American Mathematics Competitions
(AMC 8)
Preparation

Volume 4

The American Mathematics Competitions 8 is a 25-question multiple-choice contest for students in the sixth through eighth grade. Accelerated fourth and fifth graders can also take part. The AMC 8 is administered in schools in November. The American Mathematics Competitions (AMC) publishes the Achievement Roll list recognizing students in 6th grade and below who scored 15 or above, and the Honor Roll list recognizing students who score in the top 5%, and the Distinguished Honor Roll list recognizing students who score in the top 1%.

This book can be used by 5th to 8th grade students preparing for AMC 8. Each chapter consists of (1) basic skill and knowledge section with plenty of examples, (2) about 30 exercise problems, and (3) detailed solutions to all problems.

We would like to thank the American Mathematics Competitions (AMC 8 and 10) for their mathematical ideas. Many problems (marked by ☆) in this book are inspired from these tests. We only cited very few problems directly from these tests for the purpose of comparison with our own solutions.

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1. BASIC KNOWLEDGE

In this lecture, we learn how to solve the following types of problems using unconventional signs for the written notation of mathematical notions and reasoning: additions, subtractions, multiplications, divisions, exponents, and radicals.

All the rules of operations (addition, subtraction, multiplication, division, radicals, and exponents) we learnt from arithmetic and algebra are still valid with these symbols.

Fundamental law of fractions:

For any fraction $\frac{a}{b}$ and any number c ≠ 0, $\frac{a}{b} = \frac{a \times c}{b \times c}$.

Division of fractions

To divide by a fraction, we simply multiply by its reciprocal.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Power rules of exponents

$a^m \times a^n = a^{m+n}$ ⇔ $a^{m+n} = a^m \times a^n$

$(a^m)^n = a^{mn}$ ⇔ $(ab)^n = a^n b^n$

$(\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$ ⇔ $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Properties of radicals

$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ ($a > 0$, and $b > 0$)

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \qquad (a > 0, \text{ and } b > 0)$$

$$\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}$$

Properties of absolute value

$$|-x| = |x|$$

$$|x - y| = |y - x|$$

$|xy| = |x| \cdot |y|$, and $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $\quad y \neq 0.$

Square binomial

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$x^2 - y^2 = (x - y)(x + y)$$

Number of divisors

For an integer n greater than 1, let the prime factorization of n be $n = p_1^a p_2^b p_3^c \ldots p_k^m$, Where $a, b, c, \ldots$, and m are nonnegative integers, $p_1, p_2, \ldots, p_k$ are prime numbers.

The number of divisors is: $d(n) = (a+1)(b+1)(c+1)\ldots(m+1)$

Sum of the positive divisors

The sum of divisors is:

$$\sigma(n)=(\frac{p_1^{a+1}-1}{p_1-1})(\frac{p_2^{b+1}-1}{p_2-1})\dots(\frac{p_k^{m+1}-1}{p_k-1})$$

Or $\sigma(n)=(p_1^{\ a}+p_1^{\ a-1}+\dots+p_1^{\ 0})(p_2^{\ b}+p_2^{\ b-1}+\dots+p_2^{\ 0})\dots(p_k^{\ m}+p_k^{\ m-1}+\dots+p_k^{\ 0})$

Patterns of the last digit of a^n

The last digits of a^n have patterns shown in the table below.

n	1	2	3	4	Period
2^n	**2**	**4**	**8**	**6**	4
3^n	**3**	**9**	**7**	**1**	4
4^n	**4**	**6**			2
5^n	**5**				1
6^n	**6**				1
7^n	**7**	**9**	**3**	**1**	4
8^n	**8**	**4**	**2**	**6**	4
9^n	**9**	**1**			2

For example, when $a = 2$,

$2^1 = 2$ $2^2 = 4$, $2^3 = 8$, $2^4 = 16$,

$2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256,\dots$

The last digits of 2^n demonstrate a pattern: 2, 4, 8, 6, 2, 4, 8, 6, etc…

Pythagorean triples

A **Pythagorean triple** consists of three positive integers a, b, and c, such that $a^2 + b^2 = c^2$.

There are 16 primitive Pythagorean triples with $c < 100$:

(3, 4, 5)	(5, 12, 13)	(8, 15, 17)	(7, 24, 25)
(20, 21, 29)	(12, 35, 37)	(9, 40, 41)	(28, 45, 53)
(11, 60, 61)	(16, 63, 65)	(33, 56, 65)	(48, 55, 73)
(13, 84, 85)	(36, 77, 85)	(39, 80, 89)	(65, 72, 97)

2. PROBLEMS SOLVING

2. 1. Additions

☆**Example 1.** For any positive integer *n*, define $(6)n$ to be the sum of the positive factors of *n*. For example, $(6) = 1 + 2 + 3 + 6 = 12$. Find $((18))$.

(A) 39 (B) 40 (C) 48 (D) 56 (E) 60

Solution: (D).

Method 1:

First calculate $(18) = 1 + 2 + 3 + 6 + 9 + 18 = 39$.

$(39) = 1 + 3 + 13 + 39 = 56$.

Method 2:

$18 = 3^2 \times 2$.

The sum of the positive factors of 18 is $(3^2 + 3^1 + 3^0)(2^1 + 2^0) = 13 \times 3 = 39$.

The sum of the positive factors of 39 is $(3^1+ 3^0)(13^1 +13^0) = 56$.

$((18)) = 56$.

☆**Example 2.** For the positive integer *n*, let <*n*> denote the sum of all the positive divisors of *n* with the exception of *n* itself. For example, <4> = 1 + 2 = 3 and <12> = 1 + 2 + 3 + 4 + 6 = 16. What is <<<28>>>?

(A) 28 (B) 11 (C) 21 (D) 6 (E) 3

Solution: (A).

Method 1:

First calculate <28> = 1 + 2 + 4 + 7 + 14 = 28.

As a consequence, we also have <<<28>>> = 28.

Method 2:

$28 = 2^2 \times 7$.

The sum of all the positive divisors of 28 is $(2^2 + 2^1 + 2^0)(7^1 + 7^0) = 7 \times 8 = 56$.

so <28>= 56 – 28 = 28.

As a consequence, we also have <<<28>>> = 28.

Note: A positive integer whose divisors other than itself add up to that positive integer is called a perfect number. The two smallest perfect numbers are 6 and 28.

☆**Example 3.** Let ♣(x) denote the sum of the digits of the positive integer x. For example, ♣ (8) = 8 and ♣ (123) = 1 + 2 + 3 = 6. For how many two-digit values of x is ♣ (x) = 12?

(A) 3 (B) 4 (C) 6 (D) 9 (E) 7

Solution: E.

Since $x \le 99$, there are 7 values of x for which ♣ (x) = 12:

93, 39; 84, 48; 75, 57; and 66.

Example 4. If $a \blacklozenge b = a^2 + ab - b^2$, then find $(3 \blacklozenge 2) \blacklozenge 13$.

(A) 11 (B) 95 (C) – 48 (D) 59 (E) 143

Solution: B.

$3 \blacklozenge 2 = 3^2 + 3 \times 2 - 2^2 = 9 + 6 - 4 = 11$.

$(3 \blacklozenge 2) \blacklozenge 13 = 11 \blacklozenge 13 = 11^2 + 11 \times 13 - 13^2 = 95$.

Example 5. If $a \triangle b = (a + b) + ab + b$, what is $(5 \triangle 7) \triangle 3$?

(A) 54 (B) 57 (C) 219 (D) 222 (E) 232

Solution: D.

$5 \triangle 7 = (5 + 7) + 5 \times 7 + 7 = 54$.

$54 \triangle 3 = (54 + 3) + 54 \times 3 + 3 = 222$.

Example 6. For all real numbers a and b, where $b \neq 0$, the operation ★ is defined as $a \bigstar b = \dfrac{a^2 + b^2}{b^3}$. Compute the following, and express your answer as a common fraction: $(1 \bigstar 2)^2/(2 \bigstar 1)$.

(A) $\frac{1}{8}$ (B) $\frac{1}{5}$ (C) $\frac{125}{64}$ (D) $\frac{5}{64}$ (E) $\frac{25}{64}$

Solution: D.

$1\bigstar 2 = \dfrac{1^2+2^2}{2^3} = \dfrac{5}{8} \quad \Rightarrow \quad (1\bigstar 2)^2 = \dfrac{25}{64}$

$2\bigstar 1 = \dfrac{2^2+1^2}{1^3} = 5$.

$(1\bigstar 2)^2/(2\bigstar 1) = \dfrac{25}{64} / 5 = \dfrac{5}{64}$.

Example 7. If $a \bigtriangledown b = a^2 + 2ab + b^2$, what is the value of $(3 \bigtriangledown 2) \bigtriangledown 5$?
(A) 25 (B) 60 (C) 900 (D) 625 (E) 30

Solution: C.
$a \bigtriangledown b = a^2 + 2ab + b^2 = (a + b)^2$.
$3 \bigtriangledown 2 = (3 + 2)^2 = 25$.
$25 \bigtriangledown 5 = (25 + 5)^2 = 900$.

2. 2. Subtractions

☆**Example 8.** Define $x \otimes y = x^3 - y$. What is $h \otimes (h \otimes h)$?
(A) $-h$ (B) -0 (C) h (D) $2h$ (E) h^3

Solution: C.
By the definition we have $(h \otimes h) = h^3 - h$.
$h \otimes (h \otimes h) = h \otimes (h^3 - h) = h^3 - (h^3 - h) = h$.

Example 9. The operation $\oplus$ is defined as $m \oplus n = m^2 - mn - n^2$, and the operation $\bowtie$ is defined as $m \bowtie n = 2(m - n)$. Compute $(3\oplus 4) \bowtie (4\oplus 3)$.
(A) -26 (B) -28 (C) -19 (D) 28 (E) -5

Solution: B.
$3\oplus 4 = 3^2 - 3 \times 4 - 4^2 = -19$.

$4 \oplus 3 = 4^2 - 4 \times 3 - 3^2 = -5$.

$(3 \oplus 4) \bowtie (4 \oplus 3) = (-19) \bowtie (-5) = 2[-19 - (-5)] = 2 \times (-14) = -28$.

Example 10. If for positive integers a and b, $\langle ab \rangle = ab - a - b$, find the value of $a + b$ in the equation $\langle ab \rangle = 6$.

(A) 9 (B) 10 (C) 8 (D) 15 (E) 12

Solution: B.

$\langle ab \rangle = 6 \quad \Rightarrow \quad ab - a - b = 6 \quad \Rightarrow \quad (a-1)(b-1) = 7$.

We have

$a - 1 = 1$ (1)

and $b - 1 = 7$ (2)

(1) + (2): $a + b = 10$.

Note that we can have $a - 1 = 7$ and $b - 1 = 1$ with the same answer.

Example 11. Define $(x \triangle y)$ to mean $2x - 3y$. Evaluate $((4 \triangle 3) \triangle (5 \triangle 3))$.

(A) -1 (B) 1 (C) 5 (D) -5 (E) 15

Solution: D.

$4 \triangle 3 = 2 \times 4 - 3 \times 3 = -1$.

$5 \triangle 3 = 2 \times 5 - 3 \times 3 = 1$.

$((4 \triangle 3) \triangle (5 \triangle 3)) = (-1) \triangle (1) = 2 \times (-1) - 3 \times 1 = -5$.

Example 12. Given that $a \text{✦} b = a^3 - b^2$, what is the value of $4 \text{✦} (2 \text{✦} 1)$?

(A) -7 (B) -15 (C) 8 (D) 15 (E) 7

Solution: D.

$2 \text{✦} 1 = 2^3 - 1^2 = 8 - 1 = 7$.

$4 \text{✦} (2 \text{✦} 1) = 4 \text{✦} (7) = 4^3 - 7^2 = 64 - 49 = 15$.

Example 13. 10. Subfactorials, $!n$, are defined by the formula:

$$!n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

Express the following where for $x = 6$: $\frac{!x}{!(x-1)}$.

(A) $\frac{265}{44}$ (B) $\frac{11}{30}$ (C) $\frac{53}{24}$ (D) $\frac{53}{144}$ (E) 6.

Solution: A.

$$\frac{!x}{!(x-1)} = \frac{!6}{!(6-1)} =$$

$$\frac{6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)}{5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)} = \frac{6\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)}{\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)}$$

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} = \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} = \frac{11}{30}$$

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} = \frac{11}{30} + \frac{1}{720} = \frac{53}{144}.$$

$$\frac{!6}{!(6-1)} = \frac{6 \times \frac{53}{144}}{\frac{11}{30}} = \frac{265}{44}.$$

Example 14. The operation ✿ is defined by n ✿ $= n^2 - 1$. What is the value of the following: 10(3✿) – 5(4✿)?

(A) –5 (B) 99 (C) 8 (D) 15 (E) 5

Solution: E.

3✿$= 3^2 - 1 = 8$

4✿= $4^2 - 1 = 15$

10(3✿) – 5(4✿) = 10 × 8 – 5 ×15 = 80 – 75 = 5.

2. 3. Multiplications

Example 15. If 4! means 4 · 3 · 2 · 1, what is the value of $\frac{5!}{3!}$?

(A) 20 (B) 40 (C) 30 (D) 60 (E) 80

Solution: A.

$\frac{5!}{3!} = \frac{5 \times 4 \times 3!}{3!} = 5 \times 4 = 20$.

Example 16. If *a* ★ *b* is defined as (*a* + 1)(*b* + 1), find (2 ★ 3) ★ 4.

(A) 120 (B) 65 (C) 56 (D) 11 (E) 20

Solution: B.

2 ★ 3 = (2 + 1)(3 + 1) = 12.

(2 ★ 3) ★ 4 = 12 ★ 4 = (12 + 1)(4 + 1) = 65.

Example 17. Two binary operations are defined by the rules *a* ✯ *b* = $a^3 - b^3$ and *a* ▿ *b* = $(a + b)^3$. What is the value of (2 ✯ 3) ▿ 9?

(A) 1000 (B) 729 (C) – 512 (D) 1 (E) –1000

Solution: E.

2 ✯ 3 = $2^3 - 3^3 = -19$.

(2 ✯ 3) ▿ 9 = (– 19) ▿ 9 = $(-19 + 9)^3 = -1000$.

Example 18. Given $a * b = ab + 1$, evaluate: $4 * [(6 * 8) + (3 * 5)]$.
(A) 65 (B) 121 (C) 144 (D) 261 (E) 111

Solution: D.
$6 * 8 = 6 \times 8 + 1 = 49$.
$3 * 5 = 3 \times 5 + 1 = 16$.
$[(6 * 8) + (3 * 5)] = 49 + 16 = 65$
$4 * [(6 * 8) + (3 * 5)] = 4 * 65 = 4 \times 65 + 1 = 261$.

Example 19. If $a ☆ b = ab - 1$ and $a ★ b = a + b - 1$, what is the value of $4 ☆[(6 ★ 8)★(3 ☆ 5)]$?
(A) 27 (B) 104 (C) 26 (D) 103 (E) 182

Solution: D.
$6 ★ 8 = 6 + 8 - 1 = 13$
$3 ☆ 5 = 3 \times 5 - 1 = 14$
$(6 ★ 8)★(3 ☆ 5) = 13★ 14 = 13 + 14 - 1 = 26$.
$4 ☆[(6 ★ 8)★(3 ☆ 5)] = 4 ☆ 26 = 4 \times 26 - 1 = 103$.

2. 4. Divisions

☆**Example 20.** For each pair of real numbers $a \neq b$, define the operation $*$ as $(a * b) = \dfrac{a+b}{a-b}$. What is the value of $((1 * 2) * 4)$?
(A) –2/3 (B) – 1/7 (C) 0 (D) 1/2 (E) This value is not defined.

Solution: B.
First we have $(1 * 2) = \dfrac{1+2}{1-2} = -3$.
Then $((1 * 2) * 4) = (-3 * 4) = \dfrac{-3+4}{-3-4} = -\dfrac{1}{7}$.

☆**Example 21.** Define $a@b = ab - b^2$ and $a\#b = a + b - ab^2$. What is $\dfrac{6@3}{6\#3}$?

(A) –1/5 (B) – 1/4 (C) 1/8 (D) 1/4 (E) 1/2

Solution: A.

We have $6@3 = 6\times 3 - 3^2 = 9$ and $6\#3 = 6 + 3 - 6\times 3^2 = -45$.

Therefore $\dfrac{6@3}{6\#3} = -\dfrac{9}{45} = -\dfrac{1}{5}$.

☆**Example 22.** For the nonzero numbers a, b, and c, define $(a,b,c) = \dfrac{abc}{a+b+c}$.

Find (2, 5, 8).

(A) 16/3 (B) 5 (C) 15/2 (D) 6 (E) 24

Solution: A.

$(2,5,8) = \dfrac{2\times 5\times 8}{2+5+8} = \dfrac{80}{15} = \dfrac{16}{3}$.

Example 23. Express 3 ∗ (4 ∗ 5) as a common fraction given $a * b = \dfrac{ab}{a+b}$.

(A) $\dfrac{20}{9}$ (B) 1 (C) $\dfrac{60}{47}$ (D) $\dfrac{20}{47}$ (E) $\dfrac{47}{60}$

Solution: C.

$4 * 5 = \dfrac{4\times 5}{4+5} = \dfrac{20}{9}$.

$$3 * (4 * 5) = 3 * \frac{20}{9} = \frac{3\times\frac{20}{9}}{3+\frac{20}{9}} = \frac{\frac{60}{9}}{\frac{47}{9}} = \frac{60}{47}.$$

Example 24. If $a \circledast b = \dfrac{(a+b)}{b}$, compute $(3 \circledast 1) \circledast 2$.

(A) 3 (B) 4 (C) 2 (D) 1 (E) 8

Solution: A.

$3 \circledast 1 = \dfrac{(3+1)}{1} = 4$ $\quad (3 \circledast 1) \circledast 2 = 4 \circledast 2 = \dfrac{(4+2)}{2} = 3$.

Example 25. If $a \star b = \dfrac{(\frac{1}{b} - \frac{1}{a})}{(a-b)}$, express $5 \star 7$ as a common fraction.

(A) $\dfrac{35}{3}$ (B) $\dfrac{1}{35}$ (C) $\dfrac{2}{35}$ (D) $\dfrac{4}{35}$ (E) $-\dfrac{1}{35}$

Solution: B.

$$5 \star 7 = \frac{(\frac{1}{7} - \frac{1}{5})}{(5-7)} = \frac{-\frac{2}{35}}{-2} = \frac{1}{35}.$$

Example 26. Given $a * b = \dfrac{a+b}{a \cdot b}$, find $(5 * 6) * 1$.

(A) $\dfrac{30}{11}$ (B) $\dfrac{11}{30}$ (C) $\dfrac{41}{11}$ (D) $\dfrac{11}{41}$ (E) $\dfrac{985}{341}$

Solution: C.

$$5 * 6 = \frac{5+6}{5 \cdot 6} = \frac{11}{30}$$

$$\frac{11}{30} * 1 = \frac{\frac{11}{30} + 1}{\frac{11}{30} \cdot 1} = \frac{\frac{41}{30}}{\frac{11}{30}} = \frac{41}{11}.$$

Example 27. If a ✪ b is defined as $\frac{a+b}{2}$, what is the value of 6 ✪(3✪5)?

(A) 1 (B) 4 (C) 2 (D) 10 (E) 5

Solution: E

3✪5 = $\frac{3+5}{2}=4$ and 6 ✪(3✪5) = 6 ✪(4) = $\frac{6+4}{2}=5$.

2. 5. Exponents

☆**Example 28.** The operation ⊗ is defined for all nonzero numbers by $a \otimes b = a^2/b$. Determine [(1 ⊗ 2) ⊗ 4] – [1 ⊗ (2 ⊗ 4)].
(A) –15/16 (B) – 14/15 (C) 0 (D) 15/16 (E) 17/16

Solution: (A).

We have (1 ⊗ 2) ⊗ 4 = $\frac{1^2}{2}\otimes 4=\frac{1}{2}\otimes 4=\frac{(\frac{1}{2})^2}{4}=\frac{1}{16}$ and

1 ⊗ (2 ⊗ 4)= $1\otimes(\frac{2^2}{4})=1\otimes 1=\frac{1^2}{1}=1$.

Therefore [(1 ⊗ 2) ⊗ 4] – [1 ⊗ (2 ⊗ 4)] $=\frac{1}{16}-1=-\frac{15}{16}$.

Example 29. If x ✪ $y = (x^y)^x$, what is the units digit of 4 ✪ 10?

(A) 6 (B) 8 (C) 2 (D) 4 (E) 0

Solution: A.

4 ✪ 10 = $(4^{10})^4 = 4^{40}$

The pattern for the last digit of 4^n is 4, 6, 4, 6, etc.
When the exponent 40 is divided by 2, the remainder is 0. Therefore, the last digit of 4^{40} is the same as the last digit of 4^2. So the last digit is 6.

Example 30. If $a \blacklozenge b$ means $3a - 2^b$, then what value is associated with $4 \blacklozenge (2 \blacklozenge 3)$?

(A) $\frac{49}{4}$ (B) 8 (C) 16 (D) $\frac{47}{4}$ (E) – 2

Solution: D.

$2 \blacklozenge 3 = 3 \times 2 - 2^3 = 6 - 8 = -2.$

$4 \blacklozenge (2 \blacklozenge 3) = 4 \blacklozenge (-2) = 3 \times 4 - (2)^{-2} = 12 - \frac{1}{2^2} = \frac{47}{4}.$

Example 31. If $a \bigstar b = a^b + b^a$, what is $(4 \bigstar 3) \div (3 \bigstar 4)$?

(A) 256 (B) 145 (C) 1 (D) 5 (E) 2

Solution: C.

$4 \bigstar 3 = 4^3 + 3^4 = 145$

$3 \bigstar 4 = 3^4 + 4^3 = 145$

$(4 \bigstar 3) \div (3 \bigstar 4) = 145 \div 145 = 1.$

Example 32. For natural numbers a and b, $a \triangle b = b^a + 2ab$. Find the value of $(2 \triangle 3) - (3 \triangle 2)$.

(A) 1 (B) 41 (C) 37 (D) 12 (E) 0

Solution: A.

$2 \triangle 3 = 3^2 + 2 \times 2 \times 3 = 9 + 12 = 21.$

$3 \triangle 2 = 2^3 + 2 \times 2 \times 3 = 8 + 12 = 20.$

$(2 \triangle 3) - (3 \triangle 2) = 21 - 20 = 1.$

Example 33. If $x \otimes y = x^y - x + y^y$, find the value of $(4 \otimes 2) - (3 \otimes 1)$.

(A) 16 (B) 1 (C) 13 (D) 17 (E) 15

Solution: E.

$4 \otimes 2 = 4^2 - 4 + 2^2 = 16 - 4 + 4 = 16$

$3 \otimes 1 = 3^1 - 3 + 1^1 = 3 - 3 + 1 = 1$

$(4 \otimes 2) - (3 \otimes 1) = 16 - 1 = 15$.

Example 34. If $x \oplus y = (x^y)^x$, what is the units digit of $7 \oplus 5$?
(A) 7 (B) 9 (C) 3 (D) 1 (E) 6

Solution: C.
$7 \oplus 5 = (7^5)^7 = 7^{35}$.
We know that

n	1	2	3	4	Period
7^n	**7**	**9**	**3**	**1**	4

$35 = 4 \times 8 + 3$.
7^{35} has the same last digit as 7^3. The answer is 3.

Example 35. Given $a \triangle b = \dfrac{a^b}{b^a}$, and $a \square b = \dfrac{3a-b}{ab}$, find the common fraction equivalent to $(2\triangle 3)\square(2\triangle 1)$.

(A) $\frac{3}{8}$ (B) $\frac{24}{9}$ (C) $\frac{8}{9}$ (D) $\frac{8}{3}$ (E) $\frac{16}{9}$

Solution: A.

$2\triangle 3 = \dfrac{2^3}{3^2} = \dfrac{8}{9}$, $2\triangle 1 = \dfrac{2^1}{1^2} = 2$

$$(2\triangle 3)\square(2\triangle 1) = \frac{8}{9} \square 2 = \frac{3\times\frac{8}{9}-2}{\frac{8}{9}\times 2} = \frac{3}{8}.$$

2.6. Radicals

☆**Example 36.** For real numbers *a* and *b*, define $a\Diamond b=\sqrt{a^2+b^2}$. What is the value of $(8\Diamond 15)\Diamond((-15)\Diamond(-8))$?

(A) 0 (B) 13/2 (C) 15 (D) $17\sqrt{2}$ (E) 26

Solution: D.

It follows from the definition that

$(8\Diamond 15)\Diamond((-15)\Diamond(-8))$ = $\sqrt{8^2+15^2}\Diamond\sqrt{(-15)^2+(-8)^2}=17\Diamond 17=\sqrt{17^2+17^2}=17\sqrt{2}$.

Example 37. Let ▽ be defined as ▽(*a*, *b*) = $\sqrt{a^2+b^2}$, for all real numbers *a* and *b*. Find ▽ (▽ (▽(12, 5), 84), 132).

(A) 97 (B) 117 (C) 137 (D) 157 (E) 187

Solution: D.

▽(12, 5) = $\sqrt{12^2+5^2}=13$.

▽ (▽(12, 5), 84) = ▽ (13, 84) = $\sqrt{13^2+84^2}=85$.

▽ (▽ (▽(12, 5), 84), 132) = ▽ (85, 132) = $\sqrt{85^2+132^2}=157$.

Example 38. Let ☆ be defined as ☆(*a*, *b*) = $\sqrt{a^2+b^2}$, for all real numbers *a* and *b*. Find ☆(☆(16, 63), ☆(33, 56)) and express in simplest radical form.

(A) 65 (B) $65\sqrt{2}$ (C) $63\sqrt{2}$ (D) 130 (E) $56\sqrt{2}$

Solution: B.

☆(16, 63) = $\sqrt{16^2+63^2}=65$

☆(33, 56) = $\sqrt{33^2+56^2}=65$

☆(☆(16, 63), ☆(33, 56)) = ☆(65, 65) = $\sqrt{65^2+65^2}=65\sqrt{2}$

Example 39. If $x \heartsuit y = \sqrt{xy} + \frac{2}{x} - \frac{3}{2}$, find the value of (4 ♡ 4) ♡ 3. Express your answer as a common fraction.

(A) $\frac{13}{6}$ (B) 3 (C) $\frac{23}{6}$ (D) $\sqrt{17}$ (E) 7

Solution: A.

$4 \heartsuit 4 = \sqrt{4 \times 4} + \frac{2}{4} - \frac{3}{2} = 4 + \frac{1}{2} - \frac{3}{2} = 4 - 1 = 3$

$(4 \heartsuit 4) \heartsuit 3 = 3 \heartsuit 3 = \sqrt{3 \times 3} + \frac{2}{3} - \frac{3}{2} = 3 - \frac{5}{6} = \frac{13}{6}$.

3. PROBLEMS

Problem 1. Let the operations △ and □ be defined for all real numbers a and b as follows:

$a \triangle b = a + 3b$

$a \square b = a + 4b$

If $4 \triangle (5y) = (5y) \square 4$, what is the value of y?

(A) 6/5 (B) 1 (C) 2 (D) 3/5 (E) 3/5

☆**Problem 2.** For the positive integer n, let $<n>$ denote the sum of all the positive divisors of n with the exception of n itself. For example, $<4> = 1 + 2 = 3$ and $<12> = 1 + 2 + 3 + 4 + 6 = 16$. What is $<<<18>>>$?

(A) 1 (B) 11 (C) 21 (D) 6 (E) 3

Problem 3. If $A \otimes B = \frac{A}{B} + \frac{B}{A}$, what is the value of $(3 \otimes 2) - (2 \otimes 3)$?

(A) $\frac{4}{3}$ (B) $\frac{3}{4}$ (C) 0 (D) 1 (E) $\frac{12}{13}$

Problem 4. If ☆ represents an operation defined by $a ☆ b = a^3 + b$, find $(1☆2)☆3$.

(A) 3 (B) 27 (C) 30 (D) 9 (E) 732

Problem 5. If $a \diamond b = \frac{1}{a} + \frac{1}{b}$, for what decimal value of a is $a \diamond 0.2 = 10$?

(A) $\frac{1}{5}$ (B) $\frac{7}{10}$ (C) 0 (D) 1 (E) $\frac{1}{2}$

Problem 6. Given that $a \otimes b = (a^2 + b) \div 2$. What is the value of $5 \otimes 3$?

(A) 14 (B) 4 (C) 15 (D) 8 (E) 28

Problem 7. If $a \oslash b = a^2 + b$, evaluate $(4 \oslash 3) \oslash 18$.

(A) 19 (B) 361 (C) 380 (D) 324 (E) 379

☆**Problem 8.** Define $x \otimes y = x^3 - y$. What is $h \otimes (h \otimes h)$?

(A) $-h$ (B) -0 (C) h (D) $2h$ (E) h^4

Problem 9. If $x \oslash y = x^2 - y^2$, what is $(3 \oslash 2) \oslash 4$?

(A) 5 (B) 25 (C) 9 (D) 16 (E) 25

Problem 10. If $\langle ab \rangle = ab - a - b$, find the value of b in the equation $\langle 3b \rangle = 5$.

(A) 5 (B) 4 (C) 3 (D) 15 (E) 2

Problem 11. If $a \blacklozenge b = 2a - b$, what does $3 \blacklozenge 4$ equal?

(A) –2 (B) 2 (C) 10 (D) 5 (E) 12

Problem 12. Suppose that a ✪ $b = ab - b$ for all integers a and b. What is the value of 3 ✪(–2)?

(A) –4 (B) –5 (C) 8 (D) 4 (E) 7

Problem 13. If A ✭ $B = 3A^2 - 2B^3$, find 7 ✭ 3.

(A) 179 (B) –93 (C) 93 (D) 21 (E) 10

Problem 14. If $y^{\triangleleft} = y^2 - 1$, find $(9^{\triangleleft})^{\triangleleft}$.

(A) 80 (B) 6400 (C) 6399 (D) 79 (E) 81

Problem 15. If $a \blacktriangledown b = 3a - b^2$, find $2 \blacktriangledown (3 \blacktriangledown 1)$.

Problem 16. If 4! means $4 \cdot 3 \cdot 2 \cdot 1$, express $\dfrac{8!}{6!2!2!}$ in simplest form.

(A) 40320 (B) 56 (C) 28 (D) 14 (E) 120

Problem 17. Given $a \diamond b = a(a + b) + b(a + b)$, find $9 \diamond 7$.

(A) 256 (B) 225 (C) 4 (D) 16 (E) 289

Problem 18. For all values a, b, c, and d, $\begin{vmatrix} a & c \\ d & b \end{vmatrix} = ab - cd$. If $\begin{vmatrix} 5 & x \\ -2 & 6 \end{vmatrix} = 8$, what is x?

(A) 14 (B) -11 (C) -14 (D) -15 (E) 11

Problem 19. If $a \diamond b = a^2b$, find $(3 \diamond 2) - (2 \diamond 3)$.

(A) 5 (B) 6 (C) 18 (D) 12 (E) -6

Problem 20. If $x ◎ y = \frac{x}{y} + xy$, express $\frac{3}{8} ◎ \frac{3}{4}$ as a common fraction.

(A) $\frac{5}{32}$ (B) $\frac{9}{32}$ (C) $\frac{1}{2}$ (D) $\frac{32}{9}$ (E) $\frac{25}{32}$.

Problem 21. Given $a \diamond b = \frac{(a^2 - b^2)}{ab}$, express $6 \diamond 2$ as a common fraction.

(A) $\frac{8}{3}$ (B) $\frac{7}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{3}$ (E) $\frac{3}{8}$

Problem 22. Given the operations: $a ☆ b = 2a - b$ and $a ★ b = \frac{a+b}{b}$, evaluate: $6 ★ | 3☆9 |$.

(A) $\frac{1}{3}$ (B) 3 (C) $\frac{1}{4}$ (D) 4 (E) -3

Problem 23. Evaluate $3 ❄ 4$ if $a ❄ b = \frac{a^2 + b^2}{a + b}$. Express your answer as a common fraction.

(A) $\frac{5}{7}$ (B) $\frac{25}{7}$ (C) $\frac{9}{7}$ (D) $\frac{16}{7}$ (E) 1

Problem 24. The operation $*$ is defined to be $a * b = \dfrac{8a - 2b}{2ab}$. Express $3 * (3 * 3)$ as a common fraction.

(A) $\frac{31}{3}$ (B) $\frac{31}{9}$ (C) $\frac{3}{11}$ (D) $\frac{11}{3}$ (E) 1

Problem 25. If $[abc] = \dfrac{a+b}{c}$, what is the value of $[[123][231][312]]$?

(A) 3 (B) 4 (C) 2 (D) 1 (E) 5

Problem 26. If $A ☆ B = \dfrac{2A - B}{2}$, what is the value of $(3 ☆ 4) ☆ 5$? Express your answer as a common fraction.

