ELE 509: Final

Duration of the examination is 48 hours

Please provide a comprehensive description of your work. Your grades depend not only on the final answers but also on the details and proofreading of your work. You may utilize your notes, textbooks, and the internet as resources. However, seeking assistance from others to solve the questions is not permitted. Please refrain from sharing your problem-solving approach with others, and if you have any questions, kindly direct them to me.

This exam has 4 questions.

1) (**25 Mark**) Answer the following questions for each of the following functions.
   A1) Is it a characteristic functions? Justify your answer carefully.
   A2) If it is a characteristic function, find the distribution of the corresponding random variable.
   a) $\frac{1}{4-it} + \frac{1}{3}e^{-2it}\cos(t)$
   b) $e^{-t^2} + \sin^2(t)$
   c) $e^{-t^2}\cos^2(2t)$

2) (**25 Mark**)
Consider a Markov chain with the following transition matrix

$$\begin{bmatrix} . & 1 & . & . & . & . & . & . & . \\ \frac{1}{2} & . & \frac{1}{2} & . & . & . & . & . & . \\ . & \frac{1}{2} & . & \frac{1}{2} & . & . & . & . & . \\ \frac{1}{2} & . & \frac{1}{2} & . & . & . & . & . & . \\ . & . & . & \frac{1}{4}p & . & \frac{1}{4}q & . & . & \frac{3}{4} \\ . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & \frac{1}{2} & . & \frac{1}{2} & . \\ . & . & . & . & . & 1 & . & . & . \\ . & . & \frac{1}{2}p & . & \frac{1}{2} & . & \frac{1}{2}q & . & . \end{bmatrix}$$

(for readability 0 is indicated by .) where $0 \leq p \leq 1$ and $q = 1 - p$. The following questions should be answered for all values of p. Notice that in the questions, the states are labeled 1 through 9.

a) Draw the transition diagram of the chain, and find the classes of the chain. For each class indicate if it is transient or recurrent and periodic or aperiodic.
b) Find all stationary distributions.
c) Suppose that the chain starts in state 5. What is the probability that the chain will reach state 1 at some time?
d) Suppose that the chain starts in state 5. On average, how many steps will it take the chain to reach a recurrent state?
e) Find $\mathbf{A} = \lim_n \to \infty \mathbf{P}^n$ if it exists (you only need to indicate where it's non-zero).

3) (**25 Mark**)

Let X_i be independent random variables. The variable X_i has an exponential distribution with parameter

$$\lambda_i = \frac{i}{i+1}$$

Let

$$S_n = \sum_{i=1}^{n} X_i$$

a) Show that

$$\frac{S_n - n}{\sqrt{n}} \xrightarrow{D} Z$$

for some random variable Z and find the distribution of Z.

b) Show that

$$\frac{S_n}{n} \xrightarrow{p} 1$$

4) (**25 Mark**) A random sequence X_n is called a *martingale* if $E|X_n| < \infty$ and

$$E\{X_n|X_{n-1}, \ldots, X_1\} = X_{n-1}$$

Show that if random variables Y_n are independent, then their sum $X_n = Y_1 + Y_2 + \ldots + Y_n$ is a martingale.