:

$$x + (x + y + z) = 36$$

$$y + (x + y + z) = 40$$

$$z + (x + y + z) = 32$$

Notice how each equation simply consists of the sum $x + y + z$ plus an additional x, y, or z. The sum $x + y + z$ reoccurs in every equation.

How can we make another $x + y + z$ term?

We can add all 3 equations together!. This will give us

$$(x + y + z) + 3(x + y + z) = 4(x + y + z) = 36 + 40 + 32 = 108$$

. This simplifies to $x + y + z = \boxed{27}$.

Example 12.2 (EMCC)

Suppose x, y, z are real numbers that satisfy:

$$x + y - z = 5$$

$$y + z - x = 7$$

$$z + x - y = 9$$

Find $x^2 + y^2 + z^2$.

Video Solution

Example 12.3 (EMCC)

Let x, y, and z be real numbers so that $(x+y)(y+z) = 36$ and $(x+z)(x+y) = 4$. Compute $y^2 - x^2$.

Video Solution

Example 12.4 (BMMT)

Let x and y be real numbers such that $xy = 4$ and $x^2y + xy^2 = 25$. Find the value of $x^3y + x^2y^2 + xy^3$

Video Solution

12.3 Word Problems

Concept 12.3.1 (System of Equations Word Problems)
This is one of the most common topics on the AMC 8. The trick to solving these types of problems is to just assign variables to the unknowns in the problem and solve them.

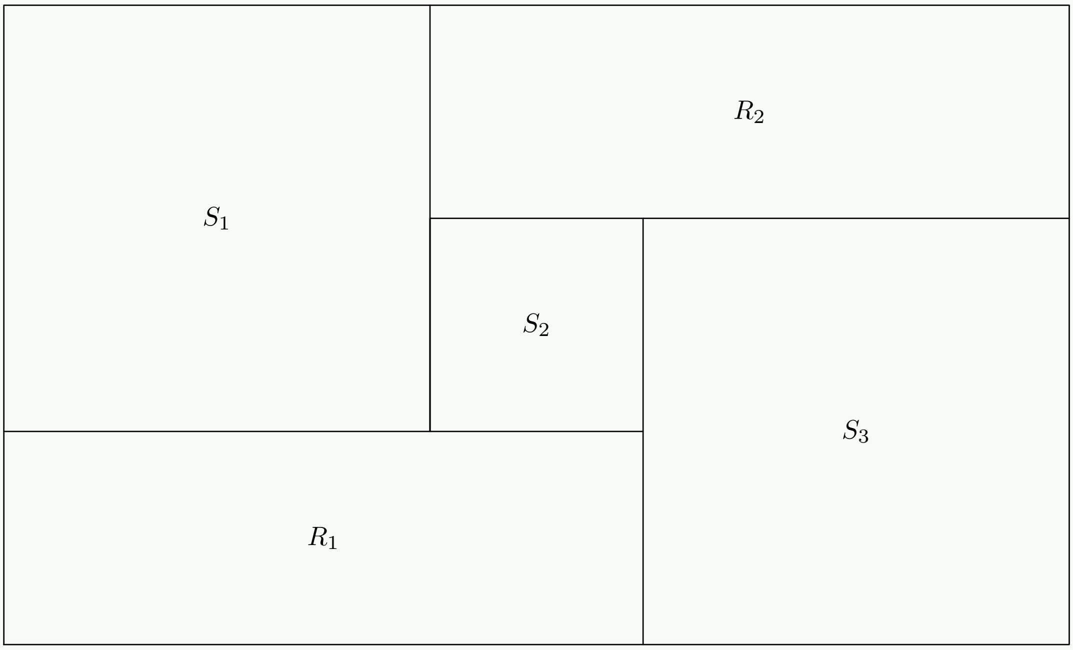
Example 12.5 (Omega Learn)

Orangey loves drinking diluted juice. Orangey takes an 8 ounce glass of orange juice and drinks some fraction of it. Then, he fills the rest with apple juice. After thoroughly mixing the glass, he drinks two thirds of the mixture. He then fills the rest with water. Orangey then finishes the whole glass. If he drank 4 times as much orange juice than apple juice, how much total liquid in ounces did he drink?

Video Solution

Example 12.6 (AMC 8)

Rectangles R_1 and R_2, and squares S_1, S_2, and S_3, shown below, combine to form a rectangle that is 3322 units wide and 2020 units high. What is the side length of S_2 in units?



Video Solution

Example 12.7 (AIME)

Jar A contains four liters of a solution that is 45% acid. Jar B contains five liters of a solution that is 48% acid. Jar C contains one liter of a solution that is $k\%$ acid. From jar C, $\frac{m}{n}$ liters of the solution is added to jar A, and the remainder of the solution in jar C is added to jar B. At the end both jar A and jar B contain solutions that are 50% acid. Given that m and n are relatively prime positive integers, find $k+m+n$.

Video Solution

12.4 Practice Problems

Problem 12.4.1 (AMC 8)

Shauna takes five tests, each worth a maximum of 100 points. Her scores on the first three tests are 76, 94, and 87. In order to average 81 for all five tests, what is the lowest score she could earn on one of the other two tests?

Video Solution

Problem 12.4.2 (AMC 8)

In a jar of red, green, and blue marbles, all but 6 are red marbles, all but 8 are green, and all but 4 are blue. How many marbles are in the jar?

Video Solution

Problem 12.4.3 (Omega Learn Math Contest)

If

$$a \times b \times c = 4$$
$$b \times c \times d = 6$$
$$c \times d \times e = 12$$
$$d \times e \times f = 18$$
$$e \times a \times b = 9$$

Find $a \times b \times c \times d \times e \times f$

Problem 12.4.4 (AMC 8)

In a mathematics contest with ten problems, a student gains 5 points for a correct answer and loses 2 points for an incorrect answer. If Olivia answered every problem and her score was 29, how many correct answers did she have?

Video Solution

Problem 12.4.5

I am thinking of 3 integers. Added two at a time their sums are 37, 41, 44. What is the product of the 3 integers?

Problem 12.4.6 (AMC 8)

Hui is an avid reader. She bought a copy of the best seller Math is Beautiful. On the first day, Hui read $\frac{1}{5}$ of the pages plus 12 more, and on the second day she read $\frac{1}{4}$ of the remaining pages plus 15 pages. On the third day she read $\frac{1}{3}$ of the remaining pages plus 18 pages. She then realized that there were only 62 pages left to read, which she read the next day. How many pages are in this book?

Video Solution

Problem 12.4.7 (AMC 8)

Before the district play, the Unicorns had won 45% of their basketball games. During district play, they won six more games and lost two, to finish the season having won half their games. How many games did the Unicorns play in all?

Video Solution

Problem 12.4.8 (AMC 8)

Ralph went to the store and bought 12 pairs of socks for a total of \$24. Some of the socks he bought cost \$1 a pair, some of the socks he bought cost \$3 a pair, and some of the socks he bought cost \$4 a pair. If he bought at least one pair of each type, how many pairs of \$1 socks did Ralph buy?

Video Solution

Problem 12.4.9 (AMC 8)

Suppose that $a * b$ means $3a - b$. What is the value of x if

$$2 * (5 * x) = 1$$

Video Solution

Problem 12.4.10 (AMC 10)

In an after-school program for juniors and seniors there is a debate team with an equal number of students from each class on the team. Among the 28 students on the program, 25% of the juniors and 10% of the seniors are on the debate team. How many juniors are in the program?

Video Solution

Problem 12.4.11 (AMC 10)

Joe has a collection of 23 coins, consisting of 5-cent coins, 10-cent coins, and 25-cent coins. He has 3 more 10-cent coins than 5-cent coins, and the total value of his collection is 320 cents. How many more 25-cent coins does Joe have than 5-cent coins?

Video Solution

Problem 12.4.12 (AMC 10)

Pablo, Sofia, and Mia got some candy eggs at a party. Pablo had three times as many eggs as Sofia, and Sofia had twice as many eggs as Mia. Pablo decides to give some of his eggs to Sofia and Mia so that all three will have the same number of eggs. What fraction of his eggs should Pablo give to Sofia?

Video Solution

Problem 12.4.13 (AMC 10/12)

The sum of two natural numbers is 17,402. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

Video Solution

Problem 12.4.14 (AMC 10)

Let $a+1=b+2=c+3=d+4=a+b+c+d+5$. What is $a+b+c+d$?

Video Solution

Problem 12.4.15 (AMC 10)

If $y+4=(x-2)^2, x+4=(y-2)^2$, and $x \neq y$, what is the value of x^2+y^2?

Video Solution

Problem 12.4.16 (EMCC)

Given that

$$a+5b+9c=1$$
$$4a+2b+3c=2$$
$$7a+8b+6c=9$$

What is $741a+825b+639c$?

Additional Problems

Problem 12.4.17 (MATHCOUNTS)

The units and tens digits of one two-digit integer are the tens and units digits of another two-digit integer, respectively. If the product of the two integers is 4930, what is their sum?

Problem 12.4.18 (MATHCOUNTS)

A car passes point A driving at a constant rate of 60 km per hour. A second car, traveling at a constant rate of 75 km per hour, passes the same point A a while later and then follows the first car. It catches the first car after traveling a distance of 75 km past point A. How many minutes after the first car passed point A did the second car pass point A?

Problem 12.4.19 (AMC 8)

Four numbers are written in a row. The average of the first two is 21, the average of the middle two is 26, and the average of the last two is 30. What is the average of the first and last of the numbers?

Problem 12.4.20 (AMC 10)

Ana and Bonita were born on the same date in different years, n years apart. Last year Ana was 5 times as old as Bonita. This year Ana's age is the square of Bonita's age. What is n?

Problem 12.4.21 (AMC 8)

On the last day of school, Mrs. Awesome gave jelly beans to her class. She gave each boy as many jelly beans as there were boys in the class. She gave each girl as many jelly beans as there were girls in the class. She brought 400 jelly beans, and when she finished, she had six jelly beans left. There were two more boys than girls in her class. How many students were in her class?

Problem 12.4.22 (AMC 10)

At Megapolis Hospital one year, multiple-birth statistics were as follows: Sets of twins, triplets, and quadruplets accounted for 1000 of the babies born. There were four times as many sets of triplets as sets of quadruplets, and there was three times as many sets of twins as sets of triplets. How many of these 1000 babies were in sets of quadruplets?

Problem 12.4.23 (AMC 10)

Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is $9:1$, and the ratio of blue to green marbles in Jar 2 is $8:1$. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

Problem 12.4.24 (MATHCOUNTS)

The product of three distinct positive integers is 144. If the sum of the three integers is 26, what is the sum of their squares?

Problem 12.4.25 (MATHCOUNTS)

If

$$\frac{3}{2-x}+\frac{2}{y-3}=1$$

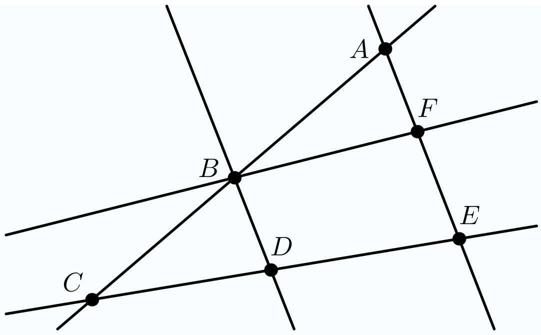
and

$$\frac{2}{2-x}+\frac{8}{y-3}=2$$

find the value of x.

Problem 12.4.26 (AMC 8)

Each of the points A, B, C, D, E, and F in the figure below represents a different digit from 1 to 6. Each of the five lines shown passes through some of these points. The digits along each line are added to produce five sums, one for each line. The total of the five sums is 47. What is the digit represented by B?



Problem 12.4.27 (AMC 12)

The state income tax where Kristin lives is levied at the rate of $p\%$ of the first \$28000 of annual income plus $(p+2)\%$ of any amount above \$28000. Kristin noticed that the state income tax she paid amounted to $(p+0.25)\%$ of her annual income. What was her annual income?

Problem 12.4.28 (AMC 10/12)

Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 P.M. How long, in minutes, was each day's lunch break?

Problem 12.4.29 (AMC 12)

A group of 12 pirates agree to divide a treasure chest of gold coins among themselves as follows. The k^{th} pirate to take a share takes $\frac{k}{12}$ of the coins that remain in the chest. The number of coins initially in the chest is the smallest number for which this arrangement will allow each pirate to receive a positive whole number of coins. How many coins does the 12^{th} pirate receive?

Hints

Answers

12.2 149

12.3 32

12.4 $\frac{561}{4}$

12.5 $\frac{46}{3}$

12.6 651

12.7 85

12.4.1 48

12.4.2 9

12.4.3 72

12.4.4 7

12.4.5 8160

12.4.6 240

12.4.7 48

12.4.8 7

12.4.9 10

12.4.10 8

12.4.11 2

12.4.12 1/6

12.4.13 $14,238$

12.4.14 $\frac{-10}{3}$

12.4.15 15

12.4.16 921

12.4.17 143

12.4.18 15

12.4.19 25

12.4.20 12

12.4.21 28

12.4.22 100

12.4.23 5

12.4.24 338

12.4.25 -3

12.4.26 5

12.4.27 32000

12.4.28 48

12.4.29 1925

Chapter 13

Speed, Distance, and Time

13.1 Speed, Distance, and Time

Video Lectures

Speed, Distance, and Time

Theorem 13.1.1

$$\text{Distance} = \text{Speed} \times \text{Time}$$

Equivalently,

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

Example 13.1

Joe needs to get to his office 40 miles away. If he drives at 60 miles per hour, how many minutes will it take him to get to work?

Solution

This problem is just a direct application of the speed, distance, and time formula.

What information do we know?

The distance is 40 miles. The speed is 60 miles per hour.

How can we calculate the number of minutes?

The number of minutes, a measure of time, can be calculated by dividing distance by speed which is $\frac{40\text{miles}}{60\text{miles per hour}} = \frac{2}{3}\text{hours}$.

How many minutes will it take?

The time we calculated was in hours because the speed was given in miles per hour, so we multiply by 60 to $\frac{2}{3}$ to get $60 \cdot \frac{2}{3} = \boxed{40}$

Theorem 13.1.2

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}}$$

Remark 13.1.3
A common mistake is to assume that average speed is the averages of all speeds (especially when the distance you are traveling at each of those speeds are the same). Remember, that's not true unless you are traveling at those speeds for the same amount of time!

Remark 13.1.4
Often, we may have to form equations with distance, speed, and time and use them to solve our problem.

Example 13.2 (Omega Learn)

Sohil is running along a 2 mile path at 8 miles per hour and back on the same path. On the way back he is tired so for after every 1 mile he runs on the way back, his speed is instantly reduced by 1 mile per hour. What is his average speed throughout the whole trip?

Video Solution

Example 13.3 (AMC 8)

Each day for four days, Linda traveled for one hour at a speed that resulted in her traveling one mile in an integer number of minutes. Each day after the first, her speed decreased so that the number of minutes to travel one mile increased by 5 minutes over the preceding day. Each of the four days, her distance traveled was also an integer number of miles. What was the total number of miles for the four trips?

Video Solution

Example 13.4 (AMC 8)

A bus takes 2 minutes to drive from one stop to the next, and waits 1 minute at each stop to let passengers board. Zia takes 5 minutes to walk from one bus stop to the next. As Zia reaches a bus stop, if the bus is at the previous stop or has already left the previous stop, then she will wait for the bus. Otherwise she will start walking toward the next stop. Suppose the bus and Zia start at the same time toward the library, with the bus 3 stops behind. After how many minutes will Zia board the bus?

Video Solution

Example 13.5 (AIME)

Points A, B, and C lie in that order along a straight path where the distance from A to C is 1800 meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at A and running toward C, Paul starting at B and running toward C, and Eve starting at C and running toward A. When Paul meets Eve, he turns around and runs toward A. Paul and Ina both arrive at B at the same time. Find the number of meters from A to B.

Video Solution

13.2 Practice Problems

Problem 13.2.1 (AMC 8)

Qiang drives 15 miles at an average speed of 30 miles per hour. How many additional miles will he have to drive at 55 miles per hour to average 50 miles per hour for the entire trip?

Video Solution

Problem 13.2.2 (AMC 10)

Chantal and Jean start hiking from a trailhead toward a fire tower. Jean is wearing a heavy backpack and walks slower. Chantal starts walking at 4 miles per hour. Halfway to the tower, the trail becomes really steep, and Chantal slows down to 2 miles per hour. After reaching the tower, she immediately turns around and descends the steep part of the trail at 3 miles per hour. She meets Jean at the halfway point. What was Jean's average speed, in miles per hour, until they meet?

Video Solution

Problem 13.2.3 (AMC 8)

Bella begins to walk from her house toward her friend Ella's house. At the same time, Ella begins to ride her bicycle toward Bella's house. They each maintain a constant speed, and Ella rides 5 times as fast as Bella walks. The distance between their houses is 2 miles, which is $10,560$ feet, and Bella covers $2\frac{1}{2}$ feet with each step. How many steps will Bella take by the time she meets Ella?

Video Solution

Problem 13.2.4 (Omega Learn Math Contest)

Jeff is driving 250 miles to Orlando at a constant speed of 50 miles per hour. He needs to reach in exactly 5 hours. After driving 100 miles, he takes a 30 minute break. He then continues driving at a new constant speed that would allow him to arrive at the same time. How much faster did he drive after his break compared to before the break in miles per hour?

Video Solution

Problem 13.2.5 (AMC 8)

Jeremy's father drives him to school in rush hour traffic in 20 minutes. One day there is no traffic, so his father can drive him 18 miles per hour faster and gets him to school in 12 minutes. How far in miles is it to school?

Video Solution

Problem 13.2.6 (AMC 8)

George walks 1 mile to school. He leaves home at the same time each day, walks at a steady speed of 3 miles per hour, and arrives just as school begins. Today he was distracted by the pleasant weather and walked the first $\frac{1}{2}$ mile at a speed of only 2 miles per hour. At how many miles per hour must George run the last $\frac{1}{2}$ mile in order to arrive just as school begins today?

Video Solution

Problem 13.2.7 (AIME)

Al walks down to the bottom of an escalator that is moving up and he counts 150 steps. His friend, Bob, walks up to the top of the escalator and counts 75 steps. If Al's speed of walking (in steps per unit time) is three times Bob's walking speed, how many steps are visible on the escalator at a given time? (Assume that this value is constant.)

Video Solution

Additional Problems

Problem 13.2.8

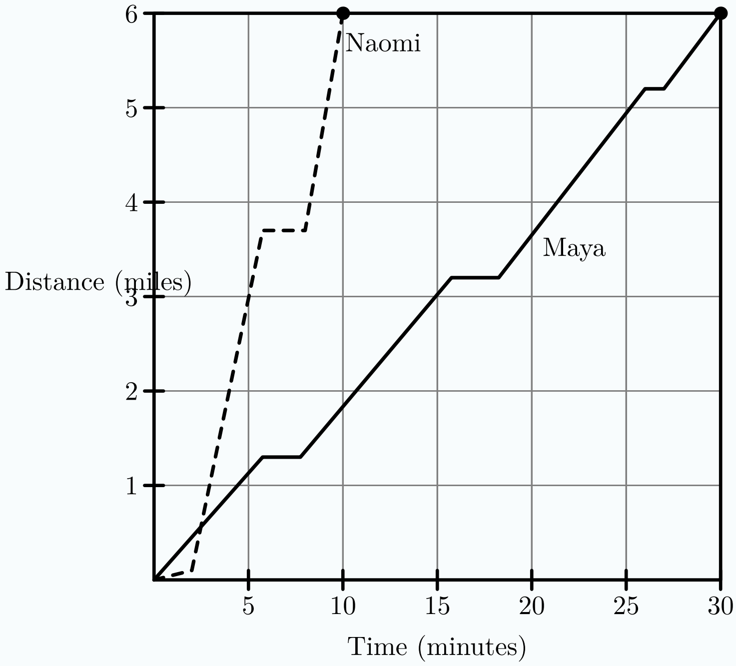
Joe drives at a speed of 30 miles per hour for 60 miles. He also drives at 45 miles per hour for 15 miles. Find the average speed of his car.

Problem 13.2.9 (MATHCOUNTS)

Jones is chasing a car 800 meters ahead of him. He is on a horse moving at 50 km/h. If Jones catches up to the car in 4 minutes, how fast was the car moving in km/h?

Problem 13.2.10 (AMC 8)

After school, Maya and Naomi headed to the beach, 6 miles away. Maya decided to bike while Naomi took a bus. The graph below shows their journeys, indicating the time and distance traveled. What was the difference, in miles per hour, between Naomi's and Maya's average speeds?



Problem 13.2.11 (AMC 10)

David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

Problem 13.2.12 (EMCC)

Jacob and Alexander are walking up an escalator in the airport. Jacob walks twice as fast as Alexander, who takes 18 steps to arrive at the top. Jacob, however, takes 27 steps to arrive at the top. How many of the upward moving escalator steps are visible at any point in time?

Problem 13.2.13 (AMC 10)

Emily sees a ship traveling at a constant speed along a straight section of a river. She walks parallel to the riverbank at a uniform rate faster than the ship. She counts 210 equal steps walking from the back of the ship to the front. Walking in the opposite direction, she counts 42 steps of the same size from the front of the ship to the back. In terms of Emily's equal steps, what is the length of the ship?

Problem 13.2.14 (AIME)

Butch and Sundance need to get out of Dodge. To travel as quickly as possible, each alternates walking and riding their only horse, Sparky, as follows. Butch begins by walking while Sundance rides. When Sundance reaches the first of the hitching posts that are conveniently located at one-mile intervals along their route, he ties Sparky to the post and begins walking. When Butch reaches Sparky, he rides until he passes Sundance, then leaves Sparky at the next hitching post and resumes walking, and they continue in this manner. Sparky, Butch, and Sundance walk at 6, 4, and 2.5 miles per hour, respectively. The first time Butch and Sundance meet at a milepost, they are n miles from Dodge, and they have been traveling for t minutes. Find $n+t$.

Hints

Answers

13.1 40

13.2 $\frac{224}{29}$

13.3 25

13.4 17

13.5 800

13.2.1 110

13.2.2 $\frac{12}{13}$

13.2.3 704

13.2.4 10

13.2.5 9

13.2.6 6

13.2.7 120

13.2.8 $\frac{225}{7}$

13.2.9 38

13.2.10 24

13.2.11 210

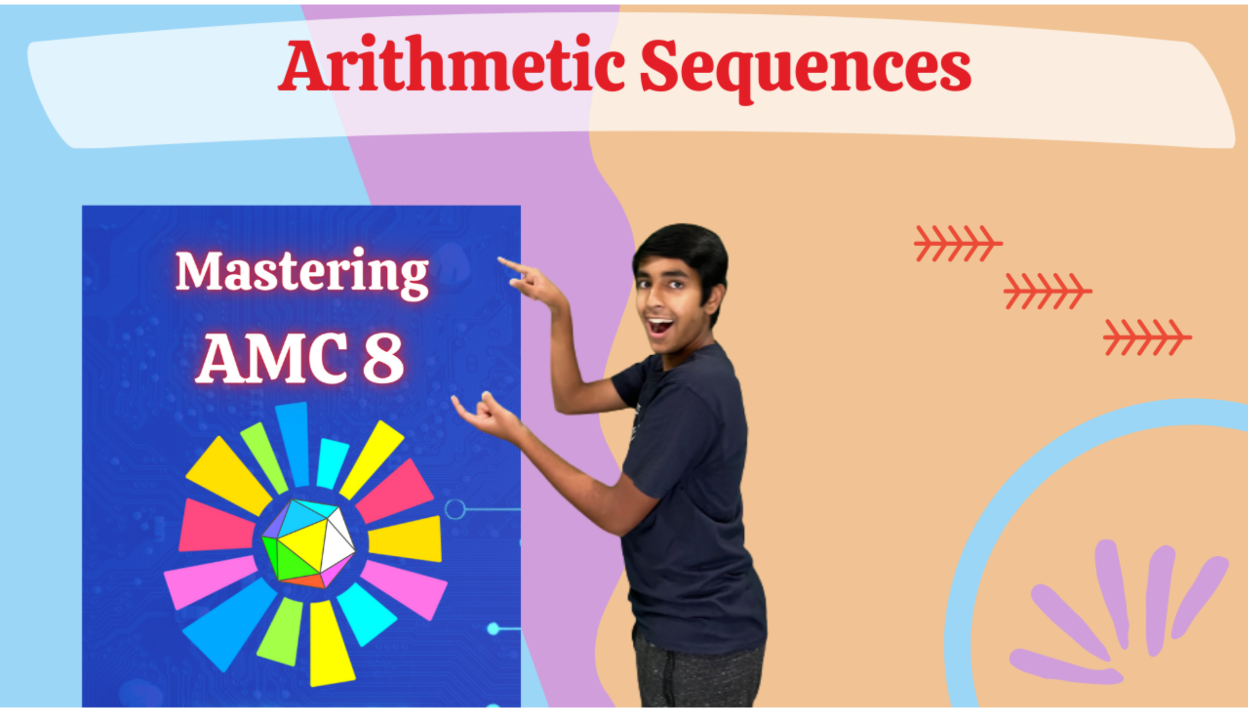
13.2.12 54

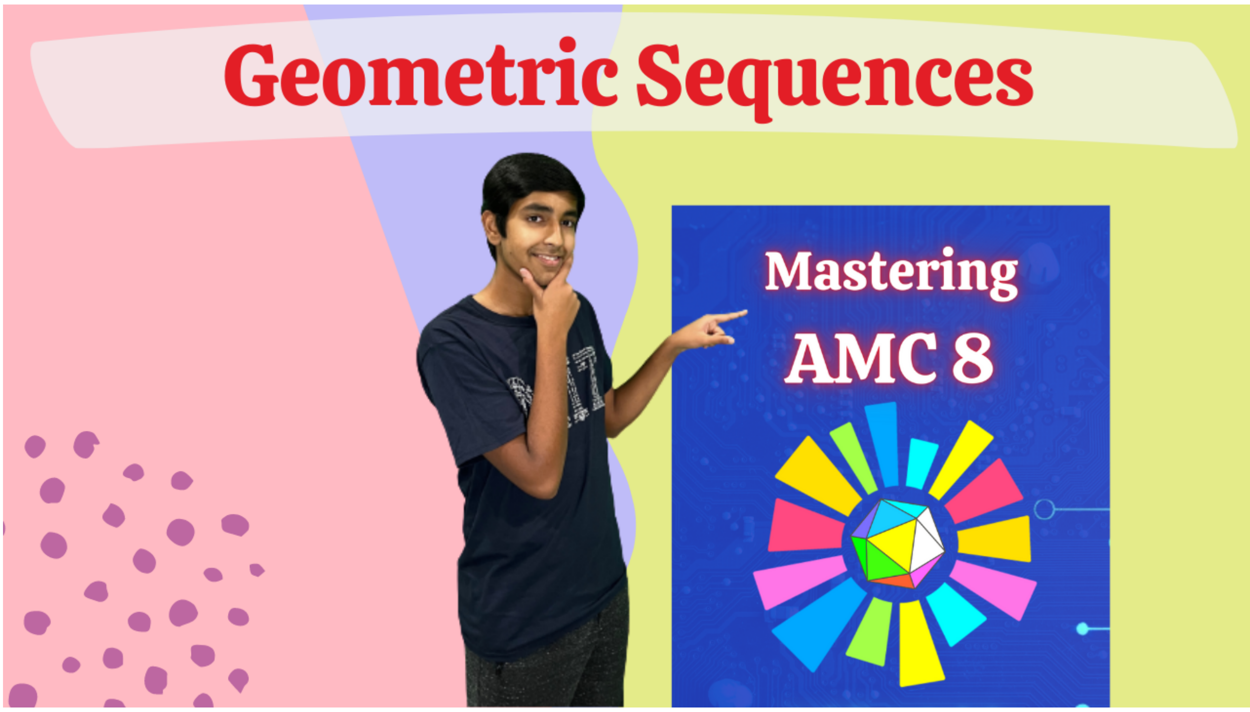
13.2.13 70

13.2.14 279

Chapter 14

Sequences and Series




Geometric Sequences
Mastering
AMC 8

14.1 Arithmetic Sequences

Definition 14.1.1 (Arithmetic Sequence)**.** An arithmetic sequence is a sequence of numbers with the same difference between consecutive terms.

$$1, 4, 7, 10, 13, \ldots, 40$$

is an arithmetic sequence because there is always a difference of 3 between consecutive terms.

Remark 14.1.2
Note that an arithmetic sequence can also have a negative common difference. For example, in the arithmetic sequence

$$40, 37, 34, \ldots, 4, 1$$

the common difference is -3.

Definition 14.1.3 (Arithmetic Sequence Notation)**.** In general, the terms of an arithmetic sequence can be represented as:

$$a_1, a_2, a_3, a_4, \ldots, a_n$$

where

- d is the common difference between consecutive terms
- n is the number of terms

Theorem 14.1.4 (nth term in an Arithmetic Sequence)

$$a_n = a_1 + (n-1)d$$

which basically means the nth term of an arithmetic sequence is equal to

$$\text{first term} + (\text{number of terms} - 1)(\text{common difference})$$

We also have that

$$a_n = a_m + (n-m)d$$

which means

$$\text{the nth term} = \text{mth term} + (\text{number of terms} - m)(\text{common difference})$$

Theorem 14.1.5 (Number of Terms in an Arithmetic Sequence)

$$n = \frac{a_n - a_1}{d} + 1$$

Essentially,

$$\text{Number of Terms} = \frac{\text{Last Term} - \text{First Term}}{\text{Common Difference}} + 1$$

Theorem 14.1.6 (Average of Terms in an Arithmetic Sequence)

$$\text{Average of Terms} = \frac{a_1 + a_n}{2}$$

$$\text{Average of Terms} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

which essentially means

$$\text{Average of Terms} = \frac{\text{First Term} + \text{Last Term}}{2}$$

$$\text{Average of Terms} = \frac{\text{Sum of all Terms}}{\text{Number of Terms}}$$

$$\text{If the number of terms is even, } x = \text{average of middle 2 terms}$$

.

$$\text{If number of terms is odd, } x = \text{middle term}$$

Theorem 14.1.7 (Sum of all Terms in an Arithmetic Sequence)

$$S_n = \frac{a_1 + a_n}{2} \times n$$

which essentially means

$$\text{Sum of All Terms} = \text{Average of Terms } \times \text{Number of Terms}$$

We can also substitute

$$a_n = a_1 + (n-1)d$$

to get

$$S_n = \frac{2a_1 + (n-1)d}{2} \times n$$

Example 14.1 (AIME)

Find the value of $a_2 + a_4 + a_6 + a_8 + \ldots + a_{98}$ if a_1, a_2, $a_3 \ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \ldots + a_{98} = 137$.

Video Solution

Example 14.2

The sum of the first 5 terms of an arithmetic sequence is 65 and the sum of the first 10 terms of the same sequence is 255. Find the sum of the first 15 terms of the sequence.

Video Solution

14.2 Special Series

Theorem 14.2.1 (Sum of Numbers Formula)

$$1+2+3+\cdots+n=\frac{(n)(n+1)}{2}$$

Theorem 14.2.2 (Sum of Odd Numbers Formula)

$$1+3+5+\cdots+(2n-1)=n^2$$

In simple terms, the

$$\text{Sum of first n odd numbers}=n^2$$

Theorem 14.2.3 (Sum of Even Numbers Formula)

$$2+4+6+\cdots+2n=n(n+1)$$

To intuitively think about it, just take 2 common from each term

$$2(1+2+3+\cdots+n)=2\frac{(n)(n+1)}{2}=n(n+1)$$

In simple terms, the

$$\text{Sum of first n even numbers}=2\times\text{sum of first n numbers}$$

Theorem 14.2.4 (Sum of Squares Formula)

$$1^2+2^2+\cdots+n^2=\frac{(n)(n+1)(2n+1)}{6}$$

Theorem 14.2.5 (Sum of Cubes Formula)

$$1^3+2^3+\cdots+n^3=\left(\frac{(n)(n+1)}{2}\right)^2$$

14.3 Geometric Sequences

Definition 14.3.1 (Geometric Sequence)**.** A geometric sequence is a sequence of numbers with the same ratio between consecutive terms.

$$1, 2, 4, 8, 16, 32 \ldots, 1024$$

is a geometric sequence because there is always a ratio of 2 between consecutive terms.

Definition 14.3.2 (Geometric Sequence Notation)**.** In general, the terms of a geometric sequence can be represented as:

$$g_1, g_2, g_3, g_4, \ldots, g_n$$

where

- r is the common ratio between consecutive terms
- n is the number of terms

Remark 14.3.3
Note that a geometric sequence can also have a negative common ratio. For example the sequence $1, -2, 4, -8, \ldots, 512, -1024$ has a common ratio of -2.

Theorem 14.3.4 (nth term in a Geometric Sequence)

$$g_n = g_1 \cdot r^{n-1}$$

which basically means

$$\text{the nth term of a geometric sequence} = \text{first term} \times (\text{common ratio})^{\text{number of terms}-1}$$

A general form for calculating the nth term

$$g_n = g_m \cdot r^{(n-m)}$$

which basically means

$$\text{the nth term} = \text{mth term} \times \text{common ratio}^{(n-m)}$$

Theorem 14.3.5 (Number of Terms in a Finite Geometric Sequence)

$$n = \log_r \left(\frac{g_n}{g_1}\right) + 1$$

Essentially,

Number of Terms $= 1$ more than the number of times we needed to multiply$r$ by g_1 to get g_n

Theorem 14.3.6 (Sum of all Terms in a Finite Geometric Sequence)

$$S_n = g_1 \frac{(1 - r^n)}{1 - r}$$

which essentially means

$$\text{Sum of All Terms} = \text{First Term} \times \frac{1 - \text{common ratio}^n}{1 - \text{common ratio}}$$

Theorem 14.3.7 (Sum of all Terms in an Infinite Geometric Sequence)
For $-1 < r < 1$,

$$S_\infty = \frac{g_1}{1 - r}$$

Remark 14.3.8
The reason the formula only works for $|r| < 1$ is because if $|r| \geq 1$ the sum will diverge or essentially be infinite. We can only find the sum of a converging geometric sequence for which the sum approaches a constant value. Some Examples:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2$$

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

Example 14.3 (AIME)

Call a 3-digit number geometric if it has 3 distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

Video Solution

Example 14.4 (AIME)

Suppose n is a positive integer and d is a single digit in base 10. Find n if

$\frac{n}{810} = 0.d25d25d25\ldots$

Video Solution

14.4 Arithmetico-Geometric Sequence

Example 14.5

Evaluate $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \ldots$

Video Solution

Concept 14.4.1

The strategy for these problems is just to break the sequence into multiple geometric sequences. Another strategy is to let the expression be S. Then, divide S by some number such that when you subtract it from S, most of the terms cancel. This strategy is faster, but harder to figure out.

14.5 Practice Problems

Problem 14.5.1 (AMC 8)

An arithmetic sequence is a sequence in which each term after the first is obtained by adding a constant to the previous term. For example, $2,5,8,11,14$ is an arithmetic sequence with five terms, in which the first term is 2 and the constant added is 3. Each row and each column in this 5×5 array is an arithmetic sequence with five terms. The square in the center is labelled X as shown. What is the value of X?

1				25
		X		
17				81

Video Solution

Problem 14.5.2 (Omega Learn Math Contest)

There are 4 positive integers A, B, C, and D. The first 3 numbers form an arithmetic sequence in that order. The last 3 numbers form a geometric sequence in that order. The value of C is 3 times the value of A. What is the smallest possible value of D?

Video Solution

Problem 14.5.3 (AMC 10)

Suppose that $\{a_n\}$ is an arithmetic sequence with

$$a_1+a_2+\cdots+a_{100}=100 \text{ and } a_{101}+a_{102}+\cdots+a_{200}=200.$$

What is the value of a_2-a_1?

Video Solution

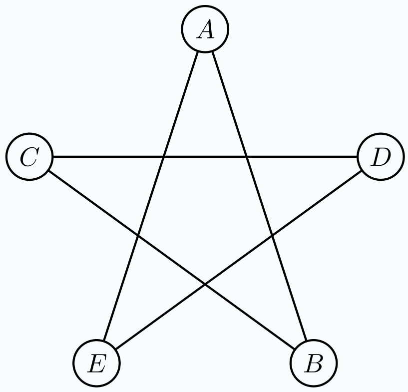
Problem 14.5.4 (AMC 10/12)

A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term in the geometric progression?

Video Solution

Problem 14.5.5 (AMC 10)

In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers 3, 5, 6, 7, and 9, although not necessarily in this order. The sums of the numbers at the ends of the line segments AB, BC, CD, DE, and EA form an arithmetic sequence, although not necessarily in this order. What is the middle term of the sequence?



Video Solution

Problem 14.5.6 (AIME)

The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

Video Solution

Additional Problems

Problem 14.5.7 (AHSME)

What is the 100th number in the arithmetic sequence $1, 5, 9, 13, 17, 21, 25, \ldots$

Problem 14.5.8 (MATHCOUNTS)

The non-negative integers a, b, c, d, and e form an arithmetic sequence. If their sum is 440, what is the largest possible value for e?

Problem 14.5.9 (AMC 8)

The sum of 25 consecutive even integers is $10,000$. What is the largest of these 25 consecutive integers?

Problem 14.5.10 (AMC 10/12)

A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

Hints

Answers

14.1 93

14.2 570

14.3 840

14.4 750

14.5 2

14.5.1 31

14.5.2 9

14.5.3 0.01

14.5.4 1

14.5.5 12

14.5.6 195

14.5.7 397

14.5.8 176

14.5.9 424

14.5.10 10

Chapter 15

Mean, Median, Mode

Video Lecture

15.1 Mean, Median, Mode Fundamentals

Definition 15.1.1 (Mean/Average).

$$\text{Mean} = \text{average of all terms} = \frac{\text{sum of all terms}}{\text{number of terms}}$$

Remark 15.1.2
Often, in these types of problems, we can simply consider the difference between each value and the mean and make sure that value sums to 0.

Definition 15.1.3 (Mode).

$$\text{Mode} = \text{Most common term(s)}$$

Remark 15.1.4
There could be multiple modes. If the problem says "unique mode", it means that there is only one mode.

Definition 15.1.5 (Median). After arranging the numbers in increasing or decreasing order:
If number of terms is odd,

$$\text{Median} = \text{middle number}$$

If number of terms is even,

$$\text{Median} = \text{average of middle two numbers}$$

Definition 15.1.6 (Range).

$$\text{Range} = \text{Largest number} - \text{Smallest Number}$$

Example 15.1

Find the sum of the mean, median, mode, and range of 1, 9, 7, 1, 3, 5, 2.

Solution

To solve this, we will apply the formulas above.

What is the first step we should do to make analyzing the numbers easier?

We should order them! Doing so, we end up with $1, 1, 2, 3, 5, 7, 9$.

What is the mean?

The mean is the sum of the numbers divided by the count of numbers. The sum of numbers is $1+1+2+3+5+7+9 = 28$. There are 7 numbers. Therefore, the mean is $\frac{28}{7} = 4$.

What is the median?

There are 7 numbers, so the median will just be the middle, or 4th number when the numbers are arranged in increasing order. The 4th number is 3, so the median is also 3.

What is the mode?

The only number that appears more than once is 1, so 1 is the mode.

What is the range?

The range will be the largest number, 9, minus the smallest number, 1, which is equal to $9 - 1 = 8$.

Therefore, the sum of the mean, median, mode, and range is $1+3+4+8 = \boxed{16}$.

Example 15.2 (AIME)

Let S be a list of positive integers–not necessarily distinct–in which the number 68 appears. The average (arithmetic mean) of the numbers in S is 56. However, if 68 is removed, the average of the remaining numbers drops to 55. What is the largest number that can appear in S?

Video Solution

Example 15.3 (AMC 10)

What is the median of the following list of 4040 numbers?

$$1, 2, 3, \ldots, 2020, 1^2, 2^2, 3^2, \ldots, 2020^2$$

Video Solution

15.2 Mean Median Mode Conditions Examples

Example 15.4 (Omega Learn)

A list of 6 distinct positive integers has a mean of 7, a median of 6, and a range of 9. Find the sum of the 2nd and 5th smallest numbers.

Solution

This problem may seem a little tricky because of how many possibilities there are. Let's first focus on the median, as there only a few cases which work.

What are the possibilities for the middle 2 numbers, or the 3rd and 4th smallest numbers?

The average of these 2 numbers must be 6, or the sum must be $2 \times 6 = 12$.

What must be the sum of the remaining 4 numbers?

The mean of all 6 numbers is 7, so their sum is $6 \times 7 = 42$. The sum of the middle 2 numbers is 12, so the sum of the remaining 4 numbers must be $42 - 12 = 30$.

Next, let's consider the cases for the 2 middle numbers that result in a sum of 12. The following possibilities exist:

Case 1: 3rd smallest number: 6, 4th smallest number 6

This is not possible because the numbers are distinct.

Case 2: 3rd smallest number: 4, 4th smallest number 8

In this case, there are very few possibilities for the lowest 2 numbers. Since they must be positive, less than 4, and distinct, the only possibilities are (1,2), (1,3), and (2, 3). In addition, the range must also be 9. Therefore, the largest number will be the lowest number plus the range, which can be $1+9=10$ or $2+9=11$. We can consider each of the cases for the largest and smallest numbers, however, we should first check if this case is even possible.

Is it possible for the sum of the 1st, 2nd, 5th, and 6th smallest numbers to be 30 in this case?

We can consider each of the cases for the largest and smallest numbers, however, we should first check if having a mean of 7 is even possible. The largest number can be at most 11, and the 2nd largest number can be at most 10 since it can't be the same as the largest number. This means the sum of these 4 numbers can be at most $2+3+10+11=26$, which isn't large enough to be 30. Therefore, this case doesn't work.

Case 3: 3rd smallest number 3, 4th smallest number 9

The smallest 2 numbers must be 1 and 2, since all of the numbers must be positive and less than 3. Then, by similar logic to the previous case, the largest value is $1+9=10$. The 2nd largest value can be at most 9. This means the sum is at most $1+2+9+10=22$, which is again, not large enough.

Case 4: 3rd smallest number 2 or 1, 4th smallest number 10 or 11, respectively

For the other cases, the 3rd smallest number will be less than 3. However, we must have 2 distinct positive values less than the 3rd smallest number (the 1st and 2nd smallest numbers). We can see that if the 3rd smallest value is less than 3 and is 1 or 2, then there are 0 or 1 distinct positive values less than this number. Therefore, these cases don't work. **Case 5:** 3rd smallest number: 5, 4th smallest number 7

Is it possible for the smallest number to be 2?

Let's see if it's possible for the sum of smallest 2 numbers and largest 2 numbers to be 30 in this case. The largest possible value for the 2nd smallest number is 4, since it has to be less than 5. The largest number must be the smallest number plus the range, which is $2+9=11$, and the 2nd largest number can be at most 10. Therefore, sum of smallest 2 numbers and largest 2 numbers is at most $2+4+10+11=27$. Therefore, the smallest number cannot be 2.

If the smallest number is 1, the sum will be even lower since the largest will be 10, and the 2nd largest number will be at most 9. This would result in a sum even lower than in the previous possibility.

Therefore, the smallest number must be 3, and since the 2nd smallest number has to be greater than the smallest number (3) and less than the 3rd smallest number (5), it must be 4. In addition, the largest number will be $3+9=12$. Then, let the 2nd largest number be x.

What value of x will result in the sum smallest 2 numbers and largest 2 numbers being 30?

Adding all of these numbers, we get that this sum is $3+4+12+x=30$. This means that the 2nd largest number is 11. This case works because this number is smaller than the largest number (12), and larger than the 4th smallest number (7).