(A) $-\frac{3}{2}$ (B) $\frac{3}{2}$ (C) $\frac{2}{3}$ (D) $-\frac{2}{3}$ (E) 1

Problem 27. If a ☆ b = $\dfrac{a+b}{2}$, what is the value of (7 ☆ 9)☆(30 ☆ 17)?

(A) $15\frac{1}{4}$ (B) $\frac{47}{2}$ (C) $15\frac{3}{4}$ (D) $\frac{11}{41}$ (E) $14\frac{3}{4}$

Problem 28. Given $a ★ b = \dfrac{a+b}{2}$, find (7★9)★12.

(A) 10 (B) 8 (C) 20 (D) 96 (E) 6

Problem 29. If $a \triangle b = (ab)^a$, find $5 \triangle 2$. Express the answer as a whole number.

(A) 100,000 (B) 50,000 (C) 20,000 (D) 96,000 (E) 6,000

Problem 30. If $a \bigstar b = a^{b^a}$, then what value is associated with $2 \bigstar 3$?

(A) 64 (B) 512 (C) 8 (D) 128 (E) 1024

Problem 31. If a ✪ b is defined as $2a - b^a$, what value is associated with (5 ✪ 2) – (3 ✪ 2)?

(A) – 24 (B) – 20 (C) – 22 (D) 22 (E) 24

Problem 32. Given a ✿ $b = b^a - ba + a^b$, find (2 ✿ 3) × (3 ✿ 2).

(A) 121 (B) 22 (C) 11 (D) 144 (E) 81

Problem 33. For natural numbers, a and b $a \boxtimes b = b^a - a + b$. Find the value of $(4 \boxtimes 2) - (2 \boxtimes 4)$.

(A) 32 (B) – 4 (C) 18 (D) 14 (E) 22

Problem 34. Given the a ✦ $b = a^b - b^a$, and $a \bigtriangledown b = (a + b)(a - b)$, what is the value of a ✦ $(a \bigtriangledown b)$ if $a = 3$ and $b = 2$?

(A) 118 (B) 15 (C) 243 (D) 125 (E) 115

Problem 35. If (a ✦ b) = $(a \times b) + a^b + b^a$, find 3 ✦ 5.

(A) 243 (B) 383 (C) 125 (D) 15 (E) 115

Problem 36. If $a \clubsuit b = (\frac{1}{a})^b + (\frac{1}{b})^a$, find $2 \clubsuit 3$.

(A) $\frac{17}{72}$ (B) $\frac{2}{17}$ (C) $1\frac{1}{9}$ (D) $\frac{1}{8}$ (E) $\frac{5}{6}$

Problem 37. If $a \diamondsuit b = \sqrt{a^2 + b^2}$, find the value of $(2\frac{1}{2}) \diamondsuit 6$ and express the result as a common fraction.

(A) $\frac{13}{2}$ (B) $\frac{17}{2}$ (C) $\frac{5}{2}$ (D) $\sqrt{13}$ (E) $\sqrt{61}$

Problem 38. Let $\triangledown$ be defined as $\triangledown(a, b) = \sqrt{a^2 + b^2}$, for all real numbers a and b. Find $\triangledown(\triangledown(8, 5), 144)$.

(A) 85 (B) 125 (C) 135 (D) 145 (E) 165

Problem 39. Given $a \star b = \sqrt{a^2 + b^2}$, find $((13 \star 84) \star (36 \star 77))$.

(A) 85 (B) $85\sqrt{2}$ (C) $83\sqrt{2}$ (D) 135 (E) $58\sqrt{2}$

Problem 40. Given that $x \diamond y = \sqrt{x + y}$, find $(6 \diamond 10) \diamond 5$.

(A) 4 (B) 3 (C) 16 (D) $\sqrt{15}$ (E) 7

Problem 41. The symbols ✦ and ∗ represent different operations, either +, −, ×, or ÷, and x is a positive integer. Find x if 17 ✦ x = 54 ∗ x.

(A) 4 (B) 3 (C) 6 (D) 5 (E) 7

4. SOLUTIONS

Problem 1. Solution: A.

$4 \triangle (5y) = 4 + 3 \times 5y$ (1)

$(5y) \square 4 = 5y + 4 \times 4$ (2)

We are given that (1) = (2). So $4 + 3 \times 5y = 5y + 4 \times 4 \Rightarrow 4 + 15y = 5y + 16 \Rightarrow 10y = 12 \Rightarrow y = 12/10 = 6/5$.

★**Problem 2.** Solution: A.

Method 1:

The positive divisors of 18, other than 18, are 1, 2, 3, 6, and 9, so <18> = 1 + 2 + 3 + 6 + 9 = 21. <21> = 1 + 3 + 7 = 11. <11> = 1.

Problem 3. Solution: 0.

$(3 \otimes 2) = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$.

$(2 \otimes 3) = \frac{2}{3} + \frac{3}{2} = \frac{13}{6}$.

The answer is $(3 \otimes 2) - (2 \otimes 3) = 0$.

Problem 4. Solution: C.

$1☆2 = 1^3 + 2 = 3$.

$3☆3 = 3^3 + 3 = 27 + 3 = 30$.

Problem 5. Solution: A.

$a \diamond 0.2 = 10 \Rightarrow \frac{1}{a} + \frac{1}{0.2} = 10 \Rightarrow \frac{1}{a} + 5 = 10 \Rightarrow \frac{1}{a} = 5 \Rightarrow a = \frac{1}{5}$.

Problem 6. Solution: A.

$5 \otimes 3 = (5^2 + 3) \div 2 = 14$.

Problem 7. Solution: E

$4 \oslash 3 = 4^2 + 3 = 19$.

$19 \oslash 18 = 19^2 + 18 = 19^2 + 19 - 1 = 19(19 + 1) - 1 = 19 \times 20 - 1 = 380 - 1 = 379$.

★**Problem 8.** Solution: C.

By the definition we have $h \otimes (h \otimes h) = h \otimes (h^3 - h) = h^3 - (h^3 - h) = h$.

Problem 9. Solution: C

$3 \oslash 2 = 3^2 - 2^2 = (3 - 2)(3 + 2) = 5$.

$(3 \oslash 2) \oslash 4 = 5 \oslash 4 = 5^2 - 4^2 = (5 - 4)(5 + 4) = 9$.

Problem 10. Solution: B.

$\langle 3b \rangle = 5 \quad \Rightarrow \quad 5 = 3b - 3 - b \quad \Rightarrow \quad 2b = 8 \Rightarrow \quad b = 4$

Problem 11. Solution: B.

$3 \blacklozenge 4 = 2 \times 3 - 4 = 2$.

Problem 12. Solution: A.

3 ✪$(-2) = 3 \times (-2) - (-2) = -6 + 2 = -4$.

Problem 13. Solution: C.

7 ✯ $3 = 3 \times 7^2 - 2 \times 3^3 = 93$.

Problem 14. Solution: C.

$9^{\triangleleft} = 9^2 - 1 = 80$

$80^{\triangleleft} = 80^2 - 1 = 6399$.

Problem 15. Solution: -58

3 ▼ 1 = 3 × 3 – 1^2 = 8.

2 ▼ (3 ▼ 1) = 2 ▼ 8 = 3 × 2 – 8^2 = –58.

Problem 16. Solution: D.

$$\frac{8!}{6!2!2!} = \frac{8\times7\times6!}{6!2!2!} = \frac{8\times7}{2!2!} = \frac{8\times7}{4} = 14.$$

Problem 17. Solution: A.

9 ◇ 7 = 9(9 + 7) + 7(9 + 7) = (9 + 7)(9 + 7) = 16^2 = 256.

Problem 18. Solution: B.

$$\begin{vmatrix} 5 & x \\ -2 & 6 \end{vmatrix} = 8 \quad \Rightarrow \quad 30-(-2)\times x = 8 \;\Rightarrow\; 2x = 8-30 = -22 \;\Rightarrow\; x = -11.$$

Problem 19. Solution: B.

3◇2 = 3^2× 2 = 18

2◇3 = 2^2× 3 = 12

(3◇2) − (2◇3) = 18 – 12 = 6.

Problem 20. Solution: E.

$$\frac{3}{8} ◎ \frac{3}{4} = \frac{\frac{3}{8}}{\frac{3}{4}} + \frac{3}{8}\times\frac{3}{4} = \frac{1}{2} + \frac{9}{32} = \frac{25}{32}.$$

Problem 21. Solution: A.

$$6 \diamond 2 = \frac{(6^2-2^2)}{6\times2} = \frac{32}{6\times2} = \frac{8}{3}.$$

Problem 22. Solution: B.

3☆9 = 2 × 3 – 9 = – 3.

| 3☆9 | = 3.

6 ★ | 3☆9 | =6 ★ 3 = $\frac{6+3}{3}=3$.

Problem 23. Solution: B.

3 ❉ 4 = $\frac{3^2+4^2}{3+4}=\frac{25}{7}$.

Problem 24. Solution: D.

3 ∗ 3 = $\frac{8\times 3-2\times 3}{2\times 3\times 3}=\frac{3\times 6}{3\times 6}=1$

3 ∗ (3 ∗ 3) = 3 ∗ 1 = $\frac{8\times 3-2\times 1}{2\times 3\times 1}=\frac{22}{6}=\frac{11}{3}$.

Problem 25. Solution: A.

$[123]=\frac{1+2}{3}=1$

$[231]=\frac{2+3}{1}=5$

$[312]=\frac{3+1}{2}=2$

$[[123][231][312]]=[152]=\frac{1+5}{2}=3$.

Problem 26. Solution: A.

3 ☆ 4 = $\frac{2\times 3-4}{2}=1$

(3 ☆ 4) ☆ 5 = 1☆ 5 = $\frac{2\times 1-5}{2}=-\frac{3}{2}$.

Problem 27. Solution: C.

7 ☆ 9 = $\frac{7+9}{2}=8$.

30 ☆ 17 = $\frac{30+17}{2}=\frac{47}{2}$.

(7 ☆ 9)☆(30 ☆ 17) = 8☆ $\frac{47}{2}$ = $\frac{8+\frac{47}{2}}{2}=\frac{\frac{63}{2}}{2}=\frac{63}{4}=15\frac{3}{4}$.

Problem 28. Solution: A

7★9 = $\frac{7+9}{2}=8$

(7★9)★12 = 8★12 = $\frac{8+12}{2}=10$.

Problem 29. Solution: A.

5△2 = $(5 \times 2)^5$ = 100, 000.

Problem 30. Solution: B.

2 ★ 3 = $2^{3^2} = 2^9$ = 512.

Problem 31. Solution: B.

5 ✪ 2 = $2 \times 5 - 2^5$ = 10 – 32 = –22.

3 ✪ 2 = $2 \times 3 - 2^3$ = 6 – 8 = – 2.

(5 ✪ 2) – (3 ✪ 2) = –22 – (– 2) = – 22 + 2 = – 20.

Problem 32. Solution: A.

2 ✿ 3 = $3^2 - 3 \times 2 + 2^3$ = 9 – 6 + 8 = 11.

3 ✿ 2 = $2^3 - 2 \times 3 + 3^2$ = 8 – 6 + 9 = 11.

(2 ✿ 3) × (3 ✿ 2) = 11 × 11 = 121.

Problem 33. Solution: B.
$4 \boxtimes 2 = 2^4 - 4 + 2 = 16 - 4 + 2 = 14.$
$2 \boxtimes 4 = 4^2 - 2 + 4 = 16 - 2 + 4 = 18.$
$(4 \boxtimes 2) - (2 \boxtimes 4) = 14 - 18 = -4.$

Problem 34. Solution: A.
$3 \triangledown 2 = (3 + 2)(3 - 2) = 5.$
$a \text{✦} (a \triangledown b) = 3 \text{✦} 5 = 3^5 - 5^3 = 118.$

Problem 35. Solution: B.
$3 \text{✦} 5 = (3 \times 5) + 3^5 + 5^3 = 15 + 243 + 125 = 383.$

Problem 36. Solution: A.
$2 \clubsuit 3 = (\frac{1}{2})^3 + (\frac{1}{3})^2 = \frac{1}{8} + \frac{1}{9} = \frac{17}{72}$

Problem 37. Solution: A.
$(2\frac{1}{2}) \diamond 6 = \sqrt{(2\frac{1}{2})^2 + 6^2} = \sqrt{\frac{25}{4} + 36} = \sqrt{\frac{169}{4}} = \frac{13}{2}.$

Problem 38. Solution: D.
$\triangledown(8, 5) = \sqrt{8^2 + 5^2} = 17.$
$\triangledown(\triangledown(8, 5), 144) = \triangledown(17, 144) = \sqrt{17^2 + 144^2} = 145.$

Problem 39. Solution: B.
$13 \star 84 = \sqrt{13^2 + 84^2} = 85$

$36 \star 77 = \sqrt{36^2 + 77^2} = 85$

$((13 \star 84) \star (36 \star 77)) = 85 \star 85 = \sqrt{85^2 + 85^2} = 85\sqrt{2}$

Problem 40. Solution: B.

$6 \diamond 10 = \sqrt{6+10} = 4$

$(6 \diamond 10) \diamond 5 = 4 \diamond 5 = \sqrt{4+5} = 3$.

Problem 41. Solution: B.

$17 ✦ x = 54 * x \quad \Rightarrow \quad 17 \times x = 54 - x \quad \Rightarrow \quad x = 3$.

1. BASIC KNOWLEDGE

Definition. For $b \geq 2$, we write $b \mid a - r$ if $a - r$ is divisible by b.

The following expressions mean the same thing:

- $b \mid a - r$
- $a - r = qb$ (q is the quotient and has an integer value)
- $a = r + qb$
- $\dfrac{a-r}{b} = q$

Example 1. When x ($x > 5$) is divided by 5, the remainder is 3. What is the sum of the first three smallest possible values of x?
(A) 29 (B) 39 (C) 41 (D) 52 (E) 38

Solution: B.
The smallest value of x is 5 + 3 = 8. The next possible value is 8 + 5 = 13. The third possible value is 13 + 5 = 18. The answer is 8 + 13 + 18 = 39.

Example 2. When $20 + x$ is divided by 11, the remainder is 7. What is the smallest possible positive value of x?
(A) 2 (B) 4 (C) 6 (D) 9 (E) 10

Solution: D.
The smallest value of $20 + x$ is 11 + 7 = 18. The next possible value is 18 + 11 = 29. So the smallest value of x is 29 – 20 = 9.

Theorem 1.
There exists a unique pair (q, r) such that

$$a = qb + r \qquad (1.1)$$

or $$\frac{a}{b} = q + \frac{r}{b} \qquad (1.2)$$

where a and b are integers, $b > 0$, q is the quotient, and r is the remainder with $0 \leq r < b$.

If $r = 0$, $\frac{a}{b} = q$, which can be written as $a = qb$. This is equivalent to saying that a is divisible by b, $b|a$, and b divides a.

Example 3. When $534 + x$ is divided by 32, the quotient is 16 and the remainder is 30. Find x.
(A) 8 (B) 13 (C) 24 (D) 25 (E) 16

Solution: A.
By (1.1), we have $534 + x = 32 \times 16 + 30 \quad \Rightarrow \quad x = 8$.

Theorem 2. (Residue Classes)
For any given positive integer m, when it is divided by n, the remainder must be one the following: 0, 1, 2, …, $n - 1$. All integers can be classified into a unique class according to their remainder when divided by n. A complete set of residue classes has n classes.

Any integer can be classified into two residue classes when divided by 2:
$2k$ (remainder is 0)
$2k + 1$ (remainder is 1)
k is an integer.

Any integer can be classified into 4 classes when divided by 4:
$4k$ (remainder is 0)
$4k + 1$ (remainder is 1)
$4k + 2$ (remainder is 2)
$4k + 3$ (remainder is 3)
k is an integer.

Example 4. When 33 is divided by a natural number x, the remainder is 5. How many values of x are there?
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution: B.
We have $33 = x \cdot q + 5 \quad (x > 5)$.
Or $33 - 5 = x \cdot q \quad \Rightarrow \quad 28 = x \cdot q$.
Since $28 = 2^2 \times 7$, the number of factors of 28 is $(2 + 1) \times (1 + 1) = 6$.

Since $x > 5$, we need to exclude 1, 2, and 4. There are $6 – 3 = 3$ such numbers (7, 14, and 28).

Theorem 3.
The largest number which divides any two given numbers leaving the same remainder equals the difference of the two numbers.

Example 5. A number N divides each of 17 and 30 with the same remainder in each case. What is the largest value of N?
(A) 11 (B) 12 (C) 13 (D) 15 (E) 16

Solution: C.
By the theorem 3, the largest $N = 30 – 17 = 13$

Theorem 4.
If $a|b$, $a|c$, then $a|\ (jb+kc)$, and j, k are any integers.

Example 6. A number N divides each of 17 with the remainder of r and 30 with the remainder $2r$ respectively. What is the largest value of N?
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution: C.
We know that N divides both $(17 - r)$ and $(30 - 2r)$.
By the Theorem 4, N also divides any linear combination of them. That is, N divides $2(17 - r) - (30 - 2r) = 4$. The greatest N is then 4.

2. PROBLEM SOLVING SKILLS

2.1. Find the divisor

Example 7. If 161 is divided by a whole number, the quotient is 14 with a remainder of 7. What is the divisor?
(A) 12 (B) 13 (C) 14 (D) 15 (E) 11

Solution: E.
Let the whole number be x.
By (1.1), we have: $161 = 14x + 7 \quad \Rightarrow \quad x = 11.$

Example 8. When the three integers 618, 343, and 277 are divided by a positive integer, d, where $d > 1$, the remainders are the same. What is the smallest possible value of d?
(A) 5 (B) 11 (C) 25 (D) 55 (E) 16

Solution: B.
Method 1:
By the Theorem (3), d should be a common factor of $343 - 277 = 66 = 2 \times 3 \times 11$ and $618 - 343 = 275 = 5 \times 5 \times 11$. So d is 11.

Method 2:
Let the remainder be r. We can write the following algebraic forms where t, w, and z are positive integers:
$618 = dt + r$ (1)
$343 = dw + r$ (2)
$277 = dz + r$ (3)
$(1) - (2) \Rightarrow \quad 275 = d(t - w)$
$(2) - (3) \Rightarrow \quad 66 = d(w - z)$
$(1) - (3) \Rightarrow \quad 341 = d(t - z)$
Adding all three equations together yields $2dt = 682 \Rightarrow dt = 341 = 11 \times 31$. The smallest value for d is 11.

Example 9. When 732 is divided by a natural number x, the remainder is 12. How many values of x are there?
(A) 20 (B) 30 (C) 40 (D) 50 (E) 60

Solution: A.
Because the remainder when 732 is divided by x is 12, we can subtract 12 from 732 to find a number that leaves a remainder of 0 when divided by 12. $732 - 12 = 720$ and $720 = 2^4 \times 3^2 \times 5$.

There are $(4 + 1) \times (2 + 1) \times (1 + 1) = 30$ factors of 720. Among them, we must subtract the factors that are less than 12, because they will leave a remainder less than 12. There are 10 factors: 1, 2, 3, 4, 5, 6, 8, 9, 10, and 12. The answer is $30 - 10 = 20$.

2.2 Find the number

Example 10. If a certain number is divided by 2, 3, 4, or 5, the remainder is 1 in each case. What is the least number that satisfies these conditions?
(A) 59 (B) 60 (C) 61 (D) 121 (E) 30

Solution: C.
Let the least number be a. $a - 1$ will be divisible by 2, 3, 4, or 5, that is, divisible by LCM (2, 3, 4, 5, 6) = 60.
The least number of $a - 1$ is 60. a will be 60 + 1 = 61.

Example 11. What is the smallest whole number such that if it is divided by 2, 3 and 4, the remainders will be 1, 2 and 3, respectively?
(A) 21 (B) 31 (C) 12 (D) 11 (E) 10

Solution: D.
Let the number be x.
By (1.1), we have:
$x = 2q_1 + 1$ (1)
$x = 3q_2 + 2$ (2)
$x = 4q_3 + 3$ (3)

Adding 1 to both sides of (1), (2), and (3):
$x + 1 = 2q_1 + 2 = 2(q_1 + 1)$ (4)
$x + 1 = 3q_2 + 3 = 3(q_2 + 1)$ (5)
$x + 1 = 4q_3 + 4 = 4(q_3 + 1)$ (6)

Therefore we know that $x + 1$ is divisible by LCM (2, 3, 4) = 12. Thus the smallest value of $x + 1$ is 12 and the smallest value of x is 12 – 1 = 11.

Example 12. The members of a band are arranged in a rectangular formation. When they are arranged in 8 rows, there are 2 positions unoccupied in the formation. When they are arranged in 9 rows, there are 3 positions unoccupied. How many members are in the band if the membership is between 100 and 200?
(A) 144 (B) 150 (C) 120 (D) 60 (E) 72

Solution: B.
Let the number be x.
By (1.1), we have:
$x = 8q_1 - 2$ (1)
$x = 9q_2 - 3$ (2)

Subtracting 6 from both sides of (1) and (2):
$x - 6 = 8q_1 - 8 = 8(q_1 - 1)$ (4)
$x - 6 = 9q_2 - 9 = 9(q_2 - 1)$ (5)

Therefore we know that $x - 6$ is divisible by $LCM\ (8, 9) = 72$. Thus the value of $x - 6$ can be 72, 144, 216,…
We know that x is between 100 and 200. So $x - 6 = 144$
is 72 and the smallest value of x is $144 + 6 = 150$.

2.3 Find the remainder

Example 13. Both x and y are integers. $x + 9y$ is divisible by 5. What is the remainder when $8x + 7y$ is divided by 5?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: A.
Method 1:
$8x + 7y = 5(2x + 5y) - 2(x + 9y)$.
Since $x + 9y$ is divisible by 5, the remainder when $8x + 7y$ is divided by 5 is 0.

Method 2:
Let $x = 1$ and $y = 1$. Therefore $x + 9y = 10$ which is divisible by 5.
$8x + 7y = 8 \times 1 + 7 \times 1 = 15$ which is also divisible by 5.
The remainder when $8x + 7y$ is divided by 5 is 0.

Example 14. A positive integer n has the remainder 2 when it is divided by 7. What is the remainder when n is multiplied by 9 and then is divided by 7?
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution: C.

$n = 7q + 2$ (1)

Multiply both sides by 9:

$9n = 9(7q + 2) = 9 \times 7q + 18 = 9 \times 7q + 7 + 7 + 4 = 7(9q + 2) + 4$

When $9n$ is divided by 7, the new quotient is $(9q + 2)$ and the remainder is 4.

Example 15. When a is divided by 5, the remainder is 1. When b is divided by 5, the remainder is 4. If $3a > b$, what is the remainder when $3a – b$ is divided by 5?

(A) 12 (B) 8 (C) 6 (D) 4 (E) 2

Solution: D.

$$a = q_1 \times 5 + 1 \quad (1)$$
$$b = q_2 \times 5 + 4 \quad (2)$$

Multiplying (1) by 3:

$$3a = q_1 \times 5 \times 3 + 3 \quad (3)$$

(3) – (2): $3a – b = q_1 \times 5 \times 3 – (q_2 \times 5 + 4) = (q_1 – q_2) \times 5 – 1 = (q_1 – q_2 – 1) \times 5 + 5 – 1 = (q_1 – q_2 – 1) \times 5 + 4$.

The remainder is 4 when $3a – b$ is divided by 5.

2.4. Calendar problems and other applications

Example 16. What day of the week will be 5000 days from Sunday?

(A) Monday (B) Tuesday (C) Wednesday (D) Thursday (E) Friday

Solution: B.

$5000 = 714 \times 7 + 2$.

On the $5000 – 2 = 4998^{th}$ day, it will be a Sunday. Two days from Sunday is a Tuesday.

Example 17. If the first day of a month is Monday, what day of the week is the twenty-third day?

(A) Monday (B) Tuesday (C) Wednesday (D) Thursday (E) Friday

Solution: B.

Method 1: $23 = 3 \times 7 + 2$. The remainder is 2 so it is a Tuesday.
Method 2:

Mon	Tue	Wed	Thu	Fri	Sat	Sun
1						
8						
15						
22	23rd					

Example 18. If March 17 falls on a Wednesday, on what day of the week will April 4 of the same year fall?
(A) Monday (B) Tuesday (C) Wednesday (D) Saturday (E) Sunday

Solution: E.

Wed	Thurs	Fri	Sat.	Sun	Mon	Tues
17						
24						
31	April 1	April 2	April 3	April 4		

Example 19. Kim's birthday was 200 days ago. Today is Wednesday. On what day of the week did his birthday fall?
(A) Monday (B) Tuesday (C) Wednesday (D) Saturday (E) Sunday

Solution: D.
$200 = 7 \times 28 + 1$. The remainder is 4. It is 4 days before today so it is a Tuesday.

Thurs	Fri	Sat.	Sun	Mon	Tues	Wed
		4	3	2	1	Today

2.5. Other applications

Example 20. Wendy noticed when she stacked her quarters in piles of 5 she had 3 left over and when she stacked them in piles of 7 she had 5 left over. If she has less than ten dollars worth of quarters, how many quarters does she have?
(A) 11 (B) 22 (C) 33 (D) 44 (E) 55

Solution: C.
Let the number of quarters be x.

By (1.1), we have:
$x = 5q_1 + 3$ (1)
$x = 7q_2 + 5$ (2)
Adding 2 to both sides of (1) and (2):
$x + 2 = 5q_1 + 5 = 5(q_1 + 1)$ (4)
$x + 2 = 7q_2 + 7 = 7(q_2 + 1)$ (5)
Therefore we know that $x + 2$ is divisible by LCM (5, 7) = 35. Thus the value of $x + 2$ can be 35, 70, 105,…The number of quarters can be: 33, 68, 103,…Since we know that the number is less than \$10, that is, less than 40 quarters, so 33 is the answer.

Example 21. Natasha has more than \$1 but less than \$10 worth of dimes. When she puts her dimes in stacks of 3, she has 1 left over. When she puts them in stacks of 4, she has 1 left over. When she puts them in stacks of 5, she also has 1 left over. How many dimes does Natasha have?
(A) 60 (B) 61 (C) 62 (D) 54 (E) 66

Solution: B.
Let the number of dimes be x.
By (1.1), we have:
$x = 3q_1 + 1$ (1)
$x = 4q_2 + 1$ (2)
$x = 5q_2 + 1$ (3)
Subtracting 1 from each side of (1), (2), and (3):
$x - 1 = 3q_1 + 1 - 1 = 3q_1$ (4)
$x - 1 = 4q_2 + 1 - 1 = 4q_2$ (5)
$x - 1 = 5q_2 + 1 - 1 = 5q_2$ (5)

Therefore we know that $x - 1$ is divisible by LCM (3, 4, 5) = 60. Thus the value of $x - 1$ can be 60, 120, 180,…The number of dimes can be: 61, 121, 181,…Since we know that the number is less than 100, so 61 is the answer.

3. MORE EXMAPLES

Example 22. If p is an integer and 3 is the remainder when $2p + 5$ is divided by 7, then p could be
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution: (E).
$2p + 5$ is odd and could be $3 + 7 + 7 = 17$, 31, 15 and p could be 6, 13, 20. E is the only choice.

Example 23. When 1270 is divided by an integer n, the quotient is 74 with a remainder of 12. Find the divisor.

(A) 17 (B) 18 (C) 16 (D) 14 (E) 12

Solution: A.
Let the divisor be x and the remainder be r. By (1.1), we have: $1270 = 74x + 12$
$\Rightarrow \quad x = 17$.

Example 24. If July 1 falls on a Monday, then August 2 of the same year falls on what day of the week?(July has 31 days.)

(A) Tuesday (B) Wednesday (C) Thursday (D) Friday (E) Saturday

Solution: D
Method 1: There are $31 + 2 = 33$ days from July 1 to August 3. $33 = 4 \times 7 + 5$. The remainder is 5 so it is a Friday.

Method 2:

Mon	Tues	Wed	Thurs	Fri	Sat.	Sun
1						
8						
15						
22						
29	30	31	1	2		

Example 25. When 1991 is divided by n, there is a remainder of 2. When 1769 is divided by n, there is a remainder of 1. What is the least natural number n?

(A) 12 (B) 18 (C) 13 (D) 14 (E) 20

Solution: C.
n should be greater than 1. $1991 - 2 = 1989$ is divisible by n, as well as $1769 - 1 = 1768$.

Therefore n is a factor of the greatest common multiple of 1989 and 1768. $(1989, 1768) = 13 \times 17$. So the smallest value of n is 13.

Example 26. When two different numbers are divided by 7, remainders of 2 and 3, respectively, are left. What is the remainder when the sum of these two numbers is divided by 7?

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

Solution: B.

Let the two numbers be a and b.

$$a = q_1 \times 7 + 2 \qquad (1)$$

$$b = q_2 \times 7 + 3 \qquad (2)$$

(1) + (2): $a + b = (q_1 \times q_1)\times 7 + 5$.

The remainder is 5 when the sum of these two numbers is divided by 7.

Example 27. Amy has fewer than 100 computer disks. When she stacks them by elevens, ten are left over. When she stacks them by tens, seven are left over, and three are left over when she stacks them by sixes. How many disks does she have?

(A) 30 (B) 60 (C) 57 (D) 77 (E) 87

Solution: E.

By (1.1), we have

$$x = 11q_1 + 10 \qquad (1)$$

$$x = 10q_2 + 7 \qquad (2)$$

$$x = 6q_3 + 3 \qquad (3)$$

Adding 3 to both sides of (2) and (3), we have

$$x + 3 = 10q_2 + 10 = 10(q_2 + 1) \qquad (4)$$

$$x + 3 = 6q_2 + 6 = 6(q_3 + 1) \qquad (5)$$

(4) and (5) mean that $x + 3$ is divisible by both 6 and 10, or divisible by the *LCM* (6, 10) = 30.

Since x is less than 100, $x + 3$ could be 30, 60, 90 and x could be 27, 57, 87. Among them, 87 has a remainder of 3 when divided by 11.

4. PROBLEMS

Problem 1. Which of the following could be the remainders when 5 consecutive positive integers are each divided by 4?

(A) 3, 0, 1, 2, 3 (B) 0, 1, 2, 3, 4 (C) 0, 1, 2, 3, 5
(D) 0, 1, 2, 0, 4 (D) 0, 2, 3, 0, 4

Problem 2. When the positive integer x is divided by 7, the remainder is 5. What is the remainder when $x + 29$ is divided by 5?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 3. In a biology class of s students, there are m microscopes available. If the instructor assigns one microscope to each student, 8 more microscopes will be needed. If the instructor had twice as many microscopes available and assigned one microscope to each student, 8 microscopes would be left over. What is the value of s?
(A) 12 (B) 16 (C) 18 (D) 24 (E) 28

Problem 4. The number n is a 2-digit number. When n is divided by 10, the remainder is 9, and when n is divided by 9, the remainder is 8. What is the value of n?
(A) 90 (B) 88 (C) 86 (D) 89 (E) 87

Problem 5. If 214 is divided by a whole number, the quotient is 23 with a remainder of 7. What is the divisor?
(A) 9 (B) 12 (C) 16 (D) 14 (E) 22

Problem 6. When 200 is divided by a natural number x, the remainder is 8. How many values of x are there?
(A) 12 (B) 8 (C) 6 (D) 4 (E) 2

Problem 7. What is the smallest number that gives a remainder of 1 when divided by 4, a remainder of 2 when divided by 5, and a remainder of 3 when divided by 6?
(A) 60 (B) 59 (C) 58 (D) 57 (E) 56

Problem 8. When a number x is divided by 8, the remainder is 5. What is the remainder when x is divided by 4?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 7

Problem 9. When two different numbers are divided by 11, remainders of 7 and 9, respectively, are left. What is the remainder when the sum of these two numbers is divided by 11?
(A) 10 (B) 8 (C) 5 (D) 4 (E) 6

Problem 10. When Rachel divides her favorite number by 7, she gets a remainder of 5. What will the remainder be if she multiplies her favorite number by 5 and then divides by 7?
(A) 1 (B) 3 (C) 6 (D) 4 (E) 2

Problem 11. K is the smallest positive integer satisfying these properties:
1) When K is divided by 5 the remainder is 4.
2) When K is divided by 8 the remainder is 2.
3) When K is divided by 11 the remainder is 1.
K is ____.
(A) 12 (B) 22 (C) 34 (D) 54 (E) 42

Problem 12. When n is divided by 5, the remainder is 1. What is the remainder when $3n$ is divided by 5?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 0

Problem 13. Find two integers between 1 and 100 such that for each:
a) if you divide by 4, the remainder is 3;
b) if you divide by 3, the remainder is 1; and
c) if you divide by 5, the remainder is 1.

