① $P(X_k = \sqrt{k}) = P(X_k = -\sqrt{k}) = \frac{1}{2}$ $\qquad Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k$

$$\phi(t) = E(e^{itX}) = \sum_k e^{itk} \, P(X_n = \sqrt{k})$$

a) $\phi_{X_k}(t) = E(e^{itX_k}) = \frac{1}{2} e^{it\sqrt{k}} + \frac{1}{2} e^{-it\sqrt{k}}$

For Y_n : Sum of independent random variables are being multiplied with their characteristic functions

$$\phi_{Y_n}(t) = \left(\phi_{X_k}\left(\frac{t}{n}\right)\right)^n = \left(\frac{1}{2} e^{i\frac{t}{n}\sqrt{k}} + \frac{1}{2} e^{-i\frac{t}{n}\sqrt{k}}\right)^n$$

$$= \left(\frac{1}{2}\left(e^{i\frac{t}{n}\sqrt{k}} + e^{-i\frac{t}{n}\sqrt{k}}\right)\right)^n$$

$$= \left(\cos\left(\frac{t}{n}\sqrt{k}\right)\right)^n$$

b) $\phi_{X_k}(t) = E(e^{itX_k})$ $\qquad F_X(x) = P(X \leq x) = \sum p(x_i)$

$\phi_{Y_n}(t) \xrightarrow[n\to\infty]{} \phi_Y(t)$: $\qquad \cos\left(\frac{t}{n}\sqrt{k}\right)^n \xrightarrow[n\to\infty]{} 1$

$\downarrow$ Continuity theorem

$X_n \xrightarrow{D} X$ $\qquad \lim_{n\to\infty} F_n(x) = \delta(x)$?

IFFT

② $S_n = \sum_{i=1}^{n} a_i X_i \qquad M_n = \frac{S_n}{\sum_{i=1}^{n} a_i}$

$a_1 = k$

$a_n = a_{n-1} + k \qquad n > 1$

$a_i = k i$

a) mean and variance of S_n and M_n:

S_n: $E[S_n] = \sum_{i=1}^{n} a_i \, E[X_i] = \sum_{i=1}^{n} a_i \cdot m = \sum_{i=1}^{n} ik\, m = mk \cdot \sum_{i=1}^{n} i = mk \cdot \frac{n(n+1)}{2}$

$Var[S_n] = \sum_{i=1}^{n} a_i^2 \, Var[X_i] = \sum_{i=1}^{n} a_i^2 \cdot \sigma^2 = \sigma^2 k^2 \cdot \left(\frac{n(n+1)(2n+1)}{6}\right)$

M_n: $E[M_n] = \frac{E[S_n]}{E[\sum_{i=1}^{n} a_i]} = \frac{\sum_{i=1}^{n} a_i m}{\sum_{i=1}^{n} a_i} = m$

$$Var[M_n] = \frac{Var[S_n]}{(\sum_{i=1}^{n} a_i)^2} = \frac{\sigma^2 \cdot \sum_{i=1}^{n} a_i^2}{(\sum_{i=1}^{n} a_i)^2} = \frac{\sigma^2 \cdot \sum_{i=1}^{n} a_i^2}{mk^2 \left(\frac{n(n+1)}{2}\right)^2} = \frac{\sigma^2 \cdot k^2 \left(\frac{n(n+1)(2n+1)}{6}\right)}{mk^2 \left(\frac{n(n+1)}{2}\right)^2}$$

$$= \frac{\sigma^2}{m} \cdot \frac{(2n+1)^2}{n(n+1) \cdot 3}$$

b) $X_n \xrightarrow{P} X$

if $P(|X_n - X| > \varepsilon) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$

$P(|M_n - m| \geq \varepsilon) \leq \frac{\sigma_m}{\varepsilon}$ Chebychev

$$\sigma_m = \sqrt{Var} = \sqrt{\frac{\sigma^2}{m} \cdot \frac{(2n+1)^2}{n(n+1) \cdot 3}} = \frac{\sigma}{\varepsilon} \cdot \sqrt{\frac{2}{3m} \cdot \frac{(2n+1)}{(n^2+n)}} \xrightarrow[n \to \infty]{} 0$$

$\to$ therefore $M_n \xrightarrow{P} m$

c) $\frac{S_n - E(S_n)}{\sqrt{Var(S_n)}} \xrightarrow{D} Z$ Z is an $N(0,1)$ r.v.