The 6 numbers (in increasing order) are 3, 4, 5, 7, 11, 12 satisfy all 3 conditions. The sum of the 2nd and 5th smallest numbers are $4+11=\boxed{15}$.

Example 15.5 (AMC 10)

The mean, median, unique mode, and range of a collection of eight integers are all equal to 8. The largest integer that can be an element of this collection is what?

Video Solution

15.3 Practice Problems

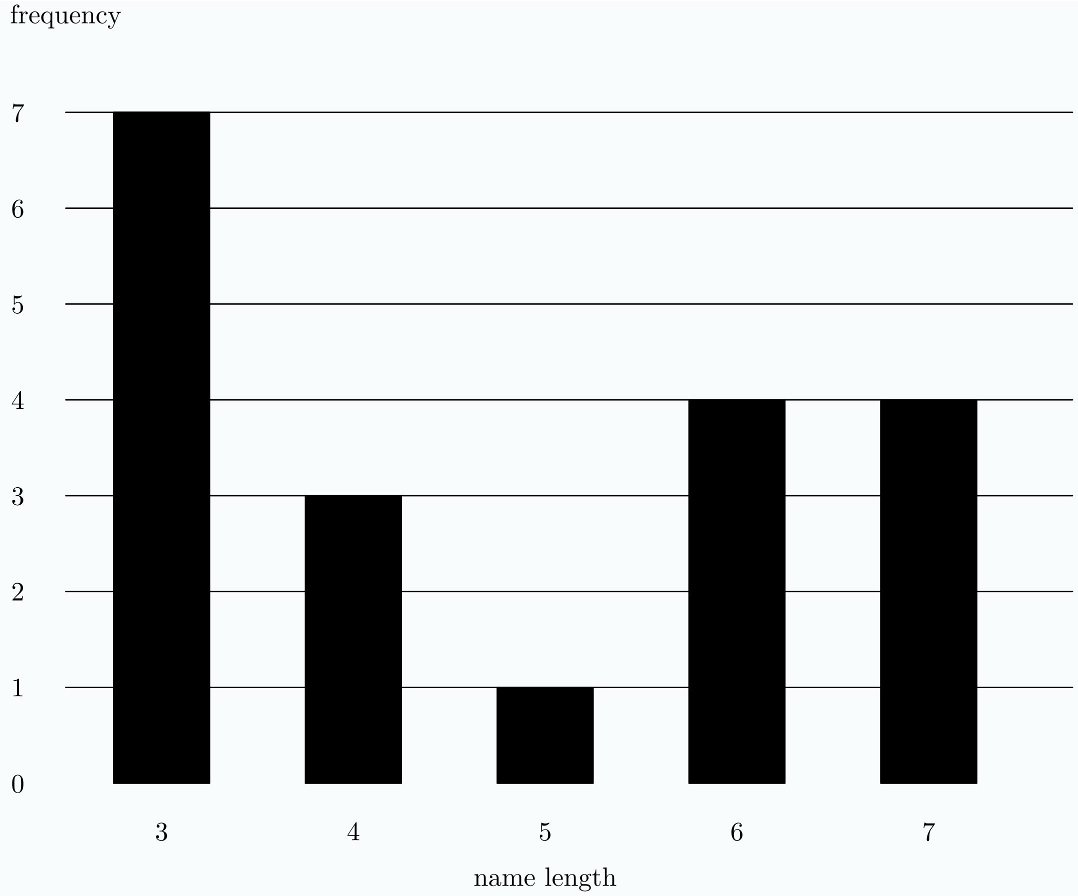
Problem 15.3.1 (AMC 8)

Four students take an exam. Three of their scores are $70, 80$, and 90. If the average of their four scores is 70, then what is the remaining score?

Video Solution

Problem 15.3.2 (AMC 8)

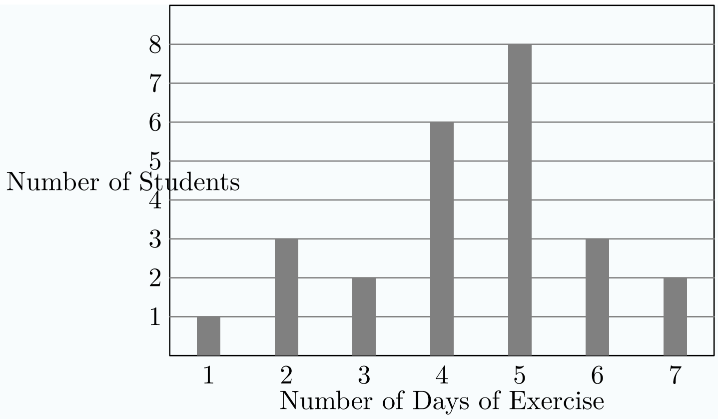
The following bar graph represents the length (in letters) of the names of 19 people. What is the median length of these names?



[Video Solution](#)

Problem 15.3.3 (AMC 8)

Mr. Garcia asked the members of his health class how many days last week they exercised for at least 30 minutes. The results are summarized in the following bar graph, where the heights of the bars represent the number of students.



What was the mean number of days of exercise last week, rounded to the nearest hundredth, reported by the students in Mr. Garcia's class?

Video Solution

Problem 15.3.4 (AMC 8)

Hammie is in the 6th grade and weighs 106 pounds. Her quadruplet sisters are tiny babies and weigh 5, 5, 6, and 8 pounds. Which is greater, the average (mean) weight of these five children or the median weight, and by how many pounds?

Video Solution

Problem 15.3.5 (AMC 8)

What is the sum of the mean, median, and mode of the numbers 2, 3, 0, 3, 1, 4, 0, 3 ?

Video Solution

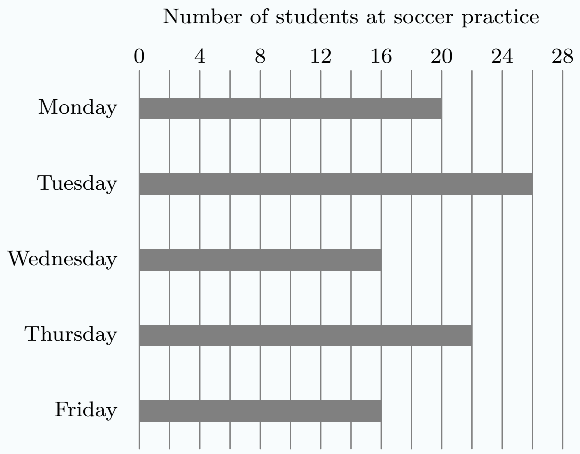
Problem 15.3.6 (AMC 8)

The mean, median, and unique mode of the positive integers 3, 4, 5, 6, 6, 7, and x are all equal. What is the value of x?

Video Solution

Problem 15.3.7 (AMC 8)

The diagram shows the number of students at soccer practice each weekday during last week. After computing the mean and median values, Coach discovers that there were actually 21 participants on Wednesday. Which of the following statements describes the change in the mean and median after the correction is made?



(A) The mean increases by 1 and the median does not change.

(B) The mean increases by 1 and the median increases by 1.

(C) The mean increases by 1 and the median increases by 5.

(D) The mean increases by 5 and the median increases by 1.

(E) The mean increases by 5 and the median increases by 5.

Video Solution

Problem 15.3.8 (AMC 8)

The mean of a set of five different positive integers is 15. The median is 18. What is the maximum possible value of the largest of these five integers?

Video Solution

Problem 15.3.9 (AMC 8)

One day the Beverage Barn sold 252 cans of soda to 100 customers, and every customer bought at least one can of soda. What is the maximum possible median number of cans of soda bought per customer on that day?

Video Solution

Problem 15.3.10 (AMC 8)

The harmonic mean of a set of non-zero numbers is the reciprocal of the average of the reciprocals of the numbers. What is the harmonic mean of 1, 2, and 4?

Video Solution

Problem 15.3.11 (AMC 8)

How many subsets of two elements can be removed from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ so that the mean (average) of the remaining numbers is 6?

Video Solution

Problem 15.3.12 (AMC 12)

When the mean, median, and mode of the list

$$10, 2, 5, 2, 4, 2, x$$

are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

Video Solution

Problem 15.3.13 (AMC 10)

Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer in S is also removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S. What is the average value of all the integers in the set S?

Video Solution

Additional Problems

Problem 15.3.14 (AMC 8 Modified)

Here is a list of the numbers of fish that Tyler caught in nine outings last summer:

$$2, 0, 1, 3, 0, 3, 3, 1, 2.$$

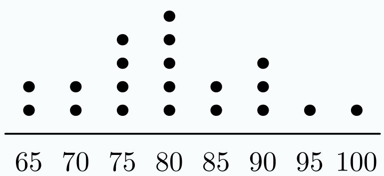
What is the ordering of the mean, median, and mode of the number of fish caught?

Problem 15.3.15 (EMCC)

The mean of the numbers 2, 0, 1, 5, and x is an integer. Find the smallest possible positive integer value for x.

Problem 15.3.16 (AMC 8)

Mr. Ramos gave a test to his class of 20 students. The dot plot below shows the distribution of test scores.



Later Mr. Ramos discovered that there was a scoring error on one of the questions. He regraded the tests, awarding some of the students 5 extra points, which increased the median test score to 85. What is the minimum number of students who received extra points?
(Note that the median test score equals the average of the 2 scores in the middle if the 20 test scores are arranged in increasing order.)

Problem 15.3.17 (AMC 10)

The mean, median, and mode of the 7 data values $60, 100, x, 40, 50, 200, 90$ are all equal to x. What is the value of x?

Problem 15.3.18 (AMC 10)

When 15 is appended to a list of integers, the mean is increased by 2. When 1 is appended to the enlarged list, the mean of the enlarged list is decreased by 1. How many integers were in the original list?

Problem 15.3.19 (AMC 10)

Melanie computes the mean μ, the median M, and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, . . . , 12 28s, 11 29s, 11 30s, and 7 31s. Let d be the median of the modes. Which of the following statements is true?

(A) $\mu < d < M$ **(B)** $M < d < \mu$ **(C)** $d = M = \mu$ **(D)** $d < M < \mu$ **(E)** $d < \mu < M$

Problem 15.3.20 (MATHCOUNTS)

The mean of a list of nine numbers is 17, and the modes are a, b and c. If a + 4, 1 + b and c - 8 are distinct numbers in the list, and none of them are modes of the list, then what is the value of 3(a + b + c)?

Problem 15.3.21 (Omega Learn)

6 positive integers have a median of 5.5, two distinct modes, a range of 5, and a mean of $\frac{35}{6}$. Find the sum of all possible values for the largest integer.

Hints

Answers

15.2 649

15.3 1976.5

15.5 14

15.3.1 40

15.3.2 4

15.3.3 4.36

15.3.4 Average, by 20

15.3.5 7.5

15.3.6 11

15.3.7 The mean increases by 1 and the median increases by 1

15.3.8 35

15.3.9 3.5

15.3.10 $\frac{12}{7}$

15.3.11 5

15.3.12 20

15.3.13 36.8

15.3.14 mean < median < mode

15.3.15 2

15.3.16 4

15.3.17 90

15.3.18 4

15.3.19 $d < \mu < M$

15.3.20 156

15.3.21 17

Chapter 16

Telescoping

Video Lecture

16.1 Telescoping

16.2 Telescoping Basics

Concept 16.2.1 (Telescoping)
Expand the first few and last few terms, and cancel out any terms you see.

Remark 16.2.2
Generally, whenever you have long expressions that seem to be hard or impossible to compute manually, telescoping is probably at play.

16.3 Telescoping Sums

Example 16.1 (EMCC)

Evaluate $(1^2 - 3^2 + 5^2 - 7^2 + 9^2 - \cdots + 2009^2) - (2^2 - 4^2 + 6^2 - 8^2 + 10^2 - \cdots + 2010^2)$

Video Solution

Example 16.2

Find $\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72}$

Video Solution

16.4 Telescoping Products

Example 16.3 (Omega Learn)

What is the value of $\frac{3}{4} \times \frac{4}{6} \times \frac{5}{8} \times \cdots \times \frac{10}{18}$? (For each fraction, the denominator is twice the numerator minus 2)

Video Solution

16.5 Telescoping Equation

Example 16.4 (BmMT)

Suppose $x_1, x_2, ..., x_{2022}$ is a sequence of real numbers such that:

$$x_1 + x_2 = 1$$

$$x_2 + x_3 = 2$$

$$\vdots$$

$$x_{2021} + x_{2022} = 2021$$

If $x_1 + x_{499} + x_{999} + x_{1501} = 222$, then what is the value of x_{2022}?

Video Solution

16.6 Practice Problems

Problem 16.6.1 (AMC 8)

What is the value of $4 \cdot (-1+2-3+4-5+6-7+\cdots+1000)$?

Video Solution

Problem 16.6.2 (AMC 8)

What is the value of the expression $\frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}{1+2+3+4+5+6+7+8}$?

Video Solution

Problem 16.6.3 (AMC 8)

What is the value of the product

$$\left(1+\frac{1}{1}\right)\cdot\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)\cdot\left(1+\frac{1}{5}\right)\cdot\left(1+\frac{1}{6}\right)?$$

Video Solution

Problem 16.6.4 (AMC 8)

Find the value of the expression

$$100-98+96-94+92-90+\cdots+8-6+4-2.$$

Video Solution

Problem 16.6.5 (AMC 8)

What is the value of the product

$$\left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{2\cdot 4}{3\cdot 3}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\cdots\left(\frac{97\cdot 99}{98\cdot 98}\right)\left(\frac{98\cdot 100}{99\cdot 99}\right)?$$

Video Solution

Additional Problems

Problem 16.6.6 (AMC 8)

What is the value of

$$1+3+5+\cdots+2017+2019-2-4-6-\cdots-2016-2018?$$

Problem 16.6.7 (AMC 8)

What is the product of $\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\cdots\times\frac{2006}{2005}$?

Problem 16.6.8 (AMC 8)

What is the value of

$$\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdots\frac{18}{20}\cdot\frac{19}{21}\cdot\frac{20}{22}?$$

Problem 16.6.9 (MATHCOUNTS)

If

$$n=10^{2020}-10^{2019}+10^{2018}-10^{2017}+\cdots+10^{2}-10^{1}$$

what is the sum of digits of the integer n?

Problem 16.6.10

Evaluate

$$\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{10\times 11}$$

Problem 16.6.11 (Omega Learn)

Evaluate $1-2-3+4+5+6-7-8-9-10\cdots+90+91$ (1 positive consecutive value, 2 negative consecutive values, 3 positive consecutive values, ...)

Problem 16.6.12 (Omega Learn)

Let the sequence a_n be defined as $a_n = 3^{1+2+3+\cdots+n}$ for all positive integers n. Sohil is trying to evaluate the following sequence: $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots$. Due to his bad handwriting, in the expression above, he accidentally writes each multiplication sign before all fractions of the form $\frac{a_n}{a_n+1}$ (for example, $\frac{3^1}{3^1+1}$, $\frac{3^{(1+2)}}{3^{(1+2)}+1}$, etc.) with a plus sign. What is the value of the new expression?

Hints

Answers

16.1 -2011

16.2 $\frac{7}{18}$

16.3 $\frac{5}{256}$

16.4 1330

16.6.1 2000

16.6.2 1120

16.6.3 7

16.6.4 50

16.6.5 $\frac{50}{99}$

16.6.6 1010

16.6.7 1003

16.6.8 $\frac{1}{231}$

16.6.9 9090

16.6.10 $\frac{9}{22}$

16.6.11 616

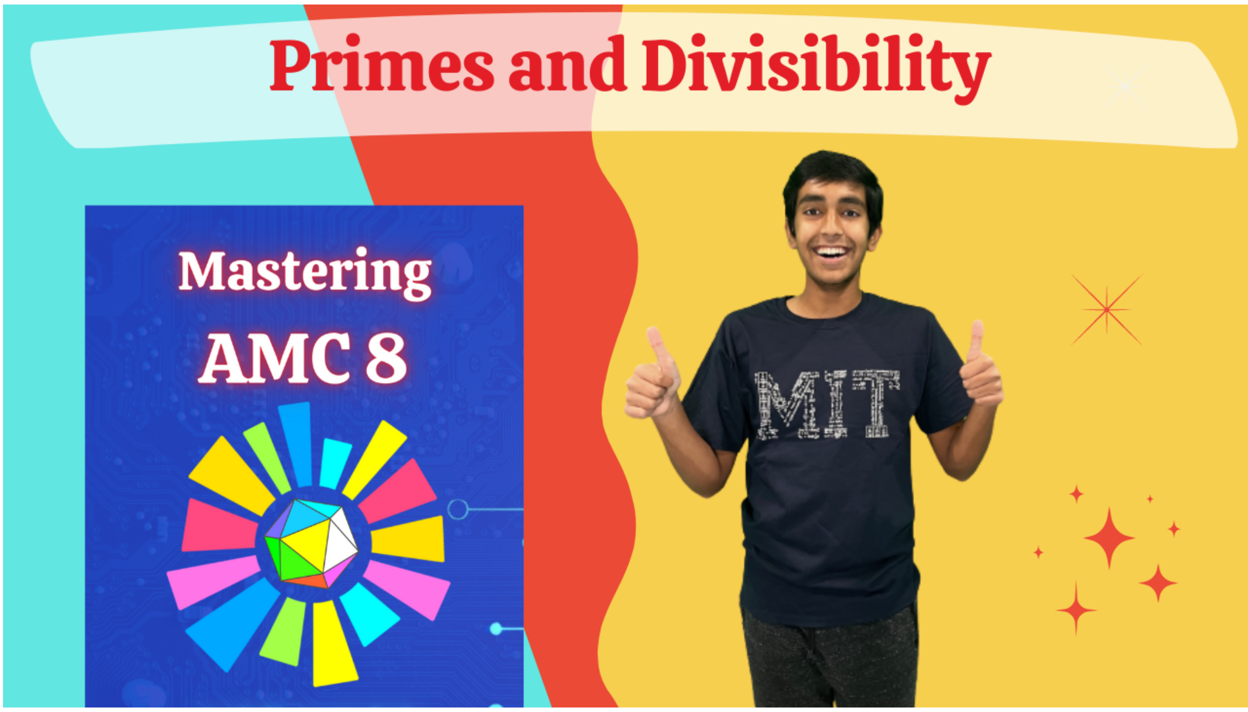
16.6.12 $\frac{1}{2}$

Number Theory

Chapter 17

Primes and Divisibility

Video Lecture



17.1 Primes

Definition 17.1.1 (Primes)**.** Primes are numbers that have exactly two factors: 1 and the number itself.
Example: 2, 3, 5, 7, 11, 13, 17, 19, 23 are all primes
Note: 1 is not a prime and 2 is the only even prime.

Remark 17.1.2
It's highly recommended to memorize first few primes as this can save some time during the contest: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37

Remark 17.1.3
In order to check whether a number n is prime, we need to check whether it can be divided by all the primes that are less than or equal to $\sqrt{n}$

17.2 Divisibility Rules

Here are some important rules to figure out whether a number is divisible by another number.

Divisibility rule for 2: Last digit is even

Example: 362
The last digit is even, so 362 is divisible by 2.

Divisibility rule for 3: Sum of digits is divisible by 3

Example: 972
The sum of digits is $9 + 7 + 2 = 18$, which is a multiple of 3

Divisibility rule for 4: The number formed by the last 2 digits is divisible by 4

Example: 172947
The number formed by last two digits is 47 which is not divisible by 4, so 172947 is also not divisible by 4

Example: 10764
The number formed by last two digits is 64 which is divisible by 4, so 10764 is divisible by 4

Divisibility rule for 5 - Last digit is 0 or 5

Example: 375
The last digit is 5, so 375 is divisible by 5.

Example: 140
The last digit is 0, so 140 is divisible by 5.

Divisibility rule for 6: The number is divisible by both 2 and 3

Example: 196
The last digit is 2, so 196 is divisible by 2. The sum of digits is $1 + 9 + 6 = 16$, which is not a multiple of 3, so 196 is not a multiple of 3. Therefore, 196 is also not a multiple of 6.

Example: 252
The last digit is 2, so 252 is divisible by 2. The sum of digits is $2 + 5 + 2 = 9$, which is a multiple of 3, so 196 is a multiple of 3. Therefore, 196 is also a multiple of 6.

Divisibility rule for 7: Take out factors of 7 until you reach a small number that is either divisible or not divisible by 7

Example: 2240
We subtract 2100, which is divisible by 7 (7×300) to get 2240 - 2100 = 140. Then, we see that 140 is also divisible by 7 as it is 7×20. Therefore, 2240 is also divisible by 7.

Example: 28146
We subtract 28000, which is divisible by 7 (7×4000) to get 28146 - 28000 = 146. Then, we subtract 140 as it is also divisible by 7 to get 6. 6 is clearly not divisible by 7, so 28146 is also not divisible by 7.

Divisibility rule for 8: The number formed by the last 3 digits is divisible by 8

Example: 423824
The number formed by the last 3 digits is 824 which is divisible by 8 (8×103), so 423824 is also divisible by 8.

Example: 4827676 The last 3 digits are 676. Using the same technique that we used to find numbers divisible by 7, we can subtract off numbers divisible by 8. Clearly, 640 is divisible by 8 (8×80), so subtracting that off, we get $676 - 640 = 36$. Then, we can subtract off $8 \times 4 = 32$ to get 4. We know that 4 is not divisible by 8. Therefore, 4827676 is also not divisible by 8.

Divisibility rule for 9: Sum of digits is divisible by 9

Example: 63198
The sum of digits is $6+3+1+9+8=27$, which is divisible by 9, so 63198 is also divisible by 9.

Example: 815419 The sum of digits is $8+1+5+4+1+9=28$, which is not divisible by 9, so 63198 is not divisible by 9.

10 - Last digit is 0

Example: 331960
The last digit is 0, so 331960 is divisible by 10.

Divisibility rule for 11: Calculate the sum of odd positioned digits (O) and even positioned digits (E). If O - E is divisible by 11, then the number is also divisible by 11. Don't forget that O - E can be negative.

Example: 1331 First, we calculate the sum of the odd positioned digits, or the sum of the 1st and 3rd digits from the left. We find $O = 1+3 = 4$.

Next, we calculate the sum of the even positioned digits, or the sum of the 2nd and 4th digits from the left. We find $E = 3+1 = 4$.

The difference of numbers, $O - E = 4 - 4 = 0$, and 0 is a multiple of 11, so 1331 is divisible by 11.

Example: 629321
First, we calculate the sum of the odd positioned digits, or the sum of the 1st, 3rd, and 5th digits from the left. We find $O = 6+9+2 = 17$. Next, we calculate the sum of the even positioned digits, or the sum of the 2nd, 4th, and 6th digits. We find $E = 2+3+1 = 6$.

The difference $O - E = 17 - 6 = 11$, which is clearly a multiple of 11, so 629321 is divisible by 11.

Divisibility rule for 12: Divisible by 3 and 4

Example: 1584
First, we check if 1584 is divisible by 3. The sum of digits is $1+5+8+4 = 18$, which is divisible by 3, so 1584 is divisible by 3.

Next, we check if 1584 is divisible by 4. The last 2 digits, 84, is divisible by 4, so 1584 is also divisible by 4.

Therefore, 1584 is divisible by 12 since it is divisible by both 3 and 4.

Example: 230792
First, we check if 230792 is divisible by 3. The sum of digits is $2+3+0+7+9+2 = 23$, which is not divisible by 3, so 1584 is not divisible by 3.

Next, we check if 230792 is divisible by 4. The last 2 digits, 92, is divisible by 4, so 230792 is divisible by 4.

Although 230792 is divisible by 4, it is not divisible by 3, so therefore cannot be divisible by 12.

Divisibility rule for 15: Divisible by 3 and 5

Example: 352305
First, we check if 352305 is divisible by 3. The sum of digits is $3+5+2+3+0+5 = 18$, which is divisible by 3, so 352305 is not divisible by 3.

Next, we check if 352305 is divisible by 5. The last digits, 5, is divisible by 5, so 235305 is also divisible by 5.

Therefore, 352305 is divisible by 15 since it is divisible by both 3 and 5.

Example: 374740
First, we check if 374740 is divisible by 3. The sum of digits is $3+7+4+7+4+0=25$, which is not divisible by 3, so 374740 is not divisible by 3.

Next, we check if 374740 is divisible by 5. The last digit, 0, is divisible by 5, so 374740 is also divisible by 5.

Although 374740 is divisible by 5, it is not divisible by 3, so it cannot be divisible by 15.
Here is a summary of all the divisibility rules:

Concept 17.2.1 (Divisibility Rules)

2	Last digit is even
3	Sum of digits is divisible by 3
4	Last 2 digits divisible by 4
5	Last digit is 0 or 5
6	Divisible by 2 and 3
7	Take out factors of 7 until you reach a small number that is either divisible or not divisible by 7
8	Last 3 digits are divisible by 8
9	Sum of digits is divisible by 9
10	Last digit is 0
11	Calculate the sum of odd digits (O) and even digits (E). If $\lvert O-E\rvert$ is divisible by 11, then the number is also divisible by 11
12	Divisible by 3 and 4
15	Divisible by 3 and 5

Remark 17.2.2
In general, if a number is divisible by 2 prime numbers or prime powers (numbers that are powers of a prime number like 3^5 or 2^4), then it will also be divisible by their product.

For example, to find whether a number is divisible by $30 = 2 \times 3 \times 5$, we just need to check whether it's divisible by each of 2, 3, and 5 using the divisibility rules.

17.3 Prime Factorization

Prime factorization is a way to express each number as a product of primes.

Examples:

The prime factorization of 21 is 3×7

The prime factorization of 60 is $2^2 \times 3 \times 5$

Example 17.1

Find the prime factorization of 117.

Solution

First, let's check if 117 is divisible by any small primes. Clearly, 117 is odd and not divisible by 2.

Is 117 divisible by 3?

We can see that since the sum of digits is 9, a multiple of 3, the number is divisible by 3. In fact, since 9 is also a multiple of 9, the number is also divisible by 9 or 3^2.

Dividing by 3^2 we get $\frac{117}{9} = 13$. Clearly, 13 is prime.

Therefore, our prime factorization is $3^2 \times 13$.

Example 17.2 (AMC 8)

Let Z be a 6-digit positive integer, such as 247247, whose first three digits are the same as its last three digits taken in the same order. Which of the following numbers must also be a factor of Z?

(A) 11 **(B)** 19 **(C)** 101 **(D)** 111 **(E)** 1111

Video Solution

Example 17.3 (MATHCOUNTS)

What four-digit number has tens digit 2 and units digit 8, is a multiple of 16, and when its digits are reversed the result is also a multiple of 16?

Video Solution

Example 17.4 (AMC 8)

A number is called flippy if its digits alternate between two distinct digits. For example, 2020 and 37373 are flippy, but 3883 and 123123 are not. How many five-digit flippy numbers are divisible by 15?

Video Solution

Example 17.5 (Omega Learn)

The 3 digit number $A3B$ is divisible by 15. What is the sum of all such possible numbers?

Solution

We can use our knowledge of divisibility rules to solve this.

For a number to be divisible by 15, what other numbers must it be divisible by?

We can see that $15 = 3 \times 5$, so for a number to be divisible by 15, it must also be divisible by 3 and 5. In order to solve this problem, we will look at the divisibility rule for 3 and 5.

Which divisibility rule should we look at first?

If we look at the divisibility rule for 3 first, we will end with many possibilities for A and B.

On the other hand, what does the divisibility rule for 5 tell us?

The divisibility rule for 5 tells us that B must be 0 or 5. This is much easier to use than the divisibility rule for 3 as now, we only have 2 possibilities to consider. Because we have already checked for the divisibility of 5, we must now check if the number is divisible by 3.

Case 1: B = 0

If $B = 0$, then the number will be $A30$. For this number to be divisible by 3, $A+3+0 = A+3$ must be divisible by 3. Because 3 is a multiple of 3, A must also be a multiple of 3. Therefore, A has 3 possibilities and can be 3, 6, or 9. The digit A can't be 0 since it has to be a 3 digit number. This gives the following possible numbers: 330, 630, 930.

Case 1: $B = 5$

If $B = 5$, then the number will be $A35$. For this number to be divisible by 3, $A+3+5 = A+8$ must be divisible by 3. Because 8 leaves a remainder of 2 when divided by 3, A must leave a remainder of 1 when divided by 3 for the sum to be a multiple of 3. Therefore, A has 3 possibilities and can be 1, 4, or 7. This gives the following possible numbers: 135, 435, 735.

Summing all numbers, we get

$$330+630+930+135+435+735 = \boxed{3195}$$

Example 17.6 (Omega Learn)

The number $3AB76$ is divisible by 264, where A and B are digits. Find $A+B$.

Solution

Let's see how we can approach this problem.

For A number to be divisible by 264, what other numbers must it be divisible by?

To figure this out, we first find the prime factorization of 264.

First, we divide out by powers of 2. Doing this repeatedly we get $264 = 2\times132 = 2^2\times66 = 2^3\times33$

Then, we can notice that the sum of digits of 33 is $3+3 = 6$, which is a multiple of 3, so 33 is a multiple of 3. Dividing by 3, we are then left with 11, which is clearly a prime number.

Therefore, the prime factorization is $2^3\times3\times11$. Therefore, for the number to be divisible by 264, it must be divisible by $2^3 (= 8)$, 3, and 11.

First, what information does $3AB76$ being divisible by 8 give us?

By the divisibility rule for 8, the last 3 digits, or $B76$, must be divisible by 8. Notice that 76 leaves a remainder of 4 when divided by 8. Therefore, $B = 0$ doesn't work.

When B is 1, then we are adding 100 to the last 3 digits. 100 also leaves a remainder of 4 when divided by 8. So $176 = 176+100$ will leave a remainder of $4+4 = 8$ or 0 when divided by 8, so 176 is divisible by 8.

Next, we can see that when B is 2, our number will be 276, which is equal to $100+176$. Since 176 is divisible by 8 and 100 leaves A remainder of 4 when divided by 8, 276 will leave A remainder of $0+4=4$ when divided by 8.

Do you notice A pattern with the remainders here?

Whenever B is even, it appears that $B76$ leaves a remainder of 4 when divided by 8, and whenever B is odd, $B76$ is divisible by 8. This is because 176 is divisible by 8, adding 100 will make the remainder 4 when divided by 8. Therefore, by adding 200, the remainder will become 4 + 4 = 8 or 0. Therefore, only odd values of B work.

Next, what does $3AB76$ being divisible by 11 tell us?

Using the divisibility rule for 11, we have that the sum of the odd positioned digits, $O = 3+B+6 = 9+B$ and the sum of the even positioned digits, $E = A+7$. Therefore, $O-E = 2+B-A$ must be divisible by 11.

What are the possible values for O - E = 2 + B - A?

Let's suppose it is equal to -11. Then $2+B-A = -11$ which means $A-B = 13$. Since A and B are digits from 0 to 9, this is clearly impossible. Any other negative multiple of 11 such as -22, -33, etc. will also mean that $A-B$ equals something larger than 9.

Next, let's suppose it is equal to 0. Then $2+B-A = 0$ which means $A-B = 2$. Since we have that B is odd from it being a multiple of 8, the only possibilities are:

1. $A = 3$, $B = 1$
2. $A = 5$, $B = 3$
3. $A = 7$, $B = 5$
4. $A = 9$, $B = 7$.

If $B = 9$, then $A = 11$, which is not possible as A needs to be 1 digit.

Finally, let's suppose O - E is equal to 11. Then $2+B-A = 11$ which means $B-A = 9$. Because A and B are digits from 0 to 9, the only possibility is when $A = 0$ and $B = 9$. Any other positive multiple of 11 such as 22, 33, etc. will mean that $B-A$ equals something larger than 9.

How do we use the fact that $3AB76$ is a multiple of 3 to finish the problem?

Notice how we only have 5 cases, so we can easily manually check if each one is a multiple of 3 by using its divisibility rule.

Case 1: $A = 3$, $B = 1$

For this case, the number is 33176. The sum of digits is $3+3+1+7+6 = 20$, which is not a multiple of 3, so this case does not work.

Case 2: $A = 5$, $B = 3$

For this case, the number is 35376. The sum of digits is $3+5+3+7+6 = 24$, which is a multiple of 3, so this case does work.

Case 3: $A = 7$, $B = 5$

For this case, the number is 37576. The sum of digits is $3+7+5+7+6 = 28$, which is not a multiple of 3, so this case does not work.

Case 4: $A = 9$, $B = 7$

For this case, the number is 39776. The sum of digits is $3+9+7+7+6 = 32$, which is not a multiple of 3, so this case does not work.

Case 5: $A = 0$, $B = 9$

For this case, the number is 30976. The sum of digits is $3+0+9+7+6 = 25$, which is not a multiple of 3, so this case does not work.

Therefore, $A = 5$ and $B = 3$ is the only case that works, so our answer is $5+3 = \boxed{8}$.

Remark 17.3.1

In these types of problems, it's almost always best to first find the prime factorization of the number, and then use the divisibility rules for each of the primes or prime powers in the prime factorization.

17.4 Legendre's Formula

Theorem 17.4.1 (Legendre's Theorem)

$$v_p(n!) = \text{Number of powers of } p \text{ in } n! = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots$$

Remark 17.4.2
This formula seems complicated, but it much easier to apply than it may seem. This basically means

$$\text{number of factors of p in } n! =$$

$$\text{the number of multiples of p} \leq n + \text{the number of multiples of } p^2 \leq n + \dots$$

Example 17.7

Find the number of 0's at the end of 100!.

Video Solution

17.5 Practice Problems

Problem 17.5.1 (AMC 8)

Which of the following numbers has the smallest prime factor?
(A) 55 (B) 57 (C) 58 (D) 59 (E) 61

Video Solution

Problem 17.5.2 (AMC 8)

What is the sum of the two smallest prime factors of 250?

Video Solution

Problem 17.5.3 (AMC 8)

How many three-digit numbers are divisible by 13?

Video Solution

Problem 17.5.4 (AMC 8)

The sum of two prime numbers is 85. What is the product of these two prime numbers?

Video Solution

Problem 17.5.5

The 7-digit numbers $\underline{7}\,\underline{4}\,\underline{A}\,\underline{5}\,\underline{2}\,\underline{B}\,\underline{1}$ and $\underline{3}\,\underline{2}\,\underline{6}\,\underline{A}\,\underline{B}\,\underline{4}\,\underline{C}$ are each multiples of 3. Which of the following could be the value of C?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 8

Video Solution

Problem 17.5.6 (AMC 8)

The 5-digit number $\underline{2}\ \underline{0}\ \underline{1}\ \underline{8}\ \underline{U}$ is divisible by 9. What is the remainder when this number is divided by 8?

Video Solution

Problem 17.5.7 (AMC 8)

What is the smallest positive integer that is neither prime nor square and that has no prime factor less than 50?

Video Solution

Problem 17.5.8 (AMC 8)

If $3^p + 3^4 = 90$, $2^r + 44 = 76$, and $5^3 + 6^s = 1421$, what is the product of p, r, and s?

Video Solution

Problem 17.5.9 (AMC 8)

For any positive integer M, the notation $M!$ denotes the product of the integers 1 through M. What is the largest integer n for which 5^n is a factor of the sum $98! + 99! + 100!$?

Video Solution

Problem 17.5.10 (AMC 8)

Three members of the Euclid Middle School girls' softball team had the following conversation.
Ashley: I just realized that our uniform numbers are all 2-digit primes.
Brittany : And the sum of your two uniform numbers is the date of my birthday earlier this month.
Caitlin: That's funny. The sum of your two uniform numbers is the date of my birthday later this month.
Ashley: And the sum of your two uniform numbers is today's date. What number does Caitlin wear?

Video Solution

Problem 17.5.11 (Omega Learn Math Contest)

The number AB962C is divisible by 792 where A, B, C are digits. What is the value of A?

Video Solution

Problem 17.5.12 (AMC 8)

The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number PQRST. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P?

Video Solution

Problem 17.5.13 (AMC 10)

The base-ten representation for 19! is $121,6T5,100,40M,832,H00$, where T, M, and H denote digits that are not given. What is $T+M+H$?

Video Solution

Problem 17.5.14 (AMC 10)

Call a positive integer an uphill integer if every digit is strictly greater than the previous digit. For example, $1357, 89$, and 5 are all uphill integers, but $32, 1240$, and 466 are not. How many uphill integers are divisible by 15?

Video Solution

Additional Problems

Problem 17.5.15

What is the prime factorization of 420?

Problem 17.5.16

What is the prime factorization of 1001?

Problem 17.5.17 (AMC 8)

The number 64 has the property that it is divisible by its units digit. How many whole numbers between 10 and 50 have this property?

Problem 17.5.18 (Omega Learn)

The number 8173ABC is divisible by 165. What is the average of all such 3 digit numbers ABC that satisfy this condition?

Problem 17.5.19 (MATHCOUNTS)

Ayasha, Beshkno, and Chenoa were all born after 2000. Each of them was born in a year after 2000 that is divisible by exactly one of the prime numbers 2, 3 or 5. Each of these primes is a divisor of one of the birth years. What is the least possible sum of their birth years?

Problem 17.5.20 (Omega Learn)

Sohil is thinking of a 5 digit number such that the first 3 digits are divisible by 63, the middle 3 digits and divisible by 8, and the last 3 digits are divisible by 11. What is the largest possible number he can be thinking of?

Problem 17.5.21 (BmMT)

How many permutations of 123456 are divisible by their last digit? For instance, 123456 is divisible by 6, but 561234 is not divisible by 4.

Answers

17.2 11

17.3 6528

17.4 4

17.7 24

17.5.1 58

17.5.2 7

17.5.3 69

17.5.4 166

17.5.5 1

17.5.6 3

17.5.7 3127

17.5.8 40

17.5.9 26

17.5.10 11

17.5.11 7

17.5.12 1

17.5.13 12

17.5.14 6

17.5.15 $2^2 \cdot 3 \cdot 5 \cdot 7$

17.5.16 $7 \cdot 11 \cdot 13$

17.5.17 17

17.5.18 522.5

17.5.19 6008

17.5.20 94561

17.5.21 648

Chapter 18

Factors

In this chapter, we will learn about interesting techniques to calculate the number, sum, and product of factors.

18.1 Number of Factors

Theorem 18.1.1 (Number of Factors of a Number)
If the prime factorization of the number is expressed as:

$$p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$$

then the number of factors of this number is

$$(e_1 + 1)(e_2 + 1) \ldots (e_k + 1)$$

Concept 18.1.2
Basically, in order to find the number of factors of a number:

1. Find the prime factorization of the number
2. Add 1 to all of the exponents
3. Multiply them together

Example 18.1

How many factors does the number 112 have?

Solution

In order to find how many factors 112 has, we must find the prime factorization of 112.

Clearly, 112 is divisible by 2. We can repeatedly divide by 2 to get

$$112 = 2 \times 56 = 2^2 \times 28 = 2^3 \times 14 = 2^4 \times 7$$

Since 7 is prime, there are no further primes to divide, so the prime factorization is $2^4 \times 7^1$

From this, how do find the number of factors of 112?

We can simply use the formula shown above to find the number of factors. In the prime factorization the exponents are 4 and 1. Adding 1 to each of the exponents, we get 5 and 2. Therefore, the number of factors is $5 \times 2 = \boxed{10}$.

Concept 18.1.3

Here is a logical way to think about this formula - for each prime p_k, the exponent can be anything from 0 to e_k. Therefore, there are $e_k + 1$ choices. Therefore, the total number of factors is just the product of all such $e_k + 1$ terms.

For example, in $112 = 2^4 \times 7$, in the factor, the power of 2 can be $2^0, 2^1, 2^2, 2^3, or 2^4$. The power of 7 can be $7^0 or 7^1$.

Therefore, in total there would be $5 \times 2 = 10$ ways just as we calculated in the previous problem. This logic will be useful when other conditions are imposed.

Example 18.2

How many positive factors does the number 144 have that are perfect squares?

Solution

First, we begin by finding the prime factorization of 144.

We first divide by factors of 2. This gives us $144 = 2^1 \times 72 = 2^2 \times 36 = 2^3 \times 18 = 2^4 \times 9$. 9 is simply 3^2, so our prime factorization is $2^4 \times 3^2$. Unlike the last problem, we can't just directly apply the number of factors formula.

How can we deal with the condition that the factor must be a perfect square?

For a number to be a perfect square, all primes in the prime factorization must be even. Let's try to find the number of exponents possible for each prime to form a perfect square factor.

How many choices are there for the exponent of the prime 2 in the prime factorization?

For the number to be a square, the exponent must be even and between 0 to 4. It can be either 0, 2, or 4. Therefore, we have 3 choices.

How many choices are there for the exponent of the prime 3 in the prime factorization?

Similarly, the exponent for 3 must be even and from 0 to 2, so it can be either 0, or 2. Therefore, we have 2 choices.
Therefore, in total, 144 has $3 \times 2 = \boxed{6}$ perfect square factors.

Example 18.3 (AMC 8)

How many positive integer factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely $1, 2, 3, 4, 6$, and 12.)

Video Solution

18.2 Sum of Factors

Theorem 18.2.1 (Sum of Factors of a Number)
If the prime factorization of the number is expressed as:

$$p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$$

then the sum of factors of this number is

$$(1 + p_1^1 + p_1^2 + \cdots + p_1^{e_1-1} + p_1^{e_1})(1 + p_2^1 + p_2^2 + \cdots + p_2^{e_2-1} + p_2^{e_2}) \ldots (1 + p_k^1 + p_k^2 + \cdots + p_k^{e_k-1} + p_k^{e_k})$$

Concept 18.2.2 (Sum of Factors of a Number)
Essentially, for a prime p in the prime factorization, first find the sum of p^k for all possible exponents k in the prime factorization. Then, we will multiply all such sums for all of the primes to get the sum of all the factors of the number. This may seem complicated, but it will make more sense with an example

Example 18.4

What is the sum of factors of 112?

Solution

We can calculate that the prime factorization of $112 = 2^4 \times 7$. We will find the sum of powers of 2 and powers of 7.

What is the sum of powers of 2?

The exponent of 2 can be anything from 0 to 4. Therefore, the sum of powers of 2 is $2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 31$.

Similarly, the sum of powers of 7 is $7^0 + 7^1 = 8$.

Therefore, the sum of factors of 112 will be the product of both of our sums, which is $31 \times 8 = \boxed{248}$.

18.3 Product of Factors

Theorem 18.3.1 (Product of Factors of a Number)
The product of factors of a number n where it has f factors (this can be calculated using the number of factors formula) is $n^{\frac{f}{2}}$

Example 18.5

Find the product of the factors of 20.

Solution

First, we must find the number of factors of 20, as this is needed in the formula. To do this, we must first find the prime factorization of 20.

We first divide by powers of 2. We have that $20 = 2^1 \times 10 = 2^2 \times 5 = 2^2 \times 5^1$. Since 5 is prime, this is the prime factorization of 20.

From here, how many factors does 20 have?

We simply add 1 to each exponent to get 3 and 2. The number of factors will be the product, which is $3 \times 2 = 6$.

Then, we have to apply the formula to get the product of factors. From above, we have that the number of factors $f = 6$. Also, we have that the number $n = 20$. Plugging this into the formula, we get $20^{\frac{6}{2}} = 20^3 = \boxed{8000}$

Example 18.6 (Omega Learn)

Let a be the product of all odd factors of 54. Let b be the sum of all even factors of 36. Find $a - b$.

Video Solution

18.4 Practice Problems

Problem 18.4.1 (AMC 8)

What is the sum of the prime factors of 2010?

Video Solution

Problem 18.4.2 (AMC 8)

How many positive factors does 23,232 have?

Video Solution

Problem 18.4.3 (AMC 8)

On June 1, a group of students is standing in rows, with 15 students in each row. On June 2, the same group is standing with all of the students in one long row. On June 3, the same group is standing with just one student in each row. On June 4, the same group is standing with 6 students in each row. This process continues through June 12 with a different number of students per row each day. However, on June 13, they cannot find a new way of organizing the students. What is the smallest possible number of students in the group?

