American Mathematics Competitions (AMC 8) Preparation

Volume 3

The American Mathematics Competitions 8 is a 25-question multiple-choice contest for students in the sixth through eighth grade. Accelerated fourth and fifth graders can also take part. The AMC 8 is administered in schools in November. The American Mathematics Competitions (AMC) publishes the Achievement Roll list recognizing students in 6th grade and below who scored 15 or above, and the Honor Roll list recognizing students who score in the top 5%, and the Distinguished Honor Roll list recognizing students who score in the top 1%.

This book can be used by 5th to 8th grade students preparing for AMC 8. Each chapter consists of (1) basic skill and knowledge section with plenty of examples, (2) about 30 exercise problems, and (3) detailed solutions to all problems.

We would like to thank the American Mathematics Competitions (AMC 8 and 10) for their mathematical ideas. Many problems (marked by ☆) in this book are inspired from these tests. We only cited very few problems directly from these tests for the purpose of comparison with our own solutions.

We wish to thank the following reviewers for their invaluable solutions, insightful comments, and suggestions for improvements to this book:
Alex Cheng (UT), Jin Cheng (CA), Felix Cui (NE), Albert Hao (CA), Sameer Khan (VA), Kathy Liu (VA), Priyo Majumdar (LA), Aadith Menon, Kalyanasundaram Seshadri (CA), Huili Shao (MA), Stephan Xie (TX), Cindy Ye (AR), Samuel Yoon (VA) and Sophia Zhang (CO).

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Please contact mymathcounts@gmail.com for suggestions, corrections, or clarifications.

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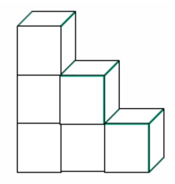
1. THREE-DIMENSIONAL FIGURE

Draw the three views of a three-dimensional figure

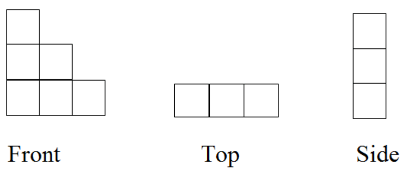
A three-dimensional figure can be represented by its three views: the front view, side view, and top view.

To draw a top view, imagine flying like a bird above the three-dimensional figure and looking straight down on it. If you flew above the pyramid and looked down, you would see more than just the top point. You would see the outline of the base of the figure as well as the four edges that lead to the corners of the base. For a bottom view, imagine lying below the three-dimensional figure and looking straight up. For side views, imagine floating around the figure and taking a picture of what you see when you face each side directly.

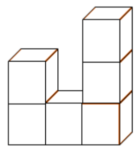
Example 1. Draw the figure shown in front, top, and side views.



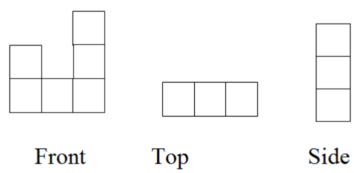
Solution:
From the top and side views, there appears to be 3 cubes on the top level. The front view shows that the figure has six cubes.



Example 2. Draw the front, top, and side views of the figure.

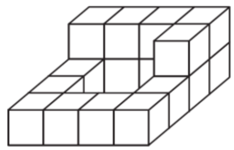


Solution:
Front: The figure looks like a row of 3 squares on the bottom with 2 squares on top of the right side and 1 square on top of the left side.

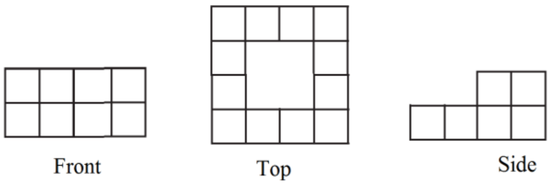


Top: The figure looks like a row of 3 squares.
Side: The figure looks like a column of 3 squares.

Example 3. Draw the front, top, and side views of the following solid.



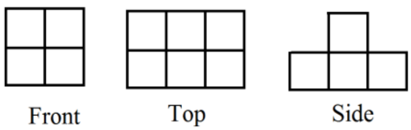
Solution:



Draw the three-dimensional figure giving three views

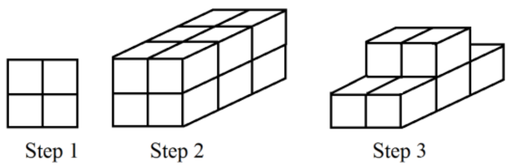
Different views of a solid figure are shown. A point of view is called a perspective. You can draw a three-dimensional figure using three different perspectives.

Example 4. The front, top, and side views of a stack of cubes are shown below. Draw the three-dimensional figure.



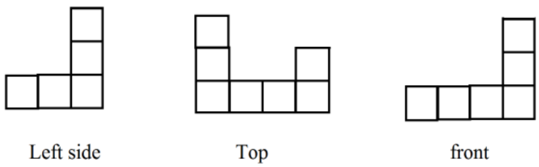
Solution:
Step 1. Use the front view to build the front side of the figure. The front view shows that the front side is a 2-by-2 square.

Step 2. Use the top view to draft the figure. The top view shows that the length of the solid is 3 units.



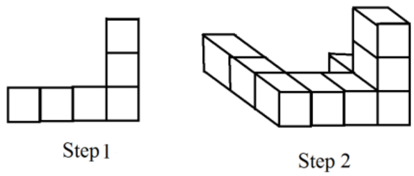
Step 3. Use the side view to complete the figure. The side view shows that the base layer has three cubes and the top layer has only one cube in the middle. So we remove four cubes from the second layer.

Example 5. A figure made up of unit cubes appears from the different views. What is the minimum number of cubes which could be used to build this figure?

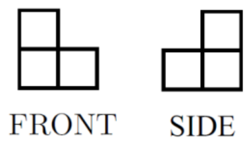


(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Solution: B.

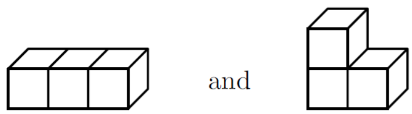


☆**Example 6.** (2003 AMC 8) A figure is constructed from unit cubes. Each cube shares at least one face with another cube. What is the minimum number of cubes needed to build a figure with the front and side views shown?

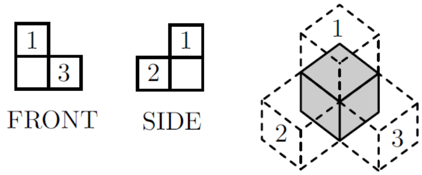


(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

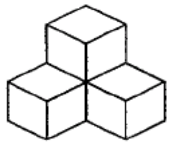
Solution: (B).
There are only two ways to construct a solid from three cubes so that each cube shares a face with at least one other:



Neither of these configurations has both the front and side views shown. The four-cube configuration has the required front and side views. Thus at least four cubes are necessary.



Example 7. The outside of this set of four cubic blocks with no space between blocks is painted. What is the number of painted square faces?



(A) 12 (B) 14 (C) 16 (D) 18 (E) 24

Solution: D.

We view this solid from three sides:

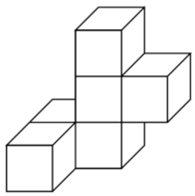
Top (and bottom): we see 3 squares. $3 \times 2 = 6$.

Front (and back): we see 3 squares. $3 \times 2 = 6$.

Left side (and right): we see 3 squares. $3 \times 2 = 6$.

The number of painted square faces is $6 \times 3 = 18$.

☆**Example 8.** (2002 AMC 8 problem 22) Six cubes, each an inch on an edge, are fastened together, as shown. Find the total surface area in square inches. Include the top, bottom and sides.



(A) 18 (B) 24 (C) 26 (D) 30 (E) 36

Solution: C.

Method 1: When viewed from the top and bottom, there are 4 faces exposed; from the left and right sides, there are 4 faces exposed and from the front and back, there are 5 faces exposed. The total is $4 + 4 + 4 + 4 + 5 + 5 = 26$ exposed faces.

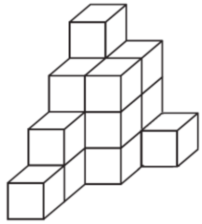
Method 2:

Before the cubes were glued together, there were $6 \times 6 = 36$ faces exposed. Five pairs of faces were glued together, so $5 \times 2 = 10$ faces were no longer exposed. This leaves $36 - 10 = 26$ exposed faces.

Drawing a base plan

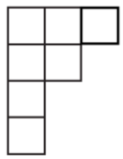
A base plan shows the shape of the base when viewed from above. The base plan indicates the height of each part of the base, usually with numbers. Base plans are particularly useful for figures with rectangular faces.

Example 9. Draw a base plan for the following solid object:



Solution:
Step 1. Draw the base as it looks from above. Since the solid object is made of cubes,
draw one square for each stack.

Step 2. Write the number of cubes in each stack inside the correct squares.



Step 1

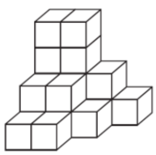
4	3	1
3	3	
2		
1		

Step 2

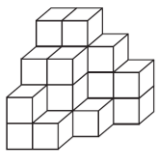
Example 10. The drawing below is a base plan of a solid figure made of stacked cubes.

4	4	3	2
3	3	1	
2	1		

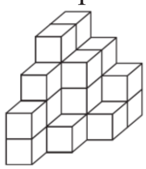
Which of the following solid figures is represented by the base plan above?



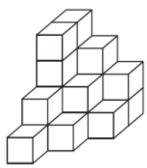
(A)



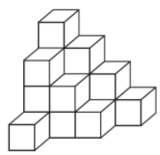
(B)



(C)



(D)



(E)

Solution: B.

☆**Example 11.** (1999 AMC 8 problem 20) Figure 1 is called a "stack map." The numbers tell how many cubes are stacked in each position. Fig. 2 shows these cubes, and Fig. 3 shows the view of the stacked cubes as seen from the front.

3	4
2	1

Figure 1

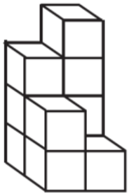


Figure 2

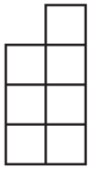
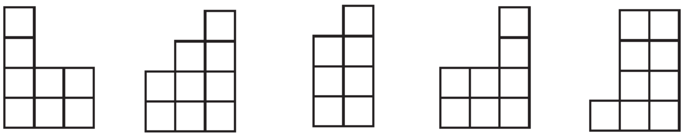


Figure 3

2	2	4
1	3	1

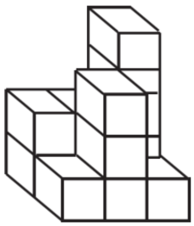
Figure 4

Which of the following is the front view for the stack map in Figure 4?

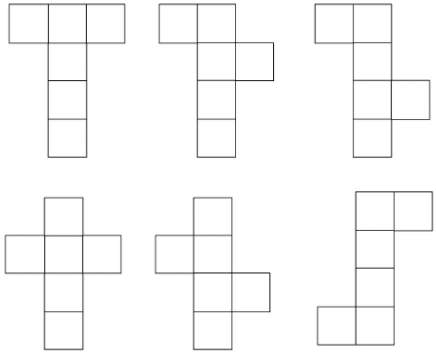


(A) (B) (C) (D) (E)

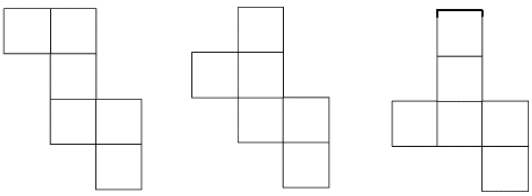
Solution: B.
The front view shows the larger of the numbers of cubes in the front or back stack in each column. Therefore the desired front view will have, from left to right, 2, 3, and 4 cubes. This is choice B.



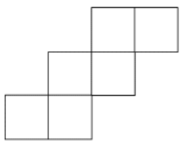
2. NET OF A CUBE (11 TOTAL)



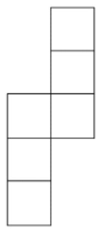
one-four-one



one-three-two

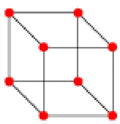


two-two-two

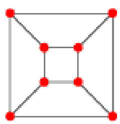


three-three

Figure (b) is a topological transformation of the edges of the cube in figure (a).



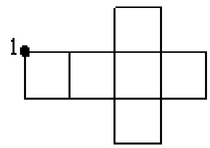
(a)



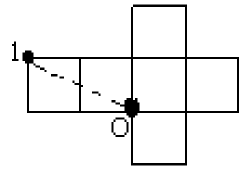
(b)

Find the other vertices that meet with a given vertex when the net is folded

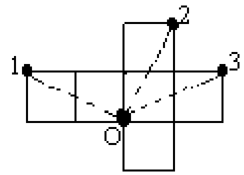
Which vertices meet with vertex 1 when the net below is folded?



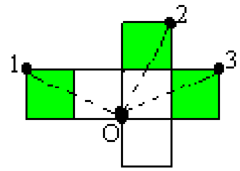
Step 1. Draw a line across **two square faces**, that is, draw a line from the given vertex 1 to point *O*.



Step 2. Find other points that are the same distance to point *O*. In this case find the points 2 and 3.



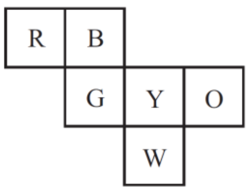
Step 3. Conclusion: Vertices 1, 2 and 3 will meet when the net is folded.



Note that for a cube, three faces share one vertex.

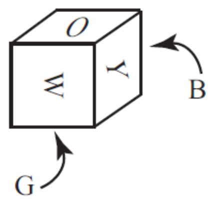
☆**Example 12.** (1999 AMC 8) Six squares are colored, front and back, (R=red, B=blue, O=orange, Y=yellow, G=green, and W=white). They are hinged together as shown, then folded to form a cube. The face opposite the white face is

(A) B (B) G (C) O (D) R (E) Y

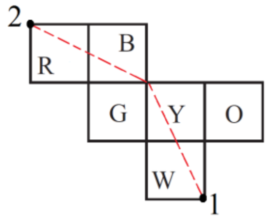


Solution: A.

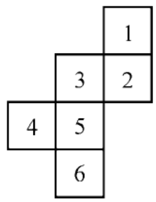
Method 1 (official solution):
When G is arranged to be the base, B is the back face and W is the front face. Thus, B is opposite W.



Method 2 (our solution):
We know that faces G, Y, and O are adjacent to face W. So we only need to examine faces R and B. We draw two line segments as shown in the figure below.
We then know that vertex 2 will meet with vertex 1 when the net is folded. So B must be the face opposite to W.



☆**Example 13.** (2003 Mathcounts Chapter) When this net of six squares is cut out and folded to form a cube, what is the product of the numbers on the four faces adjacent to the one labeled with a "1" ?

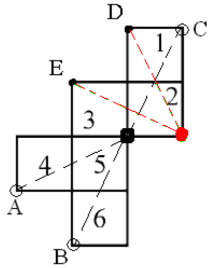


Solution: 144.
Method 1 (official solution):
With our visualizing skills, we imagine the cube folding back up. The "1" will touch 2, 3, 4, and 6. The product of the numbers on the four faces adjacent to the one labeled with a "1" is $2 \times 3 \times 4 \times 6 = 144$.

Method 2 (our solution):
We know that face 2 is adjacent to face 1. We draw three black line segments as shown in the figure below.

From these line segments, we know that vertices *A*, *B*, and *C* will meet when the cube is formed, so we have three faces (2, 4 and 6) of the four faces.

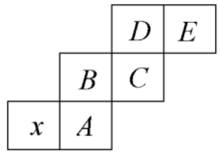
We also know that the face marked 3 is adjacent to the one labeled with "1" since E and D will meet when folded.



Therefore the product of the numbers on the four faces adjacent to the one labeled with a "1" is $2 \times 3 \times 4 \times 6 = 144$. (Can you see which face is opposite to face 1?)

☆**Example 14.** (1995 AMC 12, 2003 Mathcounts School) The figure shown can be folded into the shape of a cube. In the resulting cube, which of the lettered faces is opposite the face marked x?

(A) A (B) B (C) C (D) D (E) E

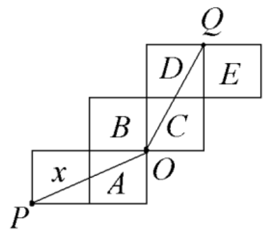


Solution: C.

Method 1 (official solution):

Think of A as the bottom. Fold B up to be the back. Then x folds upward to become the left side and C folds forward to become the right side, so C is opposite x.

Method 2 (our solution):

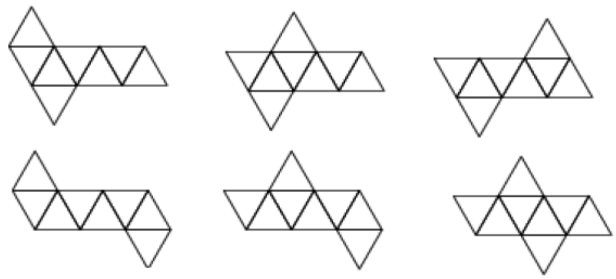


We select a point P and draw a line segment PO across two faces. We find another point which is Q as shown in the figure. We then know that P and Q will meet when folded. The face x will share the same vertex with the faces D and E, so D and E are not opposite to x. We also know that face x shares the same vertex with faces B and A. The only face left is face C, which must be opposite x.

3. NET OF AN OCTAHEDRON

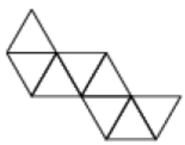
An **octahedron** is the solid with six polyhedron vertices, twelve polyhedron edges, and eight equivalent equilateral triangular faces.

There are 11 distinct nets for the octahedron.

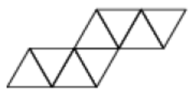


one – six – one

one – five – two

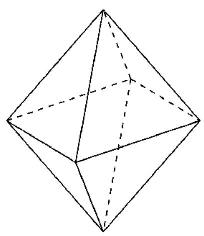


one – four – three

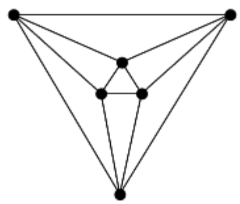


four – four

Figure (b) is a topological transformation of the edges of the octahedron in figure (a).



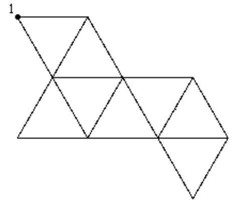
(a)



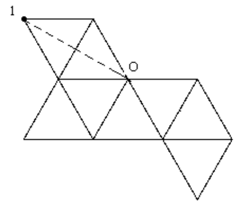
(b)

Find the other vertices that meet with a given vertex when the net is folded

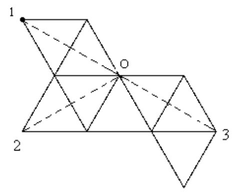
Which vertices meet with vertex 1 when the net below is folded?



Step 1. Draw a line across **two triangular faces**, i.e. draw a line from the vertex 1 to point *O*.

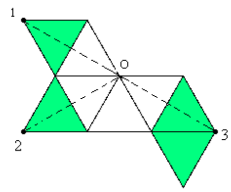


Step 2. Find other points that are the same distance to point *O* (in this case, find the points 2 and 3).

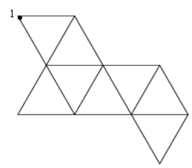


Step 3. Conclusion: Vertices 1, 2, and 3 will meet when the net is folded.

Note that for an octahedron, four faces share one vertex.

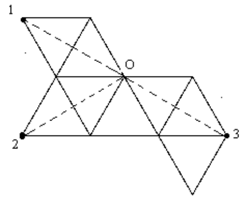


Example 15. Find the other vertices that meet with Vertex 1 when the resulting net is folded.



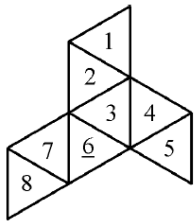
Solution: Vertices 2 and 3.
Step 1: Draw a line that crosses two triangular faces, i.e. draw a line from the vertex 1 to point *O*.

Step 2: Find other points that are the same distance to point O, in this case, points 2 and 3.



Note for an octahedron, four faces share one vertex.

Example 16. (2004 Mathcounts State Sprint #21) This net is folded into a regular octahedron. What is the sum of the numbers on the triangular faces sharing an edge with the face with a "1" on it?

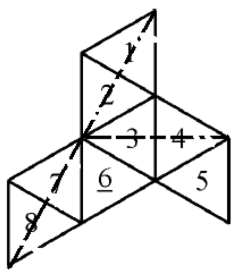


Solution: 14.
Method 1 (official solution):
The net is folded into a regular octahedron, and each triangle is equilateral.

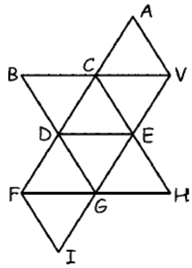
Once folded, it looks like two square pyramids connected at their square bases. The sides around the top are 1, 2, 3 and 4 and the sides around the bottom are 5, 6, 7 and 8. Side 1 is connected to 2 and 4 on the top and 8 on the bottom. 4 + 8 + 2 = 14 Ans.

Method 2 (our solution):
From the figure, it is easy to see that face 2 shares an edge with face 1. Now we just need to find the other two faces.
As shown in the figure and from our method, the faces are 8 and 4. (When folded, four faces 1, 4, 5, and 8 share one vertex. Note face 5 shares a vertex with face 1 but not an edge).

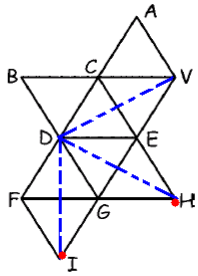
The sum of the three faces is 8 + 4 + 2 = 14.



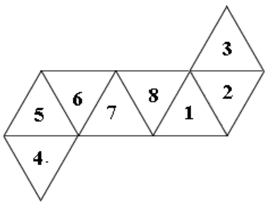
Example 17. The net below can be folded up to form an octahedron. When it is folded up, which two vertices are glued to vertex *V*?



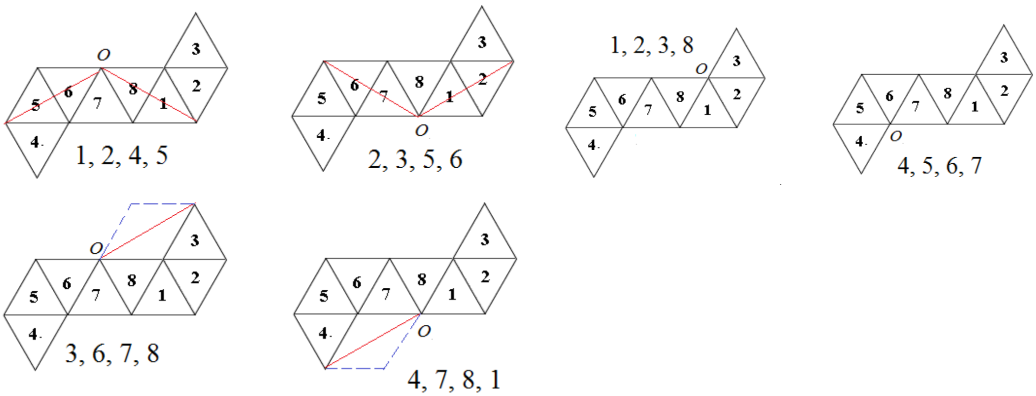
Solution: H, I.
The two vertices are *H* and *I*.



Example 18. If the strip of triangles, as shown below, is folded to form an octahedron, and each vertex is assigned the value of the sum of the four triangular faces to which it belongs, find the minimum value of a vertex.



Solution: 12.
The minimum value is on the vertex with 1, 2, 4, and 5 and the sum is 12.



4. IMPORTANT FORMULAS

(1) Euler's formula: $F + V = E + 2$

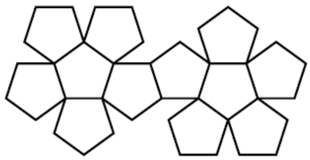
F: number of faces, V: number of vertices, and E: number of edges of a polyhedron.

Polyhedron	Faces	Vertices	Edges
Tetrahedron	4	4	6
Cube	6	8	12
Octahedron	8	6	12
Icosahedron	20	12	30
Dodecahedron	12	20	30

(2) Edges and faces formula: $E = \dfrac{F \times S}{2}$

E is the number of edges of the solid, F is the number of faces of the solid or net, and S is the number of sides of the faces.

☆**Example 19.** (2007 Mathcounts Chapter Team #6) The diagram shown is the net of a regular dodecahedron. In a regular dodecahedron, three edges come together at each vertex. When the net of this dodecahedron is put together, the solid has x vertices and y edges. What is the value of $x + y$?

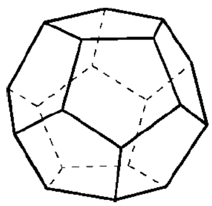


(A) 25 (B) 30 (C) 35 (D) 40 (E) 50

Solution: E.

Method 1 (official solution):

A dodecahedron, which is made up of 12 pentagons, looks like this when the net is put together.



Clearly you can see 5 vertices on the pentagon in front and, similarly, there are 5 vertices on the pentagon in the back. The other 10 pentagons are split into 5 which abut the front pentagon and 5 which abut the rear pentagon. Each of the first 5 shares their vertices with the second 5.

