American Mathematics Competitions (AMC 8) Preparation

Volume 2

The American Mathematics Competitions 8 is a 25-question multiple-choice contest for students in the sixth through eighth grade. Accelerated fourth and fifth graders can also take part. The AMC 8 is administered in schools in November. The American Mathematics Competitions (AMC) publishes the Achievement Roll list recognizing students in 6th grade and below who scored 15 or above, and the Honor Roll list recognizing students who score in the top 5%, and the Distinguished Honor Roll list recognizing students who score in the top 1%.

This book can be used by 5th to 8th grade students preparing for AMC 8. Each chapter consists of (1) basic skill and knowledge section with plenty of examples, (2) about 30 exercise problems, and (3) detailed solutions to all problems.

We would like to thank the American Mathematics Competitions (AMC 8 and 10) for their mathematical ideas. Many problems (marked by ☆) in this book are inspired from these tests. We only cited very few problems directly from these tests for the purpose of comparison with our own solutions.

We would also like to thank the following students who kindly reviewed the manuscripts and made valuable suggestions and corrections: Jamie Cheng, Albert Hao, Sameer Khan, Aadith Menon, Jeffery Shen, and Samuel Yoon.

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Please contact mymathcounts@gmail.com for suggestions, corrections, or clarifications.

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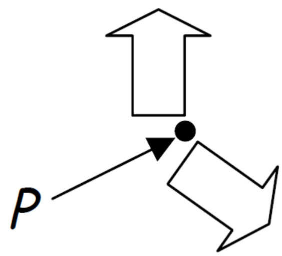
1. BASIC KNOWLEDGE

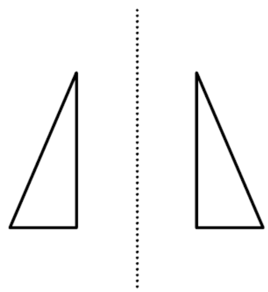
The Euclidean transformations are the most commonly used transformations. An Euclidean transformation is either a translation, a rotation, or a reflection. In an Euclidian transformations, lengths and angles are preserved.

Rotation means turning around a center. The distance from the center to any point on the shape stays the same. Every point makes a circle around the center.



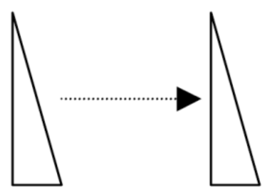
Reflection

A transformation that creates a mirror image of a figure; a flip. Reflection has the same size as the original image. The central line is called the mirror line. Mirror lines can be in any direction.



Translation

A transformation in which every point of a figure moves an equal distance in the same direction; a slide.



Translation simply means moving without rotating, resizing or anything else, just moving. To translate a figure, every point of the shape must move the same distance in the same direction.

When the transformations are done in a coordinate system, it is called the coordinate system transformation.

In this lecture, we will classify the transformations problems in AMC 8 or Mathcounts by two types: the coordinate system transformation, and the geometric objects transformation.

2. THE COORDINATE SYSTEM TRANSFORMATION

(1). $P(x_0, y_0)$ is a point. The image of P under reflections:

(*a*). In the x–axis $(x_0, -y_0)$

(*b*). In the y–axis $(-x_0, y_0)$

(*c*). In the line $x = a$ $(2a - x_0, y_0)$

(*d*). In the line $y = a$ $(x_0, 2a - y_0)$

(*e*). In the line $y = x$ (y_0, x_0)

(*f*). In the line $y = -x$ $(-y_0, -x_0)$

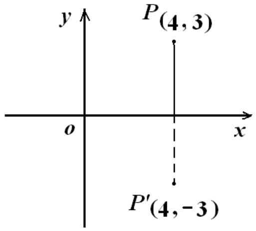
(*g*). In the line $y = x + m$ $(y_0 - m, x_0 + m)$

(*h*). In the line $y = -x + n$ $(n - y_0, n - x_0)$

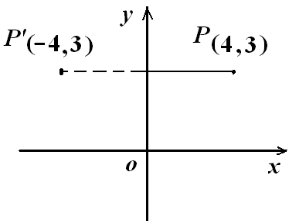
(i). In the point $A(a, b)$ $(2a - x_0, 2b - y_0)$

(j). In the line $Ax + By + C = 0$ $(x_1, y_1): \begin{cases} A \cdot \dfrac{x_0 + x_1}{2} + B \cdot \dfrac{y_0 + y_1}{2} + C = 0 \\ A(y_0 - y_1) = B(x_0 - x_1) \end{cases}$

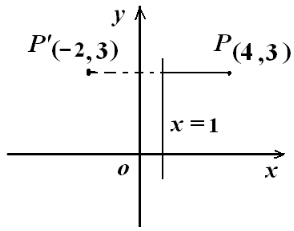
The figures for (a) to (f) are shown below using a sample point $P(4,3)$.



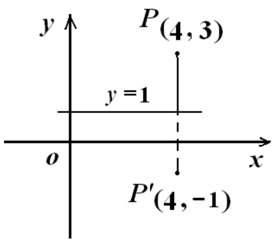
(a)



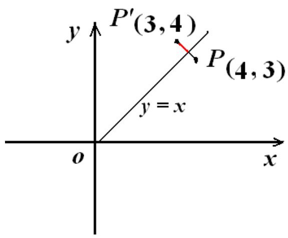
(b)



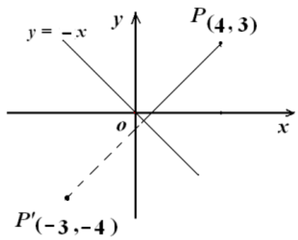
(c)



(d)



(e)



(f)

(2). The reflection of the line $Ax + By + C = 0$ in the point $P(a, b)$:
$Ax + By - (2aA + 2bB + C) = 0$

Example 1. What are the coordinates of the point which is the reflection in the y-axis of the point whose coordinates are $(5, -3)$?

(A) $(-5, -3)$ (B) $(5, -3)$ (C) $(-3, 5)$ (D) $(5, 3)$ (E) $(3, -5)$

Solution: A.

We know that by (1) (*b*), for point $P(x_0, y_0)$, the image of *P* under reflections in the *y*–axis is $(-x_0, y_0)$. So the coordinates of the point are (– 5, – 3).

Example 2. If the point whose coordinates are (– 5, 3) is reflected about the line $y = -2$, what are the coordinates of its image?
(A) (– 5, – 7) (B) (5, – 7) (C) (– 7, 5) (D) (5, 7) (E) (7, – 5)

Solution: A.
We know that by (1) (*d*), for point $P(x_0, y_0)$, the image of *P* under reflections in the line *y* = *a* is $(x_0, 2a - y_0)$. The answer is [– 5, 2 × (– 2) – 3], or (– 5, – 7).

Example 3. The point (5, 3) is reflected about the line $x = 2$. The image point is then reflected about the line $y = 2$. The resulting point is (*a*, *b*). Compute *a* + *b*.
(A) 1 (B) 0 (C) 2 (D) 4 (E) 8

Solution: B.
We know that by (1) (*c*), for point (5, 3), the image under reflections in the line *x* = *a* is $(2a - x_0, y_0)$, or (2 × 2 – 5, 3), or (– 1, 3).

We know that by (1) (*d*), for point (– 1, 3), the image under reflections in the line *y* = *a* is $(x_0, 2a - y_0)$. The answer is (– 1, 2 × 2 − 3), or (– 1, 1). So the answer is 1 – 1 = 0.

Example 4. What is the *y*-coordinate of the image when (5, 3) is reflected over the line $y = x$?
(A) 2 (B) 3 (C) 5 (D) 8 (E) 1

Solution: C.
We know that by (1) (*e*), for point $P(x_0, y_0)$, the image of *P* under reflections in the line *y* = *x* is (y_0, x_0), or (3, 5).

The y-coordinate of the image is 5.

Example 5. When the point (– 3, – 4) is reflected about the line $y = -x$, what is the y-coordinate of its image?

(A) 4 (B) 3 (C) – 4 (D) – 3 (E) – 7

Solution: B.

We know that by (1) (*f*), for point $P(x_0, y_0)$, the image of P under reflections in the $y = -x$ is $(-y_0, -x_0)$. So the coordinates of the image are (4, 3) and the y-coordinate is 3.

Example 6. The graph of the parabola $y = x^2 - 2$ is reflected with respect to the line $y = -x$. Write the equation of the resulting graph.

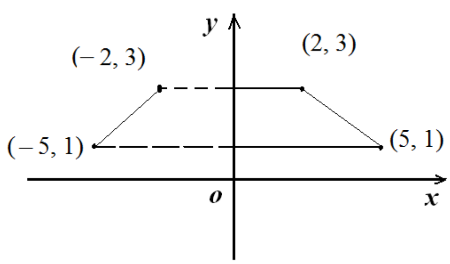
Solution: $x + y^2 = 2$.

We know that by (1) (*f*), for point $P(x_0, y_0)$, the image of P(–2, 4) under reflections in the $y = -x$ is $(-y_0, -x_0)$.

$y = x^2 - 2$ will becomes $(-x) = (-y)^2 - 2 \quad \Rightarrow \quad x + y^2 = 2$.

Example 7. The points (2, 3) and (5, 1) are reflected over the y-axis. Find the number of square units in the area of the quadrilateral whose vertices are the points and their images.

(A) 4 (B) 6 (C) 8 (D) 14 (E) 28



Solution: D.

We know that by (1) (*b*), for point $P(x_0, y_0)$, the image of P under reflections in the y–axis is $(-x_0, y_0)$.

So the coordinates of the images of the points (2, 3) and (5, 1) are (−2, 3) and (−5, 1).

The area of the quadrilateral is $\frac{(4+10)\times 2}{2}=14$.

Example 8. The triangle with vertices at A(– 2, 2), B(– 8, 2), and C(– 8, – 1) is reflected about the line $y=2x+1$. Express the coordinates of the reflection of *A* as an ordered pair.

(A) (– 2, 0) (B) (0, – 2) (C) (– 2, 8) (D) (2, – 2) (E) (2, 0)

Solution: E.

We know that by (1) (*j*), for point $P(x_0, y_0)$, the image of *P* under reflections in the line *Ax* + *By* +*C* = 0 is $(x_1, y_1): \begin{cases} A\cdot\frac{x_0+x_1}{2}+B\cdot\frac{y_0+y_1}{2}+C=0 \\ A(y_0-y_1)=B(x_0-x_1) \end{cases}$.

The line $y=2x+1$ can be written as $2x-y+1=0$.

$$\begin{cases} 2\cdot\frac{-2+x_1}{2}+(-1)\cdot\frac{2+y_1}{2}+1=0 \\ 2(2-y_1)=-1(-2-x_1) \end{cases} \Rightarrow \begin{cases} x_1+2y_1=2 \\ 2x_1-y_1=4 \end{cases} \Rightarrow \begin{cases} x_1+2y_1=2 \\ 4x_1-2y_1=8 \end{cases}$$

Solving we get $x_1 =$ 2 and $y_1 = 0$.

So the coordinates of *A* are (2, 0).

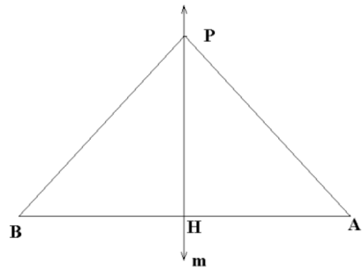
3. THE GEOMETRIC OBJECTS TRANSFORMATION

Figure reflection theorem

If a figure is determined by certain points, then its reflection image is the corresponding figure determined by the reflection images of those points.

The perpendicular bisector theorem

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.
Given: P is on the perpendicular bisector m of the segment AB. Prove: $PA = PB$.



Proof:
$\angle PBA = \angle PAB = 90°$, $BH = AH$, $PH = PH \Rightarrow \Delta PHA \cong \Delta PHB \Rightarrow PA = PB$

The reflection image of point A over the line m is the point B if and only if m is the perpendicular bisector of segment AB.

Segment symmetry theorem

A segment has exactly two symmetry lines:
(1) Its perpendicular bisector
(2) The line containing the segment.

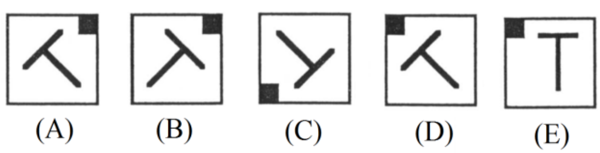
Angle symmetry theorem

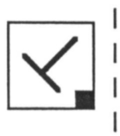
The line containing the bisector of an angle is a symmetry line of the angle.

Summary of two important properties:

(1) The symmetric line of a shape separates the shape into two congruent parts.
(2) The symmetric line is the perpendicular bisector of the line segment connecting two symmetric points.

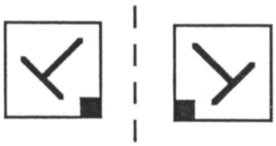
☆**Example 9.** Which of the five "T-like shapes" would be symmetric to the one shown with respect to the dashed line?



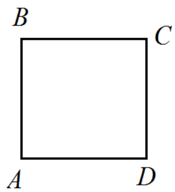


Solution: C.

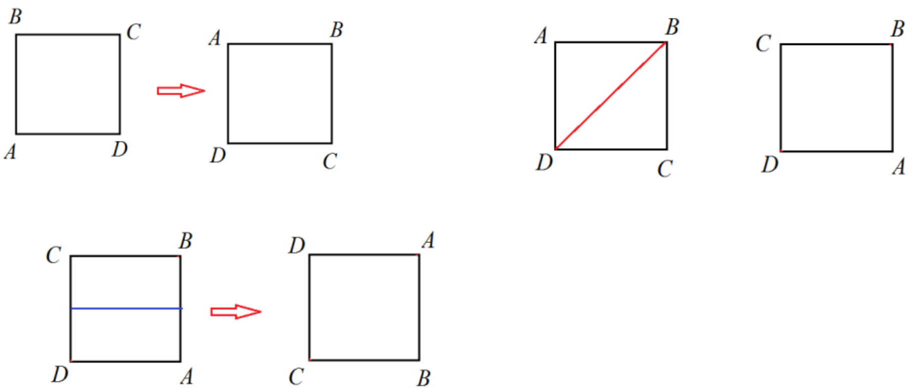
Two figures are symmetric with respect to a line if the figures coincide when the paper is folded along that line.



Example 10. Square $ABCD$ is rotated 90° clockwise about its center, and reflected over a diagonal line determined by lower left and upper right vertices. The square is then reflected over a horizontal line through the center. What point now corresponds to the position originally occupied by B?



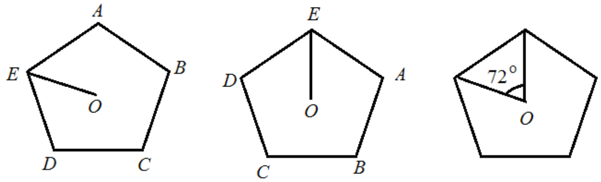
Solution: D.



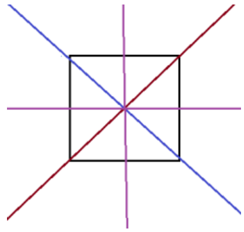
Example 11. A regular pentagon is rotated d° clockwise around its center until it coincides with its original image. What is the smallest positive measure of degrees in d?

(A) 30° (B) 45° (C) 60° (D) 72° (E) 108°

Solution: D.



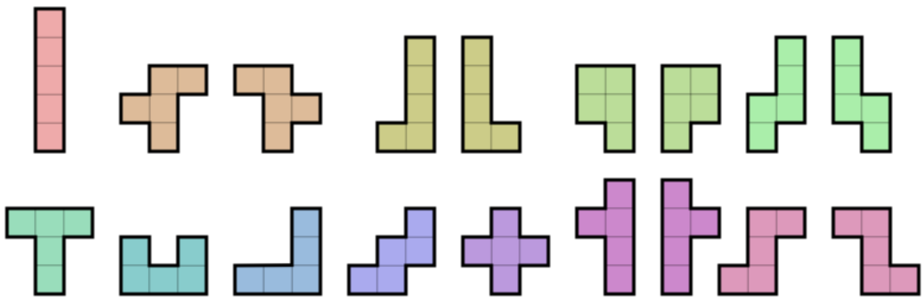
Example 12. How many lines of symmetry does a square have?



Solution: 4 (lines).

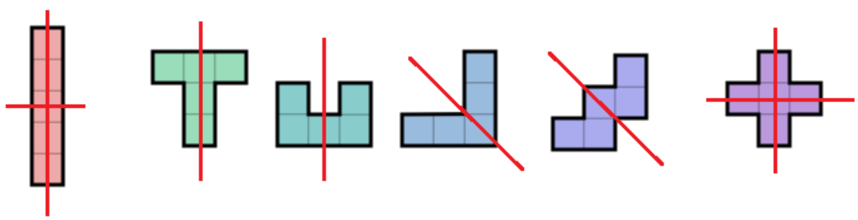
☆**Example 13.** How many of the eighteen pentominoes pictured below have at least one line of symmetry?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7



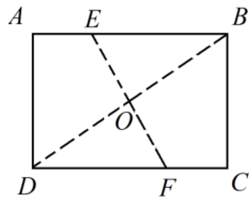
Solution: (D).

Exactly six have at least one line of symmetry. They are:



Example 14. Rectangle $ABCD$ is folded along line EF so that point B falls on point D. If $AD = 6$ and $AB = 8$, find the length of the crease $\overline{EF}$.

(A) $\frac{15}{4}$ (B) $5\frac{5}{8}$ (C) $\frac{15}{2}$ (D) $\frac{5}{2}$ (E) $11\frac{1}{4}$



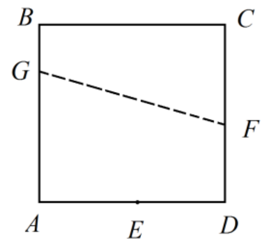
Solution: C.

$\Delta BCD \sim \Delta FOD \quad \Rightarrow \quad \frac{BC}{DC} = \frac{FO}{DO} \quad (1)$

ΔBCD is a 6-8-10 right triangle. So $BD = 10$.

$$\frac{6}{8} = \frac{FO}{5} \quad \Rightarrow \quad FO = \frac{30}{8} \quad \Rightarrow EF = 2FO = \frac{60}{8} = \frac{15}{2}.$$

Example 15. In the figure shown, $ABCD$ is a square piece of paper 6 cm on each side. Corner C is folded over so that it coincides with E, the midpoint of $\overline{AD}$. If $\overline{GF}$ represents the crease created by the fold, what is the length of $\overline{FD}$?



(A) 4/9 (B) 9/4 (C) 3/2 (D) 7/4 (E) $\frac{3\sqrt{5}}{2}$

Solution: B.

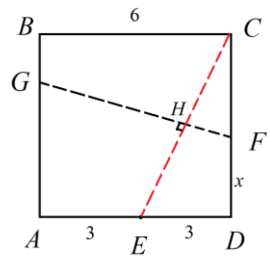
Connect CE.

GF is the symmetric line (perpendicular bisector of the line segment CE).

Applying Pythagorean Theorem to ΔCDE:

$ED^2 + CD^2 = CE^2 \Rightarrow 3^2 + 6^2 = CE^2$

$\Rightarrow \quad CE = 3\sqrt{5} \quad \Rightarrow \quad CH = \frac{3\sqrt{5}}{2}.$



We see that $\Delta CDE \sim \Delta CHF$.

So we have $\frac{CD}{CH}=\frac{CE}{CF} \quad \Rightarrow \quad \frac{6}{\frac{3\sqrt{5}}{2}}=\frac{3\sqrt{5}}{6-x} \quad \Rightarrow \quad x=\frac{9}{4}$.

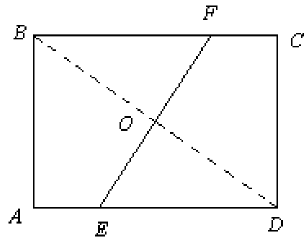
Example 16. A rectangular sheet of paper measures 12" by 9". One corner is folded onto the diagonally opposite corner and the paper is creased. What is the length in inches of the crease?

(A) $5\frac{5}{8}$ (B) 15/2 (C) 9/2 (D) 5/4 (E) $11\frac{1}{4}$

Solution: E.

Method 1:

Connect *BD* and call the intersection point of *BD* and *EF* at *O*.



We know that *EF* is the perpendicular bisector of *BD* and $EO = FO$.

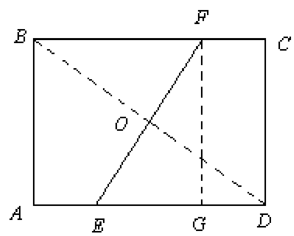
Since $AB = 9''$, $BC = 12''$, so $BD = 15''$ and $BO = 15/2''$.

$\triangle BOF \backsim \triangle BCD$ (because two angles are the same),

so $\frac{OF}{CD}=\frac{BO}{BC}$. Then $\frac{OF}{9}=\frac{\frac{15}{2}}{12}$ and $OF = 45/8$.

$EF = 2 \times 45/8 = 45/4 = 11\frac{1}{4}$.

Method 2:



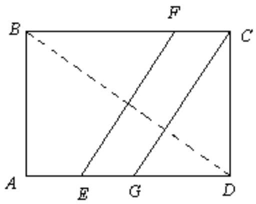
Draw $FG \perp AD$,

$\triangle FEG \backsim \triangle DBA$.

$\frac{EF}{FG}=\frac{BD}{AD}$. From here, we can solve for *EF*.

$EF=\frac{BD}{AD}\times FG=\frac{15}{12}\times 9=11\frac{1}{4}$.

Method 3:

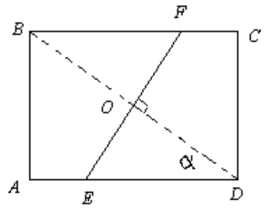


Connect BD, $BD = 15''$ and draw $CG//FE$, with G on AD.
$EFCG$ is a parallelogram, so $CG = FE$.
$\angle GCD = \angle BDA$, so $\triangle CGD \backsim \triangle DBA$.

$\frac{CG}{BD} = \frac{CD}{AD}$, so $\frac{CG}{15} = \frac{9}{12}$.

$CG = EF = \frac{15}{12} \times 9 = 11\frac{1}{4}$.

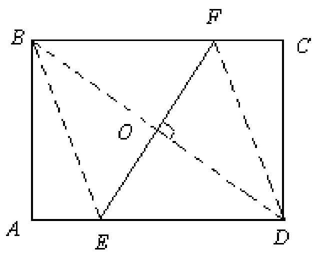
Method 4:



Draw BD ($BD = 15''$). So that it meets EF at O.
EF is the perpendicular bisector of BD.
$BD = 15''$, $OD = 15/2\,''$. Let $\angle ADB = \alpha$.
Right triangle $ADB \Rightarrow \tan\alpha = AB/AD = 9/12 = 3/4$.
Right triangle $EOD \Rightarrow \tan\alpha = OE/OD = OE/(15/2) = 2OE/15$.

So $\frac{3}{4} = \frac{2OE}{15}$. $OE = OF$, and $EF = 2\,OF = 45/4 = 11\frac{1}{4}$.

Method 5:



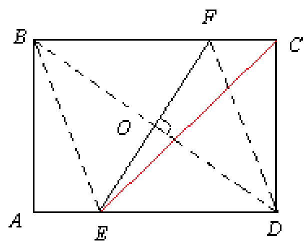
Connect BD, BE, DF.
Quadrilateral $BEDF$ is a rhombus. The area of the rhombus is $\frac{1}{2} EF \times BD = DE \times CD$,

$BE^2 = AB^2 + AE^2 = AB^2 + (AD - DE)^2$.

i.e. $DE^2 = 9^2 + (12 - DE)^2$ or $DE = 75/8$.

$\frac{1}{2} EF \times 15 = \frac{75}{8} \times 9$, or $EF = 45/4 = 11\frac{1}{4}$.

Method 6:



The area of triangle EFD is equal to ECD, so we can say

$$\frac{EF \times OD}{2} = \frac{CD \times ED}{2} \Rightarrow EF \times OD = DE \times CD.$$

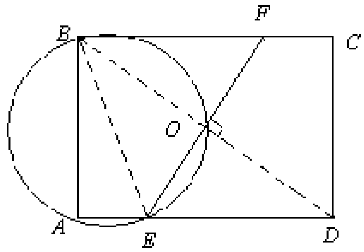
Thus $EF = \dfrac{DE \times CD}{OD} = \dfrac{\frac{75}{8}\times 9}{\frac{15}{2}} = \dfrac{45}{4} = 11\dfrac{1}{4}$.

Method 7:
We also can find DE using the following method:
B, A, E, and O are on the same circle because $\angle A + \angle O = 180°$. So,

$DE \times DA = DO \times DB$,

$$DE = \frac{DO \times DB}{DA} = \frac{\frac{15}{2}\times 15}{12} = \frac{75}{8}.$$

Then we get $EF = 11\dfrac{1}{4}$.



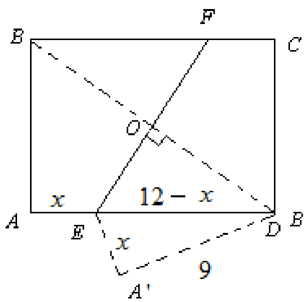
Method 8: As shown in the figure to the right,

$x^2 + 9^2 = (12 - x)^2$

Solving for x, $x = 21/8$.
Using the Pythagorean theorem again for triangle DEO to get EO:

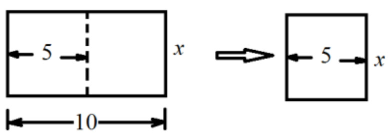
$$OE^2 + (\frac{15}{2})^2 = (12 - \frac{21}{8})^2 \quad \Rightarrow \quad OE = \frac{45}{8}$$

$2EO = EF$, so $EF = \dfrac{45}{4} = 11\dfrac{1}{4}$.



Example 17. A rectangular paper is folded along an axis of symmetry as shown. The shape of the resulting figure is similar to the shape of the original figure. Find x.

(A) 10 (B) 5 (C) 2 (D) $3\sqrt{2}$ (E) $5\sqrt{2}$

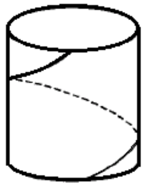


Solution: E.