(A) (11, 71) (B) (31, 91) (C) (21, 51) (D) (16, 31) (E) (46, 61)

Problem 14. Susan's March birthday is on a Saturday this year, but she doesn't celebrate it until 100 days later. On what day of the week will she celebrate?
(A) Monday (B) Tuesday (C) Wednesday (D) Saturday (E) Sunday

Problem 15. Today is a Saturday in March. What day of the week will it be one year from today since next year is not a leap year?
(A) Monday (B) Tuesday (C) Wednesday (D) Saturday (E) Sunday

Problem 16. What is the least positive integer value of p such that $7p$ divided by 11 has a remainder of 1?
(A) 10 (B) 9 (C) 8 (D) 7 (E) 6

Problem 17. Several thieves found a package containing some dollar bills. They found that when they tried to give \$3 to each thug, they ran out of money and one thug received nothing. When each thug took \$2, they had \$1 left over. What is the least possible number of dollars if the money is more than \$2000?
(A) \$2001 (B) \$2014 (C) \$2015 (D) \$2016 (E) \$2017

Problem 18. When Joyce counts the pennies in her bank by fives, she has one left over. When she counts them by threes, there are two left over. What is the least possible number of pennies in the bank?
(A) 10 (B) 11 (C) 12 (D) 17 (E) 16

Problem 19. The number of students in Teresa's graduating class is more than 50 and fewer than 100 and is 1 less than a multiple of 3, 2 less than a multiple of 4, and 3 less than a multiple of 5. How many students are in Teresa's graduating class?
(A) 60 (B) 69 (C) 80 (D) 71 (E) 62

Problem 20. I have some tables and chairs. If I place two chairs at each table, I have one extra chair. If I place three chairs at each table, I have one table with no chairs. What is the sum of the number of tables and the number of chairs?
(A) 10 (B) 11 (C) 12 (D) 13 (E) 16

Problem 21. Find the smallest positive integer that gives a remainder of 5 when divided by 6, a remainder of 6 when divided by 7, and a remainder of 7 when divided by 8.
(A) 167 (B) 166 (C) 168 (D) 169 (E) 171

Problem 22. When 51 is divided by the positive integer k, the remainder is 3. For how many different values of k is this true?

(A) Ten (B) Eight (C) Seven (D) Four (E) Five

Problem 23. Find the greatest integer that will divide 2613, 2243, 1503, and 985 and leave the same remainder.
(A) 70 (B) 74 (C) 78 (D) 77 (E) 76

Problem 24. When a positive integer n is divided by 7, which of the following CANNOT be the remainder?
(A) 1 (B) 2 (C) 3 (D) 5 (E) 7

Problem 25. When a positive integer is divided by 7, the remainder is 4. When the same integer is divided by 9, the remainder is 3. What is the smallest possible value of this integer?
(A) 39 (B) 49 (C) 58 (D) 67 (E) 76

Problem 26. When 73, 216, and 227 are divided by a positive integer b, the remainders are the same. What is the remainder when 108 is divided by b?
(A) 10 (B) 9 (C) 8 (D) 7 (E) 6

Problem 27. Both m and n are integers. $5m + 3n$ is divisible by 11. What is the remainder when $9m + n$ is divided by 11?
(A) 0 (B) 1 (C) 4 (D) 7 (E) 9

5. SOLUTIONS

Problem 1. Solution: A
We also know that the remainder must be less than the divisor by the **Theorem 2.** So the remainder must be less than 4.

Problem 2. Solution: B.
x can be 7 + 5 = 12. 12 + 29 = 41. The remainder is 1 when 41 is divided by 5.

Problem 3. Solution: D.
By (1.1), we have:
$m = s - 8$ (1)
$2m = s + 8$ (2)
(1) × 2 – (2): $s = 24$.

Problem 4. Solution: D.
By (1.1), we have:
$n = 10q_1 + 9$ (1)
$n = 9q_2 + 8$ (2)
Adding 1 to both sides of (1), and (2):
$n + 1 = 10q_1 + 9 + 1 = 10(q_1 + 1)$ (4)
$n + 1 = 9q_2 + 8 + 1 = 9(q_2 + 1)$ (5)
Therefore we know that $n + 1$ is divisible by LCM (9, 10) = 90. Thus the smallest value of $n + 1$ is 90 and the smallest value of n is 90 – 1 = 89.

Problem 5. Solution: A.
Let the whole number be x.
By (1.1), we have: $214 = 23x + 7 \Rightarrow x = 9$.

Problem 6. Solution: B.
We have $200 = x \cdot q + 8$ $(x > 8)$.
Or $200 - 8 = x \cdot q \Rightarrow 192 = x \cdot q$.
Since $192 = 2^6 \times 3$, the number of factors of 196 is $(6 + 1) \times (1 + 1) = 14$.
Since $x > 8$, we need to exclude 1, 2, 3, 4, 6, 8.
There are 14 – 6 = 8 such numbers.

Problem 7. Solution: D.

By (1.1), we have:
$x = 4q_1 + 1$ (1)
$x = 5q_2 + 2$ (2)
$x = 6q_3 + 3$ (3)
Adding 3 to both sides of (1), (2), and (3):
$x + 3 = 4q_1 + 4 = 4(q_1 + 1)$ (4)
$x + 3 = 5q_2 + 5 = 5(q_2 + 1)$ (5)
$x + 3 = 6q_3 + 6 = 6(q_3 + 1)$ (6)
Therefore we know that $x + 3$ is divisible by LCM (4, 5, 6) = 60. Thus the smallest value of $x + 3$ is 60 and the smallest value of x is 60 – 3 = 57.

Problem 8. Solution: B.
x could be 8 + 5 = 13 or 13 + 8 = 21, 29,…. The remainder when x is divided by 4 is 1.

Problem 9. Solution: C.
Let the two numbers be a and b.

$a = q_1 \times 11 + 7$ (1)

$b = q_2 \times 11 + 9$ (2)

(1) + (2): $a + b = (q_1 \times q_1)\times 11 + 16 = (q_1 \times q_1)\times 11 + 11 + 5$
The remainder is 5 when the sum of these two numbers is divided by 11.
Note: the remainder is always smaller than the divisor.

Problem 10. Solution: D.
Rachel's favorite number could be 7 + 5 = 12.
12 × 5 = 60 and 60 ÷ 7 = 8 × 7 + 4.
The remainder is 4.

Problem 11. Solution: C.
By (1.1), we have:
$K = 5q_1 + 4$ (1)
$K = 8q_2 + 2$ (2)
$K = 11q_3 + 1$ (3)
Adding 6 to both sides of (1) and (2):
$K + 6 = 5q_1 + 10 = 5(q_1 + 2)$ (4)
$K + 6 = 8q_2 + 10 = 8(q_2 + 1)$ (5)

Therefore we know that $K + 6$ is divisible by LCM (5, 8) = 40. Thus the smallest value of $K + 6$ is 40 and the smallest value of K is 40 – 6 = 34. When 34 is divided by 11 the remainder is 1 and we are done.

Problem 12. Solution: C.
The smallest n is 5 + 1 = 6. $3n$ = 18 and the remainder is 3 when 18 is divided y 5.

Problem 13. Solution: B.
Let the integer be K.
By (1.1), we have:
$K = 4q_1 + 3$ (1)
$K = 3q_2 + 1$ (2)
$K = 5q_3 + 1$ (3)
Subtracting 1 from both sides of (2) and (3):
$K - 1 = 3q_1 + 1 - 1 = 3q_1$ (4)
$K - 1 = 5q_2 + 1 - 1 = 5q_2$ (5)
Therefore we know that $K - 1$ is divisible by LCM (3, 5) = 15. Thus the values of $K - 1$ could be 15, 30, 45, 60, 75, and 90. The values of $K - 1$ could be 16, 31, 46, 61, 76, and 91. Among them, only 31 and 91 have a remainder of 3 when divided by 4.

Problem 14. Solution: A.
100 = 14 × 7 + 2. The remainder is 2 so it is 2 days from Saturday: Monday.

Problem 15. Solution: E.
365 = 52 × 7 + 1. The remainder is 1 so it is 1 day from Saturday: Sunday.

Problem 16. Solution: C.
$7p$ could be 11 + 1 = 12, 12 + 11 = 23, 34, 45, 56, 67,… Among them, the least one that is divisible by 7 is 56. So $7p$ is 8.

Problem 17. Solution: A.
Let the money be x dollars.
By (1.1), we have:
$x = 3q_1 - 3$ (1)
$x = 2q_2 + 1$ (2)
Adding 3 to both sides of (1) and (2):

$x + 3= 3q_1 – 3 + 3 = 3q_1$ (3)
$x + 3 = 2q_2 + 1 + 3 = 2(q_2 + 2)$ (4)
So $x + 3$ is divisible by $2 \times 3 = 6$. So $x + 3$ could be 6, 12,… Since x is more than 2000, and $2000 = 6 \times 333 + 2$, $x + 3 = 6 \times 333 + 2 + 4$ and $x = 6 \times 333 + 2 + 4 – 3 = 2001$.

Problem 18. Solution: B.
Let the money be x pennies.
By (1.1), we have:
$x = 5q_1 + 1$ (1)
$x = 3q_2 + 2$ (2)
Adding4 to both sides of (1) and (2):
$x + 4= 5q_1 + 1 + 4 = 5(q_1 + 1)$ (3)
$x + 4 = 3q_2 + 2 + 4 = 3(q_2 + 2)$ (4)
We see that $x + 4$ is divisible by $5 \times 3 = 15$. So $x + 4$ could be 15, 30,… The least of $x + 4$ is 15 and x is 11.

Problem 19. Solution: E.
Let the number of students be K.
By (1.1), we have:
$K = 3q_1 – 1$ (1)
$K = 4q_2 – 2$ (2)
$K = 5q_3 – 3$ (3)

Adding 2 to both sides of (2) and (3):
$K – 2 = 3q_1 – 1 – 2 = 3(q_1 – 1)$ (4)
$K – 2 = 4q_2 – 2 – 2 = 4(q_2 – 1)$ (5)
$K – 2 = 5q_3 – 3 – 2 = 5(q_3 – 1)$ (6)

Therefore we know that $K – 2$ is divisible by $LCM\ (3, 4, 5) = 60$. Thus the values of $K – 2$ could be 60, 120, …. Since $50 < K < 100$, $K – 2 = 60$ and $K = 62$.

Problem 20. Solution: D.
Let the number of chairs be x. the number of tables be q.
By (1.1), we have:
$x = 2q + 1$ (1)
$x = 3(q – 1)$ (2)

(1) – (2): $q = 4$ and $x = 9$
$x + q = 13$.

Problem 21. Solution: A.
Let the number of students be K.
By (1.1), we have:
$K = 6q_1 + 5$ (1)
$K = 7q_2 + 6$ (2)
$K = 8q_3 + 7$ (3)
Adding 1 to both sides of (1), (2), and (3):
$K + 1 = 6q_1 + 5 + 1 = 6(q_1 + 1)$ (4)
$K + 1 = 7q_2 + 6 + 1 = 7(q_2 + 1)$ (5)
$K + 1 = 8q_3 + 7 + 1 = 8(q_3 + 1)$ (6)
Therefore we know that $K + 1$ is divisible by LCM 6, 7, 8) = 168. The smallest value of $K + 1$ is 168. The smallest value of K is 167.

Problem 22. Solution: C
$51 - 3 = 48 = 2^4 \times 3$ is divisible by k. So k is a factor of 48. We also know by the **Theorem 2** that k is at least 4. There are $(4 + 1) \times (1 + 1) = 10$ factors of 48. Among them, we must subtract the factors that are less than 4. There are 3 factors less than 4: 1, 2, and 3. The answer is $10 - 3 = 7$.

Problem 23. Solution: B.
Method 1: By the Theorem (3), d should be a common factor of $2613 - 2243 = 370 = 2 \times 5 \times 37$, $2243 - 1503 = 740 = 2 \times 5 \times 37$, and $1503 - 985 = 518 = 2 \times 7 \times 37$. So the greatest value for d is $2 \times 37 = 74$.

Method 2: Let the divisor be b and the remainder be r.
$2613 = bq_1 + r$, (1)
$2243 = bq_2 + r$, (2)
$1503 = bq_3 + r$, (3)
$985 = bq_4 + r$, (4)
(1) – (2): $370 = b(q_1 - q_2)$.
(3) – (2): $518 = b(q_3 - q_4)$.
Since both 370 and 518 are divisible by b, b is the common factor of 370 and 518. The common factors of 370 and 518 are 2, 37, and 74. So the greatest divisor is 74 (and the remainder is 23).

Problem 24. Solution: E
By the **Theorem 2**, the remainder k should be less than the divisor. So it cannot be 7.

Problem 25. Solution: A.
When a positive integer is divided by 7, the remainder is 4. So the number could be $7 + 4 = 11$.
But when 11 is divided by 9, the remainder is not 3. So we keep adding 7 to 11 until the resulting number has a remainder of 3 when divided by 9. The number is $11 + 7 + 7 + 7 + 7 = 39$.

Problem 26. Solution: B.
Method 1: By the Theorem (3), b should be a factor of $227 - 216 = 11$. Since 11 is a prime number, d is 11. The remainder when 108 is divided by b is 9.

Method 2:
$73 = bq_1 + r$ (1)
$216 = bq_2 + r$ (2)
$227 = bq_3 + r$ (3)
(3) – (2): $11 = b(q_3 - q_2)$
Since 11 is a prime number, and b is not 1, so b equals 11. $108 = 11 \times 9 + 9$.
The remainder when 108 is divided by b is 9.

Problem 27. Solution: A.
$3(9m + n) = 27m + 3n = (5m + 3n) + 22m.$ Since both $5m + 3n$ and 22 are divisible by 11, $3(9m + n)$ is also divisible by 11. The remainder when $9m + n$ is divided by 11 is 0.

Method 2:
Let $m = 1$ and $n = 2$. Therefore $5m + 3n = 11$ which is divisible by 11.
Substituting in these values into $9m + n,$ we get
$9m + n = 9 \times 1 + 2 = 11$ which is also divisible by 11.
The remainder when $9m + n$ is divided by 11 is 0.

1. BASIC KNOWLEDGE

1.1. Terms

An element of a sequence is called a term of the sequence, written as a_1, a_2, a_3, ...

a_1 is called the first term.

a_n is called the general term or n^{th} term.

The sum of the first n terms is expressed as S_n. For example, S_{12} means the sum of the first twelve terms.

1.2. Arithmetic Sequences:

If any two consecutive terms in a sequence a_1, a_2, a_3, ..., a_n,…, have the same difference, the sequence is called an arithmetic sequence (or arithmetic progression).

The same difference is called the common difference (d).

A finite sequence such as 1, 2, 3, 4, 5,…, in which each term after the first is obtained by adding the preceding term by a fixed number, is an example of an arithmetic sequence.

If we insert a number c between two numbers a and b such that a, c, b are in arithmetic sequence, then c is called the arithmetic mean of a and b and $c = (a + b)/2$.

$$d = a_{n+1} - a_n \tag{1.2.1}$$

$$a_n = a_1 + (n-1)d \tag{1.2.2}$$

$$S = \frac{(a_1 + a_n)n}{2} \tag{1.2.3}$$

$$S = na_1 + \frac{(n-1)d}{2}n \tag{1.2.4}$$

1.3. Geometric Sequences:

If any two consecutive terms in a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$, have the same ratio, the sequence is called a geometric sequence (or geometric progression).

The same ratio is called the common ratio (q or r).

A finite sequence such as 2, 4, 8, 16, 32, …, in which each term after the first is obtained by multiplying the preceding term by a fixed number, is an example of a geometric sequence.

If we insert a number c between two numbers a and b such that a, c, b are in geometric sequence, then c is called the geometric mean of a and b and $c^2 = ab$, or $c = \pm\sqrt{ab}$, where $ab > 0$.

$$\frac{a_n}{a_{n-1}} = q \qquad (q \neq 0,\ n \geq 2) \qquad (1.3.1)$$

$$a_n = a_1 \cdot q^{n-1} \qquad (a_1 q \neq 0) \qquad (1.3.2)$$

$$S_n = \frac{a_1(1-q^n)}{1-q} \quad (q \neq 1) \qquad (1.3.3)$$

$$S_n = \frac{a_1 - a_n q}{1-q} \qquad (1.3.4)$$

$$S_n = na_1 \qquad (q = 1) \qquad (1.3.5)$$

2. PROBLEM SOLVING SKILLS

(2.1). The common difference (d) of arithmetic sequences

$$d = a_{n+1} - a_n \qquad (2.1.1)$$

$$d = \frac{a_m - a_n}{m-n} \qquad (2.1.2)$$

$$\frac{d}{2} = \frac{\frac{S_m}{m} - \frac{S_n}{n}}{m-n} \qquad (2.1.3)$$

Example 1. In an arithmetic sequence a_n, $a_7 - 2a_4 = -1$, $a_3 = 0$. Find the common difference d.

A. –2 B. $-\frac{1}{2}$ C. $\frac{1}{2}$ D. 2 E. 4.

Solution: B.

Method 1:

By formula (2.1.2), we have: $\frac{a_7 - a_3}{7-3} = \frac{a_4 - a_3}{4-3}$

$$\Rightarrow \quad d = \frac{a_7 - a_3}{7-3} = \frac{2a_4 - 2a_3}{2(4-3)} = \frac{a_7 - a_3 - (2a_4 - 2a_3)}{(7-3) - 2(4-3)} = \frac{a_7 - 2a_4}{2} = -\frac{1}{2}.$$

Method 2:

By formula (2.1.2), we have: $d = \frac{a_7 - a_3}{7-3} = \frac{a_4 - a_3}{4-3} \Rightarrow d = \frac{a_7}{4} = a_4$ (1)

We are given that $a_7 - 2a_4 = -1 \Rightarrow a_7 = 2a_4 - 1$.

Substituting the value of a_7 into (1), we get: $\frac{2a_4 - 1}{4} = a_4 \Rightarrow 2a_4 - 1 = 4a_4$

Therefore $a_4 = d = -\frac{1}{2}$.

(2.2). The *n*th term of arithmetic sequences

The nth term is expressed as $a_n = a_1 + (n-1)d$ (2.2.1)

Other forms: $a_n = a_m + (n-m)d$ (2.2.2)

$a_n = S_n - S_{n-1}$ (n is positive integer and $n > 1$) (2.2.3)

If $m + n = p + q$ where m, n, p, and q are positive integers, then

$a_m + a_n = a_p + a_q$ (2.2.4)

If $m + n = 2q$, then $a_m + a_n = 2a_q$ (2.2.5)

Example 2. In an arithmetic sequence a_n, if $a_1 + a_9 = 10$, find a_5.

（A）5 （B）6 （C）8 （D）10 (E) 20

Solution: (A)

By formula (2.2.5): $a_1 + a_9 = 2a_5$. Therefore $a_5 = 5$.

Example 3. In an arithmetic sequence a_n, $a_7 - 2a_4 = -1$, $a_3 = 0$. Find the common difference d.

(A) -2 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 2 (E) 4

Solution: (B)

Formula (2.2.2) gives us $a_7 = a_3 + 4d$ and $2a_4 = 2(a_3 + d)$.

Therefore, $a_7 - 2a_4 = a_3 + 4d - 2(a_3 + d) = 2d = -1 \Rightarrow d = -\frac{1}{2}$.

Example 4. In an arithmetic sequence a_n, if $a_2 + a_{10} = 20$， find a_6.

（A）5 （B）6 （C）8 （D）10 (E) 12

Solution: (D).

We know that $a_2 + a_{10} = 2a_6$, from formula (2.2.5), so $a_6 = 10$.

Example 5. In an arithmetic sequence a_n, $a_1 + a_3 + a_5 = 105, a_2 + a_4 + a_6 = 99$. Find a_{20}.

(A) -1 (B) 1 (C) 3 (D)7 (E) -2.

Solution: B.

From $a_1 + a_3 + a_5 = 105$, we get $3a_3 = 105 \Rightarrow a_3 = 35$.

Similarly from $a_2 + a_4 + a_6 = 99$, we get $a_4 = 33$.

The common difference of this arithmetic sequence is therefore $d = a_4 - a_3 = -2$.

Thus, $a_{20} = a_4 + (20 - 4) \times d = 1$.

(2.3). The sum of *n* terms in the arithmetic sequences:

$$S=\frac{(a_1+a_n)n}{2} \qquad (2.3.1)$$

$$S=na_1+\frac{(n-1)d}{2}n \qquad (2.3.2)$$

Example 6. In an arithmetic sequence a_n, the sum of first n terms is S_n. If $a_2=3$, $a_6=11$, find S_7.

(A) 13 (B) 35 (C) 49 (D) 63 (E) 64

Solution: C.

Method 1:

$$S_7=\frac{7(a_1+a_7)}{2}=\frac{7(a_2+a_6)}{2}=\frac{7(3+11)}{2}=49.$$

Note that by (2.2.4) we have $a_1+a_7=a_2+a_6$.

Method 2:

$$\begin{cases}a_2=a_1+d=3\\ a_6=a_1+5d=11\end{cases}\Rightarrow\begin{cases}a_1=1\\ d=2\end{cases},\ a_7=1+6\times 2=13.\ \text{Therefore } S_7=\frac{7(a_1+a_7)}{2}=\frac{7(1+13)}{2}=49.$$

Example 7. Find the sum of all counting numbers between 50 and 350 that have 1 as the last digit.

(A) 5050 (B) 5880 (C) 5441 (D) 5552 (E) 5338

Solution: B.

The integers with 1 as the last digit form an arithmetic sequence, where the first term is $a_1=51$, the last term is $a_n=341$, and the common difference is $d=10$.

$341=51+(n-1)\times 10 \qquad \Rightarrow \qquad n=30.$

The sum of all counting numbers between 5 and 350 that have 1 as the last digit is

$$S=\frac{(a_1+a_{30})}{2}\times n=\frac{(51+341)\times 30}{2}=5880.$$

(2.4). The *n*th term of geometric sequences

$a_n = a_1 \cdot q^{n-1}$ $(a_1 q \neq 0)$ (2.4.1)

$a_n = a_m \cdot q^{n-m}$ $(a_1 q \neq 0)$ (2.4.2)

${a_n}^2 = a_{n-k} \cdot a_{n+k}$ $(n \geq k)$ (2.4.3)

If $m + n = p + q$, $a_m \cdot a_n = a_p \cdot a_q$ (2.4.4)

If $m + n = 2t$, $a_m \cdot a_n = {a_t}^2$ (2.4.5)

Example 8. In a geometric sequence: 6, …, 768, …, 12288, 768 is the *n*th term and 12288 is the $(2n - 4)$th term. Find the common ratio q.
(A) 2 (B) 3 (C) 4 (D) 5 (E) 8

Solution: A.
By (2.4.1), we have:

$6q^{n-1} = 768 \quad \Rightarrow \quad q^{n-1} = 128 = 2^7$ (1)

$6q^{2n-5} = 12288 \quad \Rightarrow \quad q^{2n-5} = 2048 = 2^{11}$

$\Rightarrow \quad q^{2(n-1)-3} = 2^{11} \quad \Rightarrow \quad \dfrac{(q^{n-1})^2}{q^3} = 2^{11}$ (2)

Substituting (1) into (2), we have $\dfrac{(2^7)^2}{q^3} = 2^{11} \Rightarrow \quad q^3 = 2^3$. Therefore $q = 2$.

Example 9. In geometric sequence $\{a_n\}$, if $a_2 = 4$, and $a_5 = -\dfrac{1}{2}$, find the general term.

Solution:

$a_2 = 4$, $a_5 = -\dfrac{1}{2}$.

By (2.4.2), we have $a_5 = a_2 q^{5-2} \Rightarrow \quad q = -\dfrac{1}{2}$. So $a_n = a_2 q^{n-2} = 4(-\dfrac{1}{2})^{n-2}$.

(2.5). Find a term in the geometric sequence

Example 10. If five numbers are inserted between 8 and 5832, the fifth term in the geometric series formed is:
(A) 648 (B) 832 (C) 1168 (D) 1944 (E) none of these

Solution: A.
Denote the terms in the geometric progression by

$$a_1 = 8,\ \ a_2 = 8r, \cdots,\ \ a_7 = 8r^6 = 5832.$$

$\therefore\ r^6 = 729;$ $\therefore\ r = 3$ and $a_5 = 8r^4 = 648.$

Example 11. The second and fourth terms of a geometric sequence are 2 and 6. Which of the following is a possible first term?

(A) $-\sqrt{3}$ (B) $-\frac{2\sqrt{3}}{3}$ (C) $-\frac{\sqrt{3}}{3}$ (D) $\sqrt{3}$ (E) 3

Solution: (B).
Let the sequence be denoted a, ar, ar^2, ar^3, . . . , with $ar = 2$ and $ar^3 = 6$. Then $r^2 = 3$ and $r = \sqrt{3}$ or $r = -\sqrt{3}$. Therefore $a = \frac{2\sqrt{3}}{3}$ or $a = -\frac{2\sqrt{3}}{3}$.

(2.6). Find the common ratio of geometric sequences

Example 12. By adding the same constant to each of 20, 50, 100 a geometric progression results. The common ratio is

(A) $\frac{5}{3}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{1}{2}$ (E) $\frac{1}{3}$

Solution: (A).
Let a = the constant: $\frac{20+a}{50+a} = \frac{50+a}{100+a}$; $\therefore\ a = 25;$ $\therefore\ r = \frac{5}{3}.$

Example 13. $\{a_n\}$ is a positive geometric sequence. Find the common ratio q if $a_{2010} = 8a_{2007}$.
A. 2 B. 3 C. 4 D. 8 E. 16.

Solution: A.

By (2.4.2), we have $\frac{a_{2010}}{a_{2007}} = q^3 = 8 \quad \therefore q = 2$

3. MORE EXAMPLES

Example 14. The first term in the sequence of numbers 5, 7, −7, …is 5. Each even-numbered term is 2 more than the previous term and each odd-numbered term, after the first, is – 1 times the previous term. For example, the second term is 5 + 2 = 7, and the third term is (−1) × 7. What is the 255th term of the sequence?

(A) – 7 (B) – 5 (C) –1 (D) 5 (E) 7

Solution: A.

We see the pattern: 5, 7, –7, –5, 5, 7, –7, –5… It repeats every four terms. Since 255/ 4 has a remainder of 3, is the 255th term of the sequence is the same as the third term, which is – 7.

Example 15. The first term of a sequence is 3, and every term after the first term is – 3 times the preceding term. How many of the first 100 terms of this sequence are less than 2000?

(A) 52 (B) 57 (C) 53 (D) 58 (E) 30

Solution: C

$3, -3^2, 3^3, -3^4, 3^5, -3^6, 3^7 = 2187$.

All the negative numbers are less than 1000. So we have 50 of them. For positive numbers we see that $3^7 = 2187$ which is over 2000. So we have 3 of them (3, 3^3, and 3^5). The answer is 50 + 3 = 53.

Example 16. The first term of the sequence is 1 and the second term is 5, and each term after the second is 5 times the preceding term. Which of the following expressions represents the nth term of the sequence?

(A) 5^n (B) 5^{n-1} (C) n^2 (D) $(n-5)^2$ (E) $5n$.

Solution: B.

We see the pattern: $5^0, 5^1, 5^2, 5^3, \ldots, 5^{n-1}$.

Example 17. Set A consists of the numbers in the arithmetic sequence 15, 24, 33, . . . , and set B consist of the numbers in the arithmetic sequence 21, 27, 33, ... What is the sum of the three smallest numbers common to both sets?
(A) 152 (B) 157 (C) 151 (D) 153 (E) 133

Solution: D.
The common difference of the first sequence is 9 and the second is 6. LCM (6, 9) = 18. Therefore the numbers common to both sets have the difference of 18. The sum of the five numbers is 33 + (33 + 18) + (33 + 2 × 18) = 153.

Example 18. The arithmetic sequences 1, 5, 9, 13, 17, 21, 25, 29, . . . and 1, 8, 15, 22, 29, . . . have infinitely many terms in common. Calculate the sum of the first three common terms.
(A) 171 (B) 175 (C) 177 (D) 178 (E) 170

Solution: A.
The common difference of the first sequence is 4 and the second is 7. LCM (4,7) = 28. Therefore the numbers common to both sets have the difference of 28. The sum of the five numbers is 29 + (29 + 28) + (29 + 2 × 28) = 171.

Example 19. If the blanks are replaced with three numbers to create an arithmetic sequence, what is the sum of these three numbers? 18, ——, —— , ——, 54.
(A) 102 (B) 104 (C) 106 (D) 108 (E) 110

Solution: D.
Let the three numbers be x , y, and z. in order from the smallest to the greatest.
By the definition of arithmetic sequence, we have $\frac{18+54}{2} = 36 = y = \frac{x+z}{2}$ $\Rightarrow$

$x + y = 2y$

So $x + y + z = 3y = 3 \times 36 = 108$

Example 20. The third term of an arithmetic sequence is 15 and the fifth term is 23. What is the first term?
(A) 2 (B) 5 (C) 7 (D) 8 (E) 10

Solution: C.
By the formula $a_n = a_1 + (n-1)d$, we have

$15 = a_1 + (3-1)d \quad \Rightarrow \quad 15 = a_1 + 2d \quad \Rightarrow \quad 30 = 2a_1 + 4d \qquad (1)$

$23 = a_1 + (5-1)d \quad \Rightarrow \quad 23 = a_1 + 4d \qquad (2)$

(1) – (2): $a_1 = 7$.

Example 21. In an arithmetic sequence the 113th term is 786 and the 125th term is 870. Find the 150th term.
(A) 1024 (B) 1045 (C) 1053 (D) 1058 (E) 1030

Solution: B.

$$d = \frac{a_m - a_n}{m-n} = \frac{a_{125} - a_{113}}{125-113} = \frac{a_{150} - a_{113}}{150-113} \Rightarrow \frac{870-786}{12} = \frac{a_{150} - 786}{37} \Rightarrow$$

$$a_{150} = \frac{84}{12} \times 37 + 786 = 1045.$$

Example 22. In an arithmetic sequence a_n, the sum of first n terms is S_n. If $S_3 =$ 6, a_1 = 4, find the common difference d.

(A)1 (B) $\frac{5}{3}$ (C) –2 (D) 3 (E) –5

Solution: C.
Method 1:

By formula (2.1.3), we have: $\frac{d}{2} = \frac{\frac{S_3}{3} - \frac{S_1}{1}}{3-1} = \frac{\frac{6}{3} - \frac{a_1}{1}}{2} = \frac{\frac{6}{3} - \frac{4}{1}}{2} = -1 \quad \Rightarrow d = -2.$

Method 2:

We calculate $S_3 = 6 = \frac{(a_1 + a_3) \times 3}{2} \Rightarrow a_1 + a_3 = 4$.

Since $a_1 = 4$, $a_3 = 0$.

By the formula (2.1.2), we have: $d = \frac{a_3 - a_1}{3-1} = \frac{0-4}{2} = -2$.

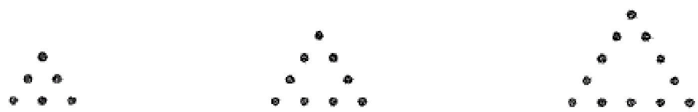
Example 23. In an arithmetic sequence a_n, $a_3 = -6$, and $a_6 = 0$. Find a_n in terms of n.
(A) $2n - 12$ (B) $2n - 8$ (C) $10 - 2n$ (D) $n - 12$ (E) $2n + 12$

Solution: A.
Let the common difference of the arithmetic sequence be d.
We are given that $a_3 = -6$, and $a_6 = 0$.
Therefore $\begin{cases} a_1 + 2d = -6 \\ a_1 + 5d = 0 \end{cases}$.
Solving we get $a_1 = -10, d = 2$.
$a_n = -10 + (n-1)\cdot 2 = 2n - 12$.

Example 24. In geometric sequence $\{a_n\}$, if $a_3 a_4 a_5 = 8$, find $a_1 a_3 a_4 a_5 a_7$.
(A) 24 (B) 26 (C) 32 (D) 58 (E) 30

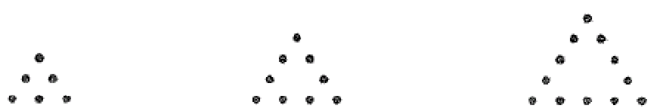
Solution: C.
By (2.4.3), we have $a_3 a_5 = a_4^2$.
$a_3 a_4 a_5 = a_4^3 = 8$.
So $a_4 = 2$.
By (2.4.5), $a_1 a_7 = a_4^2$. Therefore $a_1 a_3 a_4 a_5 a_7 = a_4^5 = 32$.

Example 25. Each new triangle shown below has one more dot per side than the previous triangle. What is the total number of dots on the triangle with 358 dots per side?



(A) 1024 (B) 1045 (C) 1053 (D) 1058 (E) 1071

Solution: E.
We see the pattern:



$3 + 2\times 1 + 1$ $4 + 2\times 2 + 1$ $5 + 3\times 2 + 1$

The formula is $(n + 2) + 2n + 1 = 3(n + 1)$.
The total number of dots on the triangle with 358 dots per side is 3(358 + 1) = 1077.