While each of the first 5 pentagons touch the second 5 pentagons at 3 vertices, one of them is shared with a neighbor. Thus, each of the 5 pentagons shares 2 vertices with the $5 \times 2 = 10$. $10 + 10 = 20$ vertices. For the edges, we have the 5 on the front pentagon and 5 on the rear pentagon. Then, there are 5 edges emanating from the vertices of the front pentagon and 5 emanating from the rear pentagon.

Finally, there are 10 more where the two sets of 5 vertices meet each other. $10 + 10 + 10 = 30$. $20 + 30 = 50$.

Method 2 (our solution):
We know that $F + V = E + 2$, which can be rearranged to: $V + E = 2E + 2 - F$
We also know that $E = \frac{F \times S}{2}$ (two sides of the net will become one edge when folded), so $E = 12 \times 5 \div 2 = 30$. $F = 12$
$x + y = V + E = 2E + 2 - F = 2 \times 30 + 2 - 12 = 50$.

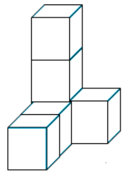
Example 20. A convex polyhedron has twelve faces and 8 vertices. There are two vertices with six edges coming together at each vertex. The rest of vertices with n edges come together at each vertex. Find n.
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: D.
We know that $F + V = E + 2 \Rightarrow V + F - E = 2 \Rightarrow 8 + 12 - E = 2$.
$E = 18$.
By the Edges and Faces formula, we have:

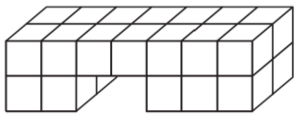
$$E = \frac{F \times S}{2} = \frac{2 \times 6 + 6 \times N}{2} = 18 \qquad \Rightarrow \qquad n = 4.$$

PROBLEMS

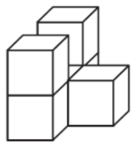
Problem 1. Draw the figure shown in front, top, and side views.



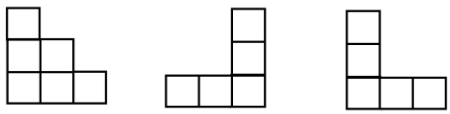
Problem 2. Draw the front, top, and side views of the following solid.



Problem 3. Draw the front, top, and side views of the following solid.



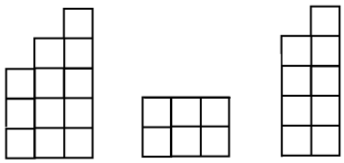
Problem 4. How many unit cubes are needed to build a stack of cubes which has the following views?



Front Top Side

(A) 8 (B) 9 (C) 10 (D) 12 (E) 14

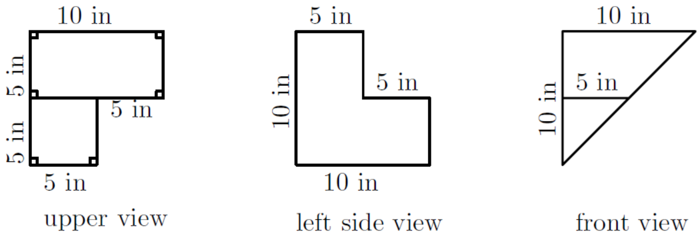
Problem 5. The front, bottom and side views of a three-dimensional figure are shown. Individual unit cubes are stacked to form the figure. What is the least possible number of cubes needed to build this figure?



Front Bottom Side

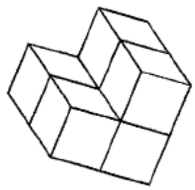
(A) 15 (B) 20 (C) 10 (D) 12 (E) 14

Problem 6. Using the following pictures, find the volume of the 3-dimensional figure.



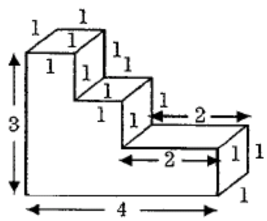
(A) 208.3 in^3 (B) 312.5 in^3 (C) 350 in^3 (D) 416.7 in^3 (E) 625 in^3

Problem 7. The length of an edge of each of the six cubes in the solid is 2 centimeters. In square centimeters, what is the total surface area of the solid?



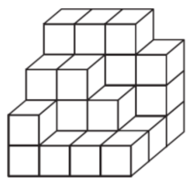
(A) 80 (B) 88 (C) 90 (D) 92 (E) 94

Problem 8. The following polyhedron with dimensions as given has a volume of 7 cubic units. How many faces does the figure contain? (Three edges and one vertex of the polyhedron are not shown in the diagram.)



(A) 8 (B) 9 (C) 10 (D) 12 (E) 14

Problem 9. The drawing shows a solid made of stacked cubes. Which of the following best represents the base plan for the solid?



4	4	4	3
3	3	2	1
2	1	1	1

(A)

4	4	4	3
3	3	2	
2	1		

(B)

3	3	3	2
3	2	2	1
2	1	1	1

(C)

4	4	4	2
3	2	1	
2	1		

(D)

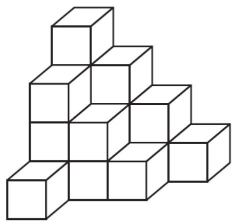
4	4	3	2
3	3	1	
2	1		

(E)

Problem 10. Draw a solid object that corresponds to the following base plan.

4	3	2
3	2	1
1		

Problem 11. Draw a base plan for the following solid object.



Problem 12. The piece of paper can be folded up to form a cube. What numbered face will be opposite the number 6 face?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

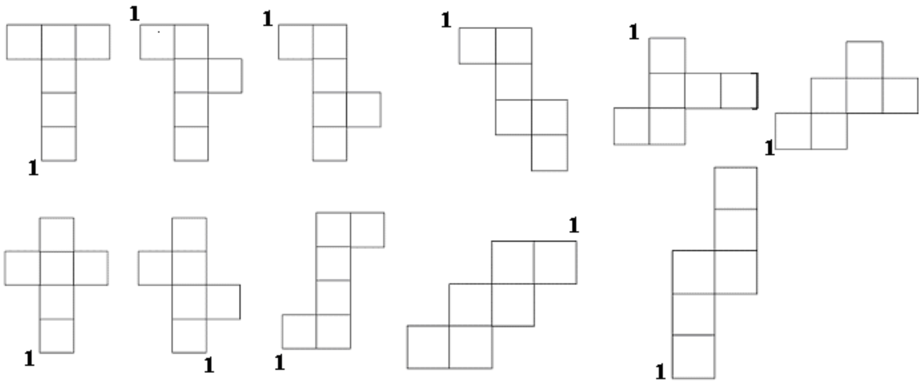
5	2	1
	3	
	4	
	6	

Problem 13. The figure may be folded along the lines shown to form a number cube. Three number faces come together at each corner of the cube. What is the largest sum of three numbers whose faces come together at a corner?

(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

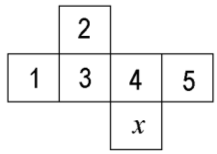
	1		
6	2	4	5
	3		

Problem 14. When folded up, find all the other vertices that meet with vertex 1 in the following nets.



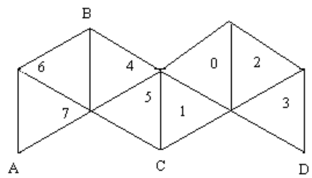
Problem 15. When folded to form a cube, what is the value in the square opposite the one marked x?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5



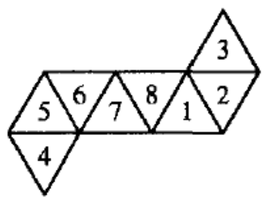
Problem 16. Find the sum of the numbers on the triangular faces that share the same vertex as A.

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18



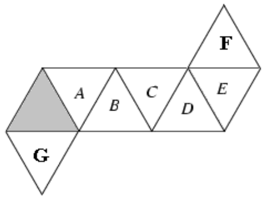
Problem 17. If the strip of triangles, as shown, is folded to form an octahedron, and each vertex is assigned the value of the sum of the four triangular faces to which it belongs, find the maximum value of a vertex.

(A) 18 (B) 19 (C) 20 (D) 22 (E) 24



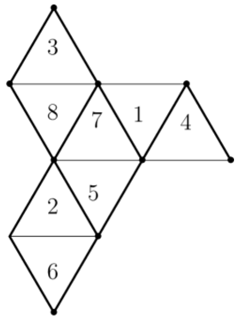
Problem 18. When the strip of triangles is folded to form an octahedron, which face is opposite the shaded one?

(A) F (B) B (C) C (D) D (E) E



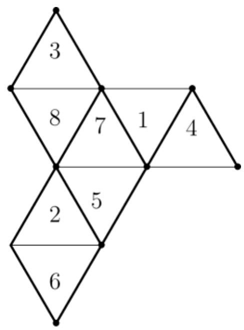
Problem 19. An octahedral net is a collection of adjoining triangles that can be folded into a regular octahedron. When the net below is folded to form an octahedron, what is the sum of the numbers on the faces adjacent to one marked with a 3?

A. 13 B. 15 C. 17 D. 18 E. 19



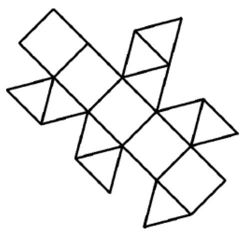
Problem 20. An octahedral net is a collection of adjoining triangles that can be folded into a regular octahedron. When the net below is folded to form an octahedron, what is the sum of the numbers on the faces adjacent to one marked with a 4?

A. 13 B. 15 C. 17 D. 18 E. 12



☆**Problem 21**. (2004 Mathcounts State Team) This net with 5 square faces and 10 equilateral triangular faces is folded into a 15-faced polyhedron. How many edges does the polyhedron have?

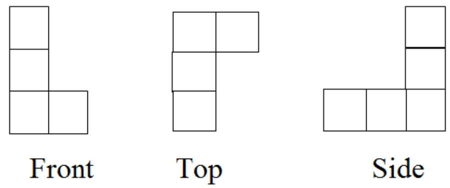
(A) 28 (B) 25 (C) 20 (D) 18 (E) 14



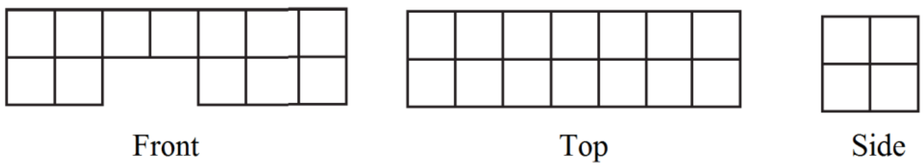
Problem 22. A convex polyhedron has 36 faces, 24 of which are triangular, and 12 of which are quadrilaterals. Find the number of space diagonals the polyhedron has. (A *space diagonal* is a line segment connecting two vertices which do not belong to the same face).
(A) 325 (B) 301 (C) 265 (D) 241 (E) 214

SOLUTIONS

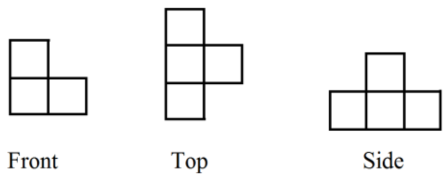
Problem 1. Solution:
From the front and top views, there appears to be 1 cube on the bottom level. The side view shows that the bottom layer has cubes.



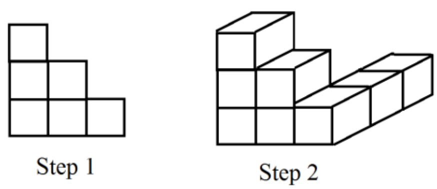
Problem 2. Solution:
Front: Treat the front face as a 2-dimensional figure.
Top: Treat the top face as a 2-dimensional figure.
Side: Treat the side face as a 2-dimensional figure.



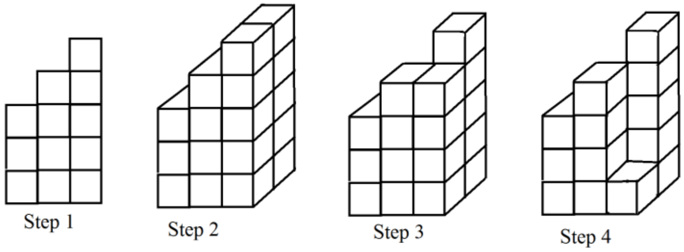
Problem 3. Solution:



Problem 4. Solution: A.

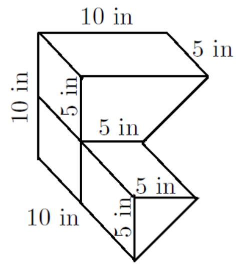


Problem 5. Solution: A.



Problem 6. Solution: B.

We draw the three-dimensional figure as shown in the figure.



The volume is the sum of volume of the right prism with a trapezoid for a base with a height of 5 inches and the volume of right triangular prism with a height of 10 inches:

$$\frac{(10+5)\times 5}{2}\times 5+\frac{(5\times 5)}{2}\times 10=312.5.$$

Problem 7. Solution: B.

We view this solid from three sides:

Top (and bottom): we see 4 squares. 4 × 2 = 8.

Front (and back): we see 3 squares. 3 × 2 = 6.

Left side (and right): we see 4 squares. 4 × 2 = 8.

The number of faces is 8 + 6 + 8 = 22.

The area of each face is 2 × 2 = 4 (cm^2).

The answer is 22 × 4 = 88 (cm^2).

Problem 8. Solution: C.

We view this solid from three sides:

Top (and bottom): 3 + 1 = 4.

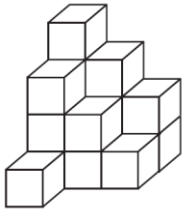
Front (and back): 1 + 1 = 2.

Left side (and right): 3 + 1 = 4.

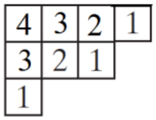
The number of faces is 4 + 2 + 4 = 10.

Problem 9. Solution: A.

Problem 10. Solution:

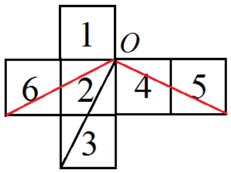


Problem 11. Solution:

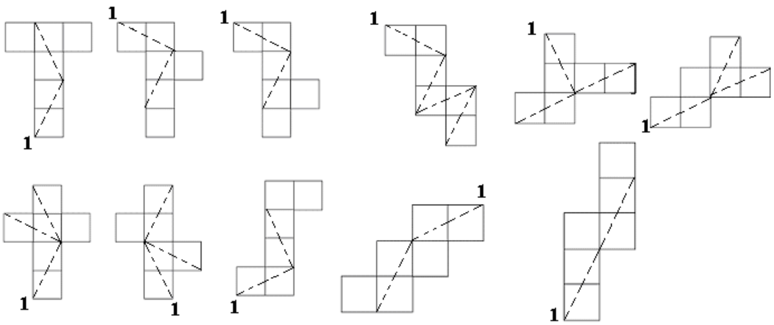


Problem 12. Solution: C.
When we fold the paper, number 1 face will be opposite the number 5 face, number 2 face will be opposite the number 4 face, and the number 3 face will be opposite the number 6 face.

Problem 13. Solution: D.
When we fold the figure, faces 6, 5, and 3 will come together at a corner. The sum is the largest: 6 + 5 + 3 = 14.

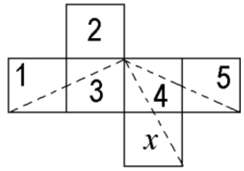


Problem 14. Solution:



Problem 15. Solution: B.

Face 1 will touch face x and faces 3, 4, and 5 are adjacent to face x, so face 2 is opposite to face x in the folded cube.

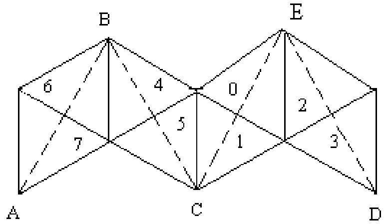


Problem 16. Solution: C.

We know that one of the numbers is 7.

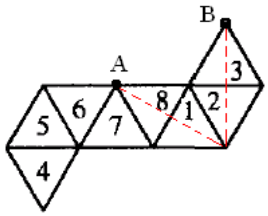
We see that A will meet with C and D.

So faces 7, 5, 1 and 3 share one vertex. The sum of the values is $7 + 5 + 1 + 3 = 16$.



Problem 17. Solution: E.

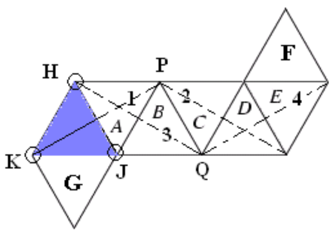
We already know three faces that meet at vertex A: 8, 7, 6, so we only need to find one more face. Vertex B will meet Vertex A when folded, so the other face is the face marked “3”. The faces marked with the numbers 8, 7, 6, and 3 form the maximum vertex. $8 + 7 + 6 + 3 = 24$.



Problem 18. Solution: C.

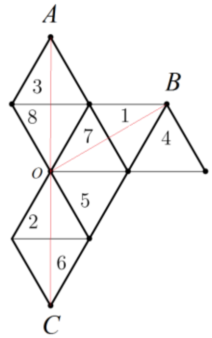
The face that is opposite the shaded one should not share any vertices or edges with the shaded triangle.

We name the three vertices of the shaded triangle in the net H, J, and K. We draw line 1 from K to P. From this line, we draw line 2 based on our method in the chapter discussion. From line 2, we see that faces D and E share the same vertices with the shaded triangle.



Next, we draw line 3 from H to Q and we get line 4. From here, we know that face F shares a vertex with the shaded triangle.

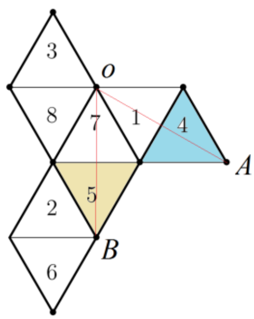
Looking at vertex J, we know that faces A, G, and B all share vertices with the shaded triangle. Therefore the only face not sharing any vertices or edges with the shaded triangle is the face C.



Problem 19. Solution: B.
From the figure below we see that vertices A, B, and C will meet. When they meet, the faces 1 and 8 will be adjacent to one marked with a 3. Since face1 is adjacent to face 7, face 4, and face 3, it must not be adjacent to face 6. Since face 3 is adjacent to face 1, it must be adjacent to face 6 (not face 4). So the sum is 8 + 6 + 1 = 15.

Or :
We see that the face 8 is adjacent to face 3. We also see that the red lines going through faces 3, 1, and 6. So face 1 and face 6 will be adjacent to face 3. So the sum is 8 + 6 + 1 = 15.

Problem 20. Solution: E.
From the figure below we see that vertices A and B will meet. So the faces marked 2, 5, 6, and 4 will share the same vertex.

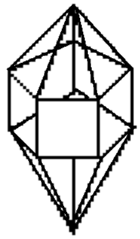


We know that face 1 is adjacent to face 4. Two faces out of three faces 2, 5, and 6 will be adjacent to face 4.

We see that the red lines going through both faces 4 and 5. So face 4 and face 5 will be adjacent.

Since face 2 is adjacent to face 5, it must not be adjacent to face 4. So face 6 is adjacent to face 4. So the sum is 1 + 5 + 6 = 12.

Problem 21. Solution: B.
Method 1 (official solution):
The net with 5 square faces and 10 equilateral triangles is folded into a 15-faced polyhedron as shown below.



Since there are five squares with four sides each, they contribute 20 sides. There are also 10 triangles with three sides each, which contribute 30 more sides. Notice that every side of a square or triangle hooks up with a side of another square or triangle to form an edge. Therefore, the 50 sides we have will form 50 ÷ 2 = 25 edges.

Method 2 (our solution):
By the Edges and Faces formula, we have:
$E = \frac{F \times S}{2} = \frac{5 \times 4 + 10 \times 3}{2} = 25$ edges.

Problem 22. Solution: D.
$F = 24 + 12 = 36$.
By the Edges and Faces formula, the number of edges of the polyhedron is

$E = \frac{F \times S}{2} = \frac{24 \times 3 + 12 \times 4}{2} = 60$.
We know that $F + V = E + 2$. The number of vertices is $V = E + 2 - F = 60 + 2 - 36 = 26$.

Each quadrilateral has two face diagonals so we get 2 × 12 = 24 diagonals for 12 quadrilaterals.

The number of segments of the convex polyhedron is $\binom{26}{2} = 325$.

The answer is 325 – 60 – 24 = 241.

1. BASIC KNOWLEDGE

1.1. Terms

FACTORS: Factors are the numbers you multiply together to get the product.

$1\times 12 = 2 \times 6 = 3 \times 4 = 12 \quad \Rightarrow \quad$ 1, 2, 3, 4, 6 and 12 are factors of 12.

DIVISOR: A divisor is a number which divides the given number without leaving any remainder.

$\frac{6}{1} = 6 \qquad \frac{6}{2} = 3 \qquad \frac{6}{3} = 2 \qquad \frac{6}{6} = 1$

The denominators 1, 2, 3, and 6 are called the divisors of the numerator 6.

Note: A factor of the number is the same as a divisor of the number.

MULTIPLE: Multiples are the products when you multiply two or more numbers.

$24 = 1 \times 24 = 2 \times 12 = 3 \times 8 = 4 \times 6 \Rightarrow$ 24 is a multiple of 1, 2, 3, 4, 6, 8, 12, and 24.

PRIME FACTORIZATIONS: Prime factorization of a number is to express the number as a product of factors that are all prime.

$12 = 2 \times 6 = 2 \times 2 \times 3 = 2^2 \times 3.$ $\qquad 1001 = 7 \times 11 \times 13.$

1.2. Fundamental theorem of arithmetic:

Any composite number, besides 0 and 1, can be written as a product of prime numbers.
And this expression is unique.

$36 = 6 \times 6 = (2 \times 3) \times (2 \times 3) = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$
$36 = 4 \times 9 = (2 \times 2) \times (3 \times 3) = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$
$36 = 3 \times 12 = 3 \times (2 \times 6) = 3 \times (2 \times 2 \times 3) = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$

$36 = 2 \times 18 = 2 \times (2 \times 9) = 2 \times (2 \times 3 \times 3) = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$
$36 = 1 \times 36 = 1 \times (2 \times 3)^2 = 2^2 \times 3^2$

1.3. Methods of prime factorization

The divisibility rules and the square root rule can be used to find the prime factorization of a number.

Example 1: What is the largest prime factor of 189?
(A) 63 (B) 3 (C) 7 (D) 13 (E) 21

Solution: C.
Method 1: Since the sum of the digits is divisible by 3, 189 is divisible by 3.
$189 = 3 \times 63 = 3 \times 3 \times 21 = = 3 \times 3 \times 3 \times 7 = 3^3 \times 7$.

Note that 189 is also divisible by 7.

Method 2: Since $13^2 = 169 < 189$ and $14^2 = 196 > 189$, the square root of 189 is between 13 and 14.

So we use the prime numbers 13, 11, 7, and 3 to test and we get: $189 = 7 \times 27$. Note that we do not use the prime number 5 in our test because we know that 189 is not divisible by 5.

1.4. Number of divisors

For an integer n greater than 1, let the prime factorization of n be

$$n = p_1^a p_2^b p_3^c \dots p_k^m$$

Where $a, b, c, \dots$, and m are nonnegative integers, $p_1, p_2, \dots, p_k$ are prime numbers.

The number of divisors is: $d(n) = (a+1)(b+1)(c+1)\dots(m+1)$

Example 2: How many factors does $2^3 \cdot 3^6 \cdot 5$ have?
(A) 56 (B) 18 (C) 12 (D) 6 (E) 46

Solution: A.
The number of factors is (3 + 1)(6 + 1)(1 + 1) = 56.

Example 3. How many distinct positive integral factors would the following product have: (12)(15)(17)?
(A) 9 (B) 6 (C) 18 (D) 36 (E) 56

Solution: D.
We do prime factorization of (12)(15)(17):
$(12)(15)(17) = 189 = 3 \times 2^2 \times 3 \times 5 \times 17 = 2^2 \times 3^2 \times 5^1 \times 17^1$
The number of divisors $d = (2 + 1)(2 + 1)(1 + 1)(1 +1) = 36$.

Example 4: How many positive integral factors does N have if $N = 6^2 \cdot 15$?
(A) 6 (B) 9 (C) 12 (D) 24 (E) 21

Solution: D.
We do prime factorization first: $N = 6^2 \cdot 15 = (2 \cdot 3)^2 \cdot 3 \cdot 5 = 2^2 \cdot 3^3 \cdot 5$
The number of divisors $d = (2 + 1)(3 + 1)(1 + 1) = 24$.

2. PROBLEM SOLVING SKILLS

2.1 Find the number of even divisors

Skill: (1) Prime factorization of the given number. (2) Take out one 2. (3) Calculate the number of factors of the remaining number.

Theorem: The number of factors of a prime number is even (2).

Example 5: How many positive integer factors of 72 are even?
(A) 3 (B) 6 (C) 9 (D) 12 (E) 18

Solution: C.
$72 = 3^2 \times 2^3 = 2(3^2 \times 2^2)$
Any factors of $3^2 \times 2$ will be a multiple of 2.
The number of positive integer factors of 72 are also multiples of 2 is: (2 + 1)(2 + 1) = 9.