Case 1: $\dfrac{x}{x} = \dfrac{10}{5} \quad \Rightarrow \quad 5 = 10$ (ignored)

Case 2: $\dfrac{10}{x}=\dfrac{x}{5}$ $\Rightarrow$ $x=5\sqrt{2}$.

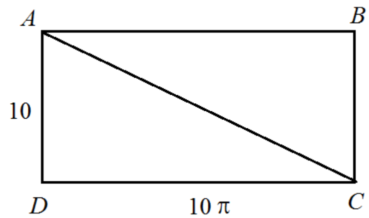
Example 18. A wire is wrapped around a cylinder forming a helix as in the picture. If the wire only goes around the cylinder once, and the height and diameter of the cylinder are both 10 cm, find the length of the wire in simplest radical form.



(A) $10\sqrt{1+\pi^2}$ (B) 10 (C) $20\sqrt{1+\pi^2}$
(D) $\pi\sqrt{10}$ (E) 10π

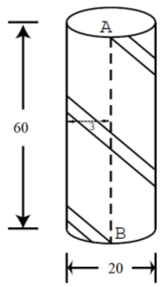
Solution: A.

Since the cylinder vertically with a single cut extending from one end of the wire to the other, Unfold the cylinder. The flat unfolded cylinder is a rectangle of height 10 cm and width 10π cm.



$AC=\sqrt{10^2+(10\pi)^2}=10\sqrt{1+\pi^2}$.

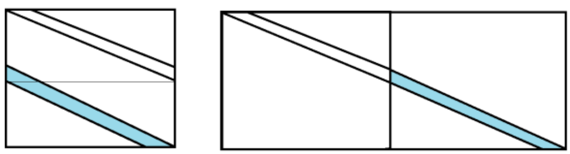
☆**Example 19.** A white cylindrical silo has a diameter of 20 feet and a height of 60 feet. A red stripe with a horizontal width of 3 feet is painted on the silo, as shown, making two complete revolutions around it. What is the area of the stripe in square feet?



(A) 120 (B) 180 (C) 240 (D) 160 (E) 480

Solution: B.

If the stripe were cut from the silo and spread flat, it would form a parallelogram 3 feet wide and 60 feet high. So the area of the stripe is 3(60) = 180 square feet.



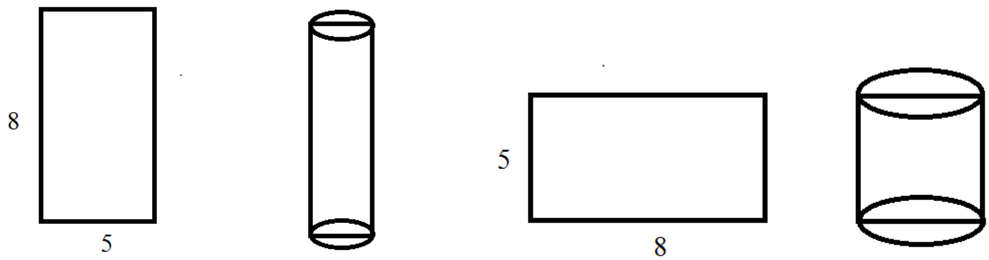
Example 20. A 5 inch by 8 inch rectangular piece of paper can be rolled up to form either of two right circular cylinders, a cylinder with a height of 8 inches or a cylinder with a height of 5 inches. What is the ratio of the volume of the 8 inch tall cylinder to the volume of the 5 inch tall cylinder?

(A) 5/8 (B) 9/4 (C) 3/2 (D) 7/4 (E) $\frac{\sqrt{5}}{2}$

Solution: A.

We know that $V_1 = \frac{1}{3}\pi \times (\frac{5}{2\pi})^2 \times 8$ and $V_2 = \frac{1}{3}\pi \times (\frac{8}{2\pi})^2 \times 5$.

$$\frac{V_1}{V_2} = \frac{\frac{1}{3}\pi \times (\frac{5}{2\pi})^2 \times 8}{\frac{1}{3}\pi \times (\frac{8}{2\pi})^2 \times 5} = \frac{5}{8}.$$



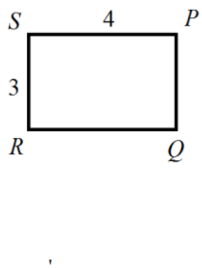
☆**Example 21.** Rectangle $PQRS$ lies in a plane with $PQ = RS = 3$ and $QR = SP = 4$. The rectangle is rotated 90° clockwise about R, then rotated 90° clockwise about the point that S moved to after the first rotation. What is the length of the path traveled by point P ?

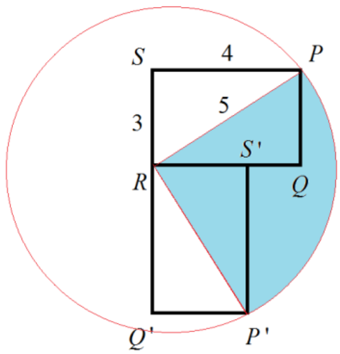
(A) $(\sqrt{2}+\sqrt{5})\pi$ (B) 6π (C) $\frac{9}{2}\pi$ (D) $(\sqrt{3}+2)\pi$ (E) $2\sqrt{10}\pi$

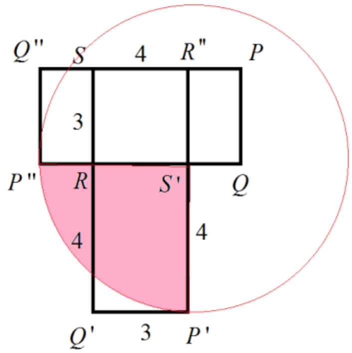
Solution: (C).

Let P' and S' denote the positions of P and S, respectively, after the rotation about R, and let P'' denote the final position of P. In the rotation that moves P to position P', the point P rotates $90°$ on a circle with center R and radius $PR = \sqrt{3^2+4^2}=5$. The length of the arc traced by P is $\frac{1}{4}(2\pi\times 5)=\frac{5\pi}{2}$. Next, P' rotates to P' through a $90°$ arc on a circle with center S' and radius $S'P' = 4$. The length of this arc is $\frac{1}{4}(2\pi\times 4)=2\pi$.

The total distance traveled by P is $\frac{5\pi}{2}+2\pi=\frac{9}{2}\pi$.







PROBLEMS

Problem 1. If the graph of the equation $y=(x+2)^2$ is reflected with respect to the y-axis, what is the equation of the resulting graph?

(A) $y=x^2-4x+4$ (B) $y=x^2-4x+2$ (C) $y=(x-2)^2$

(D) $y=x^2-4x-4$ (E) $y=x^2-2x+4$

Problem 2. P and Q are reflections of (2, – 3) across the x-axis and the y-axis, respectively. Find the length of $\overline{PQ}$ in simplest radical form.

(A) $\sqrt{97}$ (B) $2\sqrt{13}$ (C) $\sqrt{13}$ (D) 6 (E) 4

Problem 3. If the graph of the equation $y=x^2+3$ is reflected with respect to the line $y=-2$, what is the maximum value of the reflected graph?

(A) −4 (B) 7 (C) −7 (D) 3 (E) 4

Problem 4. What is the sum of the coordinates of the point obtained by first reflection (8, 8) over the line $x=3$, and then reflecting that point over the line $y=4$?

(A) −2 (B) 8 (C) −8 (D) 3 (E) 4

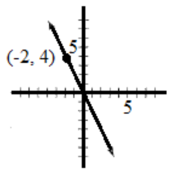
Problem 5. The point (2, 3) is reflected about the x-axis to a point P. Then P is reflected about the line $y=x$ to a point Q. What is the x-coordinate of Q?

(A) 2 (B) 5 (C) −3 (D) 3 (E) −2

Problem 6. The center of a circle has coordinates (6, – 5). The circle is reflected about the line $y=x$. What are the coordinates of the center of the image circle?

(A) (6, −5) (B) (−5, 6) (C) (5, −6) (D) (1, 0) (E) (0, 1)

Problem 7. Write the equation for the graph shown below after it is reflected about the line $y=-x$



(A) $x+2y=0$ (B) $x-2y=0$ (C) $2x+y=0$

(D) $2x-y=0$ (E) $x+y=0$

Problem 8. What is the sum of the new coordinates of point (5, – 1) when it is reflected across the line $y=-x$.

(A) −4 (B) 4 (C) −6 (D) 6 (E) 5

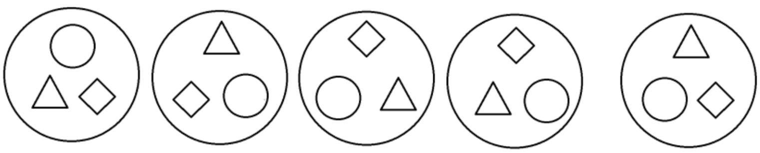
Problem 9. Point A has coordinates (– 2, 1). Point B is the image of A reflected in the line $y=3$. Point C is the image of B reflected in the line $y=x+3$. Point D is the image of C reflected in the line $x=0$. What is the distance between A and D ?

(A) $2\sqrt{5}$ (B) 4 (C) 2 (D) 0 (E) 1

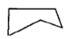
Problem 10. The point Q is the image of point P (2, 7) reflected through the line $x+2y=6$. What are the coordinates of Q?

(A) (−4, −6) (B) (−2, −1) (C) (2, −1) (D) (−2, 1) (E) (7, 2)

☆**Problem 11.** Which of the following represents the result when the figure shown is rotated clockwise 240° about its center?

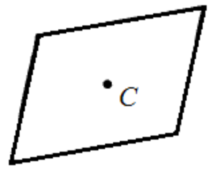


(A) (B) (C) (D) (E)

Problem 12. Which of the figures shown can be obtained from  by a rotation of the figure in the plane of the paper?

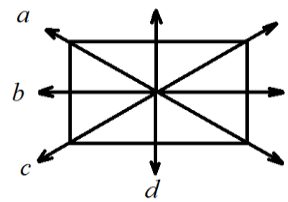
(A)  (B)  (C)  (D) 

Problem 13. If the rhombus is rotated clockwise about its center point C, what is the minimum number of degrees it must rotate before it coincides with the original shape?

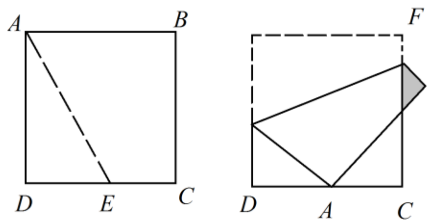


Problem 14. How many axes of symmetry does a rectangle have given that it is not a square?

Problem 15 . Which lines in the rectangle are lines of symmetry?

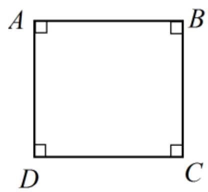


Problem 16. A square piece of paper 12 inches on each side is folded as shown, so that A falls on E, the midpoint of $\overline{DC}$. What is the number of square inches in the area of the triangular piece that extends beyond $\overline{FC}$ (the triangular piece is shaded in the diagram shown)?



(A) 8/3 (B) 9/4 (C) $1\frac{1}{2}$ (D) 7/4 (E) $\frac{\sqrt{5}}{2}$

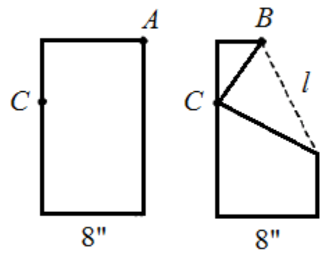
Problem 17. The square piece of paper shown is folded over so that vertex A lies on vertex C. The paper is folded again so that vertex C lies on the midpoint of $\overline{DB}$. Given that the length of one side of the square is 6 cm, find the length in centimeters of the second fold.



(A) $6\sqrt{2}$ (B) 4 (C) 3 (D) $2\sqrt{3}$ (E) $3\sqrt{2}$

Problem 18. Corner A of a rectangular piece of paper of width 8 inches is folded over so that it coincides with point C on the opposite side. If $BC = 5$ inches, find the length in inches of fold l.

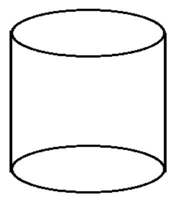
(A) $5\sqrt{5}$ (B) $\sqrt{5}$ (C) $2\sqrt{5}$ (D) $2\sqrt{3}$ (E) $3\sqrt{3}$



Problem 19. A square piece of paper is folded once so that one pair of opposite corners coincide. When the paper is unfolded, two congruent triangles have been formed. Given that the area of the original square is 49 square inches, what is the number of inches in the perimeter of one of these triangles?

(A) $14+7\sqrt{2}$ (B) $7+7\sqrt{2}$ (C) 21 (D) $21\sqrt{3}$ (E) $14+14\sqrt{2}$.

Problem 20. A can is in the shape of a right circular cylinder. The circumference of the base of the can is 12 inches, and the height of the can is 5 inches. A spiral strip is painted on the can in such a way that it winds around the can exactly once as it reaches from the bottom of the can to the top. It reaches the top of the can directly above the spot where it left the bottom. What is the length, in inches, of the strip?



(A) 17 (B) 13 (C) 25π (D) 12π (E) 60

Problem 21. A circular cylinder is formed by rolling an $8\frac{1}{2}$" × 11" paper vertically and taping it with no overlap. A second cylinder is formed by rolling an $8\frac{1}{2}$" × 11" paper horizontally and taping it with no overlap. What is the ratio of the volume of the $8\frac{1}{2}$" tall cylinder to the volume of the 11" tall cylinder?

(A) 11/8 (B) 17/11 (C) 3/2 (D) 22/17 (E) 11/17

☆**Problem 22.** Triangle OAB has $O = (0, 0)$, $B = (4, 0)$, and A in the first quadrant. In addition, $\angle ABO = 90°$ and $\angle AOB = 60°$. Suppose that OA is rotated 90° counterclockwise about O. What are the coordinates of the image of A?

(A) $(4\sqrt{3},4)$ (B) $(-4\sqrt{3},4)$ (C) $(4\sqrt{3},-4)$ (D) $(\sqrt{3},4)$ (E) $(-\sqrt{3},4)$

SOLUTIONS

Problem 1. Solution: A.
We know that by (1) (*b*), for point $P(x_0, y_0)$, the image of *P* under reflections in the *y*–axis is $(-x_0, y_0)$. So the equation becomes $y = (-x+2)^2$ or $y = x^2 - 4x + 4$.

Problem 2. Solution: B.
We know that by (1) (*a*), for point $(2,-3)$, the image *P* under reflections in the *x*–axis is $(x_0, -y_0)$, or (2, 3).

We know that by (1) (*b*), for point $(2,-3)$, the image *Q* under reflections in the *y*–axis is $(-x_0, y_0)$, or (–2, – 3).
The length of $\overline{PQ}$ is $\sqrt{(-2-2)^2 + (-3-3)^2} = 2\sqrt{13}$.

Problem 3. Solution: C.
We know that by (1) (*d*), for point $P(x_0, y_0)$, the image of *P* under reflections in the line *y* = *a* is $(x_0, 2a - y_0)$. So the equation becomes $2 \times (-2) - y = (-x)^2 + 3 \Rightarrow$ $y = -x^2 - 7$.
We see that the greatest value for y is – 7.

Problem 4. Solution: A.
We know that by (1) (*c*), for point (8, 8), the image under reflections in the line *x* = *a* is $(2a - x_0, y_0)$, or (2 × 3 – 8, 8), or (– 2, 8).

We know that by (1) (*d*), for point (– 2, 8), the image under reflections in the line *y* = *a* is $(x_0, 2a - y_0)$. The coordinates are (– 2, 2 × 4 − 8), or (– 2, 0). So the answer is 0 – 2 = – 2.

Problem 5. Solution: C.

We know that by (1) (a), for point (2, 3), the image P under reflections in the x–axis is $(x_0, -y_0)$, or (2, – 3).

We know that by (1) (e), for point $P(x_0, y_0)$, the image Q under reflections in the line $y = x$ is (y_0, x_0), or (– 3, 2).
The x-coordinate of the image Q is – 3.

Problem 6. Solution: B.
We know that by (1) (e), for point $P(x_0, y_0)$, the image Q under reflections in the line $y = x$ is (y_0, x_0), or (– 5, 6).
The $x - y$ coordinates of the center of the image circle are (– 5, 6).

Problem 7. Solution: A.
We know that by (1) (f), for point $P(x_0, y_0)$, the image of $P(-2, 4)$ under reflections in the $y = -x$ is $(-y_0, -x_0)$. So the coordinates of the image are (–4, 2).
Let the equation be $y = kx \quad \Rightarrow \; 2 = -4k \quad \Rightarrow \; k = -1/2$.
So $y = -\frac{1}{2}x \quad \Rightarrow \qquad x + 2y = 0$.

Problem 8. Solution: A.
We know that by (1) (f), for point $P(x_0, y_0)$, the image of $P(5, -1)$ under reflections in the $y = -x$ is $(-y_0, -x_0)$. So the coordinates of the image are (1, –5).
The sum of the new coordinates is 1 –5 = –4.

Problem 9. Solution: D.
We know that by (1) (d), for point $P(x_0, y_0)$, the image of P under reflections in the $y = a$ is $(x_0, 2a - y_0)$. So the coordinates of B are (– 2, 2 × 3 – 1), or (– 2, 5).

We know that by (1) (*g*), for point $P(x_0, y_0)$, the image of *P* under reflections in the line $y = x + m$ is $(y_0 - m, x_0 + m)$. So the coordinates of *C* are $(5-3, -2+3)$ or (2, 1).

We know that by (1) (*b*), for point $P(x_0, y_0)$, the image of *P* under reflections in the *y*–axis is $(-x_0, y_0)$. So the coordinates of *D* are (– 2, 1).

The distance between *A* and *D* is 0.

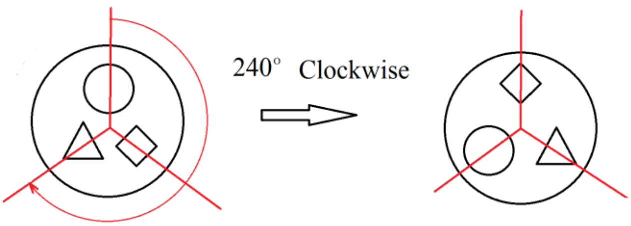
Problem 10. Solution: B.
We know that by (1) (*j*), for point $P(x_0, y_0)$, the image of *P* under reflections in the line $Ax + By + C = 0$ is $(x_1, y_1):\begin{cases} A\cdot\dfrac{x_0+x_1}{2} + B\cdot\dfrac{y_0+y_1}{2} + C = 0 \\ A(y_0 - y_1) = B(x_0 - x_1) \end{cases}$.

The line $x + 2y = 6$ can be written as $x + 2y - 6 = 0$.

$$\begin{cases} 1\cdot\dfrac{2+x_1}{2} + 2\cdot\dfrac{7+y_1}{2} - 6 = 0 \\ 1\cdot(7-y_1) = 2(2-x_1) \end{cases} \Rightarrow \begin{cases} x_1 + 2y_1 = -4 \\ 2x_1 - y_1 = -3 \end{cases} \Rightarrow \begin{cases} x_1 + 2y_1 = -4 \\ 4x_1 - 2y_1 = -6 \end{cases}$$

Solving we get $x_1 = -2$ and $y_1 = -1$.
So the coordinates of *Q* are (– 2, – 1).

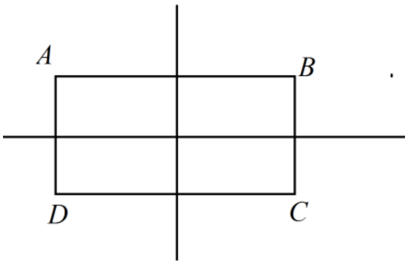
Problem 11. Solution: C.



Problem 12. Solution: A and C.

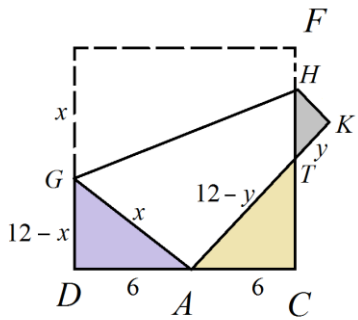
Problem 13. Solution: 180 (degrees).

Problem 14. Solution: 2 (axes).



Problem 15 . Solution: *b*, *d*.

Problem 16. Solution: C.



Applying Pythagorean Theorem to ΔADG:

$6^2 + (12-x)^2 = x^2 \quad \Rightarrow \quad x = 15/2$.

So the area of ΔADG is:

$$S_{\Delta ADG} = \frac{6(12-x)}{2} = \frac{6(12-\frac{15}{2})}{2} = \frac{27}{2}.$$

We see that $\Delta ADG \sim \Delta TCA \sim \Delta KH$.

So we have $\dfrac{AG}{AT} = \dfrac{GD}{AC} \Rightarrow \dfrac{x}{12-y} = \dfrac{12-x}{6} \Rightarrow \dfrac{\frac{15}{2}}{12-y} = \dfrac{12-\frac{15}{2}}{6} \Rightarrow y = 2$.

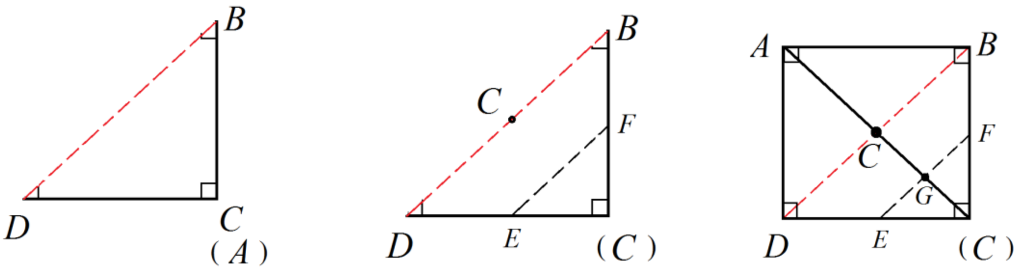
We know that $\dfrac{S_{\Delta HKT}}{S_{\Delta ADG}} = (\dfrac{KT}{AD})^2 = (\dfrac{2}{6})^2 \Rightarrow$

$$S_{\Delta HKT} = (\frac{2}{6})^2 S_{\Delta ADG} = \frac{\frac{27}{2}}{9} = \frac{27}{18} = \frac{3}{2} = 1\frac{1}{2}.$$

Problem 17. Solution: E.

ΔCDB is a $45° - 45° - 90°$ right triangle. So $DB = 6\sqrt{2}$.

It is clear that $EF = \dfrac{DB}{2} = 3\sqrt{2}$.



Problem 18. Solution: A.

ΔBCE is a $3 - 4 - 5$ right triangle.

Draw AB and AD. Connect AC. BD is the perpendicular bisector of AC.

Applying Pythagorean Theorem to ΔACE: $EC^2 + CA^2 = AC^2 \quad \Rightarrow$

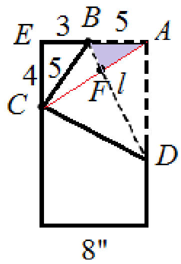
$$4^2 + 8^2 = AC^2 \quad \Rightarrow \quad AC = 4\sqrt{5}\,.$$

So $AF = FC = \dfrac{AC}{2} = 2\sqrt{5}$ and $BF^2 + AF^2 = 5^2$

$\Rightarrow \quad BF = \sqrt{25 - 20} = \sqrt{5}$.

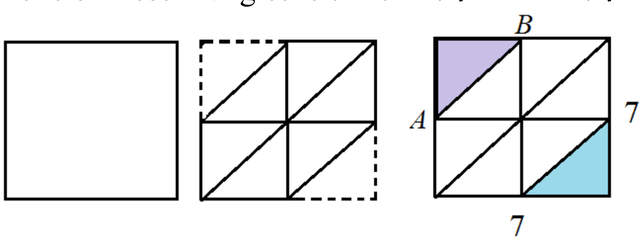
We see that $\Delta BDA \sim \Delta BAF$.

So we have $\dfrac{BF}{AB} = \dfrac{AB}{BD} \quad \Rightarrow \quad BD = \dfrac{AB^2}{BF} = \dfrac{25}{\sqrt{5}} = 5\sqrt{5}$.



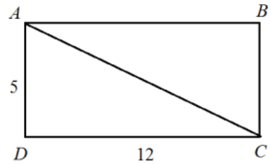
Problem 19. Solution: A.

The perimeter of one of these triangles is $7 + 7 + 7\sqrt{2} = 14 + 7\sqrt{2}$.



Problem 20. Solution: B.

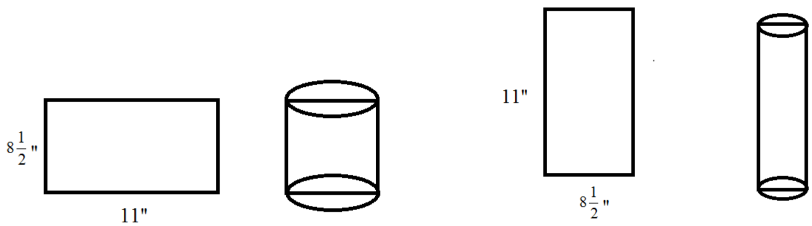
Since the cylinder vertically with a single cut extending from one end of the wire to the other, Unfold the cylinder. The flat unfolded cylinder is a rectangle of height 5 inches and width 12 inches. So ΔADC is a 5-12-13 right triangle and the length, in inches, of the strip 13.



Problem 21. Solution: D.

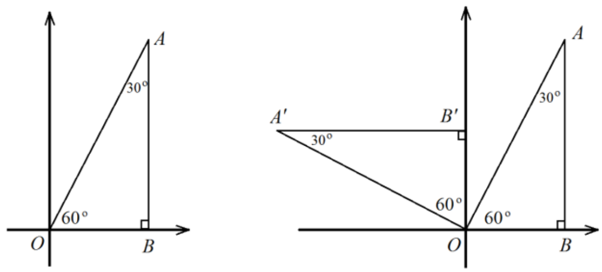
We know that $V_1 = \frac{1}{3}\pi \times (\frac{11}{2\pi})^2 \times 8\frac{1}{2}$ and $V_2 = \frac{1}{3}\pi \times (\frac{8\frac{1}{2}}{2\pi})^2 \times 11$.

$$\frac{V_1}{V2} = \frac{\frac{1}{3}\pi \times (\frac{11}{2\pi})^2 \times 8\frac{1}{2}}{\frac{1}{3}\pi \times (\frac{8\frac{1}{2}}{2\pi})^2 \times 11} = \frac{11}{8\frac{1}{2}} = \frac{22}{17}.$$



Problem 22. Solution: (B).