Example 26. Seventeen consecutive positive integers have a sum of 306. What is the sum of the seventeen consecutive positive integers that are following the previous mentioned seventeen positive integers?
(A) 565 (B) 575 (C) 585 (D) 595 (E) 47

Solution: D.
Method 1:
By the formula (2.1.3),

$$\frac{1}{2}=\frac{\frac{S_{34}}{34}-\frac{S_{17}}{17}}{34-17} \Rightarrow \frac{17}{2}=\frac{S_{34}}{34}-\frac{S_{17}}{17} \Rightarrow 17^2=S_{34}-2S_{17} \Rightarrow$$

$S_{34}-S_{17}=17^2+S_{17}=595$.

Method 2:
We know that $a_9=\frac{S}{n}=\frac{(a_1+a_{17})}{2}=\frac{306}{17}=18$.
Since these positive integers are consecutive, $a_{18}=a_9+9=18+9=27$ and $a_{34}=43$. The sum is 595.

Example 27. In an arithmetic sequence a_n, the sum of first n terms is S_n. If $S_9=72$, then $a_2+a_4+a_9$=?
(A) 24 (B) 10 (C) 15 (D) 28 (E) 30

Solution: A.
We have $S_9=72$

$\therefore S_9=9a_5$, and $a_5=8$.

$\therefore a_2+a_4+a_9=(a_2+a_9)+a_4=(a_5+a_6)+a_4=3a_5=24$.

4. PROBLEMS

Problem 1. The first term in the sequence 5, 19, 61, 187, . . . is 5, and each term after the first is determined by multiplying the preceding term by m and then adding p. What is the value of m?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 9

Problem 2. The first term of sequence P: 3, 9, 27, . . .is 3, and each term after the first is 3 times the preceding term. The first term of sequence T: 100, 200, 300, . . . is 100, and each term after the first is 100 more than the preceding term. What is the least value of n such that the nth term of sequence P is greater than the nth term of sequence T?
(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Problem 3. In the sequence 7, a, b, , . . . , the first term is 7 and the second term is a. Each term after the second is the product of the two immediately preceding terms. If $a < 0$, what is the 11th term of the sequence?
(A) -7^{24} (B) 7^{21} (C) 7 (D) -7 (E) -7^{45}

Problem 4. How many integers belong to the arithmetic sequence 93, 103, 113, . . . , 1463?
(A) 93 (B) 138 (C) 94 (D) 137 (E) 139

Problem 5. What is the tenth number in the arithmetic sequence 3, x, 11, . . .?
(A) 100 (B) 101 (C) 20 (D) 38 (E) 39

Problem 6. The first three terms of an arithmetic sequence are $x-1$, $x+1$ and $2x+3$. What is the value of x?
(A) 0 (B) 2 (C) 1 (D) 5 (E) 3

Problem 7. The first term of an arithmetic sequence is 15, and the seventh term is 57. What is the third term of the sequence?
(A) 87 (B) 29 (C) 24 (D) 45 (E) 36

Problem 8. For what value of x does $1 + 2 + 3 + 4 + 5 + . . . + x = 120$?
(A) 14 (B) 15 (C) 16 (D) 12 (E) 10

Problem 9. The arithmetic sequences 1, 4, 7, 10, . . . , and 2, 10, 18, 26, . . . , each contain 100 terms. How many numbers are common to both sequences?
(A) 15 (B) 14 (C) 13 (D) 100 (E) 99

Problem 10. The sum of consecutive even integers $2 + 4 + 6 + \cdots + m = 2550$. What is the value of m?
(A) 10 (B) 102 (C) 100 (D) 50 (E) 51

Problem 11. How many different arithmetic sequences are there with all of the following properties:
a) the first term is 119, b) the last term is 179,
c) the common difference is a whole number, and
d) the total number of terms is at least three?
(A) 11 (B) 12 (C) 14 (D) 16 (E) 10

Problem 12. What is the 4^{th} number in this geometric progression?
108, – 36, 12, —
(A) 5 (B) 4 (C) – 4 (D) –6 (E) 6

Problem 13. This sequence is generated using the rule where each term is the sum of the two preceding terms. Find the second term.
5, — , — , — , — , — , 9
(A) 2 (B) – 2 (C) 6 (D) 7 (E) – 12

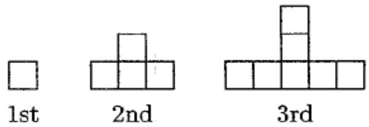
Problem 14. What is the 101^{st} term of the sequence 1, 2, 2, 3, 3, 3, . . . , in which each positive integer n occurs in blocks of n terms?
(A) 12 (B) 13 (C) 15 (D) 14 (E) 16

Problem 15. In the sequence 1, 2, 4, 8, . . . , each term is twice the term immediately before it. What is the eleventh term?
(A) 2048 (B) 1024 (C) 512 (D) 256 (E) 64

Problem 16. Each term in a sequence of whole numbers is one more than the square of the previous term Given that the fourth term is 26, what is the sum of the third and fifth terms?
(A) 615 (B) 616 (C) 649 (D) 684 (E) 682

Problem 17. In a sequence, each term is obtained by calculating the sum of the preceding two terms. The eighth term is 81, and the sixth term is 31. What is the fourth term?
(A) 12 (B) 19 (C) 31 (D) 50 (E) 76

Problem 18. How many squares are needed to build the 10th shape in the pattern?



(A) 12 (B) 28 (C) 18 (D) 72 (E) 60

Problem 19. In an arithmetic sequence a_n, if $a_3 + a_4 + a_5 = 12$, then $a_1 + a_2 + \ldots + a_7 =$
(A) 14 (B) 21 (C) 28 (D) 35 (E) 24

Problem 20. In an arithmetic sequence a_n, if the sum of first n terms is $S_n = n^2$, find a_8.
(A) 15 (B) 16 (C) 49 (D) 64 (E) 60

Problem 21. In an arithmetic sequence a_n, $a_5 = 9$, $a_{19} = 19$, and $2^{11} - 3$ is the nth term. Find n.
(A) 1023 (B) 1024 (C) 1025 (D) 1064 (E) 1060

Problem 22. In an arithmetic sequence a_n, first term a_1 and the common difference d are real numbers. The sum of first n terms is S_n. If $S_5S_6 + 15 = 0$ and $S_5 = 5$, find the sum of S_6 and a_1.
(A) 2 (B) 3 (C) 4 (D) 6 (E) 7

Problem 23. The first three terms of a geometric progression are $\sqrt{2}, \sqrt[3]{2}, \sqrt[6]{2}$. The fourth term is
(A) 1 (B) $\sqrt[7]{2}$ (C) $\sqrt[8]{2}$ (D) $\sqrt[9]{2}$ (E) $\sqrt[10]{2}$

5. SOLUTIONS

Problem 1. Solution: C.

$5m + p = 19$ (1)

$19m + p = 61$ (2)

(2) – (1): $14m = 42 \Rightarrow m = 3$.

Problem 2. Solution: A.

Sequence P:	3,	9,	27,	81,	243,	729
Sequence T:	100,	200,	300	400	500	600

Problem 3. Solution: A.

$7a = b$ (1)

$ab = 7$ (2)

(1) ÷ (2): $b^2 = 7^2$.

Since $a < 0$, and $7a = b$, $b < 0$. Therefore $b = -7$ and $a = -1$.

The terms are $7, -1, -7, 7, -7^2, -7^3, 7^5, -7^8, -7^{13}, 7^{21}. -7^{24}$.

Problem 4. Solution: B.

$a_n = a_1 + (n-1)d \Rightarrow 1463 = 93 + (n-1)\times 10 \Rightarrow n = 138$

Problem 5. Solution: E.

$x - 3 = 1 - x \Rightarrow x = 7 \Rightarrow d = 4 \Rightarrow a_{10} = 3 + (10-1)\times 4 = 39$.

Problem 6. Solution: A.

$\frac{(x-1)+(2x+3)}{2} = x+1 \Rightarrow x = 0$.

Problem 7. Solution: B.

By the formula $a_n = a_1 + (n-1)d$, we have

$a_3 = 15 + (3-1)d \Rightarrow a_3 = 15 + 2d \Rightarrow 3a_3 = 45 + 6d$ (1)

$57 = 15 + (7-1)d \Rightarrow 57 = 15 + 6d$ (2)

(1) – (2): $3a_3 - 57 = 45 - 15 \Rightarrow 3a_3 = 87 \Rightarrow a_3 = 29$.

Problem 8. Solution: B.

By the sum formula, we have $\frac{x(x+1)}{2}=120 \Rightarrow \quad x(x+1)=240$.
Since x and $x + 1$ are consecutive integers, we have $x(x+1)=240=15\times 16$. So x is 15.

Problem 9. Solution: C.
By the formula $a_n = a_1 + (n-1)d$, we have $a_{100} = 1+(100-1)\times 3 = 298$ and $b_{100} = 2+(100-1)\times 8 = 794$.
We know that LCM (3, 8) = 24.
The number of terms in common to both sequences is n and $298 = 10 + (n-1)\times 24$ $\Rightarrow n = 13$.
Note we do not need to look at the second sequence because its last term is much bigger.

Problem 10. Solution: C.
$2 + 4 + 6 + \cdots + m = 2550$ becomes $1 + 2 + 3 + \cdots + n = 2550/2$. Where $n = m/2$.
By the sum formula, we have $\frac{n(n+1)}{2}=\frac{2550}{2} \Rightarrow \quad n(n+1)=2550$.
Since n and $n + 1$ are consecutive integers, we have $n(n+1)=2550=50\times 51$. So n is 50 and m is 100.

Problem 11. Solution: A.
By the formula $a_n = a_1 + (n-1)d$, we have $179 = 119 + (n-1)d \Rightarrow (n-1)d = 60$
We know that $60 = 2^2 \times 3 \times 5$ has $(2 + 1)(1 + 1)(1 + 1) =12$ factors including 1.
Since $n - 1 \ge 2$, so $n - 1$ can be any one of the $12 - 1 = 11$ factors.

Problem 12. Solution: C.
Let x be the 4$^{\text{th}}$ number. $\frac{-36}{108}=\frac{x}{12} \Rightarrow \quad x = -4$.

Problem 13. Solution: B.
Let the numbers be 5, a, b, c, d, e, 9.
$b = a + 5$; $c = 2a + 5$; $d = 3a + 10$; $e = 5a + 15$.
Thus $9 = d + e = 3a + 10 + 5a + 15 \Rightarrow \quad 8a = 9 - 25 \Rightarrow \quad a = -2$.

Problem 14. Solution: D.
The number of terms is $1 + 2 + 3 + \ldots + n$ which is close to 101.

By the sum formula, we have $\frac{n(n+1)}{2} = 101 \Rightarrow \quad n(n+1) = 202$.

We use the calculator to find: $\sqrt{202} \approx 14$. When $n = 14$, $\frac{14(14+1)}{2} = 105$, which means that we have 4 more 14 to go until we count a 15. So the answer is 14.

Problem 15. Solution: B.
We see the pattern that

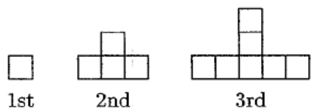
1	2	4	8	16…
2^0	2^1	2^2	2^3	2^4

Therefore the eleventh term is $2^{10} = 1024$.

Problem 16. Solution: E.
Let the first number be x. The second number will be $x^2 + 1$. The third number will be $(x^2 + 1)^2 + 1$. Thus $((x^2 + 1)^2 + 1)^2 + 1 = 26 \quad \Rightarrow \quad ((x^2 + 1)^2 + 1)^2 = 25$
$\Rightarrow (x^2 + 1)^2 + 1 = 5 \quad \Rightarrow (x^2 + 1)^2 = 4 \quad \Rightarrow \quad x^2 + 1 = 2 \quad \Rightarrow \quad x^2 = 1$.
So $x = 1$.
The sum of the third and fifth terms is $(x^2 + 1)^2 + 1 + 26^2 + 1 = 682$.

Problem 17. Solution: A.
Let the fourth term be x, the fifth term be y, and the seventh term be z.
We have: $31 + z = 81$. So $z = 50$. We also have $y + 31 = z$. So $y = 19$. Since $x + y = 31$, $x = 12$.

Problem 18. Solution: B.
We see the pattern:



,….

$1+3\times0 \quad 1 + 3\times1 \quad 1+ 3\times2, \ldots$
The 10^{th} shape will need $1 + 9 \times 3 = 28$ squares.

Problem 19. Solution: C.

Since $a_3 + a_4 + a_5 = 12$, then $a_4 = 4$.

$a_1 + a_2 + \cdots \qquad \frac{1}{2} \times 7 \times (a_1 + a_7) = 7a_4 = 28$

Problem 20. Solution: A.
We know that $a_n = S_n - S_{n-1}$ ($n \ge 2$). Therefore $a_8 = S_8 - S_7 = 64 - 49 = 15$.

Problem 21. Solution: A.
We have $a_5 = a_1 + 4d = 9$ and $a_{10} = a_1 + 9d = 19$.
Therefore $a_1 = 1,\ d = 2.$ $\qquad a_n = a_1 + (n-1)d = 2n - 1$.
We know that $2^{11} - 3 = 2(2^{10} - 1) - 1$.
Therefore $n = 2^{10} - 1$ = 1024 – 1 = 1023.

Problem 22. Solution: C.
$S_6 = -\frac{15}{S_5} = -3$. $\qquad a_6 = S_6 - S_5 = -8$.
Therefore we have

$$\left.\begin{array}{l} 5a_1 + 10d = 5 \\ a_1 + 5d = -8 \end{array}\right\}$$

Solving we get $a_1 = 7$. Therefore $a_1 = 7$ and $S_6 = -3$. The answer is 7 – 3 = 4.

Problem 23. Solution: A.
Method 1 (official solution):
In a geometric progression, each term is r times the preceding one. So in this case, $r = 2^{1/3}/2^{1/2} = 2^{1/3-1/2} = 2^{-1/6}$. Thus, the fourth term is $r \cdot 2^{1/6} = 2^{-1/6}2^{1/6} = 2^0 = 1$.

Method 2:
Let the fourth term be x. If $m + n = p + q$, $a_m \cdot a_n = a_p \cdot a_q$, we have:

$$x \times \sqrt{2} = \sqrt[3]{2} \times \sqrt[6]{2} \qquad \Rightarrow \qquad x = \frac{\sqrt[3]{2} \times \sqrt[6]{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

Method 3:
Let the fourth term be x. By (2.4.1), we have:

$$x \times \sqrt{2} = \sqrt[3]{2} \times \sqrt[6]{2} \quad \Rightarrow \quad \frac{\sqrt[3]{2}}{\sqrt{2}} = \frac{x}{\sqrt[6]{2}} \quad \Rightarrow \quad x = \frac{\sqrt[3]{2} \times \sqrt[6]{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

1. BASIC KNOWLEDGE OF FUNCTIONS

Definition

A function is a relationship between the independent variable x and dependent variable y. Each value of x is corresponding to exactly one value of y.

Note that two different values of x can have the same value of y, but one value of x cannot have two different values of y.

For example, the price of fruits in a store is a function of fruit kind. One pound of apple (x_1) and one pound of orange (x_2) can have the same price \$1.99 per pound (y). However, one pound of apple (x) cannot have two different prices at the same time (one price tag says \$1.99 per pound (y_1) and price tag says \$0.99 per pound (y_2).

Example 1. Which of these relations cannot be functions?

(A) $g(x) = \{(9, 3), (2, 4)\}$ (B) $f(x) = \{(5, 6), (3, 6)\}$ (C) $2x + 5y = 10$

(D) $\{(1, 2), (2, 4)\}$ (E) $h(x) = \{(3, 4), (3, 9)\}$

Solution: E.

	A	B	C	D	E
x	(9), (2)	(5), (3)	(0), (5)	(1), (2)	(3), (3)
y	(3), (4)	(3), (6)	(2), (0)	(2), (4)	(4), (9)

We see that $h(x)$ is not a function because one value of x has two different values of y.

Example 2. Find the value of m, expressed as a common fraction, so that this relation is not a function. $\{(-2m+1, -4), (-6m+8, 0)\}$.

A. $\frac{7}{8}$ B. $\frac{7}{4}$ C. $\frac{2}{7}$ D. $\frac{4}{7}$ E. $\frac{8}{7}$

Solution: B.

When $-2m+1=-6m+8$, or $6m-2m=8-1$ or $m=\frac{7}{4}$, this relation is not a function.

Example 3. The relations, $f, g,$ and h have the following properties:

$f:1\to 2$
$f:2\to 1$
$g:1\to 2$
$g:1\to 3$
$h:4\to 1$
$h:3\to 1$
$m:4\to 7$
$m:5\to 7$

Which of these relations cannot be functions?

A. f B. g C. h D. m E. all of them

Solution: B.

We see that g is not a function because one value of x has two different values of y.

Example 4. Let $f(x)=\begin{cases} x, & x<1 \\ x+1, & 1\le x<3 \\ x+3, & x\ge 3 \end{cases}$. Then $f(0)+f(2)+f(4)$ is equal to

A. 0 B. 9 C.10 D. 21 E. None of these

Solution: C.

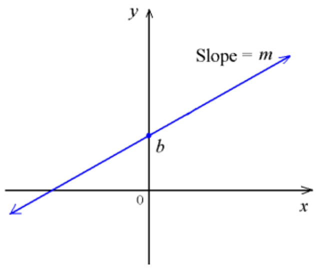
$f(0) = 0$.
$f(2) = 2 + 1 = 3$.
$f(4) = 4 + 3 = 7$.
$f(0) + f(2) + f(4) = 0 + 3 + 7 = 10$.

Example 5. If $f(1) = 2$ and $f(n + 1) = (f(n))^2$, what is the value of $f(4)$?
A. 4. B. 16. C. 64. D. 256. E. 65,536.

Solution: D.
$f(2) = f(1 + 1) = (f(1))^2 = 4$ and $f(3) = f(2 + 1) = (f(2))^2 = 16$, and $f(4) = f(3 + 1) = (f(3))^2 = 16^2 = 256$.

2. BASIC FUNCTIONS

2.1. Linear function



A linear function is a function whose graph is a straight line.

$y = mx + b$, where m is the slope and b is the y-intercept .

Example 6. A formula that relates the temperature Celsius to the temperature Fahrenheit is $F = \frac{9}{5} \cdot C + 32$. If the temperature is $140^0 F$, want is the equivalent temperature Celsius?
A. 64 B. 40 C. 64 D. 60 E. 63

Solution: D.
$$F = \frac{9}{5} \cdot C + 32 \quad \Rightarrow \quad C = \frac{5(F - 32)}{9}.$$
When the temperature is $140^0 F$, $C = \frac{5 \times (140 - 32)}{9} = 60$.

Example 7. If $f(1) = 5$, $f(2) = 8$ and $f(x) = ax + b$, what is the value of $f(3)$?
A. 10 B. 16 C. 18 D. 15 E. 11.

Solution: E.

From $f(x)=ax+b$,

$f(1)=a+b=5$ (1)

$f(2)=2a+b=8$ (2)

(2) – (1): $a=3$. So $b=2$.

$f(x)=ax+b \quad \Rightarrow \quad f(x)=3x+2$.

Thus $f(3)=3\times 3+2=11$

Example 8. If $f(x)=2x+3$, find $f(4)$.

A. 10 B. 16 C. 8 D. 11 E. 5.

Solution: D.

$f(x)=2x+3 \quad \Rightarrow \quad f(4)=2\times 4+3=11$.

Example 9. If $f(x)=2x+1$ and $g(x)=3x-5$, find $f(g(2))$.

A. 0 B. 1 C. 2 D. 3 E. 5.

Solution: D.

$g(2)=3\times 2-5=1$.

$f(g(2)) = f(1)=2\times 1+1=3$.

Example 10. $g(x)$ is a linear function such that $g(0)=5$ and $g(1)=11$. Find the value of $g(2.5)$.

A. 10 B. 15 C. 20 D. 21 E. 16.

Solution: C.

Let $g(x)=ax+b$.

Since $g(0)=5$, $g(0)=a\times 0+b=5 \quad \Rightarrow \quad b=5$.

Since $g(1)=11.$, $g(1)=a\times 1+b=11 \Rightarrow \quad a=6$.

So $g(x)=ax+b=6x+5$ and $g(2.5) = 6\times 2.5+5=20$.

Example 11. The function f is linear and satisfies $f(d+1)-f(d)=3$ for all real numbers d. What is $f(3)-f(5)$?

A. −6 B. −15 C. 10 D. 21 E. 16.

Solution: A.

Let $f(x)=ax+b$.

$f(d)=ad+b$ (1)

$f(d+1)=a(d+1)+b$ (2)

(2) – (1): $f(d+1)-f(d)=3$ or $a(d+1)+b-(ad+b)=3 \Rightarrow a=3$.

So $f(x)=3x+b$.

$f(3)-f(5)=3\times 3+b-(3\times 5+b)=9-15=-6$.

2.2. Quadratic functions

(a) The Quadratic Function

The following function is called the quadratic function:

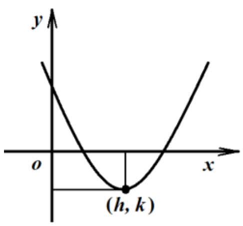
$y = f(x) = ax^2 + bx + c$ (2.1)

where a, b, and c are real numbers with $a \neq 0$.

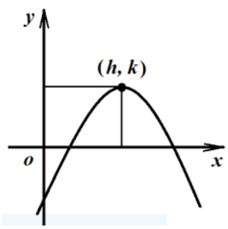
$$y = f(x) = a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} \qquad (2.2)$$

(b). Vertices

$$h=-\frac{b}{2a}, \text{ and } k=\frac{4ac-b^2}{4a} \qquad (2.3)$$



($a > 0$, $y_{\min} = k$)



($a < 0$, $y_{\max} = k$)

Example 12. Find the general expression of a quadratic function that passes through (0, 3) and (8, 3).
A. $f(x) = ax^2 - 8x + 3$ B. $f(x) = a(x^2 - 8x) + 3$ C. $f(x) = a(x^2 - 8x + 3)$
D. $f(x) = x^2 - ax + 3$ E. none of these

Solution: B.
Initially, let $f(x) = ax^2 + bx + c$. $f(0) = c = 3$. $f(8) = 64a + 8b + c = 3$
$64a + 8b = 0 \quad \Rightarrow \quad b = -8a$.
$f(x) = ax^2 + (-8a)x + 3 = a(x^2 - 8x) + 3$.

Example 13. Each spring a 12 meter × 12 meter rectangular garden has its length increased by 2 meters but its width decreased by 50 centimeters. What will be the maximum attainable area of the garden?
A. 144 m^2 B. 176 m^2 C. 189 m^2 D. 200 m^2 E. 225 m^2

Solution: E.
The area will be $(12 + 2x)(12 - 0.5x) = 144 + 24x - 6x - x^2 = 144 + 18x - x^2$, where x is the number of years. This quadratic expression has a maximum at its vertex, which occurs when $x = -b/a = -18/(-2) = 9$. The area when $x = 9$ is $144 + 18(9) - 9^2 = 225$.

Example 14. 14. If $f(x) = x^2 - 1$, and $g(x) = 2x$, find $f(g(2))$.
A. 11 B. 12 C. 13 D. 14 E. 15

Solution: E.
$g(2) = 2 \times 2 = 4$
$f(g(2)) = 4^2 - 1 = 15$.

Example 15. $f(x) = (x-3)^2$. If $f(x) = 16$ and x is negative, what is the value of x?
A. –1 B. –2 C. –3 D. –4 E. –5

Solution: A.
$(x-3)^2 = 16 \quad \Rightarrow \quad x - 3 = 4$ or $x - 3 = -4$. So $x = -1$ is the answer.

Example 16. If $f(x)=3x^2+1$, how much larger than $f(3)$ is $f(4)$?

A. 21 B. 22 C. 23 D. 24 E. 25

Solution: A.

$f(4)-f(3) = 3\times 4^2+1-(3\times 3^2+1)=21$.

Example 17. If $\{(x,h(x))$: x is an integer and $h(x)=x^2-2x+3\}$, find x so that $h(x)$ is as small as possible.

A. 1 B. 2 C. 3 D. – 1 E. – 2.

Solution: A.

$h(x)=x^2-2x+3=x^2-2x+1+2=(x-1)^2+2$.

We know that we can get the smallest value of h when $(x-1)^2=0$. So $x=1$.

Example 18. If $f(x)=3x^2+2x-k$ find the value(s) of k which $f(2)=6$.

A. 10 B. 12 C. 13 D. 14 E. 15

Solution: A.

$f(2)=3\times 2^2+2\times 2-k=6 \quad \Rightarrow \quad k=10$

Example 19. If $f(x)=3x^2-1$, find $f(2)$.

A. 11 B. 12 C. 13 D. 14 E. 15

Solution: A.

$f(2)=3\times 2^2-1=12-1=11$.

Example 20. The function $f(x)$ is defined as $f(x)=x^2-x$. For how many values of x will $f(x)=x$?

A. 1 B. 2 C. 3 D. 4 E. 5

Solution: B.

$f(x)=x \quad \Rightarrow \quad x^2-x=x \quad \Rightarrow \quad x^2-2x=0 \quad \Rightarrow \quad x(x-2)=0$.

So $x = 0$ or $x = 2$. The answer is B.

2.3. Floor functions

Definition:

$y=\lfloor x \rfloor$

(i). $\lfloor x \rfloor$ is an integer. (ii). $\lfloor x \rfloor \le x$, and (iii). $x < \lfloor x \rfloor + 1$.

$\lfloor x \rfloor$ is called the floor function. Whenever we see this notation, we take the greatest integer value not greater than x. It is also called Gaussian Function since it was introduced by Carl Friedrich Gauss in 1808 (using the square bracket notation $[x]$).

The expressions $\lfloor x \rfloor$ and $[x]$ are the same.

Example 21. Find the values of the following expressions:

(a) $\lfloor 3.14 \rfloor$. (b) $\lfloor 4.5 \rfloor$. (c) $\lfloor -0.5 \rfloor$.

Solution:

(a) $\lfloor 3.14 \rfloor = 3$ (b) $\lfloor 4.5 \rfloor = 4$ (c) $\lfloor -0.5 \rfloor = -1$ (not 0).

Example 22. How many positive integers from 992 to 1992 are multiples of 7?

A. 284 B. 143 C. 142 D. 141 E. 140.

Solution: D.

$$\left\lfloor \frac{1992}{7} \right\rfloor = 284 \text{ and } \left\lfloor \frac{991}{7} \right\rfloor = 141.$$

There are 284 – 141 =143 positive integers from 992 to 1992 that are the multiples of 7.

Example 23. In the prime factorization of 100!, what is the power of 3?

A. 33 B. 44 C. 47 D. 48 E. 100.

Solution: D.

$$\left\lfloor \frac{100}{3^1} \right\rfloor + \left\lfloor \frac{100}{3^2} \right\rfloor + \left\lfloor \frac{100}{3^3} \right\rfloor + \left\lfloor \frac{100}{3^4} \right\rfloor = 33 + 11 + 3 + 1 = 48.$$

Note that this question is solved in the same as finding how many zeros 100! ends in base 6 systems.

Example 24. $f(x) = [x]$ where "$[x]$" is "the greatest integer less than or equal to x", find the value of $\left[\dfrac{1+\frac{1}{3}}{\frac{1}{2}-1}\right]$.

A. 3 B. 4 C. 2 D. – 3 E. – 2.

Solution: D.

$$\left[\frac{1+\frac{1}{3}}{\frac{1}{2}-1}\right] = \left[\frac{\frac{4}{3}}{-\frac{1}{2}}\right] = \left[-\frac{8}{3}\right] = \left[-2-\frac{2}{3}\right] = -3$$

Example 25. The greatest integer function of x is expressed as $[x]$, and is defined to be the greatest integer less than or equal to x. Find $[\pi - 4]$.

A. 1 B. 3 C. 2 D. – 3 E. – 1.

Solution: E.

$[\pi - 4] = [\pi] - 4 = 3 - 4 = -1$.

2.4. Other nonlinear functions

Example 26. If $f(x)=4x^3-3x^2+x-10$, find $f(2)$.

A. 11 B. 12 C. 13 D. 14 E. 15

Solution: B.

$f(2)=4\times 2^3-3\times 2^2+2-10=32-12+2-10=12$.

Example 27. If $f(x)=5x^4-4x^3+3x^2-2x+1$, find $f(-1)$.

A. 11 B. 12 C. 13 D. 14 E. 15

Solution: E.

$f(-1)=5\times(-1)^4-4\times(-1)^3+3\times(-1)^2-2\times(-1)+1=5+4+3+2+1=15$.

Example 28. Given $f(x)=\dfrac{3x^2-2x+1}{2x-3}$, find $\dfrac{f(1)-f(0)}{f(2)}$.

A. $-\dfrac{5}{27}$ B. $-\dfrac{3}{7}$ C. 2 D. $\dfrac{7}{3}$ E. $\dfrac{22}{7}$

Solution: A.

$f(0)=\dfrac{0-0+1}{0-3}=-\dfrac{1}{3}$, $f(1)=\dfrac{3-2+1}{2-3}=-2$, and $f(2)=\dfrac{12-4+1}{4-3}=9$.

$$\frac{f(1)-f(0)}{f(2)}=\frac{-2-(-\frac{1}{3})}{9}=\frac{-\frac{5}{3}}{9}=-\frac{5}{27}.$$

2.5. Functional functions

Functional Equations: Equations containing unknown functions.

Example 29. If $f(2x)=\dfrac{2}{2+x}$ for all $x > 0$, then $2f(x) =$

A. $\dfrac{2}{1+x}$ B. $\dfrac{2}{2+x}$ C. $\dfrac{4}{1+x}$ D. $\dfrac{4}{2+x}$ E. $\dfrac{8}{4+x}$

Solution: E.

Let $X = 2x \quad \Rightarrow \quad x=\dfrac{X}{2}$

$$f(2x)=\frac{2}{2+x} \quad \Rightarrow \quad f(X)=\frac{2}{2+\frac{X}{2}}=\frac{2}{\frac{4}{2}+\frac{X}{2}}=\frac{4}{4+X} \Rightarrow \quad f(x)=\frac{4}{4+x}.$$

$$2f(x) = 2 \times \frac{4}{4+x}=\frac{8}{4+x}.$$

Example 30. If $f(n+1)=2f(n)+1$ for every natural number n and $f(1)=3$, find $f(11)$.

A. 4096 B. 5050 C. 4095 D. 4085 E. 4079.

Solution: C.

If $n = 1$, $f(1+1)=2f(1)+1=2\times 3+1=7$.
If $n = 2$, $f(2+1)=2f(2)+1=2\times 7+1=15$.
If $n = 3$, $f(3+1)=2f(3)+1=2\times 15+1=31$.
If $n = 4$, $f(4+1)=2f(4)+1=2\times 31+1=63$.
If $n = 5$, $f(5+1)=2f(4)+1=2\times 63+1=127$.
If $n = 6$, $f(6+1)=2f(6)+1=2\times 127+1=255$.
If $n = 7$, $f(7+1)=2f(7)+1=2\times 255+1=511$.
$f(9) = 2 \times 511 + 1 = 1023$, $f(10) = 2 \times 1023 + 1 = 2047$,
$f(11) = 2 \times 2047 + 1 = 4095$.

Example 31. If $f(n+2)=\dfrac{n}{f(n)}$ for all positive integers n and $f(2)=1$, find $f(8)$.

A. 1 B. 3 C. 2 D. 4 E. 7.

Solution: B.

If $n = 2$, $f(2+2)=\dfrac{2}{f(2)}=2$.

If $n = 4$, $f(4+2)=\dfrac{4}{f(4)}=2$.

If $n = 6$, $f(6+2)=\dfrac{6}{f(6)}=3$.

Example 32. The function $f(x)$ is called a nested function, and is defined as:

$$f(x)=\begin{cases} x-2 & \text{for } x>5 \\ x-f(f(x+3)) & \text{for } x\le 5 \end{cases}$$

Find $f(1)$.

A. 1 B. 3 C. 2 D. 4 E. 5.

Solution: A.

$f(6)=6-2=4$

$f(7)=7-2=5$

$f(8)=8-2=6$

$f(5)=5-f(f(5+3))=5-f(f(8))=5-f(6)=5-4=1$

$f(4)=4-f(f(4+3))=4-f(f(7))=4-f(5)=4-1=3$

$f(3)=3-f(f(3+3))=3-f(f(6))=3-f(4)=3-3=0$

$f(2)=2-f(f(2+3))=2-f(f(5))=2-f(1)$ (1)

$f(1)=1-f(f(1+3))=1-f(f(4))=1-f(3)=1-0=1$ (2)

The answer is 1.