Example 6: How many positive integer factors of 72 are also multiples of 4?
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution: E.

$72 = 3^2 \times 2^3 = 4(3^2 \times 2)$.
The number of positive integer factors of 72 that are also multiples of 3 is: $(2 + 1)(1 + 1) = 6$.

2.2 Find the number of odd divisors

Skill: (1) Prime factorization of the given number. (2) Take out all 2's. (3) Calculate the number of factors of the remaining number.

Theorem: The number of factors of a square number is odd.

Example 7: How many odd positive integers are factors of 100?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: D.
$100 = 5^2 \times 2^2 = 2^2(5^2)$
The number of odd factors of $2^2(5^2)$ is the same as the number of factors of (5^2) which is 3.

Example 8: How many of the positive integers from 1 to 10000 do not have an odd number of factors?
(A) 100 (B) 300 (C) 1000 (D) 9900 (E) 5000

Solution: D.
Any square number will have an odd number of factors. There are $\sqrt{10000} = 100$ square numbers from 1 to 10000. So the answer is $10000 – 100 = 9900$.

2.3 Find the number of divisors that are the multiple of *m*

Steps: (1) Prime factorization of the given number. (2) Take out one *m*. (3) Calculate the number of factors of the remaining number.

Example 9: How many positive integer factors of 56 are also multiples of 4?
(A) 6 (B) 1 (C) 2 (D) 3 (E) 4

Solution: E.
$56 = 7 \times 2^3 = 4(7 \times 2)$.

The number of factors of 7×2^3 which are also multiples of 4 is the same as the number of factors of (7×2) which is 4.

Example 10: How many positive integer factors of 56 are also multiples of 14?
(A) 1 (B) 2 (C) 3 (D) 14 (E) 4

Solution: C.
$56 = 7 \times 2^3 = 14(2^2)$.
The number of factors of 2^2 is 3.

2.4 Find the number of divisors that are square numbers

Steps: (1) Prime factorization of the given number. (2) Group all integers with an even exponent and write them in the form of N^2. (3) Take out all integers left over. (4) Calculate the number of factors of N.

Example 11: The prime factorization of a certain number is $2^2 \cdot 3^2 \cdot 5$. How many of its positive integral factors are perfect squares?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: D.
$2^2 \cdot 3^2 \cdot 5 = (2 \cdot 3)^2 \cdot 5$.
Any factors of $(2 \cdot 3)$ will be a factor of perfect square.
The answer is $(1 + 1)(1 + 1) = 4$.

Example 12: How many of the positive integer factors of 432 are perfect squares?
(A) 3 (B) 6 (C) 12 (D) 20 (E) 2

Solution: B.
$432 = 2^4 \cdot 3^3 = (2^2 \cdot 3)^2 \cdot 3$.

Any factors of ($2^2 \cdot 3$) will be a factor of perfect square.
The answer is (2 + 1)(1 + 1) = 6.

Example 13: How many odd perfect square factors does $2^4 \times 3^6 \times 5^{10} \times 7^9$ have?
(A) 60 (B) 120 (C) 30 (D) 115 (E) 20

Solution: B.
We just look at the number of perfect square factors for $3^6 \times 5^{10} \times 7^9$.
$3^6 \times 5^{10} \times 7^9 = (3^3 \times 5^5 \times 7^4)^2 \times 7$.
There are (3 + 1)(5 + 1)(4 + 1) = 120 odd perfect square factors.

Example 14: How many of the positive integers from 1 to 100 have an odd number of factors?
(A) 90 (B) 10 (C) 50 (D) 20 (E) 11

Solution: B.
Any square number will have an odd number of factors. There are $\sqrt{100} = 10$ square numbers from 1 to 100. So the answer is 10.

2.5 Find the number of divisors that are cubic numbers

Steps: (1) Prime factorization of the given number. (2) Group all integers with an odd exponent that is a multiple of 3 and write them in the form of N^3. (3) Take out all integers left over. (4) Calculate the number of factors of N.

Example 15: How many perfect cube factors does $2^4 \times 3^6 \times 5^{10} \times 7^9$ have?
(A) 30 (B) 110 (C) 24 (D) 36 (E) 96

Solution: E.
$2^4 \times 3^6 \times 5^{10} \times 7^9 = (2^3 \times 3^6 \times 5^9 \times 7^9) \times 2^1 \times 5^1 = (2^1 \times 3^2 \times 5^3 \times 7^3)^3 \times 2^1 \times 5^1$

$\Rightarrow \quad (2^1 \times 3^2 \times 5^3 \times 7^3)^3 \qquad \Rightarrow \quad N = 2^1 \times 3^2 \times 5^3 \times 7^3$

$d(N) =$ (1 + 1) (2 + 1)(3 + 1)(3 + 1) = 2 × 3 × 4 × 4 = 96.

Example 16: What is the least positive integer by which you could multiply 180 to get a product that is a perfect cube?
(A) 50 (B) 75 (C) 150 (D) 180 (E) 5

Solution: C.
Let m^3 be the perfect cube and n be the smallest positive integer.
$180 = 2^2 \cdot 3^2 \cdot 5$.
$180 \times n = 2^2 \cdot 3^2 \cdot 5 \times n = m^3$.
$n = 2 \cdot 3 \cdot 5^2 = 150$.

3. MORE EXAMPLES

Example 17. If p, r, and s are three different prime numbers greater than 2, and $n = p^2 \times r \times s$, how many positive factors, including 1 and n, does n have?
(A) 3 (B) 6 (C) 12 (D) 2 (E) 20

Solution: C.
The number of positive factors is $(2 + 1)(1 + 1)(1 + 1) = 12$.

Example 18. If one of the positive factors of 80 is to be chosen at random, what is the probability that the chosen factor will be a multiple of 10?
(A) $\frac{3}{5}$ (B) $\frac{5}{7}$ (C) $\frac{1}{5}$ (D) $\frac{2}{5}$ (E) $\frac{7}{8}$

Solution: D.
$80 = 2^4 \times 5$ which has $(4 + 1)(1 + 1) = 10$ factors.
We also know that $80 = 2^4 \times 5 = 10(2^3)$ has $(3 + 1) = 4$ factors each is a multiple of 10.
So the probability is $4/10 = 2/5$.

Example 19. A positive integer is said to be “quadruple-factorable” if it is the product of four consecutive integers. How many positive integers less than 10, 000 are quadruple -factorable?
(A) 24 (B) 2 (C) 4 (D) 6 (E) 8

Solution: E.
We know that $9 \times 10 \times 11 \times 12 = 11800$ and $8 \times 9 \times 10 \times 11 = 7920$. We also know that $1 \times 2 \times 3 \times 4 = 24$.
We see that the first number in the product changes from 1 to 8. So there are 8 of them that are quadruple -factorable.

Example 20. How many even positive integral factors does 6006 have?
(A) 3 (B) 7 (C) 11 (D) 13 (E) 16

Solution: E.
$6006 = 2 \times 3 \times 1001 = 2 \times (3 \times 7 \times 11 \times 13)$.
The number of factors equals $(1 + 1)(1 + 1)(1 + 1)(1 + 1) = 16$.

Example 21. How many even positive integral factors does N have if $N = 6^2 \times 15$?
(A) 24 (B) 6 (C) 12 (D) 16 (E) 8

Solution: D.
$N = 6^2 \times 15 = (2 \times 3)^2 \times 3 \times 5 = 2^2 \times 3^3 \times 5 = = 2(2 \times 3^3 \times 5)$.

The number of even positive integral factors of N is the same as the number of factors of $2 \times 3^3 \times 5$.

So the answer is $(1 + 1)(3 + 1)(1 + 1) = 16$.

Example 22. If $n = 2^3 \times 3^2 \times 5$, how many odd positive factors does n have?
(A) 12 (B) 6 (C) 4 (D) 8 (E) 9

Solution: B.
$n = 2^3 \times 3^2 \times 5 = 2^3 \times (3^2 \times 5)$. We only need to calculate the number of factors for $(3^2 \times 5)$, which is $(2 + 1)(1 + 1) = 6$. So the number of even factors is 6.

Example 23. How many perfect cube factors does $2^4 \times 3^6 \times 5^{10}$ have?
(A) 24 (B) 20 (C) 16 (D) 14 (E) 12

Solution: A.
$2^4 \times 3^6 \times 5^{10} = 2^3 \times (3^2)^3 \times (5^3)^3 \times 2 \times 5 = (2^1 \times 3^2 \times 5^3)^3 \times 2 \times 5$.
We only need to calculate the number of factors for $(2^1 \times 3^2 \times 5^3)$, which is $(1 + 1)(2 + 1)(3 + 1) = 24$. So the number of perfect cube factors is 24.

Example 24. How many natural numbers n will make $\frac{36}{n+1}$ a natural number?

(A) 2 (B) 4 (C) 6 (D) 8 (E) 9

Solution. D.

$36 = 6^2 = (2 \times 3)^2 = 2^2 \times 3^2$ has $3 \times 3 = 9$ factors. $n - 1$ must be one of the 9 factors in order for $\frac{36}{n+1}$ to be a natural number. Note that since n is positive integer, so $n + 1 > 1$. Among these 9 factors, the factor 1 should not be counted. So the answer is $9 - 1 = 8$.

Example 25. In the equation $x^y = 8192$, x and y are positive integers. What is the greatest possible value of $x - y$?

(A) 81 (B) 92 (C) 8190 (D) 8191 (E) 13

Solution: D.

$x^y = 8192 = 2^{13} = (2^{13})^1$.

So $x = 2^{13} = 8192$ and $y = 1$. $x - y = 8192 - 1 = 8191$.

4. PROBLEMS

Problem 1. What is the greatest three-digit integer that has a factor of 19?
(A) 999 (B) 998 (C) 989 (D) 988 (E) 969

Problem 2. Let a "*prd*" number be defined as one in which the product of the positive divisors of the number, not including the number itself, is greater than the number. Which of the following is NOT a *prd* number?
(A) 12 (B) 18 (C) 27 (D) 45 (E) 20

Problem 3. If one of the positive factors of 120 is to be chosen at random, what is the probability that the chosen factor will not be a multiple of 5?
(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{5}$ (E) $\frac{1}{6}$

Problem 4. In the equation $x^y = 8192$, x and y are positive integers. What is the smallest possible value of $x - y$?
(A) 11 (B) – 11 (C) 9 (D) – 9 (E) – 13

Problem 5. How many positive integers less than 100 have an odd number of factors?
(A) 90 (B) 50 (C) 10 (D) 9 (E) 5

Problem 6. What is the sum of all of the prime factors of 71,400?
(A) 32 (B) 34 (C) 24 (D) 27 (E) 29

Problem 7. How many odd positive integers are factors of 480?
(A) 24 (B) 12 (C) 8 (D) 6 (E) 4

Problem 8. How many positive factors of 36 are also multiples of 4?
(A) 5 (B) 1 (C) 3 (D) 2 (E) 4

Problem 9. What is the smallest positive integer by which 120 can be multiplied so that the product will be a perfect square?
(A) 120 (B) 30 (C) 50 (D) 20 (E) 40

Problem 10. Find the largest prime factor of 493.
(A) 17 (B) 19 (C) 23 (D) 29 (E) 31

Problem 11. Find the number of even factors of 14014.
(A) 18 (B) 12 (C) 6 (D) 4 (E) 2

Problem 12. What is the largest prime factor of 1463?
(A) 7 (B) 11 (C) 17 (D) 19 (E) 23

Problem 13. How many positive integral factors does $2^8 \cdot 3^4 \cdot 7^6 \cdot 11$ have?
(A) 192 (B) 630 (C) 63 (D) 540 (E) 504

Problem 14. How many positive divisors does the number $2 \times 6^2 \times 5^3$ have?
(A) 48 (B) 24 (C) 15 (D) 12 (E) 6

Problem 15. If 2^k is a divisor of 2304, what is the largest possible value for k?
(A) 12 (B) 11 (C) 8 (D) 6 (E) 4

Problem 16. If 2^N is a factor of 20!, what is the largest possible value of N?
(A) 17 (B) 19 (C) 18 (D) 12 (E) 13

Problem 17. The number 596,505 can be expressed as a product $n \cdot m \cdot p$, where each of n, m, and p are two-digit numbers. Find $n + m + p$.
(A) 255 (B) 91 (C) 164 (D) 70 (E) 95

Problem 18. How many odd perfect square factors does $2^4 \times 3^6 \times 5^{10} \times 7^9$ have?
(A) 24 (B) 60 (C) 120 (D) 124 (E) 128

Problem 19. How many perfect cube factors does $2^4 \times 3^6 \times 5^{10}$ have?
(A) 20 (B) 24 (C) 48 (D) 12 (E) 6

Problem 20. A rectangular quilt has 42 squares. How many shapes are there in which the quilt can be arranged?
(A) 1 (B) 2 (C) 3 (D) 5 (E) 4

Problem 21. What is the smallest positive integer by which 80 can be multiplied so that the product will be a perfect cube?

(A) 4 (B) 25 (C) 8 (D) 10 (E) 100

Problem 22. For how many positive integers n will $\frac{60}{n}$ also be an integer?

(A) 16 (B) 12 (C) 8 (D) 4 (E) 3

Problem 23. How many positive integer values of x are there such that $\frac{36}{x+3}$ is an integer?

(A) 9 (B) 8 (C) 6 (D) 4 (E) 3

Problem 24. In the equation $x^y = 8192$, x and y are positive integers. What is the greatest possible value of $x - y$?

(A) 81 (B) 92 (C) 8190 (D) 8191 (E) 13

Problem 25. How many positive integers less than 20 have exactly 2 factors?

(A) 20 (B) 10 (C) 9 (D) 8 (E) 7

Problem 26. How many of the positive integers from 1 to 100 have an odd number of factors?

(A) 10 (B) 9 (C) 8 (D) 5 (E) 3

Problem 27. What is the smallest positive integer by which 252 can be multiplied so that result would be a perfect cube?

(A) 8 (B) 125 (C) 294 (D) 216 (E) 343

Problem 28. How many factors of 21,600 are perfect squares?

(A) 24 (B) 20 (C) 16 (D) 14 (E) 12

5. SOLUTIONS

Problem 1. Solution: D.
We know that $1000/19 \approx 52$. So $52 \times 19 = 988$. 988 is the greatest three-digit integer that has a factor of 19.

Problem 2. Solution: C.
The divisors of 27 are 1, 3, and 9 (not including 27). $1 + 3 + 9 = 13 < 27$.

Problem 3. Solution: A.
$120 = 2^3 \times 3 \times 5$ which has $(3 + 1)(1 + 1)(1 + 1) = 16$ factors.
We also know that $120 = 5 \times (2^3 \times 3)$ has $(3 + 1)(1 + 1) = 8$ factors each is a multiple of 5.
So the probability is $8/16 = 1/2$.

Problem 4. Solution: B.
$x^y = 8192 = 2^{13} = (2)^{13}$.
So $x = 2$ and $y = 13$. $x - y = 2 - 13 = -11$.

Problem 5. Solution: D.
All the square numbers have an odd number of factors. There are 9 square numbers from 1 to 99. The answer is then 9.

Problem 6. Solution: B.
$71400 = 100 \times 7 \times 102 = 100 \times 7 \times 3 \times 34 = 100 \times 7 \times 3 \times 34 = (2 \times 5)^2 \times 7 \times 3 \times 2 \times 17$.
The prime factors are 2, 3, 5, 7, and 17 and their sum is 34.

Problem 7. Solution: E.
$480 = 2^5 \times 3 \times 5 = 2^5 \times (3^1 \times 5^1)$
The number of odd factors equals $(1 + 1)(1 + 1) = 4$.

Problem 8. Solution: C.
Since $36 = 2^2 \times (3^2)$, the number of positive factors of 36 that are also multiples of 4 is $(2 + 1) = 3$.

Problem 9. Solution: B.

Let m^2 be the perfect cube and n be the smallest positive integer.
$120 = 12 \times 10 = 3 \times 4 \times 2\times 5 = 2^3 \times 3 \times 5$.
$80 \times n = 2^3 \times 3 \times 5 \times n = m^2$.
$n = 2 \times 3 \times 5 = 30$.

Problem 10. Solution: D.
Since $22^2 = 484 < 4939$ and $23^2 = 529 > 4939$, the square root of 493 is between 22 and 23.
So we use the prime number 19, 17, 13, 11, 7, and 3 to test and we get:
$493 = 17 \times 29$.

Problem 11. Solution: B.
$14014 = 2 \times 7007 = 2 \times 7 \times 1001 = 2 \times 7 \times 7 \times 11 \times 13 = 2 \times (7^2 \times 11 \times 13)$.
There are $(2 + 1)(1 + 1)(1 + 1) = 12$ even factors.

Problem 12. Solution: D.
1463 is divisible by both 7 and 11 so it is divisible by 77.
$1463 = 77 \times 19 = 11 \times 7 \times 19$

Problem 13. Solution: B.
The number of divisors $d = (8 + 1)(4 + 1)(6 + 1)(1 + 1) = 630$.

Problem 14. Solution: A.
Note that $2 \times 6^2 \times 5^3 = 2 \times 2^2 \times 3^2 \times 5^3 = 2^3 \times 3^2 \times 5^3$.
The number of divisors $d = (3 + 1)(2 + 1)(3 + 1) = 48$.

Problem 15. Solution: C.
$2304 = 16 \times 144 = 2^4 \times (12)^2 = 2^4 \times (3 \times 2^2)^2 = 2^4 \times 3^2 \times 2^4 = 2^8 \times 3^2$. $k = 8$.

Problem 16. Solution: C.
$$N = \left\lfloor \frac{20}{2} \right\rfloor + \left\lfloor \frac{20}{2^2} \right\rfloor + \left\lfloor \frac{20}{2^3} \right\rfloor + \left\lfloor \frac{20}{2^4} \right\rfloor = 10 + 5 + 2 + 1 = 18.$$

Problem 17. Solution: A.
$596{,}505 = 5 \times 119301 = 5 \times 3 \times 39767 = 5 \times 3 \times 7 \times 5681 = 5 \times 3 \times 7 \times 13 \times 437$
$= 5 \times 3 \times 7 \times 13 \times 19 \times 23 = (3 \times 23) \times (5 \times 19) \times (7 \times 13) = 69 \times 95 \times 91$
The sum is $69 + 95 + 91 = 255$.

Problem 18. Solution: C.

$2^4 \times 3^6 \times 5^{10} \times 7^9 \quad \Rightarrow \quad 3^6 \times 5^{10} \times 7^9 \quad \Rightarrow \quad (3^3 \times 5^5 \times 7^4)^2 \times 7$

The number of odd perfect square divisors is the same as the number of divisors for $3^3 \times 5^5 \times 7^4$ which can be calculated as (3 + 1)(5 + 1)(4 + 1) = 120.

Problem 19. Solution: B.

$2^4 \times 3^6 \times 5^{10} = (2 \times 3^2 \times 5^3)^3 \times 2 \times 5$

The number of factors is (1 + 1)(2 + 1)(3 + 1) = 24.

Problem 20. Solution: E.

This is the same way of asking how many ways can 42 be expressed as a multiple of two numbers. 42 = 1 × 42 = 2 × 21 = 3 × 14 = 6 × 7.

Problem 21. Solution: E.

$80 = 2^4 \times 5$.

The smallest positive integer is $2^2 \times 5^2 = 100$.

Problem 22. Solution: B.

$60 = 6 \times 10 = 2^2 \times 3 \times 5$ has 12 factors. n must be one of them in order for $\frac{60}{n}$ to be an integer. So the answer is 12.

Problem 23. Solution: C.

$36 = 6^2 = (2 \times 3)^2 = 2^2 \times 3^2$ has 3 × 3 = 9 factors. $x + 3$ must be one of the 9 factors in order for $\frac{36}{x+3}$ to be an integer. Note that since x is positive integer, so $x + 3 \geq 4$. Among these 9 factors, three of them are less than 4. So the answer is 9 – 3 = 6.

Problem 24. Solution: D.

$x^y = 8192 = 2^{13} = (2^{13})^1$.

So $x = 2^{13} = 8192$ and $y = 1$. $x - y = 8192 - 1 = 8191$.

Problem 25. Solution: D.

Theorem: The number of factors of a prime number is 2. There are 8 prime number from 1 to 20 (2, 3, 5, 7, 11, 13, 17, 19). So the answer is 8.

Problem 26. Solution: A.
Any square number will have an odd number of factors. There are $\sqrt{100}=10$ square numbers from 1 to 100. So the answer is 10.

Problem 27. Solution: C.
Let m^3 be the perfect cube and n be the smallest positive integer.
$252 = 2^2 \times 7 \times 3^2$
$252 \times n = 2^2 \times 7 \times 3^2 \times n = m^3$.
$n = 2 \times 3 \times 7^2 = 294$.

Problem 28. Solution: E.
$21600 = 216 \times 100 = (2 \times 3)^3 \times 10^2 = 2^3 \times 3^3 \times 2^2 \times 5^2 = 2^4 \times 3^2 \times 5^2 \times 2 \times 3 = (2^2 \times 3 \times 5)^2 \times 2 \times 3$.
We only need to calculate the number of factors for $(2^2 \times 3 \times 5)$, which is $(2 + 1)(1 + 1)(1 + 1) = 12$. So the number of perfect squares factors is 12.

1. BASIC KNOWLEDGE

1. 1. TERMS

Prime number: A prime number is a positive integer greater than 1 and only divisible by 1 and itself. Another way of saying this is that a prime number is a positive integer with exactly two factors (1 and itself).

The smallest prime number: 2, which is an even number and the only even prime number.

Theorem 1: There are infinite many prime numbers.

Theorem 2: There is no greatest prime number.

Composite number: When a number has more than two factors, the number is called a composite number.

Relatively prime: If two positive integers have no common factor except 1, the two positive integers are said to be relatively prime, for example, 4 and 9 are relatively prime.

1.2. METHOD TO DETERMINE A PRIME NUMBER

Theorem 3 (The square root rule): If a is not divisible by all the prime numbers less than or equal to $\sqrt{a}$, then a is a prime number.

Example 1. How many prime numbers are there between $\sqrt{51}$ and $\sqrt{600}$?
(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

Solution: A.
Let the prime number be p.
$\sqrt{51} \le p \le \sqrt{600}$.
Since $7^2 = 49 < \sqrt{51}$, p is greater than 7.

Since $25^2 = 625 > \sqrt{600}$, p is smaller than 25.
We look at all prime number between 8 to 24: 11, 13, 17, 19, and 23. There are 5 of them.

Example 2. What is the smallest prime number greater than 120?
(A) 121 (B) 123 (C) 125 (D) 127 (E) 131

Solution: D.
121 (11×11), 123 (divisible by 3), 125 (multiple of 5). The next is 127.

We know that $11^2 = 121 < 127 < 144 = 12^2$. We only need to test if 127 is divisible by 11, 7, 5, 3, and 2. We are sure that 127 is neither divisible by 2, nor 3 nor 5. We only need to divide 127 by 7 and 11, respectively.
$127 = 7 \times 17 + 1$
$127 = 11 \times 11 + 6$
So 127 is the smallest prime number greater than 120.

1.3. TWENTY FIVE PRIME NUMBERS (UP TO 100)

There are 25 prime numbers from 1 to 100.

2		3	5	7	
	11	13		17	19
		23			29
	31			37	
	41	43		47	
		53			59
	61			67	
	71	73			79
		83			89
				97	

Example 3. Two prime numbers that differ by two are called twin primes. Find the sum of a pair of twin primes between 60 and 75.
(A) 135 (B) 138 (C) 141 (D) 144 (E) 148

Solution: D.
71 + 73 = 144.

Example 4. What is the sum of the two-digit prime numbers between 10 and 20?
(A) 33 (B) 34 (C) 56 (D) 58 (E) 60

Solution: E.
11 + 13 + 17 + 19 = 60.

2. PROBLEM SOLVING SKILLS

2.1. Property 1: A prime number *p* can only be written as *p* × 1.

In other words, a prime number p can only be divided, without a remainder, by itself and 1.

If p is a prime number and $p = mn$, then one of the two numbers m and n must be 1 and another one must be p.

Example 5. Find the value of a positive integer x such that, when 64 is taken away from it, the results is a square number, and when 25 is added to it, the result is also a square number.
(A) 1400 (B) 1600 (C) 1980 (D) 2000 (E) 1125

Solution: D.

$$\left.\begin{array}{l} x-64=n^2 \\ x+25=m^2 \end{array}\right\} \Rightarrow \quad m^2-n^2=89 \quad \Rightarrow \quad (m-n)(m+n)=89$$

Since 89 is a prime number and $m + n > m - n$,

$$\left.\begin{array}{l} m+n=89 \\ m-n=1 \end{array}\right\} \quad \Rightarrow \quad m = 45,\ n = 44.$$

$x = 45^2 - 25 = 2000.$

Example 6. Find the value of $b-c$ if $a^3 = b^2$, $c^2 = d$, and $d-a=5$, where *a, b, c*, and *d* are positive integers.

(A) 5 (B) 9 (C) 12 (D) 24 (E) 25

Solution: A.