Because $\angle OAB$ is a $30 - 60 - 90°$ triangle, $BO = 4$, $OA = 8$ and $BA = 4\sqrt{3}$. Let A' and B' be the images of A and B, respectively, under the rotation. Then $B' = (0, 4)$, $B'A'$ is horizontal, and $B'A' = BA = 4\sqrt{3}$. Hence A' is in the second quadrant and $A'(-4\sqrt{3}, 4)$.



1. BASIC KNOWLEDGE

(1) The Sum formula

The following formula can be used to compute the sum of consecutive integers, or sum of consecutive odd/even integers or the sum of a series of integers that have a common difference.

$$S = \frac{(a+b)n}{2} \qquad (1)$$

Where a is the beginning number and b is the ending number. n is the number of terms.

(2). The common difference *d*

For the list of numbers: 1, 2, 3, 4,…, the common difference is $d = 2 - 1 = 3 - 2 = 4 - 3 = 1$.
For the list numbers: 1, 3, 5, 7,…, the common difference $d = 3 - 1 = 5 - 3 = 7 - 5 = 2$.
For the list of numbers: 1, 5, 9, 13,…, the common difference is $d = 5 - 1 = 9 - 5 = 13 - 9 = 4$.

Example 1. Compute:

(1). $1 + 2 + 3 + 4 + 5 + \ldots + 20 = \frac{(a+b)n}{2} = \frac{(1+20)\times 20}{2} = 21\times 10 = 210$.

(2). $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = \frac{(1+15)\times 8}{2} = 16\times 4 = 64$.

(3). What is the sum of all positive odd multiple of 3 that are less than 100?

Solution: The smallest value is 3 and the greatest value is 99:

$3 + 9 + 15 + \ldots + 99 = \frac{(3+99)}{2}\times 17 = 867$.

(3) The number of terms *n*

n = (last term – first term) ÷ d + 1 (2)

For consecutive integers: n = (last term – first term) + 1 (3)

Example 2.

(1). How many terms are there in the sequence 1 + 2 + 3 +.. + 100?

Solution: $n = 100 - 1 + 1 = 100$.

(2). How many terms are there in the sequence 11 + 12 + 13 + .. + 100?

Solution: $n = 100 - 11 + 1 = 90$.

(3). How many terms are there in the sequence 2 + 4 + 6 + .. + 100?

Solution: $n = (100 - 2)/2 + 1 = 50$.

2. PROBLEM SOLVING SKILLS

(1). The sum of series integers

The following formula can be used to compute the sum of series integers that have a common difference.

$$S = m \times n \qquad (3)$$

Where m is the middle number and n is how many numbers in the addition.

Example 3. Compute:

(1). $37 + 38 + 39 + 40 + 41 + 42 + 43 = 40 \times 7 = 280$

(2). $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 9 \times 9 = 81$.

(3). $38 + 39 + 40 + 41 + 42 + 47 = 40 \times 5 + 47 = 200 + 47 = 247$.

(4). $38 + 39 + 40 + 41 + 42 + 43 = (\frac{40+41}{2}) \times 6 = 81 \times 3 = 243$.

☆**Example 4.** The sum of six consecutive positive integers is 2019. What is the largest of these six integers?
(A) 338 (B) 339 (C) 340 (D) 345 (E) 350

Solution: (B).
The average of the six integers is $2019/6 = 336.5$, so $2019 = 334+ 335 + 336 + 337 + 338 + 339$. The largest of the six integers is 339.

(2). The middle number method

Given the sum of a number of consecutive integers, the middle number is the arithmetic mean of these integers. If we have an odd number of consecutive integers, the middle number is an integer. If we have an even number of consecutive integers, the middle number is NOT an integer.

Example 5.

(1). The sum of three consecutive integers is 6. Find the middle number.

Solution:
The middle number is $6/3 = 2$.

(2) The sum of four consecutive integers is 10. Find the middle number.

Solution:
The middle number is $10/ 4 = 2.5$.

Note: The middle number here is the average value of the second and the third terms:

(3) The sum of five consecutive integers is 30. What is the largest of these integers?

Solution:
The third term is 30 / 5 = 6. The fourth term is 7. So the largest term (the 5th term) is 8.

(4) The sum of four consecutive integers is 22. What is the smallest term of these integers?

Solution:
The third term is 22 / 4 = 5.5. The second term is 5.5 – 0.5 = 5 (and the third term is 5.5 + 0.5 = 6). So the smallest term (the first term) is 4.

Example 6. The sum of 101 consecutive integers is 101. What is the largest integer in the sequence?
(A) 101 (B) 102 (C) 51 (D) 52 (E) None of them

Solution: 51.
The average value is 101/101 = 1, which is also the middle term (51th term). So the 101th term is 1 + 50 = 51.

Example 7. If the sum of the consecutive integers from -22 to x, inclusive, is 98, what is the value of x?
(A) 23 (B) 26 (C) 50 (D) 75 (E) 94

Solution: B.
$-22 + -21 + \ldots + 22 + 23 + 24 + 25 + 26 = 23 + 24 + 25 + 26 = 98$. So $x = 26$.

(3). Rules of consecutive integers

(1). A positive integer N can be written as the sum of consecutive integers only when this positive integer cannot be expressed as a power of 2.

(2). If a positive integer N has n odd factors, then there are $n - 1$ ways to express N as the sum of two or more consecutive positive integers.

(3). There is one way to express any odd prime number as the sum of two or more consecutive positive integers.

(4). For some positive integers *m* and *k*,

$$N = m + (m+1) + (m+2) + (m+3) + ... + (m+k-1) \\ = \frac{(m+m+k-1)k}{2} = \frac{(2m+k-1)k}{2} \qquad (4)$$

(5). $2N = (2m+k-1)k$ (5)

(6). If $m = 1$, (4) and (5) become:

$$N = 1+2+3+4+....+n = \frac{(1+n)n}{2} \qquad (6)$$

(7). $2N = (1+n)n$ (7)

(8). k and $(2m+k-1)$ have opposite parity, which means that if k is odd, then $(2m+k-1)$ is even and if k is even, then $(2m+k-1)$ is odd.

(9). $(2m+k-1) > k$ (8)

(10). The greatest number of consecutive integers in a given way of expression.

k_{max} is the greatest number of terms possible:

$$k_{\max} \le \left\lfloor \frac{1}{2}(\sqrt{8N+1}-1) \right\rfloor \qquad (9)$$

(11). The product of any k consecutive positive integers is divisible by k!

Example 8. A list consists of 1000 consecutive odd integers. What is the difference between the greatest number in the list and the least number in the list?
(A) 1000 (B) 1998 (C) 1999 (D) 999 (E) 998

Solution: B.
Method 1:
Let the first odd integer be $2n + 1$ and the last be $2n + m$.
$2n + m = 2n + 1 + (1000 - 1) \times 2$.
Solving for m we get $m = 1 + 1998 = 1999$.
The answer is 1999 – 1 =1998.
Method 2:
Let the 1000 consecutive odd integers be 1, 3, 5,…, x.

$$\frac{x-1}{2}+1=1000 \qquad \Rightarrow \qquad x = 1999.$$

The answer is 1999 – 1 =1998.

Example 9. If 165 is to be written as the sum of n consecutive positive integers, which of the following cannot be the value of n?
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution: 4.
We know that $2N = k\times(2m+k-1)$, where k and $(2m+k-1)$ have opposite parity, and $(2m+k-1) > k$.
$2 \times 165 = 2 \times (3 \times 5 \times 11) = 3 \times (2 \times 5 \times 11) = 5 \times (2 \times 3 \times 11) = 6 \times (5 \times 11)$.
So k can be 2, 3, 5, and 6 but cannot be 4.

Example 10. How many ways to write the number 1995 as the sum of two or more consecutive positive integers?
(A) 16 (B) 15 (C) 7 (D) 5 (E) 19

Solution: B.
$1995 = 5 \times 3 \times 7 \times 19$ has $2 \times 2 \times 2 \times 2 = 16$ odd factors. There are $16 – 1 = 15$ ways to express 1995 as the sum of two or more consecutive positive integers.

Example 11. The sum of n consecutive positive integers is 100. What is the greatest possible value of n?
(A) 8 (B) 25 (C) 9 (D) 26 (E) 50

Solution: A.

We are seeking for the greatest possible value of k. We know that $k < 2m + k - 1$ and k and $2m + k - 1$ have the different parity, so we set the value for k such that it is as close as possible to $2m + k - 1$: $2N = k(2m+k-1) = 2^3 \times 5^2 = 5\times 40 = 8\times 25$. So we get $k = 8$.

Example 12. Write the number 105 as the sum of ten consecutive integers. What is the largest one of the ten numbers?

(A) 15 (B) 25 (C) 14 (D) 24 (E) 21

Solution: A.

We first achieve the middle number: $105 \div 10 = 10.5$. Since 10.5 is not an integer; it is not what we want.

But we can think of 10.5 as one of the ten numbers we want. Then there are 5 numbers to the left of 10.5, where 10 is the nearest number, and 5 numbers to the right of 10.5, where 11 is the nearest one. The five numbers to the right of "10.5" are then 11, 12, 13, 14, and 15. 15 is our answer.

Example 13. The positive integer $N = (n + 4)(n + 5)(n + 6)(n + 7)$ is NOT divisible by which of the following numbers?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 24

Solution: (C).

We know that the product of any k consecutive positive integers is divisible by $k!$. So $N = (n + 4)(n + 5)(n + 6)(n + 7)$ is divisible by $4! = 24$. Thus N is not divisible by 5.

3.PROBLEMS

Problem 1. The sum of five consecutive integers is 5. What is the smallest of these integers?

(A) −1 (B) −2 (C) 1 (D) 2 (E) 5

Problem 2. The number 25 can be written as the sum of two or five consecutive whole numbers. What is the positive difference between the products of the numbers in each set?

(A) 156 (B) 25 (C) 2526 (D) 2364 (E) 2682

Problem 3. The number of sets of two or more consecutive positive integers whose sum is 100 is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 4. 1000 students are arranged in several rows (more than 16). The numbers of students in each row are consecutive positive integers. How many students are there in the first row?

(A) 20 (B) 25 (C) 28 (D) 44 (E) 80

Problem 5. The number 585 can be written as the sum of two consecutive integers, 292 and 293. What is the greatest number of consecutive positive integers whose sum is 585?

(A) 13 (B) 30 (C) 34 (D) 39 (E) 15

Problem 6. How many ways to write the number 2009 as the sum of two or more consecutive positive integers?

(A) 5 (B) 6 (C) 7 (D) 41 (E) 49

Problem 7. The number 210 can be written as the sum of consecutive positive integers in several ways. When written as the sum of the greatest possible number of consecutive positive integers, what is the largest of these integers?

(A) 21 (B) 20 (C) 15 (D) 28 (E) 60

Problem 8. Write the number 2000 as the sum of 25 consecutive even positive integers. What is the smallest number among these 25 numbers?
(A) 12 (B) 25 (C) 24 (D) 56 (E) 80

Problem 9. Express 84 as the sum of k different positive integers. What is the greatest possible value of k?
(A) 8 (B) 11 (C) 12 (D) 13 (E) 10

Problem 10. The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in the incorrect sum of 1986. What was the number of the page that was added twice?
(A) 86 (B) 25 (C) 53 (D) 62 (E) 33

Problem 11. The least integer of a set of consecutive integers is – 100. If the sum of these integers is 410, how many integers are in this set?
(A) 118 (B) 125 (C) 104 (D) 410 (E) 205

Problem 12. The sum of 5 consecutive integers is 9, 000. What is the value of the greatest of these integers??
(A) 8995 (B) 1801 (C) 1800 (D) 1802 (E) 1798

Problem 13. The median of a set of 11 consecutive integers is 98. What is the greatest of these 9 integers?
(A) 94 (B) 95 (C) 103 (D) 102 (E) 104

Problem 14. What is the smallest of 9 consecutive integers if the sum of these integers equals 495?
(A) 56 (B) 55 (C) 51 (D) 49 (E) 486

4. SOLUTIONS

Problem 1. Solution: A.
The third (middle) term is 1, so the second term is 0 and the first term is – 1.

Problem 2. Solution: D.
We can easily write out all the terms:

25 = 12 + 13 = $a + b + 5 + c + d$ = 3 + 4 + 5+ 6 + 7

The product is then 12 × 13 = 156
3 × 4 × 5 × 6 × 7 = 2520
2520 – 156 = 2364.

Problem 3. Solution: B.
$100 = 2^2 \times 5^2$. The number of odd factors of 100 is 3. The solution will be 3 – 1 = 2, or answer choice B.

Problem 4. Solution: C.
Let m be the number of students in the first row.
$m+(m+1)+(m+2)+.....+(m+k-1)=1000$
$(2m+k-1)\times k=1000\times 2=2^4\times 5^3$.
We know that $k > 16$, and $2m + k - 1$ and k must have different parity, so we can only have $(2m+k-1)\times k=1000\times 2=2^4\times 5^3=25\times 80$
So $k = 25$ and $2m + k - 1 = 80$ $\Rightarrow$ $m = 28$.

Problem 5. Solution: B.
Method 1:
If a sum of consecutive positive integers starts with a number greater than 1, then we can think of this sum as the difference between two triangular numbers. The number 595 is the 34th triangular number, and it is 10 more than our number 585. Ten is the fourth triangular number, so this sum must consist of 34 – 4 = 30 consecutive positive integers.

Method 2:
$(2m+k-1)\times k=2N$

We need to write $2N$ as the product of two integers that are as close as possible in order to obtain the greatest number.

$2N = 2\times 585 = 1170 = 2\times 3^2\times 5\times 13 = 30\times 39$

Since k is the smaller value of the two factors, $k = 30$.

Problem 6. Solution: A.

2009 has 6 odd factors. 6 – 1 = 5. There are 5 ways to write 2009 as the sum of two or more consecutive positive integers.

Problem 7. Solution: B.

We have: $2N = (2m + k - 1)k$ or $k(2m + k - 1) = 2\times 2\times 5\times 3\times 7$

We are looking for the greatest possible value of k. We know that $k < 2m + k - 1$ and that k and $2m + k - 1$ have different parity, so we set the value for k so that it is as close as possible to $2m + k - 1$: $k(2m + k - 1) = 20\times 21$.

$k = 20$ and $2m + k - 1 = 21$. We get $k = 20$ and $m = 1$.

The value of the greatest term = $m + k - 1 = 1 + 20 - 1 = 20$.

Problem 8. Solution: D.

We achieve the middle number first: 2000/25 = 80. We know that there are 12 numbers on the left of 80 and that there are also 12 numbers on the right of 80. Since the number 80 is an even number and the difference of two even numbers is 2, we get the smallest number by: 80 – 12 × 2 = 56.

Problem 9. Solution: A.

We know: $m + (m+1) + (m+2) + ... + (m + k - 1) = N$, or

$2N = (2m + k - 1)\times k$.

$(2m + k - 1)\times k = 2\times 84 = 2^3\times 3\times 7$

We know that $k < 2m + k - 1$ and that k and $2m + k - 1$ have different parity, so we set the value for k so that it is as close as possible to $2m + k - 1$:

$k(2m + k - 1) = 8\times 21$.

So $k = 8$.

Note that $2m + k - 1 = 21$ and $m = 7$. 84 = 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14.

Problem 10. Solution: E.

We calculate the possible k values first by using the formula:

$k = \left\lfloor \frac{1}{2}(\sqrt{8 \times 1986 + 1} - 1) \right\rfloor = \lfloor 62.5 \rfloor = 62$.

And then we calculate the correct sum of the numbers:

$N = \frac{(1+62) \times 62}{2} = 1953.$

Finally, we know that the number that was mistakenly added twice is 1986 – 1953 = 33.

Problem 11. Solution: E.

(– 100) + (– 99) + (– 98) + …+ 99 + 98 + 100 + 101 + 102 + 103 + 104 = 101 + 102 + 103 + 104 = 410. So there are 104 – (– 100) + 1 = 205 integers.

Problem 12. Solution: D.

The average of these 5 numbers is 9000/5 = 1800, which is the 3rd number. So the fifth number or the greatest number is 1802.

3rd	4th	5th
1800	1801	1802

Problem 13. Solution: C.

The median of 11 consecutive integers is is also the middle number, the 6th number. So the 11th number is 103, which is the greatest.

6th	7th	8th	9th	10th	11th
98	99	100	101	102	103

Problem 14. Solution: C.

The average of these 9 numbers is 495/9 = 55, which is the 5th number. So the first number or the smallest number is 51.

1st	2nd	3rd	4th	5th
51	52	53	54	55

1. BASIC KNOWLEDGE

1.1. Decimal representation:

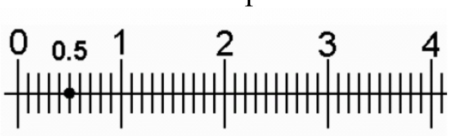
A decimal is used to represent a portion of whole. It contains three parts: an integer (which indicates the number of wholes), a decimal point (which separates the integer on the left from the decimal on the right), and a decimal number (which represents a number between 0 and 1).

The decimal 3.14159 is read "three and fourteen thousand one hundred fifty – nine hundred-thousandths". The dot is the decimal point and is read as "and".

3	**.**	**1**	**4**	**1**	**5**	**9**
Units	And	Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths

Whole numbers can also be written with one or more zeros after the decimal point:
3 = 3.0 = 3.00 = 3.000

The number line representations of decimals:



0.5

0.05

1.2. Terminating decimals and repeating decimals

Any rational number can be expressed as either a terminating decimal or a repeating decimal.
A decimal such as 0.25, which stops, is called a terminating decimal.

A rational number *a/b* in lowest terms results in a repeating decimal if a prime other than 2 or 5 is a factor of the denominator.

$\frac{1}{3} = 0.33333.... = 0.\bar{3}$. $\frac{1}{7} = 0.142857142857.... = 0.\overline{142857}$.

A rational number *a/b* in lowest terms results in a terminating decimal if the only prime factor of the denominator is 2 or 5 (or both).

$\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$, $\frac{1}{10} = 0.1$.

1.3. Repeating block

(1). If the denominator has only 2 and 5 as its factors, this fraction can become a terminating decimal. The length of the decimal part equals the greater power of 2 or 5.

Examples:

	$\frac{1}{5} = 0.2$	$\frac{1}{25} = 0.04$	$\frac{1}{125} = 0.008$	$\frac{1}{625} = 0.0016$
Length of the decimal part	1	2	3	4

(2). If the denominator has only factors other than 2 and 5, then this fraction can become repeating decimal. The length of the repeating block (period) is the smallest number of nines needed for the number containing 9's to be divisible by the denominator.

Examples: For $\frac{1}{11}$, the repeating block is 2 since $\frac{99}{11} = 9$. ($\frac{1}{11} = 0.\overline{09}$)

For $\frac{1}{37}$, the repeating block is 3 since $\frac{999}{37} = 27$. ($\frac{1}{37} = 0.\overline{027}$).

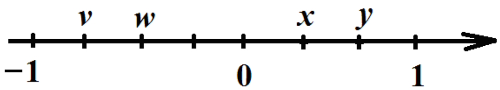
For $\frac{1}{13}$, the repeating block is 6 since $\frac{999999}{13} = 76923$. ($\frac{1}{13} = 0.\overline{076923}$).

(3) If the denominator has 2 or 5 as factors and other prime factors, this fraction can become a mixed repeating decimal. The non-repeating block length is the greater power of 2 or 5. The repeating block length (the repeating period) is the smallest number of nines needed to be divisible by the denominator.

2. PROBLEM SOLVING SKILLS

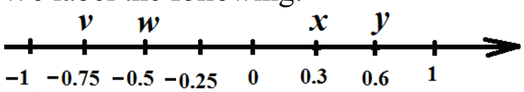
2.1. Decimal representation:

Example 1. The letters v, w, x, and y represent numbers as shown on the number line below. Which of the following expressions has the least value?



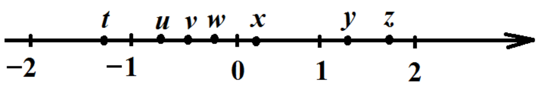
(A) $v+y$ (B) $v-y$ (C) $w+x$ (D) $v-w$ (E) $y-x$

Solution: B.
We label the following:



The least value is $v-y = -0.75 - 0.6 = -1.35$.

Example 2. On the number line above. t, u, v, w, x, y, and z are coordinates of the indicated points. Which of the following is closest in value to $|t+v|$?



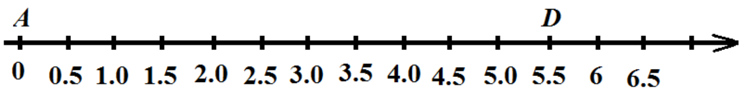
(A) t (B) w (C) x (D) y (E) z

Solution: (E).
$|t+v| = |-1.25 + (-0.5)| = 1.75$

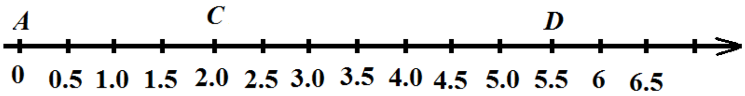
Example 3. Five different points A, B, C, D, and E lie on a line in that order. The length of $\overline{AD}$ is 5.5 and the length of $\overline{BE}$ is 4.5. If the length of $\overline{CD}$ is 3.5, what is one possible value for the length of $\overline{BC}$?

(A) –0.5 (B) 0.5 (C) 1 (D) 1.5 (E) 2

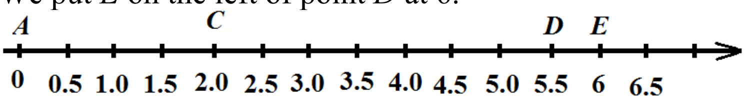
Solution: B.
We mark the number line by the interval of 0.5 and label A at 0 and D at 5.5 so $\overline{AD} = 5.5$.



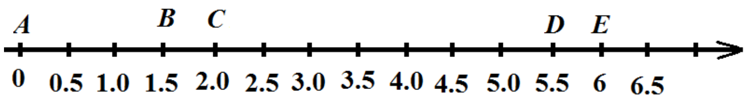
We then label C at 2.0 so that $\overline{CD}$ = 3.5.



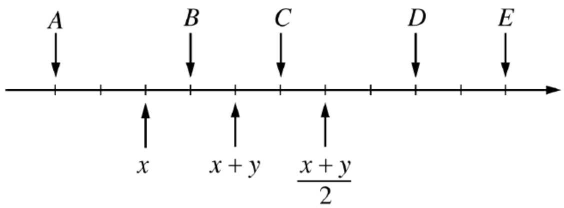
We put E on the left of point D at 6.



We count 4.5 back from E to a point B such that $\overline{BE}$ = 4.5. The value for B is 1.5. $\overline{BC}$ = 2.0 – 1.5 = 0.5.



Example 4. On the number line below, the tick marks are equally spaced. What is the value of D?



(A) –0.6 (B) 0.2 (C) 1 (D) 3 (E) 0

Solution: E.

Let the length of AE be m. So $x+y-x=\frac{2}{10}m$ and $y=\frac{1}{5}m$.

We have $\frac{x+y}{2}-(x+y)=\frac{2}{10}m \quad \Rightarrow \quad \frac{-x-y}{2}=\frac{2}{10}m \quad \Rightarrow \quad x=\frac{-3m}{5}$.

$D=x+3y=\frac{-3m}{5}+\frac{3m}{5}=0$.

2.2. Converting Repeating Decimals to Fractions

Rule 1: The fraction of any single digit repeating decimal is the digit over 9: $0.\bar{5} = \frac{5}{9}$.

Example 5. Convert the repeating decimal $0.\bar{1}$ to a fraction.

(A) $\frac{1}{9}$ (B) $\frac{2}{9}$ (C) $\frac{4}{9}$ (D) $\frac{5}{9}$ (E) $\frac{7}{9}$

Solution: A.

Method 1:

Following Rule 1, we have: $0.\bar{1} = \frac{1}{9}$.

Method 2:

Let $x = 0.\bar{1} = 0.111111\ldots$ (1)

Multiply both sides by 10: $10x = 1.11111...$ (2)

(2) – (1): $9x = 1$ $\Rightarrow$ $x = \frac{1}{9}$.

Rule 2: The fraction of any 2-digit repeating decimal is the digits over 99: $0.\overline{55} = \frac{55}{99}$.

Example 6. Convert the repeating decimal $0.\overline{25}$ to a fraction.

(A) $\frac{2}{9}$ (B) $\frac{5}{9}$ (C) $\frac{2}{99}$ (D) $\frac{5}{99}$ (E) $\frac{25}{99}$

Solution: E.

Method 1:

Following Rule 2, we have: $0.\overline{25} = \frac{25}{99}$.

Method 2:

Let $x = 0.\overline{25} = 0.2525....$ (1)

Multiply both sides of (1) by 100: $100x = 25.25252525...$ (2)

(2) – (1): $99x = 25.0000$

Solving for x: $x = \dfrac{25}{99}$.

Example 7. Express $0.\overline{3123}$ as a common fraction.

(A) $\dfrac{3123}{9999}$ (B) $\dfrac{347}{1111}$ (C) $\dfrac{31}{99}$ (D) $\dfrac{312}{999}$ (E) $\dfrac{1}{9}$

Solution: B.

$$0.\overline{3123} = \frac{3123}{9999} = \frac{347}{1111}.$$

Rule 3: Repeating decimal with non repeating part:

$$0.17\overline{857142} = 0.\underbrace{17}_{2\text{ digits}}\underbrace{\overline{857142}}_{6\text{ digits}} = \frac{17857142-17}{\underbrace{999999}_{6\ 9\text{'s}}\underbrace{00}_{2\ 0\text{'s}}}$$

(1) Count the number of digits of non repeating part (2, in this case),
(2) Count the number of digits of repeating part (6, in this case).
(3) Write the numerator and the denominator. Numerator: the decimal part (17857142, in this case) – the non repeating part (17, in this case). The denominator: the number of 9's (6, in this case since it has 6 repeating digits) followed by the number of zero's (2, in this case since it has 2 non repeating digits).