Central limit theorem: $S_n = X_1 + X_2 + \dots X_n$

$X_1, X_2, \dots$ iid $\rightarrow \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0,1)$

$\mu, \sigma^2 \neq 0$

$$E[S_n] = mk \cdot \frac{n(n+1)}{2}$$

$$Var[S_n] = \sum_{i=1}^{n} a_i^2 \, Var[X_i] = \sum_{i=1}^{n} a_i^2 \cdot \sigma^2 = \sigma^2 \cdot \sum_{i=1}^{n} (ik)^2 = \sigma^2 k^2 \cdot \left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$\frac{S_n - mk\frac{n(n+1)}{2}}{\sqrt{\sigma^2 \cdot k^2 \cdot \left(\frac{n(n+1)(2n+1)}{6}\right)}} = \frac{\sum_{i=1}^{n} iX_i - m\frac{n(n+1)}{2}}{\sqrt{\sigma^2 \cdot \left(\frac{n(n+1)(2n+1)}{6}\right)}} \xRightarrow[n\to\infty]{} \frac{n\sum_{i=1}^{n} X_i - m n^2}{\sqrt{\sigma^2 \cdot n^3}}$$

$$= \frac{\sum_{i=1}^{n} X_i - mn}{\sqrt{\sigma^2 \cdot n}} \xrightarrow{D} Z$$

c) Theorem: $X_1, X_2, \ldots$ independent $E[X_i]=0$ $Var(X_i)=\sigma_i^2$, $E|X_j^3| < \infty$

$$\frac{1}{\sigma(n)^3}\sum_{j=1}^{n} E|X_j^3| \to \infty \text{ as } n\to\infty$$

$$\left(\sigma(n)^2 = var\left(\sum_{j=1}^{n} X_j\right) = \sum \sigma_j^2\right)$$

$$\Rightarrow \frac{1}{\sigma(n)}\sum_{j=1}^{n} X_j \xrightarrow{D} N(0,1)$$

$$Z_i = f(X_i) = \frac{1}{m}\cdot a_i\cdot X_i - a_i \quad E[Z_i] = \frac{m}{m}\cdot a_i - a_i = 0 \qquad \frac{S_n - E(S_n)}{\sqrt{Var(S_n)}} \xrightarrow{D} Z$$

$$E[Z_i^3] = \frac{a_i^3}{m}\cdot E[X_i^3] - a_i$$ and as defined all moments of X_i are finite, therefore: $E[Z_i^3] < \infty$

$$Var(Z_i) = Var\left(\frac{1\cdot a_i}{m}X_i - 1\right) = \left(\frac{a_i}{m}\right)^2\cdot Var(X_i) = \frac{\sigma^2}{m^2}a_i^2$$

$$\sigma(n)^2 = var\left(\sum_{j=1}^{n} Z_i\right) = \sum_{j=1}^{n} a_j^2\frac{\sigma^2}{m^2} = \frac{\sigma^2}{m^2}\cdot k^2\cdot\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$\frac{1}{(\sigma(n))^3}\cdot\sum_{j=1}^{n} E[Z_i^3] \leq \frac{n^4}{\left(\frac{\sigma^2}{m^2}\cdot k^2\cdot\left(\frac{n(n+1)(2n+1)}{6}\right)\right)^{\frac{3}{2}}} \xrightarrow[n\to\infty]{} 0$$

because nominator bigger than denominator

$$\sum_{j=1}^{n} a_j^3 = k^3\cdot\frac{n^2(n+1)^2}{4}$$

$$\Rightarrow \frac{1}{\sigma(n)}\cdot\sum_{j=1}^{n} Z_j \xrightarrow{D} N(0,1)$$

and if $Z_j \sim N(0,1)$ then S_n is as well?

③ $f(x) = \frac{1}{2} e^{|x|}$ $\qquad S_n = \sum_{i=1}^{n} X_i$

a) $\lim_{n\to\infty} P(S_n^2 > n)$

Chebyshev's inequality: $a > 0$, μ, σ^2

$\rightarrow P(|X-\mu| > a) \leq \frac{\sigma}{a}$

Markov's inequality: $P(|X| \geq a) \leq \frac{E(|X|)}{a}$

$$P((S_n - \mu_{S_n})^2 > n) = P(S_n^2 > n) = P(S_n > \sqrt{n}) + P(S_n < -\sqrt{n}) \quad \frac{\sigma}{\sqrt{n}}$$

$$= P(S_n > \sqrt{n}) + 1 - P(S_n \geq -\sqrt{n}) = P(S_n > \sqrt{n}) + 1 - P(S_{n-1} \geq -\sqrt{n-1})$$