Since *a* and *b* are positive integers, let $a^3 = b^2 = t^{3\times 2} = t^6$, we get: $a = t^2$ and $b = t^3$.

Therefore, $d - a = 5$ can be written as $c^2 - t^2 = 5$ or $(c-t)(c+t) = 5$

Since 5 is a prime number and $c + t > c - t$, we have:

$$\left.\begin{array}{l} c+t=5 \\ c-t=1 \end{array}\right\} \quad \Rightarrow \quad c = 3,\ t = 2.$$

$b - c = 2^3 - 3 = 5.$

2.2. PROPERTY 2: If the sum of two prime numbers is an odd number, then one of the two prime numbers must be 2.

Example 7. The sum of two prime numbers is 39. What is their product?

(A) 37 (B) 74 (C) 121 (D) 169 (E) 40

Solution: B.

Since the sum is an odd number, one prime number must be 2. The other one is then $39 - 2 = 37$. The product is $2 \times 37 = 74$.

Example 8. The sum of two prime numbers is 49. What is the sum of the reciprocals of the two prime numbers?

(A) 47 (B) $\frac{49}{94}$ (C) $\frac{1}{47}$ (D) $\frac{1}{2}$ (E) 49

Solution: B.

Let the two prime numbers be x and y and $x < y$.
Since $y + x = 49$, x must be 2 and $y = 47$.
$\frac{1}{47} + \frac{1}{2} = \frac{49}{94}$.

Example 9. Three prime numbers p, q, and r satisfy the following conditions: $p + q = r$ and $1 < p < q$. Find the value for p.
(A) 2 (B) 3 (C) 5 (D) 7 (E) 23

Solution: A.
Since p, q, and r are all prime numbers, r must be an odd number and one of the two numbers p and q must be 2.
Since $1 < p < q$, $p = 2$.

Example 10. If $a + b + c = 66$ and $ab + bc + ca = 1071$, where a, b, and c are all prime numbers, find the value of abc.
(A) 1071 (B) 944 (C) 1886 (D) 958 (E) 1024

Solution: C.
Since the sum of three prime numbers is an odd number, one of the prime numbers must be 2. Let $a \le b \le c$, we get $a = 2$ and $b + c = 64$.
$ab + bc + ca = 1071 \Rightarrow 2b + bc + 2c = 1071$ or $2(b + c) + bc = 1071$.
$bc = 943$. Thus $abc = 2 \times 943 = 1886$.

2.3. How Many Prime Numbers

Example 11. For how many positive integers n is $n^2 - 3n + 2$ a prime number?
(A) none (B) one (C) two (D) more than two, but finitely many
(E) infinitely many.

Solution: B.
We factor $n^2 - 3n + 2 = (n - 1)(n - 2)$.
When $(n - 1) = 1$, $n = 2$. $n^2 - 3n + 2 = 0$ which is not a prime.
When $n - 1 = -1$, $n = 0$ which is not a positive integer.
When $(n - 2) = 1$, $n = 3$. $n^2 - 3n + 2 = 2$ which is a prime.
When $(n - 2) = -1$, $n = 1$. $n^2 - 3n + 2 = 0$ which is not a prime.

Therefore, $n^2 - 3n + 2$ is prime only when $n = 3$.

Example 12. The positive integers A, B, $A - B$, and $A + B$ are all prime numbers. The sum of these four primes is
(A) even (B) divisible by 3 (C) divisible by 5 (D) divisible by 7
(E) prime

Solution: E.
The numbers $A - B$ and $A + B$ are both odd or both even. However, they are also both prime, so they must both be odd. Therefore, one of A and B is odd and the other even. Because A is a prime between $A - B$ and $A + B$, A must be the odd prime. Therefore, $B = 2$, the only even prime. So $A - 2$, A, and $A + 2$ are consecutive odd primes and thus must be 3, 5, and 7. The sum of the four primes 2, 3, 5, and 7 is the prime number 17.

Example 13. The product of three prime numbers is five times the sum of these prime numbers and it is also divisible by 5. Find the remainder when the sum of these prime numbers is divided by 11.
(A) 9 (B) 7 (C) 6 (D) 3 (E) 1

Solution: D.
Since the product is divisible by 5, one of these prime numbers must be 5.
Let p and q be the other two prime numbers, we have: $5pq = 5(p + q + 5)$
$\Rightarrow \quad pq - p - q + 1 = 6. \quad \Rightarrow \quad (p - 1)(q - 1) = 6 = 2 \times 3 = 1 \times 6.$
If $p - 1 = 2$ and $q - 1 = 3$, q = 4 is not a prime number which is not possible.
If $p - 1 = 1$ and $q - 1 = 6$, $p = 2$ and $q = 7$.
Three prime numbers are (2, 5, 7). The sum is 14 and the answer is 3.

Example 14. Find $\dfrac{p}{3q+1}$ if both p and q are prime numbers and $3p + 5q = 31$.

(A) $\dfrac{1}{8}$ or 1 (B) $\dfrac{1}{7}$ or 2 (C) $\dfrac{1}{5}$ or 11 (D) $\dfrac{1}{13}$ or 17 (E) 5

Solution: A.

Since $3p + 5q = 31$, $3p$ or $5q$ must be even since two odd numbers do not sum to an odd integer.

Case I: If $3p$ is even, p must be 2. We then have $5q = 31 - 3 \times 2 = 25 \quad \Rightarrow q = 5$.
$\frac{p}{3q+1} = \frac{2}{3\times 5+1} = \frac{1}{8}$.

Case II: If $5q$ is even, q must be 2. We then have $3p = 31 - 5 \times 2 = 21 \quad \Rightarrow p = 7$.
$\frac{p}{3q+1} = \frac{7}{3\times 2+1} = 1$.

3. MORE EXAMPLES

Example 15. If m is the greatest prime factor of 57 and n is the greatest prime factor of 120, what is the value of $m + n$?
(A) 7 (B) 12 (C) 24 (D) 29 (E) 44

Solution: C.
Since $5 + 7 = 12$, 57 is divisible by 3. $57 = 3 \times 19$.
$120 = 12 \times 10 = 3 \times 4 \times 2 \times 5$. The greatest prime factor of 120 is 5.
$m + n = 19 + 5 = 24$.

Example 16. Jerry chose a four-digit number to be the personal identification number for his bank account. The first digit (leftmost digit) is prime, the greatest common factor of the middle two digits is 2, and the last digit is a divisor of 20. Which of the following numbers could be his personal identification number?
(A) 2463 (B) 3475 (C) 3864 (D) 5467 (E) 6216

Solution: C.
We eliminate (E) since 6 is not a prime. We eliminate (B) since the greatest common factor of 4 and 7 is not 2. Since the last digit is a divisor of 20, we eliminate (A) and (D).

Example 17. For three positive prime numbers a, b, and c, $ab = 55$ and $bc = 35$. What is the value of abc?
(A) 175 (B) 605 (C) 165 (D) 385 (E) 1925

Solution: D.
$ab = 55 = 5 \times 11$. $bc = 35 = 5 \times 7$.
So $a = 11$, $b = 5$, and $c = 7$ $\Rightarrow$ $abc = 35 \times 11 = 385$.

Example 18. How many of the prime factors of 210 are greater than 2?
(A) One (B) Two (C) Three (D) Four (E) Five

Solution: C.
$210 = 2 \times 3 \times 5 \times 7$. Only 3, 5, and 7 are greater than 2.

Example 19. What is the product of the prime numbers between 30 and 40?
(A) 1023 (B) 1108 (C) 1147 (D) 1221 (E) 1143

Solution: C.
$31 \times 37 = 1147$.

Example 20. The difference of two prime numbers is 101. What is the value of the larger prime number?
(A) 101 (B) 103 (C) 107 (D) 109 (E) 191

Solution: B.
Let the two prime numbers be x and y and $x < y$.
Since $y - x = 101$. x must be 2 and $y = 103$.

Example 21. Two prime numbers p and q satisfy the following conditions: $p = m + n$ and $q = mn$, where both m and n are positive integers. Find the value of $p^q + q^p$.
(A) 11 (B) 13 (C) 15 (D) 17 (E) 19

Solution: D.
Since q is a prime number and $q = mn$, m or n must be 1.

Let $m = 1$, we have $q = n$ and $p = 1 + n = 1 + q$
Since both p and q are prime number, $q = 2$ and $p = 3$.
$p^q + q^p = 3^2 + 2^3 = 17$.

Example 22. If $a + b + c = 68$ and $ab + bc + ca = 1121$, where a, b, and c are all prime numbers, find the value of abc.
(A) 989 (B) 1978 (C) 1292 (D) 323 (E) 1003

Solution: B.
Since the sum of three prime numbers is an odd number, one of the prime numbers must be 2. Let $a \le b \le c$, we get $a = 2$ and $b + c = 66$.
$ab + bc + ca = 1121 \Rightarrow 2b + bc + 2c = 1121$ or $2(b + c) + bc = 1121$.
$bc = 989$. $abc = 2 \times 989 = 1978$.

Example 23. For how many positive integers a is $a^2 - 3a + 2$ a prime number?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: B.
We factor: $a^2 - 3a + 2 = (a - 2)(a - 1)$.
When $(a - 2) = 1$, $a = 3$, $a^2 - 3a + 2 = 2$ (a prime number).
When $(a - 2) = -1$, $a = 1$, $a^2 - 3a + 2 = 6$, (not a prime number).
When $(a - 1) = 1$, $a = 2$, $a^2 - 3a + 2 = 0$ (not a prime number).
When $(a - 1) = -1$, $a = 0$ which is not a positive number.
So there is one positive integer a such that $a^2 - 3a + 2$ is a prime number.

Example 24. Find a positive integer such that the sum of the positive integer and 72 is a square number, and the sum of the positive integer and 55 is also a square number.
(A) 8 (B) 9 (C) 7 (D) 5 (E) 17

Solution: B.
Let x be the positive integer.

$$x + 72 = y^2 \qquad (1)$$

$$x + 55 = z^2 \qquad (2)$$

(2) – (1):

$z^2 - y^2 = 17 \quad \Rightarrow \quad (z-y)(z+y) = 17$

Since $z - y < z + y$, we have:

$$\left.\begin{aligned} z - y &= 1 \\ z + y &= 17 \end{aligned}\right\}$$

Solving we get: $z = 9$, $y = 8$, and $x = 9$.

Example 25. For how many integers n is $n^2 - 8n + 15$ a prime number?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: C.

We factor: $n^2 - 8n + 15 = (n - 3)(n - 5)$.

When $(n - 3) = 1$, $n = 4$. $n^2 - 8n + 15 = -1$ (not a prime).

When $(n - 3) = -1$, $n = 2$. $n^2 - 8n + 15 = 3$ (a prime).

When $(n - 5) = 1$, $n = 6$. $n^2 - 8n + 15 = 3$ (a prime).

When $(n - 5) = -1$, $n = 4$. $n^2 - 8n + 15 = -1$ (not a prime).

4. PROBLEMS

Problem 1. The prime number p is a factor of 70 and is also a factor of 105. How many possible values are there for p ?
(A) One (B) Two (C) Three (D) Four (E) Five

Problem 2. Which of the following is NOT an element of both the set of positive odd integers and the set of prime numbers?
(A) 7 (B) 17 (C) 37 (D) 47 (E) 57

Problem 3. How many integers greater than 20 and less than 30 are each the product of exactly two prime numbers?
(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 4. What is the sum of the first ten prime numbers?
(A) 129 (B) 137 (C) 127 (D) 133 (E) 141

Problem 5. The sum of three consecutive primes is 159. What is the largest of the three primes?
(A) 53 (B) 57 (C) 59 (D) 61 (E) 47

Problem 6. How many pairs of distinct prime numbers have a sum of 22?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 7. The sum of three prime numbers is 38. Find the largest of the primes.
(A) 17 (B) 19 (C) 37 (D) 31 (E) 41

Problem 8. The Goldbach Conjecture, which has not been proven, states that every even integer greater than 2 is expressible as the sum of two prime numbers. In how many ways can the number 36 be expressed as the sum of two primes?
(A) 4 (B) 2 (C) 3 (D) 1 (E) 5

Problem 9. How many prime numbers satisfy $\sqrt{300} < x < \sqrt{700}$?
(A) 2 (B) 0 (C) 3 (D) 4 (E) 5

Problem 10. How many ordered triples of primes (a, b, c) exist such that $a + b + c = 7$?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 11. How many ordered pairs of primes (a, b) satisfy $a + b = 20$?
(A) 1 (B) 4 (C) 3 (D) 2 (E) 5

Problem 12. How many ordered triples of primes (a, b, c) exist such that $a \le b \le c$ and $a + b + c = 26$?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 13. If $a + b + c = 28$ and $ab + bc + ca = 185$, where a, b, and c are all prime numbers, find the greatest of the three prime numbers.
(A) 21 (B) 12 (C) 17 (D) 19 (E) 26

Problem 14. The sum of three prime numbers is 40. Find the largest possible product of the two of the three prime numbers.
(A) 144 (B) 221 (C) 323 (D) 217 (E) 361

Problem 15. How many ordered triples of three prime numbers exist for which the sum of the members of the triple is 24?
(A) 6 (B) 15 (C) 12 (D) 3 (E) 5

Problem 16. If the sum of the square of a prime number and an odd positive integer is 125, find the odd positive integer.
(A) 121 (B) 131 (C) 161 (D) 125 (E) 177

Problem 17. For how many positive integers n is $n^2 - 4n - 21$ a prime number?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 18. A group of 25 pennies is arranged into three piles such that each pile contains a different prime number of pennies. What is the greatest number of pennies possible in any of the three piles?
(A) 13 (B) 15 (C) 11 (D) 17 (E) 19

Problem 19. Find the sum of all integral values of x such that $8x^2 + 2x - 55$ is prime.

(A) 0 (B) 1 (C) 2 (D) 14 (E) 51

5. SOLUTIONS

Problem 1. Solution: B.
$70 = 7 \times 5 \times 2. 105 = 7 \times 5 \times 3$. There are two values: 7 and 5,

Problem 2. Solution: E.
The sum of the digits of 57 is $5 + 7 = 12$. So 57 is divisible by 3. $57 = 3 \times 17$.

Problem 3. Solution: B.
$21 = 3 \times 7$; $22 = 2 \times 11$; $25 = 5 \times 5$; and $26 = 2 \times 13$.

Problem 4. Solution: A.
$2 + 3 +5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 = 129$.

Problem 5. Solution: C.
$159/3 = 53$ which is a prime number. So we know that the other two prime numbers are 47 and 59.

Problem 6. Solution: B.
$22 = 2 + 20 = 3 + 19 = 5 + 17 = 7 + 15 = 11 + 11$
Only (3, 19) and (5, 17) are pairs of distinct prime.

Problem 7. Solution: D.
Since the sum is even, there must be a 2 in them. The sum of the other two prime numbers is 36.

$36 = 3+ 33 = 5 + 31 = 7 + 29 = 11 + 25 = 13 + 23 = 17 + 19$.
The pairs of prime numbers are (5, 31), (7, 29), (13, 23), and (17, 19). The largest one is 31.

Problem 8. Solution: A.
$36 = 5 + 31 = 7 + 29 = 13 + 23 = 17 + 19$.

Problem 9. Solution: A.
$\sqrt{300} < x < \sqrt{700} \Rightarrow 18 \le x \le 26$. We have two prime numbers: 19 and 23.

Problem 10. Solution: C.
We see that 2 + 2 + 3 = 7. So we have 3 ordered triplets: (2, 2, 3), (2, 3, 2), and (3, 2, 2).

Problem 11. Solution: B.
Since 20 = 3 + 17 = 7 + 13, we have 4 ordered pair: (3, 17), (17, 3), (7, 13), and (13, 7).

Problem 12. Solution: C.
Since the sum is even, there must be a 2 in them. The sum of the other two prime numbers is 24.

Since 24 = 3 + 21 (not a prime) = 5 + 19 = 7 + 17 = 11 + 13, we have 3 ordered triplets: (2, 5, 19), (2, 7, 17), and (2, 11, 13).

Problem 13. Solution: D.
Since the sum of three prime numbers is an odd number, one of the prime numbers must be 2. Let $a \le b \le c$, we get $a = 2$ and $b + c = 26$. $ab + bc + ca = 2b + bc + 2c = 185$ or $2(b + c) + bc = 185$. $bc = 133$ which is divisible by 7. $bc = 7 \times 19$. The greatest of a, b, c is 19.

Problem 14. Solution: E.
Since the sum is even, there must be a 2 in them. The sum of the other two prime numbers is 38.

Since 38 = 3 + 35 (not a prime) = 5 + 33 (not a prime) = 7 + 31 = 11 + 27 (not a prime) = 13 + 25 (not a prime) = 17 + 21 (not a prime) = 19 + 19, we have 2 ordered triplets: (2, 7, 31), and (2, 19, 19). $7 \times 31 = 217$. $19 \times 19 = 361$.

Problem 15. Solution: B.
Since the sum is even, there must be a 2 in them. The sum of the other two prime numbers is 22.

Since $22 = 3 + 19 = 5 + 17 = 7 + 15$ (not a prime) $= 11 + 11$, we have 15 ordered triplets:

6 triplets (2, 3, 19), 3 triplets (2, 11, 11), and 6 triplets (2, 5, 17).

Problem 16. Solution: A.
Let the prime number be p and the odd number be $2n + 1$. $p^2 + 2n + 1 = 125$. Since 125 is an odd number, p must be 2. Thus the odd number is $2n + 1 = 125 - 4 = 121$.

Problem 17. Solution: A.
We factor: $n^2 - 4n - 21 = (n + 3)(n - 7)$.
When $(n + 3) = 1$, $n = -2$ (not positive, thus ignored).
When $(n + 3) = -1$, $n = -4$. (not positive, thus ignored).
When $(n - 7) = 1$, $n = 8$. $(n + 3)(n - 7) = 11$ (a prime).
When $(n - 7) = -1$, $n = 6$. $(n + 3)(n - 7) = -9$ (not a prime).

Problem 18. D.
$25 = 3 + 5 + 17 = 5 + 7 + 13 = 3 + 11 + 11$ (not all different) $= 3 + 3 + 19$ (not all different) $= 7 + 7 + 11$ (not all different). The greatest number is 17.

Problem 19. Solution: A.
We factor: $8x^2 + 2x - 55 = (2x - 5)(4x + 11)$.
When $(2x - 5) = 1$, $x = 3$. $(2x - 5)(4x + 11) = 23$ (a prime).
When $(2x - 5) = -1$, $x = 2$. $(2x - 5)(4x + 11)$ is negative.
When $(4x + 11) = 1$, $x = -10/4 = -5/2$ (not an integer).
When $(4x + 11) = -1$, $x = -3$. $(2x - 5)(4x + 11) = 11$ (a prime).
$3 - 3 = 0$.

1. BASIC KNOWLEDGE

(1). RATIOS

Ratios are used to compare two or more numbers.

For any two numbers *a* and *b* ($b \neq 0$), the ratio is written as $a : b = a \div b = \frac{a}{b} = a\,/\,b$.

Example 1. There are 16 girls in a class of 30 students. Find the ratio of girls to boys. Express your answer as a common fraction.
(A) 8/15 (B) 7/15 (C) 7/8 (D) 8/7 (E) 15/8.

Solution: D.
The number of boys in the class is 30 – 16 = 14 $\Rightarrow$ 16/14 = 8/7.

(2) RATES

A rate is a ratio used to compare two numbers of different units. If the second term of the ratio is 1, the rate is called a unit rate.

Example 2. Sam drove 100 miles in 2 hours. What are his rate and the unit rate?
(A) 50/1 (B) 100/1 (C) 1/50 (D) 25/2 (E) 200/1.

Solution: A.
The rate is 100 miles/2 hours and the unit rate is 50/1 or 50 miles per hour.

(3). PROPORTIONS

A proportion is an equation of two ratios. For example, $\frac{a}{b} = \frac{c}{d}$. We can find *a* if we know *b*, *c*, and *d* or we know *b* and the value of *c*/*d*.

Example 3. If refreshments cost $45 for 18 people, at the same rate how much would refreshments cost for 52 people?
(A) $90 (B) $100 (C) $120 (D) $130 (E) $150

Solution: D.

Let x be the cost of refreshment for 52 people.

$$\frac{45}{18}=\frac{x}{52} \qquad \Rightarrow \qquad x=\frac{45}{18}\times 52=130$$

2. IMPORTANT PROPERTIES

2.1. Properties of Ratios:

Property 1 : The first term of a ratio can be any number. The second term can also be any number except zero.

Property 2: If the two terms are multiplied by the same number d, the ratio does not change.

$$a:b=(a\times d):(b\times d) \qquad \Rightarrow \qquad 3:7=(3\times 5):(7\times 5)=15:35$$

Property 3: If the two terms are divided by the same number c $(c\neq 0)$, the ratio does not change.

$$a:b=(a\div c):(b\div c) \qquad \Rightarrow \qquad 10:15=(10\div 5):(15\div 5)=2:3$$

$$\frac{5}{6}:\frac{10}{13}=(1\times 6):(2\times 13)=13:12$$

Property 4: If the total number of parts is $m = A + B$, and $A : B = a :b$, then

the fractional part of a is $\frac{a}{a+b}$, and the fractional part of b is $\frac{b}{a+b}$.

$A=\frac{a}{a+b}\times m$, and $B=\frac{b}{a+b}\times m$.

Property 5: If the total number of parts is $m = A + B +C$, and $A : B : C = a :b : c$, then

the fractional part of a is $\frac{a}{a+b+c}$, the fractional part of b is $\frac{b}{a+b+c}$, and

the fractional part of c is $\frac{c}{a+b+c}$.

$A=\frac{a}{a+b+c}\times m$, $B=\frac{b}{a+b+c}\times m$, and $C=\frac{c}{a+b+c}\times m$.

Example 4. A certain paint color is created by mixing 3 parts of red with every 5 parts of blue. How many gallons of red paint are needed to mix 40 gallons of this color?

(A) 13 (B) 15 (C) 19 (D) 17 (E) 19

Solution: B

$$A=\frac{a}{a+b}\times m=\frac{3}{3+5}\times 40=15.$$

Example 5. In a group of 72 students if the ratio of boys to girls is 5: 3, how many boys are in the group?

(A) 36 (B) 40 (C) 45 (D) 21 (E) 90

Solution: C.

$$A=\frac{a}{a+b}\times m=\frac{5}{5+3}\times 72=45.$$

Example 6. Keith bought paper for making origami figure. He bought 2 packages of orange paper, 3 packages of yellow paper, and 5 packages of blue paper. What fraction of the papers was blue?

(A) 1/3 (B) 1/5 (C) 1/9 (D) 1/7 (E) 1/2

Solution: E.

$$\frac{a}{a+b+c}=\frac{5}{2+3+5}=\frac{5}{10}=\frac{1}{2}$$

Example 7. Keith bought 10 packages of paper for making origami figure. The ratio of orange paper, yellow paper, and blue paper is 2 : 3 : 5. How many packages of blue paper did he buy?
(A) 1 (B) 5 (C) 3 (D) 7 (E) 9

Solution: B.

$$A = \frac{a}{a+b+c} \times m = \frac{5}{2+3+5} \times 10 = 5$$

Example 8. Michael types 250 words in 20 minutes. How many hours will it take him to type a 7500 word paper?
(A) 10 (B) 12 (C) 14 (D) 17 (E) 15

Solution: A.
The unit rate is $250 \div 20 = 12.5$ words per minute.
The time to type 7500 words is $7500 \div 12.5 = 600$ minutes = 10 hours.

Example 9. A car gets 27 miles per gallon. How many miles will it go on 9 gallons of gas?
(A) 213 (B) 215 (C) 229 (D) 243 (E) 279

Solution: D.
The number of miles will the car go is $27 \times 9 = 243$.

Example 10. A basketball player makes 80% of the shots he attempts in each game. In a certain game, he made 20 of his shots. How many shots did he attempt in the game?
(A) 13 (B) 15 (C) 19 (D) 17 (E) 25

Solution: E.
Let x be the total number of shots he made.
$0.8 \times x = 20 \quad \Rightarrow \quad x = 25$.

Example 11. A pork roast should be cooked 50 minutes per pound. How many hours should a 6-pound roast be cooked?
(A) 3 (B) 5 (C) 9 (D) 7 (E) 4

Solution: B.
The number of hours it takes is 50 × 6 = 5 × 60 minutes = 5 hours.

2.2. Properties of Proportion:

Property 6: $\frac{a}{b}=\frac{c}{d}$ is equivalent to:

$ad=bc$, $\frac{a}{c}=\frac{b}{d}$, $\frac{d}{b}=\frac{c}{a}$, $\frac{b}{a}=\frac{d}{c}$.

$$\frac{8}{6}=\frac{4}{3} \Rightarrow 8\times 3=4\times 6 \Rightarrow \frac{8}{4}=\frac{6}{3} \Rightarrow \frac{3}{6}=\frac{4}{8} \Rightarrow \frac{6}{8}=\frac{3}{4}$$

Example 12. The ratio of length to width of a rectangular room is $\frac{4}{3}$ and the width is $8\frac{7}{10}$. What is the length?