Example 8. Express $0.72\overline{45}$ as a common fraction.

(A) $\dfrac{7245}{9999}$ (B) $\dfrac{7173}{9900}$ (C) $\dfrac{797}{1100}$ (D) $\dfrac{7173}{9999}$ (E) $\dfrac{71}{99}$

Solution: C.

Method 1: Following Rule 3: $0.72\overline{45} = \dfrac{7245-72}{9900} = \dfrac{7173}{9900} = \dfrac{797}{1100}$

Method 2: Let $x = 0.72\overline{45}$ (1)

$100x = 72.\overline{45}$ (2)

$10000x = 7245.\overline{45}$ (3)

(3) – (2): $9900x = 7173$ (4)

$x = \dfrac{7173}{9900} = \dfrac{797}{1100}$.

2.3. Operation with Repeating Decimals

Example 9. Calculate: $2.\overline{48} + 2.\overline{83}$

(A) $5.\overline{32}$ (B) $5.\overline{31}$ (C) $5.\overline{33}$ (D) $4.\overline{32}$ (E) $4.\overline{33}$

Solution: A.

Since $\begin{array}{r} 2.48 \\ +2.83 \\ \hline 5.31 \end{array}$, and the sum of the first repeating digits (4 in $2.\overline{48}$ and 8 in $2.\overline{83}$) in the addends carries 1, so the last digit of the resulting number needs to be increased by 1: $2.\overline{48} + 2.\overline{83} = 5.\overline{32}$.

Example 10. Calculate: $3.\overline{215} - 1.\overline{307}$

(A) $1.\overline{908}$ (B) $1.\overline{809}$ (C) $1.\overline{907}$ (D) $2.\overline{907}$ (E) $2.\overline{908}$

Solution: C.

Since $\begin{array}{r} 3.215 \\ -1.307 \\ \hline 1.908 \end{array}$, and the first repeating digit (2 in $3.\overline{215}$) in the top number is smaller than the corresponding digit (3 in $1.\overline{307}$) in the bottom number, so the last digit of the resulting number needs to be decreased by 1: $3.\overline{215} - 1.\overline{307} = 1.\overline{907}$.

Example 11. Calculate: $2.25 - 1.3\overline{6}$

(A) $\dfrac{795}{999}$ (B) $\dfrac{795}{900}$ (C) $\dfrac{797}{999}$ (D) $\dfrac{797}{990}$ (E) $\dfrac{7}{99}$

Solution: B.

$2.25-1.3\bar{6}=2.25\bar{0}-1.3\bar{6}=2.25\bar{0}-1.36\bar{6}=0.88\bar{3}$ or

$2.25-1.3\bar{6}=\frac{225}{100}-\frac{136-13}{90}=\frac{2025-1230}{900}=\frac{795}{900}$.

We see that both expressions are the same: $0.88\bar{3}=\frac{883-88}{900}=\frac{795}{900}$.

Example 12. 18. Evaluate $0.\overline{123}-0.\overline{12}$.

(A) $0.\overline{001912}$ (B) $0.\overline{001910}$ (C) $0.\overline{01911}$ (D) $0.\overline{001911}$ (E) $0.\overline{01912}$

Solution: D.

$$\begin{array}{r} 0.123123 \\ -\ 0.121212 \\ \hline 0.001911 \end{array}$$

Since the above subtraction holds, the solution is $0.\overline{001911}$

Example 13. What is the ratio of $0.1\bar{6}$ to $0.8\bar{3}$?

(A) $\frac{16}{83}$ (B) $\frac{1}{9}$ (C) $\frac{7}{9}$ (D) $\frac{1}{5}$ (E) $\frac{7}{99}$

Solution: D.

$0.1\bar{6}=\frac{16-1}{90}=\frac{15}{90}$, $0.8\bar{3}=\frac{83-8}{90}=\frac{75}{90}$ $\Rightarrow$ $\frac{15}{90}\div\frac{75}{90}=\frac{1}{5}$.

Example 14. How many positive integers less than 100 have reciprocals with terminating decimal representations?

(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

Solution: B.

The reciprocals of numbers can be a terminating decimal:

2	4	8	16	32	64
5	25				
10	20	40	80		
50					
1					

2.4. Decimals and Fractions

Example 15. Write the decimal equivalent of $\frac{3}{11}$.

(A) $0.\overline{27}$ (B) $0.\overline{25}$ (C) $0.\overline{23}$ (D) $0.\overline{21}$ (E) $0.\overline{20}$

Solution: A.

$\frac{1}{11} = 0.\overline{09} \quad \Rightarrow \quad 3\times\frac{1}{11} = 3\times 0.\overline{09} = 0.\overline{27}$.

Example 16. Express $\frac{17}{35}$ as a decimal rounded to the nearest tenth.

(A) 0.5 (B) 0.6 (C) 0.4 (D) 0.49 (E) 0.496

Solution: A.

$\frac{17}{35} = 0.485714 = 0.5$.

Example 17. What is the absolute value of the difference between $0.\overline{35}$ and $0.\overline{49}$?

(A). 11/90 (B). 11/99 (C) 13/99 (D). 14/99 (E). 14/90

Solution: (D).

$0.\overline{49} - 0.\overline{35} = \frac{49}{99} - \frac{35}{99} = \frac{14}{99}$.

Example 18. Express the sum $0.\overline{14} + 0.1\overline{4}$ as a common fraction.

(A) $\frac{283}{990}$ (B) $\frac{283}{999}$ (C) $\frac{143}{990}$ (D) $\frac{13}{90}$ (E) $\frac{142}{495}$

Solution: A.

Method 1: $0.\overline{14} + 0.1\overline{4} = 0.1\overline{41} + 0.1\overline{44} = 0.2\overline{85} = \frac{285-2}{990} = \frac{283}{990}$

Method 2: $0.\overline{14} + 0.1\overline{4} = \frac{14}{99} + \frac{14-1}{90} = \frac{140+143}{990} = \frac{283}{990}$.

Example 19. Express $0.\overline{1}+0.\overline{01}+0.\overline{0001}$ as a common fraction.

(A) $\frac{121}{990}$ (B) $\frac{121}{999}$ (C) $\frac{1213}{9990}$ (D) $\frac{1213}{9999}$ (E) $\frac{1213}{9900}$

Solution: D.

$0.\overline{1}+0.\overline{01}+0.\overline{0001}=\frac{1}{9}+\frac{1}{99}+\frac{1}{9999}=\frac{1213}{9999}$.

Example 20. Express 3.010101010101 . . . as a mixed number.

(A) $3\frac{1}{99}$ (B) $3\frac{1}{90}$ (C) $3\frac{2}{99}$ (D) $3\frac{4}{99}$ (E) $3\frac{1}{89}$

Solution: 3.010101010101 . . = 3 + $0.\overline{01}=3+\frac{1}{99}=3\frac{1}{99}$.

2.5. Decimals and percents

To express a decimal as a percent, you just need to move the decimal point two times to the right, then add %.

Example 21. Write the percent equivalent of 1.5.

(A) 150% (B) 15% (C) 1500% (D) 1.5% (E) 50%

Solution: A.

$1.5=\frac{1.5}{1}=\frac{150}{100}=150\%$.

Example 22. 2.5 is what percent of 40?

(A) 25% (B) 15% (C) 6.25% (D) 5% (E) 50%

Solution: C.

$2.5 = x \times 40 \Rightarrow x = 6.25\%$.

Example 23. What number is 10% of 20% of 30% of 40?

(A). 1 (B). 0.82 (C) 0.64 (D). 0.44 (E). 0.24

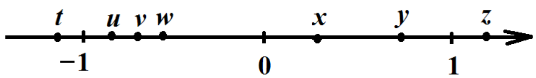
Solution: (E).
$10\% \times 20\% \times 30\% \times 40 = 0.1 \times 0.2 \times 0.3 \times 40 = 0.24$.

Example 24. Calculate: $100 \times (0.1\overline{6} - 0.16)$.
(A) 1 (B) $0.\overline{66}$ (C) $0.\overline{16}$ (D) $0.0\overline{6}$ (E) $0.\overline{6}$

Solution: E.
$100 \times (0.1\overline{6} - 0.16) = 100 \times (0.16\overline{6} - 0.16\overline{0})$ $100 \times (0.00\overline{6}) = 0.\overline{6}$.

3. MORE EXAMPLES

Example 25. On the number line below, which of the following corresponds to $|x - w|$?



(A) t (B) v (C) x (D) y (E) z

Solution: D.
$|x - w| = |0.25 - (-0.5)| = 0.75$.

Example 26. The total cost of 6 equally priced mechanical pencils is \$4.50.If the cost per pencil is increased by \$0.5, how much will 15 of these pencil cost at the new rate?
(A) \$11.25 (B) \$18.75 (C) \$19.00 (D) \$19.50 (E) \$10.00

Solution: (B).
Each pencil costs 4.5 ÷ 6 = \$0.75. When the cost per pencil is increased by \$0.5, each pencil costs \$0.75 + 0.5 = \$1.25. 15 pencils cost \$1.25 × 15 = \$18.75.

Example 27. Which of the following numbers is between 0.599 and 0.600?
I. 0.5955 II. 0.5994 III. 0.6001

(A) I only (B) II only (C) III only (D) I and II (E) II and III

Solution: (B).
0.5955 < 0.5990. 0.6001 > 0.6000.

Example 28. Calculate: $1.\overline{36}+2.\overline{375}$

Solution: $3.\overline{739011}$.
Since the first addend has 2 repeating digits and the second addend has 3 repeating digits, the sum should have 2 × 3 = 6 repeating digits (the least common multiple of 2 and 3).

$$\begin{array}{r} 1.363636 \\ +\,2.375375 \\ \hline 3.739011 \end{array}$$

So $1.\overline{36}+2.\overline{375}=1.\overline{363636}+2.\overline{375375}=3.\overline{739011}$.

Example 29. Express $1.4\overline{12}$ as a common fraction.

(A) $\dfrac{68}{165}$ (B) $\dfrac{233}{165}$ (C) $\dfrac{121}{990}$ (D) $\dfrac{68}{165}$ (E) $\dfrac{41}{330}$

Solution: B.

$$1.4\overline{12}=1+0.4\overline{12}=1+\frac{412-4}{990}=1+\frac{408}{990}=1+\frac{408\div 6}{990\div 6}=1+\frac{68}{165}=\frac{233}{165}.$$

Example 30. Express as a mixed number: $\dfrac{0.\overline{85}}{0.\overline{25}}$.

(A) $3\dfrac{1}{5}$ (B) $3\dfrac{1}{7}$ (C) $3\dfrac{2}{9}$ (D) $3\dfrac{2}{5}$ (E) $3\dfrac{1}{8}$

Solution: D.

Method 1: $\dfrac{0.\overline{85}}{0.\overline{25}}=\dfrac{17}{5}=3\dfrac{2}{5}$.

Method 2: $\dfrac{0.\overline{85}}{0.\overline{25}}=\dfrac{\frac{85}{99}}{\frac{25}{99}}=\dfrac{17}{5}=3\dfrac{2}{5}$.

Example 31. Express the following as a fraction in lowest terms:
$0.\overline{1}+0.\overline{2}\ +\ 0.\overline{01}\ +\ 0.\overline{02}$.

(A) $\frac{4}{11}$ (B) $\frac{1}{9}$ (C) $\frac{2}{9}$ (D) $\frac{3}{11}$ (E) $\frac{35}{99}$

Solution: A.

Method 1: $0.\overline{1}\ +\ 0.\overline{2}\ +\ 0.\overline{01}\ +\ 0.\overline{02}=0.\overline{11}+0.\overline{22}+0.\overline{01}+0.\overline{02}=0.\overline{36}=\frac{36}{99}=\frac{4}{11}$.

Method 2: $0.\overline{1}\ +\ 0.\overline{2}\ +\ 0.\overline{01}\ +\ 0.\overline{02}=\frac{1}{9}+\frac{2}{9}+\frac{1}{99}+\frac{2}{99}=\frac{36}{99}=\frac{4}{11}$.

Example 32. Find the product $(\frac{3}{11})(2.\overline{4})\times(0.\overline{6})$

(A) $\frac{4}{9}$ (B) $\frac{1}{9}$ (C) $\frac{2}{9}$ (D) $\frac{7}{9}$ (E) $\frac{5}{9}$

Solution: A.

$(\frac{3}{11})(2.\overline{4})\times(0.\overline{6})=\frac{3}{11}\times 2\frac{4}{9}\times\frac{6}{9}=\frac{3}{11}\times\frac{22}{9}\times\frac{6}{9}=\frac{4}{9}$.

Example 33. Calculate and express your answer as a common fraction: $\frac{0.\overline{3}+0.1\overline{2}}{0.\overline{3}-0.1\overline{2}}$.

(A) $\frac{24}{11}$ (B) $\frac{41}{19}$ (C) $\frac{11}{90}$ (D) $\frac{19}{90}$ (E) $\frac{31}{99}$

Solution: B.

We know that $0.\overline{3}=\frac{3}{9}$ and $0.1\overline{2}=\frac{12-1}{90}=\frac{11}{90}$.

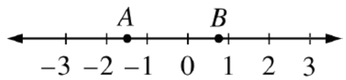
$0.\overline{3}\ +\ 0.1\overline{2}\ =\ \frac{3}{9}+\frac{11}{90}=\frac{30}{90}+\frac{11}{90}=\frac{41}{90}$, and $0.\overline{3}\ -\ 0.1\overline{2}\ =\ \frac{3}{9}\ -\ \frac{11}{90}=\frac{30}{90}-\frac{11}{90}=\frac{19}{90}$.

$\frac{0.\overline{3}+0.1\overline{2}}{0.\overline{3}-0.1\overline{2}}=\frac{\frac{41}{90}}{\frac{19}{90}}=\frac{41}{19}$.

4. PROBLEMS

Problem 1. If $6.565 = 65(x - 1)$. Then $x =$
(A) 0.11 (B) 1.1 (C) 1.899 (D) 0.899 (E) 1.001

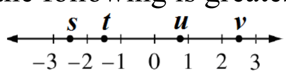
Problem 2. In the figure below, if the coordinates of points A and B are added together, the result will be the coordinate of a point between which two consecutive integers?



(A) −3 and −2 (B) −2 and −1 (C) −1 and 0 (D) 0 and 1 (E) 2 and 3

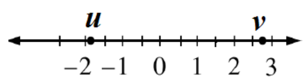
Problem 3. When 1.732 is rounded to the nearest whole number, the result is how much greater than when 1.732 is rounded to the nearest tenth?
(A) 0.3 (B) 0.1 (C) 0.2 (D) 0.4 (E) 0.5

Problem 4. If s, t, u, and v are the coordinates of the indicated points on the number line below, which of the following is greatest?



(A) $|t+s|$ (B) $|v+s|$ (C) $|t-s|$ (D) $|v-s|$ (E) $|u+s|$

Problem 5. On the number line below, which of the following is the best approximation of $|u-v|$?



(A) 2 (B) 3 (C) 3.5 (D) 4 (E) 4.5.

Problem 6. Calculate: $1.\overline{436} - 0.\overline{312}$.
(A) $1.\overline{124}$ (B) $1.\overline{125}$ (C) $1.\overline{123}$ (D) 1.124 (E) 1.125

Problem 7. Calculate: $2.\overline{35} + 0.3\overline{42}$.
(A) $2.\overline{662}$ (B) $2.6\overline{95}$ (C) $2.\overline{695}$ (D) $2.6\overline{92}$ (E) $2.\overline{692}$

Problem 8. Calculate: $0.\overline{142857} \times 3.\overline{5}$.

(A) $\frac{142857}{999999}$ (B) $\frac{15872}{111111}$ (C) $\frac{11}{90}$ (D) $\frac{19}{90}$ (E) $\frac{32}{63}$

Problem 9. Express the product $0.\overline{63} \times 3.\overline{6}$ as a common fraction.

(A) $\frac{3}{7}$ (B) $\frac{9}{5}$ (C) $\frac{2}{9}$ (D) $\frac{7}{3}$ (E) $\frac{5}{9}$

Problem 10. Express as a fraction: $0.\overline{1} + 0.\overline{001}$.

(A) $\frac{11}{90}$ (B) $\frac{37}{333}$ (C) $\frac{112}{999}$ (D) $\frac{1}{9}$ (E) $\frac{1}{999}$

Problem 11. Express as a common fraction: $0.\overline{2} \times 0.\overline{4}$.

(A) $\frac{8}{81}$ (B) $\frac{4}{9}$ (C) $\frac{2}{9}$ (D) $\frac{8}{9}$ (E) $\frac{5}{9}$

Problem 12. Express the difference between $0.\overline{5}$ and $\frac{1}{3}$ as a common fraction.

(A) $\frac{3}{9}$ (B) $\frac{9}{2}$ (C) $\frac{2}{9}$ (D) $\frac{7}{9}$ (E) $\frac{5}{9}$

Problem 13. Express as a common fraction: $(0.\overline{09})(0.\overline{7})$.

(A) $\frac{7}{90}$ (B) $\frac{7}{33}$ (C) $\frac{7}{99}$ (D) $\frac{7}{9}$ (E) $\frac{1}{11}$

Problem 14. Express 10% of 30% of 50 as decimal.

(A) 1.3 (B) 1.1 (C) 1.2 (D) 1.4 (E) 1.5

Problem 15. How many of the first 10 positive integers have reciprocals that are repeating decimals?

(A) 0 (B) 1 (C) 2 (D) 4 (E) 5

Problem 16. Express as a percent: $(1\frac{1}{3})(0.375)(1\frac{3}{5})(1.25)$.

(A) 25% (B) 15% (C) 6.25% (D) 100% (E) 50%

Problem 17. Find the product of $0.5\overline{12}$ and 33.

(A) 16.9 (B) 20.4 (C) 16.89 (D) 16.90 (E) 16.5

Problem 18. What is the 453rd digit to the right of the decimal point in the decimal expansion of $\frac{6}{13}$?

(A) 8 (B) 1 (C) 3 (D) 4 (E) 5

Problem 19. Calculate and express your answer as a common fraction: $\dfrac{0.\bar{3}+0.\overline{12}}{0.\bar{3}-0.\overline{12}}$.

(A) $\frac{7}{15}$ (B) $\frac{5}{11}$ (C) $\frac{7}{33}$ (D) $\frac{7}{9}$ (E) $\frac{15}{7}$

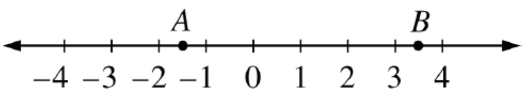
Problem 20. What is the reciprocal of $0.\bar{3}+0.25$?

(A) $\frac{12}{7}$ (B) $\frac{7}{12}$ (C) $\frac{1}{3}$ (D) $\frac{1}{11}$ (E) $\frac{14}{7}$

Problem 21. If the product of 0.6 and a number is equal to 1, what is the number?

(A) 1.7 (B) $1.\bar{6}$ (C) $1.\overline{16}$ (D) 1.16 (E) 1.6

Problem 22. Which of the following is the best estimate of the length of segment *AB* on the number line below?



(A) 3.5 (B) 4 (C) 4.5 (D) 5 (E) 6

Problem 23. Calculate: $3.\overline{14}+0.\overline{32}$.

(A) 3.46 (B) $3.\overline{46}$ (C) $3.\overline{47}$ (D) $3.\overline{45}$ (E) 3.47

Problem 24. Calculate: $0.\overline{27} \div 3.\overline{2}$

(A) $\frac{9}{29}$ (B) $\frac{27}{99}$ (C) $\frac{27}{319}$ (D) $\frac{26}{319}$ (E) $\frac{1}{99}$

Problem 25. Express $0.\overline{25} \div 0.\overline{5}$ as a common fraction.

(A) $\frac{7}{15}$ (B) $\frac{5}{11}$ (C) $\frac{3}{7}$ (D) $\frac{7}{9}$ (E) $\frac{11}{5}$

Problem 26. Express as a common fraction: $0.\overline{9} - 0.\overline{2}$.

(A) $\frac{2}{9}$ (B) $\frac{8}{9}$ (C) $\frac{7}{33}$ (D) $\frac{7}{9}$ (E) $\frac{9}{7}$

Problem 27. Express as a decimal: 222% of $\frac{1}{2}$.

(A) 1.11 (B) 1.12 (C) 1.09 (D) 2.22 (E) 2.21

Problem 28. What is the 2015th digit to the right of the decimal point in the decimal expansion of $\frac{1}{13}$?

(A) 0 (B) 7 (C) 2 (D) 3 (E) 6

5. SOLUTIONS

Problem 1. Solution: E.
6.565 = 65(x – 1) ⇒ 0.101 = x – 1 ⇒ x = 1 + 0.101 = 1.101.

Problem 2. Solution: C.
We estimate that A = – 1.5 and B = 0.8. A + B = – 1.5 + 0.8 = – 0.7 which is between – 1 and 0.

Problem 3. Solution: A.
When 1.732 is rounded to the nearest whole number: 1.732 = 2.
When 1.732 is rounded to the nearest tenth: 1.732 = 1.7. The answer is 2 – 1.7 = 0.3.

Problem 4. Solution: D.
Since s is negative, the greatest value will be $v - s \approx 2.5 - (-2.5) = 5$.

Problem 5. Solution: E.
u = – 1.8 and v = 2.7. $|u-v|=|-1.8-2.7|=4.5$.

Problem 6. Solution: A.

Since $\begin{array}{r} 1.436 \\ -0.312 \\ \hline 1.124 \end{array}$, $1.\overline{436}-0.\overline{312}=1.\overline{124}$.

Problem 7. Solution: B.

$2.\overline{35}=2.3535353....=2.3\overline{53}$ and $\begin{array}{r} 0.342 \\ +2.353 \\ \hline 2.695 \end{array}$, so $0.3\overline{42}+2.\overline{35}=0.3\overline{42}+2.3\overline{53}=2.6\overline{95}$.

Problem 8. Solution: E.

$$0.\overline{142857}\times 3.\bar{5}=\frac{142857}{999999}\times 3\frac{5}{9}=\frac{1}{7}\times 3\frac{5}{9}=\frac{1}{7}\times\frac{32}{9}=\frac{32}{63}.$$

Problem 9. Solution: D.

$0.\overline{63}\times 3.\bar{6}=\frac{63}{99}\times(3+\frac{6}{9})=\frac{63}{99}\times\frac{33}{9}=\frac{7}{3}$.

Problem 10. Solution: C.

Method 1: $0.\bar{1}+0.\overline{001}=0.\overline{111}+0.\overline{001}=0.\overline{112}=\frac{112}{999}$.

Method 2: $0.\bar{1}+0.\overline{001}=\frac{1}{9}+\frac{1}{999}=\frac{112}{999}$.

Problem 11. Solution: A.

$0.\bar{2}\times 0.\bar{4}=\frac{2}{9}\times\frac{4}{9}=\frac{8}{81}$.

Problem 12. Solution: C.

Method 1: $0.\bar{5}-\frac{1}{3}=\frac{5}{9}-\frac{3}{9}=\frac{2}{9}$.

Method 2: $0.\bar{5}-0.\bar{3}=0.\bar{2}=\frac{2}{9}$.

Problem 13. Solution: C.

Method 1: $(0.\overline{09})(0.\bar{7})=(0.\overline{09})(0.\overline{77})=0.\overline{07}=\frac{7}{99}$.

Method 2: $(0.\overline{09})(0.\bar{7})=\frac{9}{99}\times\frac{7}{9}=\frac{7}{99}$.

Problem 14. Solution: E.

10% × 30% × 50 = 0.3 × 5 =1.5.

Problem 15. Solution: D.

4 integers. 1, 2, 4, 5, 8, and 10 will generate terminating decimals. 10 – 6 = 4.

Problem 16. Solution: D.

$(1\frac{1}{3})(0.375)(1\frac{3}{5})(1.25)=\frac{4}{3}\times\frac{3}{8}\times\frac{8}{5}\times\frac{5}{4}=1=100\%$.

Problem 17. Solution: A.

$$(0.5+0.0\overline{12})\times 33=16.5+\frac{12}{990}\times 33=16.5+\frac{12}{30}=16.9.$$

Problem 18. Solution: B.

Since $\frac{1}{13}=0.\overline{076923}$, we have: $6\times\frac{1}{13}=6\times 0.\overline{076923}=0.\overline{461538}$.

The repeating block is 6 and 453 = 6 × 75 + 3.

The 453rd digit is the same as the third digit which is 1.

Problem 19. Solution: E.

$$0.\overline{3}\ +\ 0.\overline{12}\ =\ 0.\overline{33}\ +\ 0.\overline{12}\ =\ 0.\overline{45},\ 0.\overline{3}\ -\ 0.\overline{12}\ =\ 0.\overline{33}\ -\ 0.\overline{12}\ =\ 0.\overline{21}.$$

$$0.\overline{45}\ \div\ 0.\overline{21}\ =\ \frac{0.\overline{45}}{0.\overline{21}}=\frac{15}{7}.$$

Problem 20. Solution: A.

$0.\overline{3}+0.25=0.33\overline{3}+0.25\overline{0}=0.58\overline{3}=\frac{583-58}{900}=\frac{7}{12}$. The reciprocal will be $\frac{12}{7}$.

Problem 21. Solution: B.

Let the number be x. $0.6x=1\ \Rightarrow\qquad x=\frac{1}{0.6}=1.\overline{6}$.

Problem 22. Solution: D.

The length of segment AB = $B - A$ = 3.5 – (1 – 1.5) = 5.

Problem 23. Solution: B.

Since $\begin{array}{r} 3.14 \\ +0.32 \\ \hline 3.46 \end{array}$, so $3.\overline{14}+0.\overline{32}=3.\overline{46}$.

Problem 24. Solution: C.

$$0.\overline{27}\div 3.\overline{2}=\frac{27}{99}\div 3\frac{2}{9}=\frac{27}{99}\div\frac{29}{9}=\frac{27}{99}\times\frac{9}{29}=\frac{27}{319}.$$

Problem 25. Solution: B.

