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pdf $X_i$ is iid

$f(x) = \frac{1}{2} e^{-|x|}$

$S_n = \sum_{i=1}^{n} X_i \longrightarrow X \to \mu = 0$, $\sigma^2 = 2$

$\lim_{n\to\infty} P(S_n^2 > n) = ?$

$X_i \to$ iid $\Rightarrow$ Centeral Limit theorem

a)

$\Rightarrow \lim_{n\to\infty} S_n \to$ the distribution of $S_n$ will approach a normal distribution with mean $= n\mu = 0 \to$

0

* mean $_{S_n}$ $= 0$

variance: $n\sigma^2 = n(2) = 2n$

$\frac{S_n}{\sqrt{2n}} \xrightarrow{D} N(0,1)$
$n \to \infty$

$\Rightarrow S_n \to$ normal distibuted $N(0,n)$

$S_n^2 > n \Leftrightarrow |S_n| > \sqrt{n}$

$\Rightarrow \lim_{n\to\infty} P(S_n^2 > n) \longleftrightarrow P(|S_n| > \sqrt{n})$

$\Rightarrow P(|S_n| > \sqrt{n}) \to 0$ as: The probability that a standard normal variable * is outside of the $\left[-\sqrt{\frac{n}{2n}}, \sqrt{\frac{n}{2n}}\right]$ is "0" $n \to \infty$

$S_n$ is normally distributed as: $N(0,n)$

$\Rightarrow \frac{S_n}{\sqrt{2n}}$ will be a standard normal random variable as $n \to \infty$.

$\lim_{n\to\infty} P(S_n^2 > n)$: The probability that a standard normal random variable is greater than "1" or less than "-1". $P(Z > 1) + P(Z < -1)$

standard normal r.v.

$P(Z>1) = 1 - P(Z \leq 1)$. as the standard normal distribution is symmetric, $P(Z<-1) = P(Z>1) \Rightarrow P(|Z|>1) = 2P(Z>1)$

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$\Rightarrow$ Python code:

$$P(|Z|>1) = 1 - \underbrace{P(Z<1)}_{\text{norm.cdf}(1)} \rightarrow = 0.317$$

Also:

$$\text{Markov} \rightarrow \underbrace{P(|S_n^2| \geq n)}_{\text{non\_negative} \Rightarrow} \leq \frac{E[S_n^2]}{n} = 0 \quad n>0$$

$$P(|S_n^2| \geq n) = 0 \Rightarrow$$

$$\lim_{n\to\infty} P(|S_n^2| \geq n) = 0$$

$$\text{chebyshev} \rightarrow P(|\pm S_n| > \sqrt{n}) \leq 1/n \Rightarrow$$

$$\lim P(S_n^2 > n) \leq \lim_{n\to\infty} (1/n + 1/n) = \lim_{n\to\infty} (2/n) = 0$$

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4)

$$V_j = |\{n \geq 1 : X_n = j\}|$$

$\downarrow$

The number of visits of Markov chain $X_n$ to $j$.

$$r_{ij} = P(V_j = \infty \mid X_0 = i)$$

a) $r_{ii} = ?$

A state "$i$" is persistent (or recurrent) if, once visited, the probability of returning to it infinity often is $1$. $\rightarrow$ $\underbrace{E(V_i \mid X_0 = i) \text{ is infinite}}_{\text{mathematically.}}$

and it is transient if the total expected number of visits is finite.

if "$i$" is persistent $\Rightarrow$ if start from state "$i$" the probability that we will return to state "$i$" infinitly often is "$1$". $\Rightarrow r_{ii} = 1$ : The probability that we will return to state "$i$" at least once is "$1$". (Since the state is recurrent) and due to the Markov property, each time we return to state "$i$", the chain "restarts", and we have the same probability of returning to it again. since it can happen an infinite number of times, $r_{ii} = 1$

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if "i" is transient;

the expected number of visits is finite, which means there is a probability of not returning to state "i" that is greater than "0".

As we keep visiting state "i", the probability of not returning to it again increases until it becomes certain that we wont return.

In this case, the probability that we will never return to state "i" after some finite number "n" of visits is > 0, which implies that the probability of returning to state "i" infinitely often is less than "1". Since the state is transient, this probability is actually 0, $\Rightarrow \rho_{ii} = 0$

$$\Rightarrow \rho_{ii} = \begin{cases} 1 & \text{if } i \text{ is persistent} \\ 0 & \text{if } i \text{ is transient} \end{cases}$$

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b) $\rho_{ij} = ?$ $\Rightarrow$ $\rho_{ij} = \begin{cases} P(T_j < \infty \mid X_0 = i) & \text{if } j \text{ is persistent} \\ 0 & \text{if } j \text{ is transient} \end{cases}$

$\rho_{ij} \rightarrow$ The probability that starting from state "i" the markov chain will visit state "j" at least once.

$\underbrace{T_j = \min\{ n \geq 1 \mid X_n = j \}}$ is the hitting time for state "j"

if j is persistent (recurrent)

$\Rightarrow$ probability that the markov chain, starting from any state i, will eventually hit state "j" is 1.

$\Rightarrow P(T_j < \infty \mid X_0 = i) = 1$ for all i.

Since the state is persistent, it will be visited eventually, and the hitting time $T_j$ will be finite with the probability of "1".

if j is transient: $\Rightarrow$ there is a passitive probability that starting from state "i" the chain will never visit state "j" $\Rightarrow$ meaning the hitting time $T_j$ would be infinite with possitive probability.

consequently, $P(T_j < \infty \mid X_0 = i) < 1$, However since we are looking at the probability that $T_j$ is finite, this probability is "0". because a transient state, by definition, is not expected to be hit an infinite number of times.

$\Rightarrow \rho_{ij} = P(T_j < \infty \mid X_0 = i)$, when j is persistent. and

$\rho_{ij} = 0$, when j is transient

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