Video Solution

Problem 18.4.4 (Omega Learn Math Contest)

How many positive multiples of 1,001 have less than 15 factors?

Video Solution

Problem 18.4.5 (AMC 10)

Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

Video Solution

Problem 18.4.6 (AIME)

Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

Video Solution

Problem 18.4.7 (AMC 10)

For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

Video Solution

Additional Problems

Problem 18.4.8

How many positive factors does the number 60 have?

Problem 18.4.9

Find the sum of factors of 320.

Problem 18.4.10 (EMCC)

How many positive divisors of 2020 do not also divide 1010?

Problem 18.4.11 (BmMT)

What is the sum of $\frac{1}{a}$ over all positive factors a of the number 360?

Problem 18.4.12 (MATHCOUNTS)

A positive integer q is the product of a prime number and a perfect square. Additionally, q is the product of a different prime number and a perfect cube. What is the least possible value of q?

Problem 18.4.13 (MATHCOUNTS)

What is the greatest integer k such that 80! is divisible by 45^k?

Answers

18.3 7

18.6 651

18.4.1 77

18.4.2 42

18.4.3 60

18.4.4 4

18.4.5 $1:14$

18.4.6 589

18.4.7 325

18.4.8 12

18.4.9 762

18.4.10 4

18.4.11 17

18.4.12 432

18.4.13 18

Chapter 19

GCD and LCM

19.1 GCD and LCM Fundamentals

Definition 19.1.1 (Greatest Common Divisor). The Greatest Common Divisor (GCD) of two or more integers (which are not all zero) is the largest positive integer that divides each of the integers.
Note: This is also known as GCF (Greatest Common Factor), and the terms GCF and GCD are often used interchangeably.

Definition 19.1.2 (Least Common Multiple). The Least Common Multiple (LCM) of two or more integers (which are not all zero) is the smallest positive integer that is divisible by both the numbers.

Concept 19.1.3

Greatest common divisor of m and n $= GCD(m,n)$ can be found by taking the lowest prime exponents from the prime factorizations of m and n.

Least common multiple of m and n $= LCM(m,n)$ can be found by taking the highest prime exponents from the prime factorizations of m and n.

Example 19.1 (AMC 8)

What is the ratio of the least common multiple of 180 and 594 to the greatest common factor of 180 and 594?

Video Solution

19.2 GCD and LCM Product

Theorem 19.2.1

The product of GCD and LCM of two numbers is equal to the product of the two numbers:

$$GCD(m,n) \times LCM(m,n) = m \times n$$

Remark 19.2.2

Let's explore the reason why this is true. Let's say that m has p^a in it's prime factorization, and n has p^b in it's prime factorization. Then, the product mn will have p^{a+b} in it's prime factorization.

Remember from above how the gcd can be found by taking the lowest prime exponents and the lcm can be found by taking the highest prime exponents.

Therefore, if a is smaller than b, than the gcd will have p^a in it's prime factorization, and the lcm will have p^b in it's prime factorization. On the other hand, if b is smaller than a, than the gcd will have p^b in it's prime factorization, and the lcm will have p^a in it's prime factorization.

Therefore, no matter what, the product of the gcd and lcm will have p^{a+b} in it's prime factorization. This is true for all primes, so the products are equal.

Example 19.2

Suppose we have 2 numbers m and n such that $mn = 1260$ and $LCM(m,n) = 210$. Find $GCD(m,n)$.

Solution

We know the product and lcm of the numbers, we can use the identity that

$$\gcd(m,n) \times \text{lcm}(m,n) = m \times n$$

.

Then, we can simply plug in the given values to get $\gcd(m,n) \times 210 = 1260$. Dividing both sides by 210 gives $\gcd(m,n) = \boxed{6}$.

19.3 More GCD/LCM Properties

Theorem 19.3.1

$$\gcd(ac,bc) = c \cdot \gcd(a,b)$$

Theorem 19.3.2

$$\text{lcm}(ac,bc) = c \cdot \text{lcm}(a,b)$$

Theorem 19.3.3

If a number is divisible by two numbers a and b, it will also be divisible by $\operatorname{lcm}(a, b)$.

Example 19.3 (Omega Learn)

Let a be the smallest positive perfect cube that's divisible by 15, 22, and 176. If $a = b^3$, find b.

Video Solution

19.4 Euclidean Algorithm

Theorem 19.4.1 (Euclidean Algorithm)

The Euclidean algorithm states that

$$\gcd(x, y) = \gcd(x - ky, y)$$

where $x > y$ and k is a positive integer.

Example 19.4

Find the GCD of 186 and 92.

Solution

We can apply the Euclidean Algorithm multiple times to easily find the GCD of large numbers since after applying the Euclidean algorithm, we will have two smaller numbers, and we can repeatedly apply the Euclidean Algorithm again until we get two very small numbers.

$$\begin{aligned}
\gcd(186, 92) &= \gcd((186 - (2 \cdot 92)), 92) \\
&= \gcd(2, 92) \\
&= \gcd(2, (92 - (2 \cdot 46))) \\
&= \gcd(2, 0) \\
&= 2
\end{aligned}$$

Example 19.5 (Omega Learn)

3 numbers a, b, and c satisfy the conditions that $\gcd(a,b) = 4$, $\gcd(b,c) = 18$, and $\text{lcm}(a,c) = 144$. Find the value of $a+c$.

Solution

To solve this problem, let's first figure out how to use the gcd conditions.

What must be true about a and b if $\gcd(a,b) = 4$?

If $\gcd(a,b) = 4$, then both a and b must be multiples of 4.

Similarly, if $\gcd(b,c) = 18$, then both b and c must be multiples of 18.

What do both of these conditions combined tell us about b?

If b is a multiple of 4 and a multiple of 18, it also must be a multiple of the least common multiple of 4 and 18 by the above theorem. This means b is a multiple of 36. The reason this is true is because to be a multiple of 18, it must also be as multiple of 9 and 2. However, a number that is a multiple of 4 is already a multiple of 2, so therefore it just has to be a multiple of $9 \times 4 = 36$.

We know that a must be a multiple of 4. What happens if a is also a multiple of 3, or a multiple of 9?

Keep in mind that $\gcd(a,b) = 4$ means that not only both numbers share a factor of 4, but that no other factors are shared between the numbers. From earlier, we know that b is a multiple of 36. Therefore, if a has any factors of 3, then the gcd will also have factors of 3 since both a and b would then have factors of 3.

Therefore, a cannot have any factors of 3.

Similarly, what must be true about c?

We know c is a multiple of 18, which means it's a multiple of 9 and a multiple of 2. On the other hand, b is a multiple of 36. The gcd is 18, so if c is a multiple of 36, then the gcd would have been 36, which is not possible. Therefore, c cannot be a multiple of 36. We already know c is a multiple of 9, that means c is not a multiple of 4.

Finally, how do we use the condition that $\text{lcm}(a,c) = 144$?

First, let's find the prime factorization of 144.

First, we divide by factors of 2 to get $144 = 2 \times 72 = 2^2 \times 36 = 2^3 \times 18 = 2^4 \times 9$. 9 is 3^2, so the prime factorization is $2^4 \times 3^2$.

For $\operatorname{lcm}(a,c)$ to have a factor of 2^4, how many factors of 2 must a and c each have?

Remember from earlier that c is a multiple of 2, but not a multiple of 4. Therefore, c has one factor of 2. However, for the lcm to have a factor of 2^4, one of the numbers a or c must have exactly four powers of 2 in the prime factorization since the lcm is found by taking the maximum powers of each prime in the prime factorization of both numbers. Therefore, a must have exactly four powers of 2. It can't be more than four powers of 2 because then the lcm would also have more than four powers of 2. Also, c is a multiple of 2, but not 4, it will have exactly one power of 2 in it's prime factorization.

Next, for $\operatorname{lcm}(a,c)$ to have a factor of 3^2, how many factors of 3 must a and c each have?

Earlier, we found that a has no factors of 3 and c must be a multiple of 9. Therefore, c must have exactly two powers of 3 in it's prime factorization. Again, it can't have any more than two powers of 3 since that would result in lcm also having more than two powers of 3.

Finally, can either a or c have any other prime factors besides 2 or 3?

If a or c had a factor of 5, 7, or some other prime, then the lcm would also have that prime in it's prime factorization. But earlier, we found that the prime factorization $2^4 \times 3^2$ only contains factors of 2 and 3. Therefore, this is impossible.

What are the values of a and b that work?

Since a has four powers of 2, no powers of 3, and no powers of any other prime, the prime factorization is 2^4. Therefore, $a = 16$.

Similarly, since b has one power of 2, two powers of 3, and no powers of any other prime, the prime factorization is 2×3^2. Therefore, $b = 18$.

The sum is therefore $16 + 18 = \boxed{34}$.

Example 19.6 (AMC 12)

How many positive integers n are there such that n is a multiple of 5, and the least common multiple of $5!$ and n equals 5 times the greatest common divisor of $10!$ and n?

Video Solution

19.5 Practice Problems

Problem 19.5.1 (AMC 10)

Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

Video Solution

Problem 19.5.2 (AMC 8)

The least common multiple of a and b is 12, and the least common multiple of b and c is 15. What is the least possible value of the least common multiple of a and c?

Video Solution

Problem 19.5.3 (AIME)

For how many values of k is 12^{12} the least common multiple of the positive integers 6^6, 8^8, and k?

Video Solution

Problem 19.5.4 (AMC 10)

Let a, b, c, and d be positive integers such that $\gcd(a,b) = 24$, $\gcd(b,c) = 36$, $\gcd(c,d) = 54$, and $70 < \gcd(d,a) < 100$. Which of the following must be a divisor of a?

Video Solution

Additional Problems

Problem 19.5.5 (Omega Learn)

If the GCD of 2 numbers is 12 and the LCM of the same 2 numbers is 240, find the product of the numbers.

Problem 19.5.6 (Omega Learn)

Two numbers a and b satisfy the condition $lcm(gcd(24,a), gcd(a,b)) = lcm(gcd(24,b), gcd(a,b))$. If $a = 2b$, what is the smallest possible value of a + b?

Problem 19.5.7 (AIME)

Let $[r,s]$ denote the least common multiple of positive integers r and s. Find the number of ordered triples (a,b,c) of positive integers for which $[a,b] = 1000$, $[b,c] = 2000$, and $[c,a] = 2000$.

Problem 19.5.8 (Omega Learn)

Three numbers a, b, and c satisfy the conditions that $\gcd(a,b) = 20$, $\gcd(b,c) = 18$, and $\text{lcm}(a,c) = 5040$. Find the sum of all possible values of $a+c$.

Problem 19.5.9 (AMC 10)

Let n be the least positive integer greater than 1000 for which

$$\gcd(63, n+120) = 21 \quad \text{and} \quad \gcd(n+63, 120) = 60.$$

What is the sum of the digits of n?

Answers

19.1 330

19.3 660

19.6 48

19.5.1 21

19.5.2 20

19.5.3 25

19.5.4 13

19.5.5 2880

19.5.6 24

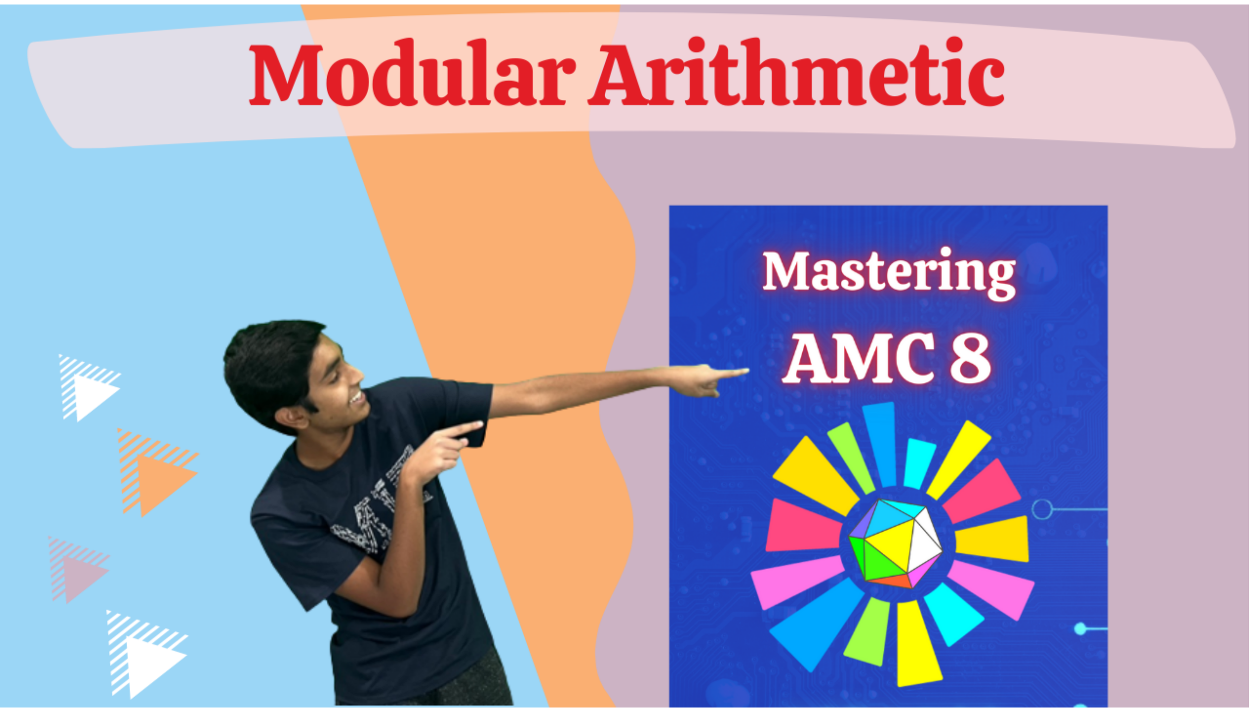
19.5.7 70

19.5.8 1470

19.5.9 18

Chapter 20

Modular Arithmetic



20.1 Modular Arithmetic Fundamentals

Definition 20.1.1.

$$n \equiv a \pmod{b}$$

means the number 'n' leaves the same remainder as 'a' when divided by b

Remark 20.1.2
Mods can also be negative. For example a number that is $5 \pmod 6$ is also $-1 \pmod 6$. This can be useful for simplifying calculations when raising this number to large powers.

20.2 Product Rule

Theorem 20.2.1
If $a = x \pmod n$ and $b \equiv y \pmod n$, then

$$ab \equiv xy \pmod n$$

Remark 20.2.2
This is useful when having to calculate the last few digits of a expression because we can then only have to consider that many digits for all of our terms.

Example 20.1 (AMC 8)

If n is an even positive integer, the *double factorial* notation $n!!$ represents the product of all the even integers from 2 to n. For example, $8!! = 2 \cdot 4 \cdot 6 \cdot 8$. What is the units digit of the following sum?

$$2!! + 4!! + 6!! + \cdots + 2018!! + 2020!! + 2022!!$$

Video Solution

Example 20.2 (AIME)

Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}}$ is divided by 1000.

Solution

The identity above allows us to simply find each term $\pmod{1000}$ and multiply them together. We can clearly see that $9 \equiv 9 \pmod{1000}$ and $99 \equiv 99 \pmod{1000}$.

What is the value of $999 \pmod{1000}$?

Since 999 is 1 less than 1000, $999 \equiv -1 \pmod{1000}$.

What is the value of $9999 \pmod{1000}$?

Notice that 9999 is again 1 less than a multiple of 1000 (10,000), so $9999 \equiv -1 \pmod{1000}$.

For all remaining terms, we can see that since it ends in 999, it will be 1 less than a multiple of 1000 or $-1 \pmod{1000}$.

In total, amongst all 999 terms, 1 term is $9 \pmod{1000}$, 1 term is $99 \pmod{1000}$, and the 997 remaining terms are all $-1 \pmod{1000}$.

Therefore, the product will be

$$9 \times 99 \times (-1)^{997} \pmod{1000} \equiv 891 \times -1 \times (-1)^{996} \equiv -891 \times ((-1)^2)^{498} \equiv -891 \times 1 \equiv \boxed{109} \pmod{1000}$$

20.3 Exponent Rule

Theorem 20.3.1

If $a \equiv x \pmod{n}$, then

$$a^m \equiv x^m \pmod{n}$$

Remark 20.3.2

This is true since a^m is just $a \times a \times \cdots \times a$ and x^m is just $x \times x \times \cdots \times x$, so it follows from the product rule above.

Example 20.3 (AMC 8)

When 1999^{2000} is divided by 5, what is the remainder?

Solution

We can simplify our expression by first evaluating 1999 mod 5.

What mod value is equivalent to $1999 \pmod 5$?

1999 leaves a remainder of 4 when divided by 5 so we could say that $1999 \equiv 4 \pmod 5$. However, then we would still have to calculate the remainder when 4^{2000} is divided by 5.

Instead, what negative mod is equivalent to $1999 \pmod 5$

Since 1999 is 1 less than 2000, a multiple of 5, we can say that $1999 \equiv -1 \pmod 5$.

Then, we have to find the remainder when -1^{2000} is divided by 5, which is

$$((-1)^2)^{1000} = 1^{1000} = 1 \pmod 5$$

so it leaves a remainder of $\boxed{1}$ when divided by 5.

Example 20.4

Find the remainder of 9^8 when it's divided by 100.

Video Solution

20.4 Multiple Modular Congruences

Concept 20.4.1
We can generally combine different mods together.

Example 20.5

Find the largest possible number less than 100 that leaves a remainder of 1 when divided by 5 and a remainder of 1 when divided by 7.

Solution

We can rewrite our conditions as $x \equiv 1 \pmod 5$ and $x \equiv 1 \pmod 7$. In order to solve the problem, we will try to combine these mods in 1 mod statement.

If $x \equiv 1 \pmod 5$ and $x \equiv 1 \pmod 7$, then what is the value of $x - 1 \pmod 5$ and $\pmod 7$?

Subtracting 1 from both sides of the equation $x \equiv 1 \pmod 5$ gives us $x - 1 \equiv 0 \pmod 5$. Similarly, we get $x - 1 \equiv 0 \pmod 7$. Therefore, $x - 1$ is a multiple of both 5 and 7.

If $x - 1$ is a multiple of both 5 and 7, what else must $x - 1$ be a multiple of?

If x - 1 is a multiple of both 5 and 7, x - 1 will also be a multiple of $5 \times 7 = 35$!. Therefore, we have $x - 1 \equiv 0 \pmod{35}$ or $x \equiv 1 \pmod{35}$.

From here, we can see that the possible values less than 100 are 1, 36, 71, 106, etc. Clearly, $\boxed{71}$ is the largest possible value less than 100.

Example 20.6

Find the smallest positive integer that leaves a remainder of 5 when divided by 6, 2 when divided by 5, 2 when divided by 7, and 2 when divided by 11.

Video Solution

20.5 Digit Cycles

Concept 20.5.1 (Digit Cycles)
To calculate large digit(s) of a number a^b, a strategy that may work is to just look for a pattern by computing the first few values of a^b and then seeing that the pattern will repeat for large values of b.

Example 20.7

Find the units digit of 2^{1026}

Solution

1026 is a very large power, so let's try to find the units digit for smaller powers.

The units digit of $2^1 = 2$ is 2.
The units digit of $2^2 = 4$ is 4.
The units digit of $2^3 = 8$ is 8.
The units digit of $2^4 = 16$ is 6.
The units digit of $2^5 = 32$ is 2.
The units digit of $2^6 = 64$ is 4.
The units digit of $2^7 = 128$ is 8.
The units digit of $2^8 = 256$ is 6.

Do you see a pattern?

We can see that the units digit cycle! In fact, they cycle for every 4 powers of 2 in the pattern 2, 4, 8, 6. After that, the units digit repeat in the same pattern.

How can we use this to find the units digit of 2^{1026}?

Since it repeats every 4, we have that $2^{1026} \equiv 2^{1022} \equiv 2^{1018} \ldots \pmod 10$. We can see that since 1026 is $2 \pmod 4$, eventually by continuously going down by 4 exponents we will have that $2^{1026} \equiv 2^2 \equiv 4 \pmod 10$. Therefore, the units digit is $\boxed{4}$.

Example 20.8

Find the remainder when

$$(15^1 + 15^2 + 15^3 + \cdots + 15^{2022}) + (34^1 + 34^2 + \cdots + 34^{2021})$$

is divided by 7.

Video Solution

20.6 Practice Problems

Problem 20.6.1 (AMC 8)

What is the units digit of 13^{2012}?

Video Solution

Problem 20.6.2

What is the units digit of $19^{19} + 99^{99}$?

Video Solution

Problem 20.6.3 (AMC 8)

What is the tens digit of 7^{2011}?

Video Solution

Problem 20.6.4

What is the smallest number greater than 2 that leaves a remainder of 2 when divided by 3, 4, 5, or 6?

Video Solution

Problem 20.6.5 (AMC 8)

The number N is a two-digit number. • When N is divided by 9, the remainder is 1. • When N is divided by 10, the remainder is 3. What is the remainder when N is divided by 11?

Video Solution

Problem 20.6.6 (AMC 8)

The product of the two 99-digit numbers 303, 030, 303,..., 030, 303 and 505, 050, 505, ..., 050, 505 has thousands digit A and units digit B. What is the sum of A and B?

Video Solution

Problem 20.6.7 (AMC 8)

How many positive three-digit integers have a remainder of 2 when divided by 6, a remainder of 5 when divided by 9, and a remainder of 7 when divided by 11?

Video Solution

Additional Problems

Problem 20.6.8

What is the units digit of $13^{17} + 17^{13}$?

Problem 20.6.9

Find the units digit of $7^{109} - 3^{336}$

Problem 20.6.10

Find the smallest positive number greater than 1 that leaves a remainder of 1 when divided by 2, by 3, by 4, by 5, and by 6.

Problem 20.6.11 (AMC 8)

How many positive integers can fill the blank in the sentence below?

"One positive integer is ________ more than twice another, and the sum of the two numbers is 28."

Problem 20.6.12 (AMC 10/12)

The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ starts with two 1s, and each term afterwards is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?

(A) 0 **(B)** 4 **(C)** 6 **(D)** 7 **(E)** 9

Problem 20.6.13 (AMC 8)

If n is an even positive integer, the double factorial notation $n!!$ represents the product of all the even integers from 2 to n. For example, $8!! = 2 \cdot 4 \cdot 6 \cdot 8$. What is the units digit of the following sum?

$$2!! + 4!! + 6!! + \cdots + 2018!! + 2020!! + 2022!!$$

Problem 20.6.14 (MATHCOUNTS)

Find the remainder when $1^2 + 2^2 + 3^2 + \cdots + 2016^2$ is divided by 17.

Problem 20.6.15 (AIME)

Let $N = 100^2 + 99^2 - 98^2 - 97^2 + 96^2 + \cdots + 4^2 + 3^2 - 2^2 - 1^2$, where the additions and subtractions alternate in pairs. Find the remainder when N is divided by 1000.

Problem 20.6.16 (Omega Learn)

In the Fibonacci sequence, how many of the first 100 terms in the sequence leave a remainder of less than 7 when divided by 8?

Answers

20.1 2

20.4 21

20.6 1157

20.8 5

20.6.1 1

20.6.2 8

20.6.3 4

20.6.4 62

20.6.5 7

20.6.6 8

20.6.7 5

20.6.8 0

20.6.9 6

20.6.10 61

20.6.11 9

20.6.12 6

20.6.13 2

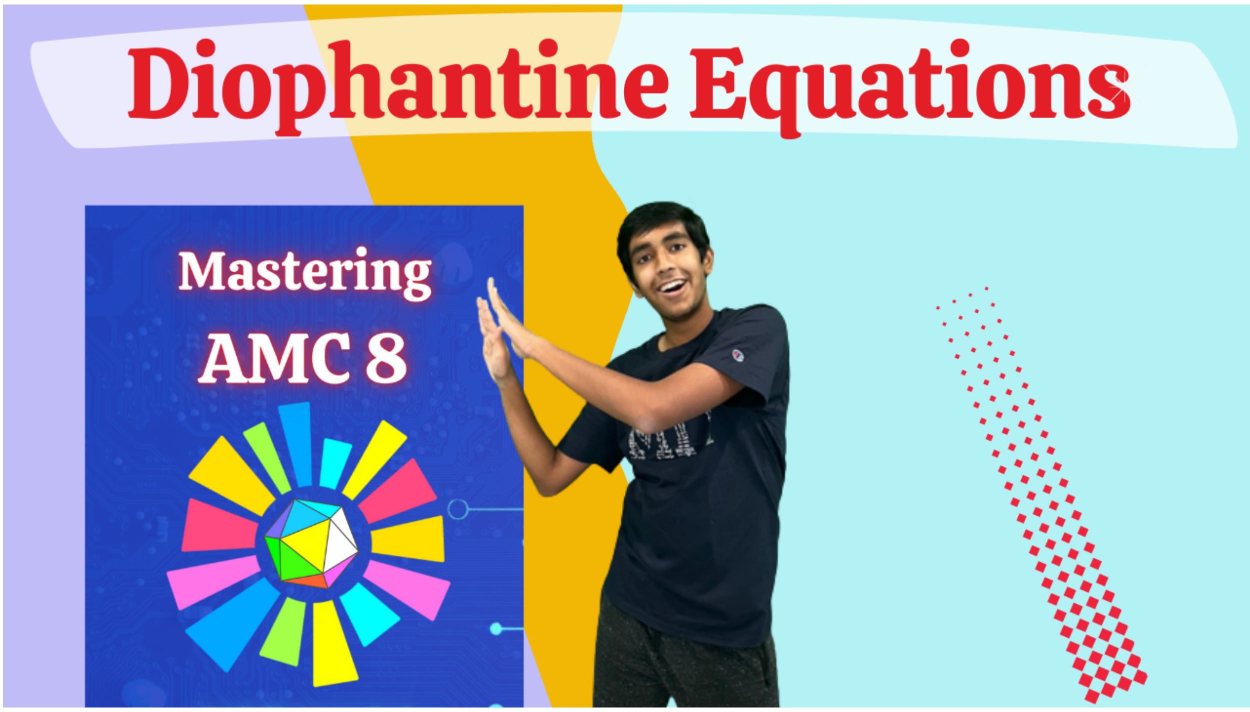
20.6.14 11

20.6.15 100

20.6.16 92

Chapter 21

Diophantine Equations



Concept 21.0.1

Diophantine equation problems often involve having to rewrite numbers and expressions into variables and combining the use of algebra and number theory. Essentially, they are equations with integer solutions.

Example 21.1 (Omega Learn)

The digits of a 3 digit number are reversed. The original number is 594 greater than the new number. What is the largest possible value for the original number?

Solution

To start, we will first try and write the problem algebraically.

How can we convert a 3 digit number to variables?

Let the hundredths digit be h, the tens digit t, and units digit u (the 3 digit number will be htu). Then, the 3 digit number is essentially $100h + 10t + u$.

What is the number in terms of h, t, and u when the digits are reversed?

Since the original hundreds digit, h, is now the units digit and the original units digit, u, is now the hundreds digit, the new number will be uth. This is essentially $100u + 10t + h$.

Now, how do we use the condition that the original number is 594 greater than the new number?

For this to happen, we must have

$$\begin{aligned} 594 &= (100h + 10t + u) - (100u + 10t + h) \\ &= (100h - h) + (10t - 10t) + (u - 100h) \\ &= 99h - 99u \\ &= 99(h - u) \end{aligned}$$

Dividing both sides by 99 gives $h - u = 6$.

What are the possibilites for t?

Notice how t is not part of the condition after simplifying, so it can be any digit.

What are the possibilites for h and u?

All the digits must be from 0 to 9, so we simply look at the possibilites for h.

If $h = 9$, then $u = 3$
If $h = 8$, then $u = 2$
If $h = 7$, then $u = 1$
If $h = 6$, then $u = 0$

These are the only cases because any lower value of h will result in a negative u, which is impossible.

What values of h, u, and t will produce the largest number?

Since t can be anything, we can let $t = 9$ to make it as large as possible. The largest values for h and u is clearly $h = 9$ and $u = 3$, since it has the largest hundreds and units digit amongst the cases for h and u. Therefore, the largest possible value for the original number is $100h + 10t + u = 100 \times 9 + 10 \times 9 + 3 = \boxed{993}$.

Example 21.2 (Omega Learn)

The digits of a 2 digit number with tens digit 7 is reversed. The digits of a 3 digit number with hundreds digit 3 and tens digit 5 is also reversed. The sum of both of the new reversed numbers is 288 more than the sum of the original 2 numbers. Find the sum of the 2 original numbers.

Video Solution

21.1 Quadratic Factorizations

Theorem 21.1.1 (Difference of Squares)

$$x^2 - y^2 = (x-y)(x+y)$$

Theorem 21.1.2 (Binomial Square Expansions)

$$(x+y)^2 = x^2 + 2xy + y^2 = (x-y)^2 + 4xy$$

$$(x-y)^2 = x^2 - 2xy + y^2 = (x+y)^2 - 4xy$$

$$(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$$

$$(x+y)^2 - (x-y)^2 = 4xy$$

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy+yz+xz)$$

Example 21.3 (Omega Learn)

How many factors does the number 9984 have?

Video Solution

Example 21.4 (Omega Learn)

How many values of n from 1 to 100 make $16^n - 2 \times 4^n + 1$ a multiple of 49?

Solution

Let's solve this problem using the formulas we learned above.

Do you notice any of the factorizations in the expression?

We see the terms 16^n and 4^n in our expression. Notice how $16^n = (4^n)^2$. Our expression can be rewritten as $(4^n)^2 - 2 \times (4^n) + 1$.

Now we can use the $(a-b)^2 = a^2 - 2ab + b^2$ identity! If we let $a = 4^n$ and $b = 1$, we can see that

$$(4^n)^2 - 2 \times (4^n) \times 1 + 1 = a^2 - 2ab + b^2 = (a-b)^2 = (4^n - 1)^2$$

When is $(4^n - 1)^2$ a multiple of 49?

Since $49 = 7^2$, for $(4^n - 1)^2$ to be a multiple of 7^2, we must have $4^n - 1$ be a multiple of 7.

Do you notice any more identities?

Again, we can notice that $4^n = (2^n)^2$. Rewriting our expression this way gives $(2^n)^2 - 1$. We can now use difference of squares! We have that

$(2^n)^2 - 1 = (2^n)^2 - 1^2 = (2^n - 1)(2^n + 1)$.

What must happen for $(2^n - 1)(2^n + 1)$ to be a multiple of 7?

7 is prime, so either $2^n - 1$ or $2^n + 1$ must be a multiple of 7. Therefore, $2^n \equiv 1, 6 \pmod 7$.

How can we analyze the values of $2^n \pmod 7$

Let's try looking at small values.

$$\begin{aligned}
2^0 &\equiv 1 \pmod 7\\
2^1 &= 2^0 \times 2 \equiv 1 \times 2 \equiv 2 \pmod 7\\
2^2 &= 2^1 \times 2 \equiv 2 \times 2 \equiv 4 \pmod 7\\
2^3 &= 2^2 \times 2 \equiv 4 \times 2 \equiv 1 \pmod 7\\
2^4 &= 2^3 \times 2 \equiv 1 \times 2 \equiv 2 \pmod 7
\end{aligned}$$

$$2^5 = 2^4 \times 2 \equiv 2 \times 2 \equiv 4 \pmod{7}$$

We can see a pattern. The values of $2^n \pmod{7}$ cycle between 1, 2, and 4.

From earlier we must have $2^n \equiv 1 or 6 \pmod{7}$ for the expression to be a multiple of 7, the only possibility is to have $2^n \equiv 1 \pmod{7}$. From the pattern, we can see that this only happens when n is a multiple of 3.

Therefore, from 1 to 100, there are $\frac{100}{3} = \boxed{33}$ multiples of 3.

21.2 Simon's Favorite Factoring Trick

Theorem 21.2.1 (Simon's Favorite Factoring Trick)

$$xy + kx + jy + jk = (x + j)(y + k)$$

Remark 21.2.2
You can generally apply this factorization when you have xy, x, and y terms. After applying the factorization, you can then find all possible values for each of your terms in your factorization (remember negatives!).

Example 21.5

Find the sum of all possible distinct values of $x + y$ such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$

Solution

This problem may seem tricky, but there is a simple technique to solve this.

How can we simplify this problem by removing the denominators?

Notice that the denominators have values of 2, x, and y, so we can multiply by $2xy$ to get rid of the fractions. We end up with $2y + 2x = xy$.

What identity or formula can we use on this simplified equation?

Notice how we have xy, x, and y terms, so we can use Simon's Favorite Factoring Trick!

We can rewrite our equation as $xy-2x-2y=0$. We must rewrite the left hand side as something of the form $(x+j)(y+k)$. Remember from the theorem above, the coefficient of the y term will be j, and the coefficient of the x term will be k.

Make sure to remember that it's not the other way around as this can be a common mistake!

Since both coefficients are -2, j and k must also both be -2. Therefore, we can rewrite as $(x-2)(y-2)$.

Are we adding any extra terms?

This product also ads an extra $(-2)\times(-2)=4$ term. Therefore, we must also subtract the extra 4 we added.

Thus, we have $(x-2)(y-2)-4=0$ or $(x-2)(y-2)=4$.

Since the right hand side is positive, we have 2 possibilities: either $x-2$ and $y-2$ are both positive, or both are negative.

Is it possible for $x-2$ and $y-2$ to be negative.

Since x and y must be positive integers (1 or more), the minimum value of $x-2$ and $y-2$ can be $1-2=-1$.

Therefore, -1 is the only negative value possible for $x-2$ and $y-2$. However, this will result in the product being $-1\times-1=1$, which isn't 4. So these values don't work.

Therefore, both $x-2$ and $y-2$ must be positive.

What are the positive possibilities for $x-2$ and $y-2$?

Since their product is 4, they must be factors of 4. Since $4=2^2$, the only possibilities for 2 positive numbers that have a product of 4 is $(2^0,2^2)$, $(2^1,2^1)$, and $(2^2,2^0)$ since the sum of the exponents of the powers of 2 must be 2 by the exponent product rule.

We can evaluate this to be $(1,4)$, $(2,2)$, and $(4,1)$. Therefore, we can split the remaining problem into 3 cases:

Case 1: (1, 4)

For this case, $x-2=1$, and $y-2=4$, so $x=3 \text{and} y=6$. This means that $x+y=3+6=9$.

Case 2: (2, 2)

For this case, $x-2=2$, and $y-2=2$, so $x=4 \text{and} y=4$. This means that $x+y=4+4=8$.

Case 1: (4, 1)

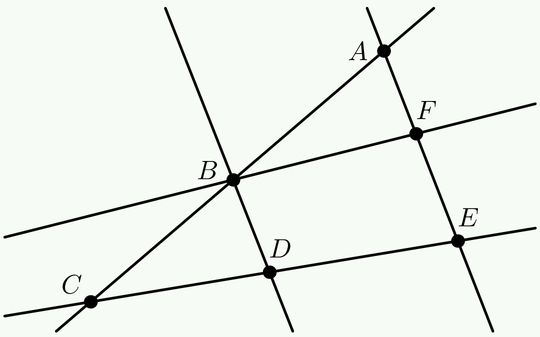
For this case, $x-2=4$, and $y-2=1$, so $x=6 \text{and} y=3$. This means that $x+y=6+3=9$.

Therefore, the distinct possible values of $x+y$ are 9 and 8. The sum of these values is $9+8=\boxed{17}$.

21.3 Interesting Examples

Example 21.6 (AMC 8)

Each of the points A, B, C, D, E, and F in the figure below represents a different digit from 1 to 6. Each of the five lines shown passes through some of these points. The digits along each line are added to produce five sums, one for each line. The total of the five sums is 47. What is the digit represented by B?



Video Solution

21.4 Cubic Factorizations (Optional)

Theorem 21.4.1 (Difference of Cubes)
$x^3-y^3=(x-y)(x^2+xy+y^2)$

Theorem 21.4.2 (Sum of Cubes)
$x^3+y^3=(x+y)(x^2-xy+y^2)$

Theorem 21.4.3 (Binomial Cube Expansions)

$$(x+y)^3=x^3+3xy(x+y)+y^3$$
$$(x+y)^3=x^3+3x^2y+3xy^2+y^3$$

$$(x-y)^3=x^3-3xy(x-y)-y^3$$
$$(x-y)^3=x^3-3x^2y+3xy^2-y^3$$

21.5 Practice Problems

Problem 21.5.1 (AMC 8)

Peter, Emma, and Kyler played chess with each other. Peter won 4 games and lost 2 games. Emma won 3 games and lost 3 games. If Kyler lost 3 games, how many games did he win?

Video Solution

Problem 21.5.2 (AMC 8)

The positive integers x and y are the two smallest positive integers for which the product of 360 and x is a square and the product of 360 and y is a cube. What is the sum of x and y?

Video Solution

Problem 21.5.3 (AMC 8)

How many digits are in the product $4^5 \cdot 5^{10}$?

Video Solution

Problem 21.5.4 (AMC 8)

For how many positive integer values of n are both $\frac{n}{3}$ and $3n$ three-digit whole numbers?

Video Solution

Problem 21.5.5 (AMC 8)

A 2-digit number is such that the product of the digits plus the sum of the digits is equal to the number. What is the units digit of the number?

Video Solution

Problem 21.5.6 (AMC 8)

In the multiplication problem below A, B, C, D and are different digits. What is $A+B$?

$$\begin{array}{cccc} & A & B & A \\ \times & & C & D \\ \hline C & D & C & D \end{array}$$

Video Solution

Problem 21.5.7 (AMC 8)

What is the correct ordering of the three numbers, 10^8, 5^{12}, and 2^{24}?

Video Solution

Problem 21.5.8 (AMC 8)

Suppose a, b, and c are nonzero real numbers, and $a+b+c=0$. What are the possible value(s) for $\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}+\frac{abc}{|abc|}$?

(A) 0 **(B)** 1 and -1 **(C)** 2 and -2 **(D)** $0, 2$, and -2 **(E)** $0, 1$, and -1

Video Solution

Problem 21.5.9 (AMC 8)

After Euclid High School's last basketball game, it was determined that $\frac{1}{4}$ of the team's points were scored by Alexa and $\frac{2}{7}$ were scored by Brittany. Chelsea scored 15 points. None of the other 7 team members scored more than 2 points. What was the total number of points scored by the other 7 team members?

Video Solution

Problem 21.5.10 (AMC 8)

A baseball league consists of two four-team divisions. Each team plays every other team in its division N games. Each team plays every team in the other division M games with $N > 2M$ and $M > 4$. Each team plays a 76 game schedule. How many games does a team play within its own division?

Video Solution

Problem 21.5.11 (AMC 10)

What is the greatest three-digit positive integer n for which the sum of the first n positive integers is *not* a divisor of the product of the first n positive integers?

Video Solution

Problem 21.5.12 (AMC 8)

How many perfect cubes lie between 2^8+1 and $2^{18}+1$, inclusive?

Video Solution

Problem 21.5.13 (AIME)

There is a prime number p such that $16p+1$ is the cube of a positive integer. Find p.

Video Solution

Additional Problems

Problem 21.5.14

After the digits of a 3 digit number are reversed, the number is 297 more than the original number. What's the largest possible value for the original number?

Problem 21.5.15 (AMC 8)

The hundreds digit of a three-digit number is 2 more than the units digit. The digits of the three-digit number are reversed, and the result is subtracted from the original three-digit number. What is the units digit of the result?

Problem 21.5.16 (AMC 12)

In multiplying two positive integers a and b, Ron reversed the digits of the two-digit number a. His erroneous product was 161. What is the correct value of the product a and b?

Problem 21.5.17 (AMC 12)

A palindrome, such as 83438, is a number that remains the same when its digits are reversed. The numbers x and $x+32$ are three-digit and four-digit palindromes, respectively. What is the sum of the digits of x?

Problem 21.5.18 (AMC 10)

A two-digit positive integer is said to be *cuddly* if it is equal to the sum of its nonzero tens digit and the square of its units digit. How many two-digit positive integers are cuddly?

Problem 21.5.19 (EMCC)

What is the largest positive 2-digit factor of $3^{2^{2011}} - 2^{2^{2011}}$

Problem 21.5.20 (MATHCOUNTS)

If m and n are positive integers where $m^2 + 14m - 32 = 3^n$, what is the value of $m+n$?

Problem 21.5.21 (AMC 10)

Which of the following is equivalent to $\sqrt{\frac{x}{1-\frac{x-1}{x}}}$ when $x < 0$?

(A) $-x$ (B) x (C) 1 (D) $\sqrt{\frac{x}{2}}$ (E) $x\sqrt{-1}$

Problem 21.5.22 (AMC 10)

In the United States, coins have the following thicknesses: penny, 1.55 mm; nickel, 1.95 mm; dime, 1.35 mm; quarter, 1.75 mm. If a stack of these coins is exactly 14 mm high, how many coins are in the stack?

Problem 21.5.23 (AMC 10)

Boris has an incredible coin-changing machine. When he puts in a quarter, it returns five nickels; when he puts in a nickel, it returns five pennies; and when he puts in a penny, it returns five quarters. Boris starts with just one penny. Which of the following amounts could Boris have after using the machine repeatedly?

(A)3.63 **(B)**5.13 **(C)**6.30 **(D)**7.45 **(E)**9.07

Answers

21.2 432

21.3 36

21.6 5

21.5.1 1

21.5.2 85

21.5.3 11

21.5.4 12

21.5.5 9

21.5.6 1

21.5.7 $2^{24} < 10^8 < 5^{12}$

21.5.8 0

21.5.9 11

21.5.10 48

21.5.11 996

21.5.12 58

21.5.13 307

21.5.14 699

21.5.15 8

21.5.16 224

21.5.17 24

21.5.18 1

21.5.19 97

21.5.20 16

21.5.21 $-x$

21.5.22 8

21.5.23 \$7.45

Chapter 22

Miscellaneous Number Theory

22.1 Palindromes

Definition 22.1.1. A palindrome is a number that reads the same forward and backward.

Video Lectures

Palindromes

22.2 Money Problems

Example 22.1 (Omega Learn)

Josh has some quarters, dimes, nickels and pennies worth 101 cents. He has a total of 9 coins, at least 1 of each type of coin, how many nickels and dimes does he have?

Solution

First we should try to handle the limiting condition to make the problem simpler.

Is there any way we can easily simplify our problem and remove the condition of must having at least 1 of each coin?

We can simply subtract off the value of 1 quarter, 1 dime, 1 nickel, and 1 penny because he must have 1 of each coin anyway.

The total value of these coins is $25+10+5+1=41$ cents. We are then left with 60 cents to assign to the rest of the coins. Because we have already used 4 coins, the remaining 5 coins must have a value of 60 cents.

Is it possible to have no more quarters?

We can see that having no more quarters is impossible because the maximum value for any other coin is 10 cents, so the maximum value of 5 coins would be 50 cents, which is less than 60 cents.

How can we split the remaining possibilities into cases?

Since quarter is the coin worth the most, we can split our problem into cases based on how many quarters we will use.

Case 1: 1 more quarter

In this case, the quarter will have a value of 25 cents, so the remaining 4 coins have a value of $60-25=35$ cents. Clearly, we can see having 4 dimes will not work. However, 3 dimes along with a nickel results in having a total value of 35 cents. Therefore, this case works.

Although it's not necessary, to show that no other cases work, we will look at what happens if we have more quarters.

Case 2: 2 more quarters

In this the 2 quarters will have a value of 50 cents, so the remaining 3 coins have a value of $60-50=10$ cents.

We can clearly see that if we have even 1 dime, it's impossible for the remaining coins to have a value of 0 cents. Similarly, having 2 nickels is impossible because that means the remaining coin will have a value of 0 cents. Therefore, we can have at most 1 nickel.