(A) 23/2 (B) 35/3 (C) 49/4 (D) 58/5 (E) 11

Solution: D.

$$\frac{L}{W}=\frac{4}{3} \Rightarrow L=\frac{4}{3}W=\frac{4}{3}\times 8\frac{7}{10}=\frac{4}{3}\times\frac{87}{10}=\frac{58}{5}.$$

Property 7: If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$ and $\frac{a-b}{b}=\frac{c-d}{d}$

$$\frac{21}{7}=\frac{30}{10} \Rightarrow \frac{21+7}{7}=\frac{30+10}{10} \text{ (= 4) and } \frac{21-7}{7}=\frac{30-10}{10} \text{ (= 2).}$$

Property 8: If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

$$\frac{21}{7}=\frac{30}{10} \Rightarrow \frac{21+7}{21-7}=\frac{30+10}{30-10} \quad (\frac{28}{14}=\frac{40}{20}=2).$$

Property 9: If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots = \frac{a_n}{b_n}$, then

$$\frac{a_1}{b_1} = \frac{a_1 + a_2}{b_1 + b_2} = \frac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3} = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{b_1 + b_2 + b_3 + \ldots + b_n}$$

Example 13. Find x if $\frac{2x - y}{5} = \frac{x + y}{10} = \frac{3}{5}$.

(A) 1 (B) 3 (C) 9 (D) 7 (E) 5

Solution: B.

$$\frac{2x - y}{5} = \frac{x + y}{10} = \frac{(2x - y) + (x + y)}{15} = \frac{3x}{15} = \frac{3}{5} \quad \Rightarrow \quad x = 3.$$

Example 14. If $\frac{y}{x - z} = \frac{x + y}{z} = \frac{x}{y}$ for three positive numbers x, y, and z, all different, then what is the value of $\frac{x}{y}$?

(A) 3 (B) 5 (C) 1 (D) 2 (E) 6

Solution: D.

$$\frac{x}{y} = \frac{y}{x - z} = \frac{x + y}{z} \quad \Rightarrow \quad \frac{x}{y} = \frac{x + y + (x + y)}{y + (x - z) + z} = \frac{2x + 2y}{x + y} = \frac{2(x + y)}{x + y} = 2.$$

3. CONTINUED RATIO

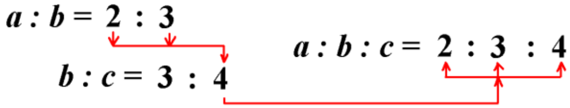
The ratio of three or more quantities is called the continued ratio. For example, $a{:}b{:}c$ is a combinations of three separated ratios $\Rightarrow$ $a{:}b$, $a{:}c$, and $b{:}c$.

Property 10: If $a : b : c = m : n : t$, then $a : b = m : n$, $b : c = n : t$, and $c : a = t : m$.

If $a : b : c = 2 : 3 : 4$, then $a : b = 2 : 3$, $b : c = 3 : 4$, and $c : a = 4 : 2$.

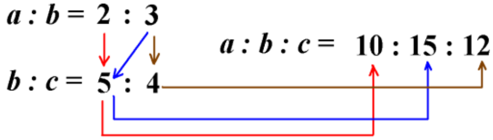
Property 11: If $a : b = m : n$, $b : c = n : t$, and $c : a = t : m$, then $a : b : c = m : n : t$.

If $a : b, = 2 : 3$, $b : c = 3 : 4$, and $c : a = 4 : 2$, then $a : b : c = 2 : 3 : 4$,



Property 12: If $a : b = m : n$, and $b : c = s : t$ (note $n \neq s$), then $a : b : c = (m \times s) : (n \times s) : (n \times t)$.

If $a : b = 2 : 3$, and $b : c = 5 : 4$ (note $3 \neq 5$), then $a : b : c = (2 \times 5) : (3 \times 5) : (3 \times 4) = 10 : 15 : 12$.



Example 15. Three numbers a, b, and c in the ratios of $a : b = 3 : 4$ and $b : c = 5 : 6$ have a sum of 118. What are the values of a, b, and c?
(A) (3, 4, 111) (B) (30, 40, 48) (C) (25, 45, 48) (D) (15, 20, 83)
(E) (27, 36, 55)

Solution: *B*.
Method 1:
$a:b=3:4=15:20$ and $b:c=5:6=20:24$. By the property of the continued ratio, we get: $a:b:c=15:20:24$.

We also know that $a+b+c=118$, so $a=\dfrac{15}{15+20+24}\times 118=30$,

Similarly, $b = 40$, and $c = 48$.

Method 2:
$a:b=3:4$ and $b:c=5:6$
By the property of the continued ratio, we get: $a : b : c = 15 : 20 : 24$.

so $a=\dfrac{15}{15+20+24}\times 118=30$, and $b = 40$, and $c = 48$.

Example 16. Machine A can fill 1 box of nails in 6 minutes. Machine B can fill 1 box of nails in 9 minutes. They started to work at the same time and they stopped also at the same time. In total they filled 100 boxes. How many were filled by machine A?
(A) 30 (B) 45 (C) 50 (D) 60 (E) 120

Solution: D.
Method 1:
Machine A would fill 3 boxes of nails in 18 minutes. Machine B would fill 2 boxes of nails in 18 minutes. So the ratio of their work is 3 : 2.
The number of boxes filled by machine A is: $\frac{3}{3+2}\times 100 = 60$.

Method 2:
Since the ratio of their work is 3 : 2, let the number of boxes filled by machines A be $3x$, and the number of boxes filled by machines B be $2x$.
$3x + 2x = 100 \quad \Rightarrow \quad x = 20 \quad \Rightarrow \quad 3x = 60$.

Example 17. Alex paid \$945 to transport his animals by ferry. The costs are \$3, \$2 and \$1 for each cats, dog, and squirrel, respectively. The ratios of cats to dogs is 2 : 9, and dog to squirrel 3 : 7. How many cats were there?
(A) 43 (B) 45 (C) 49 (D) 47 (E) 42.

Solution: E.
The ratio of the number of animals can be obtained as follows:
$c : d = 2 : 9$ and $d : s = 3 : 7 \quad \Rightarrow \quad c : d : s = 6 : 27 : 63 = 2 : 9 : 21$.

Then the ratio of the cost is then: $(3 \times 2) : (2 \times 9) : (1 \times 27) = 2 : 6 : 7$.
So the cost for cats is calculated as follows: $\frac{2}{2+6+7}\times 945 = 126$.
The number of cats is $126 \div 3 = 42$.

Example 18. If the degree measures of the angles of a convex quadrilateral are in the ratio 3:4:5:6, by how many degrees does the measure of the largest angle exceed the measure of the smallest angle?
(A) 30° (B) 45° (C) 60° (D) 75° (E) 90°.

Solution: C.
By the property 5:
$$\frac{6}{3+4+5+6}\times 360^\circ - \frac{3}{3+4+5+6}\times 360^\circ = \frac{360}{18}(6-3) = 20\times 3 = 60^\circ .$$

4. MORE EXAMPLES

Example 19. If the degree measures of the angles of a triangle are in the ratio 3:4:5, by how many degrees does the measure of the largest angle exceed the measure of the smallest angle?
(A) 25° (B) 30° (C) 35° (D) 40° (E) 45°

Solution: B.
By the property 5: $\frac{5}{3+4+5}\times 180^\circ - \frac{3}{3+4+5}\times 180^\circ = \frac{180}{12}(5-3) = 15\times 2 = 30^\circ$.

Example 20. If $\frac{x}{y} = 2$ and $\frac{z}{x} = 5$, what is the value of $\frac{x+y+z}{x}$?
(A) $4\frac{1}{2}$ (B) 5 (C) $5\frac{1}{2}$ (D) $6\frac{1}{2}$ (E) 7

Solution: D:
$$\frac{x}{y} = 2 \quad \Rightarrow \quad y = \frac{x}{2}; \qquad \frac{z}{x} = 5 \quad \Rightarrow \quad z = 5x .$$
$$\frac{x+y+z}{x} = \frac{x + \frac{x}{2} + 5x}{x} = 1 + \frac{1}{2} + 5 = 6\frac{1}{2} .$$

Example 21. The ratio of x to y is 7 to 2, and the ratio of y to z is 5 to 3. If x, y, and z are positive numbers, what is the ratio of x to $(y + z)$?
(A) 25 to 7 (B) 35 to 16 (C) 65 to 27 (D) 7 to 2 (E) 7 to 5

Solution: B.

$\frac{x}{y}=\frac{7}{2}$ $\Rightarrow$ $x=\frac{7}{2}y$; $\frac{y}{z}=\frac{5}{3}$ $\Rightarrow$ $z=\frac{3y}{5}$.

$$\frac{x}{y+z}=\frac{\frac{7}{2}y}{y+\frac{3}{5}y}=\frac{\frac{7}{2}}{1+\frac{3}{5}}=\frac{\frac{7}{2}}{\frac{8}{5}}=\frac{35}{16}.$$

Example 22. Flour, water, and salt are mixed by weight in the ratio of 7:6:1, respectively, to produce a certain type of dough. In order to make 10 pounds of this dough, what weight of salt, in pounds, is required?

(A) 1/4 (B) 2/3 (C) 3/4 (D) 1 (E) 2

Solution: B.

By the property 5: $\frac{1}{7+6+2}\times 10=\frac{2}{3}$.

Example 23. The ratio of the perimeter of a rectangle to the length of one of its sides is 14 : 3. If the area is 27 square inches, how many inches long is one of the longer sides?

(A) 3 (B) 4 (C) 5 (D) 7 (E) 6

Solution: E.

Let x be the length and y be the width.

$\frac{2(x+y)}{x}=\frac{14}{3}$ $\Rightarrow$ $\frac{x+y}{x}=\frac{7}{3}$ $\Rightarrow$ $1+\frac{y}{x}=1+\frac{4}{3}$ $\Rightarrow$ $\frac{y}{x}=\frac{4}{3}$

$\Rightarrow$ $\frac{y\times y}{x\times y}=\frac{4}{3}$

The area is $x\times y=27$ $\Rightarrow$ $\frac{y\times y}{27}=\frac{4}{3}$ $\Rightarrow$ $y\times y=36$ $\Rightarrow$ $y=6$.

Example 24. A jar contains 15 red marbles and 15 black marbles. What is the least number of marbles in the jar when the ratio of red marbles to black marbles in the jar will be 4 to 3 after removing black marble(s) from the jar and adding red marble(s) to the jar?

(A) 40 (B) 36 (C) 34 (D) 28 (E) 24

Solution: D.
Let the red marbles be x and black marbles be y (removed or added).

$$\frac{15+x}{15-y}=\frac{4}{3} \quad \Rightarrow \quad 3x+4y=15.$$

Case I: The greatest value of y is 3. It works when $y = 3$ and $x = 1$ (removing 4 black marbles and adding one red marble). The number of marbles in the jar: 30 – 3 + 1 = 28.
Case II: The smallest value of y is 0. It works when $y = 0$ and $x = 5$ (removing 0 black marble and adding 5 red marbles). The number of marbles in the jar: 30 + 5 = 35. The answer is 28.

Example 25. If the cost of a pizza is directly proportional to its area and the cost of a pizza 15 inches in diameter is \$6.00, what should a pizza 9 inches in diameter cost?
(A) \$3 (B) \$2.62 (C) \$2.16 (D) \$1.32 (E) \$1.52

Solution: C.
Method 1: The area of a pizza of 15 inches in diameter = $\frac{1}{4}\pi d^2=\frac{1}{4}\pi\times 15^2$.

$$\frac{\frac{1}{4}\pi\times 15^2}{6}=\frac{\frac{1}{4}\pi\times 9^2}{x} \qquad x=\frac{15^2}{9^2}\times 6=2.16$$

Method 2: The ratio of the areas A_1 and A_2 of two similar figures is $\frac{A_1}{A_2}=(\frac{d_1}{d_2})^2$

Since the costs C_1 and C_2 are directly proportional to the areas, so we have

$$\frac{C_1}{C_2}=\frac{A_1}{A_2}=(\frac{d_1}{d_2})^2 \quad \Rightarrow \quad C_1=C_2(\frac{d_1}{d_2})^2=6\times\left(\frac{9}{15}\right)^2=2.16$$

Example 26. If the ratio of a to b is 2:3 and the ratio of b to c is 4:7 find the ratio of $a : c$.
(A) 15 : 8 (B) 8:15 (C) 2: 3 (D) 8 : 21 (E) 4 : 7

Solution: D.
Since $a : b = 2 : 3$ and $b : c = 4 : 7$, $a : b : c = 8 : 12 : 21 \quad \Rightarrow \quad a : c = 8 : 21$.

Example 27. The ratio of cats to dogs to squirrels in Dogpatch is 4:5:12, while the ratio of squirrels to raccoons to opossums is 10:3:6. What is the ratio of dogs to opossums?

(A) 15 : 18 (B) 25: 36 (C) 25 : 30 (D) 18 : 30 (E) 40 : 50

Solution: B.

We are given that $c : d : s = 4 : 5 : 12$.

$s : r : p = 10 : 3 : 6$.

We can get: $d : s = 5 : 12$ and $s : p = 10 : 6 = 5 : 3$.

$d : s : p = 25 : 60 : 36$.

So $d : p = 25 : 36$.

Example 28. Three numbers in the ratio of 7:3:2 have a sum of 228. What is the difference between the smallest and the largest numbers?

(A) 114 (B) 95 (C) 84 (D) 98 (E) 124

Solution: B.

The largest number is A and the smallest number is B.

$$A = \frac{a}{a+b+c} \times m = \frac{7}{7+3+2} \times 228$$

$$B = \frac{b}{a+b+c} \times m = \frac{2}{7+3+2} \times 228$$

$$A - B = \frac{7}{7+3+2} \times 228 - \frac{2}{7+3+2} \times 228 = \frac{5 \times 228}{7+3+2} = 95$$

5. PROBLEMS

Problem 1. The measures of the three angles of a triangle are $8w°$, $5w°$, and $2w°$. The measure of the largest angle of the triangle is how much greater than the measure of the next largest angle?
(A) 26° (B) 36° (C) 46° (D) 56° (E) 66°

Problem 2. If $\frac{x}{y} = k$ and $k > 0$, what is $\frac{x}{x+y}$ in terms of k?
(A) $\frac{k-1}{k}$ (B) $\frac{k+1}{k}$ (C) $\frac{1}{k}$ (D) $\frac{k}{k-1}$ (E) $\frac{k}{k+1}$

Problem 3. A fruit salad is made from pineapples, pears, and peaches mixed in the ratio of 2 to 3 to 5, respectively, by weight. What fraction of the mixture by weight is pineapple?
(A) 1/5 (B) 3/10 (C) 2/5 (D) 1/2 (E) 2/3

Problem 4. If $a \neq 0$ and $\frac{11}{x} = \frac{11+a}{x+a}$, what is the value of x?
(A) −5 (B) −11 (C) 11 (D) 12 (E) 15

Problem 5. Bill made 21 out of 25 shots in a recent basketball game. What was his shooting percentage for that game? .
(A) 21% (B) 25% (C) 84% (D) 16% (E) 75%

Problem 6. The temperature in Boston is now 50°F. If it drops at a steady rate of 4° F every 3 hours, in how many hours will the temperature drop to 38°F?
(A) 12 (B) 3 (C) 14 (D) 9 (E) 6

Problem 7. The scale of a map is 1 centimeter to 50 kilometers. How many centimeters represent 225 kilometers?
(A) 3.5 (B) 4.5 (C) 5.5 (D) 2.5 (E) 6.5

Problem 8. Potatoes cost $1.75 per pound. How many dollars do four pounds cost?
(A) $3 (B) $5 (C) $7 (D) $2 (E) $6

Problem 9. The amount of water in a container triples every minute. The container is completely filled at 1:20 p.m. What fractional part of the container is filled at 1:19 p.m.?
(A) 1/3 (B) 1/5 (C) 1/34 (D) 1/2 (E) 1/6

Problem 10. A typical one-hour television program devotes 15% of each hour to commercials. At this rate, how many minutes of commercials would a 3-hour program have?
(A) 13 (B) 15 (C) 17 (D) 27 (E) 16

Problem 11. In a college of education, the ratio of math majors to English majors is 5 to 3. If there are 56 math and English majors in all (and no one majors in both math and English), how many math majors are there?
(A) 35 (B) 45 (C) 55 (D) 65 (E) 56

Problem 12. If an eight-inch square cake serves four people, how many twelve-inch square cakes are needed to provide equivalent servings to eighteen people?
(A) 3 (B) 2 (C) 1 (D) 5 (E) 6

Problem 13. The ratio of chocolate cones to vanilla cones sold at an ice cream shop is 5:4. If 63 cones were sold, how many were chocolate?
(A) 53 (B) 55 (C) 31 (D) 35 (E) 36

Problem 14. A teacher has to read 30 essays by his students. He reads five of them in the first 45 minutes. Assuming that he continues to read at the same rate, how many hours will it take him to read all 30 papers?
(A) 9/2 (B) 21/5 (C) 2 (D) 16/3 (E) 6

Problem 15. If one gram of gold can be hammered into a square sheet 10 cm by 10 cm, how many grams of gold are needed to make a sheet that is 2.25 square meters?
(A) 223 (B) 225 (C) 221 (D) 202 (E) 206

Problem 16. Of 100 citizens surveyed in a certain town, only 45% were in favor of the city council's proposition. 40% were opposed and 15% undecided. If the survey holds and 40,000 citizens vote, what percent of the undecided voters must vote in favor of the proposition in order for it to pass with 51% of the votes?

(A) 25% (B) 35% (C) 40% (D) 45% (E) 50%

Problem 17. If 2 cats can catch 3 mice in 5 days, how many mice can 20 cats catch in 10 days?
(A) 50 (B) 55 (C) 60 (D) 65 (E) 76

Problem 18. A log is cut into 4 pieces in 12 seconds using parallel slices. At this same rate, how many seconds will it take to cut the log into 6 pieces?
(A) 15 (B) 20 (C) 25 (D) 30 (E) 35

Problem 19. If a cat and a half eat a fish and a half in a day and a half, how many days will it take 14 cats to eat 14 fish?
(A) 3/2 (B) 5/3 (C) 7/3 (D) 2 (E) 3

Problem 20. A watch, set accurately at 12 noon, at 5 p.m. indicates 10 minutes to 5. How many minutes past 5 is it actually when the watch indicates 5 p.m.?
(A) $310\frac{10}{29}$ (B) $10\frac{10}{29}$ (C) 10 (D) $\frac{10}{29}$ (E) 29

Problem 21. 270 students attend a school dance. The ratio of girls to boys is 5 to 4. If twenty boys and ten girls leave, what is the new ratio of girls to boys that remain at the dance?
(A) $\frac{9}{5}$ (B) $\frac{5}{9}$ (C) $\frac{5}{4}$ (D) $\frac{7}{5}$ (E) $\frac{9}{2}$

Problem 22. The sum of three numbers is 81 and their ratio is 3:7:17. What is the value of the smallest number?
(A) 5 (B) 7 (C) 9 (D) 17 (E) 16

Problem 23. The ratio of the length to width of a rectangular block is 2:1 and the ratio of the width to height is 3:2. Find the ration of the length to height.
(A) $\frac{1}{3}$ (B) $\frac{3}{1}$ (C) $\frac{6}{5}$ (D) $\frac{5}{4}$ (E) $\frac{8}{3}$

Problem 24. Bob bought three kinds of meat: pork, beef, and chicken with the total cost of $152. The ratio of the weight of pork, beef, and chicken is 2 : 4 : 3.

The ratio of the price per pound of pork, beef, and chicken is 6 : 5 : 2. What is the sum of the last digits of the cost of each kind in dollars?

(A) 8 (B) 10 (C) 12 (D) 15 (E) 16

Problem 25. In a mixture of peanuts and cashews, the ratio by weight of peanuts to cashews is 5 to 3. How many pounds of cashews will there be in 64 pounds of this mixture?

(A) 15 (B) 20 (C) 24 (D) 26 (E) 27

Problem 26. If a recipe for a two-pound cake uses 1.5 cups of flour, at the same rate how many cups of flour are needed for a five-pound cake?

(A) $3\frac{3}{4}$ (B) $3\frac{2}{5}$ (C) 3 (D) $\frac{10}{3}$ (E) $2\frac{3}{4}$

Problem 27. Marbles are to be removed from a jar that contains 16 red marbles and 16 black marbles. What is the least number of marbles that could be removed so that the ratio of red marbles to black marbles left in the jar will be 4 to 3 ?

(A) 1 (B) 2 (C) 4 (D) 6 (E) 7

Problem 28. George earned \$7.65 for the $4\frac{1}{2}$ hours he baby-sat. At the same rate, how many dollars should he charge for 3 hours of baby-sitting?

(A) \$5.10 (B) \$6.10 (C) \$7.10 (D) \$5.60 (E) \$8.10

Problem 29. Seven black cows and 4 yellow cows give in 6 days exactly as much milk as 5 black cows and 8 yellow cows give in 5 days. Which color cow gives more milk?

(A) black (B) yellow (C) equal (D) red (E) none of the above

6. SOLUTIONS

Problem 1. Solution: B.

By the property 5: $\frac{8}{8+5+2}\times 180^\circ - \frac{5}{8+5+2}\times 180^\circ = \frac{180}{15}(8-5) = 12\times 3 = 36^\circ$.

Problem 2. Solution: E.

$$\frac{x}{x+y} = \frac{1}{\frac{x+y}{x}} = \frac{1}{1+\frac{y}{x}} = \frac{1}{1+\frac{1}{\frac{x}{y}}} = \frac{1}{1+\frac{1}{k}} = \frac{1}{\frac{k+1}{k}} = \frac{k}{k+1}.$$

Problem 3. Solution: A.

By the property 5: $\frac{2}{2+3+5} = \frac{2}{10} = \frac{1}{5}$.

Problem 4. Solution: C.

By the property 9: $\frac{11}{x} = \frac{11+a}{x+a} = \frac{11+a-11}{x+a-x} = \frac{a}{a} = 1 \Rightarrow \quad x = 11$.

Problem 5. Solution: C.

Let x be the shooting percentage for that game.

$\frac{21}{25} = \frac{x}{100} \quad \Rightarrow \quad x = 84\%$

Problem 6. Solution: D.

The total number of degrees it drops is 50 – 38 = 12.

The number of 3 hours is 12 ÷ 4 = 3.

The answer is: 3 × 3 = 9 hours.

Problem 7. Solution: B.

Let x be the centimeters representing 225 kilometers.

$\frac{1}{50} = \frac{x}{225} \quad \Rightarrow \quad x = 4.5$

Problem 8. Solution: C.

Since one pound costs \$1.75, 4 pounds will cost $1.75 \times 4 = \$7$.

Problem 9. Solution: A.
Let x be the fractional part of the container that is filled at 1:19 p.m.
$3x = 1 \quad \Rightarrow \quad x = 1/3$.

Problem 10. Solution: D.
3 hours can be converted to 180 minutes.
Let x be the minutes of commercials would a 3-hour program have.
$$\frac{15}{100} = \frac{x}{180} \quad \Rightarrow \quad x = 27$$

Problem 11. Solution: A.
The ratio of the number of members in two majors is 5 : 3.
The number of math major students is A and $A = \frac{a}{a+b} \times m = \frac{5}{5+3} \times 56 = 35$.

Problem 12. Solution: B.
The area of a 8-inches square is 64. Let x be the area of the cake needed to serve 18 people.
$$\frac{64}{4} = \frac{x}{18} \quad \Rightarrow \quad x = 288$$
The area of the 12-inches square is 144. The answer is $288 \div 144 = 2$.

Problem 13. Solution: D.
The ratio of the number of cones in two kinds is 5 : 4. The number of chocolate cones is A and $A = \frac{a}{a+b} \times m = \frac{5}{5+4} \times 63 = 35$.

Problem 14. Solution: A.
Let x be the minutes of hours needed to read all 30 papers.
$$\frac{5}{45} = \frac{30}{x} \quad \Rightarrow \quad x = 270 \text{ minutes} = \frac{270}{60} = \frac{27}{6} = \frac{9}{2} \text{ hours.}$$

Problem 15. Solution: B.
Since 1 m = 100 cm, 10 cm = 0.1 m.

Let x be the minutes of grams of gold needed to make a sheet that is 2.5 square meters. $\frac{1}{0.1\times 0.1}=\frac{x}{2.25}$ $\Rightarrow$ $x=225$.

Problem 16. Solution: C.
51% of the votes are 51% × 40000 = 20400 votes.
45% in favor means there are 45% × 40000 = 18000 votes already.
The number of votes needed is 20400 – 18000 = 2400.
15% undecided means there are 15% × 40000 = 6000 people undecided.
The ratio is 2400 ÷ 6000 = 0.4 = 40%.

Problem 17. Solution: C.
Let x be the number of mice 20 cats can catch in 10 days.

$$\frac{3}{2\times 5}=\frac{x}{20\times 10} \qquad \Rightarrow \qquad x = 60.$$

Problem 18. Solution: B.
Three cuts are needed to cut the log in 4 pieces. Each cut takes 12 ÷ 3 = 4 seconds. Five cuts are needed to cut the log in 6 pieces. The time needed is 5 × 4 = 20 seconds.