Example 33. A function is defined by $f(0)=1$ and $f(n)=f(n-1)+n+1$.
Find $f(5)$.

A. 25 B. 24 C. 12 D. 21 E. 23.

Solution: D.

$f(1)=f(1-1)+1+1=1+1+1=3$

$f(2)=f(2-1)+2+1=3+2+1=6$

$f(3)=f(3-1)+3+1=6+3+1=10$

$f(4)=f(4-1)+4+1=10+4+1=15$

$f(5)=f(5-1)+5+1=15+5+1=21$.

Example 34. The function $f(x)$ is defined as :

$$f(x)=\begin{cases} x-2 & \text{for } x\le 5, \\ x+1 & \text{for } x>5 \end{cases}$$

Find $f(3)+f(5)+(7)$.

A. 25 B. 24 C. 12 D. 21 E. 23.

Solution: C.

$f(3)+f(5)+(7)=(3-2)+(5-2)+(7+1)=12$.

3. PROBLEMS

Problem 1. Which of these relations cannot be functions?

(A) $g(x) = \{(2, 3), (2, 4)\}$ (B) $f(x) = \{(5, 8), (3, 8)\}$ (C) $3x + 5y = 15$

(D) $\{(1, 3), (2, 5)\}$ (E) $h(x) = \{(3, 4), (5, 9)\}$

Problem 2. Find the value of m, expressed as a common fraction, so that this relation is not a function. $\{(-n+2, -3), (-4n+3, 8)\}$.

A. 3 B. $\frac{1}{3}$ C. $\frac{1}{5}$ D. 1 E. $\frac{2}{3}$

Problem 3. Give the letter corresponding to the relations given which is not functions.

a) $f(x) = |\frac{x}{3}|$

b) g: $x \to 2x - 1$

c) $\{(x, g(x)) : g(x) = 3\}$

d) $\{(0,0), (1,0), (2,0), (3,0)\}$

e) $\{(0,0), (0,1), (0,2), (0,3)\}$

A. a) B. b) C. c) D. d) E. e)

Problem 4. If $f(x^2) = 4 \cdot x \cdot f(x + 2) + 3$, what is the value of $f(4)$?

A. -1 B. $-\frac{3}{7}$ C. 2 D. $\frac{7}{3}$ E. $\frac{22}{7}$

Problem 5. Let $g(x) = x^2 + b \cdot x + c$ and $g(2) = -6$. Determine $g(5)$.

A. -15 B. $2c - 9$ C. $-1.5c$ D. $2.5c - 10$ E. $-4c$

Problem 6. The formula $F = \frac{9}{5}C + 32$ is used to convert temperatures from degrees Celsius (C) to degrees Fahrenheit (F). What is the number of degrees in the Fahrenheit equivalent to $20^0\ C$?

A. 40. B. 16. C. 68 D. 56 E. 65.

Problem 7. If f is a constant function, find $f(2)$ if $f(1) = 6$.

A. 1 B. 6 C. 8 D. 5 E. 2.

Problem 8. If $f(x) = 3x + 1$ find $f(7)$.

A. 10 B. 16 C. 18 D. 20 E. 22.

Problem 9. If $f(x) = 5x + 2$ and $g(x) = 2x - 5$, find $f(g(3))$.

A. 10 B. 9 C. 8 D. 7 E. 6.

Problem 10. g is a linear function with $g(5) = 0$ and $g(0) = 10$. Find $g(10)$.

A. −10 B. −15 C. 10 D. 21 E. 16.

Problem 11. Given that f is a linear function where $f(0) = 20$ and $f(4) = 0$, what is the value of $f(10)$?

A. − 70 B. −30 C. 30 D. 20 E. 10.

Problem 12. Find the vertex of a quadratic function that has a for the x^2 coefficient and x intercepts: (2, 0) and (10, 0).

A. $(6, -16a)$ B. $(6, a)$ C. $(a, 0)$ D. $(6/a, -16)$ E. none of these

Problem 13. If the vertex of the graph of $y = x^2 + 4x - 7$ is (h, k), then what is $h + k$?

A. –11 B. –12 C. –13 D. –14 E. –15

Problem 14. If $f(x) = (2x + 3)^2$ and $g(x) = \sqrt{x} - 5$, for what value(s) of x does $g(f(x)) = 8$?

A. − 8 B. 5 C. – 5 D. (5, −8) E. (–5, 8)

Problem 15. If $k(x) = x^2 + 1$ and $g(y) = y^2 - 1$, find $k(4) - g(4)$.

A. 1 B. 2 C. 3 D. 4 E. 5

Problem 16. If $g(x) = 13x^2 + 9x$ and $h(x) = 13x^2 + 8x + 2$, find the value of $h(4) - g(4)$.

A. –1 B. –2 C. –3 D. –4 E. –5

Problem 17. For what value(s) of x does the function $f(x)=(2x-5)^2$ take on its minimum value? Express your answer as a decimal.

A. 2 B. 5 C. 2.5 D. 5.2 E. – 2.5.

Problem 18. If $k(x)=5x^2$ find $k(-1)$.

A. 2 B. 5 C. 2 D. 25 E. – 5.

Problem 19. If $f(x)=3x^2+x$ and $g(x)=f(x)+2$, find $g(1)$.

A. 6 B. 2 C. 3 D. 4 E. 5

Problem 20. Find the values of the following expressions:

(a) $\lfloor 2.71 \rfloor$. (b) $\lfloor 4.1 \rfloor$. (c) $\lfloor -0.7 \rfloor$.

Problem 21. How many positive integers from 1 to 500 are the multiples of 8?

A. 61 B. 62 C. 63 D. 64 E. 65.

Problem 22. How many zeros does 238! end in?

A. 47 B. 56 C. 57 D. 58 E. 59.

Problem 23. $f(x)=[x]+|x|$ for all x such that $-2 \le x \le 2$. Find all values of x for which $f(x)=-\frac{1}{2}$. ($[x]$ is the greatest integer less than or equal to x.)

A. (1, –1) B. (–1.5, 0.5) C. (1.5, 0.5) D. (1.5, – 0.5) E. (–1.5, –0.5).

Problem 24. If $f(x)=\lfloor x \rfloor$ where $\lfloor x \rfloor$ means "the greatest integer less than or equal to x," find $f(3\sqrt{2})$.

A. 4 B. 5 C. 7 D. 8 E. 3.

Problem 25. If $k(x)=((x+1)x-3)x+2$, find $k(4)$.

A. 47 B. 74 C. 72 D. 70 E. 79.

Problem 26. If $f(x)=8x^3-6x^2-4x+5,$ find the value of $f(-2)$.

A. 71 B. 73 C. 72 D. – 73 E. – 75.

Problem 27. Give $f(x)=\dfrac{x^2+2x+1}{x^2-1}$, find $\dfrac{f(2)}{f(0)}$.

A. 1 B. 3 C. 2 D. – 3 E. – 1.

Problem 28. If $f\left(\dfrac{x+3}{2}\right)=x,$ find $f(x)$.

A. $f(x) = x - 8$ B. $f(x) = 2x + 3$ C. $f(x) = -2x + 3$
D. $f(x) = -2x - 3$ E. $f(x)=2x-3$

Problem 29. Given $f(n+2)=3f(n)+4$ and $f(0)=2,$ find the value of $f(6)$.

A. 109 B. 105 C. 106 D. 104 E. 107.

Problem 30. A function is defined by $f(0)=0$ and $f(n)=f(n-1)+2n,$ for $n>0$. Find $f(4)$.

A. 10 B. 30 C. 20 D. 40 E. 70.

Problem 31. What is $f(f(f(3)))$?

$$f(n)=\begin{cases} n^2, & \text{if } n \text{ is even;} \\ n+1, & \text{if } n \text{ is odd.} \end{cases}$$

A. 256 B. 1024 C. 512 D. 257 E. 513.

Problem 32. The function $f(n)=(1+2+\cdots+n)+2n$ so that $f(2)=7$ and $f(3)=12$. Find $f(53)$.

A. 1532 B. 1535 C. 1537 D. 1521 E. 1523.

Problem 33. The function $f_n(x)$ is the nth digit to the right of the decimal point in the decimal representation of x. For example, $f_2(\frac{1}{7}) = f_2(0.\overline{142857}) = 4$.

Find $f_5(\frac{5}{9}+\frac{2}{3})$.

A. 5 B. 4 C. 3 D. 2 E. 1.

Problem 34. (2001 NC Algebra II) Let $h(x+1) = \dfrac{3h(x)+4}{3}$ for positive integer value x and , $h(1) = -\dfrac{2}{3}$, find $h(3)$.

A. 1 B. 0 C. 5 D. 2 E. 6

Problem 35. (2003 NC Algebra II) If $f(x) = f(x - 2) + x$, and $f(7) = 11$, find $f(5)$.

A. 10 B. 8 C. 6 D. 8 E. 4

Problem 36. If $f(x) = (x + 5)^2 + 8$, then what is the sum of the values of x for which $f(x) = 12$?

A. –10. B. –7. C. 10. D. 20. E. 297.

Problem 37. Let the function f be defined by $f(x) = x^2 + 40$. If m is a positive number such that $f(2m) = 2f(m)$ which of the following is true?

A. $0 < m \le 4$ B. $4 < m \le 8$ C. $8 < m \le 12$ D. $12 < m \le 16$ E. $16 < m$

Problem 38. Function f satisfies $f(x) + 2f(5 - x) = x$ for all real numbers x. The value of $f(1)$ is

A. 7/3 B. 3/7 C. 5/2 D. 2/5 E. None of these

Problem 39. Suppose that $f(n + 1) = f(n) + f(n - 1)$ for $n = 1, 2, \ldots$. Given that $f(6) = 2$ and $f(4) = 8$, what is $f(3) + f(5)$?

A. −18 B. −19 C. −20 D. −21 E. −22

Problem 40. The function $A(x, y)$ is defined by the following rules:

1) $A(0,n) = n+1$

2) $A(m,0) = A(m-1,1)$

3) $A(m,n) = A(m-1, A(m,n-1))$

4) m and n are natural numbers

If $A(2,3) = k$, where k is a whole number, find the value of k.

A. 5 B. 6 C. 7 D. 8 E. 9.

4. SOLUTIONS

Problem 1. Solution: A.

	A	B	C	D	E
x	(2), (2)	(5), (3)	(0), (5)	(1), (2)	(3), (5)
y	(3), (4)	(8), (8)	(5), (0)	(3), (5)	(4), (9)

We see that $g(x)$ is not a function because one value of x has two different values of y.

Problem 2. Solution: B.

When $-n+2=-4n+3$, or $4n-n=3-2$ or $n=\frac{1}{3}$, this relation is not a function.

Problem 3. **Solution:** E.

We see that e) is not a function because one value of x has four different values of y.

Problem 4. **Solution:** B.

$f(2^2) = 4 \cdot 2 \cdot f(2+2) + 3 \Rightarrow f(4) = 4 \cdot 2 \cdot f(2+2) + 3 \Rightarrow f(4) = 4 \cdot 2 \cdot f(4) + 3$
$\Rightarrow f(4) = -3/7$.

Problem 5. **Solution:** C.

$g(2) = -6 = (2)^2 + b\cdot 2 + c \Rightarrow -10 - c = 2b \Rightarrow b = -5 - c/2$.
$g(5) = (5)^2 + b\cdot 5 + c = 25 + 5(-5 - c/2) + c = -5c/2 + c = -1.5c$.

Problem 6. Solution: C.

$$F=\frac{9}{5}\cdot C+32=\frac{9}{5}\cdot 20+32=68.$$

Problem 7. Solution: B.

Since f is a constant function. $f(2) = f(1) = 6$.

Problem 8. Solution: E.
$f(x)=3x+1 \quad \Rightarrow \qquad f(7)=3\times 7+1=22$.

Problem 9. Solution: D.
$\mathrm{g}(3)=2\times 3-5=6-5=1$
$f(g(3)) = f(1)=5\times 1+2=7$.

Problem 10. Solution: A.
Let $g(x)=ax+b$.
Since $g(0)=10$, $g(0)=a\times 0+b=10 \qquad \Rightarrow \qquad b=10$.
Since $g(5)=0$, $g(5)=a\times 5+b=0 \quad \Rightarrow \qquad a=-\dfrac{b}{5}=-2$.
So $g(x)=ax+b=-2x+10$ and $g(10) = -2\times 10+10=-10$.

Problem 11. Solution: B.
Let $f(x)=ax+b$.
Since $f(0)=20$, $f(0)=a\times 0+b=20 \qquad \Rightarrow \qquad b=20$.
Since $f(4)=0$, $f(4)=a\times 4+b=0 \quad \Rightarrow \qquad a=-\dfrac{b}{4}=-5$.
So $g(x)=ax+b=-5x+20$ and $f(10) = -5\times 10+20=-30$.

Problem 12. Solution: A.
Knowing the 2 roots and the constant multiplier, we can say that the form of the quadratic function is $f(x) = a(x-2)(x-10)$ This expands to $ax^2 - 12ax + 20a$. The x-value of the vertex is $-b/2a$, which is $-(-12a)/2a = 6$. $f(6) = -16a$, making the vertex $(6, -16a)$.

Problem 13. Solution: C.
By (3.3), $h=-\dfrac{b}{2a}$, and $k=\dfrac{4ac-b^2}{4a}$.

$$h+k=-\frac{b}{2a}+\frac{4ac-b^2}{4a}=-\frac{4}{2}+\frac{4\cdot(-7)-4^2}{4}=-2-11=-13.$$

Problem 14. Solution: B.

$g(f(x))=8 \quad \Rightarrow \quad \sqrt{(2x+3)^2}-5=8 \quad \Rightarrow \quad \sqrt{(2x+3)^2}=8+5=13$

$\Rightarrow \quad (2x+3)^2=13^2$.

So $2x+3=13 \quad \Rightarrow x=5$

or $2x+3=-13 \quad \Rightarrow x=-8$ (ignored since $x\ge 0$).

Problem 15. Solution: B.

$k(4)-g(4)=4^2+1-(4^2-1)=2.$

Problem 16. Solution: B.

$h(x)-g(x)=13x^2+8x+2-(13x^2+9x)=2-x$

$h(4)-g(4)=2-4=-2$.

Problem 17. Solution: C.

We see that $(2x-5)^2$ is a square number. We know that $a^2\ge 0$ for any real number a. Thus $f(x)=(2x-5)^2$ take on its minimum value when $(2x-5)^2=0$, or $x=2.5$.

Problem 18. Solution: B.

$k(-1)=5\times(-1)^2=5$.

Problem 19. Solution: A.

$g(1)=f(1)+2=(3\times 1^2+1)+2=6$.

Problem 20. Solution:

(a) $\lfloor 2.71\rfloor=2$ (b) $\lfloor 4.1\rfloor=4$ (c) $\lfloor -0.7\rfloor=-1$.

Problem 21. Solution: B.

$\left\lfloor \frac{500}{8} \right\rfloor = \lfloor 62.5 \rfloor = 62$.

Problem 22. Solution: C.

Since in order to end in zero, a number must be divisible by 10 = 5 × 2, each zero is the result of the number having a factor of both 5 and 2. Since every other number is even there are plenty of 2's as factors, so we need to determine how many factors of 5 there are.

$$\left\lfloor \frac{238}{5^1} \right\rfloor + \left\lfloor \frac{238}{5^2} \right\rfloor + \left\lfloor \frac{238}{5^3} \right\rfloor = 47 + 9 + 1 = 57$$

There are 57 factors of 5 in 238!, therefore it ends in 57 zeros.

Problem 23. Solution: E.

We know that $-2 \le x \le 2$.

If $x = -0.5$, we have $[x] + |x| = -\frac{1}{2} \quad \Rightarrow \quad [-0.5] + |-0.5| = -1 + 0.5 = -\frac{1}{2}$.

If $x = -1.5$, we have $[x] + |x| = -\frac{1}{2} \quad \Rightarrow \quad [-1.5] + |-1.5| = -2 + 0.5 = -\frac{1}{2}$.

Problem 24. Solution: A.

$f(3\sqrt{2}) = \lfloor 3\sqrt{2} \rfloor = \lfloor 3 \times 1.414 \rfloor = 4$.

Problem 25. Solution: D.

$k(4) = ((4+1)4 - 3) \times 4 + 2 = (20 - 3) \times 4 + 2 = 68 + 2 = 70$.

Problem 26. **Solution:** E.

$f(-2) = 8 \times (-2)^3 - 6 \times (-2)^2 - 4 \times (-2) + 5 = -64 - 24 + 8 + 5 = -75$.

Problem 27. Solution: D.

$$f(0)=\frac{0+0+1}{0-1}=-1$$

$$f(2)=\frac{2^2+2\times 2+1}{2^2-1}=\frac{9}{3}=3.$$

$$\frac{f(2)}{f(0)}=\frac{3}{-1}=-3.$$

Problem 28. Solution: E.

Let $X=\frac{x+3}{2}\Rightarrow \qquad x=2X-3$

$f\left(\frac{x+3}{2}\right)=x \quad\Rightarrow \qquad f(X)=2X-3\Rightarrow \qquad f(x)=2x-3.$

Problem 29. Solution: C.

If $n=0$, $f(0+2)=3f(0)+4=3\times 2+4=10$.

If $n=2$, $f(2+2)=3f(2)+4=3\times 10+4=34$.

If $n=4$, $f(4+2)=3f(4)+4=3\times 34+4=106$.

Problem 30. Solution: C.

If $n=1$, $f(1)=f(1-1)+2=2$.

If $n=2$, $f(2)=f(2-1)+2\times 2=f(1)+4=2+4=6$.

If $n=3$, $f(3)=f(3-1)+2\times 3=f(2)+6=6+6=12$.

If $n=4$, $f(4)=f(4-1)+2\times 4=f(3)+8=12+8=20$.

Problem 31. Solution: A.

$f(3)=3+1=4$

$f(f(3))=f(4)=4^2=16$

$f(f(f(3)))=f(16)=16^2=256.$

Problem 32. Solution: C.

$f(n) = (1+2+\cdots+n)+2n = \frac{(1+n)n}{2}+2n = \frac{(1+n)n+4n}{2} = \frac{(5+n)n}{2}$.

$f(53) = \frac{(5+n)n}{2} = \frac{(5+53)53}{2} = 1537$.

Problem 33. Solution: D.

$\frac{5}{9}+\frac{2}{3}=\frac{11}{9}=1.\bar{2}$.

$f_5(\frac{5}{9}+\frac{2}{3})$ = 2.

Problem 34. Solution: D.

$h(x+1)=\frac{3h(x)+4}{3}=h(x)+\frac{4}{3}$, so each term is just four-thirds greater than the one before it. So $h(1)=-\frac{2}{3}$, $h(2)=-\frac{2}{3}+\frac{4}{3}=\frac{2}{3}$, and $h(3)=\frac{2}{3}+\frac{4}{3}=2$.

Problem 35. Solution: E.

Since $f(x) = f(x - 2) + x$, $f(7) = f(5) + 7$ $\Rightarrow$ $11 = f(5) + 7$ $\Rightarrow f(5) = 4$.

Problem 36. Solution: A.

$12 = (x + 5)^2 + 8$ $\Rightarrow$ $4= (x+5)^2$ $\Rightarrow$ $\pm 2 = (x + 5)$, so $x = -3$ or $x = -7$.

So $-3 + (-7) = -10$.

Problem 37. Solution: B.

Note that $f(2m) = (2m)^2 + 40 = 4m^2 + 40$ and $2f(m) = 2(m^2 + 40) = 2m^2 + 80$. It follows that $2m^2 = 40$, so $4 < m \le 8$.

Problem 38. Solution: A.

We know that $f(x) + 2f(5 - x) = x$.

$f(1) + 2f(5 - 1) = 1$ (1)

$f(4) + 2f(5 - 4) = 4$ (2)

(2) × 2 – (1): $3f(1) = 7$ $\Rightarrow$ $f(1) = 7/3$.

Problem 39. Solution: C.

$f(6) = f(5) + f(4) = f(4) + f(3) + f(3) + f(2) = 5f(2) + 3f(1) = 2$ (1)

$f(4) = f(3) + f(2) = f(2) + f(1) + f(2) = 2f(2) + f(1) = 8$ (2)

(2) × 3 – (1): $f(2) = 22$. Thus $f(1) = 8 - 44 = -36$.

$f(3) + f(5) = f(2) + f(1) + f(4) + f(3) = f(2) + f(1) + f(3) + f(2) + f(2) + f(1)$
$= 4f(2) + 3f(1) = 4 \times 22 + 3 \times (-36) = -20$.

Problem 40. Solution: E.

$A(2,3) = A(2-1, A(2,3-1)) = A(1, A(2,2))$ (1)

$A(2,2) = A(2-1, A(2,2-1)) = A(1, A(2,1))$ (2)

$A(2,1) = A(2-1, A(2,1-1)) = A(1, A(2,0))$ (3)

$A(2,0) = A(2-1,1) = A(1,1)$. (4)

$A(1,1) = A(1-1, A(1,1-1)) = A(0, A(1,0))$ (5)

$A(1,0) = A(1-1,1) = A(0,1)$

$A(0,1) = 1+1 = 2$.

Substituting the value for $A(0,1)$ into (5): $A(1,1) = A(0,2) = 2+1 = 3$

So we know that $A(2, 0) = A(1, 1) = 3$

Substituting the value for $A(2,0)$ into (3): $A(2,1) = A(1,3) = A(0, A(1,2))$ (6)

$A(1,2) = A(1-1, A(1,2-1)) = A(0, A(1,1)) = A(1,1) + 1 = 3+1 = 4$

So $A(2,1) = A(1,3) = A(0, A(1,2)) = A(0,4) = 4+1 = 5$.

Substituting the value for $A(2,1)$ into (2):

$A(2,2) = A(1,5) = A(1-1, A(1,5-1)) = A(0, A(1,4))$

$A(1,4) = A(1-1, A(1,4-1)) = A(0, A(1,3)) = A(1,3) + 1$
$= A(0, A(1,2)) + 1 = A(0,4) + 1 = 6$.

So $A(2,2) = A(0, A(1,4)) = A(0,6) = 6+1 = 7$.

Therefore $A(2,3) = A(1, A(2,2)) = A(1,7) = A(1-1, A(1,7-1)) = A(0, A(1,6))$
$= A(1,6) + 1$.

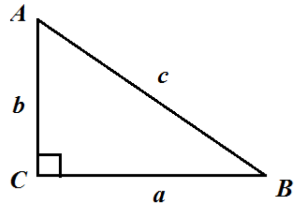
Here $A(1,6) = A(1-1, A(1,6-1)) = A(0, A(1,5)) = A(1,5) + 1 = A(0, A(1,4)) + 1$
$= A(1,4) + 1 + 1 = A(1,4) + 2 = 6+2 = 8$.

So $A(2,3) = A(1,6) + 1 = 8+1 = 9$.

1. PYTHAGOREAN THEOREM

(For right triangles only): $a^2 + b^2 = c^2$ (1.1)

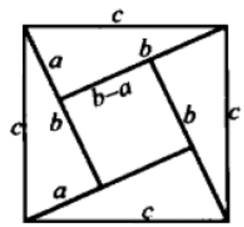
(a and b are two legs. c is the hypotenuse).



Proof:

Method 1: (Chinese way):

Arrange four congruent right triangles to form a square as show in the figure.



The area of the four triangles is $\frac{1}{2}a \times b \times 4 = 2ab$ (1)

The area of the smaller square is $(b-a)^2 = b^2 - 2ab + a^2$ (2)

The area of the large square is c^2 (3)

(3) = (1) + (2)

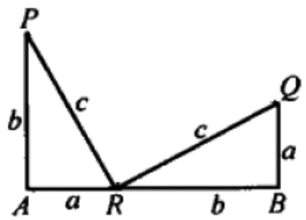
$$c^2 = \frac{1}{2}a \times b \times 4 + (b-a)^2 = 2ab + b^2 - 2ab + a^2 \quad \Rightarrow \quad c^2 = a^2 + b^2$$

Method 2: (U.S. President Garfield's way):

Arrange two congruent right triangles as show in the figure.

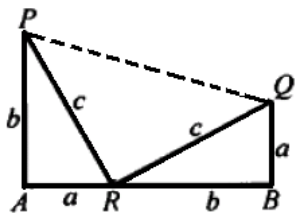
Connect PQ. Quadrilateral $ABQP$ is a trapezoid.

Since $\angle PRA + \angle QRB = 90°$, so $\angle PRQ = 90°$ and ΔPRQ is a right triangle.



The two bases of the trapezoid are a and b.

The height is ($a + b$). The area of trapezoid $ABQP$ is



$$S_{ABQP} = \frac{(a+b)\times(a+b)}{2} \quad (1)$$

The sum of the areas of three right triangles

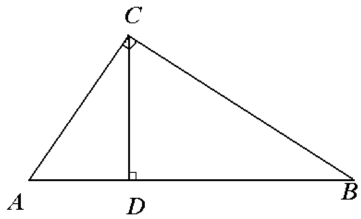
$$S = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2 \quad (2)$$

We know that (1) = (2).

$$\frac{(a+b)\times(a+b)}{2}=\frac{1}{2}ab+\frac{1}{2}ab+\frac{1}{2}c^2 \quad\Rightarrow\quad a^2+2ab+b^2=ab+ab+c^2 \Rightarrow$$
$$c^2=a^2+b^2$$

Method 3 (the simplest way).

In right triangle ABC, draw $CD \perp AB$. From lecture 27, we know that $\Delta ABC \sim \Delta ACD \sim \Delta CBD$ and:



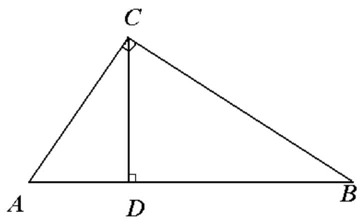
$$AC^2 = AB \times AD \qquad (1)$$
$$BC^2 = AB \times BD \qquad (2)$$

(1) + (2):

$$AC^2+BC^2 = AB\times AD + AB\times BD = AB(AD+BD) = AB\times AB = AB^2$$

Method 4:

In right triangle ABC, draw $CD \perp AB$. From the lecture 26, we know that $\Delta ABC \sim \Delta ACD \sim \Delta CBD$.



We also know that $S_{\Delta CBD} + S_{\Delta ACD} = S_{\Delta ABC} \quad\Rightarrow$

$$\frac{S_{\Delta CBD}}{S_{\Delta ABC}}+\frac{S_{\Delta ACD}}{S_{\Delta ABC}}=1$$

$$\left(\frac{BC}{AB}\right)^2+\left(\frac{AC}{AB}\right)^2=1 \quad\Rightarrow\quad \left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1$$

$$\Rightarrow \qquad a^2+b^2=c^2$$

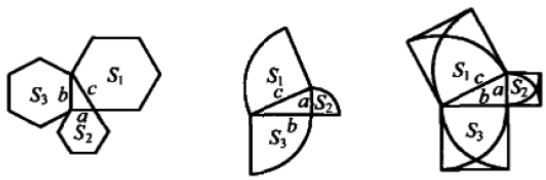
2. SOME THEOREMS

Theorem 1.

When we draw similar figures on the two legs and the hypotenuse of a right triangle, the following formula is true:

$$S_1 = S_2 + S_3 \qquad (2.1)$$

Proof:



$$\frac{S_2}{S_1} = \left(\frac{a}{c}\right)^2 \qquad (1)$$

$$\frac{S_3}{S_1} = \left(\frac{b}{c}\right)^2 \qquad (2)$$

$$S_2 + S_3 = \frac{a^2 + b^2}{c^2} S_1 = S_1 .$$

Theorem 2.

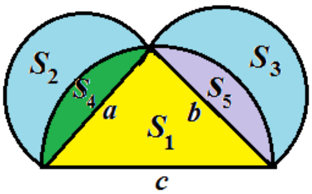
Draw semicircles along the sides of a right triangle, using the sides of the right triangle as the diameters, the following relationship is true:

$$S_1 = S_2 + S_3 \qquad (2.2)$$

Proof:

Since triangle ABC is a right triangle, we have:

$$a^2 + b^2 = c^2 \qquad (1)$$



Multiplying every term of (1) by $\frac{1}{2}\pi\left(\frac{1}{2}\right)^2$:

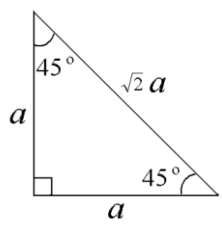
$$\frac{1}{2}\pi\left(\frac{a}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 = \frac{1}{2}\pi\left(\frac{c}{2}\right)^2 \qquad (2)$$

Subtract $(S_4 + S_5)$ in each side of (2):

$$\frac{1}{2}\pi\left(\frac{a}{2}\right)^2 - S_4 + \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 - S_5 = \frac{1}{2}\pi\left(\frac{c}{2}\right)^2 - S_4 - S_5 \qquad \Rightarrow \qquad S_2 + S_3 = S_1 .$$

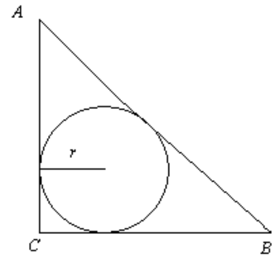
Theorem 3.

In a $45^{\circ} - 45^{\circ} - 90^{\circ}$ right triangle, or *right isosceles* triangle, the length of the hypotenuse is $\sqrt{2}$ times of the length of each leg.



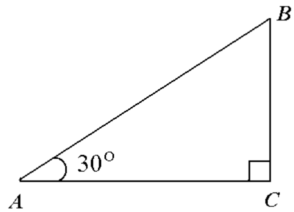
Theorem 4. If triangle ABC is a right triangle, then the radius of the inscribed circle can be calculated by:

$$r = \frac{AC + BC - AB}{2} \qquad (2.3)$$



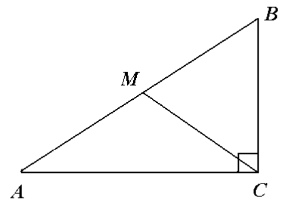
Theorem 5. For a right triangle, if $\angle A = 30°$, then

$$BC = \frac{1}{2} AB$$



Theorem 6. The length of the median to the hypotenuse of a right triangle equals one-half the length of the hypotenuse.

$AM = MB = MC$



3. PYTHAGOREAN TRIPLES

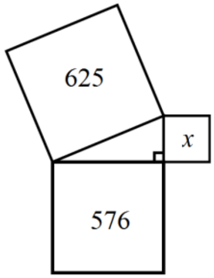
A Pythagorean triple is an ordered triple (a, b, c) of three positive integers such that $a^2 + b^2 = c^2$. If $a, b,$ and c are relatively prime, then the triple is called primitive.

Integral values of a, b, and c, where a, b, and c are relatively prime:

a	b	c	a	b	c
3	4	5	5	12	13
8	15	17	7	24	25
20	21	29	12	35	37
9	40	41	11	60	61
13	84	85	15	112	113

☆**Example 1.** Given the areas of the three squares in the figure, what is the area of the interior triangle?

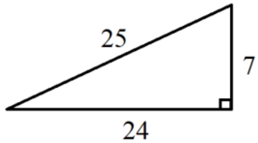
(A) 13 (B) 300 (C) 60 (D) 84 (E) 100



Solution: D.

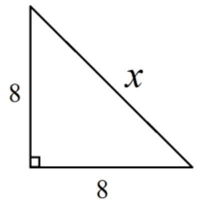
The interior triangle is a 7 – 24 – 25 right triangle.

$A = \frac{1}{2} \times 24 \times 7 = 84$.



Example 2. Find x.

(A) $8\sqrt{2}$ (B) $16\sqrt{2}$ (C) 8 (D) $10\sqrt{2}$ (E) 9



Solution: A.

By Theorem 3: $c = \sqrt{2}a = 8\sqrt{2}$

Example 3. The measure of the hypotenuse of the right triangle is 10 cm. Semicircles are drawn on the sides of the triangle as shown. Find the number of square centimeters in the sum of the shaded areas.

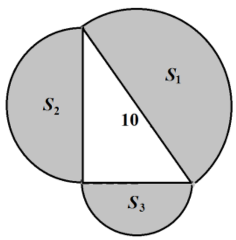
(A) 10π (B) 14π (C) 25π (D) 16π (E) 20π

Solution: C.

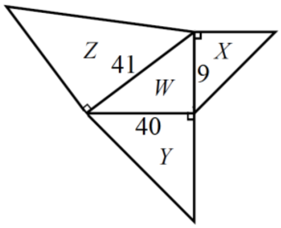
By Theorem 1, $S_1 = S_2 + S_3$.

We see that $S_1 = \frac{\pi r^2}{2} = \frac{\pi \times 5^2}{2}$

The shaded area is $S_1 + S_2 + S_3 = 2S_1 = 2 \times \frac{\pi \times 5^2}{2} = 25\pi$.



☆**Example 4.** Right isosceles triangles are constructed on the sides of a triangle of sides 9, 40, and 41 as shown. A capital letter represents the area of each triangle. Which one of the following is true?