If we have 1 nickel and the remaining 5 cents worth of coins are pennies, there will be a total of $1+5=6$ coins, which isn't allowed. If we have no nickels, there will 10 pennies, which also isn't possible for the same reason.

Case 3: 3 more quarters

This is impossible because the total value would be 75 cents, which is above the 60 cents we need.

Therefore, the only case is when we have 1 additional quarter, 3 additional dimes, and 1 additional nickel. Since we also have the include the 1 penny, 1 nickel, 1 dime, and 1 quarter we had in the beginning, there will be 4 dimes and 2 nickels.

Therefore, the total number of nickels and dimes is $4+2=\boxed{6}$.

Example 22.2 (MATHCOUNTS)

A restaurant sells three sizes of drinks: small for \$1.20, medium for \$1.30 and large for \$1.80. Each person at a table of ten ordered one drink, for a total cost of \$14.90, before sales tax. How many people ordered a large drink?

Video Solution

22.3 Integer Operations

Example 22.3 (MATHCOUNTS)

John inserts some number of parentheses into the expression shown to create a valid mathematical expression. What is the smallest possible integer value of John's expression?

$$1 \div 2 \div 3 \div 4 \div 5 \div 6 \div 7 \div 8 \div 9 \div 10$$

Video Solution

Example 22.4 (Omega Learn)

Sohil has received a new TI-84 Calculator from MATHCOUNTS. He is very excited, so he enters the following expression $2^6 \times 3^5 \times 5^7$ into the calculator. On any given operation, Sohil will divide the number by it's largest prime factor, p. Then, he will multiply the number by $p-1$. He continues this process until he reaches a final number of 1. How many operations, including the final operation, will it take for this to happen?

Solution

Let's analyze this step by step.

To start, what prime number will divide our number first?

The largest prime number in the prime factorization is 5, so 5 will divide first. Then, we will have to multiply by $5-1=4$ or 2^2.

After how many operations will all the factors of 5 be gone?

Each operation removes 1 power of 5 in the prime factorization, so after 7 operations, in the prime factorization, all the factors of 5 will be gone.

However, we will also have had to multiply by 4 after each operation. How does this change the prime factorization?

We will also have had to multiply by $4 = 2^2$, 7 times, or 2^14. Therefore, after 7 operations, in the prime factorization, we will have 14 additional factors of 2. The prime factorization will therefore be $2^6 \times 2^{14} \times 3^5 = 2^{20} \times 3^5$.

Now, what prime factor do we have to divide by?

The largest prime factor is clearly 3, so we have to divide by this now, instead of 5. Then, we will have to multiply by $3-1 = 2$.

We will need 5 operations to divide by all factors of 3. Doing so, we will also have to multiply the number by 2, 5 times, or by 2^5. Therefore, after 5 more operations the prime factorization will $2^{20} \times 2^5 = 2^{25}$.

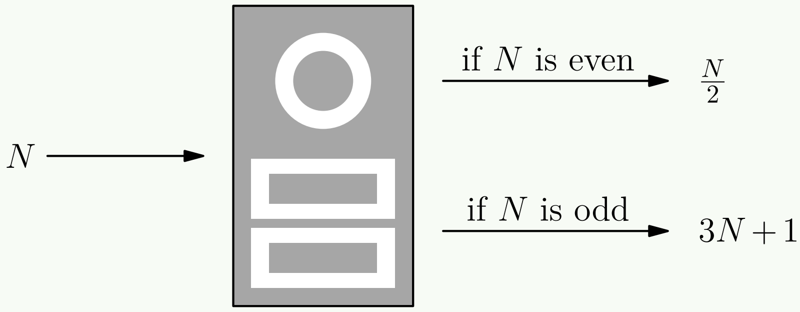
Now, what prime factor do we have to divide by?

Because the only prime factor is 2, we must divide by 2. Doing so, we will also have to multiply our number by 1, which will not change the number. Therefore, because there are 25 factors of 2, we will have to do the operations 25 more times for the number to become 1.

Therefore, the total number of operations needed is $7+5+25 = \boxed{37}$.

Example 22.5 (AMC 8)

When a positive integer N is fed into a machine, the output is a number calculated according to the rule shown below.



For example, starting with an input of $N = 7$, the machine will output $3 \cdot 7 + 1 = 22$. Then if the output is repeatedly inserted into the machine five more times, the final output is 26.

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26$$

When the same 6-step process is applied to a different starting value of N, the final output is 1. What is the sum of all such integers N?

$$N \to \_\_ \to \_\_ \to \_\_ \to \_\_ \to \_\_ \to 1$$

Video Solution

22.4 Chicken McNugget Theorem

Theorem 22.4.1 (Chicken McNugget Theorem)

The maximum value that cannot be expressed as the sum of non-negative multiples of a and b is $ab - a - b$ if a and b are relatively prime.

For relatively prime positive integers a, b ,there are exactly $\frac{(a-1)(b-1)}{2}$ positive integers which cannot be expressed in the form $ma + nb$ where m and n are positive integers.

Remark 22.4.2

This theorem is useful in finding solutions to problems like ”the maximum amount of money that can’t be created with 3 cent and 5 cent coins”.

22.5 Practice Problems

Problem 22.5.1 (AMC 8)

Loki, Moe, Nick and Ott are good friends. Ott had no money, but the others did. Moe gave Ott one-fifth of his money, Loki gave Ott one-fourth of his money and Nick gave Ott one-third of his money. Each gave Ott the same amount of money. What fractional part of the group's money does Ott now have?

Video Solution

Problem 22.5.2 (AMC 8)

You have nine coins: a collection of pennies, nickels, dimes, and quarters having a total value of \$1.02, with at least one coin of each type. How many dimes must you have?

Video Solution

Problem 22.5.3 (AMC 10A)

The average value of all the pennies, nickels, dimes, and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

Video Solution

Problem 22.5.4 (AMC 10A)

Joe has a collection of 23 coins, consisting of 5-cent coins, 10-cent coins, and 25-cent coins. He has 3 more 10-cent coins than 5-cent coins, and the total value of his collection is 320 cents. How many more 25-cent coins does Joe have than 5-cent coins?

Video Solution

Problem 22.5.5 (AMC 8)

Isabella must take four 100-point tests in her math class. Her goal is to achieve an average grade of 95 on the tests. Her first two test scores were 97 and 91. After seeing her score on the third test, she realized she can still reach her goal. What is the lowest possible score she could have made on the third test?

Video Solution

Problem 22.5.6 (AMC 8)

Laila took five math tests, each worth a maximum of 100 points. Laila's score on each test was an integer between 0 and 100, inclusive. Laila received the same score on the first four tests, and she received a higher score on the last test. Her average score on the five tests was 82. How many values are possible for Laila's score on the last test?

Video Solution

Problem 22.5.7 (AMC 10A)

Mr. Patrick teaches math to 15 students. He was grading tests and found that when he graded everyone's test except Payton's, the average grade for the class was 80. After he graded Payton's test, the test average became 81. What was Payton's score on the test?

Video Solution

Problem 22.5.8 (AMC 8)

Each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 is used only once to make two five-digit numbers so that they have the largest possible sum. Which of the following could be one of the numbers?

(A) 76531 **(B)** 86724 **(C)** 87431 **(D)** 96240 **(E)** 97403

Video Solution

Problem 22.5.9

Laura added two three-digit positive integers. All six digits in these numbers are different. Laura's sum is a three-digit number S. What is the smallest possible value for the sum of the digits of S?

Video Solution

Problem 22.5.10 (AMC 8)

What is the largest power of 2 that is a divisor of $13^4 - 11^4$?

Video Solution

Problem 22.5.11 (AMC 8)

Students guess that Norb's age is $24, 28, 30, 32, 36, 38, 41, 44, 47$, and 49. Norb says, "At least half of you guessed too low, two of you are off by one, and my age is a prime number." How old is Norb?

Video Solution

Problem 22.5.12 (Omega Learn Math Contest)

Josh went to the store with a number of dollars between 50 and 100, both inclusive. He spent a third of his money on lunch. He then tipped 15 dollars. After that, he deposited two-thirds of his remaining money in his bank account. After that, he donated a fifth of his remaining money to charity. He was then left with an integer number of dollars. How many dollars did he have at the beginning?

Video Solution

Problem 22.5.13 (AMC 8)

In a tournament there are six teams that play each other twice. A team earns 3 points for a win, 1 point for a draw, and 0 points for a loss. After all the games have been played it turns out that the top three teams earned the same number of total points. What is the greatest possible number of total points for each of the top three teams?

Video Solution

Problem 22.5.14 (Omega Learn Math Contest)

Bryan chooses a number between 1 and 1000. He gives a few conditions about the number:

1. The number has no prime factors greater than 5
2. The number is not a perfect power (of anything greater than 1)
3. The number has 6 positive factors
4. The number leaves a remainder of 5 when divided by 7

Find the sum of all possible numbers Bryan could choose based on the conditions.

Video Solution

Additional Problems

Problem 22.5.15 (MATHCOUNTS)

In eight games this season, Kelly's basketball team scored 22, 30, 33, 44, 50, 55, 61 and 66 points, respectively. They exactly tripled their opponent's score three times and exactly doubled their opponent's score three times. They lost two games by 4 points each. How many points did their opponents score altogether?

Problem 22.5.16 (MATHCOUNTS)

What fraction of the first 100 triangular numbers are evenly divisible by 7? Express your answer as a common fraction.

Problem 22.5.17 (AIME)

Find the arithmetic mean of all the three-digit palindromes. (Recall that a palindrome is a number that reads the same forward and backward, such as 777 or 383.)

Problem 22.5.18 (AMC 8)

The grid below is to be filled with integers in such a way that the sum of the numbers in each row and the sum of the numbers in each column are the same. Four numbers are missing. The number x in the lower left corner is larger than the other three missing numbers. What is the smallest possible value of x?

-2	9	5
		-1
x		8

Problem 22.5.19 (AMC 8)

A scientist walking through a forest recorded as integers the heights of 5 trees standing in a row. She observed that each tree was either twice as tall or half as tall as the one to its right. Unfortunately some of her data was lost when rain fell on her notebook. Her notes are shown below, with blanks indicating the missing numbers. Based on her observations, the scientist was able to reconstruct the lost data. What was the average height of the trees, in meters?

Tree 1	__ meters
Tree 2	11 meters
Tree 3	__ meters
Tree 4	__ meters
Tree 5	__ meters
Average height	__ .2 meters

Problem 22.5.20 (MATHCOUNTS)

If C is a digit such that the product of the three-digit numbers 2C8 and 3C1 is the five-digit number 90C58, what is the value of C?

Problem 22.5.21 (Omega Learn)

I am thinking of a number between 1000 and 9999, inclusive. It satisfies the following conditions

1. The digits strictly increase from left to right
2. The sum of the 1st and 2nd digits equals the 3rd digit
3. The 2nd digit is 1 or 2 more than the first digit
4. The sum of the first 3 digits is greater than the 4th digit
5. When the number is squared, the units digit is 1
6. The number is divisible by 37

What number am I thinking of?

Answers

22.2 4

22.3 7

22.5 83

22.5.1 $\frac{1}{4}$

22.5.2 1

22.5.3 0

22.5.4 2

22.5.5 92

22.5.6 4

22.5.7 95

22.5.8 87431

22.5.9 4

22.5.10 32

22.5.11 37

22.5.12 90

22.5.13 24

22.5.14 87

22.5.15 225

22.5.16 $\frac{7}{25}$

22.5.17 550

22.5.18 8

22.5.19 24.2

22.5.20 5

22.5.21 3589

Geometry

Chapter 23

Angle Chasing

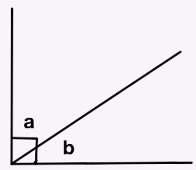
Video Lecture

23.1 Angle Chasing Basics

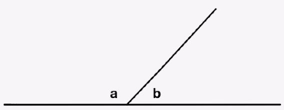
Concept 23.1.1 (Angle Chasing Tricks)

- Sum of Angles in Triangle is 180
- A triangle with 2 angles equal will have their corresponding sides equal and a triangle with 2 sides equal will have their corresponding angles equal (isosceles triangle)
- Opposite angles in intersecting lines are equal
- Corresponding angles in parallel lines are equal
- The angle made by the arc at the center of the circle is double the angle made by the arc at the boundary of the circle

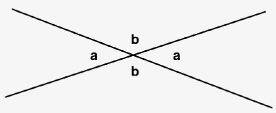
Concept 23.1.2 (Complementary Angle)
Complementary angles are a pair of angles with the sum of 90 degrees



Concept 23.1.3 (Supplementary Angle)
Supplementary angles are a pair of angles with the sum of 180 degrees

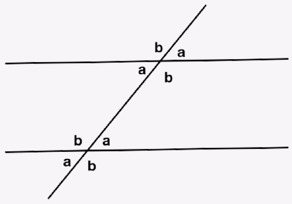


Concept 23.1.4 (Intersecting lines)
When two lines intersect, the vertical angles are equal. Vertical angles are each of the pairs of opposite angles made by two intersecting lines. ”Vertical” in this case means they share the same Vertex (corner point), not the usual meaning of up-down.



Concept 23.1.5 (Parallel Lines)

Corresponding angles equal

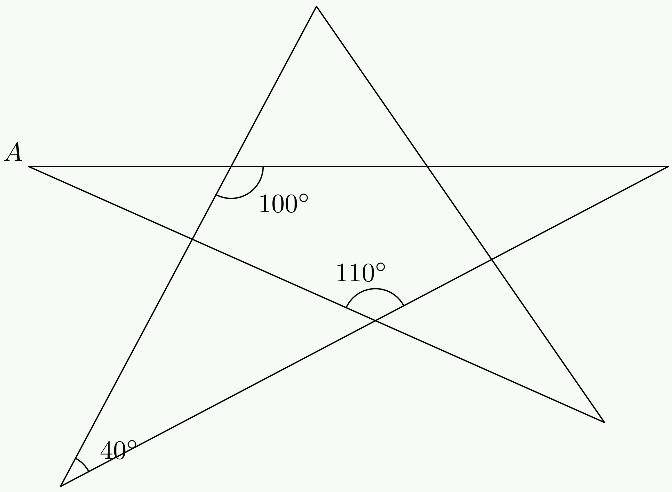


Remark 23.1.6

Using this property along with supplementary angles, we can derive many other angle relations.

Example 23.1 (AMC 8)

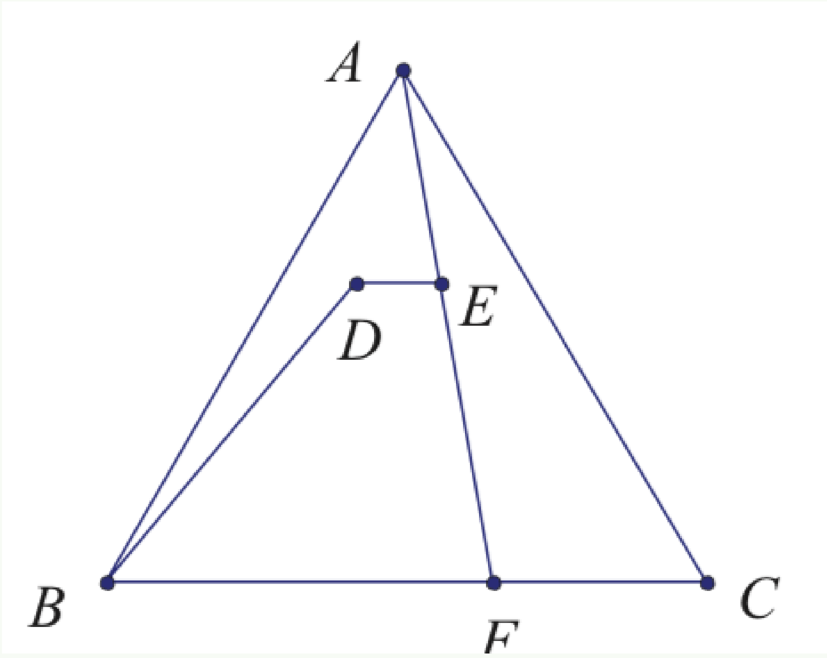
The degree measure of angle A is



[Video Solution]

Example 23.2 (EMCC)

In acute $\triangle ABC$, D and E are points inside triangle ABC such that $DE \parallel BC$, B is closer to D than it is to E, $\angle AED = 80°$, $\angle ABD = 10°$, and $\angle CBD = 40°$. Find the measure of $\angle BAE$, in degrees.



Video Solution

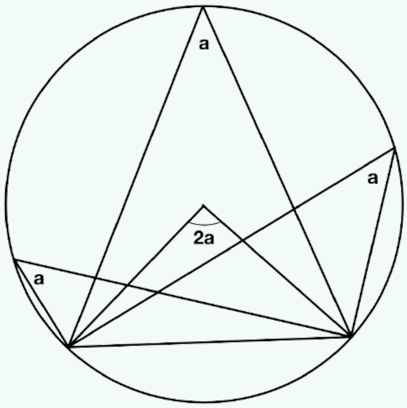
Video Lectures

Angle Chasing Basics
Angle Chasing Advanced

23.2 Inscribed Angles

Theorem 23.2.1 (Inscribed Arc Theorem)

The angle formed by an arc in the center or the arc angle is double of the angle formed on the edge.



Remark 23.2.2

Circles are really useful for angle chasing so keep an eye out for the inscribed arc theorem that can be used in many angle chasing problems.

Remark 23.2.3

A useful trick to solving angle chasing problems with regular polygons is to draw a circle around the polygon and use the inscribed arc theorem.

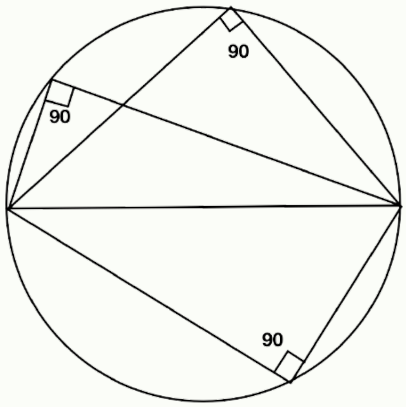
Example 23.3

In regular octagon ABCDEFGH, Find the measure of $\angle ACF$.

Video Solution

Corollary 23.2.4 (Inscribed Right Triangle)

Inscribed triangle with diameter as one side is always a right triangle.



23.3 Polygons

Theorem 23.3.1

$$\text{Sum of interior angle of a polygon} = (n-2)\cdot 180$$

$$\text{Interior angle of a regular polygon} = \frac{(n-2)}{n}\cdot 180$$

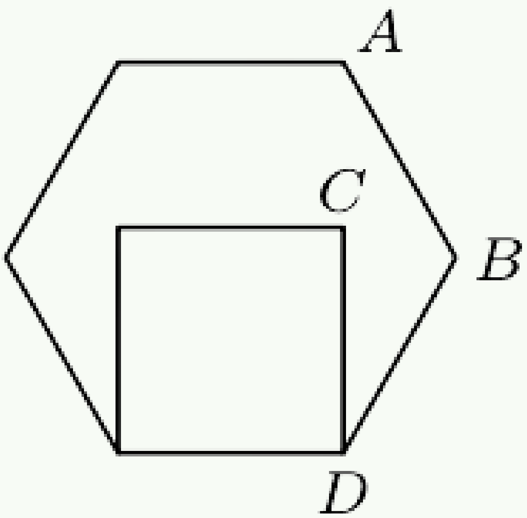
$$\text{Exterior angle of a regular polygon} = \frac{360}{n}$$

Fact 23.3.2. Important Interior Angles

Number of sides in regular polygon	Interior Angle of regular polygon
3	60
4	90
5	108
6	120
8	135
9	140
10	144

Example 23.4 (MATHCOUNTS)

A square is located in the interior of a regular hexagon, and certain vertices are labeled as shown. What is the degree measure of $\angle ABC$?



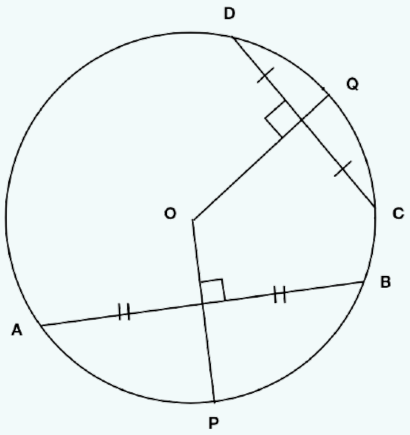
Video Solution

23.4 Advanced Circle Angle Chasing Theorems (Optional)

Definition 23.4.1 (Chord)**.** A Chord is a line segment between any two distinct points on the circle. The diameter of the circle is the longest chord in the circle.

Theorem 23.4.2

The perpendicular bisector of any chord passes through the center. In the figure below, the perpendicular bisectors of AB and CD intersect at the center O.



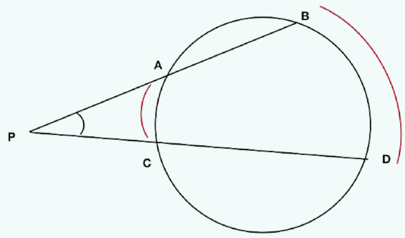
Corollary 23.4.3

- Congruent chords are equidistant from the center of a circle.
- If two chords in a circle are congruent, then their intercepted arcs are congruent.
- If two chords in a circle are congruent, then they determine two central angles that are congruent.

Theorem 23.4.4

The angle marked in the diagram is half of the difference of the 2 red arcs.

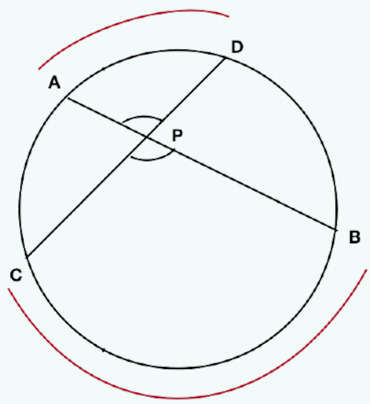
$$\angle APC = \frac{\overset{\frown}{\mathrm{BD}} - \overset{\frown}{\mathrm{AC}}}{2}$$



Theorem 23.4.5

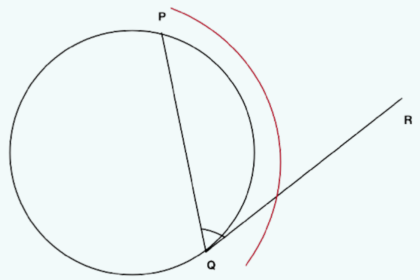
If two chords AB and CD intersect at P, then the $\angle BPC$ and $\angle APD$ are equal to the average of the two arcs.

$$\angle BPC = \angle APD = \frac{\overset{\frown}{\mathrm{BC}} + \overset{\frown}{\mathrm{AD}}}{2}$$



Theorem 23.4.6

If a tangent R intersects the circle at Q, and a chord QP is drawn, then the $\angle RQP$ is equal to half the arc angle



Theorem 23.4.7

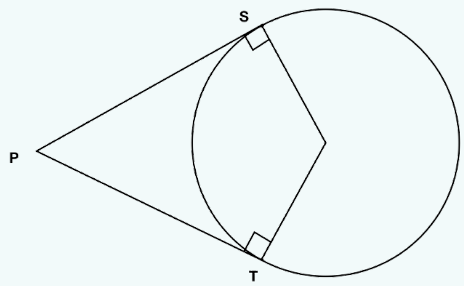
Equal chords mark out equal arcs

This basically means that if you have 2 chords of the same length, the sector of the circle they mark out will be equal

Definition 23.4.8 (Tangent)**.** A tangent is any line from a point external to the circle that just touches the circle.

Theorem 23.4.9 (Right Angle Tangency Point)

If you connect the center of a circle to the point where the circle and a line are tangent, they will form a right angle.



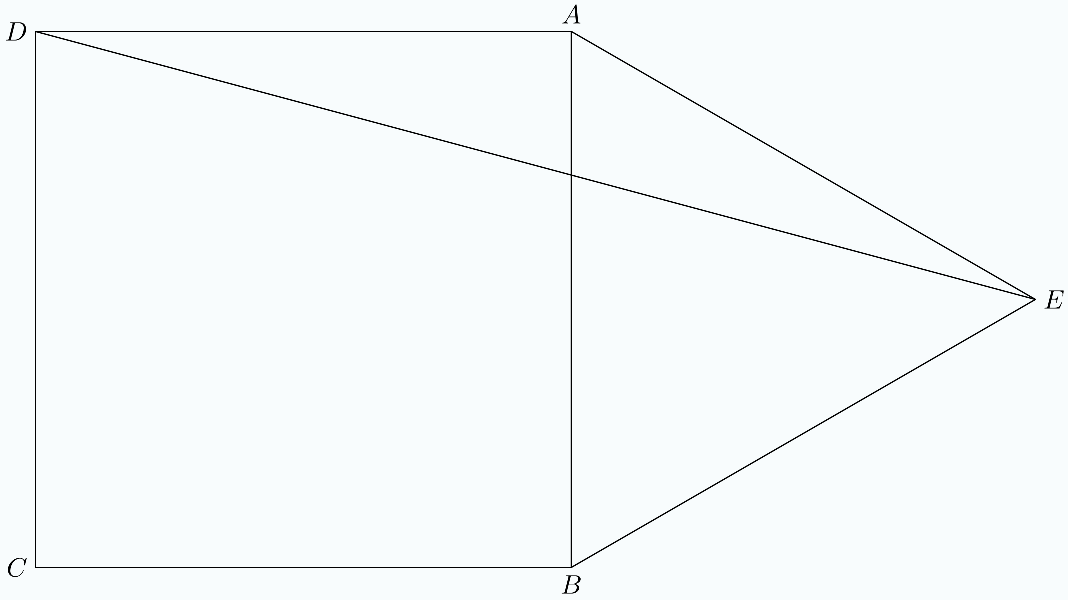
Remark 23.4.10

This property is very useful in circle problems as it allows us to work with right angles. In addition, another helpful technique is drawing useful radii to various points in your diagram as that opens up new information to work with.

23.5 Practice Problems

Problem 23.5.1

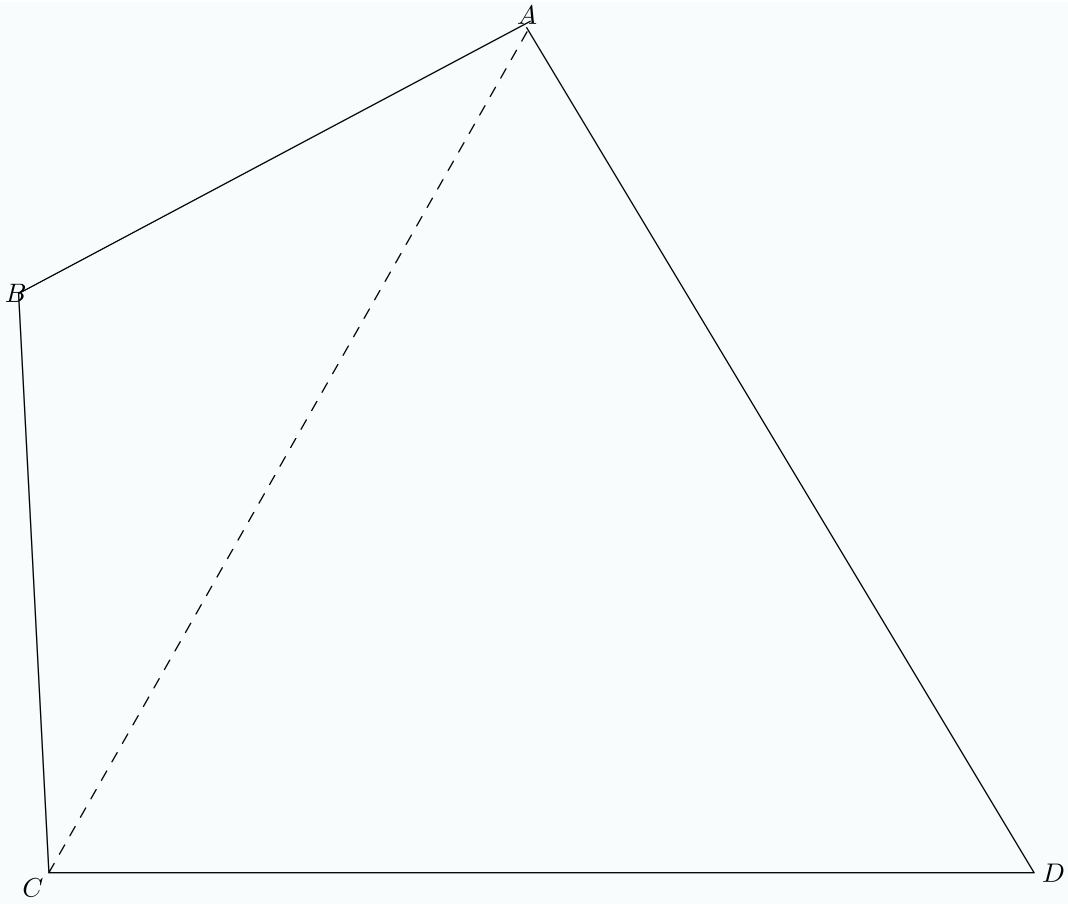
In the adjoining figure, $ABCD$ is a square, ABE is an equilateral triangle and point E is outside square $ABCD$. What is the measure of $\measuredangle AED$ in degrees?



Video Solution

Problem 23.5.2 (AMC 8)

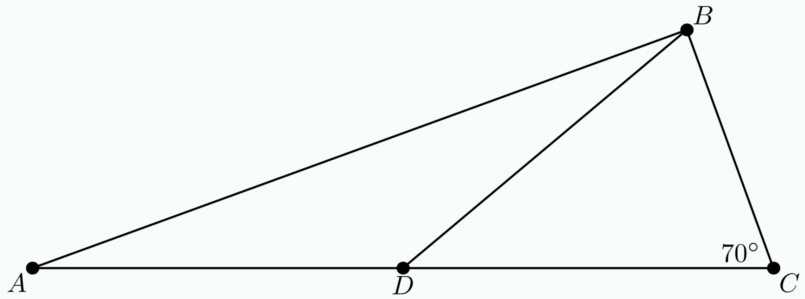
In quadrilateral $ABCD$, sides $\overline{AB}$ and $\overline{BC}$ both have length 10, sides $\overline{CD}$ and $\overline{DA}$ both have length 17, and the measure of angle ADC is $60°$. What is the length of diagonal $\overline{AC}$?



[Video Solution]

Problem 23.5.3 (AMC 8)

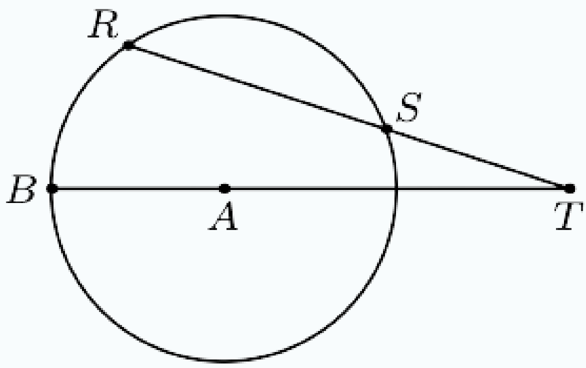
In $\triangle ABC$, D is a point on side $\overline{AC}$ such that $BD = DC$ and $\angle BCD$ measures 70°. What is the degree measure of $\angle ADB$?



Video Solution

Problem 23.5.4

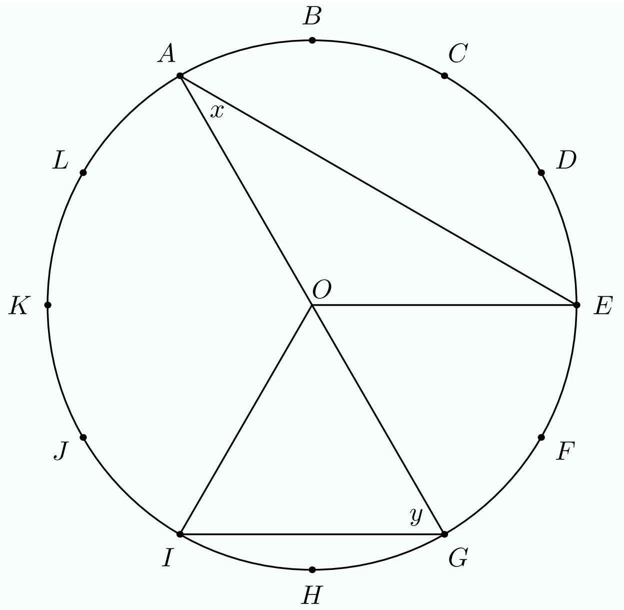
In the figure, point A is the center of the circle, the measure of angle RAS is 74°, and the measure of angle RTB is 28°. What is the measure of minor arc BR, in degrees?



Video Solution

Problem 23.5.5 (AMC 8)

The circumference of the circle with center O is divided into 12 equal arcs, marked the letters A through L as seen below. What is the number of degrees in the sum of the angles x and y?



Video Solution

Problem 23.5.6 (AMC 8)

Two angles of an isosceles triangle measure $70°$ and $x°$. What is the sum of the different possible values of x?

Video Solution

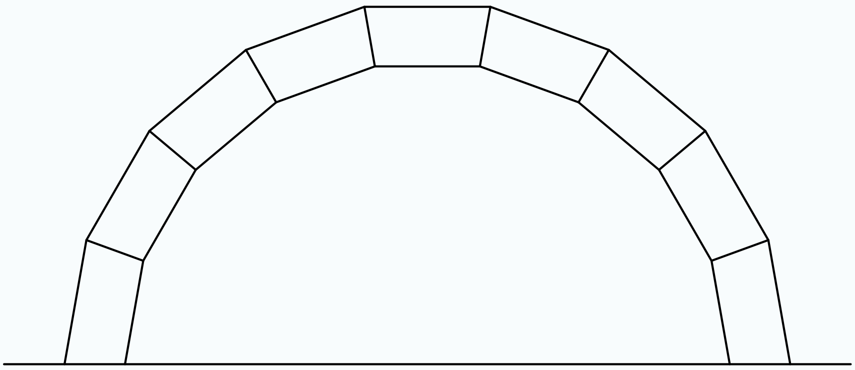
Problem 23.5.7 (AMC 8)

Two congruent circles centered at points A and B each pass through the other circle's center. The line containing both A and B is extended to intersect the circles at points C and D. The circles intersect at two points, one of which is E. What is the degree measure of $\angle CED$?

Video Solution

Problem 23.5.8 (AMC 10)

The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measure in degrees of the larger interior angle of the trapezoid. What is x?



Video Solution

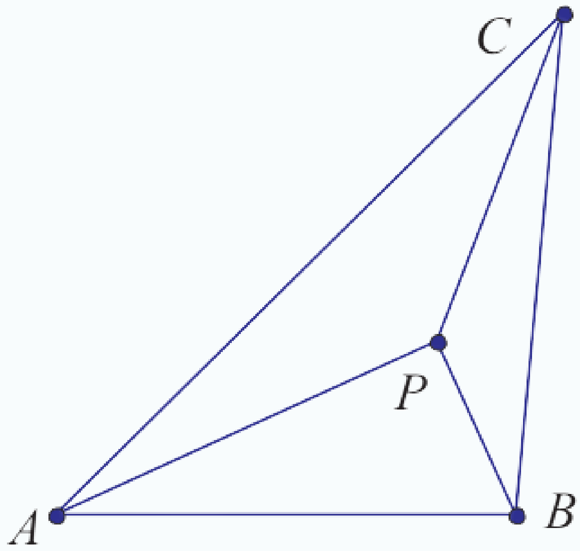
Additional Problems

Problem 23.5.9

In triangle ABC, AB $=$ AC and $\angle A = 40°$. The bisector from $\angle B$ intersects AC at point D. What is $\angle BDC$?

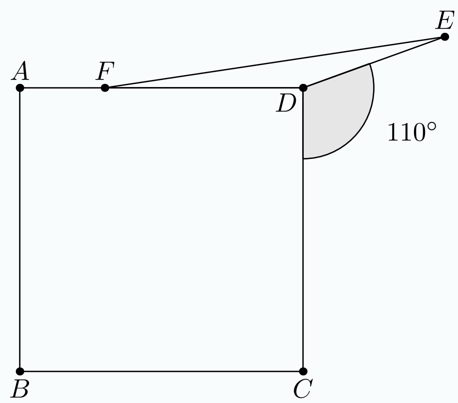
Problem 23.5.10 (EMCC)

Point P lies inside triangle ABC such that $\angle PBC = 30°$ and $\angle PAC = 20°$. If $\angle APB$ is a right angle, find the measure of $\angle BCA$ in degrees.



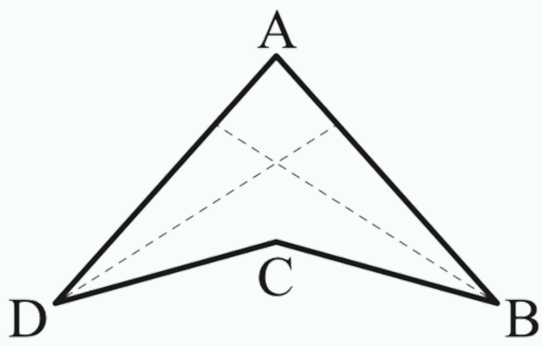
Problem 23.5.11 (AMC 10/12)

As shown in the figure below, point E lies on the opposite half-plane determined by line CD from point A so that $\angle CDE = 110°$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



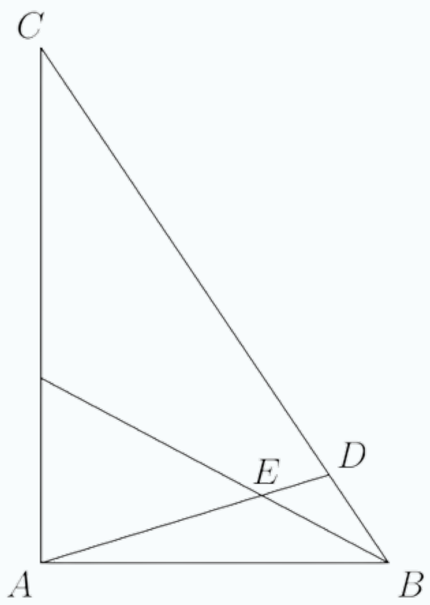
Problem 23.5.12 (MATHCOUNTS)

Concave quadrilateral ABCD is symmetric about the line AC. The measures of angles DAB and ABC are 84 degrees and 32 degrees, respectively. The dashed line segments bisect angles ABC and ADC. What is the degree measure of the acute angle at which the two dashed line segments intersect?



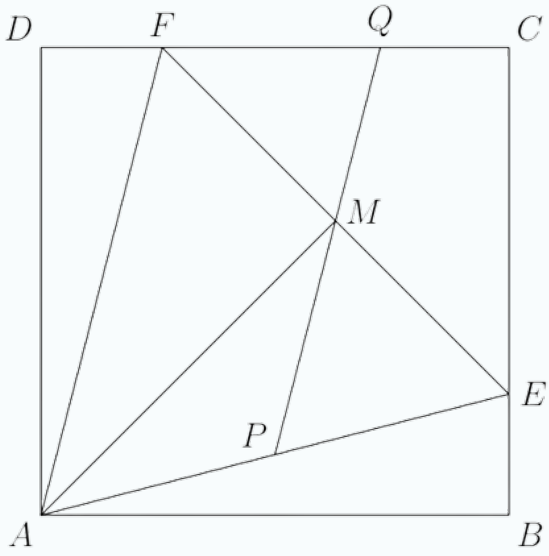
Problem 23.5.13 (EMCC)

In triangle ABC, $\angle BAC$ is a right angle and $\angle ACB$ measures 34°. Let D be a point on segment BC for which $AC = CD$, and let the angle bisector of $\angle CBA$ intersect line AD at E. What is the measure of $\angle BED$?



Problem 23.5.14 (EMCC)

In square ABCD, point E lies on side BC and point F lies on side CD so that triangle AEF is equilateral and inside the square. Point M is the midpoint of segment EF, and P is the point other than E on AE for which PM = FM. The extension of segment PM meets segment CD at Q. What is the measure of ∠CQP, in degrees?



Answers

23.1 30

23.2 50°

23.3 $\frac{135}{2}$

23.4 45°

23.5.1 15

23.5.2 17

23.5.3 140

23.5.4 81°

23.5.5 90

23.5.6 95

23.5.7 120

23.5.8 100

23.5.9 75

23.5.10 40°

23.5.11 170

23.5.12 64

23.5.13 45

23.5.14 105

Chapter 24

Triangles

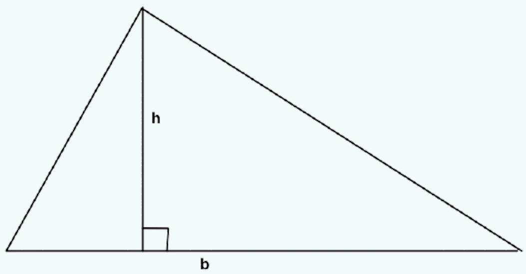
Video Lecture

24.1 Area of a Triangle

There are many ways to calculate the area of a triangle. Here are some of the most useful formulas for calculating the area of a triangle:

Theorem 24.1.1 (Using base and height)
A triangle with base b and height h has an area of

$$\frac{1}{2} \cdot b \cdot h$$



Theorem 24.1.2 (Heron's Formula)
A triangle with sides a, b, c and semiperimeter s has an area of

$$\sqrt{s(s-a)(s-b)(s-c)}$$

Example 24.1 (EMCC)

Given that the heights of $\triangle ABC$ have lengths $\frac{15}{7}$, 5, and 3, what is the square of the area of $\triangle ABC$?

Video Solution

Definition 24.1.3 (Inradius)**.** The inradius of a triangle is the radius of the inscribed circle in the triangle.

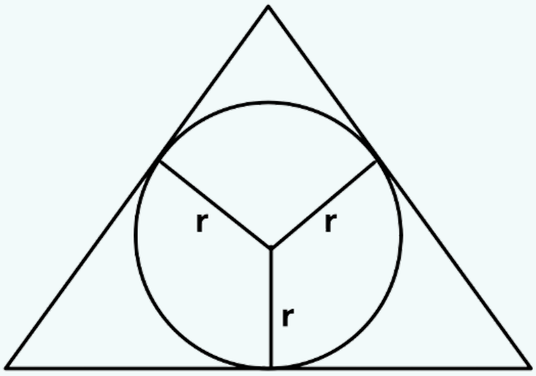
Definition 24.1.4 (Incenter)**.** The incenter of a triangle is the intersection of all the angle bisectors. This point is also the center of the incircle, and equidistant from all the three sides.

Theorem 24.1.5 (Area of a triangle using inradius)

A triangle with inradius r (the radius of the circle that can be inscribed in a triangle) and semiperimeter s has an area of:

$$\text{inradius} \cdot \text{semiperimeter}$$

$$rs$$



Remark 24.1.6

Note that if we know the area of the triangle and it's semi-perimeter, we can apply the inradius formula to find the inradius of the triangle.

Theorem 24.1.7

Inradius r of a right triangle:

$$r = \frac{1}{2}(a+b-c)$$

where a and b are the legs of the triangle, and c is the hypotenuse.

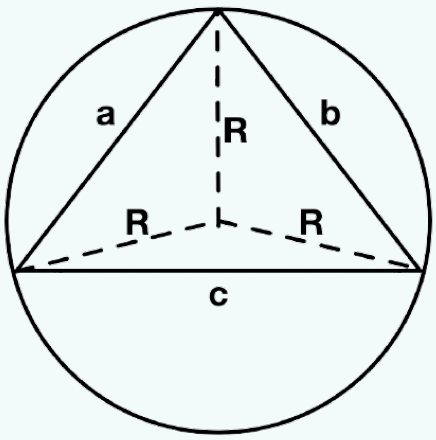
Definition 24.1.8 (Circumradius)**.** The circum-radius of a triangle is the radius of circle that a triangle is inscribed in.