Problem 19. Solution: A.
Let x be the number of days needed.

$$\frac{1+\frac{1}{2}}{(1+\frac{1}{2})\times(1+\frac{1}{2})}=\frac{14}{14\times x} \qquad \Rightarrow \qquad x=\frac{3}{2}.$$

Problem 20. Solution: B.
Since the watch is 10 minutes late in 5 hours, it is 2 minutes slow each hour.
When the standard watch goes 60 minutes, it goes 58 minutes.
Let x be the minutes it actually takes for the watch to go 5 hours (300 minutes).

$\frac{60}{58}=\frac{x}{300}$ $\Rightarrow$ $x=310\frac{10}{29}$. That is $10\frac{10}{29}$ minutes past 5.

Problem 21. Solution: D.

There are $G = \dfrac{g}{b+g} \times m = \dfrac{5}{5+4} \times 270 = 150$ girls and 270 – 150 = 120 boys.
If twenty boys and ten girls leave, the number of girls is 150 – 10 = 140 and the number of boys is 120 – 20 = 100.
The new ratio $= \dfrac{140}{100} = \dfrac{14}{10} = \dfrac{7}{5}$.

Problem 22. Solution: C.
The smallest number is A.

$$A = \frac{a}{a+b+c} \times m = \frac{3}{3+7+17} \times 81 = 9.$$

Problem 23. Solution: B.
Let the length be L, width be W and height be H.
We are given: $L : W = 2 : 1$, $W : H = 3 : 2$.
By the property of the proportion, we get:
$L : W : H = 6 : 3 : 2 \quad \Rightarrow \quad L : H = 6 : 2 = 3 : 1$.

Problem 24. Solution: C
Pork: \$48, beef: \$80, and chicken: \$24.
Let x , y, and z be the weight of pork, beef, and chicken, respectively.
$x : y : z = 2 : 4 : 3$ (1)
The ratio of the price will be $(6x) : (5y) : (2z)$
The costs of pork, beef, and chicken are A, B, and C, respectively.

$$A = \frac{6x}{6x+5y+2z} \times 152 \qquad (2)$$

From (1), we get: $y = 2x$, and $z = \dfrac{3}{2}x$. (3)

(2) becomes: $A = \dfrac{6x}{6x+5(2x)+2\times\dfrac{3}{2}x} \times 152 = \dfrac{6x}{19x} \times 152 = 48$

Similarly, $B = \dfrac{5y}{6x+5y+2z} \times 152 == \dfrac{5y}{19x} \times 152 = \dfrac{10x}{19x} \times 152 = 80$

And $C=\frac{2z}{6x+5y+2z}\times 152 == \frac{2z}{19x}\times 152=\frac{3x}{19x}\times 152=24$.

The answer is 8 + 0 + 4 = 12.

Problem 25. Solution: C.

By the property 5: $\frac{3}{5+3}\times 64=24$.

Problem 26. Solution: A.

Let *x* be the number of cups of flour are needed for a five-pound cake.

$\frac{2}{1.5}=\frac{5}{x} \qquad \Rightarrow \qquad x=\frac{5\times 1.5}{2}=\frac{7.5}{2}=3\frac{3}{4}$

Problem 27. Solution: C.

Let the number of black marbles removed be *x*.

$\frac{16}{16-x}=\frac{4}{3} \qquad \Rightarrow \qquad x=4$.

Problem 28. Solution: A.

Let *x* be the dollars should he charge for 3 hours of baby-sitting.

$\frac{7.65}{4\frac{1}{2}}=\frac{x}{3} \qquad \Rightarrow \qquad x=5.10$.

Problem 29. Solution: B.

Let *x* be the milk given by a black cow and *y* be the milk given by a yellow cow in a day. $6(7x + 4y) = 5(5x + 8y) \Rightarrow 42x + 24y = 25x + 40y \Rightarrow 17x = 15y$

$\frac{x}{y}=\frac{15}{17}<1 \qquad \Rightarrow \qquad$ *x* is less than *y*.

1. BASIC KNOWLEDGE

Notation: Given two positive integers a and b, the greatest common factor of a and b is written as (a, b) and the least common multiple of a and b is written as $[a, b]$.

The Greatest Common Factor (abbreviated *GCF*) of a group of natural numbers is the greatest natural number that is a factor of every number in the group.

Example 1. Bob has 45 football cards and 36 baseball cards. He wants to place them in stacks on a table so that each stack has the same number of cards, and no stack has different types of cards within it. Find the largest number of cards that he can have in each stack.

(A) 12 (B) 3 (C) 5 (D) 7 (E) 9

Solution: E.

Factors of 45:	1	3,	5,	9,	15,	45			
Factors of 36:	1,	2,	3	4,	6,	9,	12,	18,	36

The common factors of 45 and 36 are: 1, 3, and 9. *GCF* is the largest one of them and *GCF* (45, 36) = 9.

Properties of the Greatest Common Factor:

If $(a, b) = d$, and n is positive integer, then $(na, nb) = nd$.

$(a, 1) = 1$; $(a, a) = a$; $(a, 0) = a$; $(a, b) = (b, a)$; $(a, b) = (b, a - b)$.

The Least Common Multiple (abbreviated *LCM*) of a group of natural numbers is the smallest natural number that is a multiple of every number in the group.

Example 2. Both Alex Chan and Betty Chan work at a fast-food restaurant. Alex has every sixth day off and Betty has every fourth day off. If they are both off on Wednesday of this week, what will be the day of the week that they are next off together?

(A) Monday (B) Tuesday (C) Wednesday (D) Thursday (E) Friday

Solution: A.

Alex:	0	6	12	18	24	30	36	42	48	
Betty:	0	4	8	12	16	20	24	28	32	36

The common multiples of 6 and 4 are: 12, 24, 36, ... *LCM* is the smallest one of them and *LCM* (6, 4) = 12.

12 days from Wednesday of this week is Monday.

The *LCM* of Two Fractions

$$LCM\left(\frac{a}{b},\frac{c}{d}\right)=\frac{ac}{GCF(ad,bc)}=\frac{LCM(ad,bc)}{bd} \qquad (1.1)$$

Example 3. Find *LCM* of two fractions $\frac{10}{3}$ and $\frac{8}{5}$.

(A) 10 (B) 20 (C) 30 (D) 40 (E) 59

Solution: D.

$$LCM\left(\frac{10}{3},\frac{8}{5}\right)=\frac{10\times 8}{(50,24)}=\frac{80}{2}=40.$$

2. PROBLEM SOLVING SKILLS

2.1. Finding The Greatest Common Factor

(2.1.1). Prime factorization method

Step 1: Prime factorization of each number
Step 2: Find all primes common to all numbers
Step 3: Find all terms with the smallest exponent common to all factorizations
Step 4: Multiply all terms in step 3 and the product is the greatest common factor

Example 4. Find the *GCF* of 18, 27, and 36.

(A) 18 (B) 9 (C) 6 (D) 3 (E) 1

Solution: B.
Step 1: Prime factorization of each number:

$$18 = 2 \times 9 = 2 \times 3^2$$
$$27 = 3^3$$
$$36 = 6^2 = (2\times3)^2 = 2^2 \times 3^2$$

Step 2: Find all primes common to all numbers:

$$18 = 2 \times 9 = 2 \times 3^2$$
$$27 = 3^3$$
$$36 = 6^2 = (2\times3)^2 = 2^2 \times 3^2$$

Step 3: Find all terms with the smallest exponent common to all numbers:

$$18 = 2 \times 9 = 2 \times 3^2$$
$$27 = 3^3$$
$$36 = 6^2 = (2\times3)^2 = 2^2 \times 3^2$$

Step 4: Multiply all terms in step 3 and the product is the greatest common factor $3^2 = 9$.

(2.1.2) Euclidean Algorithm

If $a = bq + r$, $(0 \le r \le b)$, then $(a, b) = (b, r) = (b, a - bq)$ (2.1)

Example 5. Find the greatest common factor of 48 and 80.
(A) 6 (B) 8 (C) 15 (D) 16 (E) 19

Solution: D.
$80 = 48 \times 1 + 32 \quad \Rightarrow \quad (48, 80) = (48, 32)$
$48 = 32 \times 1 + 16 \quad \Rightarrow \quad (48, 32) = (16, 32)$
$32 = 16 \times 1 + 16 \quad \Rightarrow \quad (16, 32) = (16, 16) = 16.$

2.2. Finding the Least Common Multiple (Prime factorization method)

Step 1: Prime factorization of each number

Step 2: Find all primes belonging to any factorization
Step 3: Find all terms with the largest exponent in any factorization
Step 4: Multiply all terms in step 3 and the product is the least common multiple.

Example 6. Find the least common multiple of 21, 24, and 48.
(A) 336 (B) 326 (C) 325 (D) 441 (E) 419

Solution: A.
Step 1: Prime factorization of each number:

$21 = 3 \times 7$
$24 = 3 \times 8 = 3 \times 2^3$
$48 = 6 \times 8 = 2 \times 3 \times 2^3 = 3 \times 2^4$

Step 2: Find all primes belonging to any factorization

$21 = 3 \times 7$
$24 = 3 \times 8 = 3 \times 2^3$
$48 = 6 \times 8 = 2 \times 3 \times 2^3 = 3 \times 2^4$

Step 3: Find all terms with the largest exponent in any factorization

$21 = 3 \times 7$
$24 = 3 \times 8 = 3 \times 2^3$
$48 = 6 \times 8 = 2 \times 3 \times 2^3 = 3 \times 2^4$

Step 4: Multiply all terms in step 3 and the product is the least common multiple.
The answer is $3 \times 7 \times 2^4 = 336$.

2.3. Finding the *LCM/GCF* of Algebra Expressions

The use of methods to find *LCM/GCF* can also serves as a bridge to algebra, where the primes will be replaced with variables x and y.

Example 7. Find the *GCF* and *LCM* of (1). $7x^4y^5$, $14x^3y^6$ and (2). $10a^8b^5$, $6a^{10}b^3$.

Solutions:

(1). $7x^4y^5 = 7 \times x \times x \times x \times x \times y \times y \times y \times y \times y$

$14x^3y^6 = 2 \times 7 \times x \times x \times x \times y \times y \times y \times y \times y \times y$

The common factors are 7, x^3, and y^5, so the *GCF* is $7x^3y^5$.

$7x^4y^5 = 7 \times x^4 \times y^5$

$14x^3y^6 = 14 \times x^3 \times y^6$

The smallest multiple must contain 14, x^4, and y^6, so the *LCM* is $14x^4y^6$.

(2). The common factors are 2, a^8, and b^3, so the *GCF* is $2a^8b^3$.
The smallest multiple must contain 30 (the *LCM* of 10 and 6), a^{10}, and b^5, so the *LCM* is $30a^{10}b^5$.

2.4. Relationship of *LCM* and *GCF*

(Least common multiple of a and b) × (Greatest common factor of a and b) = $a \times b$.

$$(a,b)\times[a,b]=ab \qquad (2.2)$$

Example 8. The greatest common factor of two numbers is 7 and their least common multiple is 70. One of the numbers is 14. What is the other number?
(A) 14 (B) 7 (C) 15 (D) 35 (E) 9

Solution: D.
Let the other number be a.
By the formula $(a,b)\times[a,b]=ab$, we have: $7 \times 70 = a \times 14 \quad \Rightarrow \quad a = 35$.

Example 9. The least common multiple (*LCM*) of two positive integers is 1260. If one of the numbers is 60, what is the smallest possible value of the other number?
(A) 32 (B) 63 (C) 65 (D) 67 (E) 69

Solution: B.
Let the other number be a.

$1260 = 2^2 \times 3^2 \times 5 \times 7 = 60 \times 21$. So we know that a is a multiple 21.
$60 = 2^2 \times 3 \times 5$.
$a = 21 \times 3 = 3^2 \times 7 = 63$. So the smallest value for a is 63.

2.5. What we are looking for, *LCM* or *GCF*?

When we try to decide that we need to find the LCM or GCF, we have the following general rules to guide us:

Rule 1. For two distinct natural numbers m and n, $LCM\ (m, n) > GCF\ (m,n)$
For example, $LCM\ (3, 6) = 6 > GCF\ (3, 6) = 3$.

Rule 2. The $GCF\ (a, b)$ is a factor of a and b, so generally The $GCF \le a$ and the $GCF \le b$.

Rule 3. The $LCM\ (a, b)$ is a multiple of a and b, so generally The $LCM \ge a$ and the $LCM \ge b$.

Rule 4. When we are looking for a number that is bigger than a or b, we are looking for the least common multiple of a and b.

Rule 5. When we are looking for a number that is smaller than a or b, we are looking for the greatest common factor of a and b.

Example 10. Alex and Bob are training for the Chicago Marathon. Alex can go around the park in his wheelchair in 3 minutes; Bob can go the same distance in 4 minutes. If they start at the same time side-by-side, when will they be side-by-side again?
(A) 12 (B) 13 (C) 15 (D) 17 (E) 19

Solution: A.
We are looking for a number that is bigger than 3 or 4. So we are looking for the least common multiple.

$LCM(3, 4) = 12$. They will be side-by-side again in 12 minutes.

Example 11. Betsy wants to create snack bags for a trip she is going on. She has 36 granola bars and 63 pieces of dried fruit. If the snack bags should be identical without any food left over, what is the greatest number of snack bags Betsy can make?

(A) 12 (B) 13 (C) 11 (D) 7 (E) 9

Solution: E.

We are looking for a number that is smaller than 36 or 63. So we are looking for the greatest common factor.

By the formula (2.1), we have $GCF(36, 63) = GCF(36, 27) = GCF(9, 27) = GCF(9, 9) = 9$. The greatest number of snack bags Betsy can make is 9.

3. MORE EXAMPLES

Example 12. What is the difference of the least common multiple of 60 and 102 and the greatest common factor of 12 and 30?

(A) 60 (B) 6120 (C) 1020 (D) 1014 (E) 720

Solution: D.

$60 = 2^2 \times 3 \times 5$

$102 = 2 \times 3 \times 17$

$LCM\ (30, 102) = 2^2 \times 3 \times 5 \times 17 = 1020.$

$12 = 2^2 \times 3$

$30 = 2 \times 3 \times 5$

$GCF\ (12, 30) = 2 \times 3 = 6.$

$1020 - 6 = 1014.$

Example 13. If k is divisible by 3, 4, and 15, which of the following is also divisible by these numbers?

(A) 12 (B) 15 (C) 65 (D) 120 (E) 245

Solution: D.

Since k is divisible by 3, 4, and 15, k must be a multiple of 60, as 60 is the least common multiple of 3, 4, and 15. Some multiples of 60 are 0, 60, 120, and 180.

Example 14. What is the largest three-digit number divisible by 3, 4 and 5?

(A) 660 (B) 760 (C) 860 (D) 960 (E) 916

Solution: D.
The three-digit should be divisible by the *LCM* of 3, 4, and 5. *LCM* (3, 4, 5) = $3 \times 4 \times 5 = 60$.
$60 \times 16 = 960$. So 960 is the answer.

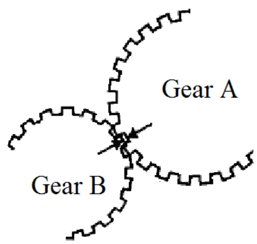
Example 15. The least common multiple of 12, 15, 20 and k is 420. What is the least possible value of k?
(A) 2 (B) 3 (C) 5 (D) 7 (E) 9

Solution: D.
$12 = 2^2 \times 3$
$15 = 3 \times 5$
$20 = 2^2 \times 5$
$420 = 2^2 \times 3 \times 5 \times 7$. We miss a 7. So $k = 7$.

Example 16. A crate contains 62 oranges, 46 apples, and 94 pears. If 2 more of each type of fruit were added to the crate, each of the three types of fruit could be divided equally among a group of people. What is the greatest possible number of fruit in such a group?
(A) 8 (B) 12 (C) 15 (D) 16 (E) 32

Solution: D.
(Rule 5).
Let the greatest possible number of people in such a group be n. n divides evenly $62 + 2 = 64$, $46 + 2 = 48$, and $94 + 2 = 96$.
$48 = 3 \times 16$
$64 = 4 \times 16$
$96 = 5 \times 16$
The greatest common factor of 64, 48, and 96 is 16.

Example 17. The figure shows parts of two circular gears whose teeth interlock when the gears turn. Gear A has 72 teeth and gear B has 48 teeth. What is the least positive number of rotations gear B must make before the arrows will again be aligned in the same way?



(A) 1 (B) 2 (C) 3 (D) 7 (E) 5

Solution: C.
(Rule 4).
$72 = 2^3 \times 3^2$
$48 = 2^4 \times 3$
The $LCM = 2^4 \times 3^2 = 144$
The least positive number of rotations gear B must make is 144 / 48 = 3.

Example 18. The least common multiple of two integers is 36, and 6 is their greatest common divisor. What is the product of the two numbers?
(A) 216 (B) 108 (C) 72 (D) 16 (E) 6

Solution: (A).
By (2.2), we know that $LCM\ (a, b) \times GCF\ (a, b) = a \times b = 36 \times 6 = 216$.

Example 19. The greatest common divisor of 21 and some number between 50 and 60 is 7. What is the number?
(A) 51 (B) 53 (C) 56 (D) 58 (E) 59

Solution: C.
The number between 50 and 60 that has a factor of 7 is 56.

Example 20. Two traffic lights turn red together at exactly 5:00 p.m. One light is on a 36-second cycle from red back to red while the other is on a 48-second cycle. At what time will they again turn red together? Give the exact time, including seconds.

Solution: 5:02:24.
(Rule 4).
$36 = 2^2 \times 3^2$, and $48 = 2^4 \times 3$.

$LCM\ (36, 48) = 2^4 \times 3^2 = 144$. The Least Common Multiple tells us that every 144 seconds, the two lights both turn red.
144 seconds is equivalent to 2 minutes and 24 seconds.
The first time that both lights are red is 2 minutes and 24 seconds after 5, or 5:02:24.

Example 21. The lengths in feet of three pieces of timber are 48, 72, and 40. The sawmill operator needs to cut the timber into logs of equal length with no waste. How many feet long is the greatest possible length she can cut?

(A) 9 (B) 6 (C) 5 (D) 16 (E) 8

Solution: E.
(Rule 5).
We need to find the greatest number that will divide 48, 72, 40 evenly, that is the *GCF*(48,72,40).
$72 = 2^3 \times 3^2$, $40 = 2^3 \times 5$ and $48 = 2^4 \times 3$.
GCF $(9 \times 8, 5 \times 8, 6\times 8) = 8$.

Example 22. Alex can jog 120 meters in 1 minute, Bob can jog 80 meters in 1 minute, and Charlie can jog 70 meters in 1 minute. The circular path has a perimeter of 400 meters. They start running together at 9:00 a.m. at point *A*. At what time will they first all be together again at point *A*?

Solution: 9:40.
(Rule 4).
Time needed for Alex to catch Bob is $400 \div (120 - 80) = 10$ minutes
Time needed for Alex to catch Charlie is $400 \div (120 - 70) = 8$ minutes
Time needed for Bob to catch Charlie is $400 \div (80 - 70) = 40$ minutes
LCM $(10, 8, 40) = 40$.
So at 9:40 they will first all be together again at point *A*.

Example 23. Two circles, one of radius 5 inches, the other of radius 2 inches, are tangent at point *P*. Two bugs start crawling at the same time from point *P*, one crawling along the larger circle at 3π inches per minute, the other crawling along the smaller circle at 2.5π inches per minute. How many minutes is it before their next meeting at point *P*?

(A) 42 (B) 43 (C) 40 (D) 47 (E) 49

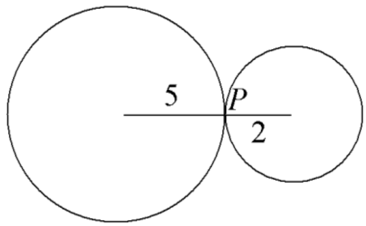
Solution: C.
(Rule 4).

At 3π inches per minute, the time needed for one rotation for the first bug to crawl along the larger circle is $\frac{2\pi \times 5}{3\pi} = \frac{10}{3}$.

At 2.5π inches per minute, the time needed for one rotation for the bug to crawl along the smaller circle is $\frac{2\pi \times 2}{2.5\pi} = \frac{8}{5}$.

By the formula (1.1), $LCM\left(\frac{10}{3}, \frac{8}{5}\right) = \frac{10 \times 8}{(50,24)} = \frac{80}{2} = 40$.

It takes 40 minutes before their next meeting at point *P*.



4. PROBLEMS

Problem 1. What is the greatest common factor (*GCF*) of 90, 135 and 270?
(A) 3 (B) 5 (C) 9 (D) 15 (E) 45

Problem 2. A group of people is dividing fruits in a crate containing 64 oranges, 48 apples, and 96 pears. If 2 more people were added to the group, each of the three types of fruit could be divided equally among them. What is the greatest possible number of people in such a group?
(A) 8 (B) 12 (C) 15 (D) 14 (E) 30

Problem 3. The eggs in a certain basket are either white or brown. If the ratio of the number of white eggs to the number of brown eggs is 3/4, each of the following could be the number of eggs in the basket except
(A) 7 (B) 12 (C) 14 (D) 28 (E) 63

Problem 4. What is the greatest 2-digit number that is divisible by both 2 and 3, but not by 4?
(A) 96 (B) 90 (C) 84 (D) 78 (E) 72

Problem 5. A bag of candy can be divided in equal shares among 2, 3, 4, 5, or 6 friends. What is the least number of pieces of candy that the bag could contain?
(A) 60 (B) 30 (C) 24 (D) 20 (E) 36

Problem 6. What is the smallest positive integer greater than 1 such that division by each of 4, 5, 6, 9 and 10 gives a remainder of 1?
(A) 180 (B) 181 (C) 124 (D) 120 (E) 136

Problem 7. What is the product of the greatest common factor and the least common multiple of 10 and 35?
(A) 310 (B) 320 (C) 340 (D) 350 (E) 400

Problem 8. The *LCM* of a pair of whole numbers is 450, and the *GCF* of the numbers is 6. One of the numbers is 18. What is the other number?
(A) 150 (B) 160 (C) 170 (D) 180 (E) 116

Problem 9. Two natural numbers have a *GCF* of 6 and an *LCM* of 36. If the sum of the two numbers is 30, what is the value of the smaller of the numbers?
(A) 8 (B) 16 (C) 14 (D) 10 (E) 12

Problem 10. Find the result if the greatest common factor (*GCF*) of 40 and 24 is subtracted from the least common multiple (*LCM*) of 15 and 6.
(A) 18 (B) 24 (C) 20 (D) 22 (E) 64

Problem 11. Find the result when the least common multiple (*LCM*) of 12 and 15 is multiplied by the greatest common factor (*GCF*) of 18 and 45.
(A) 470 (B) 412 (C) 540 (D) 528 (E) 560

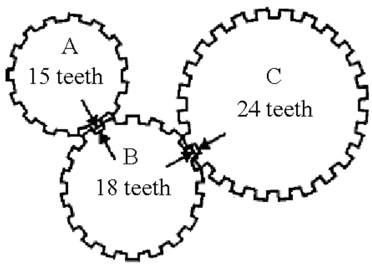
Problem 12. Ioana has three ropes whose lengths are 39 inches, 52 inches and 65 inches. She wants to cut the ropes into equal length pieces for magic tricks. No rope is to be wasted. What is the greatest number of inches possible in the length of each piece?
(A) 13 (B) 12 (C) 14 (D) 17 (E) 15

Problem 13. A school band found they could arrange themselves in rows of 6, 7, or 8 with no one left over. What is the minimum number of students in the band?
(A) 167 (B) 162 (C) 164 (D) 168 (E) 163

Problem 14. A neon sign display has three sets of flashing light. At 1:00, all lights are off. Set *A* remaining on for 15 seconds, then off for one second. Set *B* remaining on for 20 seconds, then off for one second. Set *C* remaining on for 6 seconds, then off for one second. How many seconds after 1:00 will all 3 sets be off again for the first time?
(A) 336 (B) 313 (C) 314 (D) 328 (E) 363

Problem 15. Two natural numbers have a *GCF* of 6 and an *LCM* of 36. If the sum of two numbers is 30, what is the value of the smaller of the numbers?
(A) 17 (B) 12 (C) 15 (D) 23 (E) 62

Problem 16. Susan can run a lap in 1 minute, 15 seconds; George can run a lap in 1 minute, 30 seconds; and Jennifer can run a lap in 1 minute, 40 seconds. They start running together at 9:00 a.m. at point A. At what time will they first all be together again at point A?
(A) 9:15 a.m. (B) 9:14 a.m. (C) 9:13 a.m. (D) 9:13 p.m. (E) 9:15 p.m.