(A) $X + Z = W + Y$ (B) $W + X = Z$

(C) $3X + 4Y = 5Z$ (D) $X + W = \frac{1}{2}(Y + Z)$

(E) $X + Y = Z$

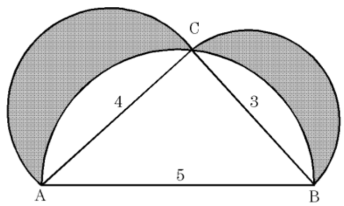
Solution: E.

Note that the triangle with the sides of 9, 40, and 41 is a right triangle ($9^2 + 40^2 = 41^2$).

By the formula (2.1), $X + Y = Z$.

Example 5. Consider the region formed by the intersections of three semi-circles whose diameters, of lengths 3, 4, and 5, form a triangle. What is the area of the region (dark shading in figure)?

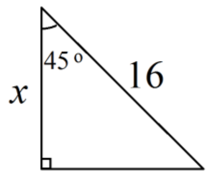
A. 6 B. 12 C. 3π D. 6π E. $(\sqrt{12} - \sqrt{5})\pi$).



Solution: A.

By Theorem 2, the shaded areas are the same as the area of the right triangle *ABC*. The answer is 6.

Example 6. Find x.



(A) $8\sqrt{2}$ (B) $16\sqrt{2}$ (C) 16 (D) $10\sqrt{2}$ (E) 8

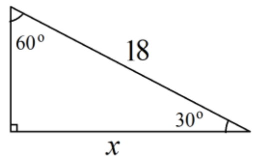
Solution: $8\sqrt{2}$

By Theorem 3: $c = \sqrt{2}a \quad \Rightarrow \quad 16 = \sqrt{2}a \quad \Rightarrow$

$$a = \frac{16}{\sqrt{2}} = \frac{16\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{16}{2}\sqrt{2} = 8\sqrt{2}$$

Example 7. Find x.

(A) $8\sqrt{2}$ (B) $9\sqrt{3}$ (C) 9 (D) $10\sqrt{2}$ (E) 9



Solution: B.

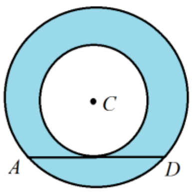
By Theorem 5: $y = \frac{1}{2} \times 18 = 9$

By Pythagorean Theorem $c^2 = a^2 + b^2 \Rightarrow \quad 18^2 = x^2 + 9^2$

$16 = \sqrt{2}a \quad \Rightarrow \quad x = \sqrt{18^2 - 9^2} = \sqrt{243} = 9\sqrt{3}$

☆**Example 8.** The two circles as shown have the same center *C*. Chord *AD* is tangent to the inner circle at *B*. Find the shaded area if $AD =$ 24.

(A) 121π (B) 144π (C) 25π (D) 169π (E) 24π

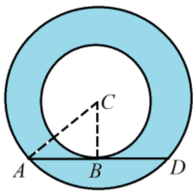


Solution: B.

Connect *AB*. Draw $CB \perp AD$ at *B*.

We know that triangle *ABC* is a right triangle. Applying Pythagorean Theorem we get $AC^2 - BC^2 = AB^2$ (1)

S, the shaded area = the area of the larger circle – the area of the smaller circle:



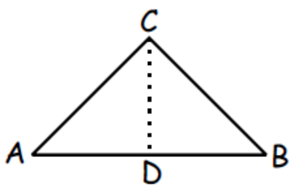
$S = \pi \times AC^2 - \pi \times BC^2 = \pi(AC^2 - BC^2)$ (2)

Substituting (1) into (2):

$S = \pi \times AB^2 = \pi \times 12^2 = 144\pi$.

Example 9. Triangle *ABC* is isosceles with *AC* = *BC*. Angle A measures 45 degrees. Segment *CD* is the perpendicular bisector of segment *AB*. If segment *AD* measures three meters, how long is segment *AC*?

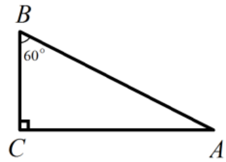
(A) $3\sqrt{2}$ (B) 6 (C) $2\sqrt{2}$ (D) $\sqrt{3}$ (E) 3



Solution: A.

Since $\angle DCA = 45°$ and $CD \perp AD$, $\angle DCA = 45°$.
$AD = DC$.
By Theorem 3: $AC = \sqrt{2}AD = 3\sqrt{2}$

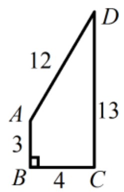
Example 10. If $m\angle B = 60°$ in right triangle ABC, and $BC = \sqrt{3}$, find AC.
(A) $2\sqrt{2}$ (B) $2\sqrt{3}$ (C) 3 (D) $3\sqrt{2}$ (E) 9



Solution: C.
We know that $\angle A = 30°$. $AB = 2BC = 2\sqrt{3}$

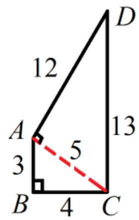
$$AC^2 + BC^2 = AB^2 \quad \Rightarrow \quad AC^2 = AB^2 - BC^2 = \left(2\sqrt{3}\right)^2 - \left(\sqrt{3}\right)^2 = 9 \;\Rightarrow\; AC = 3.$$

Example 11. As shown in the figure, $\angle B = 90°$, $AB = 3$, $BC= 4$, $CD =13$, $AD =12$. Find the area of quadrilateral $ABCD$.
(A) 32 (B) 33 (C) 34 (D) 36 (E) 30

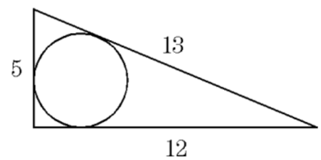


Solution: D.
ΔACD is a right triangle and $\angle DAC$=90°.
Area of $ABCD$ = Area of ΔABC + Area of ΔACD

$$\frac{1}{2}\times 3\times 4+\frac{1}{2}\times 12\times 5 = 6 + 30 = 36 \text{ square units.}$$



Example 12. In the figure shown, a circle is inscribed in a right triangle with sides of length 5, 12, 13. The radius of the circle is
A. π B. 12/5 C. $\sqrt{8}$ D. 2 E. $\sqrt{3}$



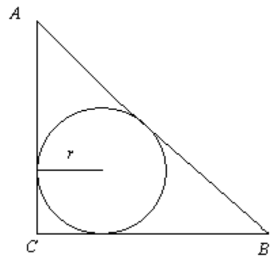
Solution: D.

By Theorem 4: $r = \dfrac{12+5-13}{2} = 2$.

Example 13. In triangle ABC, $\angle C = 90°$. $AC = 6$, $BC = 8$. Find the area of the regions outside the circle but inside the triangle.

(A) $12\sqrt{2}$ (B) $24 - 4\pi$ (C) 24 (D) $24 - 2\pi$ (E) 14

Solution: B.



Since $AC = 6$ and $BC = 8$, $AB = 10$.

By Theorem 4: $r = \dfrac{AC + BC - AB}{2} = \dfrac{6+8-10}{2} = 2$

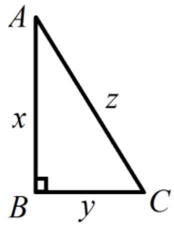
The area of the circle is $\pi r^2 = 4\pi$

The area of the triangle is $\dfrac{AC \times BC}{2} = \dfrac{6 \times 8}{2} = 24$

The answer is $24 - 4\pi$.

Example 14. Right triangle ABC with $AC = 2$, $AB + BC = \sqrt{6}$. Find the area of ΔABC.

(A) $3\sqrt{2}$ (B) 6 (C) 1 (D) 2 (E) 1/2



Solution: E.

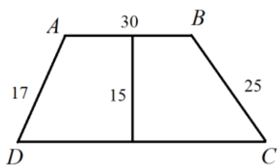
$(x+y)^2 = \left(\sqrt{6}\right)^2 \quad \Rightarrow \quad x^2 + 2xy + y^2 = 6$

By the Pythagorean Theorem: $x^2 + y^2 = 4$

So $2xy = 2 \quad \Rightarrow \quad \dfrac{xy}{2} = \dfrac{1}{2}$.

☆**Example 15.** Quadrilateral $ABCD$ is a trapezoid, $AD = 17$, $AB = 30$, $BC = 25$, and the altitude is 15. Find the area of the trapezoid.

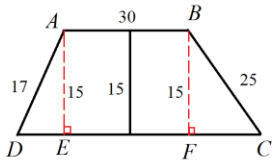
(A) 440 (B) 550 (C) 660 (D) 1320 (E) 450



Solution:

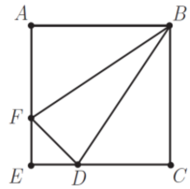
Let E and F be the feet of the perpendicular from A and B

to DC, respectively. Applying Pythagorean Theorem to right ΔAED, $DE^2 = AD^2 - AE^2 = 17^2 - 15^2 = 8^2$. So DE = 8.



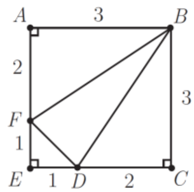
Applying Pythagorean Theorem to right ΔBFC,
$CF^2 = BC^2 - BF^2 = 25^2 - 15^2 = 20^2$.
So CF = 20. Thus DC = 8 + 30 + 20 = 58.
The trapezoid has area $\dfrac{30+58}{2}\times 15 = 660$.

☆**Example 16.** (2008 AMC 8 Problem 23) In square $ABCE$, $AF = 2FE$ and $CD = 2DE$. What is the ratio of the area of ΔBFD to the area of square $ABCE$?



(A) 1/6 (B) 2/9 (C) 5/18 (D) 1/3 (E) 7/20

Solution: (C).
Method 1 (official solution):



Because the answer is a ratio, it does not depend on the side length of the square. Let AF = 2 and FE = 1. That means square $ABCE$ has side length 3 and area 3^2 = 9 square units. The area of ΔBAF is equal to the area of ΔBCD = $\dfrac{1}{2}\times 3\times 2 = 3$ square units.
Triangle DEF is an isosceles right triangle with leg lengths $DE = FE$ = 1. The area of ΔDEF is $\dfrac{1}{2}\times 1\times 1 = \dfrac{1}{2}$ 2 square units. The area of ΔBFD is equal to the area of the square minus the areas of the three right triangles: 9 − (3 + 3 + 1/2) = 5/2 . So the ratio of the area of ΔBFD to the area of square $ABCE$ is 5/2 / 9 = 5/18.

Method 2 (our solution):
$BF = \sqrt{2^2+3^2} = \sqrt{13}$, $BD = \sqrt{2^2+3^2} = \sqrt{13}$, $BD = \sqrt{1^2+1^2} = \sqrt{2}$.
$$s = \frac{\sqrt{13}+\sqrt{13}+\sqrt{2}}{2} = \sqrt{13} + \frac{\sqrt{2}}{2}.$$

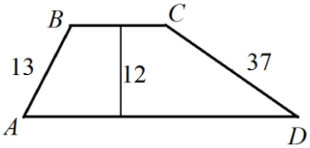
By the Heron's Formula, the area of ΔBFD =

$$\sqrt{(\sqrt{13}+\frac{\sqrt{2}}{2})(\sqrt{13}+\frac{\sqrt{2}}{2}-\sqrt{13})(\sqrt{13}+\frac{\sqrt{2}}{2}-\sqrt{13})(\sqrt{13}+\frac{\sqrt{2}}{2}-\sqrt{2})}=\frac{5}{2}.$$

So the ratio of the area of ΔBFD to the area of square $ABCE$ is 5/2 / 9 = 5/18.

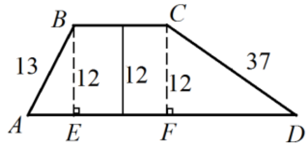
☆**Example 17.** The area of trapezoid $ABCD$ is 318 cm^2. The altitude is 12 cm, AB is 13 cm, and CD is 37 cm. What is BC, in centimeters?

(A) 13 (B) 10 (C) 12 (D) 15 (E) 6.5



Solution: E.

Let E and F be the feet of the perpendicular from B and C to AD, respectively. Applying Pythagorean Theorem to right ΔABE, $AE^2 = AB^2 - BE^2 = 13^2 - 12^2 = 5^2$. So AE = 5.



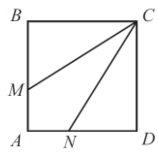
Applying Pythagorean Theorem to right ΔDCF,

$DF^2 = DC^2 - CF^2 = 37^2 - 12^2 = 35^2$. So DF = 35.

The trapezoid has area $\frac{BC+AD}{2}\times 12 = 318 \quad\Rightarrow\quad BC+AE+EF+DF=53$

$\Rightarrow\quad BC+5+BC+35=53 \Rightarrow \quad BC=6.5$.

☆**Example 18.** Square $ABCD$ has sides of length 3. Segments CM and CN divide the square's area into three equal parts. How long is the segment connecting M and N?

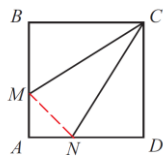


Solution: $\sqrt{2}$.

One-third of the square's area is 3, so triangle MBC has area 3 =

$\frac{1}{2}\times MB\times BC \quad\Rightarrow\quad MB = 2.$

So $AM = AN = 1$.

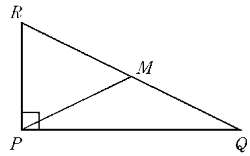


Applying Pythagorean Theorem to right ΔAMN,

$MN^2 = AM^2 + AN^2 = 1^2 + 1^2 = 2$. So $MN = \sqrt{2}$.

Example 19. If ΔQPR is a right triangle where M is the midpoint of RQ, and $MP = 5$, the length of RQ is:

A. 5 B. 10 C. $5\sqrt{2}$ D. $10\sqrt{2}$ E. 15



Solution: B.

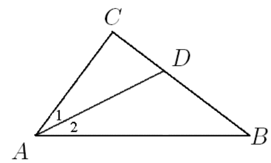
We know that M is the midpoint of RQ

By Theorem 6, $RM = MQ = MP$.

The answer is $RQ = RM + MQ = 2MP = 10$.

Example 20. In triangle ABC, $\angle C = 90°$. $\angle 1 = \angle 2$. $CD = 15$ mm, $BD = 25$ mm. Find AC.

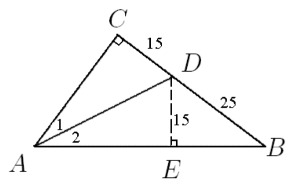
(A) $12\sqrt{2}$ (B) 25 (C) 40 (D) 30 (E) 50



Solution: D.

Method 1: By the angle bisector theorem, we have:

$$\frac{AC}{CD} = \frac{AB}{BD} \quad \Rightarrow \quad \frac{AC}{15} = \frac{AB}{25}$$



$$AB = \frac{AC}{15} \times 25 = \frac{5}{3} AC .$$

$\angle C = 90°$. By the Pythagorean theorem:

$$AC^2 + BC^2 = AB^2 \quad \Rightarrow \quad AB^2 - AC^2 = BC^2$$

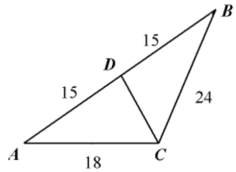
$$\Rightarrow \left(\frac{5}{3} AC\right)^2 - AC^2 = (15+25)^2 \Rightarrow \quad \frac{16}{9} AC^2 = 40^2 \Rightarrow \quad AC = 30.$$

Method 2:

Draw $DE \perp AB$ and meets AB at E. ΔCAD and ΔAED are congruent. $DE = CD = 15$ mm. ΔDBE is a $15 - 20 - 25$ right triangle and is similar to ΔABC.

$$\frac{AC}{CB} = \frac{DE}{EB} \quad \Rightarrow \quad \frac{AC}{15+25} = \frac{15}{25} \quad \Rightarrow \quad AC = 30.$$

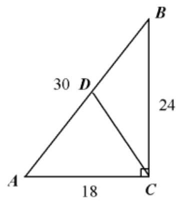
Example 21. Triangle ABC has sides AC, BC and AB measuring 18, 24 and 30 units, respectively. If D is the midpoint of segment AB, what is the length of segment CD?



(A) $10\sqrt{2}$ (B) 30 (C) 15 (D) 24 (E) 33

Solution: C.

Note that three sides of the triangle is a Pythagorean triple.



By Theorem 6, $CD = AD = DB = \frac{1}{2}AB = 15$.

Example 22. A right triangle with integer side lengths a, b, and c satisfies $a < b < c$ and $a + c = 49$. What is the area of the right triangle?

A. 176. B. 210. C. 224. D. 225. E. 232.

Solution: B.

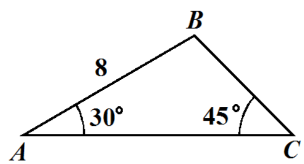
We have the following Pythagorean Triples:

a	b	c
20	21	29
12	35	37

The area of the right triangle is then $\frac{1}{2}\times 20\times 21 = \frac{1}{2}\times 12\times 35 = 210$.

Example 23. Triangle ABC is shown with measures indicated. BC equals:

A. 4 B. $4\sqrt{2}$ C. $4\sqrt{3}$ D. 5 E. $3\sqrt{5}$



Solution: B.

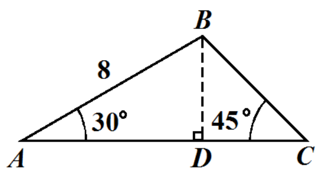
We draw the height from B to AC to meet AC at D.

ΔABD is a 30-60-90 right triangle.

By Theorem 2, $BD = 4$.

ΔCBD is an isosceles right triangle. So $DC = 4$.

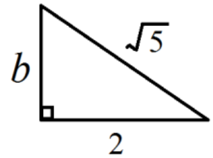
By Theorem 3, $BC = 4\sqrt{2}$.



4. PROBLEMS

Problem 1. What is the length of side b in the figure shown?

A. 1 B. 2 C. 3 D. 4 E. 5



Problem 2. The sides of a right triangle have lengths $x - y$, x, and $x + y$ where $x > y > 0$. The ratio of x to y is:

A. 3 : 2 B. 2 : 1 C. 3 : 1 D. 4 : 1 E. 4 : 3

Problem 3. Given a right triangle with an altitude drawn to the hypotenuse If this altitude divides the hypotenuse into segments measuring 16 and 4 inches, the length of this altitude is:

A. 12 inches B. 8 inches C. $4\sqrt{5}$ inches D. 10 inches E. none of the above

Problem 4. The diagonal of a rectangle is 13 inches. The width of the rectangle is 5 inches. What is the length of the rectangle?

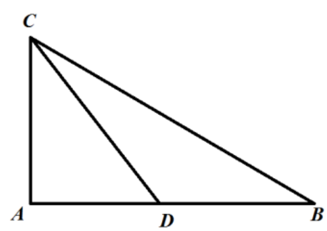
A. 12 B. 11 C. 10 D. 9 E. 8

Problem 5. The diagonal of a square is 12 inches. The length of the side of the square is:

A. 4 inches B. 12 inches C. 6 inches D. $6\sqrt{2}$ inches E. $2\sqrt{6}$ inches

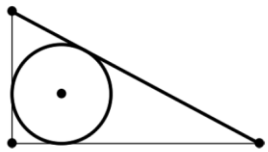
Problem 6. In the figure, $\angle CAB$ is a right angle, D lies in the line segment AB, the length of AC is 6, the length of BC is 10, and the length of BD is 4. What is the length of CD?

A. $4\sqrt{3}$ B. $2\sqrt{13}$ C. $2\sqrt{14}$ inches D. 8. E. $6\sqrt{2}$



Problem 7. Find the radius of the circle inscribed in a triangle whose sides are 8, 15, and 17.

A. 2 B. 3 C. 4 D. 5 E. 6

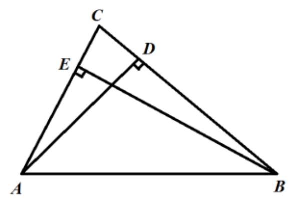


Problem 8. A right triangle has the property that the lengths of its sides form a geometric progression, (i.e. the ratio of shorter leg to the longer leg is the same as the ratio of the longer leg to the hypotenuse.) What is the ratio of the hypotenuse to the shorter leg?

A. 2 B. 5/3 C. $\sqrt{5}/2$ D. $\frac{1+\sqrt{5}}{2}$ E. $\sqrt{\frac{1+\sqrt{5}}{2}}$

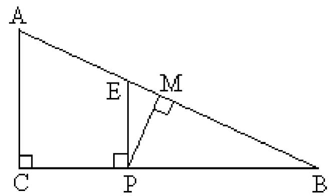
Problem 9. In the acute triangle *ABC*, the line segments *AD* and *BE* are altitudes. If the length of *AB* is 10, the length of *CD* is 2, and the length of *AD* is 6, what is the length of *BE*?

A. 8. B. 12. C. $2\sqrt{10}$. D. $3\sqrt{10}$. E. Cannot be determined.



Problem 10. Given the right triangle *ABC* as shown with $EB \perp CB$, $PM \perp AB$, and *M* as the midpoint of *AB*. If $AC = 6$ and $CB = 8$, what is the length of *EP*?

A. 5 B. $\sqrt{14}$ C. 75/16 D. 4 E. none of these

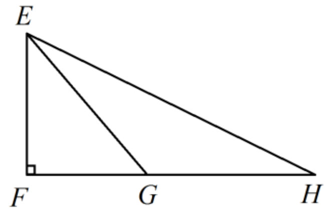


Problem 11. A triangle has sides of lengths 14, 11, and 7. Find the length of the altitude of the triangle drawn to the longest side.

A. $\frac{6\sqrt{10}}{7}$. B. $\frac{12\sqrt{10}}{7}$. C. $\frac{24\sqrt{10}}{7}$. D. $\frac{12\sqrt{5}}{7}$. E. $\frac{10\sqrt{10}}{7}$.

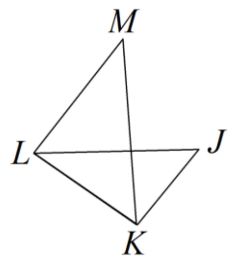
Problem 12. In right triangle *EFH*, $EF = 1$, $FG = 1$, and $EG = GH$. Find the length of *EH*.

A. 4. B. $\sqrt{5}$. C. $\sqrt{4+2\sqrt{2}}$ D. $2 + 2\sqrt{2}$. E. $4 + 2\sqrt{2}$



Problem 13. *MK* and *LJ* are the hypotenuses of overlapping right triangles *KLM* and *JKL*. $MK \perp LJ$, the length of *MK* is $6\sqrt{5}$, the length of *LK* is $6\sqrt{3}$. Find the length of *JK*.

A. $9/\sqrt{2}$. B. $18/\sqrt{5}$. C. $12/\sqrt{5}$ D. $6\sqrt{3}$. E. $6\sqrt{6}/\sqrt{5}$



Problem 14. If the hypotenuse of a right triangle is 30 inches long and one angle measures 30º, then one leg must have a length of:

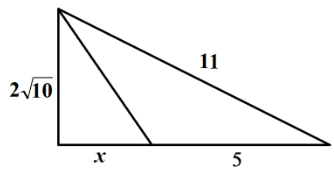
A. $\sqrt{30}$ in. B. 10 in. C. 20 in. D. $\sqrt{20}$ E. 15 in.

Problem 15. Triangle *PQR* has a right angle at P. If $QR = 16$, what is the length of median *PS*?

A. 4 B. $4\sqrt{3}$ C. 8 D. $8\sqrt{3}$ E.$16\sqrt{3}/3$.

Problem 16. Find *x* in the right triangle (not drawn to scale):

A. 14 B. 7 C. 8 D. 9 E. 4

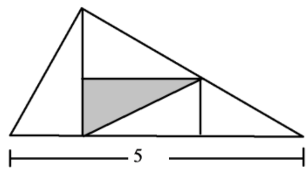


Problem 17. Triangle *ABC* has a right angle at *A*. *AD* is the interior altitude and $BD = 8$ and $CD = 4$. Find *AD*.

A. $2\sqrt{3}$. B. $4\sqrt{2}$. C. 12. D. $4\sqrt{3}$. E. 32.

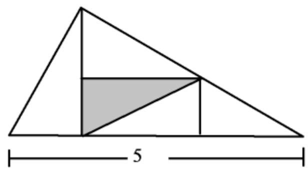
Problem 18. A right triangle is partitioned into five congruent triangles as shown in the figure. If the hypotenuse has length 5, find the length of the shorter leg.

A. $\sqrt{5}$ B. $\dfrac{5-\sqrt{5}}{2}$ C. 3 D. $\dfrac{1+\sqrt{5}}{2}$ E. $5-\sqrt{5}$



Problem 19. Consider a right triangle that can be partitioned into 5 congruent parts as shown. What is the length of the shorter leg of the smaller triangle?

A. $\sqrt{2}$ B. $\sqrt{3}$ C. 2 D. 1 E. $(1 + \sqrt{5})$



Problem 20. A right triangle with integer sides has area equal to 30. What is the length of its hypotenuse?

A. 10 B. 12 C. 13 D. 15 E. none of these

Problem 21. Given a right triangle with sides of length *a, b*, and *c* and area, $a^2 + b^2 - c^2$. Find *c/b* the ratio of the legs of the right triangle.

A. 1 B. $\sqrt{3}/2$ C. 4 D. 1/4 E. none of the above

Problem 22. Find the perimeter of a right triangle whose hypotenuse is 2 and whose area is 1.

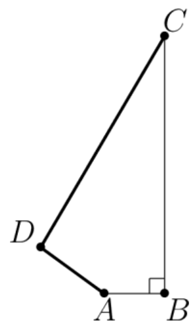
A. $2 + \sqrt{6}$ B. $2 + \sqrt{2}$ C. $2 + 2\sqrt{2}$ D. $2 + \sqrt{5}$ E. none of these

Problem 23. For what positive value of x is there a right triangle with sides $x + 1$, $4x$, and $4x + 1$?

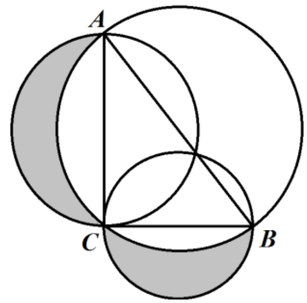
A. 4 B. 6 C. 8 D. 10 E. 12

Problem 24. Consider a quadrilateral $ABCD$ with $AB = 4$, $BC = 10\sqrt{3}$ and $\angle DAB = 150°$, $\angle ABC = 90°$, and $\angle BCD = 30°$. Find DC.

A. 16 B. 17 C. 18 D. 19 E. 20



Problem 25. Circles are constructed on each of the three sides of a right triangle ABC using the sides as diameters. In each case, the center of the circle is the midpoint of the side, with the side being a diameter of the circle. If the area of the triangle is 12 square units, what is the total area of the two smaller circles that lies outside the largest circle and is shaded in the figure?



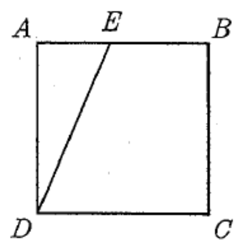
A. 3π B. 9 C. 12 D. 4π E. 5π.

Problem 26. One leg of a right triangle is 2 inches longer than the other leg. If the area of the triangle is 24 square inches, how long is the shorter leg?

A. $\sqrt{3}$ B. 4 C. 6 D. 8 E. $6\sqrt{3}$.

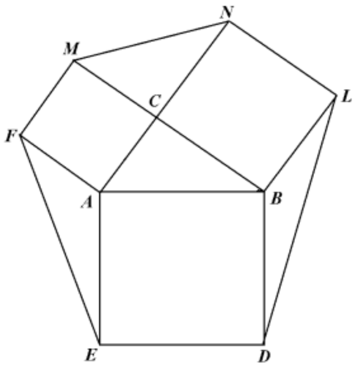
Problem 27. $ABCD$ is a rectangle and $DE = DC$. Given $AD = 5$ and $BE = 3$, find DE.

(A) $\frac{17}{3}$ (B) $\frac{16}{3}$ (C) $\frac{34}{7}$ (D) $\frac{7}{3}$ (E) 6



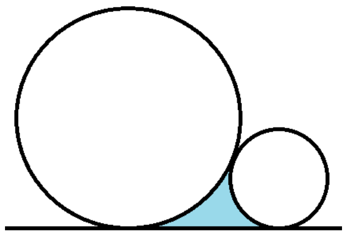
Problem 28. An irregular hexagon *DEFMNL* is drawn as follows: We start with a right triangle *ABC*, draw the squares on the legs and hypotenuse, and then join in sequence the free vertices of the squares. If the hypotenuse of triangle ABC has length 13 and the triangle *ABC* has area 30, find the area of the hexagon *DEFMNL*.

A. 458 B. 146 C. 210 D. 512 E. $264\sqrt{2}$.



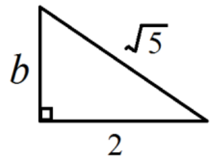
Problem 29. The two tangent circles with the radii 3 and 1, respectively, have an external common tangent as shown. Find the shaded area.

(A) $4\sqrt{3}-\frac{11}{6}\pi$ (B) $4\sqrt{3}$ (C) $\frac{11}{6}\pi$ (D) 2π (E) $4\sqrt{3}-\frac{7}{6}\pi$



5. SOLUTIONS

Problem 1. Solution: A.



By Pythagorean Theorem $c^2 = a^2 + b^2 \Rightarrow \quad (\sqrt{5})^2 = b^2 + 2^2$

$b^2 = 1 \Rightarrow \quad b = 1$

Problem 2. Solution: D.

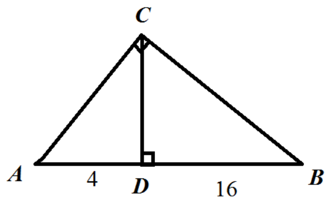
Since $x > y > 0$, $x + y$ is the longest side.

So we have $\quad (x+y)^2 = x^2 + (x-y)^2 \quad (1)$

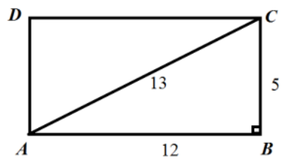
Dividing both sides of (1) by y^2: $(\frac{x}{y}+1)^2 = (\frac{x}{y})^2 + (\frac{x}{y}-1)^2 \quad (2)$

Let $m = \frac{x}{y}$. (2) becomes: $(m+1)^2 = m^2 + (m-1)^2 \quad \Rightarrow \quad m^2 = 4m \quad \Rightarrow \quad m = 4$.

Problem 3. Solution: B.



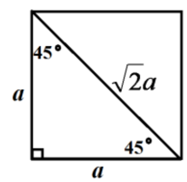
$CD^2 = AD \times BD \quad \Rightarrow \quad CD^2 = 4 \times 16 = 8^2 \quad \Rightarrow$
$CD = 8$.



Problem 4. Solution: A.

As shown in the figure, ΔABC is a 5-12-13 right triangle. So the answer is A.

Problem 5. Solution: D.



By **Theorem 3,** $c = \sqrt{2}a \quad \Rightarrow \quad 12 = \sqrt{2}a \quad \Rightarrow$

$a = \frac{12}{\sqrt{2}} = 6\sqrt{2}$.

Problem 6. Solution: B.

ΔABC is a 6-8-10 right angle. So $AB = 8$. We know that $BD = 4$. So $AD = 4$.
Applying the Pythagorean Theorem to ΔACD: $AC^2 + AD^2 = CD^2 \Rightarrow$

$$6^2 + 4^2 = CD^2 \Rightarrow \qquad CD = 2\sqrt{13}.$$

Problem 7. Solution: B.

By Theorem 4: $r = \dfrac{8+15-17}{2} = 3$.

Problem 8. Solution: D.

Let three sides be *a, b*, and *c*. *c* is the hypotenuse.
We have

$$a^2 + b^2 = c^2 \qquad (1)$$

$$\frac{a}{b} = \frac{b}{c} \qquad \Rightarrow \qquad b^2 = ac \qquad (2)$$

Substituting (2) into (1): $a^2 + ac = c^2$ (3)

We divide each term of (3) by ac: $\dfrac{a}{c} + 1 = (\dfrac{c}{a})^2$ (4)

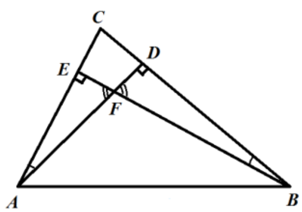
Let $m = \dfrac{a}{c}$.

(4) becomes: $m^2 - m - 1 = 0 \Rightarrow \qquad m = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4\times 1\times(-1)}}{2} = \dfrac{1+\sqrt{5}}{2}$.

Problem 9. Solution: D.