Definition 24.1.9 (Circumcenter)**.** The circumcenter of a triangle is the intersection of all 3 perpendicular bisectors (the line that bisects a segment and is perpendicular to it). This point is also the center of the circumcircle and equidistant from all the three vertices.

Theorem 24.1.10 (Area of a triangle using circumradius)

A triangle with circumradius R (the radius of the circle that the triangle can be inscribed in) and sides a, b, c has an area of

$$\frac{abc}{4R}$$



Remark 24.1.11

Similar to the inradius problem, if we know all 3 sides of a triangle, we can apply Heron's and easily calculate the circumradius of the triangle.

24.2 Special Triangles

24.2.1 Equilateral Triangle

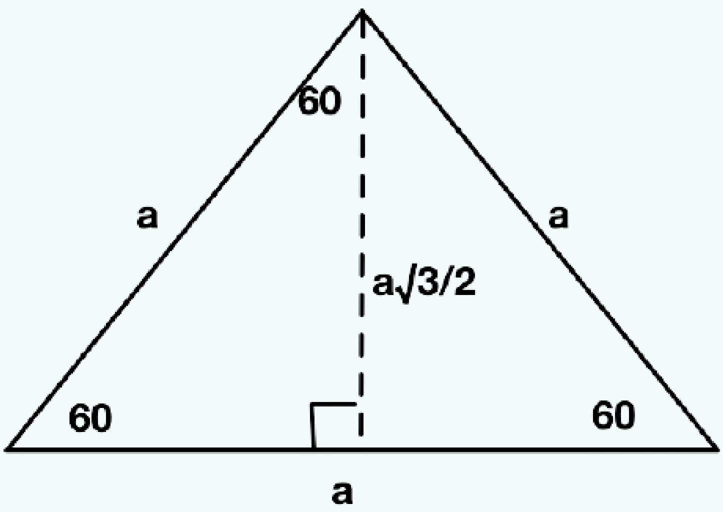
Theorem 24.2.1

If the side length of an equilateral triangle is a

$$\text{Height of the triangle} = \frac{\sqrt{3}}{2}a$$

This follows directly from the $30-60-90$ triangle.

$$\text{Area of the triangle} = \frac{\sqrt{3}}{4}a^2$$



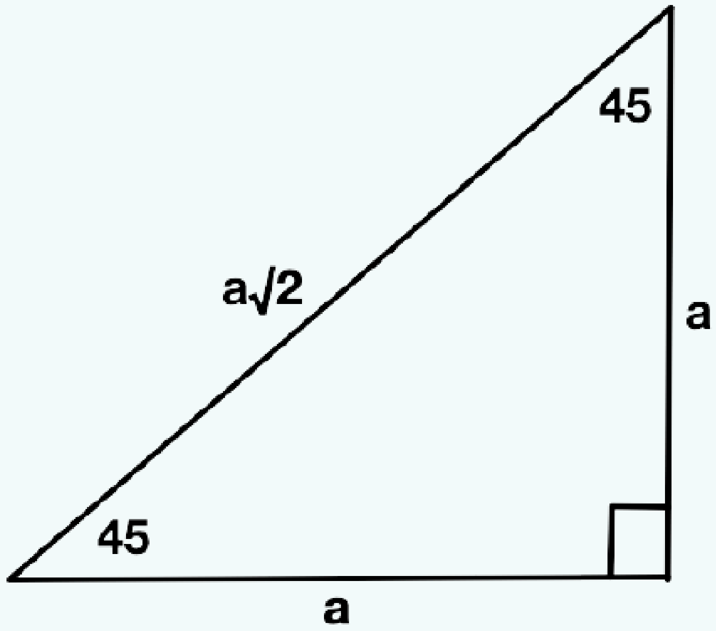
24.2.2 45-45-90 Triangle

Theorem 24.2.2

If the side length of a 45-45-90 triangle is a

$$\text{hypotenuse of the triangle} = \sqrt{2} \times \text{side length} = \sqrt{2}a$$

$$\text{Area of the triangle} = \frac{1}{2} \times \text{side length}^2 = \frac{1}{2}a^2$$



24.2.3 30-60-90 Triangle

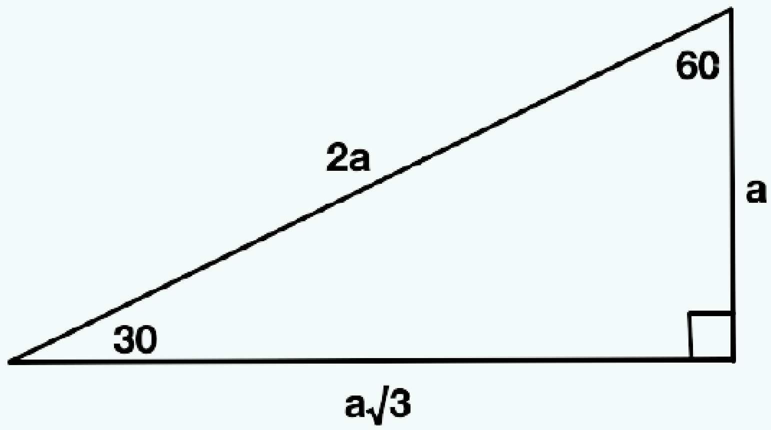
Theorem 24.2.3

If the short leg length of a 30-60-90 triangle is a

$$\text{Long Leg of the triangle} = \sqrt{3} \times \text{short leg} = \sqrt{3}a$$

$$\text{hypotenuse of the triangle} = 2 \times \text{short leg} = 2a$$

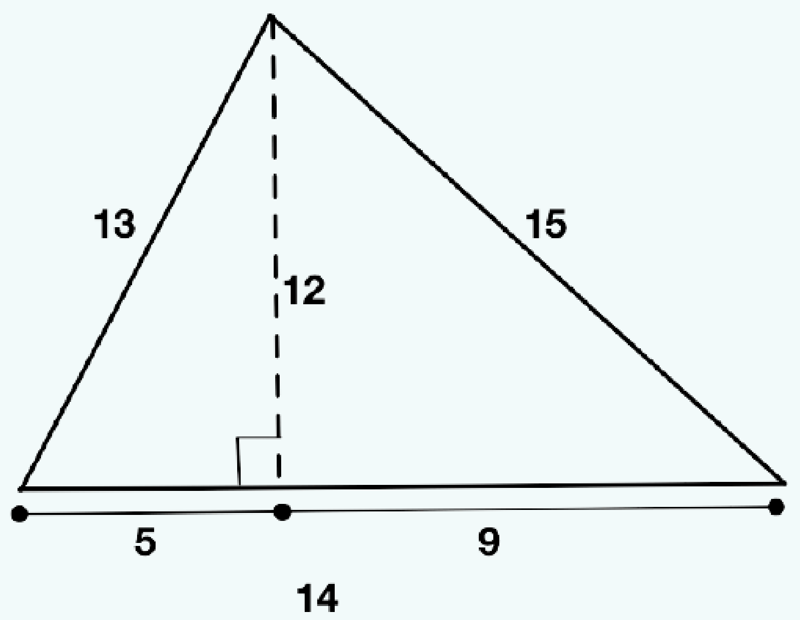
$$\text{Area} = \frac{\sqrt{3}}{2} \times \text{short leg}^2 = \frac{\sqrt{3}}{2}a^2$$



24.2.4 13-14-15 Triangle

Theorem 24.2.4

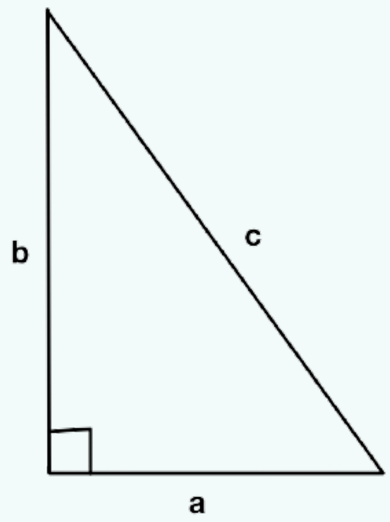
If the three sides of a triangle are 13, 14, and 15, it can be divided into two right triangles with side lengths:
5, 12, 13 and 9, 12, 15
Area of this triangle $= \frac{1}{2} \times 14 \times 12 = 84$



24.3 Pythagorean Theorem

Theorem 24.3.1 (Pythagorean Theorem)
A right triangle with legs a and b and hypotenuse c satisfies the following relation:

$$c^2 = a^2 + b^2$$



Fact 24.3.2. Important Pythagorean Triples
3, 4, 5
5, 12, 13
7, 24, 25
8, 15, 17
9, 40, 41
20, 21, 29

If all numbers in a pythagorean triple are multiplied by a constant, the resulting numbers still form a pythagorean triple.

For example: These are all pythagorean triples:
3, 4, 5
6, 8, 10
9, 12, 15
12, 16, 20
15, 20, 25

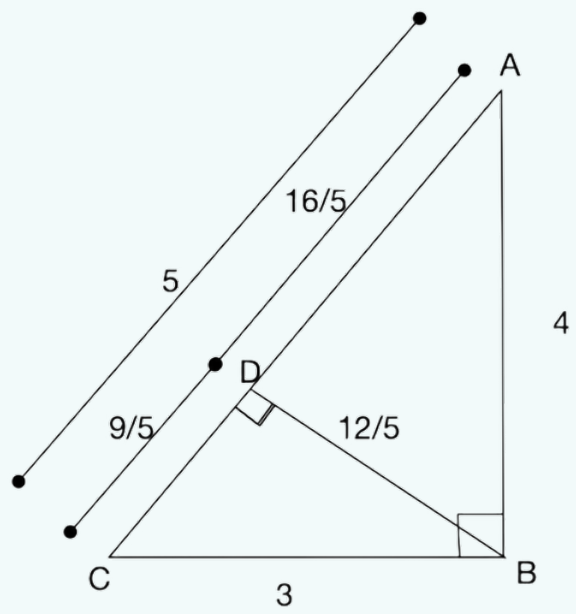
Theorem 24.3.3 (Special Properties of right triangles)
In a right triangle ABC where B is the right angle, the following triangles are similar

$$\triangle ABC \sim \triangle ADB \sim \triangle BDC$$

Length of the perpendicular to the hypotenuse (BD) $= \frac{AB \cdot BC}{AC}$

Also note that:

$$\mathrm{AD} \cdot \mathrm{CD} = \mathrm{BD}^2$$
$$\mathrm{AD} \cdot \mathrm{AC} = \mathrm{AB}^2$$
$$\mathrm{CD} \cdot \mathrm{CA} = \mathrm{CB}^2$$



24.4 Triangle Properties

Definition 24.4.1 (Cevian)**.** A cevian is any line from any vertex of a triangle to the opposite side. Medians and angle bisectors are special cases of cevians.

Definition 24.4.2 (Median)**.** A median is a line connecting a point to the midpoint of the opposite side.

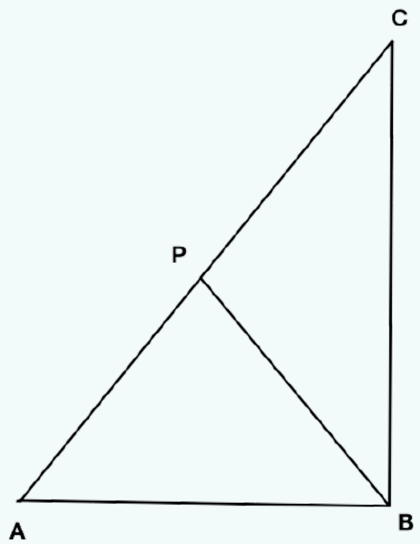
Definition 24.4.3 (Centroid)**.** In a triangle, the intersection of all 3 medians in a triangle is the centroid.

Theorem 24.4.4

The centroid of a triangle is on the median and it is $\frac{2}{3}$ of the way from from one of vertices to the midpoint of the opposite side.

Theorem 24.4.5 (Median in a Right Triangle)

In a right triangle ABC, let the median from point B intersect AC at a point P. Then $AP = BP = CP$.
Basically, in a right triangle AC is the diameter of the circumcircle, and PC, PA, and PB are radii of the circumcircle.

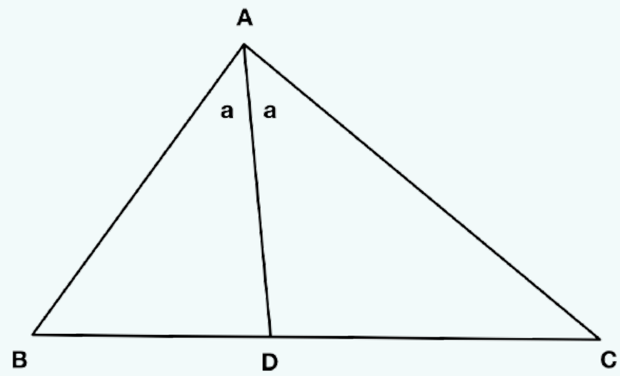


24.5 Angle Bisector Theorem

Theorem 24.5.1 (Angle Bisector Theorem)

If the line AD bisects angle A, then

$$\frac{AB}{BD} = \frac{AC}{CD}$$

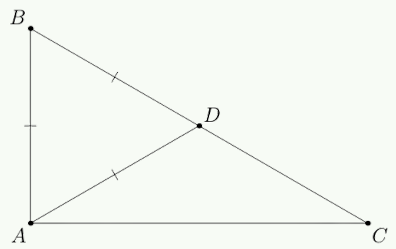


Video Lectures

More on Angle Bisector Theorem

Example 24.2 (BmMT)

In triangle △ABC, the angle trisector of ∠BAC closer to AC than AB intersects BC at D. Given that triangle △ABD is equilateral with area 1, compute the area of triangle △ABC.

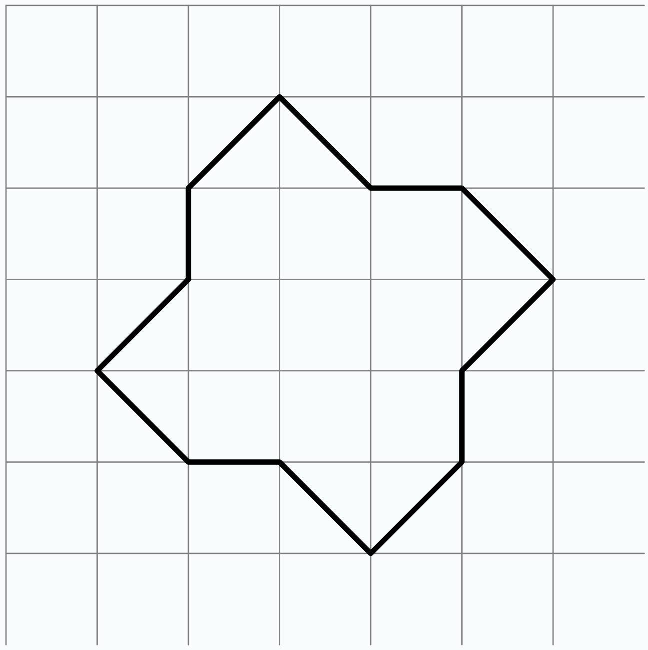


Video Solution

24.6 Practice Problems

Problem 24.6.1 (AMC 8)

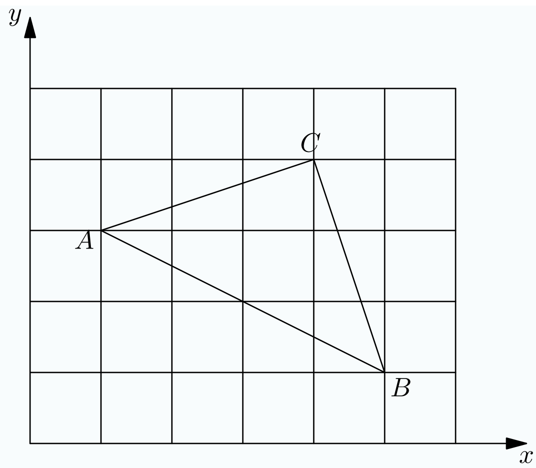
The twelve-sided figure shown has been drawn on 1 cm × 1 cm graph paper. What is the area of the figure in cm^2?



Video Solution

Problem 24.6.2 (AMC 8)

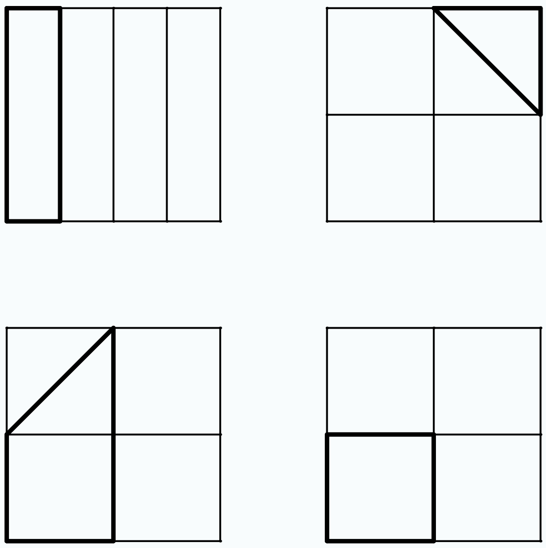
A triangle with vertices as $A = (1,3)$, $B = (5,1)$, and $C = (4,4)$ is plotted on a 6×5 grid. What fraction of the grid is covered by the triangle?



Video Solution

Problem 24.6.3 (AMC 8)

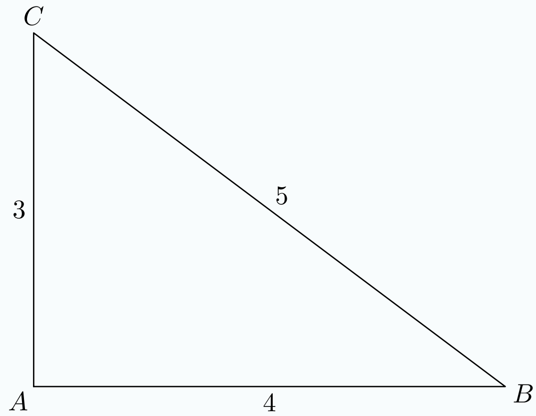
Each of the following four large congruent squares is subdivided into combinations of congruent triangles or rectangles and is partially bolded. What percent of the total area is partially bolded?



Video Solution

Problem 24.6.4 (AMC 8)

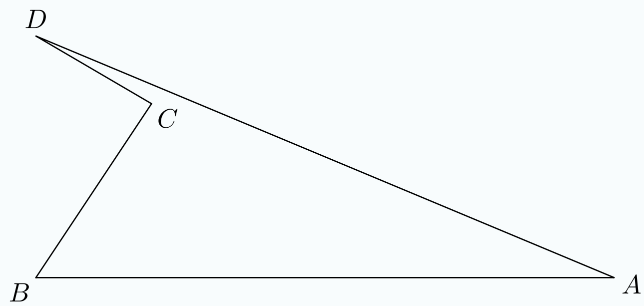
In the figure below, choose point D on $\overline{BC}$ so that $\triangle ACD$ and $\triangle ABD$ have equal perimeters. What is the area of $\triangle ABD$?



Video Solution

Problem 24.6.5 (AMC 8)

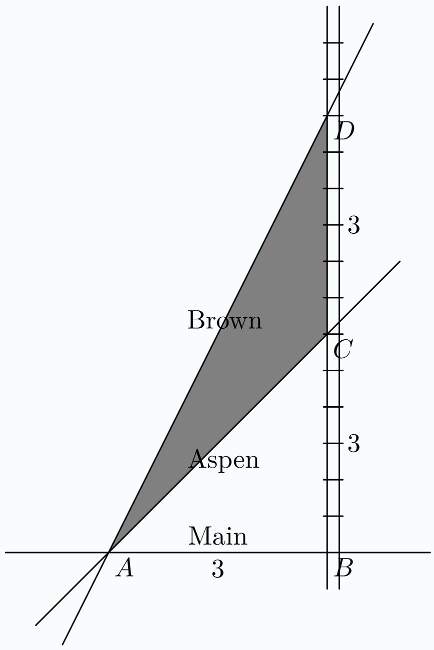
In the non-convex quadrilateral $ABCD$ shown below, $\angle BCD$ is a right angle, $AB = 12$, $BC = 4$, $CD = 3$, and $AD = 13$. What is the area of quadrilateral $ABCD$?



Video Solution

Problem 24.6.6 (AMC 8)

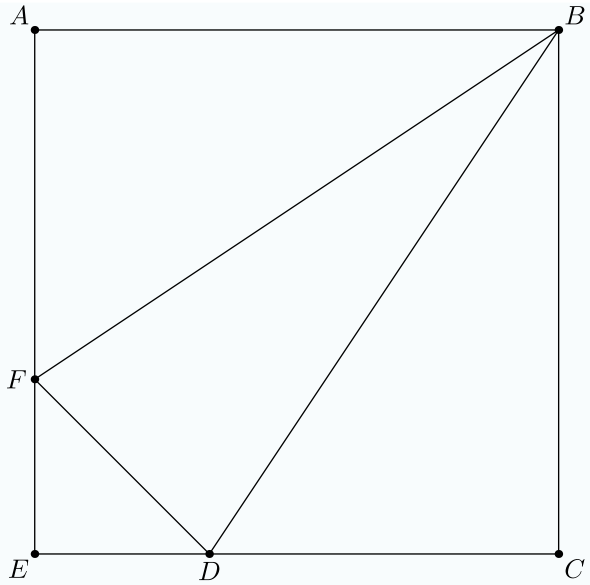
The triangular plot of ACD lies between Aspen Road, Brown Road and a railroad. Main Street runs east and west, and the railroad runs north and south. The numbers in the diagram indicate distances in miles. The width of the railroad track can be ignored. How many square miles are in the plot of land ACD?



Video Solution

Problem 24.6.7 (AMC 8)

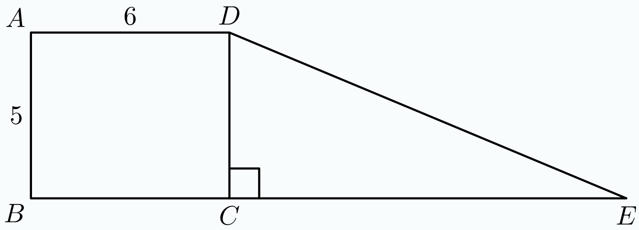
In square $ABCE$, $AF = 2FE$ and $CD = 2DE$. What is the ratio of the area of $\triangle BFD$ to the area of square $ABCE$?



Video Solution

Problem 24.6.8 (AMC 8)

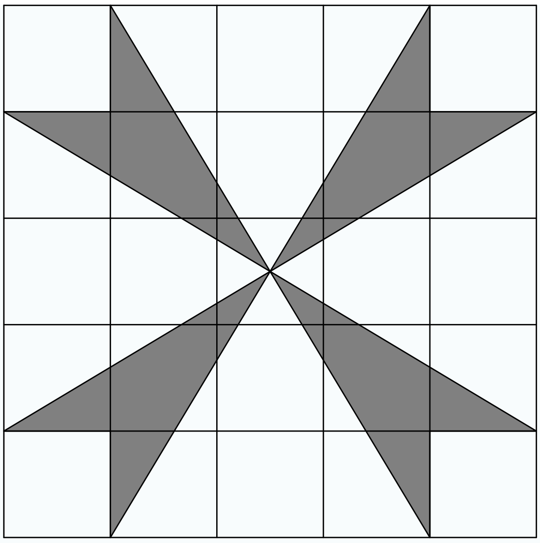
Rectangle $ABCD$ and right triangle DCE have the same area. They are joined to form a trapezoid, as shown. What is DE?



Video Solution

Problem 24.6.9 (AMC 8)

What is the area of the shaded pinwheel shown in the 5×5 grid?



Video Solution

Problem 24.6.10 (AMC 8)

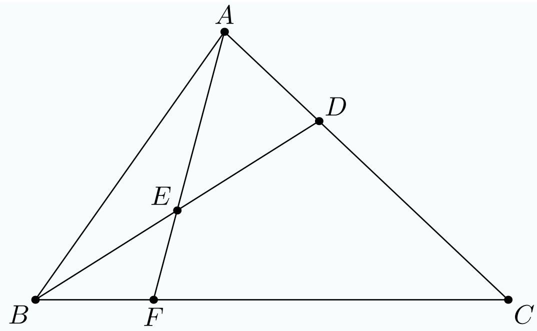
What is the area of the triangle formed by the lines $y = 5$, $y = 1 + x$, and $y = 1 - x$?

Video Solution

Additional Problems

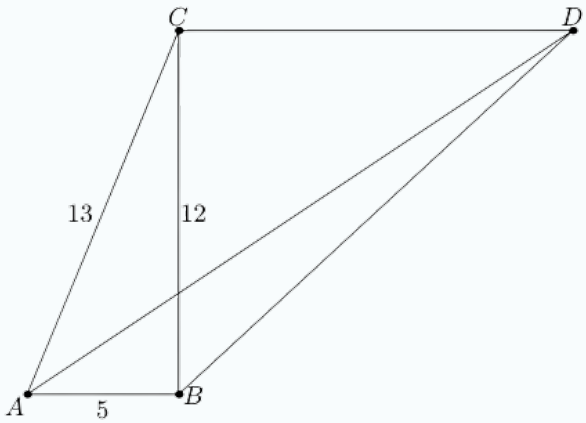
Problem 24.6.11 (AMC 8)

In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



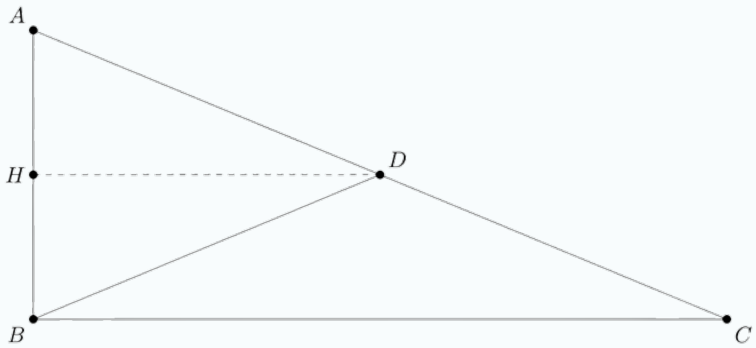
Problem 24.6.12 (BmMT)

Right triangle $\triangle ABC$ has AB = 5, BC = 12, and CA = 13. Point D lies on the angle bisector of $\angle BAC$ such that CD is parallel to AB. Compute the length of BD.



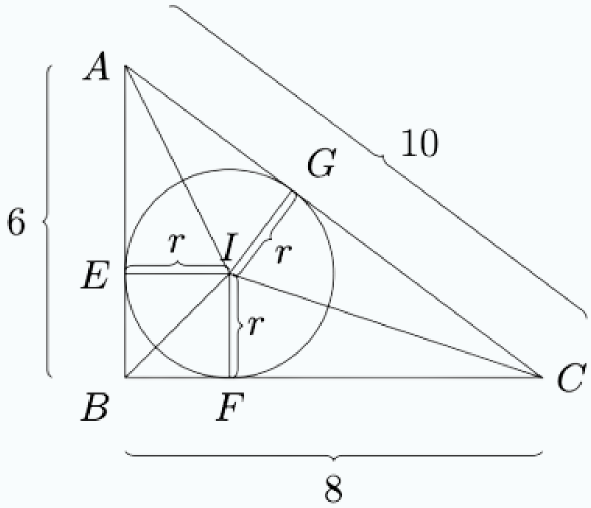
Problem 24.6.13 (BmMT)

In right triangle $\triangle ABC$ with AB = 5, BC = 12, and CA = 13, point D lies on CA such that AD = BD. The length of CD can then be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute m + n.



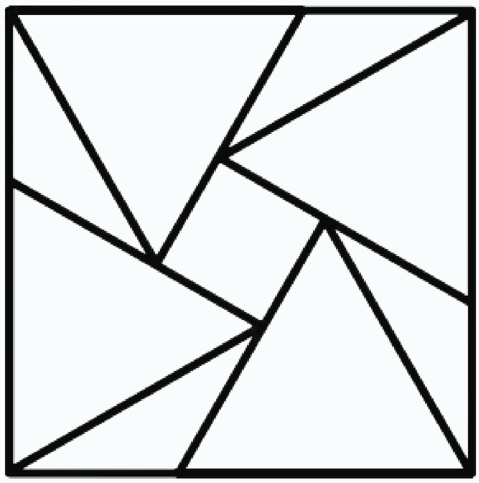
Problem 24.6.14

What is the radius of a circle inscribed in a triangle with side lengths 6, 8, and 10?



Problem 24.6.15 (MATHCOUNTS)

A unit square contains four congruent non-overlapping equilateral triangles as shown in the figure. What is the largest possible side-length of one of the triangles?

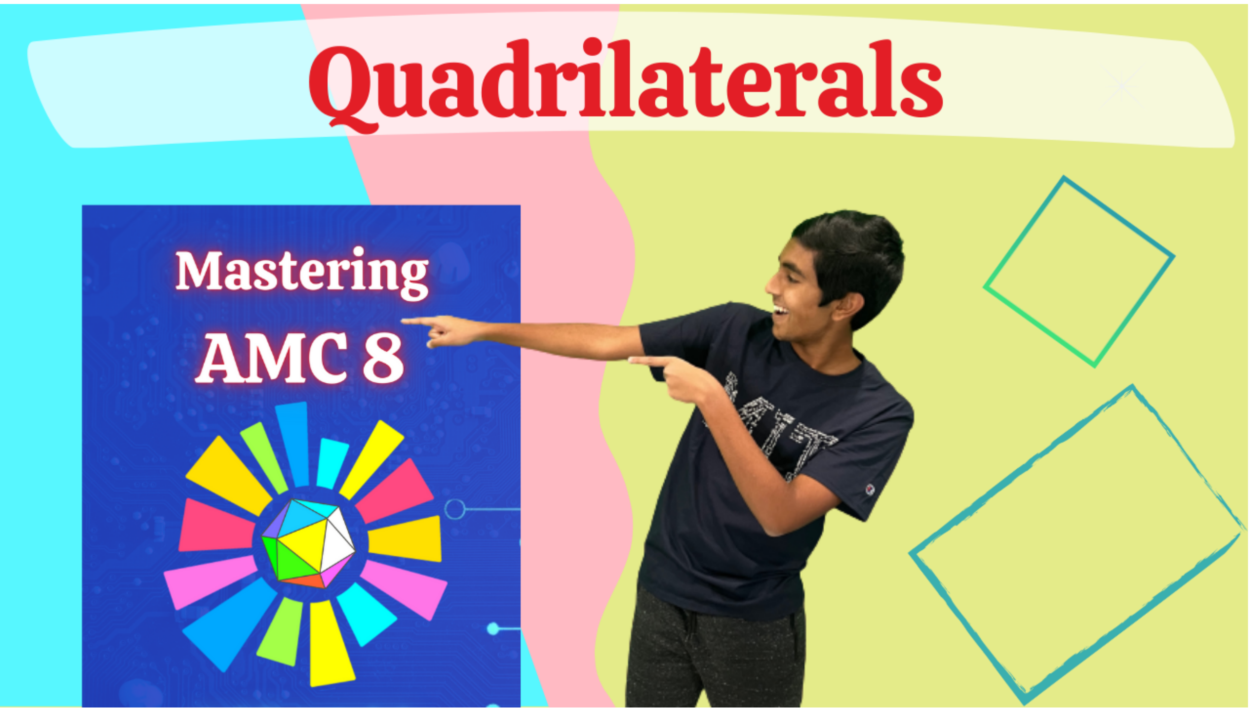


Hints

Chapter 25

Quadrilaterals

Video Lecture



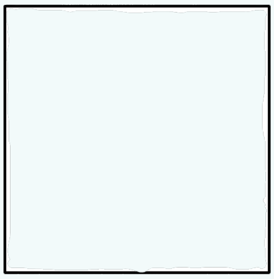
25.1 Square

Theorem 25.1.1 (Area of a Square)
Any square with side length s has an area of

$$s^2$$

and a perimeter of

$$4s$$



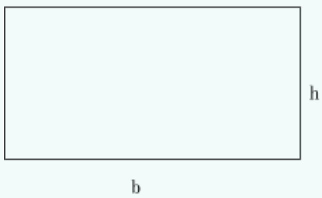
25.2 Rectangle

Theorem 25.2.1 (Area of a Rectangle)
Any rectangle with base b and height h has an area of

$$bh$$

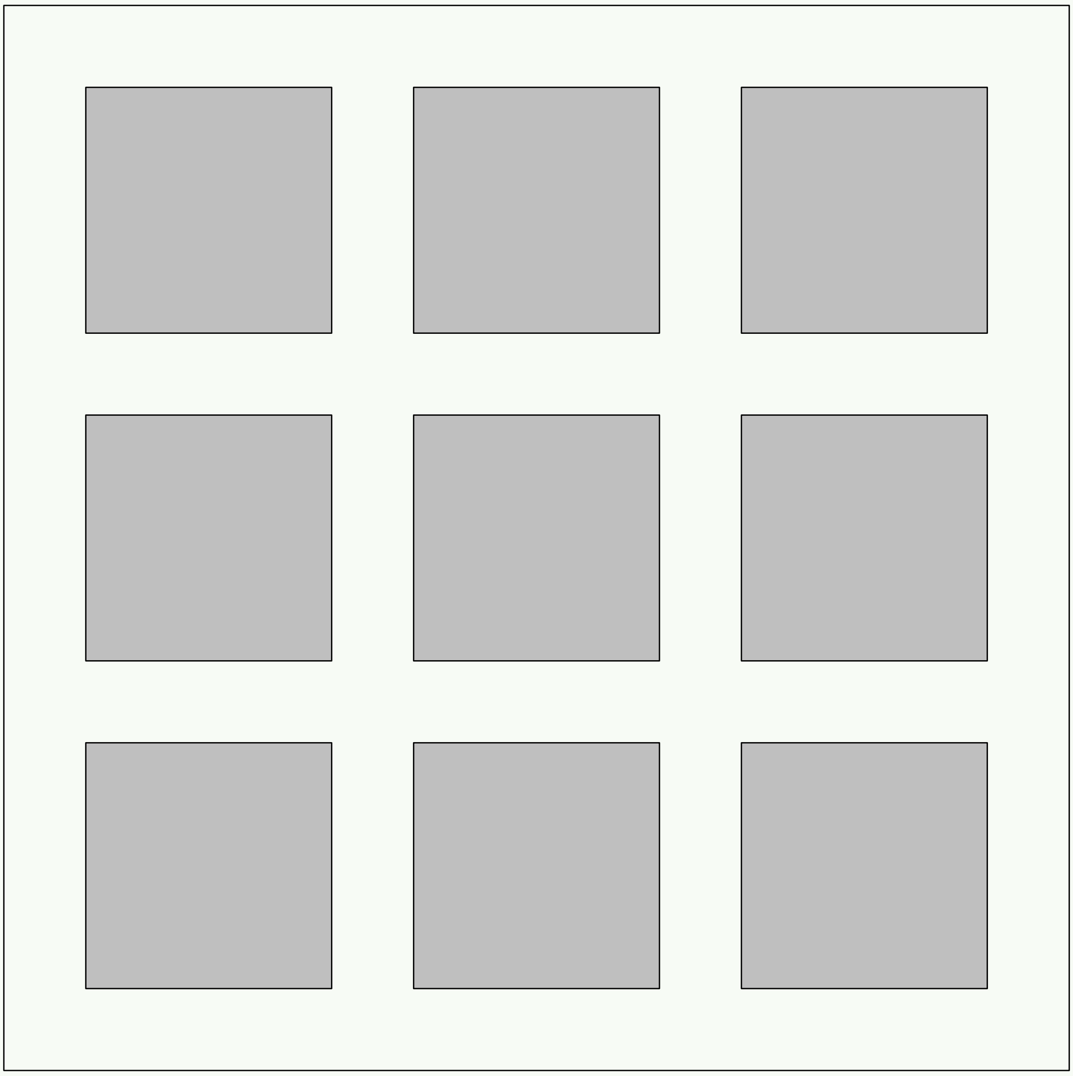
and a perimeter of

$$2b+2h$$



Example 25.1 (AMC 8)

A large square region is paved with n^2 gray square tiles, each measuring s inches on a side. A border d inches wide surrounds each tile. The figure below shows the case for $n = 3$. When $n = 24$, the 576 gray tiles cover 64% of the area of the large square region. What is the ratio $\frac{d}{s}$ for this larger value of n?



Video Solution

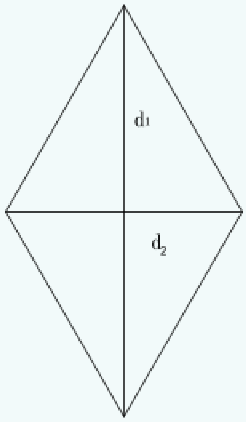
25.3 Rhombus

Theorem 25.3.1 (Area of a Rhombus)
A rhombus with diagonals d_1 and d_2 has an area of

$$\frac{1}{2}d_1d_2$$

and a perimeter of

$$2 \times \sqrt{d_1^2 + d_2^2}$$

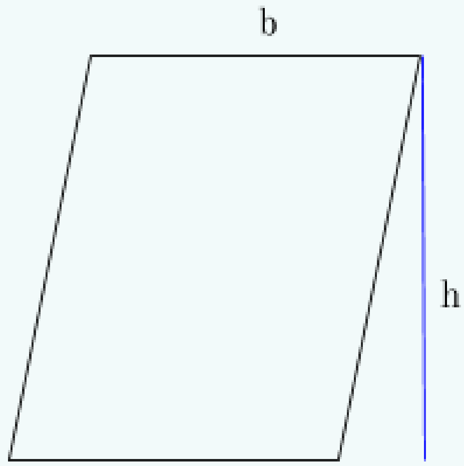


25.4 Parallelogram

Theorem 25.4.1 (Area of a Parallelogram)

A parallelogram with base b and height h has an area of

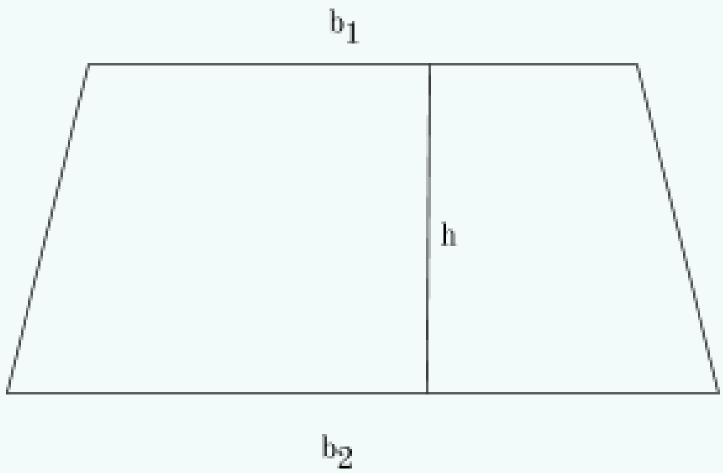
$$bh$$



25.5 Trapezoid

Theorem 25.5.1 (Area of a Trapezoid)
A trapezoid with 2 bases b_1 and b_2 and a height h has an area of

$$\frac{b_1 + b_2}{2} \cdot h$$



Example 25.2 (EMCC)

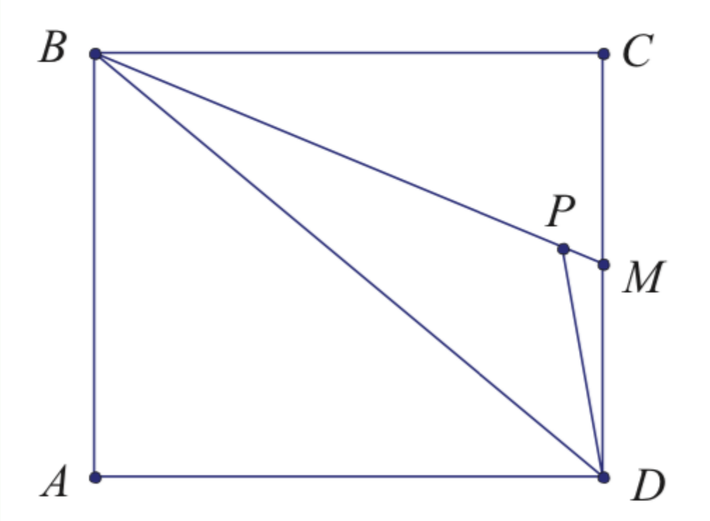
In quadrilateral PEAR, $\mathrm{PE} = 21$, $\mathrm{EA} = 20$, $\mathrm{AR} = 15$, $\mathrm{RE} = 25$, and $\mathrm{AP} = 29$. Find the area of the quadrilateral.

Video Solution

25.6 Breaking Quadrilaterals into Triangles

Example 25.3 (EMCC)

Let ABCD be a rectangle with AB = 10 and BC = 12. Let M be the midpoint of CD, and P be the point on BM such that BP = BC. Find the area of ABPD.

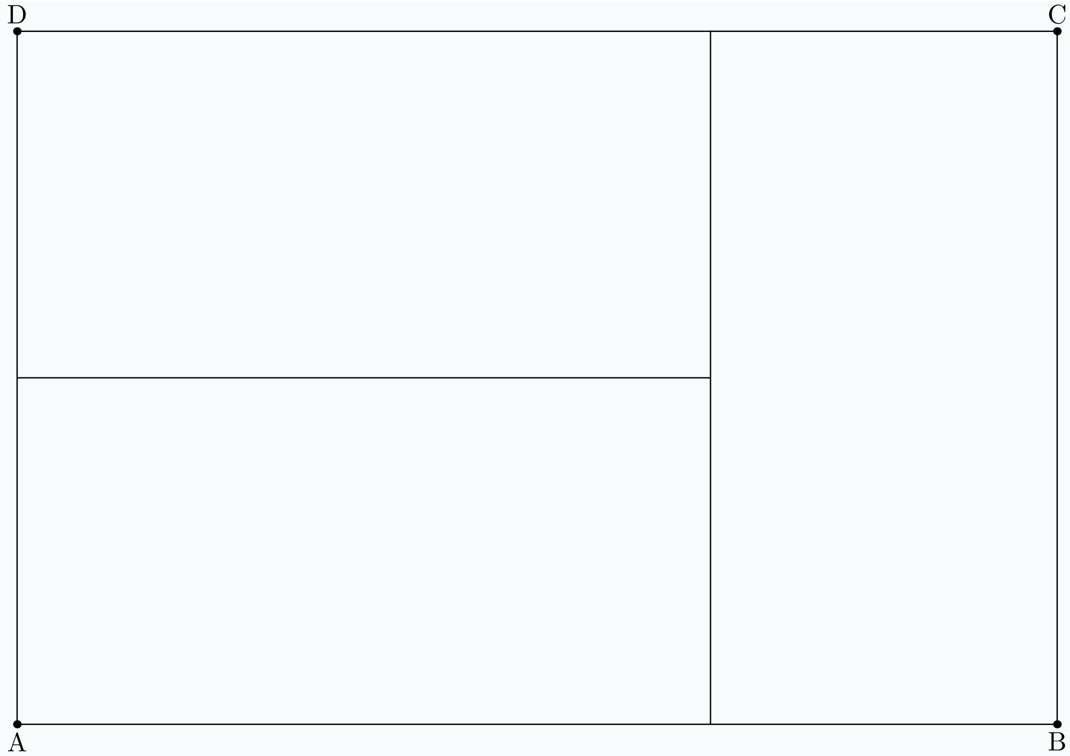


Video Solution

25.7 Practice Problems

Problem 25.7.1 (AMC 8)

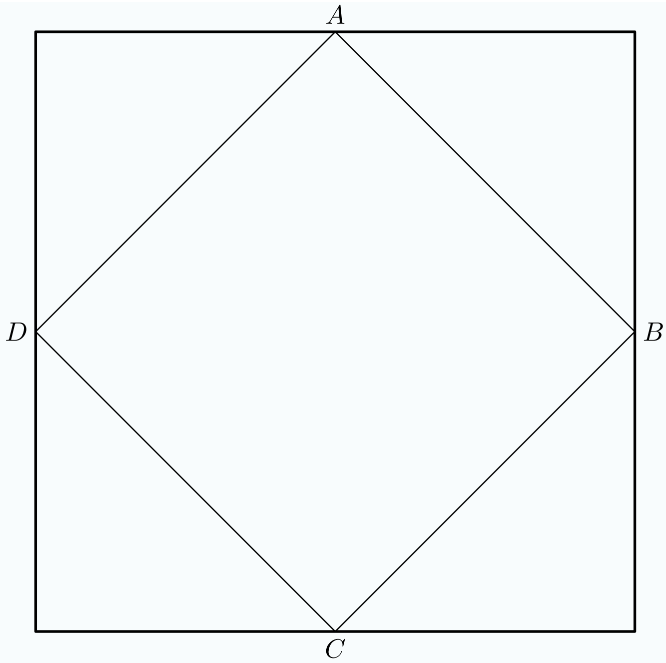
Three identical rectangles are put together to form rectangle $ABCD$, as shown in the figure below. Given that the length of the shorter side of each of the smaller rectangles is 5 feet, what is the area in square feet of rectangle $ABCD$?