Method 1: $0.\overline{25} \div 0.\overline{5} = \frac{25}{99} \div \frac{5}{9} = \frac{25}{99} \times \frac{9}{5} = \frac{5}{11}$.

Method 2: $0.\overline{25} \div 0.\overline{5} = 0.\overline{25} \div 0.\overline{55} = \frac{0.\overline{25}}{0.\overline{55}} = \frac{5}{11}$.

Problem 26. Solution: D.

Method 1: $0.\overline{9} - 0.\overline{2} = 0.\overline{7} = \frac{7}{9}$.

Method 2: $0.\overline{9} - 0.\overline{2} = \frac{9}{9} - \frac{2}{9} = \frac{7}{9}$.

Problem 27. Solution: A.

Method 1: 222% × $\frac{1}{2} = \frac{222}{100} \times \frac{1}{2} = \frac{111}{100} = 1.11$.

Method 2: 222% × $\frac{1}{2} = 2.22 \times \frac{1}{2} = 1.11$.

Problem 28. Solution: C.

For $\frac{1}{13}$, the repeating block is 6 since $\frac{999999}{13} = 76923$ or $\frac{1}{13} = 0.\overline{076923}$.

2015 = 335 × 6 + 5. So the 2015th digit is the same as the 5th digit. The answer is 2.

1. BASIC KNOWLEDGE

Definitions

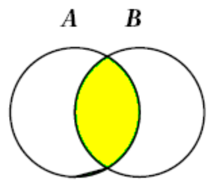
Sets: A set is any well-defined collection of objects. Individual objects are called the elements or members of the set.

Subsets: A subset is a sub-collection of a set. We denote set B as subset of A by the notation $B \subseteq A$.

Proper subset: If B is a subset of A and B is not equal to A, B is a proper subset of A, written as $B \subset A$.

Universal set: denoted by U, is the set that contains all elements considered in a given discussion.

Intersection of Sets



$A \cap B$

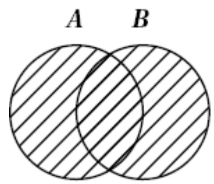
The intersection of sets A and B, written as $A \cap B$, is the set of all elements belonging to both A and B.
$A \cap B \Rightarrow A$ and B
* One simple way to remember this:
note $\cap$ is like the second letter in the word "and."

Example 1. Find $A \cap B$ if Let $A = \{1, 2, 3, 4, 5, 6\}$, and $B = \{1, 3, 5, 7, 9\}$.

Solution:
$A \cap B = \{1, 3, 5\}$.

Union of sets

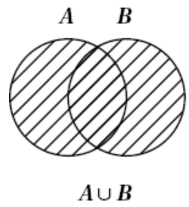
The union of sets A and B, written as $A \cup B$, is the set of all elements belonging to either A or B.
*Note the symbol $\cup$ is like the first letter in the word "Union".



$A \cup B$

Example 2. Find $A \cup B$ if Let $A = \{1, 2, 3, 4, 5, 6\}$, and $B = \{1, 3, 5, 7, 9\}$.

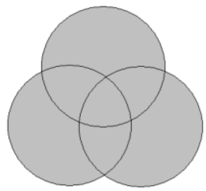
Solution:
$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}$.



The Union Formula for Two Events
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

The Union Formula for Three Events
The union of sets A, B, and C, written as $A \cup B \cup C$, is the set of all elements belonging to A, or B, or C.

$n(A \cup B \cup C) = n(A) + n(B) + n(C)$
$- n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$



2. PROBLEM SOLVING SKILLS

(1). Calculation of the number of subsets

The number of subsets of a set with n elements is 2^n.

Example 3. How many subsets of {C, H, E, N , P} have an even number of elements?

Solution:
$2^5/2 = 16$.

☆ **Example 4.** (2008 AMC 10A) Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20 (B) 40 (C) 60 (D) 160 (E) 320

Solution: (B).

Method 1 (official solution):

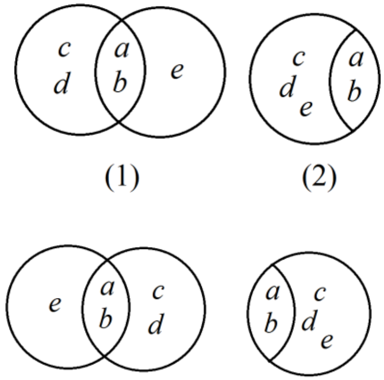
Let the two subsets be A and B. There are $\binom{5}{2} = 10$ ways to choose the two elements common to A and B. There are then $2^3 = 8$ ways to assign the remaining three elements to A or B, so there are 80 ordered pairs ($A;B$) that meet the required conditions. However, the ordered pairs (A,B) and (B, A) represent the same pair $\{A, B\}$ of subsets, so the conditions can be met in $80 \div 2 = 40$ ways.

Method 2 (our solution):
We have two cases
Case 1: 2 + 2 + 1 = 5.

$$\binom{5}{2}\binom{3}{2}\binom{1}{1} = 30$$

Case 2: 3 + 2 = 5.

$$\binom{5}{3}\binom{2}{2} = 10.$$



(1) (2) (3) (4)

The answer is 30 + 10 = 40.
Note that the ordered pairs (A,B) and (B, A) represent the same pair $\{A, B\}$ of subsets, i.e. case 3 is the same as case 1 and case 4 is the same as case 2.

(2). Calculation of the number of proper subsets

The number of proper subsets of a set with n elements is $2^n - 1$.

Example 5. How many proper subsets are there for {J, U, L, I, A}?

Solution:
$2^5 - 1 = 31$.

(3). Calculation of the number of elements

The number of positive integers from 1 to n (not bigger than n) that are divisible by d is $\left\lfloor \frac{n}{d} \right\rfloor$.

$\lfloor x \rfloor$ is called the floor function. Whenever we see this notation, we take the greatest integer value that does not exceed x. For examples,
$\lfloor 3.14 \rfloor = 3,\ \lfloor 2.7 \rfloor = 2, \text{and} \lfloor 9.9 \rfloor = 9.$

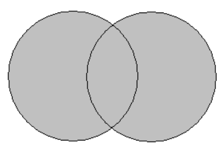
Example 6. Consider the set A = (1, 2, 3, … 57). How many are neither divisible by 3 nor 5?
(A) 30 (B) 27 (C) 25 (D) 21 (E) 20

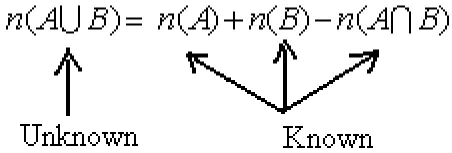
Solution: A.
Let $A_3 \subset A$ be the set of those integers divisible by 3 and $A_5 \subset A$ be the set of those integers divisible by 5. $n\ (A_3 \cup A_5) = n\ (A_3\ ) + n\ (\ A_5) - n\ (A_3 \cap A_5) =$
$$\left\lfloor \frac{57}{3} \right\rfloor + \left\lfloor \frac{57}{5} \right\rfloor - \left\lfloor \frac{57}{15} \right\rfloor = 19 + 11 - 3 = 27.$$
$n\ (A) - n\ (A_3 \cup A_5) = 57 - 27 = 30.$

(4). Problems Involved Two Events

Example 7. Every member of a math club is taking algebra or geometry and 8 are taking both. If there are 17 taking algebra and 13 taking geometry, how many members are in the club?
(A) 30 (B) 25 (C) 22 (D) 21 (E) 20

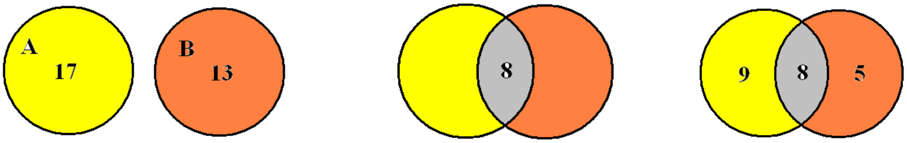
Solution: C.
The question asks to find the number of students in the math club.
This is the same as finding the union of two sets.
Set A: number of students taking algebra (17).
Set B: number of students taking geometry (13).
In the formula:





We have: $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 17 + 13 - 8 = 22$.

The Venn diagrams are as follows:

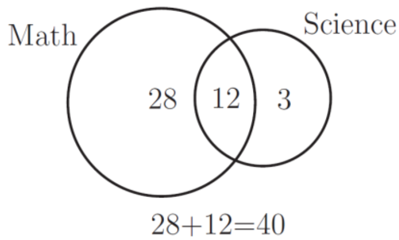


☆**Example 8.** At Asheville Middle High School, 30% of the students in the Math Club are in the Science Club, and 80% of the students in the Science Club are in the Math Club. There are 15 students in the Science Club. How many students are in both Clubs?

(A)40 (B) 12 (C) 30 (D) 36 (E) 43

Solution: E.

Since 80% of the Science Club members are also in the Math Club, there are $0.8 \times 15 = 12$ students common to both clubs. Because 30% of the students in the Math Club are also in the Science Club, there are $12 \div 0.3 = 40$ students in the Math Club.



We have: $n(M \cup S) = n(M) + n(S) - n(M \cap S) = 40 + 15 - 12 = 43$.

(5). Problems Involving Three Events

Example 9. Of the 400 eighth-graders at Pascal Middle school, 117 take algebra, 109 take advanced computer, 114 take industrial technology. Furthermore, 70 take both algebra and advanced computer, 34 take both algebra and industrial technology, and 29 take both advanced computer and industrial technology. Finally, 164 students take none of these courses. How many students take all three courses?

(A) 28 (B) 29 (C) 34 (D) 35 (E) 63

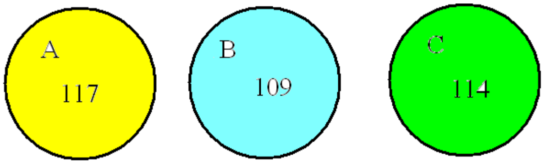
Solution: B.

We are given:

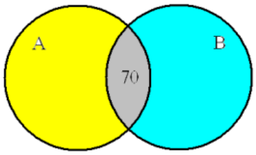
Set A: number of students taking Algebra (117).

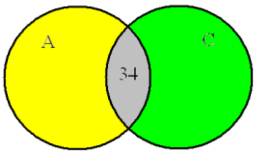
Set B: number of students taking advanced computer (109).

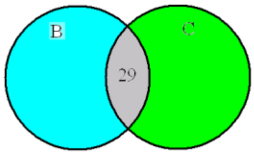
Set C: number of students taking industrial technology (109).



We also know that $n(A\bigcap B) = 70$, $n(A\bigcap C) = 34$, and $n(B\bigcap C) = 29$.

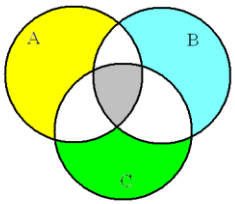






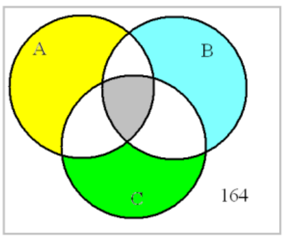
We like to find the number of students who are taking three subjects.

This is the same as finding $n(A\bigcap B\bigcap C)$, the intersection of sets *A, B* and *C*.



We first need to find the total number of students who participated in the events:

$n(A\bigcup B\bigcup C) = 400 - 164 = 236$



By the formula:

$236 = 117 + 109 + 114 - 70 - 34 - 29 + n(A \cap B \cap C) \quad \Rightarrow$
$n(A \cap B \cap C) = 236 - 207 = 29$

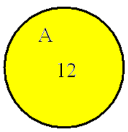
Example 10. Out of 22 students surveyed on ice cream flavors, 12 like chocolate, 5 like only strawberry, and 6 liked vanilla. If 3 liked chocolate and vanilla, how many students did not like any of these flavors?

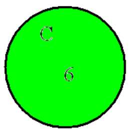
(A) 2 (B) 9 (C) 3 (D) 5 (E) 6

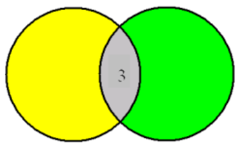
Solution: A.

We are given:

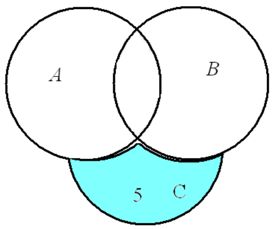
Set A: number of students favoring chocolate (12), Set C: number of students favoring vanilla (6), and the intersection of sets A and C (3). The universal set U = 22.



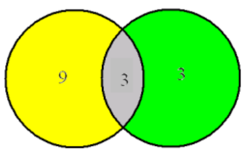




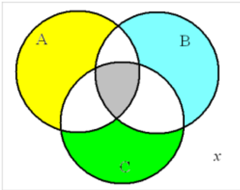
We are also given the number of students favoring only strawberry (5).



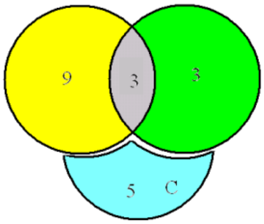
From $n(A \cup C) = n(A) + n(C) - n(A \cap C) \quad \Rightarrow \quad 80 = 48 + 45 - n(A \cap B)$
$n(A \cup C) = 12 + 6 - 3 = 15$:



We want to find x, the number of students who did not like any of these flavors:
$x = 22 - n(A \cup B \cup C)$.



The union of A B, and C can be found this way:
$n(A \cup B \cup C) = 9 + 3 + 3 + 5 = 20$.
The answer is $22 - 20 = 2$.

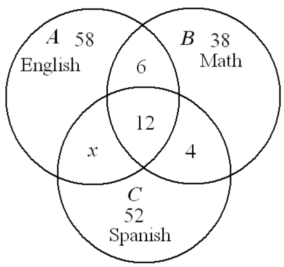


Example 11. There are 100 5th graders in Hope Middle School. 58 like English, 38 like Math, and 52 like Spanish. 6 students like Math and English only (not Spanish), 4 students like Math and Spanish only (not English), and 12 students like all three subjects. Each student likes at least one subject. How many students only like English?

(A) 18 (B) 19 (C) 24 (D) 25 (E) 26

Solution: E.
Let x be the number of students who like English and Spanish (but not Math).
By the formula:
$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

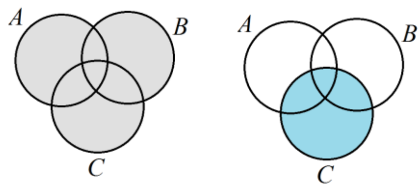
$100 = 58 + 52 + 38 - (6 + 12 + 12 + x + 12 + 4) + 12$
$\Rightarrow \quad x = 14$.
The answer is $58 - 6 - 12 - 14 = 26$.
The Venn diagram is shown on the right.



☆ **Example 12.** (AMC 10) How many numbers between 1 and 2005 are integer multiples of 3 or 4 but not 12?
(A) 501 (B) 668 (C) 835 (D) 1002 (E) 1169

Solution: (C).
Method 1 (official solution):
Between 1 and 2005, there are 668 multiples of 3, 501 multiples of 4, and 167 multiples of 12. So there are (668 − 167) + (501 − 167) = 835 numbers between 1 and 2005 that are integer multiples of 3 or of 4 but not of 12.

Method 2(our method):
Let A be the number of integers of multiple of 3.
Let B be the number of integers of multiple of 4.
Let C be the number of integers of multiple of 12.
The answer will be $n(A \cup B \cup C) - n(C)$



$$n(A \cup B \cup C) - n(C) = n(A) + n(B) + n(C)$$
$$- n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C) - n(C)$$

$$= \left\lfloor \frac{2005}{3} \right\rfloor + \left\lfloor \frac{2005}{4} \right\rfloor + \left\lfloor \frac{2005}{12} \right\rfloor - \left\lfloor \frac{2005}{lcm(3,4)} \right\rfloor - \left\lfloor \frac{2005}{lcm(3,12)} \right\rfloor - \left\lfloor \frac{2005}{lcm(4,12)} \right\rfloor$$
$$+ \left\lfloor \frac{2005}{lcm(3,4,12)} \right\rfloor - \left\lfloor \frac{2005}{12} \right\rfloor$$
$$= \left\lfloor \frac{2005}{3} \right\rfloor + \left\lfloor \frac{2005}{4} \right\rfloor + \left\lfloor \frac{2005}{12} \right\rfloor - \left\lfloor \frac{2005}{12} \right\rfloor - \left\lfloor \frac{2005}{12} \right\rfloor - \left\lfloor \frac{2005}{12} \right\rfloor + \left\lfloor \frac{2005}{12} \right\rfloor - \left\lfloor \frac{2005}{12} \right\rfloor$$
$$= \left\lfloor \frac{2005}{3} \right\rfloor + \left\lfloor \frac{2005}{4} \right\rfloor - 2 \times \left\lfloor \frac{2005}{12} \right\rfloor = 668 + 501 - 2 \times 167 = 835.$$

(6). Problems Involving "At Least" Or "At Most"

Example 13. Of the 200 students at Lakeview High School, 170 are taking science and 175 are taking mathematics. What is the fewest number of students at Lakeview who could be taking both?
(A) 245 (B) 370 (C) 145 (D) 30 (E) 116

Solution: C.
The universal set $U = 200$.
Let x be the number of students taking neither classes, and y be the number of students taking both.
By the Two Events Union Formula:
$n(A \bigcup B) = n(A) + n(B) - n(A \bigcap B)$, we have
$200 - x = 170 + 175 - y \quad \Rightarrow \quad y = 145 + x$.
Since we want to have the smallest value for y, we let x be zero and get $y = 145$.

Example 14. 8th grade classes are surveyed. 78% of the students like swimming, 80% like computer games, 84% like playing chess and 88% like reading books. At least how many percent of students like all four activities?
(A) 30 (B) 66 (C) 83 (D) 12 (E) 22

Solution: A.
Let the number of students be 100.
Number of students does not like swimming: 100 – 78 = 22.
Number of students does not like computer games: 100 – 80 = 20.
Number of students does not like playing chess: 100 – 84 = 16.
Number of students does not like reading books: 100 – 88 = 12.
At most, the number of students does not like at least one of the three activities is: 22 + 20 + 16 + 12 = 70.
At least 100 – 70 = 30 or 30 percent of students like all four activities.

Example 15. There are 100 students in a class. 75 of them like to play basketball. 80 like to play chess. 92 like to sing. 85 like to swim. At least how many students like all the 4 activities?
(A) 55 (B) 77 (C) 35 (D) 32 (E) 19

Solution: D.
Method 1:
Number of students who like to play basketball and chess: 75 + 80 = 155.

There are at least 155 – 100 = 55 students who like to play both basketball and chess.
Similarly, there are at least 92 + 85 – 100 = 77 students who like to sing and swim.
There are at least 55 + 77 – 100 = 32 students who like all 4 activities.

Method 2: **The tickets method**
Step 1: Give each student a ticket for each activity he or she likes. There are 75 + 80 + 92 + 85 = 332 tickets given out.

Step 2: Take away the tickets from them. Students who have 3 or more tickets will give back 3 tickets. Students who have less than 3 tickets will give back all the tickets.

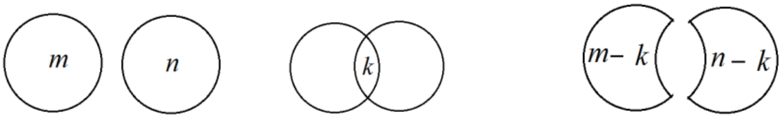
Step 3: Calculate the number of tickets taken back: at most 3 × 100 = 300 tickets were taken back.

Step 4: Calculate the number of tickets that are still in the students hands.
332 – 300 = 32.
At this moment, any student who has the ticket will have only one ticket. These students are the ones who like 4 activities. The answer is 32.

MORE EXAMPLES

Example 16. Set M has m elements and set N has n elements. Set W consists of all elements that are in either set M or set N with the exception of the k common elements ($k > 0$). Which of the following represents the number of elements in set W?
(A) $x + y + 2k$ (B) $x + y - k$ (C) $x + y + 2k$ (D) $x + y - 2k$ (E) $2x + 2y - 2k$

Solution: (D).



M N W

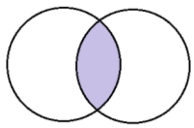
$w = (m - k) + (n - k) = m + n - 2k.$

Example 17. Set S consists of the positive multiples of 6 that are less than 100, and set T consists of the positive multiples of 8 that are less than 100. How many numbers do sets S and T have in common?

(A) None (B) One (C) Two (D) Four (E) Eight

Solution: D.

Method 1:



Let the shaded area be x. $x = \left\lfloor \frac{100}{lcm(6,8)} \right\rfloor = \left\lfloor \frac{100}{24} \right\rfloor = \lfloor 4.17 \rfloor = 4$.

There are 4 numbers in common to both sets.

Method 2:

$6 = 2 \times 3$

$8 = 2^3$.

The least common multiple is 24.

There are four numbers (24, 48, 72, and 96) in common to both sets.

Example 18. Set S consists of m integers, and the difference between the greatest integer in S and the least integer in S is 800. A new set of m integers, set T, is formed by multiplying each integer in S by 3 and then adding 10 to the product. What is the difference between the greatest integer in T and the least integer in T?

(A) 800 (B) 2400 (C) 2500 (D) 1600 (E) 2403

Solution: B.

Let S_G be the greatest integer and S_L be the least integer in S.

Let T_G be the greatest integer and T_L be the least integer in T.

W e know that $S_G - S_L = 800$ (1)
Since $T_G = 3(S_G + 10)$ and $T_L = 3(S_L + 10)$, we have
Since $T_G - T_L = 3(S_G + 10) - 3(S_L + 10) = 3(S_G - S_L)$ (2)
Substituting (1) into (2), we get $T_G - T_L = 3 \times 800 = 2400$.

Example 19. In a class of 42 students, 18 students are in the Math Club, 5 students are in both the Math Club and the Science Club, and 14 are in neither. How many students are in the Science Club?
(A) 8 (B) 14 (C) 15 (D) 16 (E) 24

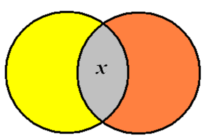
Solution: C.
There are 42 – 14 = 28 students participated in the two clubs. Let S be the number of students in the Science Club.
We know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we have $28 = 18 + S - 5$. So $S = 15$.

Example 20. A and B are sets. If A contains 6 elements, B contains 8, and together A and B contain 10, how many elements in A are also in B?
(A) 4 (B) 2 (C) 5 (D) 6 (E) 9

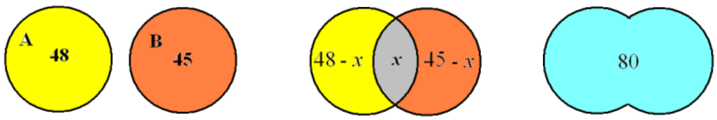
Solution: A.
Let x be the number of elements in A are also in B.
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we have $10 = 6 + 8 - x$. $x = 4$.

Example 21. Ninety-six girls were surveyed at Euclid High School. There were 48 softball players and 45 track athletes. If $\frac{1}{6}$ of the girls surveyed did not play either sport, how many girls played both sports?
(A) 12 (B) 13 (C) 8 (D) 4 (E) 5

Solution: (B).
The question asks to find the number of students who played both sports.

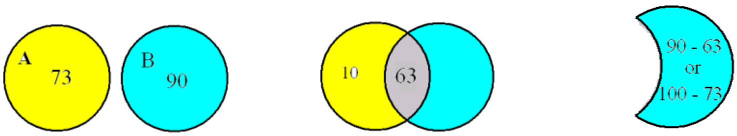


This is the same as to find the intersection of sets A and B.
Set A: number of softball players (48).
Set B: number of track athletes (45).
In the formula: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we are able to find $n(A \cup B)$, the number of students participating in the events. We know that there are total 96 girls but only 96 – $\frac{1}{6} \times 96 = 96 - 16 = 80$ girls participated in the two events, so $n(A \cup B) = 80$.
From $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ $\Rightarrow$ $80 = 48 + 45 - n(A \cap B)$.
The answer is: $n(A \cap B) = 48 + 45 - 80 = 13$.
The Venn diagram is as follows:



Example 22. One hundred students at a certain school take science or math. If 73 students take science and 90 take math, how many take math but not science?
(A) 63 (B) 27 (C) 23 (D) 17 (E) 5

Solution: (B).
$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 5 + 4 - 2 = 7$
100 = 73 + 90 – $n(A \cap B)$ $\Rightarrow$ $n(A \cap B) = 63$.
We have two ways to calculate the number of students taking math only: 100 – 73 = 27 or 90 – 63 = 27. The Venn diagram is as follows:



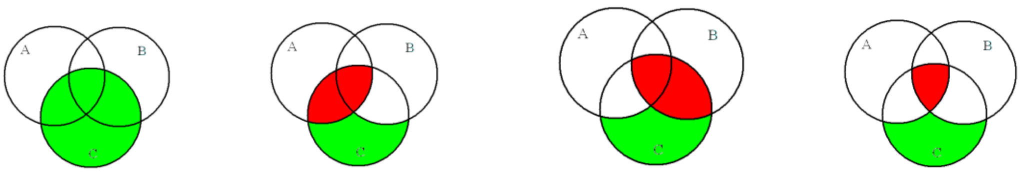
Example 23. In a class of 500 students, every student liked at least one for three kinds of music.
260 liked classical music

260 liked jazz music
75 liked classical and rock music
115 liked rock and jazz music
130 liked classical and jazz music
45 liked classical, jazz, and rock music
How many of the students in this class liked only rock music?
(A) 190 (B) 110 (C) 255 (D) 390 (E) 45

Solution: B.
Set *A*: number of students who liked classical music (260). Set *B*: number of students who liked jazz music (260). Set *C*: number of students who liked rock music (x).

Method 1:
The union of sets *A*, *B*, and *C* is:
$$n(A\bigcup B\bigcup C)= n(A)+n(B)+n(C)-n(A\bigcap B)-n(B\bigcap C)-n(C\bigcap A)+n(A\bigcap B\bigcap C)$$

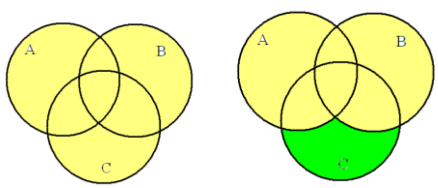
Substituting all constants into the formula:
$500 = 260 + 260 + x - (75 + 115 + 130) + 45 \qquad \Rightarrow \qquad x = 255$
We are able to draw the following Venn diagrams:



(*a*) (*b*) (*c*) (*d*)

The answer is *(a)* –*(b)* – *(c)* + *(d)* = 255 – 75 – 115 + 45 = 110.