Applying the Pythagorean Theorem to ΔACD: $AD^2 + CD^2 = AC^2 \Rightarrow$

$$6^2 + 2^2 = AC^2 \Rightarrow \qquad AC = 2\sqrt{10}.$$



ΔADB is a 6–8–10 right triangle. So $DB = 8$.
We also see from the figure that $\angle EAF + \angle F = \angle DBF + \angle F = 90°$. So $\angle EAF = \angle DBF$. Thus ΔBCE is similar to ΔACD and $\dfrac{AC}{BC} = \dfrac{AD}{BE} \qquad \Rightarrow \qquad \dfrac{2\sqrt{10}}{10} = \dfrac{6}{BE} \Rightarrow$

$$BE = 3\sqrt{10}.$$

Problem 10. Solution: C.

We know that ΔABC is similar to ΔPBM. So $\frac{AB}{PB}=\frac{CB}{MB} \quad \Rightarrow$

$$\frac{10}{PB}=\frac{8}{5} \Rightarrow PB=\frac{25}{4}.$$

We also know that ΔABC is similar to ΔPBP.

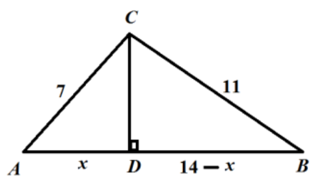
So $\frac{AC}{EP}=\frac{CB}{PB} \quad \Rightarrow \quad \frac{6}{EP}=\frac{8}{\frac{25}{4}} \Rightarrow \quad EP=\frac{75}{16}$.

Problem 11. Solution: B.

Method 1:

We draw the figure. Applying the Pythagorean Theorem to ΔACD:

$AC^2-AD^2=CD^2 \quad \Rightarrow \quad 7^2-x^2=CD^2 \qquad (1)$



Applying the Pythagorean Theorem to ΔBCD:

$BC^2-DB^2=CD^2 \quad \Rightarrow \quad 11^2-(14-x)^2=CD^2 \quad (2)$

(1) – (2): $7^2-x^2=11^2-(14-x)^2 \quad \Rightarrow \quad x=\frac{31}{7}$.

Substituting $x=\frac{31}{7}$ into (1): $7^2-(\frac{31}{7})^2=CD^2 \quad \Rightarrow \quad CD=\frac{12\sqrt{10}}{7}$.

Method 2:

The semi perimeter of the triangle is (7 + 11 + 14)/2 = 16.

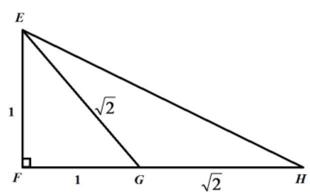
By the Heron Formula, the area of the triangle is

$\sqrt{16(16-7)(16-11)(16-14)}=12\sqrt{10}$.

Also the area can be calculated by $\frac{1}{2}\times 14\times CD$.

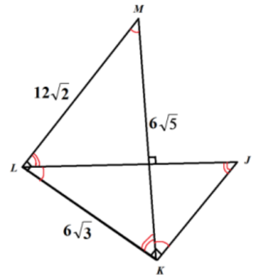
So $\frac{1}{2}\times 14\times CD=12\sqrt{10} \quad \Rightarrow \quad CD=\frac{12\sqrt{10}}{7}$.

Problem 12. Solution: C.



We know that $EG=GH=\sqrt{2}$. Applying the Pythagorean Theorem to ΔEFH: $EF^2+FH^2=EH^2 \Rightarrow$

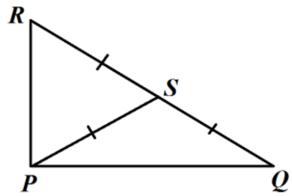
$1^2+(1+\sqrt{2})^2=EH^2 \Rightarrow \quad EH=\sqrt{4+2\sqrt{2}}$.



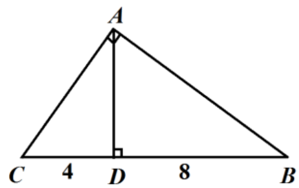
Problem 13. Solution: A.
Applying the Pythagorean Theorem to ΔKLM:
$ML^2 + KL^2 = MK^2 \Rightarrow ML^2 = (6\sqrt{3})^2 - (6\sqrt{3})^2$
$\Rightarrow ML = 12\sqrt{2}$.
We know that $\Delta\ KLM$ is similar to ΔJKL.
So $\dfrac{ML}{KL} = \dfrac{KL}{JK} \Rightarrow \dfrac{12\sqrt{2}}{6\sqrt{3}} = \dfrac{6\sqrt{3}}{JK} \Rightarrow JK = \dfrac{108}{12\sqrt{2}} = \dfrac{9}{\sqrt{2}}$.

Problem 14. Solution: E.
By **Theorem 5**, one leg must be $\frac{1}{2} \times 30 = 15$. The answer is E.

Problem 15. Solution: C.
By **Theorem 6**, $PS = RS = SQ = \frac{1}{2} \times QR = 8$.

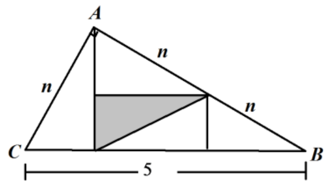


Problem 16. Solution: E.
Applying the Pythagorean Theorem to the right triangle: $(2\sqrt{10})^2 + (x+5)^2 = 11^2 \Rightarrow x = 4$.

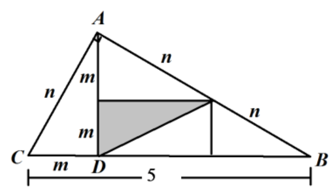


Problem 17. Solution: B.
$AD^2 = CD \times DB = 32 \qquad \Rightarrow \qquad AD = 4\sqrt{2}$.

Problem 18. Solution: A.
We label the figure as follows:
Applying the Pythagorean Theorem to the right triangle: $(n)^2 + (2n)^2 = 5^2 \Rightarrow n = \sqrt{5}$.



Problem 19. Solution: D.
We label the figure as follows:
Applying the Pythagorean Theorem to the right triangle ABC: $(n)^2 + (2n)^2 = 5^2 \Rightarrow n = \sqrt{5}$.
Applying the Pythagorean Theorem to the right triangle



$ACD: (m)^2 + (2m)^2 = n^2 \Rightarrow 5m^2 = (\sqrt{5})^2 \Rightarrow m = 1$.

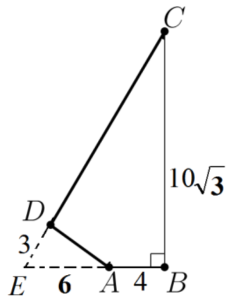
Problem 20. Solution: C.
The Pythagorean Triple 5, 12, 13 work since 5 × 12 / 2 = 30. So C is the answer.

Problem 21. Solution: C.
By the Pythagorean Theorem, we have $b^2 + c^2 = a^2$ (1)
We are given that $\frac{1}{2}bc = a^2 + b^2 - c^2 \Rightarrow bc = 2a^2 + 2b^2 - 2c^2$ (2)
Substituting (1) into (2): $4b = c \Rightarrow c/b = 4$.

Problem 22. Solution: C.
By the Pythagorean Theorem, we have $a^2 + b^2 = c^2$ (1)
We are given that $\frac{1}{2}ab = 1 \Rightarrow 2ab = 4$ (2)
(1) + (2): $(a + b)^2 = 4 + c^2 \Rightarrow (a + b)^2 = 8 \Rightarrow a + b = 2\sqrt{2}$
The perimeter is $2 + 2\sqrt{2}$.

Problem 23. Solution: B.
Let the hypotenuse be $4x + 1$.
By the Pythagorean Theorem, $(x + 1)^2 + (4x)^2 = (4x + 1)^2 \Rightarrow x = 6$ ($x = 0$ ignored).

Problem 24. Solution: B.
First we see that the angle D is 90° = 360° − 150° − 90° − 30°.
Extend CD and BA to meet at E. We see that $\angle E = 60°$.
By Theorem 5 in ΔEBC, $EB = 10$ and $EA = 6$.
By Theorem 5, in ΔEAD, $ED = 3$.
Applying the Pythagorean Theorem to the right triangle
$EBC: BC^2 + EB^2 = EC^2 \Rightarrow (10\sqrt{3})^2 + 10^2 = (3 + DC)^2 \Rightarrow DC = 17$.



Problem 25. Solution: C.
By **Theorem 2**, the shaded areas are the same as the area of the right triangle ABC. The answer is 12.

Problem 26. Solution: C.

Let the length of the other leg be x. We have $\frac{1}{2}x(x+2)=24 \quad \Rightarrow \quad x=6$.

Problem 27. Solution: A.

$AE = AB - BE = DC - 3$.

ΔDAE is a right triangle. $DE^2 = AD^2 + AE^2$

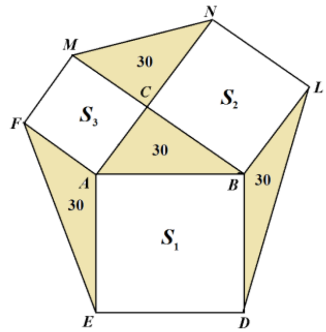
Since $DE = DC$ and $AE = DC - 3$, we have $DC^2 = 5^2 + (DC-3)^2 \Rightarrow$

$DC^2 = 5^2 + DC^2 - 6DC + 9 \quad \Rightarrow \quad 6DC = 5^2 + 9 = 34 \quad \Rightarrow DC = \frac{34}{6} = \frac{17}{3}$.

Problem 28. Solution: A.

As shown in the figure, four triangles have the same area and $S_1 = S_2 + S_3$.

The answer is $30 \times 4 + 13^2 \times 2 = 458$.



Problem 29. Solution: A.

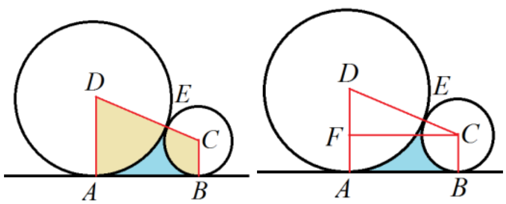
We see that $DF = DA - AF = DA - BC = 3 - 1 = 2$.

$DC = 3 + 1 = 4$.

Thus triangle DCF is a $30° - 60° - 90°$ right triangle.

The shaded area is the area of the trapezoid $ABCD$ – the areas of sectors ADE and ECB.

The answer is $\frac{(3+1)\times\sqrt{4^2-2^2}}{2} - \frac{\pi\times 3^2}{6} - \frac{\pi\times 1^2}{3} = 4\sqrt{3} - \frac{11}{6}\pi$.



BASIC KNOWLEDGE

1. Properties of probability:

(1). The probability of an event is between 0 and 1.
(2). The probability of an impossible event is 0.
(3). The probability of a certain event is 1.
(4). The probability that an event will occur is equal to one minus the probability that it will not occur.

2. Basic probability formula

$$\text{Probability} = \frac{\text{number of ways that a certain outcome can occur}}{\text{total number of possible outcomes}}$$

Example 1. A bag contains 4 red chips, 2 blue chips, and 3 white chips. If one chip is drawn at random, what is the probability that the chip will not be red?

(A) $\frac{5}{9}$ (B) $\frac{7}{9}$ (C) $\frac{5}{9}$ (D) $\frac{4}{9}$ (E) $\frac{2}{9}$

Solution: A.
There are total 4 + 3 + 2 = 9 chips. There are 4 red chips. The probability to draw a red chip is then 4/9. By property 4, the probability that the chip will not be red is 1 – 4/9 = 5/9.

Example 2. There are 2 blue marbles, 6 yellow marbles, and 5 green marbles in a bag. One at a time, two marbles are drawn randomly from the bag, with replacement after each drawing. What is the probability that all two are green?

(A) $\frac{5}{13}$ (B) $\frac{25}{169}$ (C) $\frac{5}{169}$ (D) $\frac{10}{169}$ (E) $\frac{20}{169}$

Solution: B.

There are total 2 + 6 + 5 = 13 marbles. There are 5 green marbles. The probability to draw a green marble is then 5/13. By the fundamental counting principle, the probability that all three are green is $(5/13) \times (5/13) = 25/169$.

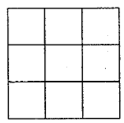
Example 3. Mark has 3 marbles in his pocket. Two marbles are yellow and one is blue. If he randomly selects two marbles, what is the probability that they are the same color?

(A) $\frac{2}{9}$ (B) $\frac{4}{9}$ (C) $\frac{2}{3}$ (D) $\frac{2}{5}$ (E) $\frac{1}{3}$

Solution: E.

There are total 3 ways to select two marbles. There is only 1 way to select 2 marbles s of the same color (two yellow marbles). The probability to select 2 marbles that are the same color is $\frac{1}{3}$.

Example 4. Three darts are thrown at the figure given, each landing in a different square. What is the probability that the squares they land in form a row, either horizontally, or vertically?

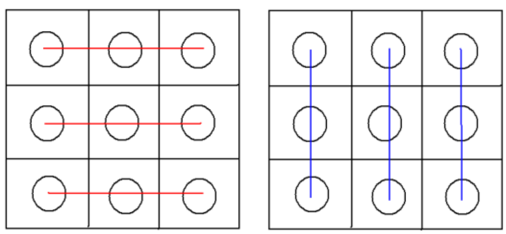


(A) $\frac{3}{28}$ (B) $\frac{1}{14}$ (C) $\frac{2}{21}$ (D) $\frac{2}{5}$ (E) $\frac{2}{9}$

Solution: B.

There are $\binom{9}{3} = 84$ ways that three darts can be thrown at the figure given.

There are 6 ways that they land in a row, either horizontally, or vertically.



The probability that the squares they land in form a row, horizontally or vertically is $6/84 = \frac{1}{14}$.

Example 5. Peter rolls a pair of dice with the integers 1 through 6 on the faces of each die. What is the probability that the sum of the integers on the top faces is 9?

(A) $\frac{2}{9}$ (B) $\frac{4}{9}$ (C) $\frac{2}{3}$ (D) $\frac{1}{9}$ (E) $\frac{1}{3}$

Solution: D.

When two dice are rolled, there are 36 outcomes. Four outcomes show that the sum of the integers on the top faces is 9.

D_1 \ D_2	1	2	3	4	5	6
1	*2*	*3*	*4*	*5*	*6*	*7*
2	*3*	*4*	*5*	*6*	*7*	*8*
3	*4*	*5*	*6*	*7*	*8*	*9*
4	*5*	*6*	*7*	*8*	*9*	*10*
5	*6*	*7*	*8*	*9*	*10*	*11*
6	*7*	*8*	*9*	*10*	*11*	*12*

The probability is 4/ 36 = 1/9.

Example 6. What is the probability that exactly two heads will come up when three coins are flipped?

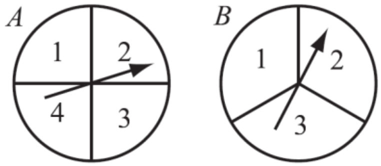
(A) $\frac{3}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $\frac{5}{8}$ (E) $\frac{7}{8}$

Solution: A.

The total number of outcomes is $2 \times 2 \times 2 = 8$.

We have three cases that exactly two heads will come up when three coins are flipped:
HHT, THH, and HTH.
The probability is 3/8.

☆**Example 7.** (2004 AMC 8 Problem 21) Spinners *A* and *B* are spun. On each spinner, the arrow is equally likely to land on each number. What is the probability that the product of the two spinners' numbers is even?



(A) 1/4 (B) 1/3 (C) 1/2 (D) 2/3 (E) 3/4

Solution: D.
Method 1 (official solution, indirect way):
To get an odd product, the result of both spins must be odd. The probability of odd is 1/2 on Spinner *A* and 1/3 on Spinner *B*. So the probability of an odd product is (1/2)(1/3) = 1/6. The probability of an even product, then, is (1 – 1/3) = 2/3.

Method 2 (our solution, direct way):
To get an even product, we have three cases:

	Spinner *A*	Spinner *B*.
Case 1:	even	even
Case 2:	even	odd
Case 3:	odd	even

The probability is

$$P=\frac{2}{4}\times\frac{1}{3}+\frac{2}{4}\times\frac{2}{3}+\frac{2}{4}\times\frac{1}{3}=\frac{8}{12}=\frac{2}{3}$$

☆**Example 8.** Angie, Bridget, Carlos, and Diego are seated at random around a square table, one person to a side. What is the probability that Angie and Carlos are seated next to each other?
(A) 1/4 (B) 1/3 (C) 1/2 (D) 2/3 (E) 3/4

Solution: D.
If Angie sits down first, there are three equally likely places for Carlos to sit. Two of these is next to Angie. Thus the probability is 2/3.

☆**Example 9.** A fair six-sided die is rolled twice. What is the probability that the first number that comes up is less than or equal to the second number?
(A) 1/6 (B) 5/12 (C) 1/2 (D) 7/12 (E) 5/6

Solution: D.
In 6 of the 36 possible outcomes the two numbers are equal. The first number is less than the second in half of the remaining 30 outcomes, so the first number is less than or equal to the second in 6 + 15 = 21 outcomes. The probability is 21/36 = 7/12.

D_1 \ D_2	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 8
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

☆**Example 10.** One of the three-digit numbers is randomly selected. What is the probability that the number is divisible by 13?
(A) 23/300 (B) 23/243 (C) 23/333 (D) 790 (E) 19/225

Solution: A.
The smallest three-digit number divisible by 13 is 13 × 8 = 104. The greatest three-digit number divisible by 13 is 13 × 76 = 988. Therefore, there are 76 − 7 = 69 three-digit numbers divisible by 13. The answer is 69/900 = 23/300.

3. Basic geometric probability formula

"Geometric probability" is exactly the same as basic probability, except that we are dealing with the geometric figures instead of the "numbers".

The basic probability formula becomes:

$$P = \frac{\text{measure of geometric figure representing desired outcomes in the event}}{\text{measure of geometric figure representing all outcomes in the same space}}$$

Some examples of **geometric measures** are lengths, areas, angle measures, and volumes.

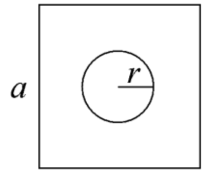
For example, a probability determined by comparing the area of a given section to that of a total available region, is:

$$P = \frac{\text{measure of area of favorable region}}{\text{measure of area of total region}}$$

Example 11. If a dart hits the square board ($a = 10$) below, what is the probability that it will land in the circle ($r = 4$)?

(A) $\frac{2\pi}{25}$ (B) $\frac{\pi}{16}$ (C) $\frac{4\pi}{25}$ (D) $\frac{3\pi}{16}$ (E) $\frac{3\pi}{8}$

Solution: C.



$$P = \frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi r^2}{a^2} = \frac{16\pi}{100} = \frac{4\pi}{25}$$

Example 12. Three concentric circles have radii of 1, 4, and 9. If a point is randomly selected from the interior of the largest circle, what is the probability that it is in the region bounded by the two smaller circles?

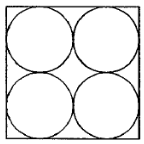
(A) $\frac{5}{27}$ (B) $\frac{4}{9}$ (C) $\frac{2}{3}$ (D) $\frac{1}{9}$ (E) $\frac{2}{9}$

Solution: A.

$$P = \frac{\text{measure of area of favorable region}}{\text{measure of area of total region}}$$

$$P = \frac{\text{Area of ring}}{\text{Area of largest circle}} = \frac{\pi r_1^2 - \pi r_2^2}{\pi r_3^2} = \frac{r_1^2 - r_2^2}{r_3^2} = \frac{16-1}{81} = \frac{15}{81} = \frac{5}{27}$$

Example 13. In the figure shown, four circles are tangent to each other and to the sides of the square as shown. A dart randomly hits the figure. What is the probability that it lands inside one of the circular regions?



(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{10}$ (C) $\frac{\pi}{16}$ (D) $\frac{3\pi}{16}$ (E) $\frac{\pi}{4}$

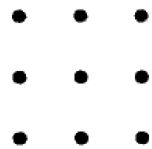
Solution: E.

$$P = \frac{\text{measure of area of favorable region}}{\text{measure of area of whole region}}$$

$$P = \frac{\text{Area of four circle}}{\text{Area of square}} = \frac{4\pi r^2}{(4r)^2} = \frac{4\pi r^2}{16 r^2} = \frac{\pi}{4}$$

Example 14. Find the probability that four randomly selected points on the geoboard below will be the vertices of a square.

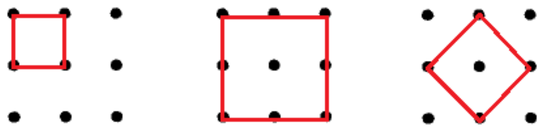
(A) $\frac{5}{126}$ (B) $\frac{1}{21}$ (C) $\frac{2}{9}$ (D) $\frac{1}{9}$ (E) $\frac{2}{21}$



Solution: B.

The number of ways to select four vertices from the nine vertices is $\binom{9}{4}=126$.

The number of squares is 6.



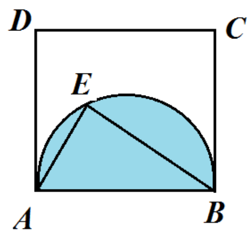
The probability is 6/126 = 1/21.

Example 15. A point E is chosen at random from within square $ABCD$. What is the probability that ΔABE is obtuse?

(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{16}$ (D) $\frac{3\pi}{16}$ (E) $\frac{3\pi}{8}$

Solution: A.

If E is inside the semicircle that has AB as its diameter, then ΔABE will be obtuse.



$$P=\frac{\text{Area of the semicircle}}{\text{Area of square}}=\frac{\frac{1}{2}\pi r^2}{(2r)^2}=\frac{\pi}{8}$$

☆**Example 16.** A complete cycle of a traffic light takes 60 seconds. During each cycle the light is green for 25 seconds, yellow for 5 seconds, and red for 30 seconds. At a randomly chosen time, what is the probability that the light will NOT be yellow?

(A) 11/12 (B) 7/12 (C) 5/12 (D) 1/2 (E) 1/3

Solution: A.

$$P = \frac{\text{time not yellow}}{\text{total time}} = \frac{r+r}{r+r+y} = \frac{25+30}{60} = \frac{11}{12}.$$

4. General addition rule of probability:

(1). If A and B are any two events:

$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(2). If A and B are mutually exclusive ($P(A \cap B) = 0$):

$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$

Note: Any two events that cannot both occur at the same time are called mutually exclusive.

(3). General addition rule of probability (three events):

$P(A \cup B \cup C) = \mathrm{P}(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Example 17. A bag contains 3 red chips, 4 blue chips, and 2 white chips. If a chip is drawn at random, what is the probability that the chip is red or white?

(A) $\frac{5}{9}$ (B) $\frac{4}{9}$ (C) $\frac{1}{3}$ (D) $\frac{1}{9}$ (E) $\frac{7}{9}$

Solution: A.

The probability that the chip is red: 3/9 = 1/3.

The probability that the chip is white: 2/9.

The probability that the chip is red or white can be calculated using the formula:

$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Since the two events (to draw a red chip and to draw a white chip) are mutually exclusive (when you draw a chip, if it is red, it cannot be white at the same time), $P(A \cap B) = 0$

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}.$$

Example 18. Two standard dice are rolled and their face values multiplied. What is the probability that the product is prime or ends in 0?

(A) $\frac{1}{3}$ (B) $\frac{1}{6}$ (C) $\frac{2}{3}$ (D) $\frac{1}{9}$ (E) $\frac{2}{9}$

Solution: A.

$\mathbf{D_1}$ / $\mathbf{D_2}$	1	2	3	4	5	6
1	*1*	*2*	*3*	*4*	*5*	*6*
2	*2*	*4*	*6*	*8*	*10*	*12*
3	*3*	*6*	*9*	*12*	*15*	*18*
4	*4*	*8*	*12*	*16*	*20*	*24*
5	*5*	*10*	*15*	*20*	*25*	*30*
6	*6*	*12*	*18*	*24*	*30*	*36*

The probability that the product is prime: 6/36 = 1/6.
The probability that the product ends in 6: 6/36 = 1/6.

The probability that the product is prime or ends in 6 can be calculated using the formula: $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Since the two events (the product is prime or ends in 6) are mutually exclusive (see the table above), $P(A \cap B) = 0$

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$

Example 19 Two standard dice are rolled and their face values multiplied. What is the probability that the product is prime or ends in 5?

(A) $\frac{1}{4}$ (B) $\frac{5}{36}$ (C) $\frac{1}{6}$ (D) $\frac{1}{18}$ (E) $\frac{2}{9}$

Solution: A.

D_2 \ D_1	1	2	3	4	5	6
1	***1***	***2***	***3***	***4***	***5***	***6***
2	***2***	***4***	***6***	***8***	***10***	***12***
3	***3***	***6***	***9***	***12***	***15***	***18***
4	***4***	***8***	***12***	***16***	***20***	***24***
5	***5***	***10***	***15***	***20***	***25***	***30***
6	***6***	***12***	***18***	***24***	***30***	***36***

The probability that the product is prime: 6/36 is 1/6.
The probability that the product is ends in 5 is 5/36.
The probability that the product is prime or ends in 5 is 2/36 = 1/18.
The probability that the product is prime or ends in 5 can be calculated using the formula:

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$=\frac{1}{6}+\frac{5}{36}-\frac{2}{36}=\frac{9}{36}=\frac{1}{4}$.

Example 20. Three coins are flipped. What is the probability, expressed as a common fraction, that all are heads or all are tails?

(A) $\frac{3}{4}$ (B) $\frac{3}{8}$ (C) $\frac{1}{6}$ (D) $\frac{1}{8}$ (E) $\frac{1}{4}$

Solution: E.

There are total $2 \times 2 \times 2 = 8$ ways to flip three coins.
There is only 1 way to flip three heads.
There is only 1 way to flip three tails.
The probability of all heads: 1/8.
The probability of all tails: 1/8.

Since the two events are mutually exclusive (you can't flip a coin and get both head and tail), $P(A \cap B)=0$

The probability that all are heads or all are tails can be calculated using the formula:

$$P(A \text{ or } B)=P(A \cup B)=P(A)+P(B)=\frac{1}{8}+\frac{1}{8}=\frac{2}{8}=\frac{1}{4}$$

Example 21. If four coins are tossed, what is the probability that exactly three of them show heads or exactly three of them show tails?

(A) $\frac{1}{2}$ (B) $\frac{1}{16}$ (C) $\frac{3}{16}$ (D) $\frac{1}{8}$ (E) $\frac{1}{4}$

Solution: A.

There are total $2 \times 2 \times 2 \times 2 = 16$ ways to flip four coins.

There are 4 ways to get exactly three heads (HHHT): $\frac{4!}{3! \times 1!}=4$.

There are 4 ways to get exactly three tails (TTTH): $\frac{4!}{3!\times 1!}=4$.
The probability of getting exactly three heads: 4/16 = 1/4.
The probability of getting exactly three tails: 4/16 = 1/4.

Since the two events are mutually exclusive (you can't flip a coin and get both heads and tails), $P(A\cap B)=0$.

The probability that exactly three of them show heads or exactly three of them show tails can be calculated using the formula is:
$P(A \text{ or } B)=P(A\cup B)=P(A)+P(B)=\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$.

6. General multiplication rule of probability:

(1). If A and B are any two events:

$P(\text{A and } B)=P(A\cap B)=P(A)\times P(B|A)$, where $P(B|A)$ is the probability of B happening under the condition of event A.

(2). If A and B are two independent events:

If the outcome of event A does not affect the outcome of event B, A and B are called independent events. $P(B|A)=P(B)$

$P(\text{A and } B)=P(A\cap B)=P(A)\times P(B)$

Example 22. What is the probability of rolling doubles on a pair of fair dice?
(A) $\frac{1}{6}$ (B) $\frac{1}{12}$ (C) $\frac{3}{18}$ (D) $\frac{1}{8}$ (E) $\frac{1}{4}$

Solution: A.

The probability of rolling a number from 1 to 6 (event A) is 1.
The probability of rolling a number that will match the number rolled before (event B) is 1/6.

Since the two outcomes are independent events, $P(B|A) = P(B)$.

The probability of rolling doubles on a pair of fair dice:

$$P(A \text{ and } B) = P(A) \times P(B) = 1 \times \frac{1}{6} = \frac{1}{6}.$$

Example 23. The probability that Chris will win the first set of a tennis match is $\frac{2}{5}$ and that he will win the second is $\frac{1}{2}$. Assuming independence of the two sets, what is the probability that he wins both sets?

(A) $\frac{1}{5}$ (B) $\frac{1}{6}$ (C) $\frac{2}{3}$ (D) $\frac{1}{10}$ (E) $\frac{2}{9}$

Solution: A.
The probability that he wins both sets:

$$P(\text{A and } B) = P(A) \times P(B) = \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}.$$

Example 24. Fifty cards numbered from 1 to 30 are placed in a box. If a card is selected at random, what is the probability that the card is a prime number and a multiple of seven?

(A) $\frac{7}{30}$ (B) $\frac{1}{10}$ (C) $\frac{2}{25}$ (D) $\frac{1}{25}$ (E) $\frac{1}{30}$

Solution: E.
There are 30 numbers and 10 of them are prime numbers (2, 3, 5, 7, 11, 13, 17, 19, 23, 29).

There is one number that is both a prime number and a multiple of seven (7).

The probability to select a prime number (event A) is 10/30= 1/3.

The probability to select a multiple of seven from all these prime numbers (event B under event A) is 1/10.

The probability that the card is a prime number and a multiple of seven

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B|A) = \frac{1}{3} \times \frac{1}{10} = \frac{1}{30}.$$

Example 25. Cards are randomly drawn one at a time, without replacement, from a standard deck of playing cards. What is the probability that the first three cards chosen are clubs?

(A) $\frac{11}{50}$ (B) $\frac{11}{850}$ (C) $\frac{13}{52}$ (D) $\frac{4}{14}$ (E) $\frac{1}{5}$

Solution: B.

Method 1:

There are 52 cards in a standard deck and 13 cards in a set clubs.

The probability to draw a club the first time: 13/52.

The probability to draw a club the second time (under the condition that one club was drawn the first time): 12/51.

The probability to draw a club the third time (under the condition that two clubs were drawn and only 10 of them are left): 11/50.

The probability that the first three cards chosen are clubs is:

$$P = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} = \frac{11}{850}$$

Method 2:

There are total 22100 ways to draw three cards from a standard deck of 52 cards:

$$\binom{52}{3} = 22100.$$

There are total 286 ways to draw three clubs from a set of 13 clubs: $\binom{13}{3} = 286$.

The probability that the first three cards chosen are clubs: $P = \frac{286}{22100} = \frac{11}{850}$.

Example 26. The figure below represents five offices that will be assigned randomly to four employees, one employee per office. If Karen and Tina are two of the four employees, what is the probability that each will be assigned an office indicated with an X?

X				X

(A) 1/10 (B) 1/12 (C) 1/6 (D) 1/4 (E) 1/2

Solution: A.
The probability that each will be assigned an office with an X is

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B|A) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$$

7. Total probability

Law of total probability (Marginal Probability)

Let A_1, A_2, A_3, …, and A_n be mutually exclusive and exhaustive events. Then for any other event B,

$$P(B) = P(A_1B_1) + P(A_2B_2) + P(A_3B_3) + ... + P(A_nB_n)$$
$$= P(A_1)P(B_1|A_1) + P(A_2)P(B_2|A_2) + P(A_3)P(B_3|A_3) + ... + P(B_n|A_n)$$

Note: Mutually exclusive means that $P(A_iB_j) = 0$

Exhaustive rule: $P(A_1 \text{ or } A_2 \text{ or } A_3 \text{ or....} A_n) = 1$

Then since A_1, A_2, A_3, …, and A_n exhaustive, if B occurs, it must be in conjunction with exactly one of A_i's. That is B = (A_1 and B_1) or (A_2 and B_2) or ….(A_n and B_n).

Example 27. Digit d is randomly selected from the set {4, 5, 6, 7}. Without replacement of d, another digit e is selected. What is the probability that the two-digit number $\underline{de}$ is a multiple of 8?

(A) $\frac{1}{3}$ (B) $\frac{1}{6}$ (C) $\frac{2}{3}$ (D) $\frac{1}{9}$ (E) $\frac{2}{9}$

Solution: B.

Method 1: The following cases are only ones that work: (5, 6), and (6, 4).

Let A_1 be the event of selecting the digit 5 as the digit d.
B_1 be the event of selecting the digit 6 as the digit e.
A_2 be the event of selecting the digit 6 as the digit d.
B_2 be the event of selecting the digit 4 as the digit e.
B be the event that the two-digit number $\underline{de}$ is a multiple of 8.

$$B = A_1B_1 + A_2B_2$$

$$P(B) = P(A_1B_1) + P(A_2B_2)$$

$$P(A_1) = \frac{1}{4}; \qquad P(A_2) = \frac{1}{4}.$$

$$P(B|A_1) = \frac{1}{3}; \qquad P(B|A_2) = \frac{1}{3}.$$

$$P(B) = \frac{1}{4}\times\frac{1}{3} + \frac{1}{4}\times\frac{1}{3} = \frac{2}{12} = \frac{1}{6}.$$

Method 2: There are a total of 12 ways to select two digits (four ways to select first digit and 3 ways to select second digit).