$$\leq \frac{\sigma}{\sqrt{n}} + 1 - \frac{\sigma}{\sqrt{n-1}} = \sqrt{2} + 1 - \sqrt{2} = \boxed{1}$$

$$\mu_X = \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} \cdot x \, dx = 0 \qquad \mu_{S_n} = E\left(\sum_{i=1}^{n} X_i\right) = 0$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2} e^{-|x|} \, dx = 2$$

$$\sigma_{S_n}^2 = \mathrm{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{var}(X_i) = 2 \cdot n \qquad \sigma_{S_{n-1}}^2 = 2 \cdot (n-1)$$

$$\lim_{n\to\infty} P(S_n > \sqrt{n}) \leq \frac{\sqrt{2n}}{\sqrt{n}} = \sqrt{2}$$

a) $P(|S_n - \mu_{S_n}|^2 > n) = P(S_n^2 > n) = P(S_n > \sqrt{n}) + P(S_n < -\sqrt{n})$

Chernoff bounds:

$\leq M(t) \cdot e^{-t\sqrt{n}}$ $t > 0$ $+ M(t)\, e^{+t\sqrt{n}}$ $t < 0$

$M_{S_n}(t) = (M_{X_i}(t))^n = \left(\frac{1}{1-t^2}\right)^n$

$= \left(\frac{1}{1-t^2}\right)^n \cdot e^{-t\sqrt{n}}$ $t>0$ $+ \left(\frac{1}{1-t^2}\right)^n e^{+t\sqrt{n}}$ $t<0$

$\lim_{n \to \infty} = 0$

b) $\lim_{n\to\infty} P(S_n^2 > n^2)^{\frac{1}{n}} = \lim_{n\to\infty} P(S_n > n)^{\frac{1}{n}}$

$= \lim_{n\to\infty} \left(P(S_n > n)^{\frac{1}{n}} + P(S_n < -n)^{\frac{1}{n}} \right)$

Chernoff bounds:

$\leq \inf_{t>0} \left(M(t) \cdot e^{-tn} \right)^{\frac{1}{n}} \quad (t>0) + \inf_{t<0} \left(M(t)\, e^{+tn} \right)^{\frac{1}{n}} \quad (t<0)$

MGF of f:

$$M_{X_i}(t) = \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t} \right) = \frac{1}{2}\left(\frac{1+t+1-t}{1-t^2} \right) = \frac{1}{1-t^2}$$

$$M_{S_n}(t) = \left(M_{X_i}(t) \right)^n = \left(\frac{1}{1-t^2} \right)^n$$

$t>0 \qquad t<0$

$$= \inf_{t>0} \left(\frac{1}{1-t^2} \right)^{\frac{n}{n}} \cdot e^{-t\frac{n}{n}} + \inf_{t<0} \left(\frac{1}{1-t^2} \right)^{\frac{n}{n}} \cdot e^{t\frac{n}{n}}$$

$= \quad 0$

(4) # visits: $V_j = |\{n \geq 1 : X_n = j\}|$ $\quad \mu_{ij} = P(V_j = \infty \mid X_0 = i)$

a) $\mu_{ii} = \begin{cases} 1 & i \text{ persistent} \\ 0 & i \text{ transient} \end{cases}$

Definition of persistent: $P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = 1$

the probability of eventual return to i, having started in i, is 1.
if this is less than 1, the state is called transient

$\mu_{ii} = P(V_i = \infty \mid X_0 = i)$

since V_i are the number of visits, if $P(V_i = 1 \mid X_0 = i) = 1$ then the state is persistent, since the return is eventual, and if one visit is eventual then ∞ are eventual as well therefore μ_{ii} is persistent

if $P(V_i = 1 \mid X_0 = i) = 0$ that mean there is no eventual return for some $n \geq 1$ steps and $P(V_i = \infty \mid X_0 = i)$ is also 0 if we can't even get back one time, then we also can't get back more than once therefore $\mu_{ii} = 0$

b) $$N_{ij} = \begin{cases} P(T_j < \infty \mid X_0 = i) & \text{if } j \text{ is persistent} \\ 0 & \text{if } j \text{ is transient} \end{cases}$$

$T_j = \min(n \geq 1 \mid X_n = j)$ $\Rightarrow$ the first time the MC reaches state j
the number of steps

so if j is persistent, that means that an eventual return/reach of state j is certain, therefore the number of steps is finite and

$$N_{ij} = P(T_j < \infty \mid X_0 = i)$$

if j is transient on the other hand means that the chain may never reach state j therefore

$$N_{ij} = 0$$