Problem 17. Gears are aligned as shown, and each gear has the number of teeth indicated. What is the least positive number of rotations gear *C* must make before the arrows will again be aligned in the same way?
(A) 11 (B) 12 (C) 17 (D) 22 (E) 15

Problem 18. Three boats are circling a small island of circumference of 15 km. Boat *A* has a speed 6 km/h; boat *B* 5 km/h; boat *C* 3 km/h. They start at the same time and same point. How long does it take for them to have their first meeting all together at the starting point?
(A) 15 (B) 17 (C) 900 (D) 20 (E) 11

Problem 19. The sum of two numbers is 70 and their greatest common factor is 7. What is the greatest difference between these two numbers?
(A) 52 (B) 56 (C) 27 (D) 63 (E) 55

Problem 20. The positive difference of two odd natural numbers is 2 and the positive difference of the greatest common factor and the least common multiple is 142. Find the larger of the two numbers.
(A) 10 (B) 9 (C) 11 (D) 13 (E) 15

Problem 21. One gear turns 33 1/3 times in a minute. Another gear turns 45 times in a minute. Initially, a mark on each gear is pointing due north. After how many seconds will the two gears next have both their marks pointing due north?
(A) 36 (B) 18 (C) 9 (D) 22 (E) 16

Problem 22. Alice can run a lap in $\frac{141}{10}$ seconds; Betsy can run a lap in $\frac{235}{14}$ seconds; and Casey can run a lap in $\frac{94}{7}$ seconds. They start running together at the same time at point A. When Alice, Betsy, and Casey meet for the first time at point A, how many laps would Alice have ran?

(A) 110 (B) 122 (C) 112 (D) 200 (E) 100

5. SOLUTIONS

Problem 1. Solution: E.
Use prime factorization to rewrite 45, 135 and 270 as follows
$90 = 3^2 \times 5 \times 2$
$135 = 3^3 \times 5$
$270 = 2 \times 3^3 \times 5$
The $GCF = 5 \times 3^2 = 45$

Problem 2. Solution: D.
(Rule 5).
Let the the greatest possible number of people in such a group be n. $n + 2$ divides evenly 64 , 48, and 96.
$48 = 3 \times 16$
$64 = 4 \times 16$
$96 = 6 \times 16$
The greatest common factor of 64, 48, and 96 is 16. That is, $n + 2 = 16 \Rightarrow$
$n = 14$.

Problem 3. Solution: B.
Let the number of white eggs be $3a$ and the number of brown eggs be $4a$. The total number of eggs in the basket will be $7a$, that is a multiple of 7. So the answer is (B).

Problem 4. Solution: B.
The greatest 2-digit number that is divisible by $2 \times 3 = 6$ is 96. Since 96 is also divisible by 4, so we look at next number $96 - 6 = 90$, which is not divisible by 4. So it is the answer.

Problem 5. Solution: A.
(Rule 5).
The least number should be divisible by $LCM\ (2, 3, 4, 5, 6) = 60$.

Problem 6. Solution: B.
When 1 is subtracted from the number, the result should be divisible by LCM (4, 5, 6, 9) = 180. So the smallest resulting number is 180 and the answer is $180 + 1 = 181$.

Problem 7. Solution: D.
By the formula (2.2), the product is the same as the product of two numbers. $35 \times 10 = 350$.

Problem 8. Solution: A.
By the formula (2.2), the other number is $450 \times 6/18 = 150$.

Problem 9. Solution: E.
Let two numbers be a and b with $a > b$. We have $a + b = 30$.
By the formula (2.2), $a \times b = 6 \times 36 = 216 = 18 \times 12$. So the smaller number is 12.

Problem 10. Solution: D.
By the Euclidean Algorithm (3.1), GCF (40, 24) = GCF (16, 24) = GCF (16, 8) = GCF (8, 8) = 8.
LCM (15, 6) = 30. Therefore $30 – 8 = 22$.

Problem 11. Solution: C.
By the Euclidean Algorithm (2.1), GCF (18, 45) = GCF (18, 9) = GCF (9, 9) = 9.
LCM (15, 12) = 60. Therefore $60 \times 9 = 540$.

Problem 12. Solution: A.
(Rule 5).
Method 1: $39 = 3 \times 13$; $52 = 4 \times 13$; $65 = 5 \times 13$. GCF (39, 52, 65) = 13.

Method 2: By the Euclidean Algorithm (2.1), GCF (39, 52, 65) = GCF (39, 13, 13) = GCF (13, 13, 13) = 13.

Problem 13. Solution: D.
(Rule 4).
The minimum number of students is divisible by the LCM (6, 7, 8) = $3 \times 7 \times 8 = 168$.

Problem 14. Solution: A.
(Rule 4).
Set A has a cycle of $15 + 1 = 16$ seconds; Set B has a cycle of $20 + 1 = 21$ seconds; and Set C has a cycle of $6 + 1 = 7$ seconds.

Factor 16 and 21: $16 = 2^4$, and $21 = 3 \times 7$. The $LCM\,(16, 21, 7) = 2^4 \times 7 \times 3 =$ 336.

Problem 15. Solution: B.
Let two numbers be a and b. By the formula (2.2), we can write $a \times b = 36 \times 6$. We know that $a + b = 30$, so it is easy to see that $a = 12$ and $b = 18$. The smaller number of the two is 12.

Problem 16. Solution: A.
(Rule 4).
We know that we need to find a number (time in seconds) that is the smallest common multiple of 75 seconds, 90 seconds, and 100 second, so in other words, $LCM\,(75, 90, 100)$.
$75 = 5^2 \times 3$, and $90 = 3^2 \times 2 \times 5$, and $100 = 2^2 \times 5^2$.
$LCM\,(75, 90, 100) = 3^2 \times 2^2 \times 5^2 = 900$ seconds = 15 minutes.
So the answer is 9:15 A.M.

Problem 17. Solution: E.
(Rule 4).
We find the number of teeth needed for three gears to align as shown again.
$LCM\,(15, 18, 24) = 360$.
Gear C must make $360/24 = 15$ rotations.

Problem 18. Solution: A.
(Rule 4).
Time needed for Boat A to circle the island: $15/6 = 2.5\ h = 150$ min.
Time needed for Boat B to circle the island: $15/5 = 3\ h = 180$ min.
Time needed for Boat C to circle the island: $15/3 = 5\ h = 300$ min.
Time needed for three boats to meet again: $LCM\,(150,180,300) = 900$ min. = 15 hours.

Problem 19. Solution: B.
Since two numbers have the GCF of 7, both of them can be expressed as a multiple of 7. Let two numbers be a and b. We have $a = 7x$ and $b = 7y$. Thus $a + b = 7(x + y) = 70$. So $x + y = 10$. Since we want to have the greatest difference of two numbers, so we let $x = 9$ and $y = 1$. The greatest difference: $7(x - y) = 7 \times 8 =$ 56.

Problem 20. Solution: D.
Let one number be x and the other number be $x + 2$. It is seen that x and $x + 2$ are two consecutive odd numbers so they have no common factors. That is, $GCF(x, x + 2) = 1$. therefore $LCM(x, x + 2) - 1 = 142$. $LCM(x, x + 2) = 143$. By the formula (2.2), $1\times 143 = x(x+2)$ or $x(x+2) = 11\times 13$. The two numbers are 11 and 13 and the answer is 13.

Problem 21. Solution: A.
(Rule 4).
The first gear has its mark face north every $\dfrac{60}{33\frac{1}{3}} = \dfrac{60}{\frac{100}{3}} = \dfrac{9}{5}$ seconds.

The second gear has its mark face north every $\dfrac{60}{45} = \dfrac{4}{3}$ seconds.

The answer will then be: $[\dfrac{9}{5}, \dfrac{4}{3}] = \dfrac{9\times 4}{gcf(9\times 3, 4\times 5)} = \dfrac{36}{1} = 36$ seconds.

Problem 22. Solution: E.
(Rule 4).
This is a problem of finding the LCM of 141/10, 235/14, and 94/7.

From the formula: $[\dfrac{a}{b}, \dfrac{c}{d}] = \dfrac{ac}{gcf(ad, bc)}$,and $[a, b, c] = [[a, b], c]$,we have:
[141/10, 235/14] = 705/2 and [705/2, 94/7] = 1410

So after 1410 seconds they will meet the first time at point A again.

The number of laps Alice will have run will then be: 1410/ (141/10) = 100.

1. TERMS

Equation: An equation is a mathematical sentence in which the equal sign "=" connects two algebraic expressions.

The following are equations:

$2^{10} = 1024$; $\frac{1}{7} = 0.\overline{142857}$; $3\pi = 5x - 6$; $A = \pi r^2$

Open sentence: An equation that contains one or more variables is called an open sentence. For example: $2x + 2 = 8$. The sentence is neither true nor false.

Solution: A solution is a value of the variable that makes the equation true (Other names: root, and zero).

For the equation $x + 5 = 9$, $x = 4$ is the solution since $4 + 5 = 9$.

When the variable in the equation is replaced with a constant so that the equation becomes true, the equation has been solved.
We will learn the following skills:

(1) get rid of the denominators
(2) add parentheses
(3) remove the parentheses
(4) isolate the variable
(5) combine like terms.

We will learn how to solve the following equations:

(1) one-variable linear equations
(2) literal equations
(3) quadratic equations
(4) system of linear equations
(5) system of nonlinear equations

2. SOLVING ONE-VARIABLE LINEAR EQUATIONS

One-variable: one unknown in the equation, such as in $3x = 9$. x is the only unknown.

Linear: the variable has the power of 1, such as in $3x = 9$. x has the power of 1 ($x = x^1$).

The simplest form of this kind of equations: $ax = b$ ($a \neq 0$), where both a and b are constant. x is the variable (unknown).

The solution is $x = \frac{b}{a}$ (divide both sides of the equation by a).

Basic skills to solving one-variable linear equations

Example 1. Solve equation: $\frac{1}{11}x = 35$

(A) 11 (B) 35 (C) 35/11 (D) 11/35 (E) 385

Solution: E.

Get rid of the denominators:

Multiply each side by 11: $\frac{1}{11}x \times 11 = 35 \times 11$ $\Rightarrow$ $x = 385$.

Example 2. Solve equation: $x - \frac{5-x}{3} = 1 - \frac{x-2}{6}$

(A) 1 (B) 2 (C) 5 (D) 4 (E) 6

Solution: B.

Multiplying each side by 6: $6(x - \frac{5-x}{3}) = 6(1 - \frac{x-2}{6})$ $\Rightarrow$

$$6x - 6 \times \frac{5-x}{3} = 6 - 6 \times \frac{x-2}{6}$$

Why must we multiply each side by 6? Because we want to get rid of the denominator. There are a lot of numbers you could multiply in order to get rid of the denominator, and in this case, 6 is the smallest one.

Add parentheses:

$$6x-\cancel{6}\times\frac{5-x}{\cancel{3}}=6-\cancel{6}\times\frac{x-2}{\cancel{6}} \qquad \Rightarrow \qquad 6x-2\times(5-x)=6-(x-2)$$

Notice that the fraction line in $\frac{5-x}{3}$ acts as the parentheses. $5-x$ is one unit and should be treated as $(5-x)$.

Remove the parentheses:

$$6x-2\times5+2x=6-x+2$$

Note that the x on the left hand side changed its sign and the x on the right hand side does not change signs.

Note also $6x-10-2x=6-x-2$ is incorrect.

Isolate the variable:

$$6x-2\times5+2x=6-x+2 \qquad \Rightarrow 6x+2x+x=6+2+10$$

Move all the unknowns to the left hand side and all numbers to the right hand side. When you do so, remember to change the sign of each term to the opposite sign.

For example, when you move – 10 to the right hand side, it is changed from – 10 to + 10.

When "$-x$" is moved from the right hand side to the left hand side, it changed from "$-x$" to "$+x$".

Notice that the purpose of moving the terms is to isolate the variable. Terms that do not move to other side do not change their sign.

Combine like terms:

$\Rightarrow\ 6x+2x+x=6+2+10 \quad \Rightarrow\ 9x=18$

Divide both sides by 9: $x=2$

Example 3. Solve for *x*: $\frac{1}{3}(x-1)+\frac{3}{4}(x+1)=\frac{1}{2}(x-1)+\frac{2}{3}(x+1)$

(A) 4 (B) 5 (C) 3 (D) 6 (E) 10

Solution: C.

$\frac{1}{3}(x-1)-\frac{1}{2}(x-1)=\frac{2}{3}(x+1)-\frac{3}{4}(x+1)$

$\frac{1}{6}(x-1)=\frac{1}{12}(x+1) \quad \Rightarrow \quad 2(x-1)=x+1 \quad \Rightarrow \quad x=3$.

☆**Example 4.** In a far-off land five fish can be traded for three loaves of bread and a loaf of bread can be traded for three bags of rice. How many bags of rice is one fish worth?

(A) 1/5 (B) 1/2 (C) 3/4 (D) $1\frac{4}{5}$ (E) $\frac{5}{9}$

Solution: D.

$5F = 3B$ (1)

$B = 3R$ (2)

Substituting (2) into (1): $5F = 3 \times 3R \quad \Rightarrow \quad F = \frac{9}{5}B = 1\frac{4}{5}B$.

☆**Example 5.** Before district play, the Unicorns had won 45% of their basketball games. During district play, they won six more games and lost two, to finish the

season having won half their games. How many games did the Unicorns play in all?
(A) 48 (B) 50 (C) 52 (D) 54 (E) 60

Solution: A.
Let n be the number of Unicorn games before district play. Then $0.45n + 6 = 0.5(n + 8)$. Solving for n yields
$0.45n + 6 = 0.5n + 4$.
$2 = 0.05n$
$40 = n$
So the total number of games is 40 + 8 = 48.

3. SOLVING LITERAL EQUATIONS

A literal equation is an equation that contains one or more letters. These letters are constants but are not fixed values.
General case: x is the variable in the equation.
Case I. When $a \neq 0$, the equation has a unique solution: $x = \frac{b}{a}$.
Case II. When $a = b = 0$, the equation has infinite many solutions.
Case III. When $a = 0$ and $b \neq 0$, the equation has no solutions.

Examples 6. In the metric system, temperature is measured in degree Celsius (°C) in stead of degree Fahrenheit (°F). The formula is as follows: $C = \frac{5(F-32)}{9}$.
Solve for F.

Solution:
Multiply both sides by 9: $9C = 5(F-32)$
Remove the parentheses: $9C = 5F - 5\times 32$
Isolate the variable: $5F = 9C + 5\times 32$
Divide both sides by 5: $F = \frac{9}{5}C + 32$

Example 7. Solve for x: $a^2+ax=1-x$

Solution:

Isolate the variable by moving terms: $ax+x=1-a^2$

Combine the like terms: $(a+1)x=1-a^2$

Case I. When $(a + 1) \neq 0$, the equation has the unique solution: $x=\dfrac{1-a^2}{a+1}$.

Case II. When $(a + 1) = 0$, $a = -1$ and $1-a^2=1-1=0$, the equation has infinite many solutions.

Example 8. If $x=\dfrac{a}{b+c}=\dfrac{b}{c+a}=\dfrac{c}{a+b}$, find the value for x.

Solution:

Case I. If $a + b + c = 0$

$b + c = -a$

$$x=\frac{a}{b+c}=\frac{a}{-a}=-1$$

Case II. If $a + b + c \neq 0$

$$x=\frac{a}{b+c}=\frac{b}{c+a}=\frac{c}{a+b}=\frac{a+b+c}{b+c+c+a+a+b}=\frac{a+b+c}{2(a+b+c)}=\frac{1}{2}$$

4. SOLVING QUADRATIC EQUATIONS

The following equation is called quadratic equation:

$ax^2 + bx + c = 0$

where a, b, and c are real numbers with $a \neq 0$.

Square root property: The solution to $x^2 = k$ is $x=\sqrt{k}$ and $x=-\sqrt{k}$.

Example 9. Solve the equation $x^2 = 11$.

Solution:

The solutions are $x = \sqrt{11}$ and $x = -\sqrt{11}$.

Example 10. Solve the equation $(x - 4)^2 = 12$.

Solution:

The solutions are $x - 4 = \sqrt{12} \quad \Rightarrow \quad x = 4 + \sqrt{12} = 4 + 2\sqrt{3}$

and $x - 4 = -\sqrt{12} \quad \Rightarrow \quad x = 4 - \sqrt{12} = 4 - 2\sqrt{3}$

or $x = 4 \pm 2\sqrt{3}$.

QUADRATIC FORMULA:

We will derive the quadratic formula below. The method used is called "completing the square" method. The method works for all quadratic equations.

$ax^2 + bx + c = 0$

Since $a \neq 0$, we can divide both sides by a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Now we complete the square:

$$x^2 + 2 \times x \times \frac{b}{2a} + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a} = 0$$

$$x^2 + 2 \quad x \quad y \quad + \quad y^2 = (x + y)^2.$$

$$x^2 + 2 \times x \times \frac{b}{2a} + (\frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a}$$

We can write the left side as a perfect square, and the right side as a single fraction:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root of each side: $x+\dfrac{b}{2a}=\pm\sqrt{\dfrac{b^2-4ac}{4a^2}}$

Solve for x: $x_{1,2}=\dfrac{-b}{2a}\pm\sqrt{\dfrac{b^2-4ac}{4a^2}}=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Simplify: $x_{1,2}=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Example 11. Solve $x^2 + 6x = 40$

Solution:

$x^2 + 6x = 40 \qquad \Rightarrow \qquad x^2 + 6x - 40 = 0$

$a = 1$, $b = 6$, and $c = -40$.

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-6\pm\sqrt{6^2-4\times1\times(-40)}}{2\times5}=\frac{-6\pm14}{2}=-3\pm7$$

$x_1 = -3 + 7 = 4$ and $x_2 = -3 - 7 = -10$.

Example 12. Solve $5x^2 = 10x - 4$ using the quadratic formula.

Solution:

$5x^2 = 10x - 4 \qquad \Rightarrow \qquad 5x^2 - 10x + 4 = 0$

$a = 5$, $b = -10$, and $c = 4$.

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-10)\pm\sqrt{(-10)^2-4\times5\times4}}{2\times5}=\frac{10\pm\sqrt{100-80}}{10}=\frac{5\pm\sqrt{5}}{5}.$$

5. SOLVING SYSTEM OF LINEAR EQUATIONS

A group (two or more) of equations is called a system of equations. The solutions of a system of equations should satisfy all the equations. The most commonly used methods are (1) substitution method, and (2) elimination method.

Example 13. If $x + y = 12$ and $x - y = 8$, what is the value of $2x - xy$?
(A) 4 (B) 0 (C) 2 (D) 5 (E) 6

Solution: B.
Method 1 (Substitution method):
$x + y = 12$ (1)
$x - y = 8$ (2)
From (2), we get: $x = 8 + y$ (3)
Substitute (3) into (1): $8 + y + y = 12 \Rightarrow 2y = 12 - 8 = 4 \Rightarrow y = 2$.
Then from (1) we have $x + 2 = 12 \Rightarrow x = 12 - 2 = 10$.
$2x - xy = 2 \times 10 - 10 \times 2 = 20 - 20 = 0$.

Method 2 (Elimination method):
$x + y = 12$ (1)
$x - y = 8$ (2)
(1) + (2): $2x = 20 \Rightarrow x = 10$
Substitute $x = 10$ into (1): $y = 2$
$2x - xy = 2 \times 10 - 10 \times 2 = 20 - 20 = 0$.

Example 14. If $3a + b = 17$ and $a + 1 = b$ what is the value of $a \cdot b$?
(A) 40 (B) 50 (C) 20 (D) 18 (E) 16

Solution: C.
$3a + b = 17$ (1)
$a + 1 = b$ (2)
(1) + (2): $4a + b + 1 = 17 + b \Rightarrow 4a = 17 - 1 = 16 \Rightarrow a = 4$
So $b = a + 1 = 4 + 1 = 5$
$a \times b = 4 \times 5 = 20$

Example 15. If $x + y = 10$ and $2x - y = 11$, find the value of $x^2 + y^2$.
(A) 49 (B) 9 (C) 20 (D) 58 (E) 16

Solution: D.

$x + y = 10$ (1)

$2x - y = 11$ (2)

(1) × 2 – (2):

$3y = 9 \Rightarrow \quad y = 3$

From (1) we get $x + 3 = 10 \Rightarrow \quad x = 7$.

$x^2 + y^2 = 7^2 + 3^2 = 49 + 9 = 58$.

Example 16. Solve the system of equations and find the value of $z - y$.

$$\left.\begin{aligned} \frac{x}{2} &= \frac{y}{3} \\ \frac{y}{2} &= \frac{z}{3} \\ x + y + z &= 38 \end{aligned}\right\}$$

(A) 10 (B) 20 (C) 24 (D) 18 (E) 19

Solution: D.

From the given equations, we have

$$\frac{x}{4} = \frac{y}{6} = \frac{z}{9} = \frac{x+y+z}{4+6+9} = \frac{38}{19} = 2.$$

$\frac{x}{4} = 2 \Rightarrow \quad x = 8$.

Similarly, we get $y = 12$ and $z = 18$. $z - y = 18 - 12 = 6$.

Example 17. If $\frac{5}{x+1} = \frac{5}{2x-1}$, what is the value of x?

(A) – 2 (B) – 1 (C) 0 (D) 1 (E) 2

Solution: E.

$5(2x-1) = 5(x+1) \quad \Rightarrow \quad 2x - 1 = x + 1 \quad \Rightarrow \quad x = 2$.

Example 18. If $a > 0$ in the equations $x = 4a$ and $y = 4a^2 + 1$, find y in terms of x.
(A) $\frac{1}{4}x^2 + 1$ (B) $\frac{1}{4}x^2 + 4$ (C) $\frac{1}{2}x^2 + 1$ (D) $x^2 + 1$ (E) $x^2 + 4$

Solution: A.
Squaring both sides of the equation $x = 4a$: $x^2 = 16a^2$ $\Rightarrow$ $x^2/4 = 4a^2$
$y = 4a^2 + 1 = \frac{1}{4}x^2 + 1$.

Example 19. In the equation $6(x - 7)(x - 2) = k$, k is a constant. If the roots of the equation are 7 and 2, what is the value of k ?
(A) 0 (B) 2 (C) 3 (D) 7 (E) 14

Solution: A.
If 7 is the root of the equation $6(x - 7)(x - 2) = k$, $6(7 - 7)(x - 2) = k$ $\Rightarrow$ $k = 0$.

Example 20. Find the value of $y + z$ if $5x + 2y + 2z = 21$ and $5x + y + z = 11$.
(A) 10 (B) 12 (C) 3 (D) 7 (E) 11

Solution: A.
$5x + 2y + 2z = 21$ (1)
$5x + y + z = 11$ (2)
(1) – (2): $y + z = 10$.

Example 21. Solve:

$$\begin{cases} \frac{1}{x} + \frac{1}{y} = 1 & (1) \\ \frac{1}{x} + \frac{1}{z} = 2 & (2) \\ \frac{1}{y} + \frac{1}{z} = \frac{3}{2} & (3) \end{cases}$$

Solution: $x = \frac{4}{3}$, $y = 4$, $z = \frac{4}{5}$.

(1) + (2) + (3): $2(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}) = \frac{9}{2}$

or $\Rightarrow$ $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{9}{4}$ (4)

(4) – (3): $\frac{1}{x} = \frac{3}{4}$ $\Rightarrow$ $x = \frac{4}{3}$

(4) – (2): $\frac{1}{y} = \frac{1}{4}$ $\Rightarrow$ $y = 4$

(4) – (1): $\frac{1}{z} = \frac{5}{4}$ $\Rightarrow$ $z = \frac{4}{5}$

☆**Example 22.** In a jar of red, green, and blue marbles, all but 12 are red marbles, all but 16 are green, and all but 8 are blue. How many marbles are in the jar?

(A) 12 (B) 16 (C) 18 (D) 20 (E) 36

Solution: C.

Let g, b, and r be the number of green, blue, and red marbles respectively.

$g + b = 12$ (1)

$r + b = 16$ (2)

$r + g = 8$ (3)

(1) + (2 + (3): $2g + 2r + 2b = 36$ $\Rightarrow$ $g + r + b = 18$.

Example 23. Find the sum of the x-coordinates of the points of intersection of the graphs of the equations $y = |2x| - 2$ and $y = -|2x| + 2$.

(A) 1 (B) 6 (C) 8 (D) 0 (E) 3

Solution: D.

$y = |2x| - 2$ (1)

$y = -|2x| + 2$ (2)

Let (1) = (2):

$|2x| - 2 = -|2x| + 2 \quad \Rightarrow \quad 2|2x| = 4 \quad \Rightarrow \quad 4|x| = 4 \quad \Rightarrow \quad |x| = 1$

$x = 1$ or $x = -1$

So the sum of the x-coordinates is $1 - 1 = 0$.

Example 24. In the following system of equations, x can be expressed as m/n, where m and n are positive integers relatively prime. What is the value of $m + n$?

$$\begin{cases} \dfrac{xy}{x+y} = 3 & (1) \\ \dfrac{yz}{y+z} = 4 & (2) \\ \dfrac{zx}{x+z} = 5 & (3) \end{cases}$$

(A) 130 (B) 143 (C) 127 (D) 120 (E) 167

Solution: A.

$$\frac{xy}{x+y} = 3 \quad \Rightarrow \quad \frac{x+y}{xy} = \frac{1}{3} \quad \Rightarrow \quad \frac{1}{x} + \frac{1}{y} = \frac{1}{3} \quad (1)$$

$$\frac{yz}{y+z} = 4 \quad \Rightarrow \quad \frac{y+z}{yz} = \frac{1}{4} \quad \Rightarrow \quad \frac{1}{y} + \frac{1}{z} = \frac{1}{4} \quad (2)$$

$$\frac{zx}{x+z} = 5 \quad \Rightarrow \quad \frac{x+z}{xz} = \frac{1}{5} \quad \Rightarrow \quad \frac{1}{x} + \frac{1}{z} = \frac{1}{5} \quad (3)$$

(1) + (2) + (3): $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{1}{2}\left(\dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5}\right) = \dfrac{47}{120}$ (4)

(4) – (2): $\dfrac{1}{x} = \dfrac{47}{120} - \dfrac{1}{4} = \dfrac{17}{210} \quad \Rightarrow \quad x = \dfrac{120}{17}$.

