1,

midterm 2

a)

$$\phi_X(t) = E[e^{itX}]$$

$$X_k \in \{-\sqrt{k}, \sqrt{k}\}$$

$$P(X_k = \sqrt{k}) = P(X_k = -\sqrt{k}) = \frac{1}{2}$$

$$Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k$$

$$\phi_{X_k}(t) = E[e^{itX_k}] = \underbrace{P(X_k = \sqrt{k})}_{\frac{1}{2}} \cdot e^{it\sqrt{k}} + \underbrace{P(X_k = -\sqrt{k})}_{\frac{1}{2}} \cdot e^{-it\sqrt{k}}$$

$$\Rightarrow \boxed{\phi_{X_k}(t) = \frac{1}{2}\left(e^{it\sqrt{k}} + e^{-it\sqrt{k}}\right) = \cos(t\sqrt{k})}$$

$$\phi_{Y_n}(t) = E[e^{itY_n}] = E\left[e^{it(\frac{1}{n}\sum_{k=1}^{n} X_k)}\right] \xrightarrow{X_k: \text{iid}}$$

$$\phi_{Y_n}(t) = \prod_{k=1}^{n} E\left[e^{it(\frac{1}{n}X_k)}\right] = \prod_{k=1}^{n} \phi_{X_k}\left(\frac{t}{n}\right) \Rightarrow$$

$$\phi_{X_k}(t) = \cos(t\sqrt{k}) \longrightarrow \boxed{\phi_{Y_n}(t) = \prod_{k=1}^{n} \cos\left(\frac{t}{n}\sqrt{k}\right)}$$

since each X_k, has the same distribution $\Rightarrow$

$$\phi_{Y_n}(t) = \left(\phi_{X_1}\left(\frac{t}{n}\right)\right)^n = \left(\cos\left(\frac{t}{n}\sqrt{k}\right)\right)^n$$

if all X_k are identically distributed with $k=1$

$$\Rightarrow \phi_{Y_n} = \left(\cos\frac{t}{n}\right)^n$$

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b) $\cdot \lim_{n \to \infty} \phi_{Y_n}(t) = ?$

$\cdot$ continuity theorem to identify the distribution

characteristic function $\Rightarrow$

X_K has the same

$$\phi_{X_K}(t) = \cos(t\sqrt{K}) \longrightarrow \text{iid} \Rightarrow$$

$$\phi_{Y_n}(t) = \left(\cos\left(\frac{t}{n}\sqrt{K}\right)\right)^n$$

assume $K = 1$: since K is not specified to change with each X_K.

$$\lim_{n \to \infty} \phi_{Y_n}(t) = \lim_{n \to \infty}\left(\cos\left(\frac{t}{n}\sqrt{K}\right)\right)^n$$

$$\lim_{n \to \infty} \cos(t/n) = \cos(\cdot) = 1$$

Taylor: $\cos(x) = 1 - x^2/2!$ (for small x) $\Rightarrow$

$$\lim_{n \to \infty}(\cos(t/n))^n = \lim_{n \to \infty}\left(1 - \frac{t^2}{2!n^2}\right)^n \Rightarrow$$

limit definition of the exponential function $\Rightarrow \underbrace{e^{-t^2/2} \to N(0,1)}_{\text{standard Normal distribution}}$

$\Rightarrow$ central limit theorem $\Rightarrow$ Y_n converges in distribution to a normal distribution as: $n \to \infty$

$\Rightarrow$ CDF (normal distribution) $= \Phi(x) \to$ standard normal CDF.

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b)

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

- $\lim_{n\to\infty} \phi_{Y_n}(t) = ?$

- continuity theore

X_K: has the same characteristic function ⇒

$$\phi_{X_K}(t) = \cos(t\sqrt{K}) \longrightarrow \text{iid} \Rightarrow$$

$$\phi_{Y_n}(t) = \left(\cos\left(\frac{t}{n}\sqrt{K}\right)\right)^n$$

assume $K=1$; since k is not specified to change with each X_K.

$$\lim_{n\to\infty} \phi_{Y_n}(t) = \lim_{n\to\infty}\left(\cos\left(\frac{t}{n}\sqrt{K}\right)\right)^n$$

$$\lim_{n\to\infty} \cos(t/n) = \cos(0) = 1$$

Taylor: $\cos(x) = 1 - x^2/2!$ (for small x) ⇒

$$\lim_{n\to\infty}(\cos(t/n))^n = \lim_{n\to\infty}\left(1 - \frac{t^2}{2!n^2}\right)^n \Rightarrow$$

limit definition of the exponential function $\Rightarrow \underbrace{e^{-t^2/2} \to N(0,1)}_{\text{standard normal distribution}}$

⇒ central limit theorem ⇒ Y_n converges in distribution to a normal distribution as: $n \to \infty$

⇒ CDF (normal distribution) $= \Phi(x) \to$ standard normal CDF.

$$\Rightarrow \lim_{n \to \infty} F_n(x) = \Phi(x)$$

CDF of Y_n

for the values from -3 to $+3$:

$\Rightarrow \Phi(-3) = 0.0013$

$\Phi(-2) = 0.0228$

$\Phi(-1) = 0.1587$

$\Phi(0) = 0.5$

$\Phi(1) = 0.8413$

$\Phi(2) = 0.9772$

$\Phi(3) = 0.9987$

represent the probability that a standard normal r.v. is less than or equal to each corresponding x value.