Method 2:



The solution will be $n(A\bigcup B\bigcup C)-n(A\bigcup B)$.
$n(A\bigcup B\bigcup C)=500$ and $n(A\bigcup B) = 260 + 260 - 130 = 390$.
The answer is 500 – 390 = 110.

3. PROBLEMS

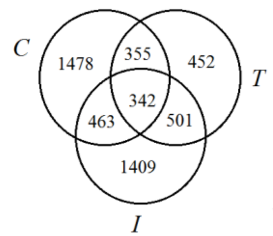
Problem 1. In a survey, 95 television viewers offered their opinions about program A and program B. Of those viewers, 28 liked neither program A nor program B, while 20 liked both program A and program B. If 25 viewers liked program A only, how many viewers liked program B only?

(A) 3 (B) 10 (C) 15 (D) 22 (E) 35

Problem 2. In an assortment of cereals, 43 types contain oats and 55 types contain rice. Some of these cereals contain both oats and rice. If 15 cereals in this assortment contain oats but not rice, how many cereals contain rice but not oats?

(A) 28 (B) 27 (C) 15 (D) 12 (E) 55

Problem 3. In a survey, 5000 students selected their usual methods of communication with friends from the following three options: calling (C), text messaging (T), or instant messaging (I). The Venn diagram above shows the results of the survey. How many students selected exactly one of the three methods of communicating?



(A) 342 (B) 1319 (C) 1478 (D) 452 (E) 3339

Problem 4. Sets P and Q are shown below. If x is a member of set P and y is a member of set Q, which of the following CANNOT be equal to the product xy ?

$P = \{1, 3, 5, 6, 11\}$

$Q = \{2, 4, 6, 7, 9, 13\}$

(A) 39 (B) 143 (C) 16 (D) 21 (E) 24

Problem 5. Of 40 people, 28 smoke and 16 chew tobacco. It is also known that 10 both smoke and chew. How many among the 40 neither smoke nor chew?

(A) 10 (B) 8 (C) 2 (D) 6 (E) 4

Problem 6. How many integers between 1 and 600 inclusive are not divisible by neither 3, nor 5, nor 7?

(A) 325 (B) 105 (C) 275 (D) 200 (E) 300

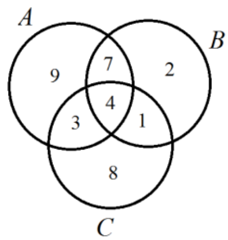
Problem 7. Of 10 boxes, 5 contain pencils, 4 contain pens, and 2 contain both pens and pencils. How many boxes contain neither pens nor pencils?
(A) 3 (B) 7 (C) 6 (D) 8 (E) 5

Problem 8. In a class of 50 students, 28 take science, 21 take French, and 5 students take both classes. How many students take neither class?
(A) 6 (B) 11 (C) 45 (D) 7 (E) 23

Problem 9. In a survey of 120 eighth graders, 84 liked math, 73 liked science, and 23 liked math but not science. How many students disliked both subjects?
(A) 26 (B) 31 (C) 40 (D) 24 (E) 25

Problem 10. A class of 31 students was polled about which soft drink was their favorite. Seventeen students liked Fizz-Up, eighteen liked Fizz-Down, and six students didn't like either. How many students liked both?
(A) 4 (B) 6 (C) 10 (D) 0 (E) 16

Problem 11. The number of elements in sets *A, B*, and *C* are shown in the diagram. How many elements belong to *A* or *B* but not *C*?
(A) 14 (B) 16 (C) 18 (D) 10 (E) 24



☆ **Problem 12.** How many numbers between 1 and 2015 are integer multiples of 3 or 5 but not 15?
(A) 403 (B) 671 (C) 835 (D) 268 (E) 806

Problem 13. 8^{th} graders have three after-class activities: Tennis, Drawing, and Art. In Tennis club, there are 25 students; Drawing: 24 students; and Art: 30 students. 5 students are in both Tennis and Art clubs; 2 students are in both Tennis and Drawing clubs; 4 students are in both Art and Drawing clubs; and 1 student is in all of the three clubs. How many students in 8^{th} grade are involved in all three after-class activities?

(A) 64 (B) 66 (C) 69 (D) 54 (E) 56

Problem 14. 85 people subscribed to three magazines A, B, and C. 49 people subscribed to A, 62 subscribed to B, and 41 subscribed to C. 24 subscribed to A and B. 22 subscribed to B and C. 25 subscribed to A and C. How many people subscribed to all three?

(A) 6 (B) 4 (C) 10 (D) 20 (E) 16

Problem 15. In a class of 25 students, 18 like pizza and 12 like spaghetti. What is the greatest possible number of students who dislike both pizza and spaghetti?

A) 7 (B) 9 (C) 6 (D) 20 (E) 12

Problem 16. Alex, Bob and Charlie are watering 100 flowerpots. Alex watered 76 pots, Bob watered 69 pots, and Charlie watered 85 pots. At least how many flowerpots have been watered three times?

A) 20 (B) 30 (C) 50 (D) 38 (E) 23

Problem 17. Every camper at camp EKO is required to take exactly two of the three crafts classes offered. One summer, 47 campers took basket weaving, 59 took cabinet making, and 34 took pottery. How many campers attended camp EKO that summer?

A) 70 (B) 48 (C) 60 (D) 30 (E) 16

Problem 18. In a class of twenty students, five like to go camping and ten like to ride bikes. What is the greatest possible number of students that like to do neither?

A) 10 (B) 8 (C) 6 (D) 12 (E) 14

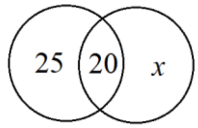
Problem 19. There are 52 students in a class. 30 of them can swim. 35 can ride bicycle. 42 can play table tennis. At least how many students can do all three sports?

A) 7 (B) 8 (C) 6 (D) 3 (E) 30

4. SOLUTIONS

Problem 1. Solution: D.

We have 95 – 28 = 67 viewers. Let x be the number of people who liked program *B* only. By the Venn diagram we get 95 – 28 = 67 = 25 + 20 + x $\Rightarrow x = 22$.



Problem 2. Solution: B.

There are 43 – 15 = 28 types that contains both oats and rice.
Therefore there are 55 – 28 = 27 types that contains rice but not oats.

Problem 3. Solution: E

1478 + 452 + 1409 = 3339.

Problem 4. Solution C.

3 × 13 = 39. 11 × 13 = 143. 3 × 7 = 21. 6 × 4 = 24.

Problem 5.Solution: D.

Let A denote the set of smokers and B the set of chewers. Then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ = 28 + 16 - 10 = 34; meaning that there are 34 people that either smoke or chew (or possibly both). Therefore the number of people that neither smoke nor chew is 40 - 34 = 6.

Problem 6. Solution C.

There are 200 + 120 + 85 – 28 – 21 – 17 + 5 = 325 integers in [1, 600] divisible by at least one of 3, 5, or 7. Those not divisible by these numbers are a total of 600 – 325 = 275.

Problem 7. Solution: A.

By the formula: $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 5 + 4 - 2 = 7$

There are 7 boxes containing pens, pencils or both.
The number of boxes containing neither pens nor pencils is 10 – 7 = 3.

Problem 8. Solution: A.

By the Formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we have $50 - x = 28 + 21 - 5$.

$x = 6$.

Problem 9. Solution: D.

By the Two Events Union Formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we have $120 - x = 84 + 73 - (84 - 23)$. $x = 24$.

Problem 10. Solution: C.

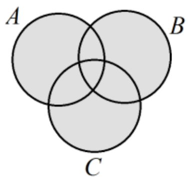
By the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we have $31 - 6 = 17 + 18 - x$.

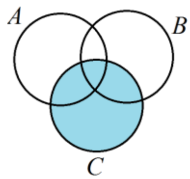
$x = 10$.

Problem 11. Solution: C.

$n(A \cup B \cup C) - n(C) = 9 + 7 + 2 + 3 + 4 + 1 + 8 - (3 + 4 + 1 + 8) = 9 + 7 + 2 = 18$.

Problem 12. Solution: E.





Let A be the number of integers of multiple of 3.
Let B be the number of integers of multiple of 5.
Let C be the number of integers of multiple of 15.
The answer will be $n(A \cup B \cup C) - n(C)$

$$n(A \cup B \cup C) = n(A) + n(B) + -n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

$$= \left\lfloor \frac{2015}{3} \right\rfloor + \left\lfloor \frac{2015}{5} \right\rfloor + \left\lfloor \frac{2015}{15} \right\rfloor - \left\lfloor \frac{2015}{\text{lcm}(3,5)} \right\rfloor - \left\lfloor \frac{2015}{\text{lcm}(3,15)} \right\rfloor - \left\lfloor \frac{2015}{\text{lcm}(5,15)} \right\rfloor + \left\lfloor \frac{2015}{\text{lcm}(3,5,15)} \right\rfloor$$

$$= \left\lfloor \frac{2015}{3} \right\rfloor + \left\lfloor \frac{2015}{5} \right\rfloor + \left\lfloor \frac{2015}{15} \right\rfloor - \left\lfloor \frac{2015}{15} \right\rfloor - \left\lfloor \frac{2015}{15} \right\rfloor - \left\lfloor \frac{2015}{15} \right\rfloor + \left\lfloor \frac{2015}{15} \right\rfloor$$

$$= \left\lfloor \frac{2015}{3} \right\rfloor + \left\lfloor \frac{2015}{5} \right\rfloor - \left\lfloor \frac{2015}{15} \right\rfloor$$

$$n(A \cup B \cup C) - n(C) = \left\lfloor \frac{2015}{3} \right\rfloor + \left\lfloor \frac{2015}{5} \right\rfloor - 2 \times \left\lfloor \frac{2015}{15} \right\rfloor = 671 + 403 - 2 \times 134 = 806.$$

Problem 13. Solution: C.

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

$= 25 + 24 + 30 - 5 - 2 - 4 + 1 = 69.$

Problem 14. Solution: B.

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

$85 = 49 + 62 + 41 - 24 - 22 - 25 + x \quad \Rightarrow \quad x = 4.$

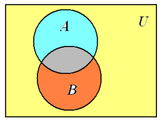
Problem 15. Solution: A.

Set A: number of students who like pizza (18).

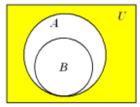
Set B: number of students who like 12).

The universal set $U = 25$.

First we draw a Venn diagram of two events (Figure *a*).



a



b

Since we want to have the greatest number of students who dislike both pizza and spaghetti, we maximize the shared region by modifying the above Venn diagram (Figure *b*). The answer is 25 – 18 = 7.

Problem 16. Solution: B.

At least 76 + 69 + 85 – 100 × 2 = 30 flowerpots have been watered three times.

Problem 17. Solution: (A). 70 (campers).

First we draw a Venn diagram as follows for three events (Figure *a*):

We know that nobody took three classes. The Venn diagram becomes (Figure *b*).

We also know that everyone took two classes. The Venn diagram finally becomes (Figure *c*).

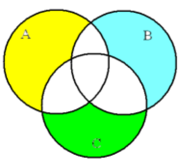


Figure *a*

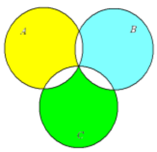


Figure *b*

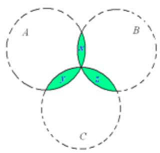


Figure *c*

We then are able to write the following system of equations:

$$\left.\begin{aligned} x + y &= 47 \\ y + z &= 59 \\ z + x &= 34 \end{aligned}\right\}$$

Add the above three equations together, we have $2(x + y + z) = 140$.

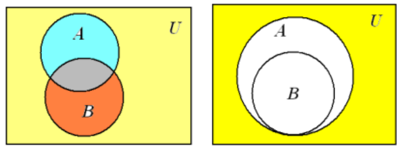
The answer is: $x + y + z = 70$.

Problem 18. Solution: A.

Set A: number of students who like to ride bikes (10).

Set B: number of students who like to go camping (5).

The universal set U = 20.



a b

First we draw a Venn diagram of two events (Figure a).

Since we want to have the greatest number of students that like to do neither, we maximize the shared region by modifying the above Venn diagram (Figure b).

The answer is $20 - 10 = 10$.

Problem 19. Solution: D.

Method 1: Number of students who cannot swim: $52 - 30 = 22$.

Number of students who cannot ride bicycle: $52 - 35 = 17$.

Number of students who cannot play tennis: $52 - 42 = 10$.

At most $22 + 17 + 10 = 49$ students cannot play at least one of the three activities.

At least $52 - 49 = 3$ students can do all three sports.

Method 2: **The tickets method**

Step 1: Give each student a ticket for each activity he or she likes. $30 + 35 + 42 = 107$ tickets are given out.

Step 2: Take away the tickets from them. Students who have 2 or more tickets will give back 2 tickets. Students who have less than 2 tickets will give back all the tickets.

Step 3: Calculate the number of tickets taken back: at most $2 \times 52 = 104$ tickets were taken back.

Step 4: Calculate the number of tickets that are still in the students hands. $107 - 104 = 3$.

At this moment, any student who has the ticket will have only one ticket. These students are the ones who like 3 activities. The answer is 3.

1. BASIC KNOWLEDGE

1. Terms

A permutation is an arrangement or a listing of things in which order is important.

A combination is an arrangement or a listing of things in which order is not important

2. Definition

The symbol ! (factorial) is defined as follows:

$$0! = 1,$$

and for integers $n \ge 1$,

$$n! = n \cdot (n-1) \ \cdots \ 1.$$

$1! = 1,$
$2! = 2 \cdot 1 = 2,$
$3! = 3 \cdot 2 \cdot 1 = 6,$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24,$
$5! = 5 \cdot 4 \cdot 3 \cdot 1 \cdot 1 = 120,$
$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$

3. Permutations

(1). Different elements, with no repetition. Take r elements each time from n distinct elements ($1 \le r \le n$).

Number of permutations $P(n,r) = \dfrac{n!}{(n-r)!}$

(2). n distinct objects can be permutated in $n!$ permutations.

We let $n = r$ in (1) to get $P(n, n) = n!$

Example 1. In how many ways can the letters of the word MATH be arranged?

(A) 4 (B) 8 (C) 12 (D) 24 (E) 1

Solution: D.
$4! = 4 \times 3 \times 2 \times 1 = 24$.

4. Combinations

Definition Let *n, r* be non-negative integers such that $0 \le r \le n$. The symbol $\binom{n}{r}$ (read "*n* choose *m*") is defined and denoted by

$$\binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!(n-r)!}$$

Remember: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, and $\binom{n}{n} = 1$

Since $n - (n - r) = r$, we have $\binom{n}{r} = \binom{n}{n-r}$

Unlike Permutations, Combinations are used when order does not matter. If we have *n* different elements, and it doesn't matter which order we take the elements, the number of ways to take *m* elements where $1 \le m \le n$, is $\binom{n}{m}$.

Example 2. In a group of 2 cats, 3 dogs, and 10 pigs in how many ways can we choose a committee of 6 animals?
(A) 1001 (B) 2002 (C) 3003 (D) 4004 (E) 5005

Solution: E.
There are 2 + 3 + 10 = 15 animals and we must choose 6. The order does not matter. Thus, there are $\binom{15}{6}$ = 5005 possible committees.

☆**Example 3.** Consider the set of 5-digit positive integers. How many have at least one 7 in their decimal representation?
(A) 52488 (B) 52484 (C) 37512 (D) 90000 (E) 45000

Solution: C.

The difference of the total number of 5-digit positive integers and the number of 5-digit integers that do not have a 7 in their decimal representations: 90000 − 52488 = 37512.

5. The Sum Rule

If an event E_1 can happen in n_1 ways, event E_2 can happen in n_2 ways, event E_k can happen in n_k ways, and if any event E_1, E_2,.. or E_k happens, the job is done, then the total ways to do the job is $n_1 + n_2 + \cdots + n_k$.

Example 4. In how many ways can one book be selected from a book shelf of 5 paperback books and 3 hardcover books?
(A) 5 (B) 3 (C) 8 (D) 11 (E) 19

Solution: C.
The event is to select one book. No matter if it is hardcover or paperback, as long as we select one book, our job is done. So we can use the sum rule to get the number of ways:
5 + 3 = 8.

6. The Product Rule (Fundamental Counting Principle)

When a task consists of k separate parts, if the first part can be done in n_1 ways, the second part can be done in n_2 ways, and so on through the k^{th} part, which can be done in n_k ways, then total number of possible results for completing the task is given by the product:

$$n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

Example 5. How many ways are there to arrange 5 people in a row of 5 seats?
(A) 5 (B) 10 (C) 12 (D) 120 (E) 720

Solution: D.
5! = 120.

Example 6. A girl has 5 shirts, 4 skirts, and 3 pairs of shoes. How many different outfits can she create?
(A) 120 (B) 60 (C) 15 (D) 12 (E) 20

Solution: B.
We see three steps to finish the job (assembling an outfit). If one step is missing, the job is not done. Only when 3 parts are done, the job is done.
She has 5 ways to select a shirt, 4 ways to select a skirt, and 3 ways to select a pair of shoes.
By the product rule, we have $5 \times 4 \times 3 = 60$ outfits.

☆**Example 7.** How many triples of three positive integers (x, y, z) have the product of 24?
(A) 20 (B) 24 (C) 27 (D) 30 (E) 21

Solution: (D).
The possible ways of expressing 24 as a product of 3 positive integers are
$(1 \cdot 1 \cdot 24)$, $(1 \cdot 2 \cdot 12)$, $(1 \cdot 3 \cdot 8)$, $(1 \cdot 4 \cdot 6)$, $(2 \cdot 2 \cdot 6)$, $(2 \cdot 3 \cdot 4)$.

The number of positive integers can be formed:
$3 + 6 + 6 + 6 + 3 + 6 = 30$.

2. PROBLEM SOLVING SKILLS

(2.1). Counting Using Charts

Example 8. Two dice are rolled. How many ways are there to roll a sum of 5?

Solution:

D_2 \ D_1	1	2	3	4	5	6
1	*2*	*3*	*4*	*5*	*6*	*7*
2	*3*	*4*	*5*	*6*	*7*	*8*
3	*4*	*5*	*6*	*7*	*8*	*9*
4	*5*	*6*	*7*	*8*	*9*	*10*
5	*6*	*7*	*8*	*9*	*10*	*11*
6	*7*	*8*	*9*	*10*	*11*	*12*

We have four ways of rolling a sum of 5.

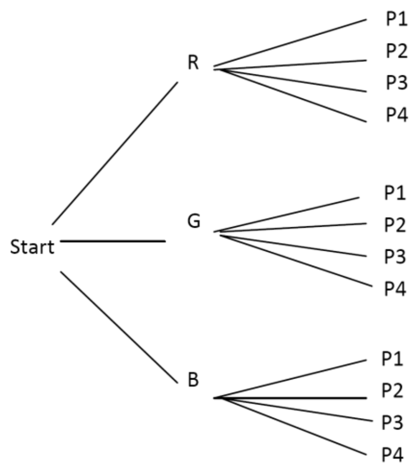
(2.2). Counting Using Tree Diagram

You can use a tree diagram to list all the possible outcomes of events.

Example 9. A designer has 3 fabric colors he may use for a dress: red, green, and blue. Four different patterns are available for the dress. If each dress design requires one color and one pattern, how many different dress designs are possible?

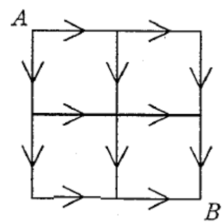
(A) 10 (B) 24 (C) 12 (D) 14 (E) 20

Solution: C.



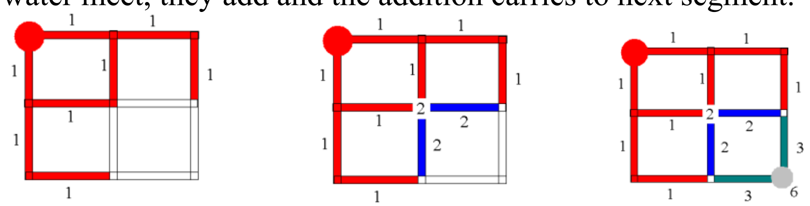
(2.3). Counting using water pipes

Example 10: Using only the line segments given in the indicated direction, how many paths are there from *A* to *B*?



Solution: 6.

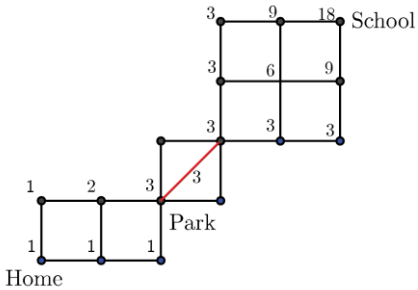
Think about water flowing in pipes (line segments). The starting point is the source of the water. When two or more branches of water meet, they add and the addition carries to next segment.



☆**Example 11.** (2013 AMC 8 problem 20) Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?

(A) 3 (B) 6 (C) 9 (D) 12 (E) 18

Solution: E.



(2.4). Counting with restriction

Example 12. In how many ways can 5 books be arranged on a shelf if two of the books must remain together, but may be interchanged?

(A) 12 (B) 24 (C) 48 (D) 96 (E) 5

Solution: C.

We tie these two books, say book A and book B together, and treat them as one book. Our problem becomes to arrange 4 books on a shelf without restriction:

$$\underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 4! = 24.$$

Then we un-tie these two books and switch their position ($AB \Rightarrow BA$), the answer will then be: $2 \times 24 = 48$.

Example 13. How many ways can 5 distinct paperback books and 1 hardcover book be arranged on a shelf if the hardcover must be the rightmost book on the shelf?

(A) 20 (B) 60 (C) 90 (D) 120 (E) 180

Solution: D.

The rightmost book must be hardcover so there is only one way. The next five books have no restrictions so they can be any of the remaining books arranged.
$1 \times 5 \times 4 \times 3 \times 2 \times 1 = 120$

☆**Example 14.** How many four –digit whole numbers do not contain the digit 1? No digit is allowed to use more than once in any such 4-digti number.
(A) 4032 (B) 2016 (C) 1024 (D) 8064 (E) 2688

Solution: E.
We have the following digits available: 0, 2, 3, 4, 5, 6, 7, 8, and 9.
There are $8 \cdot 8 \cdot 7 \cdot 6 = 2688$ four-digit integers that do not contain the digit 1.

(2.5) . Grouping

THEOREM 1: Let the number of different objects be n. Divide n into r groups $A_1, A_2, ..., A_r$ such that there are n_1 objects in group A_1, n_2 objects in group A_2, ..., n_r objects in the group A_r, where $n_1 + n_2 + \cdot\cdot\cdot + n_r = n$. The number of ways to do so is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

THEOREM 2: Let there be r types of objects: n_1 of type 1, n_2 of type 2; etc. The number of ways in which these $n_1 + n_2 + \cdot\cdot\cdot + n_r = n$ objects can be rearranged is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

Example 15. In how many ways may we distribute 6 different books to Alex, Bob, and Catherine such that each person gets 2 books?
(A) 720 (B) 120 (C) 90 (D) 60 (E) 24

Solution: C.

$$\frac{n!}{n_1!n_2!\cdots n_r!} = \frac{6!}{2!2!2!} = 90.$$

Example 16. In how many ways may we distribute 7 different books to Alex, Bob, and Catherine such that Alex and Bob each get 2 books and Catherine gets 3 books?

(A) 2520 (B) 840 (C) 630 (D) 210 (E) 120

Solution: D.

$$\frac{n!}{n_1!n_2!\cdots n_r!} = \frac{7!}{2!2!3!} = 210.$$

Example 17. In how many different ways can all the letters in INDIANA be arranged in a line? Assume that duplicate letters are indistinguishable.

A. 5040 B. 2520 C. 1260 D. 630 E. none of these

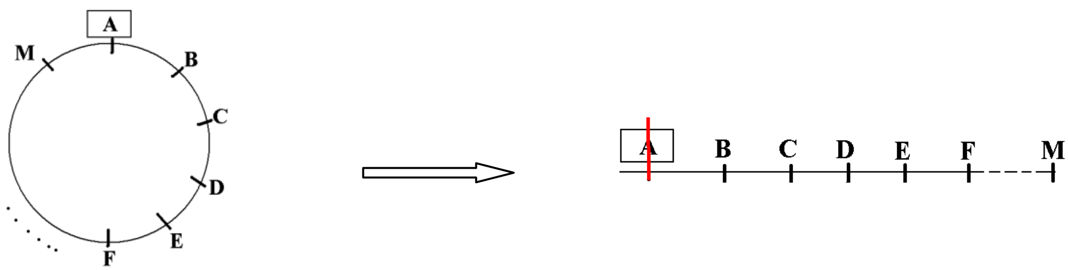
Solution: D.

$$N = \frac{7!}{2!2!2!} = 630.$$

(2.6) . Circular Permutations

THEOREM 3: The number of circular permutations (arrangements in a circle) of n distinct objects is $(n-1)!$.

We can think of this as n people being seated at a round table. Since a rotation of the table does not change an arrangement, we can put person A in one fixed place and then consider the number of ways to seat all the others. Person B can be treated as the first person to seat and M the last person to seat. The number of ways to arrange persons A to M is the same as the number of ways to arrange persons B to M in a row. So the number of ways is $(n-1)!$.



Example 18. In how many ways is it possible to seat seven people at a round table?

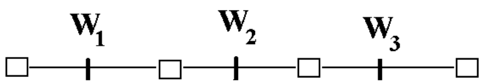
Solution:
$(n-1)! = (7-1)! = 6! = 720$.

Example 19. In how many ways is it possible to seat seven people at a round table if Alex and Bob must not sit in adjacent seats?
(A) 720 (B) 480 (C) 120 (D) 60 (E) 90

Solution: B.
The number of ways to sit 7 people at a round table is 720.
We find the number of ways Alex and Bob sit together by seeing them as a unit. There are $(6-1)! = 5!$ ways. The result is multiplied by 2 if we alter Alex's and Bob's positions. The solution is then $720 - 2 \times 5! = 720 - 2 \times 120 = 480$.