[Video Solution]

Problem 25.7.2 (AMC 8)

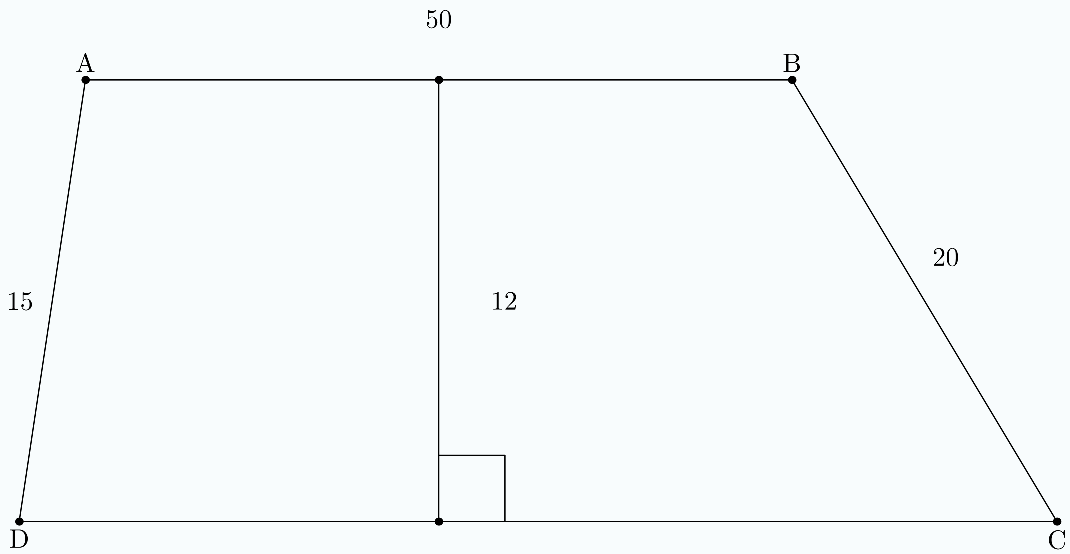
Points A, B, C and D are midpoints of the sides of the larger square. If the larger square has area 60, what is the area of the smaller square?



[Video Solution]

Problem 25.7.3 (AMC 8)

Quadrilateral $ABCD$ is a trapezoid, $AD = 15$, $AB = 50$, $BC = 20$, and the altitude is 12. What is the area of the trapezoid?



Video Solution

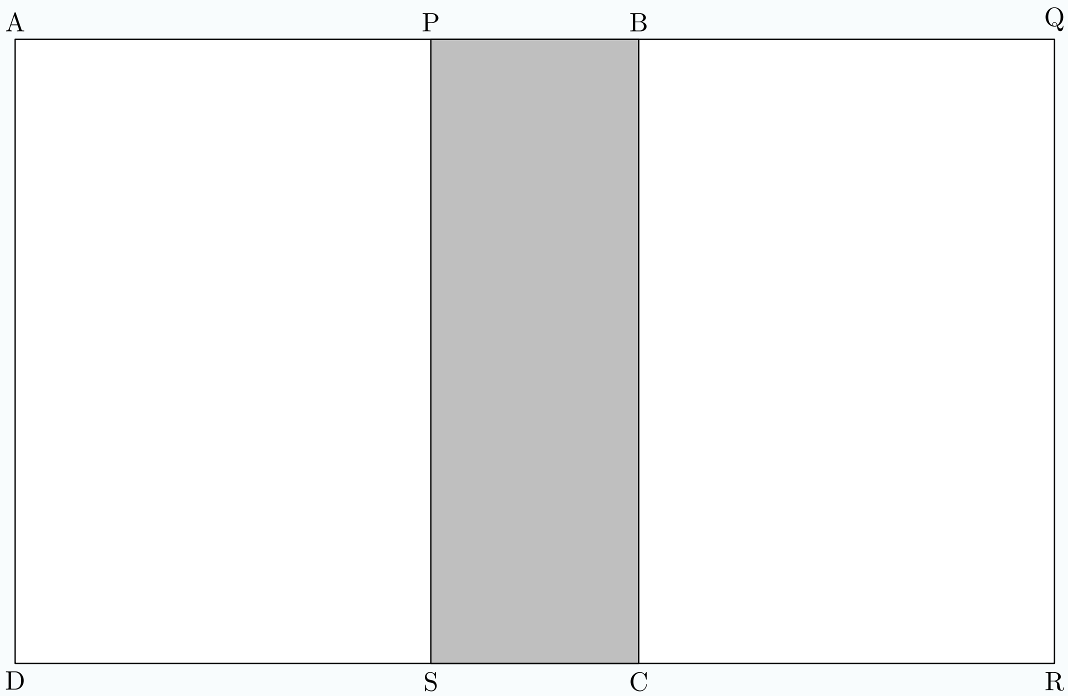
Problem 25.7.4 (AMC 8)

The midpoints of the four sides of a rectangle are $(-3,0),(2,0),(5,4)$, and $(0,4)$. What is the area of the rectangle?

Additional Problems

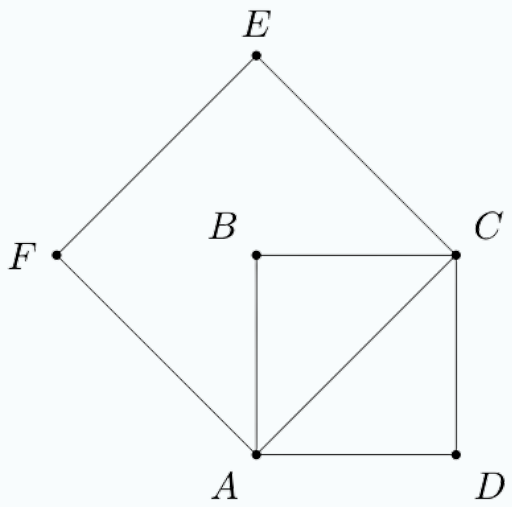
Problem 25.7.5 (AMC 8)

Two congruent squares, $ABCD$ and $PQRS$, have side length 15. They overlap to form the 15 by 25 rectangle $AQRD$ shown. What percent of the area of rectangle $AQRD$ is shaded?



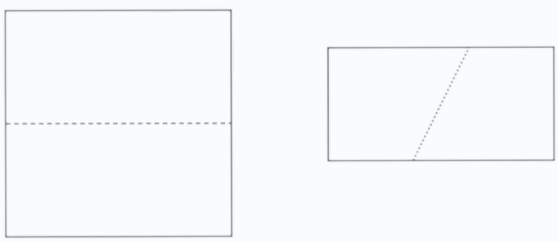
Problem 25.7.6 (BmMT)

Square ABCD has side length 2. Square ACEF is drawn such that B lies inside square ACEF. Compute the area of pentagon AFECD?



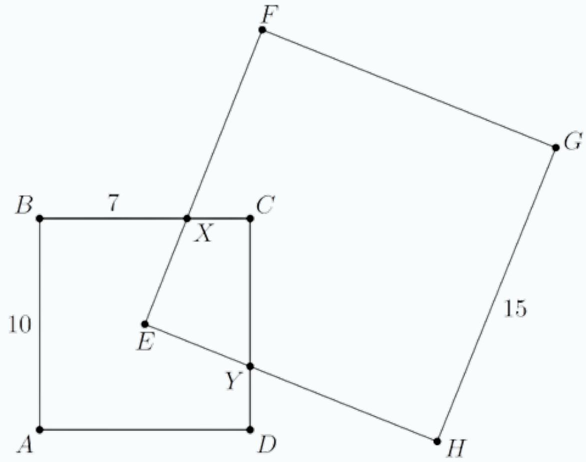
Problem 25.7.7 (BmMT)

Karthik has a paper square of side length 2. He folds the square along a crease that connects the midpoints of two opposite sides (as shown in the left diagram, where the dotted line indicates the fold). He takes the resulting rectangle and folds it such that one of its vertices lands on the vertex that is diagonally opposite. Find the area of Karthik's final figure.



Problem 25.7.8 (BmMT)

Let $ABCD$ be a square of side length 10. Let $EFGH$ be a square of side length 15 such that E is the center of $ABCD$, EF intersects BC at X, and EH intersects CD at Y. If $BX = 7$, what is the area of quadrilateral $EXCY$?



Answers

25.1 $\frac{6}{25}$

25.2 360

25.3 $\frac{1140}{13}$

25.7.1 150

25.7.2 30

25.7.3 750

25.7.4 40

25.7.5 20

25.7.6 10

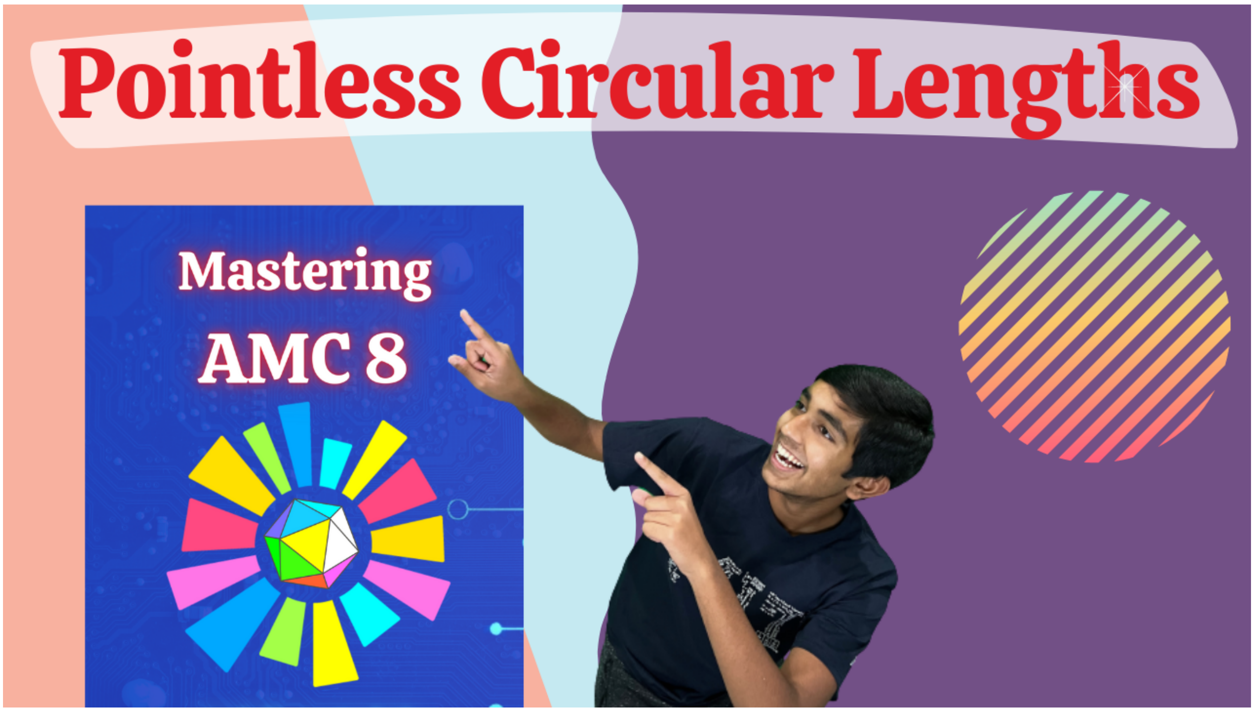
25.7.7 $\frac{11}{8}$

25.7.8 25

Chapter 26

Circles

Video Lecture


Pointless Circular Lengths
Mastering
AMC 8

26.1 Circle Properties

Theorem 26.1.1 (Area and Circumference)
A circle with radius r has

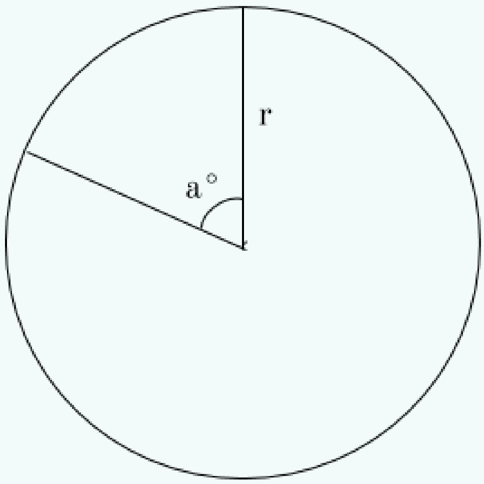
$$\text{Area} = \pi r^2$$

$$\text{Circumference} = 2\pi r$$

Theorem 26.1.2 (Arcs of a circle)
An arc of a circle with radius r and angle $a°$

$$\text{Area of a sector} = \pi r^2 \times \frac{a°}{360} = \pi \times \text{radius}^2 \times \text{fraction of circle in sector}$$

$$\text{Length of the arc} = 2\pi r \times \frac{a°}{360} = 2\pi \times \text{radius} \times \text{fraction of circle in sector}$$

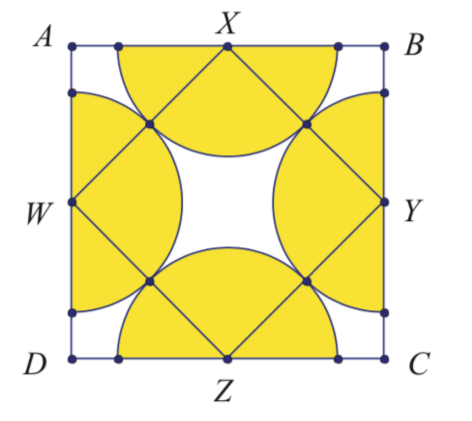


Definition 26.1.3 (Angle of an arc)**.** This is the angle that the arc makes at the center of the circle.

26.2 Circular Area

Example 26.1 (EMCC)

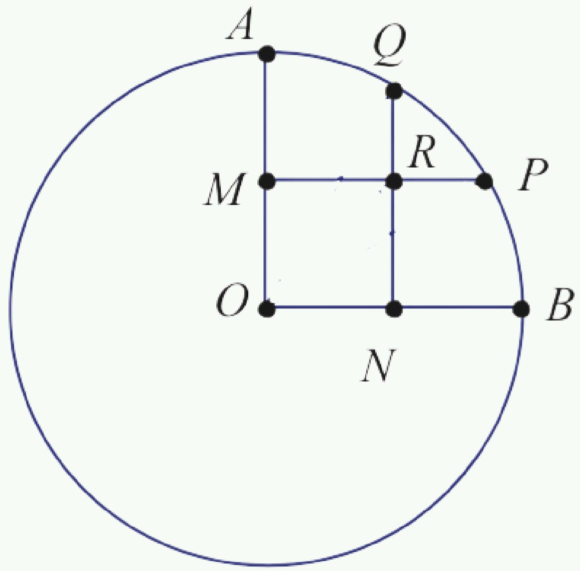
Four congruent semicircles are drawn within the boundary of a square with side length 1. The center of each semicircle is the midpoint of a side of the square. Each semicircle is tangent to two other semicircles. Region R consists of points lying inside the square but outside of the semicircles. The area of R can be written in the form $a - b\pi$, where a and b are positive rational numbers. Compute $a + b$.



Video Solution

Example 26.2 (EMCC)

Suppose points A and B lie on a circle of radius 4 with center O, such that $\angle AOB = 90°$. The perpendicular bisectors of segments OA and OB divide the interior of the circle into four regions. Find the area of the smallest region.

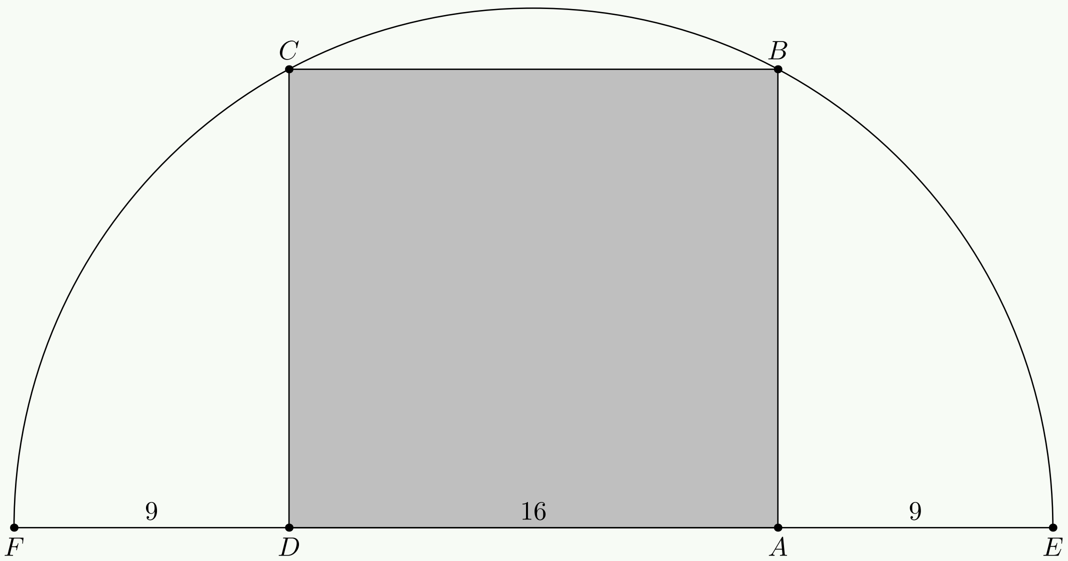


Video Solution

26.3 Length inside Circles

Example 26.3 (AMC 8)

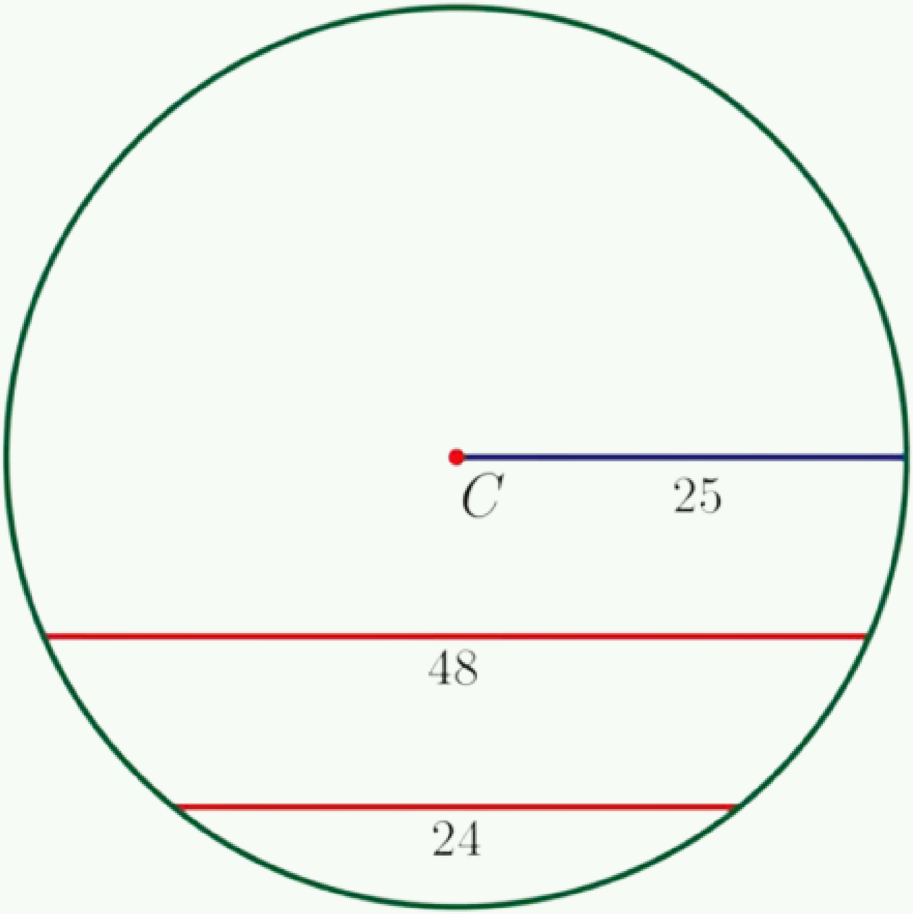
Rectangle $ABCD$ is inscribed in a semicircle with diameter $\overline{FE}$, as shown in the figure. Let $DA = 16$, and let $FD = AE = 9$. What is the area of $ABCD$?



Video Solution

Example 26.4 (Science Bowl)

A circle, centered at C, of radius of 25 has 2 parallel chords with lengths shown. To the nearest integer, what is the distance between these chords?

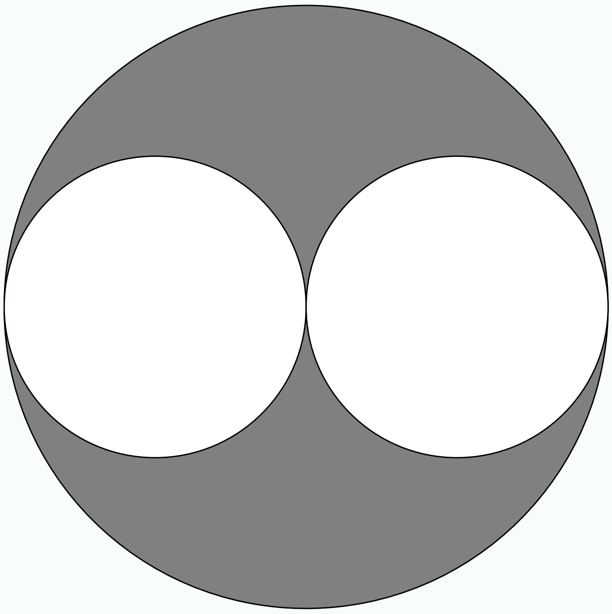


Video Solution

26.4 Practice Problems

Problem 26.4.1 (AMC 8)

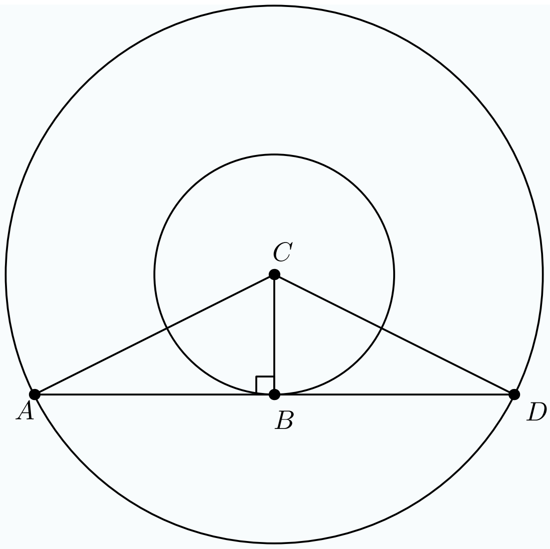
In the diagram below, a diameter of each of the two smaller circles is a radius of the larger circle. If the two smaller circles have a combined area of 1 square unit, then what is the area of the shaded region, in square units?



Video Solution

Problem 26.4.2 (AMC 8)

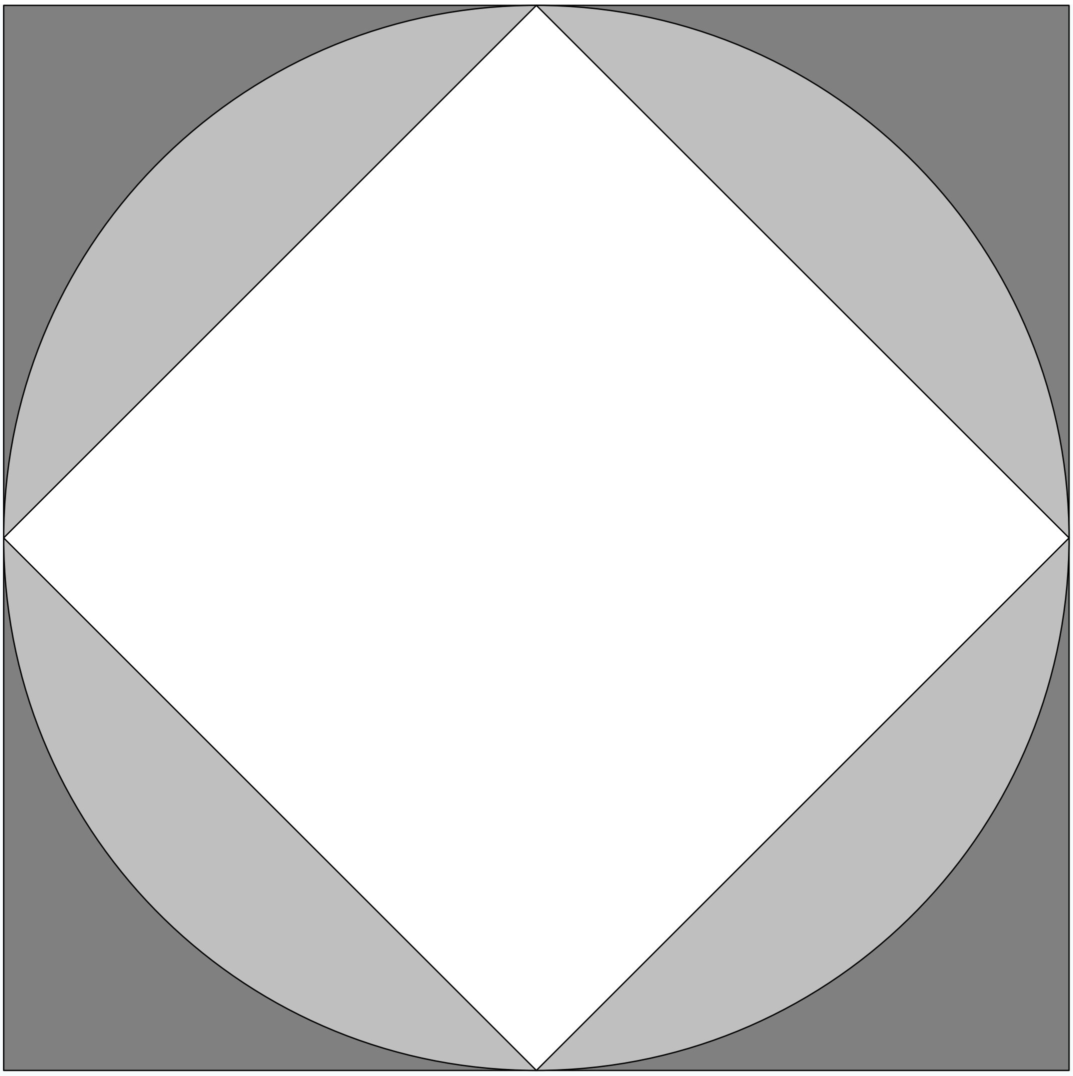
The two circles pictured have the same center C. Chord $\overline{AD}$ is tangent to the inner circle at B, AC is 10, and chord $\overline{AD}$ has length 16. What is the area between the two circles?



Video Solution

Problem 26.4.3 (AMC 8)

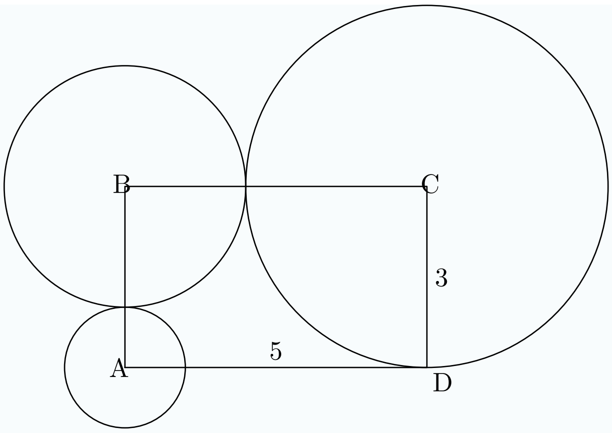
A circle with radius 1 is inscribed in a square and circumscribed about another square as shown. Which fraction is closest to the ratio of the circle's shaded area to the area between the two squares?



Video Solution

Problem 26.4.4 (AMC 8)

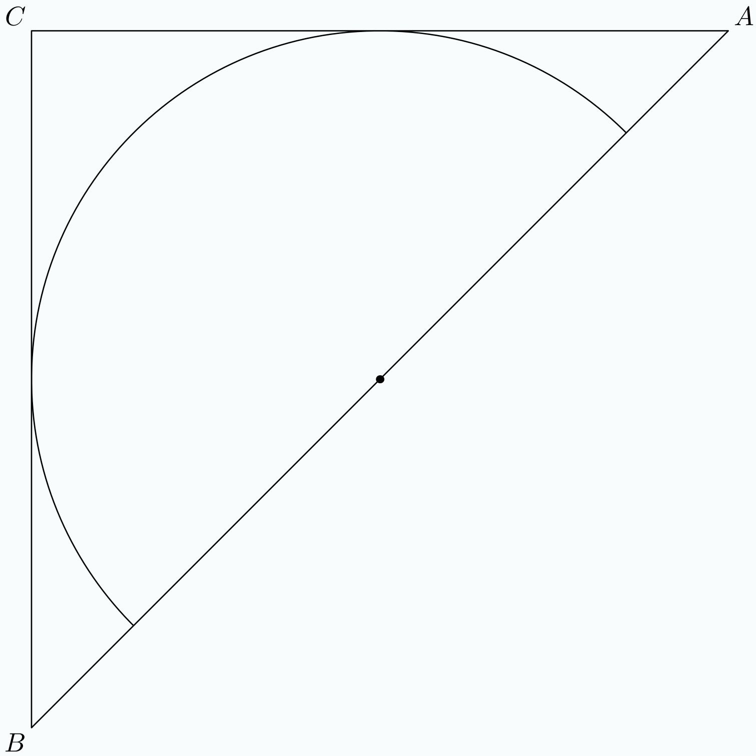
Rectangle $ABCD$ has sides $CD = 3$ and $DA = 5$. A circle with a radius of 1 is centered at A, a circle with a radius of 2 is centered at B, and a circle with a radius of 3 is centered at C. Which of the following is closest to the area of the region inside the rectangle but outside all three circles?



Video Solution

Problem 26.4.5 (AMC 8)

Isosceles right triangle ABC encloses a semicircle of area 2π. The circle has its center O on hypotenuse $\overline{AB}$ and is tangent to sides $\overline{AC}$ and $\overline{BC}$. What is the area of triangle ABC?



Video Solution

Problem 26.4.6 (AMC 8)

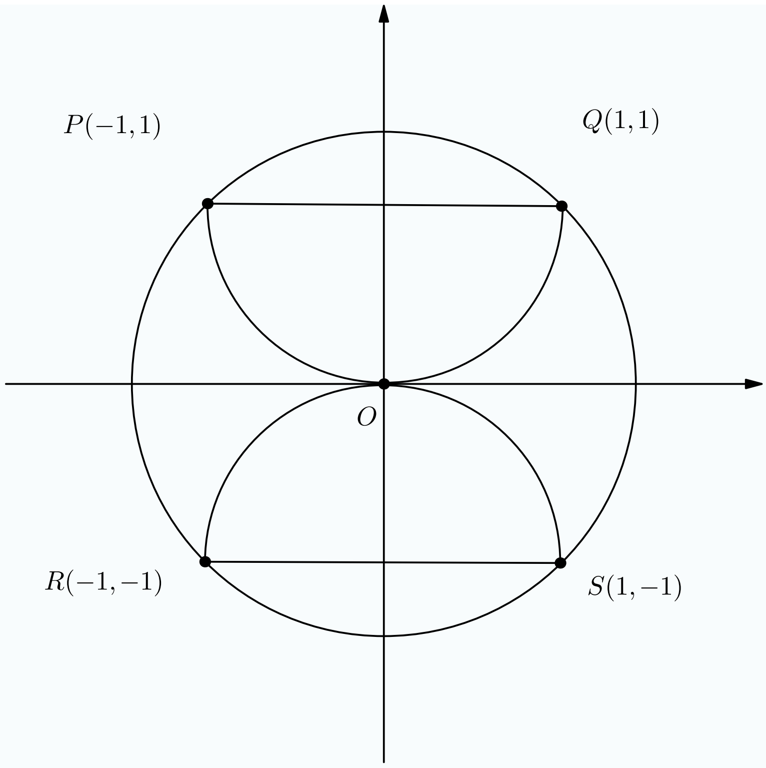
Margie's winning art design is shown. The smallest circle has radius 2 inches, with each successive circle's radius increasing by 2 inches. Which of the following is closest to the percent of the design that is black?

(A) 42 **(B)** 44 **(C)** 45 **(D)** 46 **(E)** 48

Video Solution

Problem 26.4.7 (AMC 8)

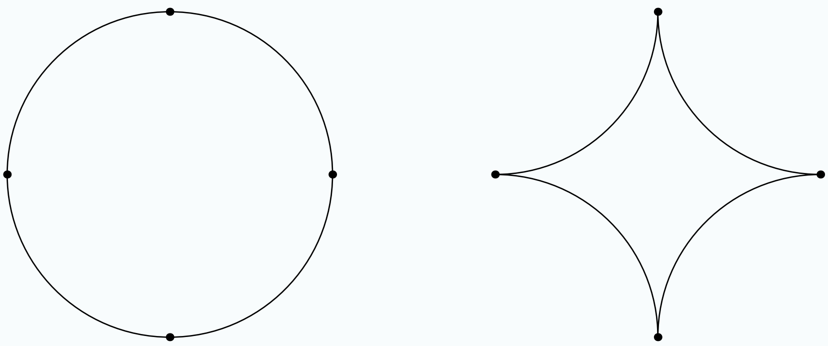
Semicircles POQ and ROS pass through the center of circle O. What is the ratio of the combined areas of the two semicircles to the area of circle O?



[Video Solution]

Problem 26.4.8 (AMC 8)

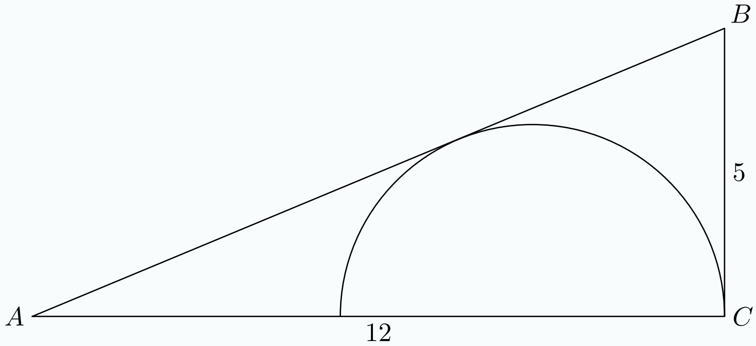
A circle of radius 2 is cut into four congruent arcs. The four arcs are joined to form the star figure shown. What is the ratio of the area of the star figure to the area of the original circle?



Video Solution

Problem 26.4.9 (AMC 8)

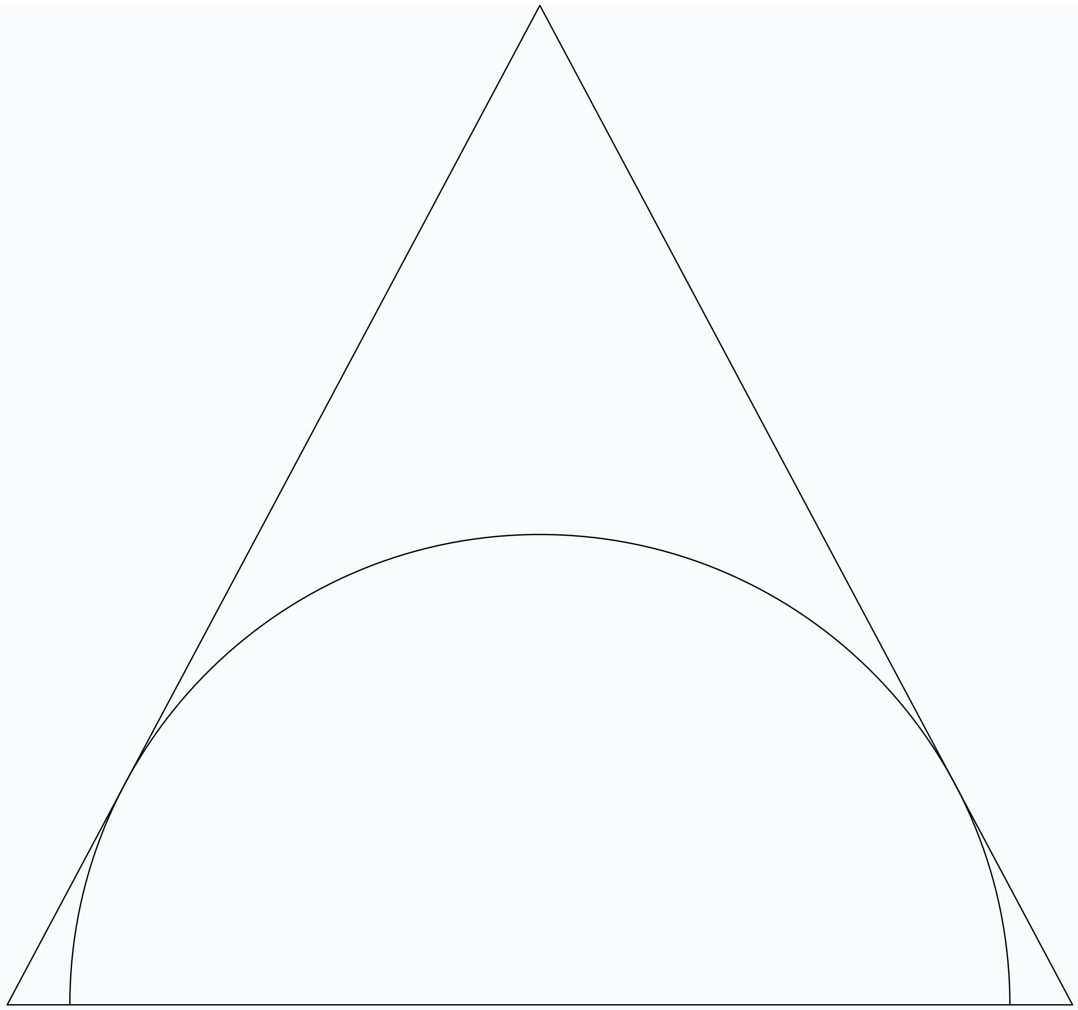
In the right triangle ABC, $AC = 12$, $BC = 5$, and angle C is a right angle. A semicircle is inscribed in the triangle as shown. What is the radius of the semicircle?



Video Solution

Problem 26.4.10 (AMC 8)

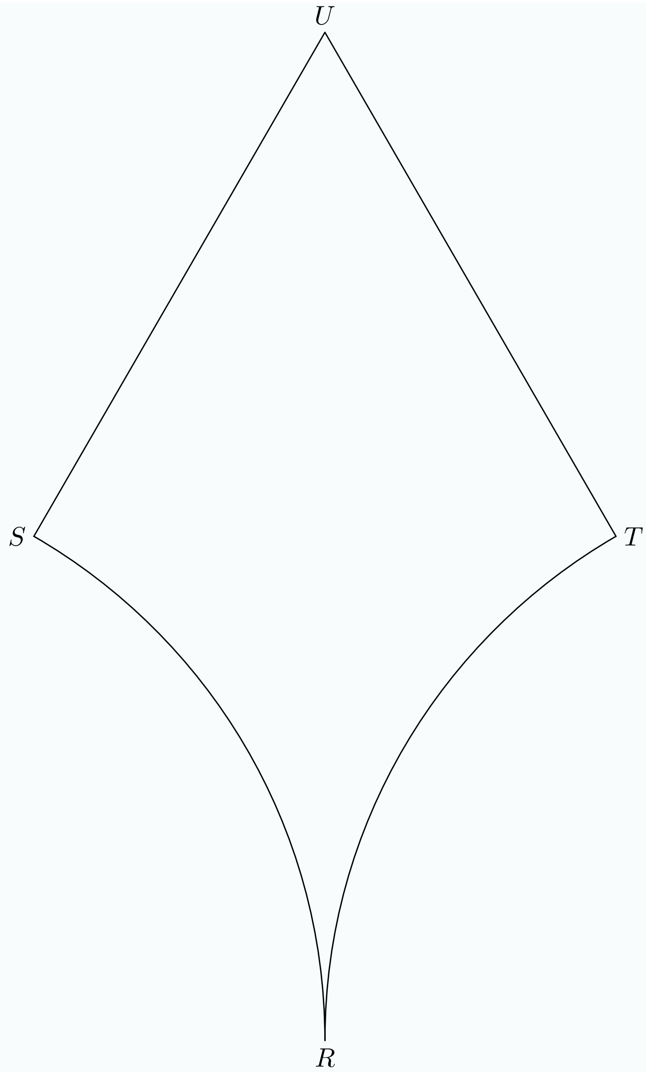
A semicircle is inscribed in an isosceles triangle with base 16 and height 15 so that the diameter of the semicircle is contained in the base of the triangle as shown. What is the radius of the semicircle?



[Video Solution]

Problem 26.4.11 (AMC 8)

In the figure shown, $\overline{US}$ and $\overline{UT}$ are line segments each of length 2, and $m\angle TUS = 60°$. Arcs $\overset{\frown}{\text{TR}}$ and $\overset{\frown}{\text{SR}}$ are each one-sixth of a circle with radius 2. What is the area of the region shown?



Video Solution

Problem 26.4.12 (AMC 12)

Three equally spaced parallel lines intersect a circle, creating three chords of lengths $38, 38$, and 34. What is the distance between two adjacent parallel lines?

Video Solution

Additional Problems

Problem 26.4.13 (EMCC)

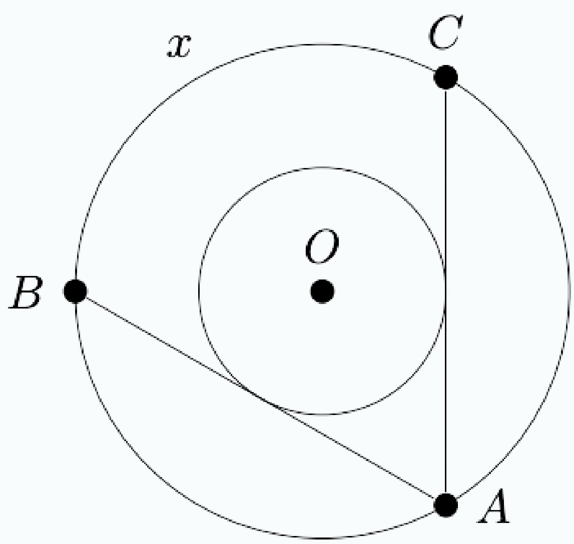
Two circles of radius 6 intersect such that they share a common chord of length 6. The total area covered may be expressed as $a\pi + \sqrt{b}$, where a and b are integers. What is $a + b$?