There are four favorable ways (5, 6), and (6, 4) such that the two-digit number is a multiple of 38.

The probability is $P = \frac{2}{4 \times 3} = \frac{1}{6}$.

☆**Example 28.** A number cube has its faces numbered 1, 2, 3, 4, 5, and 6. A second cube has its faces numbered 2, 4, 6, 8, 10, and 12. If the cubes are rolled, what is the probability that the sum of the numbers showing is 10?

(A) $\frac{1}{3}$ (B) $\frac{1}{6}$ (C) $\frac{1}{12}$ (D) $\frac{1}{9}$ (E) $\frac{2}{9}$

Solution: C.
Method 1:
For the sum to be 8, the pair of numbers on the two dice can be (2, 8), (4, 6), or (6, 4).

Let A_1 be the event that the first number is 2.
B_1 be the event that the second number is 8.
A_2 be the event that the first number is 4.
B_2 be the event that the second number is 6.
A_3 be the event that the first number is 6.
B_3 be the event that the second number is 4.
B be the event that the sum of the numbers showing is 10.

$$B = A_1B_1 + A_2B_2 + A_3B_3$$

Since the selection of the second number is independent to the selection of the first number, so

$$P(B) = P(A_1B_1) + P(A_2B_2) + P(A_3B_3) = P(A_1) \times P(B_1) + P(A_2) \times P(B_2) + P(A_3) \times P(B_3)$$

$$P(A_1) = \frac{1}{6}; \qquad P(A_2) = \frac{1}{6}; \qquad P(A_3) = \frac{1}{6}.$$

$P(B_1) = \frac{1}{6}$; $P(B_2) = \frac{1}{6}$; $P(B_3) = \frac{1}{6}$.

$P(B) = \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} = \frac{3}{36} = \frac{1}{12}$.

Method 2:
There are a total of 36 outcomes rolling two dice.
There are three favorable ways: (2, 8), (4, 6), and (6, 4).
The probability is $P = \frac{3}{36} = \frac{1}{12}$.

Example 29. An ordered pair (a, b) is determined by choosing a number a and then a number b at random without replacement from the set $\{1, 2, 3, 4, 5\}$. What is the probability that $\frac{a}{b}$ is an integer?

(A) $\frac{1}{12}$ (B) $\frac{1}{8}$ (C) $\frac{1}{6}$ (D) $\frac{1}{5}$ (E) $\frac{1}{4}$

Solution: E.

Method 1: If $\frac{a}{b}$ is an integer, the following cases are the only ones that satisfy the requirement: (5, 1), (4, 2), (4, 1), (3, 1), and (2, 1).

Let A_1 be the event that a is 5.
B_1 be the event that b is 1
A_2 be the event that a is 4.
B_2 be the event that b is 2.
A_3 be the event that a is 4.
B_3 be the event that b is 1.
A_4 be the event that a is 3.
B_4 be the event that b is 1.
A_5 be the event that a is 2.
B_5 be the event that b is 1.

Let B be the event that $\frac{a}{b}$ is an integer.

$B = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4 + A_5B_5$

$P(B) = P(A_1B_1) + P(A_2B_2) + P(A_3B_3) + P(A_4B_4) + P(A_5B_5)$

$P(A_1) = \frac{1}{5}$; $P(A_2) = \frac{1}{5}$; $P(A_3) = \frac{1}{5}$; $P(A_4) = \frac{1}{5}$, $P(A_5) = \frac{1}{5}$.

$P(B|A_1) = \frac{1}{4}$; $P(B|A_2) = \frac{1}{4}$; $P(B|A_3) = \frac{1}{4}$; $P(B|A_4) = \frac{1}{4}$, $P(B|A_5) = \frac{1}{4}$.

$P(B) = \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} = \frac{5}{20} = \frac{1}{4}$.

Method 2:
There are a total of $5\times 4 = 20$ ways to select two digits.

There are five cases that work: (5, 1), (4, 2), (4, 1), (3, 1), and (2, 1).
The probability is $P = \frac{5}{20} = \frac{1}{4}$.

☆**Example 30.** Harry selected an even positive integer less than 10, and Jim selected an odd positive integer less than 10. What is the probability that the number selected by Jim is greater than the number selected by Harry?

(A) $\frac{1}{2}$ (B) $\frac{1}{8}$ (C) $\frac{1}{6}$ (D) $\frac{1}{5}$ (E) $\frac{1}{4}$

Solution: A.
Say that Harry picks up a number from the set {2, 4, 6, 8} and Jim picks up a number from the set {1, 3, 5, 7, 9}.

Let B be the event that the number selected by Jim is greater than the number selected by Harry,

A_1 be the event that Harry picks up the number 2,
B_1 be the event that Jim picks up a number that is larger than Harry's number after A_1
A_2 be the event that Harry picks up the number 4,
B_2 be the event that Jim picks up a number that is larger than Harry's number after A_2
A_3 be the event that Harry picks up the number 6,
B_3 be the event that Jim picks up a number that is larger than Harry's number after A_3
A_4 be the event that Harry picks up the number 8,
B_4 be the event that Jim picks up a number that is larger than Harry's number after A_4

Then $B = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4$

$$P(B) = P(A_1B_1) + P(A_2B_2) + P(A_3B_3) + P(A_4B_4)$$

$$= P(A_1)P(B_1|A_1) + P(A_2)P(B_2|A_2) + P(A_3)P(B_3|A_3) + P(A_4)P(B_4|A_4)$$

Or

$$P(B) = \frac{1}{4}\times\frac{4}{5} + \frac{1}{4}\times\frac{3}{5} + \frac{1}{4}\times\frac{2}{5} + \frac{1}{4}\times\frac{1}{5} = \frac{1}{4}\times\frac{10}{5} = \frac{1}{2}.$$

☆**Example 31.** Tim selects three different numbers at random from the set {1, 7, 9, 11} and adds them. Cathy takes two different numbers at random from the set { 3, 4, 5, 6} and multiply them. What is the probability that Tim's result is less than Cathy's result?

(A) $\frac{1}{12}$ (B) $\frac{7}{12}$ (C) $\frac{1}{6}$ (D) $\frac{1}{5}$ (E) $\frac{5}{12}$

Solution: E.
Tim can get the numbers 1 + 7 + 9 = 17, 1 + 7 + 11 = 19, 1 + 9 + 11 = 21, or 7 + 9 + 11 = 27.

Cathy can get $3 \times 4 = 12$, $3 \times 5 = 15$, $3 \times 6 = 18$, $4 \times 5 = 20$, $4 \times 6 = 24$, or $5 \times 6 = 30$.
Let B be the event that Tim's result is less than Cathy's result,

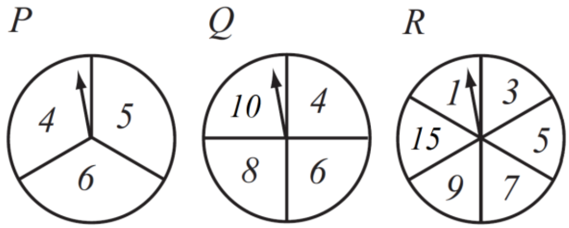
A_1 be the event that Tim picks up the number 17,
B_1 be the event that Tim picks up a number that is less than Cathy's number after A_1
A_2 be the event that Tim picks up the number 19,
B_2 be the event that Tim picks up a number that is less than Cathy's number after A_2
A_3 be the event that Tim picks up the number 21,
B_3 be the event that Tim picks up a number that is less than Cathy's number after A_3
A_4 be the event that Tim picks up the number 27,
B_4 be the event that Tim picks up a number that is less than Cathy's number after A_4
Then $B = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4$

$$P(B) = P(A_1B_1) + P(A_2B_2) + P(A_3B_3) + P(A_4B_4)$$
$$= P(A_1)P(B_1|A_1) + P(A_2)P(B_2|A_2) + P(A_3)P(B_3|A_3) + P(A_4)P(B_4|A_4)$$

Or $P(B) = \frac{1}{4} \times \frac{4}{6} + \frac{1}{4} \times \frac{3}{6} + \frac{1}{4} \times \frac{2}{6} + \frac{1}{4} \times \frac{1}{6} = \frac{1}{4} \times \frac{10}{6} = \frac{5}{12}$.

PROBLEMS

☆**Problem 1.** James rotates spinners *P, Q* and *R* and adds the resulting numbers. What is the probability that his sum is an odd number?



(A) 1/4 (B) 1/3 (C) 1/2 (D) 2/3 (E) 3/4

Problem 2. A number is selected at random from 101 through 900. What is the probability that the number selected is a perfect square?

(A) $\frac{1}{40}$ (B) $\frac{21}{800}$ (C) $\frac{1}{80}$ (D) $\frac{1}{5}$ (E) $\frac{5}{12}$

Problem 3. A bag contains 3 white, 4 blue, and 5 red marbles. What is the probability that a marble selected at random is blue?

(A) $\frac{1}{12}$ (B) $\frac{1}{3}$ (C) $\frac{1}{6}$ (D) $\frac{1}{5}$ (E) $\frac{5}{12}$

Problem 4. A bag contains 5 blue marbles, 4 white marbles, and 3 red marbles. If three marbles are randomly selected from the bag, what is the probability that the marbles selected will be of the same color?

(A) $\frac{1}{22}$ (B) $\frac{1}{110}$ (C) $\frac{1}{6}$ (D) $\frac{1}{5}$ (E) $\frac{3}{44}$

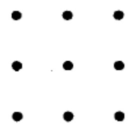
Problem 5. All three-digit numbers that have only the digits 1, 3, 5, or 7 in each position are recorded on slips of paper and placed in a container. If a slip of paper is picked at random from the container, what is the probability that it contains the number 153?

(A) $\frac{1}{12}$ (B) $\frac{1}{24}$ (C) $\frac{1}{64}$ (D) $\frac{1}{4}$ (E) $\frac{5}{24}$

Problem 6. Two fair cubical dice are tossed. What is the probability that the sum of the numbers showing on the dice will be four?

(A) $\frac{1}{12}$ (B) $\frac{1}{9}$ (C) $\frac{1}{36}$ (D) $\frac{1}{4}$ (E) $\frac{1}{18}$

Problem 7. What is the probability that three randomly selected points on the geoboard shown will be vertices of a triangle?

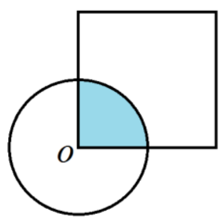


(A) $\frac{19}{21}$ (B) $\frac{1}{9}$ (C) $\frac{1}{36}$ (D) $\frac{1}{14}$ (E) $\frac{2}{21}$

Problem 8. Two different prime numbers are selected at random from among the first ten prime numbers. What is the probability that the sum of the two primes is 24?

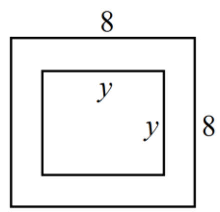
(A) $\frac{1}{45}$ (B) $\frac{2}{45}$ (C) $\frac{1}{15}$ (D) $\frac{1}{9}$ (E) $\frac{1}{5}$

Problem 9. In the figure shown, the circle with center O has a radius of 8 and the square has side length of 16. If a point is selected at random from within the region determined by the circle and the square, what is the probability that it will be within the shaded region?



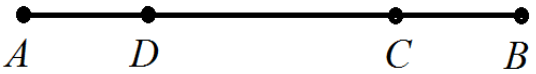
(A) $\frac{\pi}{\pi+16}$ (B) $\frac{\pi}{3\pi+4}$ (C) $\frac{\pi}{16}$ (D) $\frac{\pi}{3\pi+16}$ (E) $\frac{2\pi}{3\pi+16}$

Problem 10. Darts thrown at a board are equally likely to hit anywhere within a region on the board. If 75% of the darts land inside the small square, what is the value of y?



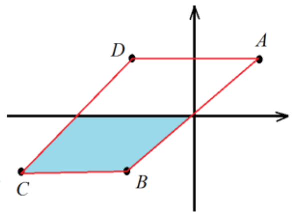
(A) $4\sqrt{3}$ (B) $3\sqrt{3}$ (C) $2\sqrt{3}$ (D) $\sqrt{3}$ (E) 7

Problem 11. Points A, B, C, and D, are located on $\overline{AB}$ such that $AB = 4AD = 5BC$. If a point is selected at random on $\overline{AB}$, what is the probability that it is between C and D?



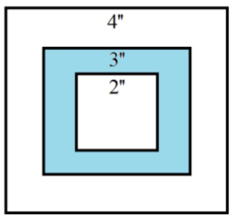
(A) $\frac{1}{2}$ (B) $\frac{11}{20}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$ (E) $\frac{9}{20}$

Problem 12. Parallelogram $ABCD$ has vertices $A(3, 3)$, $B(-3, -3)$, $C(-9, -3)$, and $D(-3, 3)$. If a point is selected at random from the region determined by the parallelogram, what is the probability that the point is not above the x-axis?



(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{5}$ (D) $\frac{1}{4}$ (E) $\frac{2}{3}$

Problem 13. A dart is thrown at the square target shown. Assuming the dart hits the target at a random location, what is the probability that it will be in the shaded region?



(A) $\frac{5}{16}$ (B) $\frac{3}{16}$ (C) $\frac{4}{4}$ (D) $\frac{3}{4}$ (E) $\frac{9}{16}$

Problem 14. When four fair coins are tossed, what is the probability that the outcome will consist of two heads and two tails?

(A) $\frac{1}{2}$ (B) $\frac{11}{20}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$ (E) $\frac{3}{8}$

Problem 15. Four coins are tossed. What is the probability that the outcome will be four heads or four tails?

(A) $\frac{3}{4}$ (B) $\frac{3}{8}$ (C) $\frac{1}{6}$ (D) $\frac{1}{8}$ (E) $\frac{1}{4}$

Problem 16. If a marble is chosen from a bag that contains 10 red marbles, 5 blue marbles, and 15 white marbles, what is the probability that the marble chosen is blue or red?

(A) $\frac{1}{2}$ (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$ (E) $\frac{2}{3}$

Problem 17. If you roll a pair of dice, what is the probability expressed as a common fraction that both numbers are 2's or that the sum is less than 6?

(A) $\frac{1}{2}$ (B) $\frac{5}{18}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$ (E) $\frac{2}{3}$.

Problem 18. In his locker, Andrew has 2 history books and 3 math books. In his rush to get to class, he grabs 1 book, then a second book, without stopping to look. What is the probability that he pulls a math book out first and a history book out second?

(A) $\frac{1}{2}$ (B) $\frac{5}{18}$ (C) $\frac{1}{3}$ (D) $\frac{3}{10}$ (E) $\frac{2}{3}$

Problem 19. A dime, 2 nickels, and 3 pennies are in a container. Assume that it is equally likely to shake out any one coin. What is the probability of shaking out a penny each of 4 times if the coin is returned after each shake?

(A) $\frac{1}{2}$ (B) $\frac{1}{8}$ (C) $\frac{1}{16}$ (D) $\frac{3}{10}$ (E) $\frac{2}{3}$

Problem 20. What is the probability that, in three single draws without replacement, two red marbles and one blue marble will be drawn in that order from a bag containing six red marbles, eight yellow marbles, and seven blue marbles?

(A) $\frac{1}{8}$ (B) $\frac{1}{18}$ (C) $\frac{3}{38}$ (D) $\frac{1}{4}$ (E) $\frac{1}{38}$.

Problem 21. A bag contains 6 marbles and each is red or blue. If 2 marbles are randomly selected, the chance that they are both blue is $\frac{1}{5}$. How many red marbles are in the bag?

(A) 3 (B) 5 (C) 7 (D) 9 (E)11.

Problem 22. What is the probability that, when Alex selects a positive even integer less than twenty and Bob picks a positive multiple of 3 less than thirty, they pick the same number?

(A) $\frac{1}{3}$ (B) $\frac{1}{9}$ (C) $\frac{2}{27}$ (D) $\frac{1}{27}$ (E) $\frac{1}{38}$.

Problem 23. Two distinct numbers are chosen at random from {1, 2, 3, 4, 5, 6}. What is the probability that the quotient of the smaller number divided by the larger number is a terminating decimal?

(A) $\frac{2}{5}$ (B) $\frac{3}{5}$ (C) $\frac{1}{6}$ (D) $\frac{1}{7}$ (E) $\frac{1}{8}$.

Problem 24. Digit d is randomly selected from the set {4, 5, 6, 7}. Without replacement of d, another digit e is selected. What is the probability that the two-digit number $\underline{de}$ is a multiple of 3?

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{7}$ (E) $\frac{1}{8}$.

Problem 25. What is the probability that David and Tim randomly select the same number if David selects a positive divisor of 64 and Tim selects a multiple of 3 that is less than 64?

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{7}$ (E) 0.

Problem 26. A bag contains only red marbles, blue marbles, and yellow marbles. The probability of randomly selecting a red marble from this bat is $\frac{1}{3}$, and the probability of randomly selecting a blue marble is $\frac{1}{5}$. Which of the following could be the total number of marbles in the bag?

(A) 10 (B) 15 (C) 18 (D) 20 (E) 32

Problem 27. A box contains wood beads, red glass beads, and blue glass beads. The number of glass beads is 7 times the number of wood beads. If one bead is to be chosen at random from the box, the probability that a red glass bead will be chosen is 6 times the probability that a blue glass bead will be chosen. If there are 24 red glass beads in the box, what is the total number of beads in the box?

(A) 32 (B) 42 (C) 48 (D) 60 (E) 64

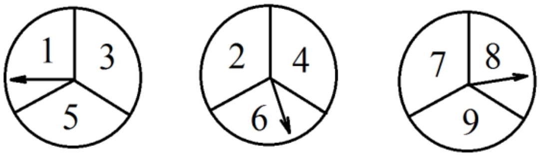
Problem 28. A list consists of all possible three-letter arrangements formed by using the letters A, B, C, D, E, F, G, H such that the first letter is D and one of the remaining letters is A. If no letter is used more than once in an arrangement in the list and one three-letter arrangement is randomly selected from the list, what is the probability that the arrangement selected will be DCA?

(A) 1/5 (B) 1/6 (C) 1/9 (D) 1/10 (E) 1/12

Problem 29. There are x (either hardback or paperback) books on a shelf. If one book is to be selected at random, the probability that a paperback will be selected is $\frac{5}{12}$. In terms of x, how many of the books are hardbacks?

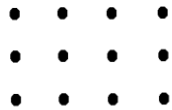
(A) $\frac{5x}{12}$ (B) $\frac{7x}{12}$ (C) $\frac{12x}{5}$ (D) $\frac{12x}{7}$ (E) $5x$

Problem 30. Anna is to spin each of the three spinners shown and then add the resulting numbers. What is the probability that the sum of the three numbers will be odd?



(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{7}$ (E) 1.

Problem 31. In the 3×4 grid shown, the points are one unit apart horizontally and vertically. Given that two points are randomly selected from the grid, what is the probability that the distance between them is $\sqrt{2}$?



(A) $\frac{1}{3}$ (B) $\frac{2}{11}$ (C) $\frac{1}{4}$ (D) $\frac{1}{7}$ (E) 0.

Problem 32. Two girls and three boys are to be seated in a row of five desks. What is the probability that the students at the ends of the row are both boys?

(A) $\frac{1}{3}$ (B) $\frac{2}{11}$ (C) $\frac{1}{4}$ (D) $\frac{3}{10}$ (E) $\frac{1}{5}$.

SOLUTIONS

☆**Problem 1. Solution:** D.

Because the sum of a number from spinner Q and a number from spinner R is always odd, the sum of the numbers on the three spinners will be odd exactly when the number from spinner P is even. Because 4 and 6 are even number on spinner P, the probability of getting an odd sum is 2/3.

Problem 2. Solution: A.

There are 900 – 101 + 1 = 800 numbers.

There are $\lfloor\sqrt{900}\rfloor-\lfloor\sqrt{101}\rfloor=30-10=20$ square numbers.

The probability that the number selected is a perfect square is 20/800 = 1/40.

Problem 3. Solution: B.

There are total 3 + 4 + 5 = 12 marbles. There are 4 blue marbles. The probability to draw a blue marble is then 4/12 = 1/3.

Problem 4. **Solution:** E.

There are total 5 + 4 + 3 = 12 marbles. If three marbles are randomly selected from the bag, the possible outcomes will be:

blue, blue, blue

white, white, white

red, red, red.

$$P=\frac{\binom{5}{3}+\binom{4}{3}+\binom{3}{3}}{\binom{12}{3}}=\frac{10+4+1}{220}=\frac{15}{220}=\frac{3}{44}.$$

Problem 5. Solution: C.

There are 4 × 4 × 4 = 64 such 3-digit numbers. There is only one 3-digit number of 153. So the probability is $\frac{1}{64}$.

Problem 6. Solution: A.

When two dice are rolled, there are 36 outcomes. Three outcomes show that the sum of the integers on the top faces is 4.

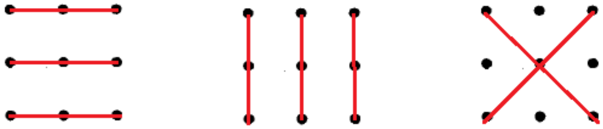
D_1 / D_2	1	2	3	4	5	6
1	***2***	***3***	***4***	***5***	***6***	***7***
2	***3***	***4***	***5***	***6***	***7***	***8***
3	***4***	***5***	***6***	***7***	***8***	***9***
4	***5***	***6***	***7***	***8***	***9***	***10***
5	***6***	***7***	***8***	***9***	***10***	***11***
6	***7***	***8***	***9***	***10***	***11***	***12***

The probability is 3/ 36 = 1/12.

Problem 7. Solution: A.

The number of ways to select three vertices from the nine vertices is $\binom{9}{3} = 84$.

We must exclude from our count those sets of three points that are collinear.



The probability is (84 – 8)/84 = $\dfrac{19}{21}$.

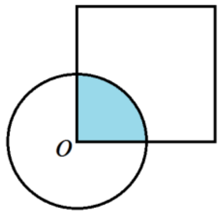
Problem 8. Solution: C.

Let us list the first ten prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

We have 3 ways to get a sum of 24: 5 + 19 = 24; 7 + 17 = 24; and 11 + 13 = 24.

So we have $\binom{10}{2} = 45$ sums and the probability is 3/45 = 1/15.

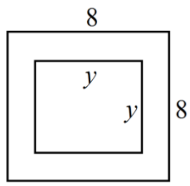
Problem 9. Solution: D.



$$P = \frac{\text{measure of area of favorable region}}{\text{measure of area of total region}}$$

$$P = \frac{\frac{1}{4}\pi \times 8^2}{\frac{3}{4}\pi \times 8^2 + 16^2} = \frac{16\pi}{3 \times 16\pi + 16^2} = \frac{\pi}{3\pi + 16}.$$

Problem 10. Solution: A.



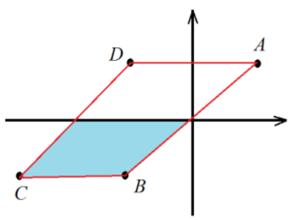
$$P = \frac{y^2}{8^2} = \frac{75}{100} = \frac{3}{4} \quad \Rightarrow \quad y^2 = \frac{3}{4} \times 8^2 \quad \Rightarrow$$

$$y = 4\sqrt{3}$$

Problem 11. Solution: B.

$$P = \frac{CD}{AB} = \frac{AB - BC - AD}{AB} = \frac{4AD - \frac{4}{5}AD - AD}{4AD} = \frac{4 - \frac{4}{5} - 1}{4} = \frac{11}{20}.$$

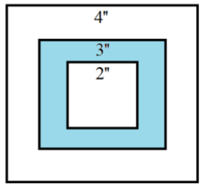
Problem 12. Solution: A.



$$P = \frac{\frac{1}{2}S_{ABCD}}{S_{ABCD}} = \frac{1}{2}.$$

Problem 13. Solution: A.

$$P = \frac{\text{measure of area of favorable region}}{\text{measure of area of total region}}$$



$$P = \frac{\text{Area of shaded}}{\text{Area of largest square}} = \frac{3^2 - 2^2}{4^2} = \frac{5}{16}$$

Problem 14. **Solution:** E.

There are total $2 \times 2 \times 2 \times 2 = 16$ ways to flip four coins.

There are 4 ways to get exactly two heads and two tails (HHTT): $\frac{4!}{2! \times 2!} = 6$.

The probability that the outcome will consist of two heads and two tails is: 6/16 = 3/8.

Problem 15. **Solution:** D.

There are total $2 \times 2 \times 2 \times 2 = 16$ ways to flip three coins.

There is only 1 way to flip four heads.

There is only 1 way to flip four tails.

The probability of all heads: 1/16.

The probability of all tails: 1/6.

Since the two events are mutually exclusive (you can't flip a coin and get both head and tail), $P(A \cap B) = 0$

The probability that all are heads or all are tails can be calculated using the formula: $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$.

Problem 16. **Solution:** A.

There are total 10 + 5 + 15 = 30 marbles. There are 5 blue and 10 red marbles.

The probability to draw a blue marble is 5/30 = 1/6.

The probability to draw a red marble is 10/30 = 1/3.

The probability that the marble chosen is blue or red can be calculated using the formula: $P(A \text{ or } B) = P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$

$$= \frac{1}{6} + \frac{1}{3} - 0 = \frac{3}{6} = \frac{1}{2}.$$

Problem 17. Solution: B.

In 1 of the 36 possible outcomes the two numbers are 2's. There are 9 outcomes that the sum is less than 6. The probability is (1 + 9)/36 = 10/36 = 5/18.

D_1 \ D_2	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 8
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Problem 18. Solution: D.

$$P = \frac{\binom{3}{1} \times \binom{2}{1}}{5 \times 4} = \frac{6}{20} = \frac{3}{10}.$$

Problem 19. Solution: C.

There are total 1 + 2 + 3 = 6 coins. There are 5 blue and 10 red marbles.

The probability of shaking out a penny is 3/6 = 1/2.

The probability of shaking out a penny each of 4 times if the coin is returned after each shake is then $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$.

Problem 20. **Solution:** E.

There are total 6 + 8 + 7 = 21 marbles.

$$P = \frac{\binom{6}{1} \times \binom{5}{1} \times \binom{7}{1}}{21 \times 20 \times 19} = \frac{210}{7980} = \frac{1}{38}.$$

Problem 21. **Solution:** A.

$$P = \frac{\binom{b}{2}}{\binom{6}{2}} = \frac{1}{5} \quad \Rightarrow \binom{b}{2} = \frac{15}{5} = 3 \quad \Rightarrow \quad \frac{b(b-1)}{2} = 3 \quad \Rightarrow b^2 - b - 6 = 0 \qquad \Rightarrow$$

$(b+2)(b-3) = 0$. So $b = 3$.

Problem 22. **Solution:** D.

Alex can select 2, 4, 6, 8, 10, 12, 14, 16, and 18

Bob can select 3, 6, 9, 12, 15, 18, 21, 24, and 27.

They can select the following three numbers: 6, 12, and 18.

The probability that Alex select these three numbers is $P_A = \frac{3}{9} = \frac{1}{3}$.

The probability that Bob matches Alex selection is $P_B = \frac{1}{9}$.

The answer is $P = \frac{1}{3} \times \frac{1}{9} = \frac{1}{27}$.

Problem 23. **Solution:** B.

The fraction can be expressed as a terminating decimal if the denomination contains only factors of 2 or 5.

We have $\binom{6}{2} = 15$ ways to select two numbers.

The following fractions are not terminating:
(1, 3), (2, 3), (1,6), (2, 6), (4, 6), (5, 6).

The probability that the quotient of the smaller number divided by the larger number is a repeating decimal is $P_r = \frac{6}{15} = \frac{2}{5}$.

The probability that the quotient of the smaller number divided by the larger number is a terminating decimal is $P_t = 1 - P_r = 1 - \frac{2}{5} = \frac{3}{5}$.

Problem 24. **Solution:** A.

Method 1: If a two-digit number is divisible by 3, then the sum of its digits must be divisible by 3. The following cases are only ones that work: (4, 5), (5, 4), (7, 5), and (5, 7).

Let A_1 be the event of selecting the digit 4 as the digit *d*.
B_1 be the event of selecting the digit 5 as the digit *e*.
A_2 be the event of selecting the digit 5 as the digit *d*.
B_2 be the event of selecting the digit 4 as the digit *e*.
A_3 be the event of selecting the digit 7 as the digit *d*.
B_3 be the event of selecting the digit 5 as the digit *e*.
A_4 be the event of selecting the digit 5 as the digit *d*.
B_4 be the event of selecting the digit 7 as the digit *e*.
B be the event that the two-digit number $\underline{de}$ is a multiple of 3.

$B = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4$

$P(B) = P(A_1B_1) + P(A_2B_2) + P(A_3B_3) + P(A_4B_4)$

$P(A_1) = \frac{1}{4}$; $P(A_2) = \frac{1}{4}$; $P(A_3) = \frac{1}{4}$; $P(A_4) = \frac{1}{4}$.

$P(B|A_1) = \frac{1}{3}$; $\quad P(B|A_2) = \frac{1}{3}$; $\quad P(B|A_3) = \frac{1}{3}$; $\quad P(B|A_4) = \frac{1}{3}$.

$$P(B) = \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{3} = \frac{4}{12} = \frac{1}{3}.$$

Method 2: There are a total of 12 ways to select two digits (four ways to select first digit and 3 ways to select second digit).

There are four favorable ways (4, 5), (5, 4), (5, 7), and (7, 5) such that the two-digit number is a multiple of 3.

The probability is $P = \frac{4}{4 \times 3} = \frac{1}{3}$.

Problem 25. **Solution:** E.

The positive divisors of 64 are: 1, 2, 4, 8, 16, 32, 64.

The multiple of 3 includes: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63.

No number is belonging to two sets. So the probability is 0.

Problem 26. Solution: B.

Let the total number of marbles be t, the number of red marbles be r, and the number of blue marbles be b.

$$\frac{\binom{r}{1}}{t} = \frac{1}{3} \qquad \Rightarrow \qquad 3r = t \qquad (1)$$

$$\frac{\binom{b}{1}}{t} = \frac{1}{5} \qquad \Rightarrow \qquad 5b = t \qquad (2)$$

Solving (1) and (2): $3r = 5b$ (3)

We know that r and b are both positive integers. So we have $b = 3$ and $r = 5$.

$t = 5b = 15$ or a multiple of 15. So B is the only answer.

Problem 27. Solution: A.

Let the number of red glass beads be r_g, and the number of blue glass beads be b_g.

The number of wood beads will be $\frac{1}{7}(r_g + b_g)$.

$$\frac{r_g}{r_g + b_g + \frac{1}{7}(r_g + b_g)} = 6 \times \frac{b_g}{r_g + b_g + \frac{1}{7}(r_g + b_g)} \qquad \Rightarrow \qquad r_g = 6b_g$$

We know that $r_g = 24$. So $b_g = 4$, and $\frac{1}{7}(r_g + b_g) = \frac{1}{7}(24 + 4) = 4$.

The answer is 24 + 4 + 4 = 32.

Problem 28. Solution: E.

We have two cases:

Case 1: *DAX*

Case 2: *DXA*

X represents the letter undetermined.

For each case, we have 6 ways to put another letter.

We only have one way to get *DCA*.

So the probability is 1/12.

Problem 29. Solution: B.

Let the number of paperback books be *p*.

$$\frac{p}{x} = \frac{5}{12} \qquad \Rightarrow \qquad p = \frac{5}{12}x.$$

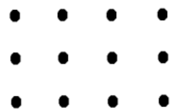
The number of hardback books is $x - p = x - \frac{5}{12}x = \frac{7}{12}x$.

Problem 30. Solution: A.

The sum of the first and the second spinners is odd. If the final sum is odd, the third spinner needs to show an even number.

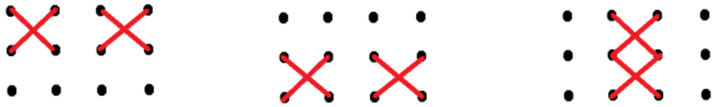
The probability that the third spinner shows an even number is 1/3. So the answer is 1/3.

Problem 31. Solution: B.



We have $\binom{12}{2} = 66$ line segments.

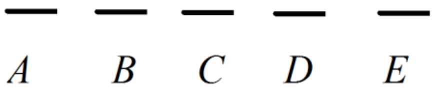
We have 12 line segments with the distance of $\sqrt{2}$:



The answer is then 12/66 = 2/11.

Problem 32. Solution: D.

We have five positions as follows.



The total number of ways to seat them is 5! = 120.

For position A, we can seat boy 1, boy 2 or boy 3. So we get 3 ways.

For position E, after position A is occupied, we have 2 ways.

After positions A and E are occupied, we have 3! Ways to seat one boy and two girls.

So we get 3 × 2 × 3! = 36 ways such that the students at the ends of the row are both boys.

The answer is then P = 36/120 = 3/10.

A

B

C

D

E

F

G

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