Example 20. In how many ways can four men and four women be seated at a round table if no two men are to be in adjacent seats?
(A) 120 (B) 144 (C) 121 (D) 60 (E) 72

Solution: B.
We seat four women first. There are $(4-1)! = 3!$ ways to sit them. After the ladies are seated, we have $4!$ ways to seat four men in the small rectangles as shown in the figure below. $4! \times 3! = 144$.



Example 21. In how many ways can a family of six people be seated at a round table if the youngest kid must sit between the parents?
(A) 6 (B) 7 (C) 10 (D) 12 (E) 24

Solution: D.
We link two parents and the youngest kid together to form a unit. There are $(4-1)!$ ways to seat them at the table. The result must be multiplied by 2 since we can switch the positions of the two parents. The solution is $(4-1)! \times 2 = 12$.

(2.7). Combinations with Repetitions

THEOREM 4: Let n be a positive integer. The number of positive integer solutions to $x_1 + x_2 + \cdots + x_r = n$ is $\binom{n-1}{r-1}$.

☆**Example 22.** Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen?
(A) 6 (B) 7 (C) 10 (D) 12 (E) 24

Solution: C.
Since each person must have at least one pencil, no one has zero pencils. In other words, we are looking for the number of positive integer solutions to the equation $x_1 + x_2 + x_3 = 6$. Our answer is $\binom{n-1}{r-1} = \binom{6-1}{3-1} = \binom{5}{2} = 10$.

Example 23. In how many ways may we write the number 9 as the sum of three positive integer summands? Here order counts, so, for example, 1 + 7 + 1 is to be regarded different from 7 + 1 + 1.
(A) 16 (B) 27 (C) 30 (D) 14 (E) 28

Solution: E.
We are seeking positive integral solutions to $x_1 + x_2 + x_3 = 9$. The number of solutions is thus $\binom{n-1}{r-1} = \binom{9-1}{2-1} = \binom{8}{2} = 28$.

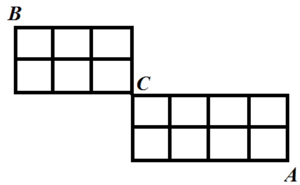
THEOREM 5: Let n be a positive integer. The number of non-negative integer solutions to $y_1 + y_2 + \cdots + y_r = n$ is $\binom{n+r-1}{n}$ or $\binom{n+r-1}{r-1}$.

☆**Example 24.** Three friends have a total of 6 identical pencils. In how many ways can this happen?
(A) 10 (B) 12 (C) 28 (D) 24 (E) 16

Solution: C.
Since each person could get zero pencil, we are looking for the number of nonnegative integer solutions to the equation $x_1+x_2+x_3=6$. Our answer is
$\binom{n+r-1}{r-1}=\binom{6+3-1}{3-1}=\binom{8}{2}=28$.

THEOREM 6: The number of ways to walk from one corner to another corner of an *m* by *n* grid can be calculated by the following formula: $N=\binom{m+n}{n}$, where *m* is the number of rows and *n* the number of column.

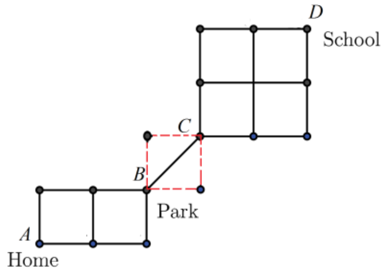
Example 25. How many ways are there to get from *A* to *B* if you can only go north or west?
(A) 120 (B) 130 (C) 140 (D) 150 (E) 160



Solution: D.
We divide this problem into two smaller problems. We first find the number of ways from *A* to *C* and then find the number of ways from *C* to *B*. Using the Fundamental Counting Principle, we can obtain our final answer by multiplying our two discoveries together.
Traveling from *A* to *C* we have $\binom{6}{2}$ ways. Travelling from *C* to *B* we have $\binom{5}{2}$ ways.
Using the fundamental counting principle as noted above, there are $\binom{5}{2}\binom{6}{2}=150$ ways.

☆**Example 26.** (2013 AMC 8 problem 20) Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the

park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?
(A) 3 (B) 6 (C) 9 (D) 12 (E) 18



Solution: E.

Traveling from A to B we have $\binom{3}{1}$ ways.

Travelling from B to C we have 1 way, and

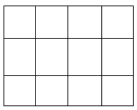
Traveling from C to D we have $\binom{4}{2}$ ways.

Using the fundamental counting principle as noted above, the number of ways from A to D: $3\times1\times6=18$ ways.

THEOREM 7: Counting How Many Rectangles

For a rectangular grid with m vertical lines and n horizontal lines, the total number of rectangles that can be counted is $N=\binom{m}{2}\times\binom{n}{2}$.

Example 27. Consider the figure shown below. How many rectangles are there?



(A) 30 (B) 60 (C) 90 (D) 120 (E) 180

Solution: B.
To count the number of rectangle bounded by gridlines, notice that each rectangle is determined by its top and bottom lines together with its left and right bounding lines.

There are $\binom{4}{2}=6$ ways to pick the upper and lower boundaries, and $\binom{5}{2}=10$ ways to pick the left and right boundaries, so there are $6\times10=60$ total rectangles.

THEOREM 8: Rising (Increasing) Number

A rising number, such as 34689, is a positive integer where each digit is larger than the one to its left.

The number of integers with digits in increasing order can be calculated by $\binom{9}{n}$.

Example 28. How many 3-digit increasing numbers are there?
(A) 34 (B) 64 (C) 84 (D) 92 (E) 120

Solution: C.

In our case, $n = 3$, we have $\binom{9}{n} = 84$ three-digit increasing numbers.

THEOREM 9: Falling (Decreasing) Number

A falling number is an integer whose decimal representation has the property that each digit except the units digit is larger than the one to its right. For example, 96521 is a falling number but 89642 is not. The number of integers with digits in decreasing order can be calculated: $\binom{10}{n}$.

Example 29. How many 3-digit falling numbers are there?
(A) 60 (B) 90 (C) 120 (D) 240 (E) 180

Solution: C.

The number of three-digit falling numbers is $\binom{10}{3} = 120$.

THEOREM 10: Palindrome Numbers

A **palindrome number** is a number that is the same when written forwards or backwards. The first few palindrome numbers are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 101, 111, and 121.

The number of palindromes with n digits can be found by $9\times 10^{\left\lfloor \frac{n-1}{2} \right\rfloor}$ for $n > 1$.
$\lfloor x \rfloor$ is the floor function.

Example 30. How many 3-digit palindromes are there?
(A) 30 (B) 60 (C) 90 (D) 120 (E) 180

Solution: C.

The number of 3-digit falling numbers is $9\times 10^{\left\lfloor \frac{n-1}{2} \right\rfloor} = 9\times 10^{\left\lfloor \frac{3-1}{2} \right\rfloor} = 90$.

3. PROBLEMS

Problem 1. Any 2 points determine a line. If there are 7 points in a plane, no 3 of which lie on the same line, how many lines are determined by pairs of these 7 points?
(A) 15 (B) 18 (C) 21 (D) 30 (E) 36

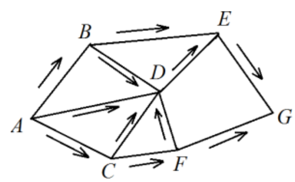
Problem 2. An electrician is testing 7 different wires. For each test, the electrician chooses 2 of the wires and connects them. What is the least number of tests that must be done so that every possible pair of wires is tested?
(A) 7 (B) 14 (C) 16 (D) 18 (E) 21

Problem 3. A snack machine has buttons arranged as shown above. If a selection is made by choosing 2 letters followed by 3 digits, what is the greatest number of different selections that could be made?

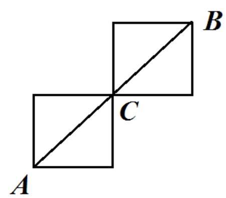
A	D		1	4	7
B	E		2	5	8
C	F		3	6	9

(A) 15 (B) 84 (C) 120 (D) 1260 (E) 1024

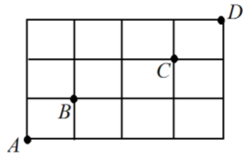
Problem 4. In the figure above, each line segment represents a one-way road with travel permitted only in the direction indicated by the arrow. How many different routes from A to G are possible?
(A) One (B) Six (C) Three (D) Four (E) Seven



Problem 5. A boy is walking along the line starting from point *A* to point *B*. Any point of intersection and line cannot be walked twice in one trip.
How many ways are there?
(A) 15 (B) 12 (C) 10 (D) 9 (E) 6

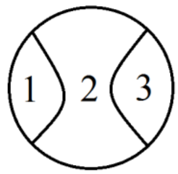


Problem 6. In the figure shown, a path from point *A* to point *D* is determined by moving upward or to the right along the grid lines. How many different paths can be drawn from *A* to *D* that must include both *B* and *C*?



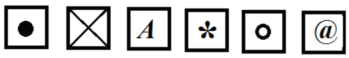
(A) 12 (B) 16 (C) 10 (D) 20 (E) 6

Problem 7. As shown above, a circular shape is divided into 3 regions. A certain design is to be painted such that two regions next to each other must have the different colors. If 6 different colors are available for the design, how many differently pained designs are possible?



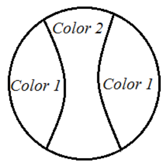
(A) 120 (B) 80 (C) 100 (D) 150 (E) 160

Problem 8. If the 6 cards shown above are placed in a row so that the card with a *A* on it is never at either end, how many different arrangements are possible?



(A) 2400 (B) 480 (C) 960 (D) 720 (E) 460

Problem 9. As shown above, a certain design is to be painted using 2 different colors. If 6 different colors are available for the design, how many differently pained designs are possible?

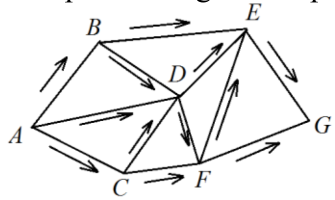


(A) 10 (B) 20 (C) 25 (D) 30 (E) 120

Problem 10. How many different even natural numbers each containing three distinct digits can be written using just the digits 0, 2, 3, 5, and 8?
(A) 6 (B) 12 (C) 20 (D) 30 (E) 36

Problem 11. In how many ways can four married couples be seated at a round table if no two men, as well as no husband and wife are to be in adjacent seats?
(A) 6 (B) 12 (C) 24 (D) 48 (E) 96

Problem 12. In the figure above, each line segment represents a one-way road with travel permitted only in the direction indicated by the arrow. How many different routes from A to G that pass through E are possible?



(A) Four (B) Five (C) Six (D) Seven (E) Eight

Problem 13. Ten points are placed on a circle. What is the greatest number of different lines that can be drawn so that each line passes through two of these points?
(A) 15 (B) 25 (C) 35 (D) 45 (E) 90

Problem 14. Four lines are drawn in a plane so that there are exactly three different intersection points. Into how many nonoverlapping regions do these lines divide the plane?
(A) Seven (B) Nine (C) Eleven (D) Thirteen (E) Fifteen

Problem 15. The four distinct points P, Q, R and S lie on a line l: the five distinct points T, U, V, W, X, and Y lie on a different line that is parallel to line l. What is the total number of different lines that can be drawn so that each line contains exactly two of the seven points?
(A) 9. (B) 12. (C) 10. (D) 20. (E) 40.

Problem 16. A number is called increasing if each of its digits is greater than the digit immediately to its left, if there is one. How many increasing numbers are there between 100 and 200?
(A) 100 (B) 101 (C) 20 (D) 28 (E) 30

Problem 17. In a volleyball league with 5 teams, each team plays exactly 3 games with each of the other 4 teams in the league. What is the total number of games played in this league?
(A) 15 (B) 20 (C) 12 (D) 25 (E) 30

Problem 18. Meredith has a red hat, a blue hat, yellow hat, and a white hat. She also has four sweaters—one red, one blue, one yellow, and one white—and four pairs of jeans—one red, one blue, one yellow, and one white. Meredith wants to wear a red, white, a yellow, and blue outfit consisting of one hat, one sweater, and one pair of jeans. How many different possibilities does she have?
(A) 64 (B) 46 (C) 24 (D) 42 (E) 27

Problem 19. What is the all possible number of points of intersection of one circle and one square?
(A) One, two, three, and four
(B) One, two, three, four, and five
(C) One, two, three, four, five, and six
(D) One, two, three, four, five, six and seven
(E) One, two, three, four, five, six, seven, and eight

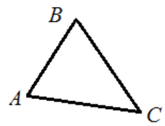
Problem 20. Four circles lie in the same plane. What is the maximum number of points of intersection of them?
(A) 6 (B) 12 (C) 24 (D) 48 (E) 96

Problem 21. Two circles and two straight lines lie in the same plane. If neither the circles nor the lines are coincident, what is the maximum possible number of points of intersection?
(A) 7 (B) 11 (C) 12 (D) 14 (E) 16

Problem 22. How many positive three-digit integers are there such that each digit is an odd number less than 8?
(A) 24 (B) 64 (C) 16 (D) 48 (E) 90

Problem 23. Jesse finds that by selecting from the different combinations of the jackets, pants, and shirts that he owns, he can create up to 150 different outfits, each consisting of one jacket, one pair of pants, and one shirt. If he owns 3 jackets and 5 shirts, how many pairs of pants does Jesse own?
(A) 4 (B) 5 (C) 6 (D) 12 (E) 10

Problem 24. A circle (not drawn) passes through point A in the figure above. What could be the total number of points of intersection of this circle and ΔABC?



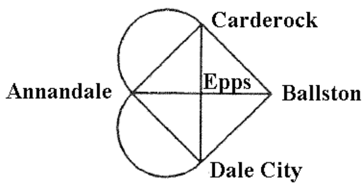
I. 1 II. 2 III. 3 IV. 4 V. 5

(A) I. II only (B) I, II and III only (C) I, II, III, and IV only
(D) II, III and IV only (E) I, II, III, IV, and V.

Problem 25. Three girls and four boys are to be seated in a row containing seven chairs. If the chairs at both ends of the row must be occupied by girls, in how many different ways can the children be seated?
(A) 120 (B) 720 (C) 5040 (D) 2520 (E) 240

Problem 26. How can three-digit natural numbers are there of the form $\underline{htu}$ where $h > t > u$?
(A) 120 (B) 28 (C) 900 (D) 720 (E) 60

Problem 27. How many different ways can Marco travel from Annandale to Ballston? He may not visit any city more than once, and he must stay on the roads.



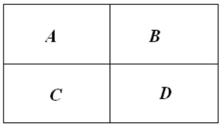
(A) 15 (B) 13 (C) 11 (D) 9 (E) 6

Problem 28. How many four-digit odd integers greater than 6000 can be formed from the digits 0, 1, 3, 5, 6 and 8, if no digit may be used more than once?
(A) 36 (B) 72 (C) 90 (D) 70 (E) 60

Problem 29. How many 3-digit numbers can be formed using the digits of 0, 1, 2, 3, and 4? No digit can be used twice in any such 3-digit number.
(A) 24 (B) 64 (C) 16 (D) 48 (E) 90

Problem 30. Five students are sitting in a row. Alex and Bob are not sitting next to each other. How many arrangements are there?
(A) 12 (B) 72 (C) 50 (D) 25 (E) 24

Problem 31. Use four different colors to color the four rectangles *A, B, C*, and *D* as shown in the figure. No two rectangles sharing the same edge can be the same color. How many ways are there to color the rectangles?



(A) 120 (B) 100 (C) 84 (D) 64 (E) 24

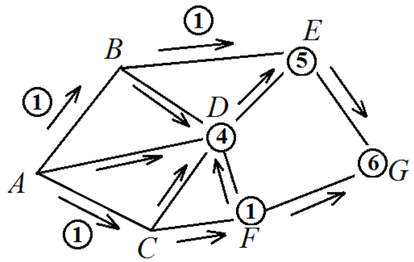
4. SOLUTIONS

Problem 1. Solution: C.
Assume that there are n points in a plane. The maximum number of lines by connecting these points is N and $N = \binom{n}{2} = \binom{7}{2} = \frac{7 \times 6}{2} = 21$

Problem 2. Solution: E.
$N = \binom{n}{2} = \binom{7}{2} = \frac{7 \times 6}{2} = 21$.

Problem 3. Solution: D.
We have $\binom{6}{2} = 15$ ways to select two letters and $\binom{9}{3} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$ ways to select three numbers. By the Fundamental Counting Principle, we get the answer $15 \times 84 = 1260$ different selections.

Problem 4. Solution: B.



Problem 5. Solution: D.
The job (walking from A to B) is done only if he finishes two steps: walking from A to C, and walking from C to B.
The boy has 3 ways from A to C, and 3 ways from C to B. The job (walking from A to B) can be done in $3 \times 3 = 9$ ways.

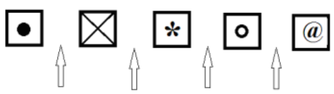
Problem 6. Solution: A.
We have $\binom{2}{1} = 2$ ways from A to B. We have $\binom{3}{1} = 3$

ways from B to C. We have $\binom{2}{1} = 2$ ways from C to D. By the Fundamental Counting Principle, we get $2 \times 3 \times 2 = 12$ ways.

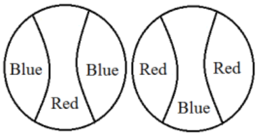
Problem 7. Solution: D.
We have 2 cases:
Case I: Region 1 and region 3 have the different color.
We have 6 colors to use to paint the region 1. We then have 5 colors to use to paint region 2 since region 2 needs to have a different color. We have 4 colors to use to paint region 3 since region 3 has a different color as regions 1 and 2. We have $6 \times 5 \times 4 = 120$ ways to paint.

Case II: Region 1 and region 3 have the same color.
We have 6 colors to choose to paint region 1. then have 5 colors to use to paint region 2. Since region 3 has the same color as region one, we have 1 color to use. Therefore the number of different paints are $6 \times 5 \times 1 = 30$.
By the sum rule, our answer is $120 + 30 = 150$.

Problem 8. B.
Solution: We arrange the other five cards first and we have $5! = 120$ ways.
Then we insert card marked with A. We have $\binom{4}{1} = 4$ ways to insert. By the Fundamental Counting Principle, we get the answer $120 \times 4 = 480$ different arrangements.



Problem 9. Solution: D.
We select 2 colors from 6 colors. We have $\binom{6}{2} = 15$ ways to so. After we select 2 colors (say, we select red and blue), we have two ways to arrange them (see figure below)

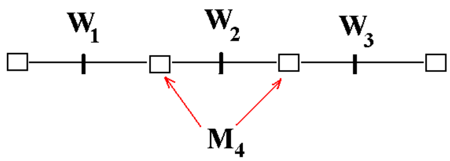


Therefore the answer will be $15 \times 2 = 30$.

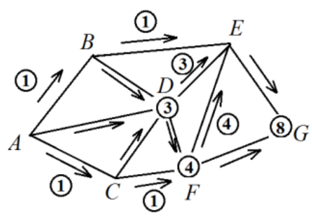
Problem 10. Solution: D.
We know that "0" cannot be the first digit in a number, but we first pretend that it can be. If the number is even then its last digit must be even. There are only three even digits to choose from. Then we arrange the remaining digits. $4 \times 3 \times 3 = 36$.

We now have to subtract the numbers that have "0" in the hundreds digit place. So we first put "0" in the hundreds place. Then we have 2 remaining digits that are even to choose from to put in the units digit. Lastly for the tens digit we can choose any of the remaining numbers: $1 \times 3 \times 2 = 6$. Subtracting 6 from 36 we get 30.

Problem 11. Solution: B.
We already know from the last problem that there are $(4 - 1)! = 3!$ to seat four women. After the ladies are seated, person M_4 (whose wife is not shown in the figure below) has two ways to sit. After he is seated in any one of the two possible seats, the other men have only one way to sit in the remaining seats.
The solution is $3! \times 2 = 12$.



Problem 12. Solution: E.



Problem 13. Solution: D.

We know that any two points will from a line. So we have $\binom{10}{2} = \frac{10\times 9}{2} = 45$ lines.

Problem 14. Solution: D.
Maximum number of regions *n* lines can divide the plane is *N* and
N=$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}=\frac{n(n+1)}{2}+1=\frac{4(5+1)}{2}+1=13$.

Problem 15. Solution: D.
We select one point on l and one point on the line parallel to l to get the number of lines:
$\binom{4}{1}\times\binom{5}{1}=20$ lines.

Problem 16. Solution: D.
Method 1: We list: We have 9 – 3 + 1 = 7 such numbers from 123, 124, …129.
We have 9 – 4 + 1 = 6 such numbers from 134, 135, …139. And so on until we get the last one: 189.
The answer will be 7 + 6 + 5 + 4 + 3 + 2 + 1 = 28.

Method 2: We go about indirectly to solve this problem.
The number of *n*-digit increasing numbers can be calculated by $N=\binom{9}{n}$.

In our case, *n* = 3, and there are a total of
$\binom{9}{3}=84$ three-digit increasing numbers.
However, we must subtract the three-digit increasing numbers from 200 to 1000.
We have the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 available, but we cannot use the digit 1.
There are $\binom{9-1}{3}=56$ increasing numbers between 200 and 1000.
The desired answer is then 84 – 56 = 28.

Problem 17. Solution: E.
Since each team plays with each of the other 4 teams in the league, there will be $\binom{5}{2}=10$ games if each team plays exactly 1 game with each of the other 4 teams in the league.

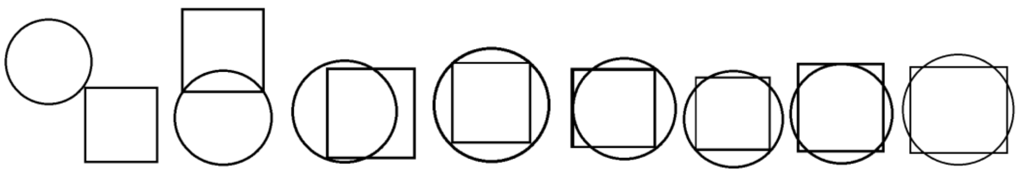
Since each team plays exactly 3 games with each of the other 4 teams, the total number of games played is $3 \times 10 = 30$.

Problem 18. Solution: C.
If she wears a red hat, she can have 6 possibilities as follows:

$$R\begin{cases} B\begin{cases} Y & W\\ W & Y\end{cases}\\ W\begin{cases} Y & B \\ B & Y\end{cases}\\ Y\begin{cases} W & B\\ B & W\end{cases}\end{cases}$$

So the total number of possibilities is $4 \times 6 = 24$.

Problem 19. Solution: E.

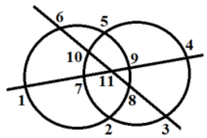


Problem 20. Solution: B.
Two intersecting circles have 2 intersection points. The third circle will at most intersect each circle twice. So the third circle will add $2 \times 2 = 4$ points of intersection. And the fourth circle will add $2 \times 3 = 6$ more circles. The total number of points of intersection will be $2 + 4 + 6 = 12$.

Problem 21. Solution: B.
Method 1: Two lines will have 1 point of intersection. One circle will have at most 2 points of intersection with each line. Each circle will add 4 more points and 2 circles will add 8 points of intersection with the lines.

However, two circles will have at most 2 points of intersection with themselves. This will add 2 points of intersection. The total points of intersection are $1 + 8 + 2 = 11$.

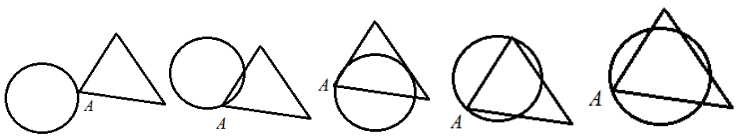
Method 2:
We draw the figure and count 11 regions.



Problem 22. Solution: B.
We have four digits (1, 3, 5, and 7) available for the each of the three digits. By the Fundamental Counting Principle, we get $4 \times 4 \times 4 = 64$ integers.

Problem 23. Solution: E.
By the Fundamental Counting Principle, we get $3 \times 5 \times x = 150 \quad \Rightarrow \quad x = 10$.

Problem 24. Solution: E.



Problem 25. Solution: B.
We first must select the girls that are going to be sitting on the two ends. Let us say, we have g_1, g_2, and g_3 three girls.
If g_1 is on the left end, g_2 or g_3 can be on the right end. We have two ways.
If g_2 is on the left end, g_1 or g_3 can be on the right end. We have two ways.
If g_3 is on the left end, g_1 or g_2 can be on the other end. We have two ways.
Total 6 ways.

Once we fix the positions for both ends, other five kids can be seated in $5! = 120$ ways.
By the product rule, the final answer is $6 \times 120 = 720$.

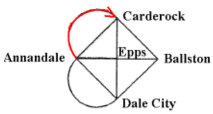
Problem 26. Solution: A.
Method 1: When $h = 9$, if t is 8, u has eight values (7, 6, 5, 4, 3, 2, 1, and 0); if t is 7, u can have seven values (6, 5, 4, 3, 2, 1, and 0). $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$.

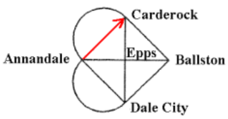
When $h = 8$, if t is 7, u has seven values (6, 5, 4, 3, 2, 1, and 0); if t is 6, u can have six values (5, 4, 3, 2, 1, and 0). $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$.

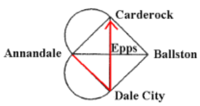
Similarly, we have $36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 = 120$ numbers.
Method 2: The formula to calculate the number of falling numbers is $\binom{10}{n}$ where n is the number of digits. The number of three-digit falling numbers is $\binom{10}{3} = 120$.

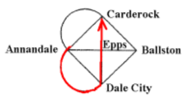
Problem 27. Solution: A.
One important way to solve this kind of problems is to go back one step. In this case, we just count the number of ways from Annandale to Carderock, Epps, and Dale City.

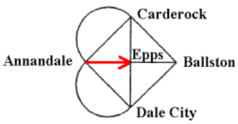
The number of ways from Annandale to Carderock is 5.











Similarly, we count the number of ways from Annandale to Epps: 5. The number of ways from Annandale to Dale City is 5. By the sum rule, the answer is $5 + 5 + 5 = 15$.