Problem 26.4.14 (BmMT)

A rectangle of height 10 and width 24 is inscribed in a circle. What is the circumference of that circle? Express your answer in terms of π.

Problem 26.4.15 (BmMT)

Let C1 and C2 be two circles centered at point O of radii 1 and 2, respectively. Let A be a point on C2. We draw the two lines tangent to C1 that pass through A, and label their other intersections with C2 as B and C. Let x be the length of minor arc BC, as shown. Compute x.



Answers

26.1 $\frac{5}{4}$

26.2 $\frac{4\pi}{3} + 4 - 4\sqrt{3}$

26.3 240

26.4 15

26.4.1 1

26.4.2 64π

26.4.3 $\frac{1}{2}$

26.4.4 4.0

26.4.5 8

26.4.6 42

26.4.7 $\frac{1}{2}$

26.4.8 $\frac{4-\pi}{\pi}$

26.4.9 $\frac{10}{3}$

26.4.10 $\frac{120}{17}$

26.4.11 $4\sqrt{3} - \frac{4\pi}{3}$

26.4.12 6

26.4.13 1032

26.4.14 26π

26.4.15

$$\frac{4\pi}{3}$$

Chapter 27

Similar Triangles

Video Lecture


Savage Similar Triangles
Mastering
AMC 8

27.1 Congruent Triangles

Theorem 27.1.1

For congruent triangles:

1. All the angles of the triangles are same
2. All corresponding sides are equal
3. Area, perimeter, inradius, circumradius, etc. are equal

Concept 27.1.2 (Congruence Test)

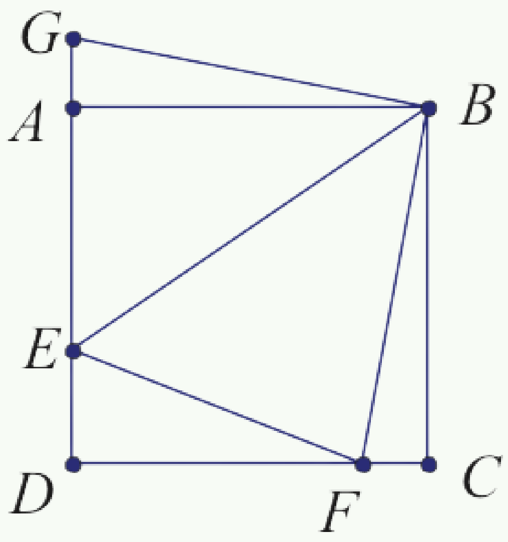
Two triangles are congruent if the three angles and sides in the triangle are the same. In other words, the triangles are the same (not necessarily same orientation). In general, triangles are congruent if:

- AAS congruence: Two angles of the triangles are same and the 1 side next to the 2 angles are equal
- SAS Congruence (Side Angle Side): Two sides are equal and the angle between the sides are equal
- SSS congruence (Side Side Side): All three sides are congruent
- HL congruence (Hypotenuse Leg): In a right triangle, hypotenuse and leg are equal
- LL congruence (LL Leg): In a right triangle, the two legs are equal

Warning: SSA is not a valid triangle congruence: if 2 corresponding sides are congruent and the angle which is not between the 2 congruent sides is equal, then the triangles are not necessarily congruent

Example 27.1 (EMCC)

In square ABCD, points E and F lie on segments AD and CD, respectively. Given that $\angle EBF = 45°$, $DE = 12$, and $DF = 35$, compute AB.



Video Solution

27.2 Similar Triangles

Concept 27.2.1 (Similarity Test)

Two triangles are similar if the three angles in the triangle are the same. In other words, the triangles are the same shape multiplied by a scale factor.

In general, triangles are similar if:

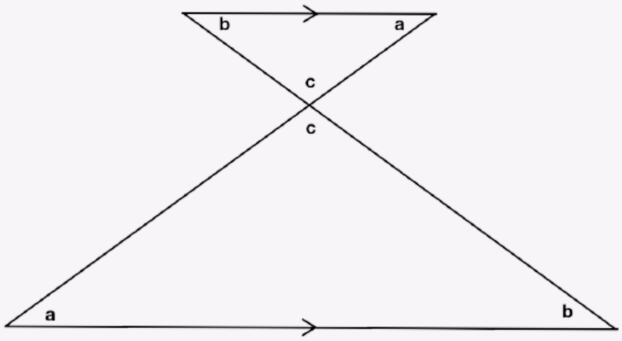
- AA similarity: Two angles of the triangles are same, which basically means that the third angle will be equal)
- SAS similarity (Side Angle Side): Two sides are proportional and the angle between the sides is equal
- SSS similarity (Side Side Side): All three sides are proportional
- HL similarity (Hypotenuse Leg): In a right triangle, the hypotenuse and leg are

proportional

- LL similarity (LL Leg): In a right triangle, the two legs are proportional

Warning: SSA does not mean triangles are similar

An easy way to detect similar triangles is if bases of triangles are parallel and the sides of the triangles are collinear (see figure below)



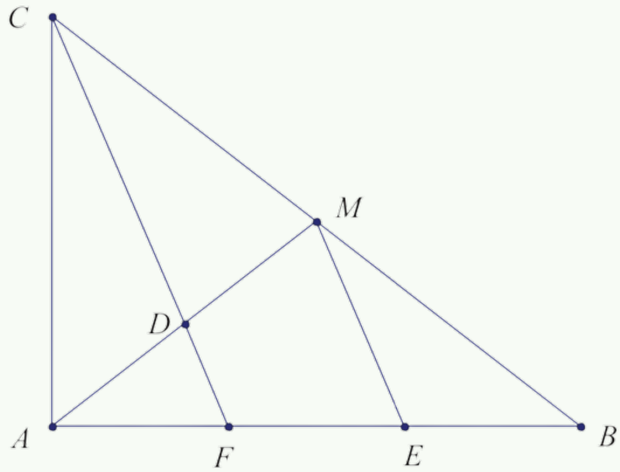
Theorem 27.2.2

For similar triangles:

1. All the angles of the triangles are same
2. All corresponding sides have same ratio
3. Area ratio is the square of side length ratio

Example 27.2 (EMCC)

In triangle ABC, $\angle BAC = 90°$, and the length of segment AB is 2011. Let M be the midpoint of BC and D the midpoint of AM. Let E be the point on segment AB such that $EM \parallel CD$. What is the length of segment BE?



Video Solution

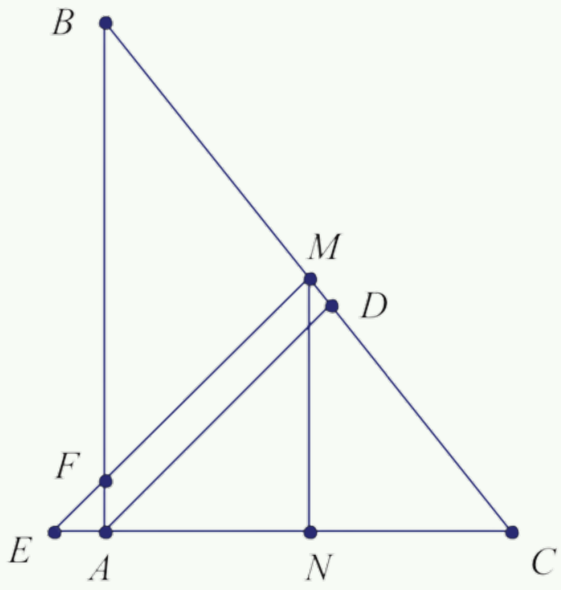
Example 27.3 (EMCC)

In triangle ABC with $BC = 5$, $CA = 13$, and $AB = 12$, Points E and F are chosen on sides AC and AB, respectively, such that $EF \parallel BC$. Given that triangle AEF and trapezoid $EFBC$ have the same perimeter, find the length of EF.

Video Solution

Example 27.4 (EMCC)

In $\triangle ABC$, $AC = 8$ and $AC < AB$. Point D lies on side BC with $\angle BAD = \angle CAD$. Let M be the midpoint of BC. The line passing through M parallel to AD intersects lines AB and AC at F and E, respectively. If $EF = \sqrt{2}$ and $AF = 1$, what is the length of segment BC?

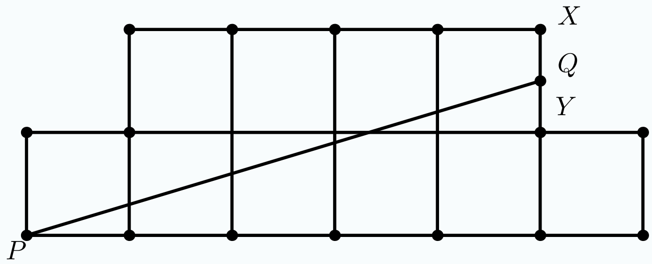


Video Solution

27.3 Practice Problems

Problem 27.3.1 (AMC 8)

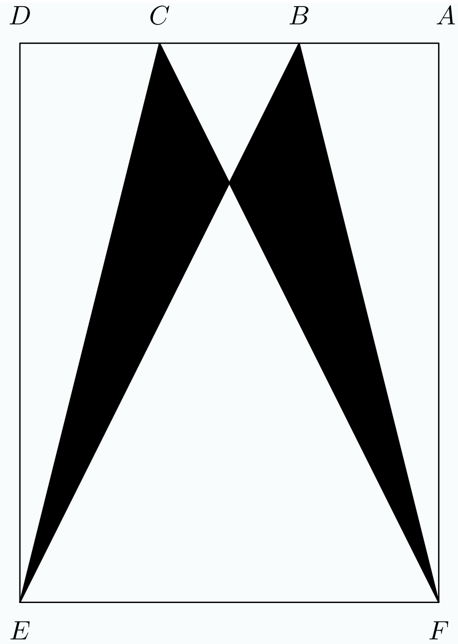
The diagram shows an octagon consisting of 10 unit squares. The portion below $\overline{PQ}$ is a unit square and a triangle with base 5. If $\overline{PQ}$ bisects the area of the octagon, what is the ratio $\frac{XQ}{QY}$?



Video Solution

Problem 27.3.2 (AMC 8)

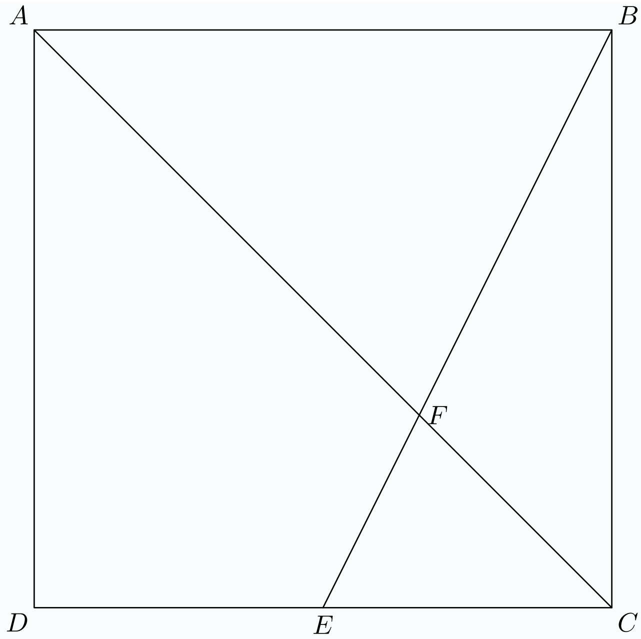
Rectangle $DEFA$ below is a 3×4 rectangle with $DC = CB = BA = 1$. The area of the "bat wings" (shaded area) is



Video Solution

Problem 27.3.3 (AMC 8)

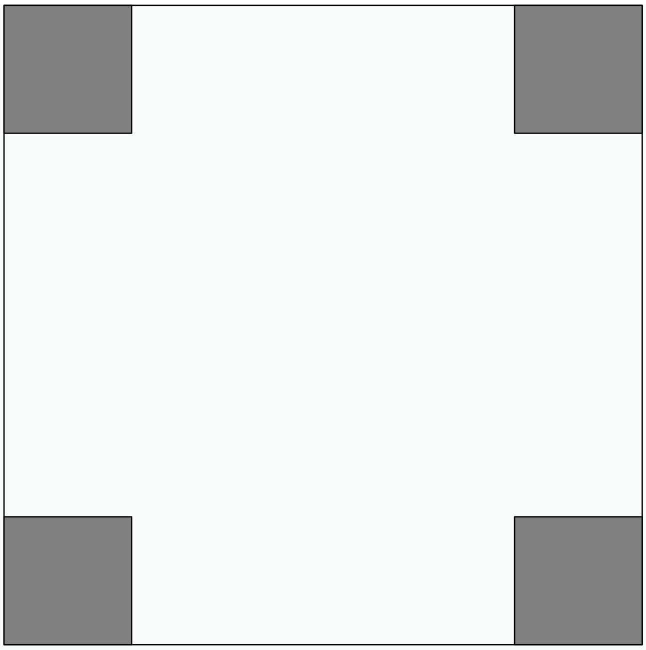
Point E is the midpoint of side $\overline{CD}$ in square $ABCD$, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$?



Video Solution

Problem 27.3.4 (AMC 8)

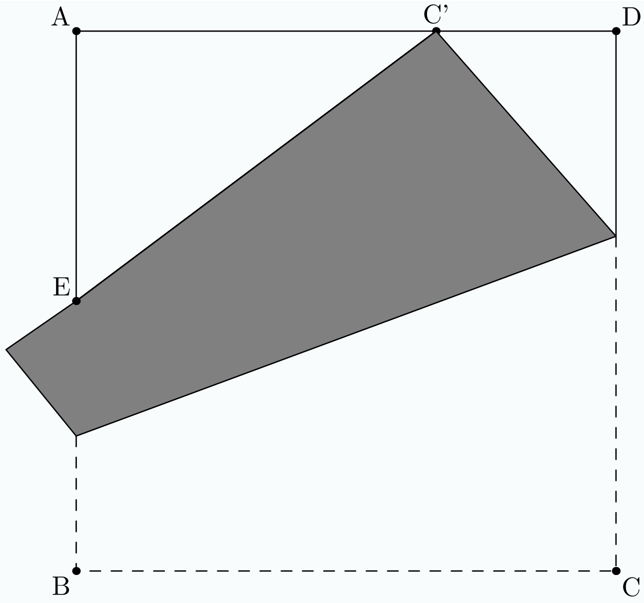
One-inch squares are cut from the corners of this 5 inch square. What is the area in square inches of the largest square that can fit into the remaining space?



Video Solution

Problem 27.3.5 (AMC 10)

A square piece of paper has side length 1 and vertices A, B, C, and D in that order. As shown in the figure, the paper is folded so that vertex C meets edge $\overline{AD}$ at point C', and edge $\overline{AB}$ at point E. Suppose that $C'D = \frac{1}{3}$. What is the perimeter of triangle $\triangle AEC'$?



Video Solution

Problem 27.3.6 (AMC 10B)

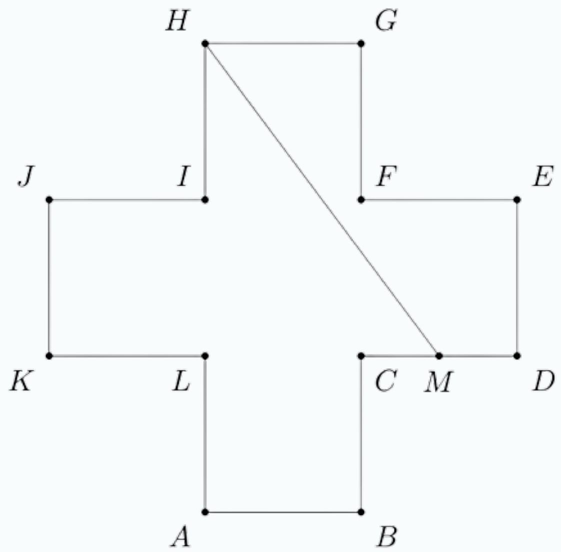
Right $\triangle ABC$ has $AB = 3, BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in $\triangle ABC$ with X and Y on $\overline{AC}, W$ on $\overline{AB}$, and Z on $\overline{BC}$. What is the side length of the square?

Video Solution

Additional Problems

Problem 27.3.7 (BmMT)

Let ABCDEFGHIJKL be the equilateral dodecagon shown below, and each angle is either 90° or 270 °. Let M be the midpoint of CD, and suppose HM splits the dodecagon into two regions. The ratio of the area of the larger region to the area of the smaller region can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute m + n.



Problem 27.3.8 (EMCC)

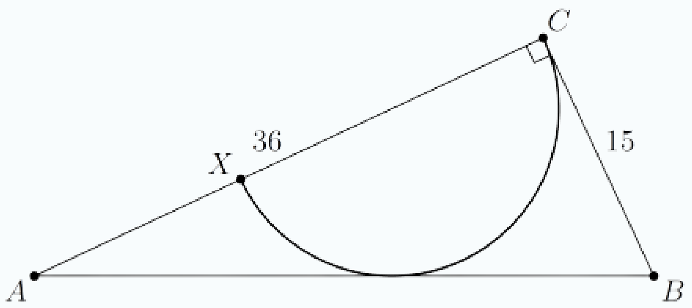
In trapezoid ABCD, points E and F lie on sides BC and AD, respectively, such that AB $\parallel$ CD $\parallel$ EF. Given that AB $=3$, EF $=5$, and CD $=6$, the ratio [ABEF] / [CDEF] can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$. (Note: [F] denotes the area of F.)

Problem 27.3.9 (EMCC)

Let ABCD be a convex quadrilateral with AB $=$ BC $=$ CA. Suppose that point P lies inside the quadrilateral with AP $=$ PD $=$ DA and $\angle$PCD $=30°$. Given that CP $=2$ and CD $=3$, compute CA.

Problem 27.3.10 (BmMT)

Let ABC be a right triangle with hypotenuse AB such that AC $=36$ and BC $=15$. A semicircle is inscribed in ABC as shown, such that the diameter XC of the semicircle lies on side AC and that the semicircle is tangent to AB. What is the radius of the semicircle?



Problem 27.3.11 (AMC 10/12)

A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

Answers

27.1 42

27.2 $\frac{2011}{3}$

27.3 3

27.4 $2\sqrt{41}$

27.3.1 $\frac{2}{3}$

27.3.2 3

27.3.3 108

27.3.4 15

27.3.5 2

27.3.6 $\frac{60}{37}$

27.3.7 10

27.3.8 27

27.3.9 $\sqrt{13}$

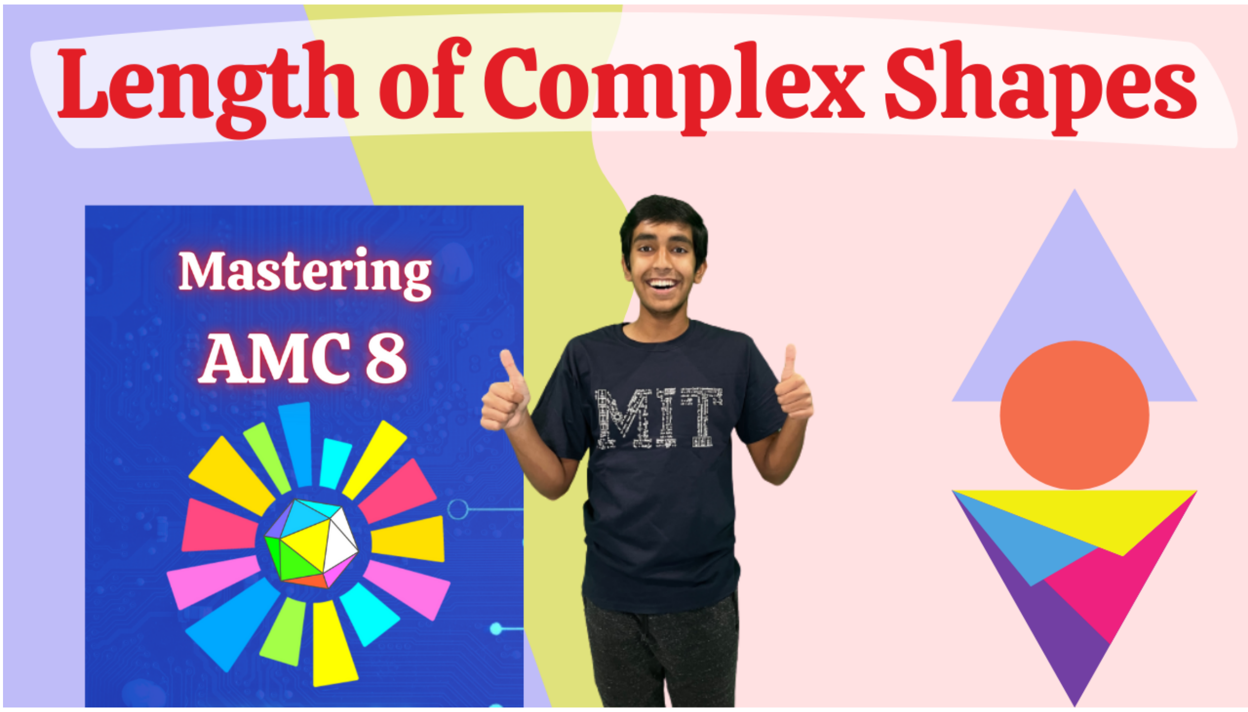
27.3.10 10

27.3.11 $\frac{37}{35}$

Chapter 28

Area and Length of Complex Shapes

Video Lecture


Length of Complex Shapes
Mastering
AMC 8
MIT

28.1 Hexagon

Theorem 28.1.1

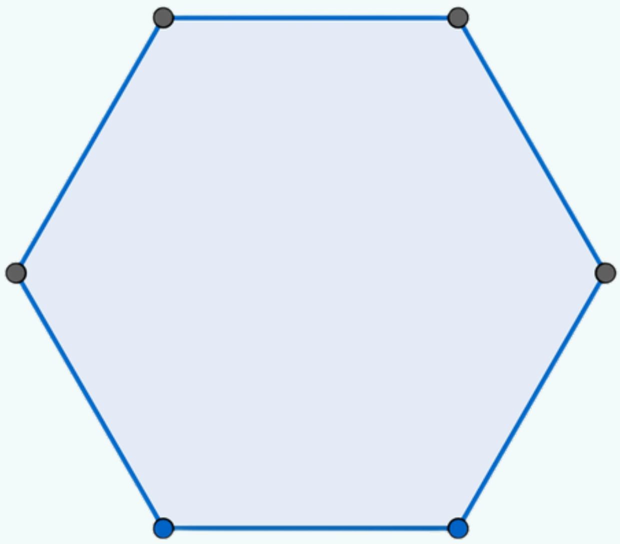
$$\text{Sum of interior angle of a regular hexagon} = (6-2)\cdot 180 = 720$$

$$\text{Interior angle of a regular hexagon} = \frac{(6-2)}{6}\cdot 180 = 120$$

$$\text{Exterior angle of a regular hexagon} = \frac{360}{6} = 60$$

$$\text{Area of a regular hexagon} = 6\cdot\frac{\sqrt{3}}{4}s^2$$

$$\text{Length of the diagonal of a regular hexagon} = 2s$$



Remark 28.1.2
A regular hexagon can be divided into 6 congruent equilateral triangles.

Example 28.1 (EMCC)

Vincent the Bug draws all the diagonals of a regular hexagon with area 720, splitting it into many pieces. Compute the area of the smallest piece.

Video Solution

Example 28.2 (EMCC)

In equilateral triangle ABC, points P and R lie on segment AB, points I and M lie on segment BC, and points E and S lie on segment CA such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PS = 2$, $RI = 3$, and $ME = 5$, compute the area of hexagon $PRIMES$.

Video Solution

28.2 Octagon

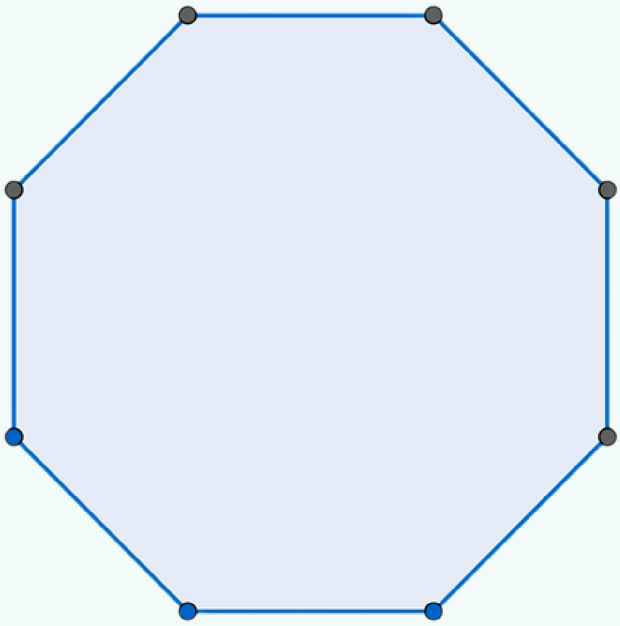
Theorem 28.2.1

$$\text{Sum of interior angle of a regular octagon} = (8-2)\cdot 180 = 1080$$

$$\text{Interior angle of a regular octagon} = \frac{(8-2)}{8}\cdot 180 = 135$$

$$\text{Exterior angle of a regular octagon} = \frac{360}{8} = 45$$

$$\text{Area of a regular octagon} = 2(1+\sqrt{2})s^2$$



28.3 Area of Complex Shapes

Concept 28.3.1

Tricks to finding the area of complex shapes

- Divide the shape into "nicer" areas which are easier to calculate
- Extend Lines
  - You generally want to extend lines when they form nicer shapes/areas to work with, such as triangles
- Break up areas
  - A common way to do so is to drop altitudes as doing so generally allows you to form right triangles

Remark 28.3.2
A common technique is to find the area of shapes and then find the area of a shape in terms of a variable (like altitude, inradius, circumradius, etc.) and then solve for that variable.

Example 28.3 (EMCC)

In rectangle $TRIG$, points A and L lie on sides TG and TR respectively such that $TA = AG$ and $TL = 2LR$. Diagonal GR intersects segments IL and IA at B and E respectively. Suppose that the area of the convex pentagon with vertices $TABLE$ is equal to 21. What is the area of $TRIG$?

Video Solution

28.4 Length of complex shapes

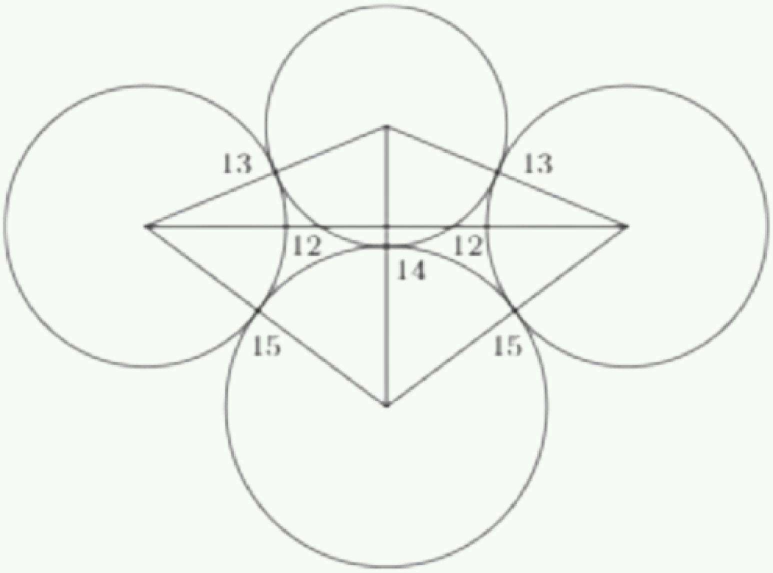
Concept 28.4.1
Finding Length of Complex Shapes

- Having equal angles means equal lengths and vice versa
- Be on the lookout for 90 degree angles, as you can use Pythagorean theorem
- Split the length into multiple components by using some of these techniques
  - Drawing extra lines
  - Dropping Altitudes

- Extending lines to create similar triangles, special triangles, etc. and then subtracting the extra length

Example 28.4 (EMCC)

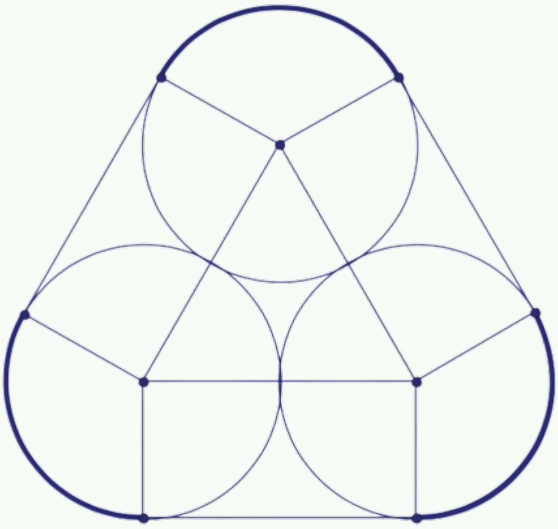
Two circles, with radius 6 and radius 8, are externally tangent to each other. Two more circles, of radius 7, are placed on either side of this configuration, so that they are both externally tangent to both of the original two circles. Out of these 4 circles, what is the maximum distance between any two centers?



Video Solution

Example 28.5 (EMCC)

David decides he wants to join the West African Drumming Ensemble, and thus he goes to the store and buys three large cylindrical drums. In order to ensure none of the drums drop on the way home, he ties a rope around all of the drums at their mid sections so that each drum is next to the other two. Suppose that each drum has a diameter of 3.5 feet. David needs m feet of rope. Given that $m = a\pi + b$, where a and b are rational numbers, find $a + b$.

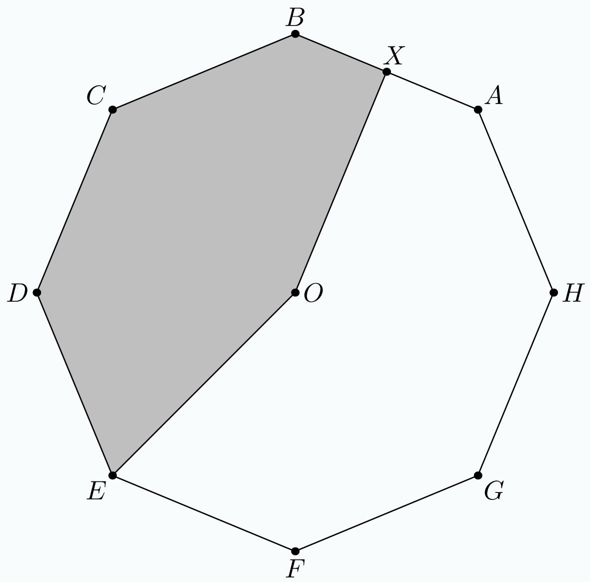


[Video Solution]

28.5 Practice Problems

Problem 28.5.1 (AMC 8)

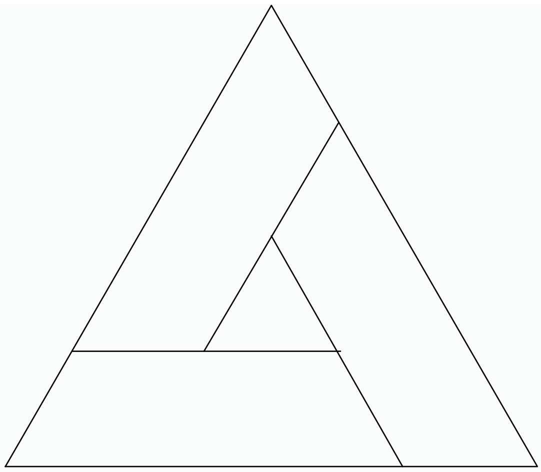
Point O is the center of the regular octagon $ABCDEFGH$, and X is the midpoint of the side $\overline{AB}$. What fraction of the area of the octagon is shaded?



Video Solution

Problem 28.5.2 (AMC 8)

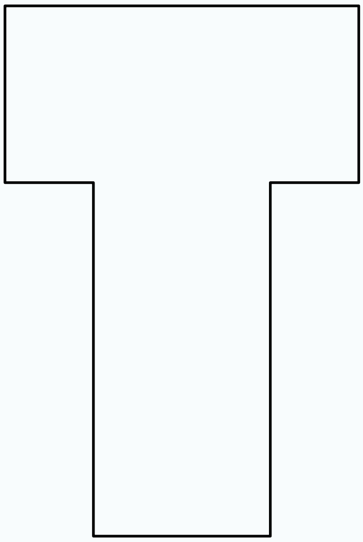
In the figure, the outer equilateral triangle has area 16, the inner equilateral triangle has area 1, and the three trapezoids are congruent. What is the area of one of the trapezoids?



Video Solution

Problem 28.5.3 (AMC 8)

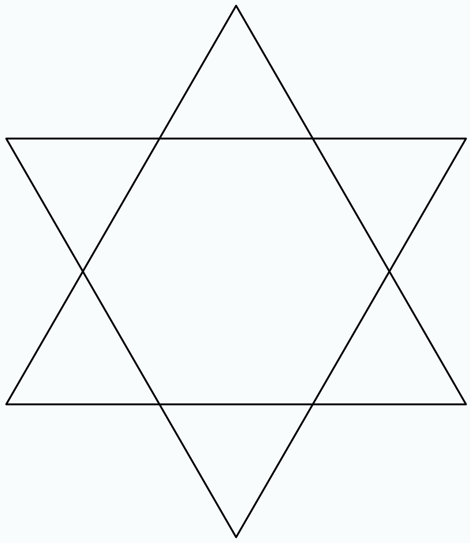
The letter T is formed by placing two 2×4 inch rectangles next to each other, as shown. What is the perimeter of the T, in inches?



Video Solution

Problem 28.5.4 (AMC 8)

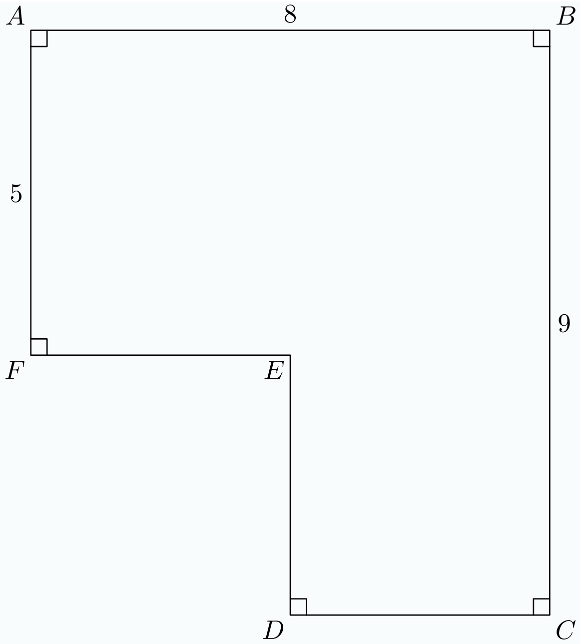
A unit hexagram is composed of a regular hexagon of side length 1 and its 6 equilateral triangular extensions, as shown in the diagram. What is the ratio of the area of the extensions to the area of the original hexagon?



Video Solution

Problem 28.5.5 (AMC 8)

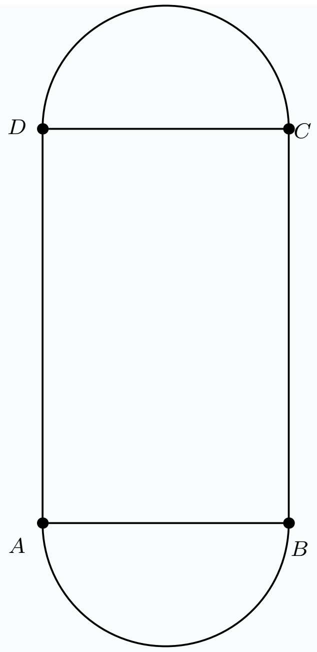
The area of polygon $ABCDEF$ is 52 with $AB = 8$, $BC = 9$ and $FA = 5$. What is $DE + EF$?



Video Solution

Problem 28.5.6 (AMC 8)

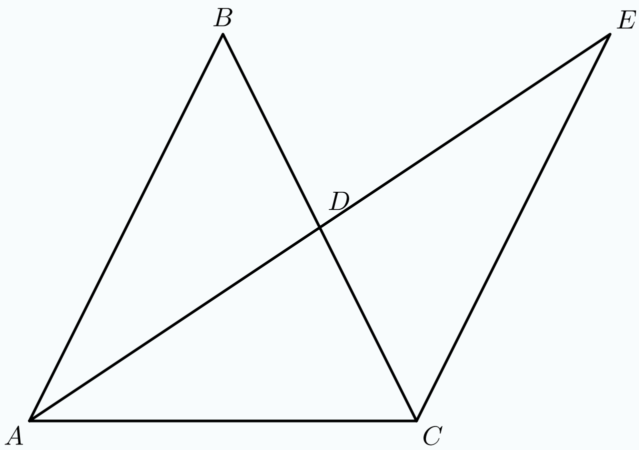
A decorative window is made up of a rectangle with semicircles on either end. The ratio of AD to AB is $3:2$, and AB is 30 inches. What is the ratio of the area of the rectangle to the combined areas of the semicircles?



Video Solution

Problem 28.5.7 (AMC 8)

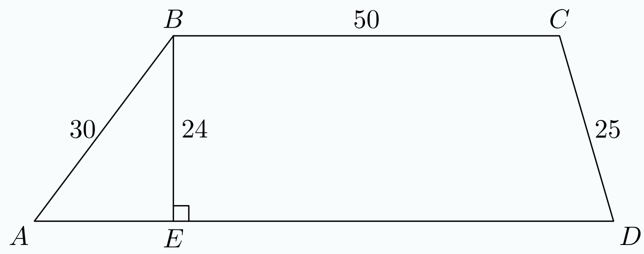
Triangle ABC is an isosceles triangle with $\overline{AB} = \overline{BC}$. Point D is the midpoint of both $\overline{BC}$ and $\overline{AE}$, and $\overline{CE}$ is 11 units long. Triangle ABD is congruent to triangle ECD. What is the length of $\overline{BD}$?



Video Solution

Problem 28.5.8 (AMC 8)

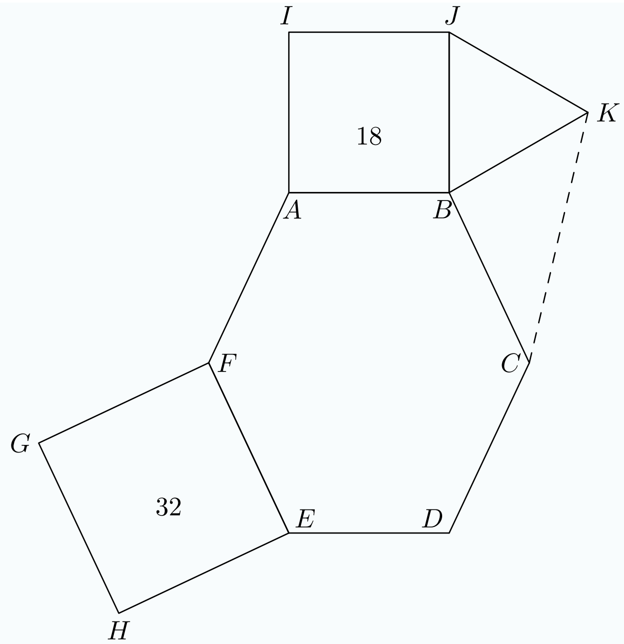
What is the perimeter of trapezoid $ABCD$?



Video Solution

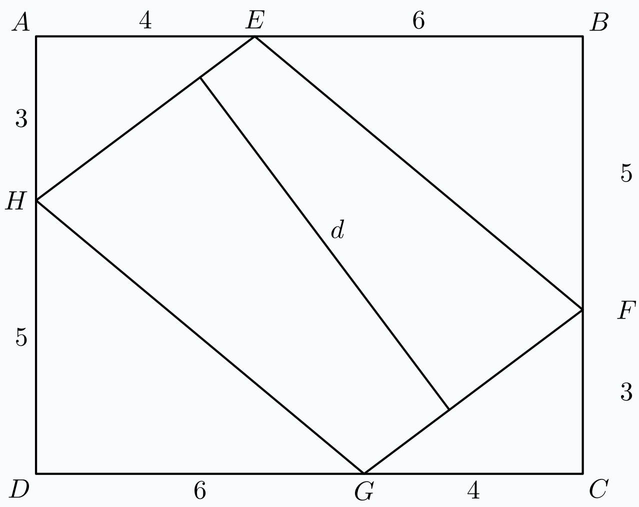
Problem 28.5.9 (AMC 8)

In the given figure hexagon $ABCDEF$ is equiangular, $ABJI$ and $FEHG$ are squares with areas 18 and 32 respectively, $\triangle JBK$ is equilateral and $FE = BC$. What is the area of $\triangle KBC$?



Problem 28.5.10 (AMC 8)

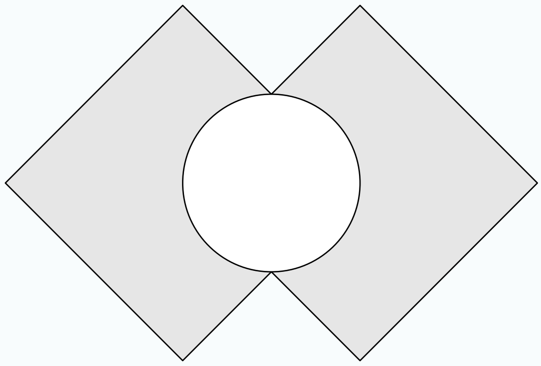
In the figure, $ABCD$ is a rectangle and $EFGH$ is a parallelogram. Using the measurements given in the figure, what is the length d of the segment that is perpendicular to $\overline{HE}$ and $\overline{FG}$?



Video Solution

Problem 28.5.11 (AMC 8)

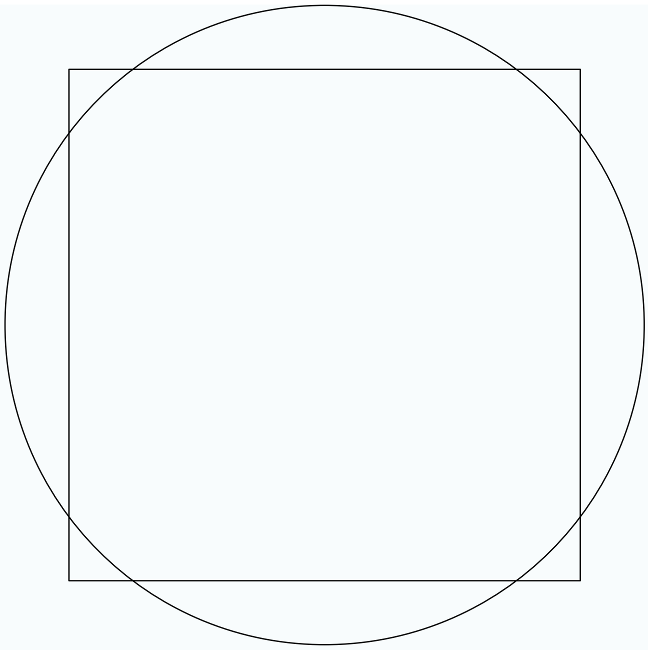
Two 4×4 squares intersect at right angles, bisecting their intersecting sides, as shown. The circle's diameter is the segment between the two points of intersection. What is the area of the shaded region created by removing the circle from the squares?



Video Solution

Problem 28.5.12 (AMC 8)

A square with side length 2 and a circle share the same center. The total area of the regions that are inside the circle and outside the square is equal to the total area of the regions that are outside the circle and inside the square. What is the radius of the circle?