The answer is 120 + 17 = 130.

6. SOLVING SYSTEM OF NONLINEAR EQUATIONS

A system of equations is called nonlinear system of equations if at least one equation is nonlinear.

Example 25. Solve the system:

$$\left.\begin{array}{l} x+y=5 \\ xy=4 \end{array}\right\}$$

Solution:

Method 1:

First solve either equation for one variable. Let's solve $x + y = 5$ for x:

$x = 5 - y$ (1)

Now substitute the result for x in equation $xy = 4$.

$(5 - y)y = 4 \quad \Rightarrow \quad 5y - y^2 = 4 \quad \Rightarrow \quad y^2 - 5y - 4 = 0 \quad \Rightarrow \quad (y - 1)(y - 4)$
$= 0$

$y = 1$ or $y = 4$.

The solutions are

$$\left.\begin{array}{l} x=1 \\ y=4 \end{array}\right\} \qquad \left.\begin{array}{l} x=4 \\ y=1 \end{array}\right\}$$

Method 2:

In the quadratic equation $ax^2 + bx + c = 0$, the sum of the root is $-b/a$, and the product of the roots is c/a. Therefore, x and y are two roots of the quadratic equation:

$t^2 - 5t + 4 = 0 \quad \Rightarrow \quad (t - 1)(t - 4) = 0 \qquad t = 1$ or $t = 4$.

The solutions are

$$\left.\begin{array}{l} x=1 \\ y=4 \end{array}\right\} \qquad \left.\begin{array}{l} x=4 \\ y=1 \end{array}\right\}$$

Example 26. Solve the system:

$$\left.\begin{aligned} 2x - 3y &= 11 \\ xy &= -5 \end{aligned}\right\}$$

Solution:
Write the equations as:

$$\left.\begin{aligned} 2x + (-3y) &= 11 \\ 2x \times (-3y) &= 30 \end{aligned}\right\}$$

In the quadratic equation $ax^2 + bx + c = 0$, the sum of the root is $-b/a$, and the product of the roots is c/a.

So $2x$ and $-3y$ are the roots of the quadratic equation:
$t^2 - 11t + 30 = 0 \quad \Rightarrow \quad (t-5)(t-6) = 0 \quad \Rightarrow \quad t = 5$ or $t = 6$.
$2x = 5$ and $-3y = 6$ or $2x = 6$ and $-3y = 5$.
The solutions of (x, y) are $(5/2, -2)$ or $(3, -5/3)$.

Example 27. Solve the system:

$$\begin{cases} x^2 + y^2 = 25 & (1) \\ x + y = 7 & (2) \end{cases}$$

Solution:
$(2)^2 - (1)$:
$xy = 12 \qquad (3)$

We know from (2) and (3) that x and y are the roots of the quadratic equation:
$t^2 - 7t + 12 = 0 \quad \Rightarrow \quad (t-3)(t-4) = 0 \quad \Rightarrow \quad t = 3$ or $t = 4$.
The solutions of (x, y) are $(3, 4)$ or $(4, 3)$.

☆**Example 28.** Find $x^2 + y^2$ if (x, y) is a solution to the system $xy = 6$ and $x^2y + y^2x + x + y = 63$.
(A) 81 (B) 140 (C) 69 (D) 63 (E) 67

Solution: C.

$x^2y + y^2x + x + y = xy(x + y) + (x + y) = (x + y)(xy + 1) = 63$

or $(x + y)(6 + 1) = 63 \Rightarrow x + y = 9 \Rightarrow (x + y)^2 = 81 \Rightarrow x^2 + y^2 + 2xy = 81$

$x^2 + y^2 = 81 - 2xy = 81 - 12 = 69$.

☆**Example 29.** How many distinct points common to the curve $3x^2 + y^2 = 13$ and $x^2 + 3y^2 = 115$?

(A) 1 (B) 2 (C) 3 (D) 0 (E) 4

Solution: B.

$3x^2 + y^2 = 12$ (1)

$x^2 + 3y^2 = 4$ (2)

(1) + (2):

$4x^2 + 4y^2 = 16 \Rightarrow x^2 + y^2 = 4$ (3)

(2) – (3): $2y^2 = 0$

So equation (2) becomes: $x^2 = 4 \Rightarrow x = \pm 2$.

There are 2 points common: (2, 0) and (– 2, 0).

☆**Example 30**. Find $x^2 + y^2$ if x and y are positive integers such that

$xy + x + y = 19$ and $x^2y + xy^2 = 84$.

(A) 193 (B) 25 (C) 153 (D) 103 (E) 74

Solution: B.

$xy + x + y = 19$

$x^2y + xy^2 = xy(x + y) = 84$.

Let $xy = m$, $x + y = n$.

$m + n = 19$

$mn = 84$

So $m = 12$ and $n = 7$.

Then $x = 3$ and $y = 4$. $x^2 + y^2 = 25$.

PROBLEMS

Problem 1. Find *n,* the root of the equation: $n+(n+1)+(n+2)=-75$.
(A) –3 (B) – 25 (C) –75 (D) – 26 (E) 26

Problem 2. Find the value of y which makes the following true: $\frac{3(6+8y)}{10}=9$.
(A) 3 (B) 5 (C) 7 (D) 6 (E) 2

Problem 3. For what value(s) of x is the equation $11x-4(2x-3)=24$ true?
(A) 3 (B) 4 (C) 5 (D) 6 (E) 11

Problem 4. What is the value of x? $\frac{x-1}{3}-1=\frac{2x-1}{5}+1$.
(A) –32 (B) – 23 (C) –27 (D) – 45 (E) 32

Problem 5. Solve for d : $\frac{3d-1}{4d-4}=\frac{2}{3}$.
(A) –3 (B) – 2 (C) –7 (D) –5 (E) 3

Problem 6. Solve the equation $a=-3(x-5b)$ for x.

Problem 7. Solve for x: $\frac{1}{3}m(x-n)=\frac{1}{4}(x+2m)$.

Problem 8. Find the greatest positive integer n such that $n^2-26n+30$ is at most 30.
(A) 19 (B) 25 (C) 26 (D) 13 (E) 27

Problem 9. One root of the equation $5x^2 + kx = 4$ is 2. What is the other?

(A) 10 (B) $-\frac{4}{5}$ (C) $-\frac{2}{5}$ (D) $\frac{2}{5}$ (E) $-\frac{2}{10}$

Problem 10. Given that a and b are positive numbers such that $a^2 + b^2 = 52$ and $a^2 - b^2 = 20$, what is the value of b ?

(A) – 4 (B) 6 (C) –6 (D) 16 (E) 4

Problem 11. How many different points of intersection are there for $x + y = 7$ and $y = x^2 - 7$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 12. Four distinct integers a, b, c, and d have the property that when added in pairs the sums 16, 19, 20, 21, 22 and 25 are obtained. What is the sum of four integers?

(A) 36 (B) 39 (C) 41 (D) 43 (E) 47

Problem 13. Given that $7x + 3y = 54$ and $3x + 7y = 46$, what is the value of $x + y$?

(A) 14 (B) 16 (C) 27 (D) 23 (E) 10

Problem 14. The sum of two numbers is 22. Their product is 72. What is the greater of the two numbers?

(A) 20 (B) 18 (C) 16 (D) 14 (E) 12

Problem 15. The product of two whole numbers is 60. If the difference between the two numbers is 11, what is the greater of the two numbers?

(A) 4 (B) 18 (C) 15 (D) –4 (E) –15

Problem 16. The difference between two positive integers is 45, and their product is 196. What is their sum?

(A) 49 (B) 53 (C) 784 (D) –53 (E) –49

Problem 17. The difference between two numbers is 1, and the sum of their squares is 141. What is the product of the numbers?
(A) 4970 (B) 4899 (C) 4830 (D) 994 (E) 2485

Problem 18. The sum of two numbers is 24. Their difference is 16. What is the larger number?
(A) 16 (B) 18 (C) 20 (D) 24 (E) 22

Problem 19. If $4x + y = 24$ and $x - y = 1$, what is the value of $x + y$?
(A) 6 (B) 9 (C) 12 (D) 14 (E) 22

Problem 20. Four dogs and 3 puppies weigh 74 pounds while 3 dogs and 4 puppies weigh 66 pounds. How many pounds does a dog plus a puppy weigh?
(A) 33 (B) 37 (C) 20 (D) 24 (E) 22

Problem 21. If $x + y = 5$, $x + z = 8$, and $y + z = 11$, what is the value of $x + y + z$?
(A) 8 (B) 10 (C) 12 (D) 14 (E) 20

Problem 22. If $x + y = 5$ and $xy = 3$, what is the value of $x^2 + y^2$?
(A) 25 (B) 19 (C) 12 (D) 14 (E) 6

Problem 23. Find xy such that $x + y = 10$ and $x^2 + y^2 = 178$.
(A) 39 (B) 33 (C) –78 (D) –39 (E) –49

Problem 24. Given that $\frac{1}{a}+\frac{1}{b}=\frac{7}{24}$ and $a+b=14$, what is the product of a and b?
(A) 24 (B) 31 (C) 48 (D) 39 (E) 49

Problem 25. Given 3 positive integers $a, b,$ and c such that $a \times b = 10, b \times c = 6,$ $c \times a = 15.$ Find $a \times b \times c.$
(A) 25 (B) 16 (C) 21 (D) 30 (E) 49

Problem 26. Three pencils and two erasers cost \$0.60. Two pencils and three erasers cost \$0.55. How much will it cost to buy seven pencils and seven erasers?
(A) \$1.15 (B) \$1.61 (C) \$1.48 (D) \$1.39 (E) \$1.49

Problem 27. Suppose $a, b,$ and c are positive integers such that $ab = 18, bc = 24$ and $ac = 48.$ Find $a + b + c$ to the nearest integer.
(A) 12 (B) 24 (C) 21 (D) 17 (E) 18

Problem 28. Three friends arrange to rent a summer cabin. Harry pays twice as much as Mary, and Mary pays twice as much as Larry. If the total rent is \$350, how many dollars does Harry pay?
(A) 350 (B) 300 (C) 250 (D) 200 (E) 175

Problem 29. Several boys bought a canoe, each paying an equal amount. If there had been two fewer boys, each would have paid \$3.00 more. If there had been one boy more, each would have paid \$1.00 less. How many boys were there?
(A) 12 (B) 14 (C) 10 (D) 8 (E) 11

☆**Problem 30.** If the line $y = mx + 1$ intersects the ellipse $x^2 + 4y^2 = 3$ exactly once, what is the value of m^2?
(A) 1/12 (B) 3/4 (C) 2/5 (D) 1/6 (E) 1/7

Problem 31. Find the real solutions of the equation:

$$\begin{cases} x + y = 2 & (1) \\ xy - z^2 = 1 & (2) \end{cases}$$

(A) (1, 1, 0) (B) (1, 0, 1) (C) (2, 1, −1) (D) (0,1, 1) (E) (−1, −1, 0)

SOLUTIONS

Problem 1. Solution: D.

$n+(n+1)+(n+2)=-75 \quad \Rightarrow \quad 3n+3=-75 \quad \Rightarrow \quad 3n=-76-3 \quad \Rightarrow$

$3n=-78 \Rightarrow \quad n=-26$

Problem 2. Solution: A.

$\frac{3(6+8y)}{10}=9 \quad \Rightarrow \quad 3(6+8y)=9\times 10 \quad \Rightarrow \quad 18+24y=90 \Rightarrow$

$24y=90-18 \quad \Rightarrow \quad 24y=72 \quad \Rightarrow \quad y=\frac{72}{24}=3$.

Problem 3. Solution: B.

$11x-4(2x-3)=24 \quad \Rightarrow \quad 11x-8x+12=24 \quad \Rightarrow \quad 3x=24-12 \Rightarrow$

$3x=12 \Rightarrow \quad x=\frac{12}{3}=4$.

Problem 4. Solution: A.

$\frac{x-1}{3}-1=\frac{2x-1}{5}+1 \quad \Rightarrow \quad \frac{x-1}{3}=\frac{2x-1}{5}+2 \quad \Rightarrow \quad \frac{x-1}{3}=\frac{2x+9}{5}$

$\Rightarrow \quad 5(x-1)=3(2x+9) \quad \Rightarrow \quad 5x-5=6x+27 \quad \Rightarrow$

$-27-5=6x-5x \Rightarrow \quad x=-32$.

Problem 5. Solution: D.

$\frac{3d-1}{4d-4}=\frac{2}{3} \quad \Rightarrow \quad \frac{3d-1}{4d-4}=\frac{2}{3} \quad \Rightarrow \quad 3(3d-1)=2(4d-1)$

$\Rightarrow \quad 9d-3=8d-8 \quad \Rightarrow \quad 9d-8d=-8+3 \quad \Rightarrow \quad d=-5$.

Problem 6. Solution:

Remove the parentheses: $a=-3x+15b$

Isolate the variable: $3x=15b-a$

Divide both sides by 3: $x = \dfrac{15b - a}{3}$

Problem 7. Solution:

Case I: when $m \neq 3/4$, the equation has one solution. $x = \dfrac{m(4n+6)}{4m-3}$

Case II: when $m = 3/4$ and $n = -3/2$, the equation has infinite many solutions.

Case III: when $m = 3/4$ and $n \neq -3/2$, the equation has no solutions.

Problem 8. Solution: C.

$n^2 - 26n + 30 \le 30 \quad \Rightarrow \quad n^2 - 26n \le 0 \quad \Rightarrow \quad n(n-26) \le 0$

$\Rightarrow \quad 0 \le n \le 26$.

The greatest value of n is 26.

Problem 9. Solution: C.

Since one root of the equation $5x^2 + kx = 4$ is 2, we have $5 \times (2)^2 + k(2) = 4$

$\Rightarrow \quad 20 + 2k = 4 \quad \Rightarrow \quad 2k = 4 - 20 \quad \Rightarrow \quad 2k = -16 \quad \Rightarrow$
$k = -8$.

The original equation can be written as $5x^2 - 8x = 4 \quad \Rightarrow$
$5x^2 - 8x - 4 = 0$

$$\Rightarrow \quad x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 5 \times (-4)}}{2 \times 5} = \frac{-(-8) \pm \sqrt{12^2}}{10} = \frac{8 \pm 12}{10}.$$

The root is $x == \dfrac{8-12}{10} = -\dfrac{2}{5}$.

Problem 10. Solution: E.

$a^2 + b^2 = 52 \qquad (1)$

$a^2 - b^2 = 20 \qquad (2)$

(1) – (2): $2b^2 = 32 \quad \Rightarrow \quad b^2 = 16 \quad \Rightarrow \quad b = 4$ or $b = -4$ (ignored).

Problem 11. Solution: C.

$x + y = 7$ (1)

$y = x^2 - 7$ (2)

(2) – (1): $x^2 + x - 14 = 0 \quad \Rightarrow \quad x^2 + x - 14 = 0$ (3)

Since $\Delta = 12 - 4 \times 1 \times (-14) > 0$, quadratic equation 3 has two solutions.

So the answer is 2.

Problem 12. Solution: C.

Let $a < b < c < d$.

$a + b = 16$ (1)

$c + d = 25$ (2)

(1) + (2): $a + b + c + d = -16 + 25 = 41$.

Problem 13. Solution: E.

$7x + 3y = 54$ (1)

$3x + 7y = 46$ (2)

(1) + (2): $10x + 10y = 100 \quad \Rightarrow \quad x + y = 10$.

Problem 14. Solution: B.

Let two numbers be x and y.

$x + y = 22$ (1)

$xy = 72 \quad \Rightarrow \quad -4xy = -288$ (2)

Squaring both sides of (1): $(x + y)^2 = 22^2 \quad \Rightarrow \quad x^2 + 2xy + y^2 = 22^2$ (3)

(3) + (2): $x^2 - 2xy + y^2 = 14^2 \quad \Rightarrow \quad (x - y)^2 = 14^2$

$\Rightarrow \quad x - y = 14$ (4)

or $\quad \Rightarrow \quad x - y = -14$ (5)

Solve the system of equations (1) and (4): $x = 18$, $y = 4$.

Solve the system of equations (1) and (5): $x = 4$, $y = 18$.

The greater of the two numbers is 18.

Problem 15. Solution: C.
Let two numbers be x and y.
$x - y = 11$ (1)
$xy = 60 \Rightarrow 4xy = 240$ (2)
Squaring both sides of (1): $(x - y)^2 = 11^2 \Rightarrow x^2 - 2xy + y^2 = 11^2$ (3)
(3) + (2): $x^2 + 2xy + y^2 = 19^2 \Rightarrow (x + y)^2 = 19^2$
$\Rightarrow x + y = 19$ (4)
or $\Rightarrow x + y = -19$ (5)
Solve the system of equations (1) and (4): $x = 15$, $y = 4$.
Solve the system of equations (1) and (5): $x = -4$, $y = -15$.
The greater of the two numbers is 15.

Problem 16. Solution: B.
Let two numbers be x and y.
$x - y = 45$ (1)
$xy = 196 \Rightarrow 4xy = 784$ (2)
Squaring both sides of (1): $(x - y)^2 = 45^2 \Rightarrow x^2 - 2xy + y^2 = 45^2$ (3)
(3) + (2): $x^2 + 2xy + y^2 = 53^2 \Rightarrow (x + y)^2 = 53^2$
$\Rightarrow x + y = 53$ (4)
or $\Rightarrow x + y = -53$ (ignored)

Problem 17. Solution: A.
Let two numbers be x and y.
$x - y = 1$ (1)
$x + y = 141$ (2)
(1) + (2): $2x = 142 \Rightarrow x = 71$
So $y = 71 - 1 = 70$.
$xy = 71 \times 70 = 4970$.

Problem 18. Solution: C.
Let two numbers be x and y.

$x - y = 16$ (1)
$x + y = 24$ (2)
(1) + (2): $2x = 40 \quad \Rightarrow \quad x = 20$
So $y = 20 - 4 = 16$.
The answer is 20.

Problem 19. Solution: B.
$4x + y = 24$ (1)
$x - y = 1$ (2)
(1) + (2): $5x = 25 \quad \Rightarrow \quad x = 5$.
So $y = 5 - 1 = 4$.
The answer is $5 + 4 = 9$.

Problem 20. Solution: C.
$4d + 3p = 74$ (1)
$3d + 4p = 66$ (2)
(1) + (2):
$7d + 7p = 140 \quad \Rightarrow \quad d + p = 20$

Problem 21. Solution: C
$x + y = 5$ (1)
$x + z = 8$ (2)
$y + z = 11$ (3)
(1) + (2) + (3): $2(x + y + z) = 24 \quad \Rightarrow \quad x + y + z = 12$.

Problem 22. Solution: B.
$x + y = 5$ (1)
$xy = 3 \quad \Rightarrow \quad -2xy = -6$ (2)
Squaring both sides of (1): $(x + y)^2 = 25 \quad \Rightarrow \quad x^2 + 2xy + y^2 = 25$ (3)
(3) + (2): $x^2 + y^2 = 19$.

Problem 23. Solution: D.

$x + y = 10$ (1)

$x^2 + y^2 = 178$ (2)

Squaring both sides of (1): $(x + y)^2 = 100 \quad \Rightarrow \quad x^2 + 2xy + y^2 = 100$ (3)

(3) − (2): $2xy = -78 \quad \Rightarrow \quad xy = -39$.

Problem 24. Solution: C.

$$\frac{1}{a}+\frac{1}{b}=\frac{7}{24} \quad \Rightarrow \quad \frac{a+b}{ab}=\frac{7}{24} \quad \Rightarrow \quad \frac{14}{ab}=\frac{7}{24} \quad \Rightarrow$$

$$\frac{2}{ab}=\frac{1}{24} \quad \Rightarrow \quad ab=48.$$

Problem 25. Solution: D.

$a \times b = 10$ (1)

$b \times c = 6$ (2)

$c \times a = 15$ (3)

(1) × (2) × (3): $(a \times b \times c)^2 = 10 \times 6 \times 15 \quad \Rightarrow \quad a \times b \times c = \sqrt{10 \times 6 \times 15} = 30$.

Problem 26. Solution: B.

$3P + 2E = 60$ (1)

$2P + 3E = 55$ (2)

(1) + (2): $5P + 5E = 115 \Rightarrow P + E = 23 \Rightarrow 7(P + E) = 23 \times 7 = 161 =$ \$1.61.

Problem 27. Solution: D.

$ab = 18$ (1)

$bc = 24$ (2)

$ac = 48$ (3)

(1) × (2) × (3): $(a \times b \times c)^2 = 18 \times 24 \times 48 \Rightarrow a \times b \times c = \sqrt{18 \times 24 \times 48} = 144$ (4)

(4) ÷ (1): $c = 8$

(4) ÷ (2): $a = 6$

(4) ÷ (3): $b = 3$

$a + b + c = 6 + 3 + 8 = 17$.

Problem 28. Solution: D.

$H = 2M \quad \Rightarrow \quad M = \frac{H}{2}$ (1)

$M = 2L \quad \Rightarrow \quad L = \frac{M}{2} = \frac{H}{4}$ (2)

$M + H + L = 350$ (3)

Substituting (1) and (2) into (3): $\frac{H}{2} + H + \frac{H}{4} = 350 \quad \Rightarrow \quad \frac{7H}{4} = 350$

$\Rightarrow \quad H = 200.$

Problem 29. Solution: D.

Let x be the number of boys and c be the cost of the canoe.

Each person pays \$ c/x.

$\frac{c}{x-2} = \frac{c}{x} + 3 \quad \Rightarrow \quad c(\frac{1}{x-2} - \frac{1}{x}) = 3$ (1)

$\frac{c}{x+1} = \frac{c}{x} - 1 \quad \Rightarrow \quad c(\frac{1}{x+1} - \frac{1}{x}) = -1$ (2)

(1) ÷ (2): $\dfrac{\frac{1}{x-2} - \frac{1}{x}}{\frac{1}{x+1} - \frac{1}{x}} = \frac{3}{-1} \quad \Rightarrow \quad -(\frac{1}{x-2} - \frac{1}{x}) = 3(\frac{1}{x+1} - \frac{1}{x}) \quad \Rightarrow$

$-\frac{1}{x-2} + \frac{1}{x} = \frac{3}{x+1} - \frac{3}{x} \quad \Rightarrow \quad \frac{4}{x} = \frac{3}{x+1} + \frac{1}{x-2} \quad \Rightarrow$

$\frac{3}{x+1} + \frac{1}{x-2} - \frac{4}{x} = 0 \quad \Rightarrow \quad \frac{3x(x-2) + x(x+1) - 4(x+1)(x-2)}{(x+1)(x-2)x} = 0$

$\Rightarrow \quad 3x(x-2) + x(x+1) - 4(x+1)(x-2) = 0 \quad \Rightarrow$

$3x^2 - 6x + x^2 + x - 4x^2 + 4x + 8 = 0 \quad \Rightarrow \quad -x + 8 = 0 \quad \Rightarrow \quad x = 8.$

☆**Problem 30.** Solution: A.

$y = mx + 1$ (1)

$x^2 + 4y^2 = 3$ (2)

Substitute (1) into (2):

$x^2 + 4(mx + 1)^2 = 3 \quad \Rightarrow x^2 + 4(m^2x^2 + 2\,mx + 1) = 3 \Rightarrow x^2 + 4m^2x^2 + 8mx + 4 = 3$

$\Rightarrow \qquad x^2(4m^2 + 1) + 8mx + 1 = 0$ (3)

Since these two lines intersect only once, (3) should have double roots, meaning that the discriminant equals 0.

$\Delta = 0 \quad \Rightarrow \ (8m)^2 - 4 \times (4m^2 + 1) \times 1 = 0 \qquad \Rightarrow \ 16m^2 - 4m^2 - 1 = 0$

$\Rightarrow \ 12m^2 - 1 = 0 \qquad \Rightarrow \ m^2 = 1/12.$

Problem 31. Solution: A.

Write (2) as $xy = 1 - z^2$ (3)

We se that from (1) and (3) that x and y are the roots of the quadratic equation:

$t^2 - 2t + (1 + z^2) = 0$ (4)

The discriminant $\Delta = 4 - 4(1 + z^2) = -4z^2 \le 0$

Only when $z^2 = 0$ or $z = 0$ does the equations have real roots.

(4) becomes $t^2 - 2t + 1 = 0 \quad \Rightarrow \quad (t - 1)^2 = 0 \quad \Rightarrow \quad t = 1.$

The solution of (x, y, z) is (1, 1, 0).

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