Problem 28. Solution: B.
Since the integers need to be greater than 6000, the thousand digit can only be 6 or 8 (2 choices).

Since the integers need to be odd, the units digit can only be 1, 3, or 5 (3 choices). After the thousand and the units digits are determined, there are 4 choices for the hundreds digit and 3 choices for the tens digit. By the product rule, the answer is 2 × 4 × 3 × 3= 72.

Problem 29. Solution: D.
Method 1: There are 4 choices (1, 2, 3 and 4) for the hundred digit. Once the hundreds digit is chosen, there are 4 choices for tens digit and 3 choices for the units digit.

Using the product rule: There are 4 × 4 × 3 = 48 3-digit number.

Method 2:
There are 5 × 4 × 3 = 60 arrangements for these digits.
There are 1 × 4 × 3 = 12 arrangements where "0" is the hundreds digit.
So the number of 3-digit positive integers is 60 – 12 = 48.

Problem 30. Solution: B.
There are 5 × 4 × 3 × 2 × 1 = 120 ways without any restrictions.
There are 2× 4 × 3 × 2 × 1 = 48 ways such that Alex and Bob are sitting next to each other.
The answer is 120 – 48 = 72.

Problem 31. Solution: C.
Case I: A and D are different colors.
There are 4 ways to color A, 3 ways to color D, 2 ways to color B, and 2 ways to color B. 4 × 3 × 2 × 2 =48.
Case II: A and D are the same color.
There are 4 ways to color A, 1 way to color D, 3 ways to color B, and three ways to color B. 4 × 1 × 3 × 3 =36.
The total number of ways to color the four rectangles is 48 + 36 = 84.

1. BASIC KNOWLEDGE

"Divisible by" means "when you divide one number by another number, the result is a whole number." "Divisible by" and "can be evenly divided by" mean the same thing.

The expressions $\overline{abc}$, $\underline{abc}$, and abc are the same. $\overline{abc} = \underline{abc} = 100a + 10b + c$. They represent a three-digit number such as $\overline{234} = \underline{234} = 234$.

2.1. Divisibility rule for 2, 4, 8, and 16:

A number is divisible by 2 if the last digit of the number is divisible by 2 (2^1).
A number is divisible by 4 if the last two digits of the number are divisible by 4 (2^2).
A number is divisible by 8 if the last three digits of the number are divisible by 8 (2^3).
A number is divisible by 16 if the last four digits of the number are divisible by 16 (2^4).

2.2. Divisibility rule for 5, 25, 125, and 225:

A number is divisible by 5 if the last digit of the number is divisible by 5 $\mathbf{(5^1)}$.
A number is divisible by 25 if the last two digits of the number form a number that is divisible by 25 $\mathbf{(5^2)}$.
A number is divisible by 125 if the last three digits of the number form a number that is divisible by 125 $\mathbf{(5^3)}$.
A number is divisible by 625 if the last four digits of the number form a number that is divisible by 625 $\mathbf{(5^4)}$.

2.3. Divisibility rule for 3 and 9

A number is divisible by 3 if the sum of the digits of the number is divisible by 3.
A number is divisible by 9 if the sum of the digits of the number is divisible by 9.

2.4. Divisibility rule for 7, 11, and 13:

(1) If you double the last digit and subtract it from the rest of the number and the answer is divisible by 7, the number is divisible by 7. You can apply this rule to that answer again if necessary.

(2) To find out if a number is divisible by 11, add every other digit, and call that sum "x." Add together the remaining digits, and call that sum "y." Take the positive difference of x and y. If the difference is zero or a multiple of eleven, then the original number is a multiple of eleven.
(3) Delete the last digit from the number, and then subtract 9 times the deleted digit from the remaining number. If what is left is divisible by 13, then so is the original number. Repeat the rule if necessary.

(4) If the positive difference of the last three digits and the rest of the digits is divisible by 7, 11, or 13, then the number is divisibly by 7, 11, or 13, respectively.

2.5. Divisibility rule for 6, 10, 12, 14, 15, 18, 24, and 36:

A number is divisible by 6 if the number is divisible by both 2 and 3.
A number is divisible by 10 if the number is divisible by both 2 and 5.
A number is divisible by 12 if the number is divisible by both 3 and 4.
A number is divisible by 14 if the number is divisible by both 2 and 7.
A number is divisible by 15 if the number is divisible by both 3 and 5.
A number is divisible by 18 if the number is divisible by both 2 and 9.
A number is divisible by 24 if the number is divisible by both 3 and 8.
A number is divisible by 36 if the number is divisible by both 4 and 9.

NOTE: If a number is divisible by two numbers that are relatively prime, then it is divisible by the product of those two numbers. "Relatively prime" means two numbers have no common factor other than 1. For example, 3 and 4 are relatively prime.

2. PROBLEM SOLVING SKILLS

(1). Divisibility rule for 2, 4, 8, and 16:

Example 1. If the three-digit number $\underline{78N}$ is divisible by 4, how many possible values of N are there?
(A) 3 (B) 6 (C) 5 (D) 4 (E) 8

Solution: A.
We have five numbers where the last digit is even: 80, 82, 84, 86, and 88. Among them, only 80, 84, and 88 are divisible by 4. So N has 3 values: 0, 4, and 8.

(2). Divisibility rule for 5, 25, 125, and 225:

Example 2. ☆ A three-digit integer contains one of each of the digits 3, 4 and 5. What is the probability that the integer is divisible by 5?
(A) 1/6 (B) 1/3 (C) 1/2 (D) 2/3 (E) 5/6

Solution: B.
The number is equally likely to end in 3, 4 or 5. The number is divisible by 5 only if it ends in 5, so the probability is 1/3.

Example 3. The six-digit number $\overline{713EF5}$ is divisible by 125. How many such six-digit numbers are there?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: E.
(713125, 713375, 713625, and 713875).
If the given number is divisible by 125, $\overline{EF5}$ should be divisible by 125.
125 × 1 = 125, 125 × 3 = 375, 125 × 5 = 625, and 125 × 7 = 875.

(3). Divisibility rule for 3 and 9

Example 4. What is the sum of all possible digits which could fill the blank in 47,__21 so that the resulting five-digit number is divisible by 3?
(A) 3 (B) 6 (C) 9 (D) 12 (E) 18

Solution: D.
Let the digit be x,
$4 + 7 + x + 2 + 1 = 14 + x$.
$x = 1, 4$, and 7 in order for $14 + x$ to be divisible by 3
The sum is $1 + 4 + 7 = 12$.

Example 5. Find the least possible value of digit d so that $\underline{437,d03}$ is divisible by 9.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: A.
$4 + 3 + 7 + d + 0 + 3 = 17 + d \quad \Rightarrow \quad d = 1$ (only value possible).

(4). Divisibility rule for 7, 11, and 13:

Example 6. Which number is not divisible by 7? 616, 567, 798, or 878.

Solution: 878.
616: $61 - 6 \times 2 = 61 - 12 = 49$ (divisible by 7)
567: $56 - 7 \times 2 = 56 - 14 = 42$ (divisible by 7)
798: $79 - 8 \times 2 = 79 - 16 = 63$ (divisible by 7)
878: $87 - 8 \times 2 = 87 - 16 = 71$ (not divisible by 7).

Example 7. What is the largest integer less than 100 and evenly divisible by 7?

Solution: 98.
$7 \times 15 = 105 \quad \Rightarrow \quad 7 \times 14 = 98$

Example 8. Which digit should replace a in the units place so that $9867542a$ is divisible by 11?

(A) 3 (B) 6 (C) 9 (D) 4 (E) 8

Solution: A.
Let $x = 9 + 6 + 5 + 2$, $y = 8 + 7 + 4 + a$.
$x - y = (9 + 6 + 5 + 2) - (8 + 7 + 4 + a) = 22 - 19 - a = 3 - a. \quad \Rightarrow \quad a = 3.$

(5). Divisibility rule for 6, 10, 12, 14, 15, 18, 24, and 36:

A number is divisible by 6 if the number is divisible by both 2 and 3.
A number is divisible by 10 if the number is divisible by both 2 and 5.
A number is divisible by 12 if the number is divisible by both 3 and 4.
A number is divisible by 14 if the number is divisible by both 2 and 7.
A number is divisible by 15 if the number is divisible by both 3 and 5.
A number is divisible by 18 if the number is divisible by both 2 and 9.
A number is divisible by 24 if the number is divisible by both 3 and 8.
A number is divisible by 36 if the number is divisible by both 4 and 9.

NOTE: If a number is divisible by n numbers that are relatively prime, then it is divisible by the product of those n numbers.

Example 9. What is the greatest three-digit number that is divisible by 6?
(A) 999 (B) 998 (C) 997 (D) 996 (E) 995

Solution: D.
The number must be divisible by both 2 and 3. So the number is even and the sum of its digits is divisible by 3. If the number is $\underline{99x}$, $x = 6$ works.

Example 10. Given the 4-digit base-ten number $\underline{77A4}$. For what value of the nonzero digit A will this 4-digit number be divisible by 3 and by 4?
(A) 3 (B) 6 (C) 9 (D) 1 (E) 7

Solution: B.
The number is divisible by 3: $7 + 7 + A + 4 = 18 + A$. A can be 0, 3, 6, and 9.
The number is divisible by 4: A must be even and nonzero $\Rightarrow$ $A = 6$.

Example 11. Find the value of x such that the four-digit number $\underline{x15x}$ is divisible by 18.
(A) 1 (B) 2 (C) 4 (D) 6 (E) 8

Solution: D.
Since $\underline{x15x}$ is divisible by 18, it must be divisible by both 2 and 9.
$x + 1 + 5 + x = 2x + 6 \qquad \Rightarrow \qquad x = 6$ (only value possible).

Example 12. Find distinct digits A and B such that $\underline{A47B}$ is as large as possible and divisible by 36. Name the number.
(A) 5472 (B) 6471 (C) 5470 (D) 3474 (E) 6470

Solution: A.
$\underline{A47B}$ is divisible by 36 and it is also divisible by 9 and 4.
$A + 4 + 7 + B = A + B + 11 \quad \Rightarrow \quad A + B = 7$ and $A + B = 16$
Since A and B are distinct, $A + B = 7$.
$\underline{7B}$ needs to be divisible by 4 and we want the greatest value of $\underline{A47B}$.
$A = 5$ and $B = 2$. $\underline{A47B} = 5472$.

Example 13. If k is a positive integer divisible by 3, and if $k < 80$, what is the greatest possible value of k?
(A) 75 (B) 76 (C) 77 (D) 78 (E) 79

Solution: (D).
The sum of the digits must be divisible by 3. 7 + 8 = 15 which is divisible by 3.

Example 14. which of the following numbers can be used to show that the statement below is FALSE?
<u>All numbers that are divisible by both 2 and 6 are also divisible by 12.</u>
(A) 8 (B) 12 (C) 18 (D) 24 (E) 36

Solution: C.
18 is divisible by both 2 and 6 but not divisible by 12.

Example 15. On a square gameboard that is divided into n rows of n squares each, k of these squares not lie along the boundary of the gameboard. If k is one of the four numbers below, what is the possible value for n?
(I) 10 (II) 25 (III) 34 (IV) 52

(A) 10 (B) 12 (C) 14 (D) 16 (E) 18

Solution: C.
The number of squares not lying along the boundary of the gameboard is

$n^2 - (n - 2)^2 = 4(n - 1)$ that is divisible by 4. So we know that $k = 52$. When $k = 52$, that is, $4(n - 1) = 52$. Solving for n: $n = 14$.

Example 16. A student practices the four musical notes as shown, starting with the note furthest left and continuing in order from left to right. If the student plays these notes over and over according to this pattern and stops immediately after playing the shaded note, which of the following could be the total number of notes played?

(A) 50 (B) 51 (C) 52 (D) 53 (E) 64

Solution: A
When 2 is added to the total number, the result is divisible by 4. $50 + 2 = 52 = 4 \times 13$.

Example 17. Which of the following statement can be used to determine if a number is divisible by 54 or not?
I. The number must be divisible by both 6 and 9.
II. The number must be divisible by both 3 and 18.
III. The number must be divisible by both 2 and 27.

(A) I only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

Solution: (B)
Two numbers must be relatively prime.

Example 18. For what digit(s) x will the 7-digit number $3xx6xx2$ be divisible by 4?
(A) 3 (B) 6 (C) 2 (D) 5 (E) 8

Solution: D.
1, 3, 5, 7, and 9.
The two-digit number $\underline{x2}$ needs to be divisible by 4 by the divisibility rules. The following numbers work: 12, 32, 52, 72, and 92.

Example 19. What is the largest digit which can replace b to make the number 437,b32 divisible by 3?
(A) 3 (B) 6 (C) 2 (D) 5 (E) 8

Solution: E.
$4 + 3 + 7 + b + 3 + 2 = 19 + b \Rightarrow \quad b = 2, 5$, and 8. The greatest value of b is 8.

Example 20. The three-digit number 2a3 is added to the number 326 to give the three-digit number 5b9. If 5b9 is divisible by 9, then $a + b$ equal:
(A) 2 (B) 4 (C) 6 (D) 8 (E) 9

Solution: (C).
The sum of the digits of 5b9 must be divisible by 9. $5 + b + 9 = 14 + b$. b must be 4.
$2a3 + 326 = 549 \quad \Rightarrow \quad 2a3 = 549 - 326 = 223 \quad a = 2$
$a + b = 2 + 4 = 6$.

Example 21. A and B are non-zero digits for which $\underline{A468B05}$ is divisible by 11. What is $A + B$?
(A) 6 (B) 8 (C) 10 (D) 14 (E) 12

Solution: E.
Let $x = A + 6 + B + 5$, $y = 4 + 8 + 0$.
$x - y = (A + 6 + B + 5) - (4 + 8 + 0) = 11 + B + A - 12 = B + A - 1 \Rightarrow \quad B + A = 12$.

Example 22. If the 4-digit number $\underline{273X}$ is divisible by 12, what is the value of X ?
(A) 3 (B) 6 (C) 5 (D) 4 (E) 8

Solution: B.
The 4-digit number $\underline{273X}$ is divisible by 3 and 4.
$2 + 7 + 3 + X = 12 + X$ must be divisible by 3, so X can be 0, 3, 6, and 9. On other hand, X must be even so that the number is divisible by 4, so we have $X = 0$ or 6. 3X must also be divisible by 4, so $X = 6$ is the only value that works.

Example 23. What value can a have to make $\underline{a74a}$ divisible by 36?

(A) 2 (B) 4 (C) 6 (D) 8 (E) 9

Solution: D.

The number is divisible by 4 and 9. For $\underline{a74a}$ to be divisible by 4, a could be 4 or 8. For $\underline{a74a}$ to be divisible by 9, $2a + 7 + 4 = 2a + 11$ should be divisible by 9. The only possible value for a is 8.

3. PROBLEMS

Problem 1. If k is a positive integer divisible by 9, and if $k < 200$, what is the greatest possible value of k?
(A) 99 (B) 189 (C) 197 (D) 198 (E) 199

Problem 2. which of the following numbers can be used to show that the statement below is FALSE?
All numbers that are divisible by both 6 and 9 are also divisible by 54.

(A) 162 (B) 108 (C) 9 (D) 72 (E) 54

Problem 3. On a rectangular gameboard that is divided into n rows of m squares each, k of these squares not lie along the boundary of the gameboard. Which of the following is a possible value for k?
(A) 15 (B) 25 (C) 35 (D) 49 (E) 52

Problem 4. When the positive integer s is divided 12, the remainder is 6. When the positive integer t is divided by 12, the remainder is 9. What is the remainder when the product st is divided by 9?
(A) 1 (B) 3 (C) 5 (D) 7 (E) 0

Problem 5. If x is an integer and $3x$ is divisible by 15, which of the following must be true?
I. x is divisible by 15.
II. x is divisible by 5.
III. x is an odd number.

(A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

Problem 6. If x is divisible by 7 and y is divisible by 8. Which of the following must be divisible by 56?
I. xy II. $7x + 8y$ III. $8x + 7y$

(A) I only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

Problem 7. If a, b, c and d are different positive integers such that a is divisible by b, b is divisible by c, and c is divisible by d, which of the following statements must be true?
I. a is divisible by cd. II. a has at least 4 positive factors. III. $a = bcd$

(A) I only (B) II only (C) I and II (D) I and III only (E) I, II, and III

Problem 8. The four-digit number $\overline{6BB5}$ is divisible by 25. How many such four-digit numbers are there?
(A) 0 (B) 1 (C) 3 (D) 2 (E) 4

Problem 9. The five-digit number $\underline{31d26}$ is divisible by 3. Find the sum of all possible values of d.
(A) 18 (B) 16 (C) 15 (D) 14 (E) 8

Problem 10. The three-digit number $\underline{6x4}$ is divisible by 7. What is the value of x?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 11. When Rachel divides her favorite number by 7, she gets a remainder of 5. What will the remainder be if she multiplies her favorite number by 5 and then divides by 7?
(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Problem 12. For what digit n is the five-digit number $3n85n$ divisible by 6?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 13. What digit should replace the tens digit d so that the seven-digit number $5{,}376{,}5d4$ is divisible by 24?
(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Problem 14. How many 2-digit numbers are not divisible by 13?
(A) 90 (B) 83 (C) 13 (D) 7 (E) 84

Problem 15. How many numbers less than 100 and divisible by 3 are also divisible by 4?
(A) 96 (B) 42 (C) 8 (D) 25 (E) 33

Problem 16. There are 24 four-digit numbers which use each of the digits 1, 2, 3, 4. How many of these are divisible by 11?
(A) 10 (B) 6 (C) 5 (D) 4 (E) 8

Problem 17. Find a digit d that makes the three-digit number $\underline{2\ d\ 6}$ a multiple of 22.
(A) 1 (B) 4 (C) 5 (D) 8 (E) 2

Problem 18. A four-digit number uses each of the digits 1, 2, 3 and 4 exactly once. What is the probability that the number is a multiple of 4?
(A) 1/2 (B) 1/3 (C) 1/5 (D) 1/4 (E) 3/8

Problem 19. What is the greatest three-digit number that is divisible by 6?
(A) 999 (B) 998 (C) 997 (D) 996 (E) 993

Problem 20. If the four-digit number $\underline{5{,}7d\ 2}$ is divisible by 18, what is d?
(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Problem 21. What digit can replace K in the number $\underline{9K73K0}$ so that $\underline{9K73K0}$ will be divisible by 60?
(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Problem 22. How many different 4-digit numbers can be formed using the digits 2, 4, 5, 6, and 7 such that no digits repeat and the number is divisible by 4?
(A) 24 (B) 36 (C) 22 (D) 31 (E) 120

Problem 23. Given that m and n are digits, what is the sum of the values for m and n such that the five-digit number $m6{,}79n$ is divisible by 72?
(A) 4 (B) 3 (C) 7 (D) 10 (E) 5

☆**Problem 24.** (2011 AMC Problem 22) What is the tens digit of 7^{2011}?
(A) 0 (B) 1 (C) 3 (D) 4 (E) 7

4. SOLUTIONS

Problem 1. Solution: D.
The sum of the digits must be divisible by 9. 1 + 9 + 8 = 18 which is divisible by 9.

Problem 2. D.
72 is divisible by both 6 and 9 but not divisible by 54.

Problem 3. E
$nm - (n - 2)(m - 2) = 2(n + m - 2)$ which must be divisible by 2.

Problem 4. Solution: E.
Solution: Let *s* be 18 and *t* be s divided by 21. Then $st = 378$. By the divisibility rule for 9, 3 + 7 + 8 = 18 which is divisible by 9. So *st* is divisible by 9. The remainder is 0.

Problem 5. Solution: B.
x must be a multiple of 5 so it is divisible by 5. Let us say $x = 10$. $3x = 30$ that is divisible by 15. So we see that both I and III are not true.

Problem 6. Solution: D.
Since x is divisible by 7 and y is divisible by 8, xy must be divisible by 56. For the same reason, $8x$ is divisible by 56 as well as $7y$. Therefore $8x + 7y$ is divisible by 56.

Problem 7. Solution: C.
We know that both I and II are true by letting $a = 8$, $b= 4$, $c = 2$, and $d = 1$.
We know that III is not true by letting $a = 16$, $b = 4$, $c = 2$, and $d = 1$.

Problem 8. Solution: D.
(6225, and 6775)
If the given number is divisible by 25, $\overline{B5}$ should be divisible by 25.
25 × 1 = 25, and 25 × 3 = 75.

Problem 9. Solution: A.
$3+1+d+2+6=12+d \quad \Rightarrow \quad d = 0, 3, 6, \text{ and } 9.$

The sum of all possible values of d is 0 + 3 + 6 + 9 = 18.

Problem 10. Solution: D.
By the divisibility rule for 7, $\underline{6x} - 4 \times 2 = 60 + x - 8 = 52 + x \quad \Rightarrow \quad x = 4$.

Problem 11. Solution: A.
The smallest such number could be 7 + 5 = 12.
$12 \times 5 = 60 \quad \Rightarrow \quad 60 \div 7 = 8\ r\ 4$.

Problem 12. Solution: E.
$3 + n + 8 + 5 + n = 16 + 2n$ is divisible by 3 when n = 1, 4, or 7. Since n is even, n = 4.

Problem 13. Solution: E.
If $5{,}376{,}5d4$ is divisible by 24, it must be divisible by both 3 and 8.
$5 + 3 + 7 + 6 + 5 + d + 4 = 30 + d \quad \Rightarrow \quad d$ = 0, 3, 6, or 9.
$5d4$ is also divisible by 8. d = 0 is the only value that works.

Problem 14. Solution: B.
There are ninety 2-digit numbers. There are seven 2-digit numbers divisible by 13 with the first one being 13 × 1 = 13 and the last one being 13 × 7 = 91.
The answer is 90 – 7 = 83.

Problem 15. Solution: C.
Method 1:
The number is divisible by 12. There are 8 such numbers less than 100 with the first one being 12 × 1 = 12 and the last one being 12 × 8 = 96.

Method 2:
$\left\lfloor \frac{100}{12} \right\rfloor = 8$. The answer is 8.

Problem 16. Solution: E.
Let the four-digit number be $\underline{abcd}$
We have: $a + c = b + d$
$a + c = 5$ and $b + d = 5$.

$\Rightarrow$ $a = 4$, $c = 1$, and $b = 3$, $d = 2$.
We also have to consider the different orderings of the numbers. We get eight numbers that are divisible by 11: 4213, 4312, 1243, 1342, 3421, 3124, 2431, 2134.

Problem 17. Solution: D.
The given number is already divisible by 2, so we only need to consider the case when it is divisible by 11. Therefore, we must have $d = 2 + 6 = 8$. We see that $286 = 11 \times 26$.

Problem 18. Solution: D.
The number formed by last two digits must be divisible by 4.
We have the following cases for the last two digits:

1	2
2	4
3	2

For all three cases we can produce two such four-digit numbers.

The solution is then $6/24 = 1/4$.

Problem 19. Solution: D.
The last digit should be even and the sum of the digits should be divisible by 3. So the greatest three-digit number is 996.

Problem 20. Solution: A.
The number should be even and divisible by 9. Since the last digit is already even, we only need to consider the sum of the digits. $d = 4$.

Problem 21. Solution: A.
The number should be divisible by 3, 5, and 4.
Since the last digit is 0, it will be divisible by 5.
The sum of the digits should be divisible by 3, so K can be 1, 4, or 7.
The last two-digit should be divisible by 4, so the only value for K is 4.

Problem 22. Solution: B.
If a number is divisible by 4, the last two digits must be divisible by 4.
We have the following six cases:

__ __ 5 2 __ __ 7 2

__ __ 2 4 __ __ 6 4

__ __ 5 6 __ __ 7 6

We have $3 \times 2 = 6$ ways to fill each two empty spaces. There are $6 \times 6 = 36$ such 4-digit numbers.

Problem 23. Solution: E.
$72 = 8 \times 9 \Rightarrow m6{,}79n$ must be divisible by both 8 and 9.
When $m6{,}79n$ is divisible by 8, the three-digit number $79n$ must divisible by 8 and n must be even. This yields $n = 2$.
When $m6{,}79n$ is divisible by 9, $m + 6 + 7 + 9 + 2 = m + 24$ must be divisible by 9.
Note that $0 \le m \le 9$, so m must be 3.
$m + n = 3 + 2 = 5$.

Problem 24. Solution: D.
Method 1(official solution):
The tens digit of a power of 7 is determined by the last two digits of the previous power of 7. The pattern for the last two digits of successive powers of 7 is 01, 07, 49, 43, 01, 07, 49, 43, 01, 07, 49, 43, 01,… Since $2011 = 4 \times 502 + 3$, the last two digits of 7^{2011} are 43 and the tens digit is 4.

Method 2 (our solution)
$$7^{2011} = (7^2 \times 7^2)^{502} \times 7^3 = [(50-1)^2]^{502} \times 343 = (2500-100+1)^{502} \times 343$$
When $(2500-100+1)^{502} \times 343$ is divided by 100, the remainder is 43. So the answer is D.

Method 3 (our solution)
We try to find the last two digit of 7^{2011}.
$$7^{2011} \equiv r \quad \text{mod } 100 \qquad (1)$$
(1) can be written as

$7^{2011} \equiv r_1 \mod 4 \quad \Rightarrow \quad 7^{2011} \equiv 3 \mod 4 \qquad (2)$

$7^{2011} \equiv r_2 \mod 25 \quad \Rightarrow \quad (7^2)^{1005} \times 7 \equiv r_2 \mod 25 \qquad (3)$

(3) can be written as

$(49)^{1005} \times 7 \equiv r_2 \mod 25 \quad \Rightarrow \quad (-1)^{1005} \times 7 \equiv r_2 \mod 25$

$\Rightarrow \quad -7 \equiv r_2 \mod 25 \quad \Rightarrow \quad 18 \equiv r_2 \mod 25$

Then we have

$7^{2011} \equiv 3 \mod 4$

$7^{2011} \equiv 18 \mod 25$

Or

$7^{2011} \equiv 43 \mod 4$

$7^{2011} \equiv 43 \mod 25$

Or $\quad 7^{2011} \equiv 43 \mod 100$

So the last two digit is 43 and the answer is D.