Video Solution

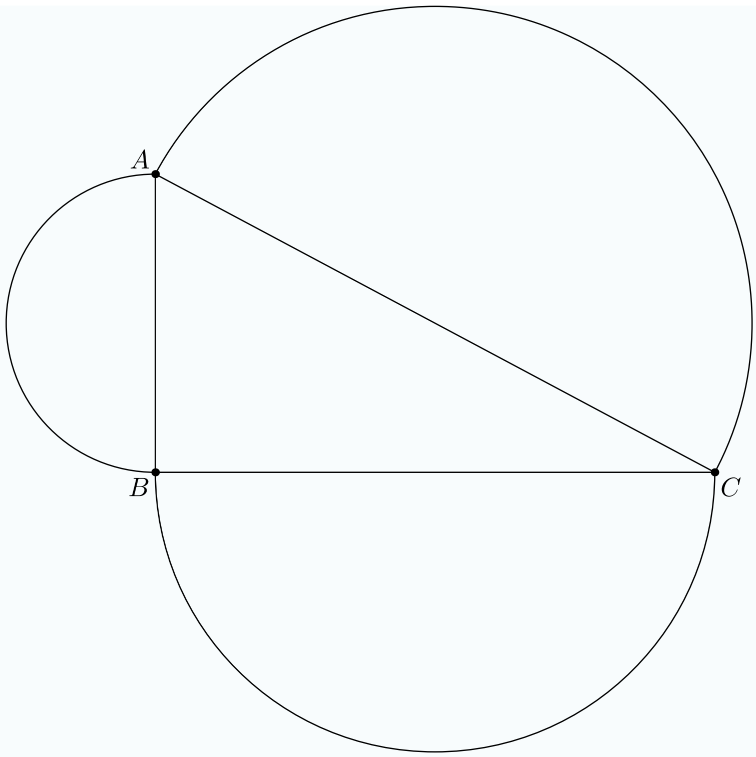
Problem 28.5.13 (AMC 8)

An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is 4, what is the area of the hexagon?

Video Solution

Problem 28.5.14 (AMC 8)

Angle ABC of $\triangle ABC$ is a right angle. The sides of $\triangle ABC$ are the diameters of semicircles as shown. The area of the semicircle on $\overline{AB}$ equals 8π, and the arc of the semicircle on $\overline{AC}$ has length 8.5π. What is the radius of the semicircle on $\overline{BC}$?

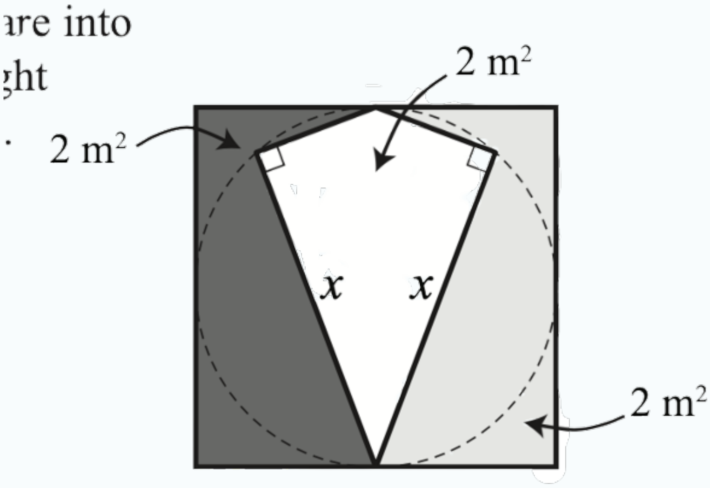


Video Solution

Additional Problems

Problem 28.5.15 (MATHCOUNTS)

In the figure, a kite with two right angles is circumscribed by a circle. The circle is then circumscribed by a square such that the diagonals of the kite are parallel to the sides of the square as shown. The longer sides of the kite each have length x meters. The sides of the kite divide the square into three regions, shaded dark gray, white and light gray as shown, with each region of area 2. If $x^2 = a + b$, what is the value of $a + b$?



Problem 28.5.16 (AMC 10)

A regular hexagon of side length 1 is inscribed in a circle. Each minor arc of the circle determined by a side of the hexagon is reflected over that side. What is the area of the region bounded by these 6 reflected arcs?

Problem 28.5.17 (AMC 10)

Three identical square sheets of paper each with side length 6 are stacked on top of each other. The middle sheet is rotated clockwise 30° about its center and the top sheet is rotated clockwise 60° about its center, resulting in the 24-sided polygon shown in the figure below. The area of this polygon can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. What is $a + b + c$?

Answers

28.1 20

28.2 $\frac{83\sqrt{3}}{4}$

28.3 56

28.4 24

28.5 14

28.5.1 $\frac{7}{16}$

28.5.2 5

28.5.3 20

28.5.4 $1:1$

28.5.5 9

28.5.6 $6:\pi$

28.5.7 5.5

28.5.8 180

28.5.9 12

28.5.10 7.6

28.5.11 $28-2\pi$

28.5.12 $\frac{2}{\sqrt{\pi}}$

28.5.13 6

28.5.14 7.5

28.5.15 8

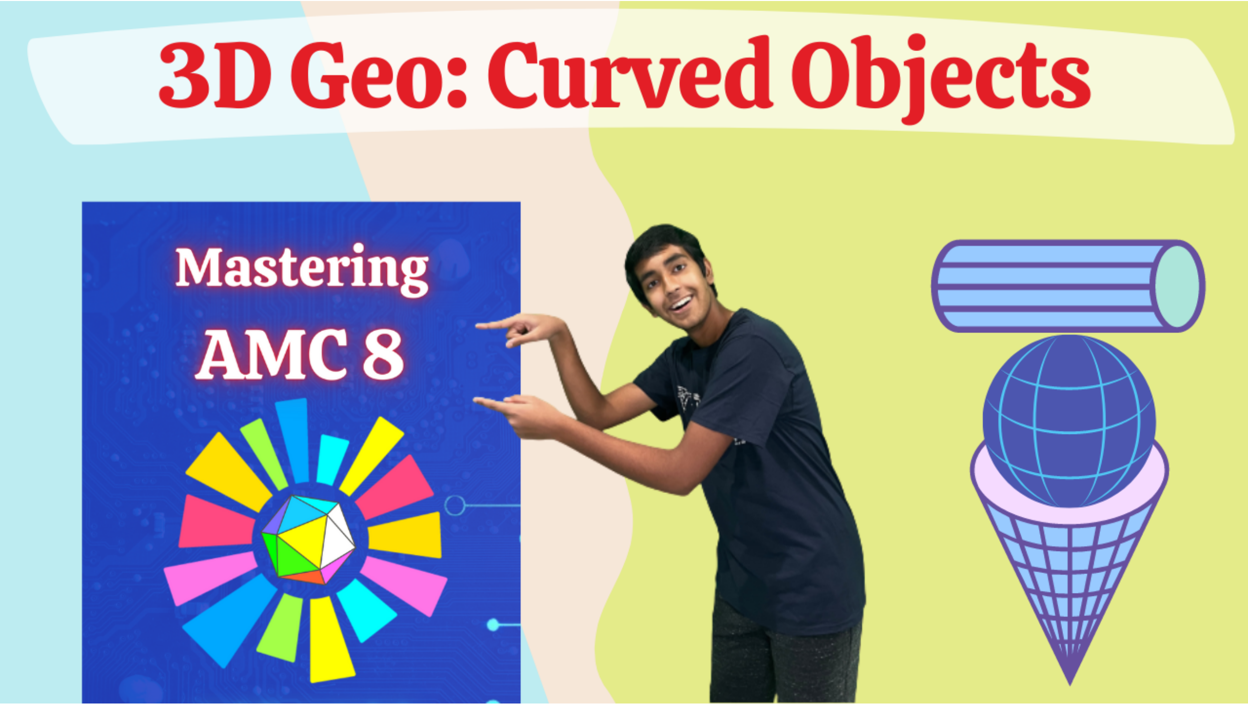
28.5.16 $3\sqrt{3}-\pi$

28.5.17 147

Chapter 29

3D Geometry

Video Lecture


3D Geo: Curved Objects
Mastering
AMC 8

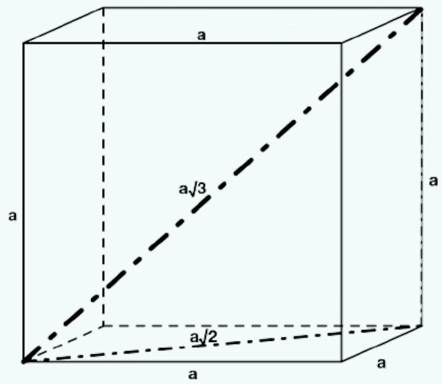

Similar Triangles in 3D
Mastering
AMC 8

29.1 Cube

Theorem 29.1.1 (Volume and Surface Area of a cube)

$$\text{Volume of a cube} = (\text{side length})^3 = a^3$$
$$\text{Surface area of a cube} = 6 \times (\text{side length})^2 = 6a^2$$

$$\text{Length of space diagonal of a cube} = \sqrt{3} \times \text{side length} = \sqrt{3}a$$



Example 29.1 (AMC 12)

What is the volume of a cube whose surface area is twice that of a cube with volume 1?

Video Solution

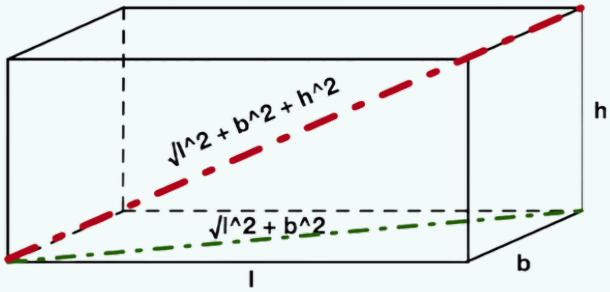
29.2 Prism

Theorem 29.2.1 (Volume and Surface Area of a rectangular prism)

$$\text{Volume of a rectangular prism} = l \times b \times h = \text{product of all three dimensions}$$

$$\text{Surface area of a rectangular prism} = 2(lb + bh + lh)$$

$$\text{Length of space diagonal of a rectangular prism} = \sqrt{l^2 + b^2 + h^2}$$



Theorem 29.2.2 (Volume of a Prism)

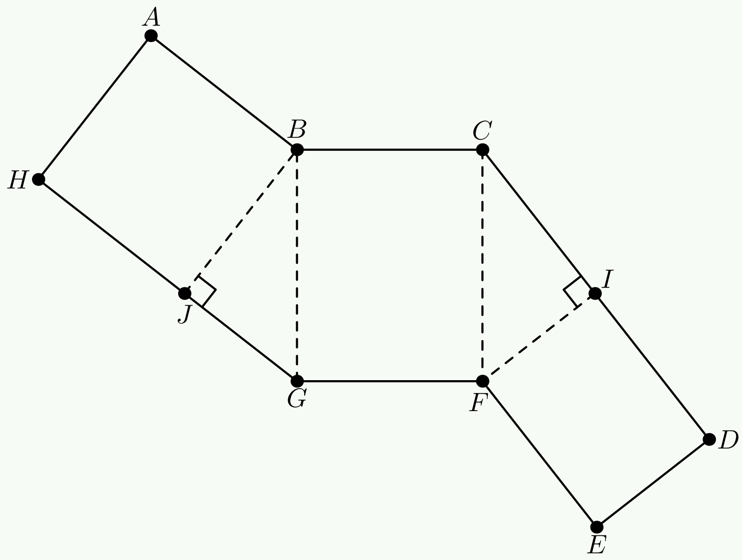
$$\text{Volume of a Prism} = \text{base area} \cdot \text{height}$$

Theorem 29.2.3 (Surface Area of a Prism)

$$\text{Surface Area of a Prism} = 2 \cdot \text{base area} + \text{base perimeter} * \text{height}$$

Example 29.2 (AMC 8)

The figure below shows a polygon $ABCDEFGH$, consisting of rectangles and right triangles. When cut out and folded on the dotted lines, the polygon forms a triangular prism. Suppose that $AH = EF = 8$ and $GH = 14$. What is the volume of the prism?

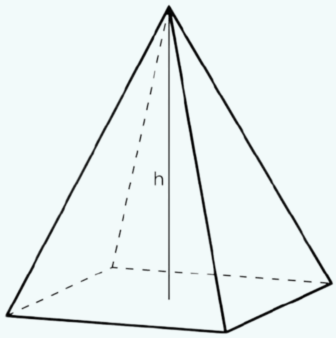


Video Solution

29.3 Pyramid

Theorem 29.3.1 (Volume of a Pyramid)

$$\text{Volume of any pyramid} = \frac{1}{3} \cdot \text{base area} \cdot \text{height}$$



Theorem 29.3.2 (Volume of a Regular Square Pyramid (all sides equal))
When the pyramid has a square base, and all the sides are equal

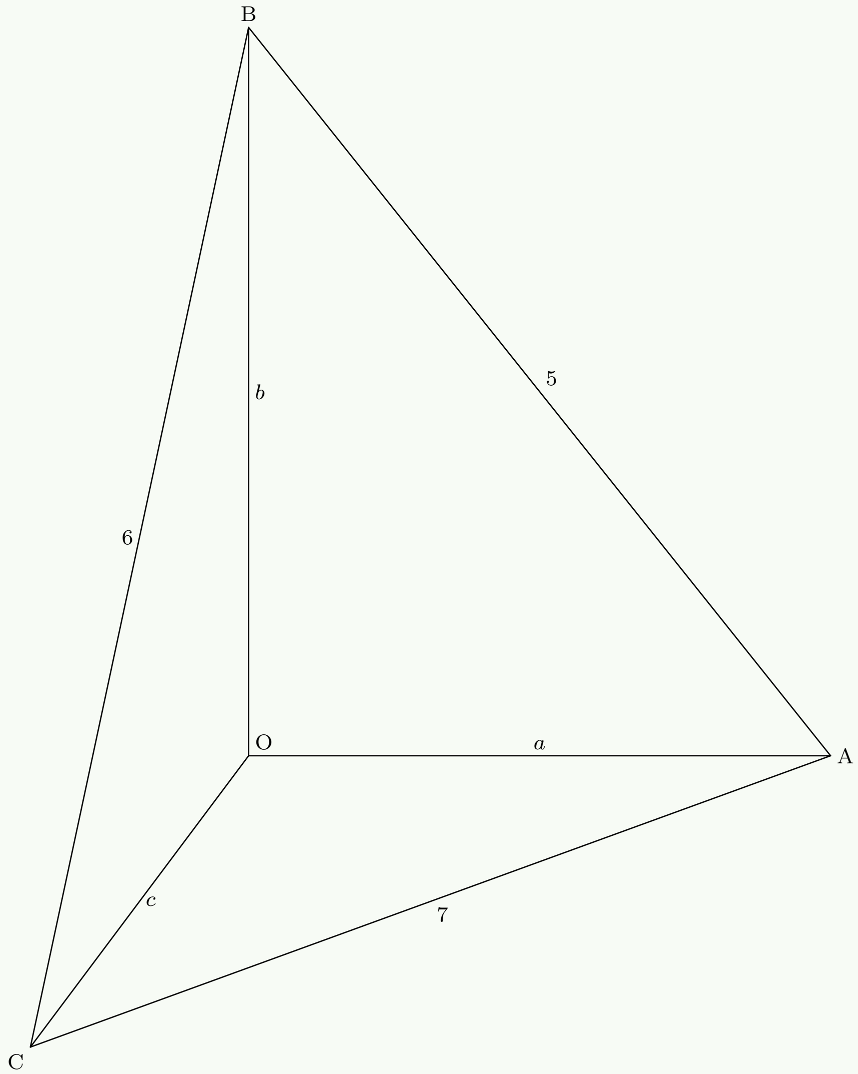
$$\text{Volume of a regular pyramid} = \frac{\sqrt{2}}{6} s^3$$

Theorem 29.3.3 (Volume of a Regular Tetrahedron (all sides equal))

$$\text{Volume of a regular tetrahedron} = \frac{\sqrt{2}}{12} s^3$$

Example 29.3 (AMC 12)

Triangle ABC, with sides of length 5, 6, and 7, has one vertex on the positive x-axis, one on the positive y-axis, and one on the positive z-axis. Let O be the origin. What is the volume of tetrahedron $OABC$?



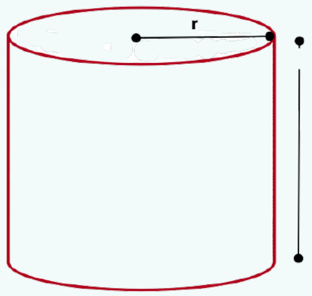
Video Solution

29.4 Cylinder

Theorem 29.4.1 (Volume and Surface Area of a cylinder)

$$\text{Volume of a cylinder} = \pi r^2 h$$

$$\begin{aligned}\text{Surface area of a cylinder} &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r(r+h)\end{aligned} \tag{29.1}$$



29.5 Cone

Theorem 29.5.1 (Volume and Surface Area of a cone)

$$\text{Volume of a cone} = \frac{1}{3}\pi r^2 h$$

which basically means

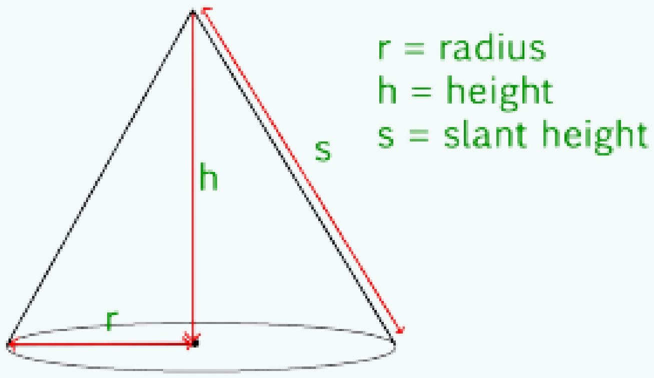
$$\text{Volume of a Cone} = \frac{1}{3}\pi \cdot \text{radius}^2 \cdot \text{height}$$

$$\text{Surface area of a cone} = \pi r^2 + \pi r s = \pi r(r+s)$$

where s is the lateral or slant height

which can also be written as

$$\pi \cdot \text{radius}^2 + \pi \cdot \text{radius} \times \text{slant height}$$



Remark 29.5.2
The slant height s can be calculated by the following formula

$$s = \sqrt{r^2 + h^2}$$

or

$$\text{slant height} = \sqrt{\text{radius}^2 + \text{height}^2}$$

Concept 29.5.3 (Cone Unfolding)
The sides of a cone can be unfolded to form a sector a circle as well. The circumference

of the base of the cone is equal to the length of the arc of the sector and the slant height of the cone is equal to the radius of the sector.

Example 29.4 (EMCC)

An ant sits on the circumference of the circular base of a party hat (a cone without a circular base for the ant to walk on) of radius 2 and height $\sqrt{5}$. If the ant wants to reach a point diametrically opposite of its current location on the hat, what is the minimum possible distance the ant needs to travel?

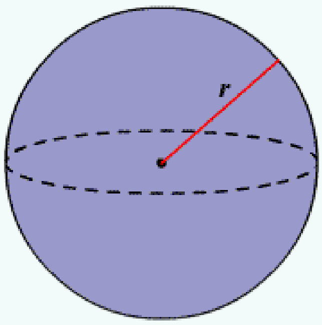
Video Solution

29.6 Sphere

Theorem 29.6.1 (Volume and Surface Area of a sphere)

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

$$\text{Surface area of a sphere} = 4\pi r^2$$



Example 29.5

Each of 4 spheres of radius 1 are tangent to 2 of the other spheres such that all 4 of their centers lie on a plane. What is the radius of the smallest sphere that contains these 4 spheres?

Video Solution

Example 29.6 (AMC 12)

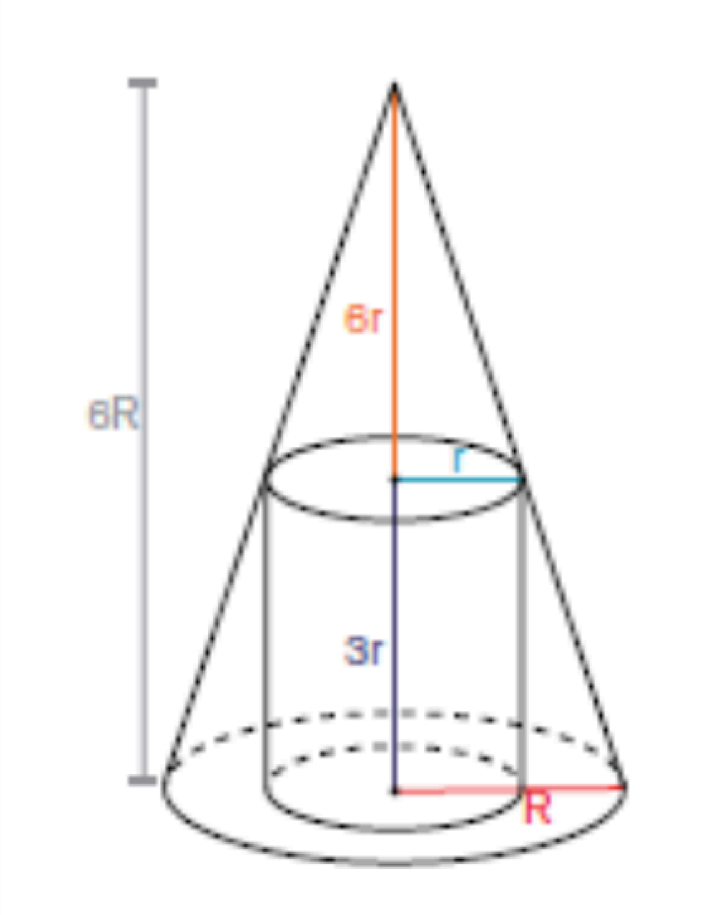
Eight spheres of radius 1, one per octant, are each tangent to the coordinate planes. What is the radius of the smallest sphere, centered at the origin, that contains these eight spheres?

Video Solution

29.7 Similar Triangles in 3D

Example 29.7 (MATHCOUNTS)

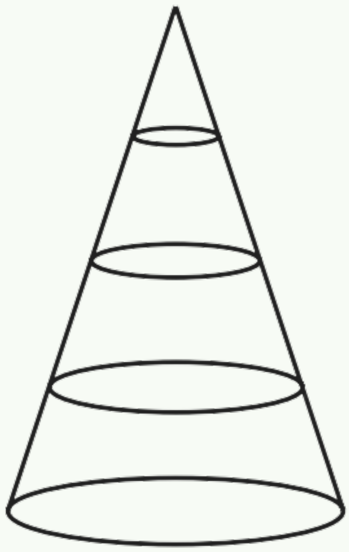
A cylinder whose height is 3 times its radius is inscribed in a cone whose height is 6 times its radius. What fraction of the cone's volume lies inside the cylinder? Express your answer as a common fraction.



Video Solution

Example 29.8 (MATHCOUNTS)

A right circular cone is sliced into four pieces by planes parallel to its base, as shown in the figure. All of these pieces have the same height. What is the ratio of the volume of the second-largest piece to the volume of the largest piece? Express your answer as a common fraction.



Video Solution

29.8 Practice Problems

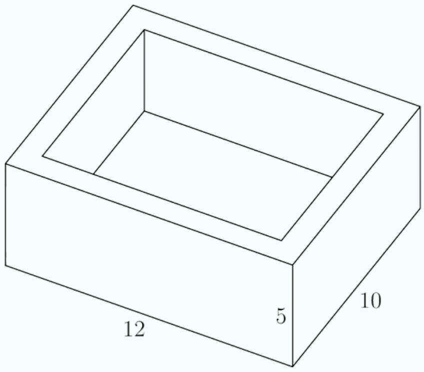
Problem 29.8.1 (AMC 8)

Alex and Felicia each have cats as pets. Alex buys cat food in cylindrical cans that are 6 cm in diameter and 12 cm high. Felicia buys cat food in cylindrical cans that are 12 cm in diameter and 6 cm high. What is the ratio of the volume of one of Alex's cans to the volume one of Felicia's cans?

Video Solution

Problem 29.8.2 (AMC 8)

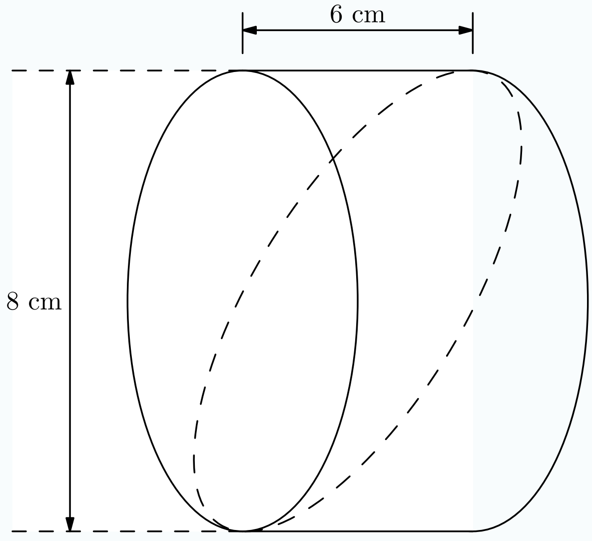
Isabella uses one-foot cubical blocks to build a rectangular fort that is 12 feet long, 10 feet wide, and 5 feet high. The floor and the four walls are all one foot thick. How many blocks does the fort contain?



Video Solution

Problem 29.8.3 (AMC 8)

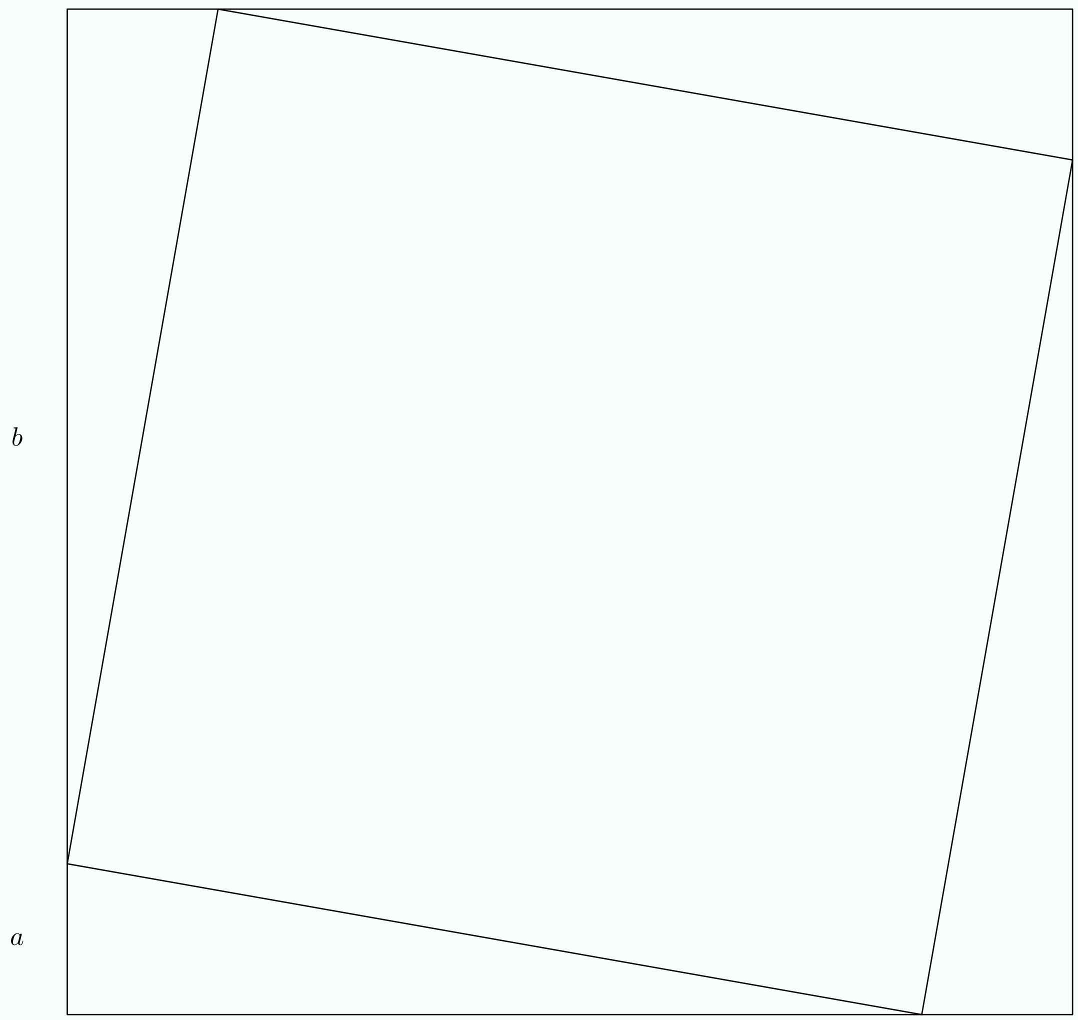
Jerry cuts a wedge from a 6-cm cylinder of bologna as shown by the dashed curve. Which answer choice is closest to the volume of his wedge in cubic centimeters?



Video Solution

Problem 29.8.4 (AMC 8)

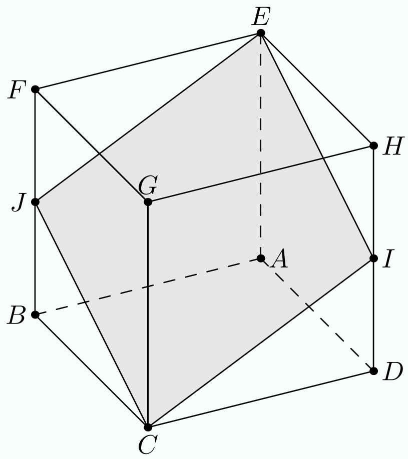
A square with area 4 is inscribed in a square with area 5, with one vertex of the smaller square on each side of the larger square. A vertex of the smaller square divides a side of the larger square into two segments, one of length a, and the other of length b. What is the value of ab?



Video Solution

Problem 29.8.5 (AMC 8)

In the cube $ABCDEFGH$ with opposite vertices C and E, J and I are the midpoints of edges $\overline{FB}$ and $\overline{HD}$, respectively. Let R be the ratio of the area of the cross-section $EJCI$ to the area of one of the faces of the cube. What is R^2?



Video Solution

Problem 29.8.6 (AMC 10A)

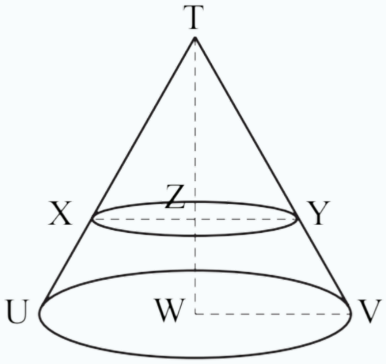
Seven cubes, whose volumes are 1, 8, 27, 64, 125, 216, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

Video Solution

Additional Problems

Problem 29.8.7 (MATHCOUNTS)

The two cones shown have parallel bases and common apex T. TW = 32m, WV = 8m, and ZY = 5m. What is the volume of the frustum with circle W and circle Z as its bases? Express your answer in terms of π.

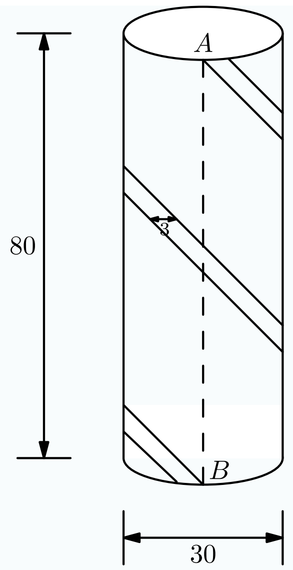


Problem 29.8.8 (EMCC)

The tiny island nation of Konistan is a cone with height 12 meters and base radius 9 meters, with the base of the cone at sea level. If the sea level rises 4 meters, what is the surface area of Konistan that is still above water, in square meters?

Problem 29.8.9 (AMC 10A)

A white cylindrical silo has a diameter of 30 feet and a height of 80 feet. A red stripe with a horizontal width of 3 feet is painted on the silo, as shown, making two complete revolutions around it. What is the area of the stripe in square feet?

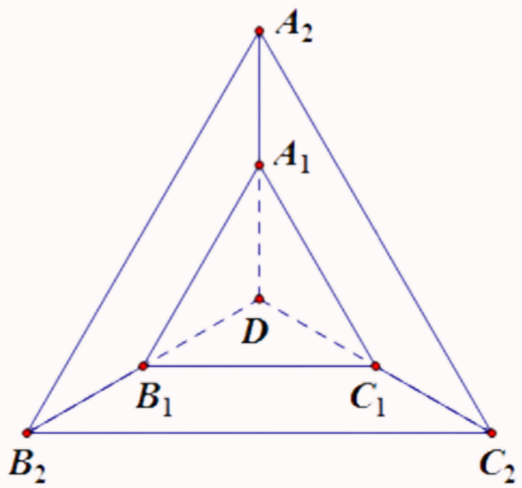


Problem 29.8.10 (BmMT)

Shivani is planning a road trip in a car with special new tires made of solid rubber. Her tires are cylinders that have width 6 inches and have diameter 26 inches, but need to be replaced when the diameter is less than 22 inches. The tire manufacturer claims that 0.12π cubic inches of its tire will wear away with every single rotation. Assuming that the tire manufacturer is correct about the wear rate of their tires, and that the tire maintains its cylindrical shape and width (losing volume by reducing radius), how many revolutions can each tire make before she needs to replace it?

Problem 29.8.11 (EMCC)

A piece of paper is in the shape of an equilateral triangle ABC with side length 12. Points A_B and B_A lie on segment AB, such that $AA_B = 3$, and $BB_A = 3$. Define points BC and CB on segment BC and points CA and AC on segment CA similarly. Point A_1 is the intersection of $ACBC$ and $ABCB$. Define B_1 and C_1 similarly. The three rhombi — $AA_BA_1A_C$, $BB_CB_1B_A$, $CC_AC_1C_B$ — are cut from triangle ABC, and the paper is folded along segments A_1B_1, B_1C_1, C_1A_1, to form a tray without a top. What is the volume of this tray?



Answers

29.1 $2\sqrt{2}$

29.2 192

29.3 $\sqrt{95}$

29.4 $3\sqrt{3}$

29.5 $\sqrt{2}+1$

29.6 $\sqrt{3}+1$

29.7 $\frac{4}{9}$

29.8 $\frac{19}{37}$

29.8.1 $1:2$

29.8.2 280

29.8.3 151

29.8.4 $\frac{1}{2}$

29.8.5 $\frac{3}{2}$

29.8.6 658

29.8.7 516π

29.8.8 60π

29.8.9 240

29.8.10 2400

29.8.11 $\frac{63\sqrt{2}}{4}$

Additional Info

Chapter 30

Additional Techniques and Strategies

30.1 Meta-solving Techniques

Video Lectures

Meta-solving Techniques

Definition 30.1.1. Meta-solving is finding the answer to a problem without actually solving it.

Remark 30.1.2
These techniques may not work for all problems. These techniques are especially useful when the problem provides answer choices.

Remark 30.1.3 (Meta-Solving Warning)
Don't get too carried away with these techniques to the point where you don't even try to solve the problem legitimately.

Concept 30.1.4 (Engineering Induction)
Engineering Induction is the process of trying and finding the value to small cases and assuming it's true for larger ones.

Steps for Engineering Induction Problems:

1. Try small cases

2. Look for a pattern amongst those small cases (there may not always be one)

3. Assume the pattern can continue for larger cases and find the answer

Remark 30.1.5
We can try to apply engineering induction when we see the values in the problem seem hard/impossible to compute.

Concept 30.1.6 (Looking for unique properties of numbers)
Rather than computing the exact answer, we can compute unique properties of your answer choices so that you can eliminate all answer choices that don't satisfy the property and so that you will be left with 1 answer choice (or possibly more in which case you can just guess from the remaining ones)

Some unique properties you can look for in your answer choices and try to compute are

- Units Digit
- Last 2 digits
- Parity (Even, Odd)
- Perfect square or not one
- Prime/composite
- Modulus (remainders when divided by 3, 4, 5, etc.)
- Denominators of common fractions (or what they must divide)
- Multiples/Factors of numbers
- etc.

Remark 30.1.7
These last 2 properties are especially useful in combinatorics problems as you can easily find numbers you have to multiply with each other to get your answer.

Concept 30.1.8
Look for the option choices that are the "odd one out" or that are different from all others

- Look for outliers (primes, large/small numbers, odd/even numbers, powers of 2, etc)

Concept 30.1.9 (Trying all the Option choices)
In some problems, you can

- Try all the option choices into the conditions in the problem
- Look at the conditions in the problem and see which of the option choices could

work

- etc.

After doing so, you will either have a better guess or exact answer.

Concept 30.1.10 (Elimination of Option Choices)
Tying closely to the previous technique, you can also usually eliminate option choices based on

- Unique properties of Numbers (See Above)
- How large or small the number must be

Concept 30.1.11 (Estimation of Answer)
Often times in problems (especially geometry) you can easily find an approximate answer and see which of the option choices most closely matches what you got.

In geometry, a common strategy to do so is to mark out areas approximately equal to those of areas you know.

Concept 30.1.12 (Using Freedom in Problems)
Assuming facts when you have freedom in the problem statement can be very useful.

Essentially, as long as the problem is not telling you "this fact is not true" (so, whatever assumption you want to make will satisfy the problem's conditions) you can assume the fact is true to simplify your problem and make it really easy to solve.

For example, if you are asked to find some universal ratio in a triangle and you aren't specifically told that the triangle isn't equilateral, you can just assume the triangle is equilateral and solve the remaining problem from there.

Remark 30.1.13
Make sure not to assume false information! Be very careful that your assumption can be true.
In our previous example, if we were told the triangle had 2 sides of length 7 and 8, then our assumption would be false, so it wouldn't work then.

30.2 Silly Mistakes

Silly mistakes are very common and can really lower your score on the AMC contests. Here are some tips on how to avoid the different kinds of silly mistakes:

Concept 30.2.1 (Avoiding Arithmetic Errors)
A good way to avoid arithmetic errors is to

- Do your computation 2 ways (e.g. If you have to do $87 \cdot 93$, you can multiply them with 87 on the top and with 93 on top)
- Be more organized, and write more steps

Concept 30.2.2 (Avoiding Mathematical Errors)
An easy way to avoid mathematical errors is to

- Do your work neatly!
- Make boxes on scratch paper per problem
- Don't skip steps
- Check your work, following the tip above will make it easier to do so
- Do your steps methodically
- Try to substitute your answer back into the problem (if you can)
- Try an alternate solution to confirm your answer
- Estimate what the answer has to be, and see if your answer is close to what your estimate is

Concept 30.2.3 (Avoid Reading Errors)
Reading the problem wrong is one of the most common mistakes. Often times, you might forget about important key words like

- inclusive, except
- even, odd
- prime, composite
- integer, natural, real, complex

- non-negative, positive (Remember, non-negative includes 0 while positive doesn't)

Some strategies to avoid them are

- After solving the problem, reread at least the question part of the problem to make sure you are answering what the question asks for!
- Underline key words while reading (you cannot do that if the test is online, but as an alternative you can take note of the important words on your scratch paper)

Concept 30.2.4 (Avoid Missing Final Step Errors)
Sometimes in problems, you might be so caught up in moving forward in the test that you might forget an important step at the end.

For example, in a problem you might think "I'll multiply by 5 to whatever answer I get" and then you find that answer but forget to multiply by 5. A way to avoid this is:

- Write "Remember ..." big and bold on your scratch or the question paper

Concept 30.2.5 (Avoid Casework Errors)
Many times, in casework problems you might

- Undercount possibilities
- Overcount possibilities
- Miss edges/extreme cases

Some strategies to avoid them are

- Be methodical in your casework
- Make sure all your cases work
  - An easy way to do this is just to try a few examples in your case to see if they actually work
  - Especially, make sure to check if extremes work
  - Make sure all your cases are disjoint and that you are not overcounting anything that's common between the cases
  - Make sure your cases cover all possibilities that the problem asks for

  - Solve the problem in multiple ways (for example, by both casework and complementary counting)

Remark 30.2.6

I know that some of these strategies may take a lot of extra time to follow so we recommend analyzing how you are making silly mistakes and from there see which strategies you will want to follow.

30.3 Other Strategies To Maximize Your Score

Concept 30.3.1 (Plan Your Time)
Make sure to plan your time.

- Questions 1-10 are generally easy, 11-20 are medium, 21-25 are hard
- Sometimes one of the early questions can be hard or bashy, or a later question can be easy
- Don't get stuck on a question. Move on and come back to it later.
- Star any question you are unsure about but you feel you can solve it, or any questions that you solved but are not confident of your answer
- Budget your time properly
- Leave enough time for last 10 problems
- Leave some time to review starred problems and check your work

Concept 30.3.2 (Guessing Strategies)
If you want to guess, make sure to keep these things in mind:

- Try using meta-solving strategies above
- In AMC 8, there is no penalty for guessing, so remember to make an educated guess for every problem

Concept 30.3.3 (Test Day Strategies)
Here are some test day strategies:

- Don't try to cram or study on the day before the test
  - Just review a few formulas or strategies from this book
- Be relaxed
  - Get a good night's sleep
  - Eat a healthy meal
  - Meditate
  - Eat dark chocolate

  - Listen to music
  - Or whatever else makes you feel relaxed

Concept 30.3.4 (Problem Solving Strategies)
Try to remember these problem solving strategies:

- When you are stuck, try to use the information in the problem you haven't used yet
- When solving a problem, think of what technique would likely be at play
- Don't get too stuck with one approach to a problem, move on, and come back to the problem with a fresh perspective
- In problems that seem complex, try small cases to look for a pattern that can allow you to figure out how to approach the problem and what patterns may exist (this is similar to engineering induction)

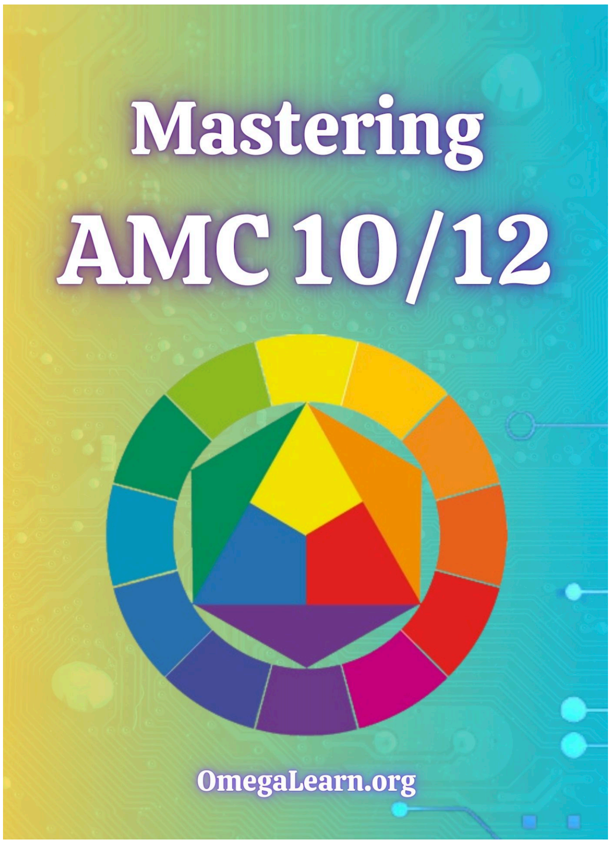
Remark 30.3.5
Last, but most important, don't stress out too much about how you will do! It's just a math contest, and you'll have many more opportunities in the future.

Good luck to you on your math competitions. We hope you found this book useful! We really appreciate your feedback and can be reached at omegalearn.info@gmail.com. Thanks!

Chapter 31

Mastering AMC 10/12

If you have completed the Mastering AMC 8 book, and looking to take your mathematical knowledge to the next level, check out the **Mastering AMC 10/12 Book** to learn the concepts needed for the AMC 10 and AMC 12 exams.



Chapter 32

The Book of Mathematical Formulas & Strategies

Check out **The Book of Mathematical Formulas & Strategies** for a collection of the most important formulas and strategies