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$\{x_i\}_{i=1}^{\infty} \rightarrow$ iid , all moment exists and are finite.

$$m = E[x_i]$$

$$\sigma^2 = var[x_i]$$

$$S_n = \sum_{i=1}^{n} a_i X_i$$

$$M_n = \frac{S_n}{\sum_{i=1}^{n} a_i}$$

$\{a_i\}_{i=1}^{\infty} \rightarrow$ possitive constants

$$a_1 = K$$

$$a_n = a_{n-1} + K \quad , \quad n > 1$$

a)

$$E[S_n] = \sum_{i=1}^{n} i\, Km = Km \sum_{i=1}^{n} i = Km\left(\frac{n(n+1)}{2}\right)$$

$X_i \rightarrow$ iid $\Rightarrow$ $Var[S_n] = Var[\sum_{i=1}^{n} a_i X_i] =$

$$\sum_{i=1}^{n} a_i^2 \, Var[X_i] = \sigma^2 \sum_{i=1}^{n} a_i^2 = \sigma^2 \sum_{i=1}^{n} (iK)^2 = \sigma^2 K^2 \sum_{i=1}^{n} i^2$$

$$= \sigma^2 K^2 \left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$\Rightarrow E[M_n] = \frac{E[S_n]}{\sum_{i=1}^{n} a_i} = \frac{Km\, \frac{n(n+1)}{2}}{Kn\left(\frac{n+1}{2}\right)} = m$$

$$Var[M_n] = \frac{Var[S_n]}{\left(\sum_{i=1}^{n} a_i\right)^2} = \frac{\sigma^2 \cancel{K^2} \frac{\cancel{n(n+1)}(2n+1)}{6}}{\left(Kn \cdot \frac{n+1}{2}\right)^2} = \frac{2\sigma^2 (2n+1)}{n(n+1) \cdot 3}$$

$$\frac{\cancel{K^2} n^2 \cdot \cancel{(n+1)} (n+1)}{4}$$

$$Var\left[\frac{S_n}{\sum_{i=1} a_i}\right]$$

$\hookrightarrow$ is a constant

④

b)

Chebyshev Inequality

$M_n \xrightarrow{p} m$

$$\boxed{\lim_{n\to\infty} P(|M_n - m| \geq \epsilon) \leq \lim_{n\to\infty} \frac{var[M_n]}{\epsilon^2}} = \lim_{n\to\infty} \frac{2\sigma^2(2n+1)}{\epsilon^2 n(n+1)\cdot 3} = 0$$

$\Rightarrow M_n \xrightarrow{p} m$, Also, using Markov's Inequality $\Rightarrow$

$$P(|M_n - m| \geq \epsilon) \leq \frac{E[M_n - m]}{\epsilon} = \frac{E[M_n] - m}{\epsilon} = \frac{m - m}{\epsilon} = 0$$

$\Rightarrow M_n \xrightarrow{p} m$

c)

$\cancel{\frac{S_n - E[S_n]}{\sqrt{var[S_n]}} \xrightarrow{D} Z,\ r.v.\ N(0,1)}$

~~I should show that the standardized version of S_n converges in distribution to a standard normal distribution Z. This follows from the central limit theorem.~~

~~The central limit theorem states that as $n \to \infty$~~

~~The central limit theorem states that the standardized sum of iid r.v's converge in distribution to a standard normal r.v Z.~~

$$\cancel{\frac{S_n - E[S_n]}{\sqrt{var[S_n]}} = \frac{\sum_{i=1}^{n} a_i X_i - k\frac{n(n+1)}{2}}{\sigma k \frac{n(n+1)(2n+1)}{6}}}$$

~~$n \to \infty$ this ratio converges in distribution to a standard normal r.v Z, which is $N(0,1)$~~

$$\cancel{\frac{S_n - E[S_n]}{\sqrt{var[S_n]}} \xrightarrow{D} N(0,1)}$$

2/

c)

$a_1 = k$,

$a_n = a_{n-1} + k$

$a_2 = a_1 + k = 2k$

$\Rightarrow a_i = ik$

$a_3 = a_2 + k = 3k$

$X_i \to iid\ (m, \sigma^2)$ and the covariances are "0" $\Rightarrow$

$E[S_n] = E[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i E[X_i] = \sum_{i=1}^{n} ikm = km\left(\frac{n(n+1)}{2}\right)$

$var[S_n] = var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 var(X_i) =$

$\sum_{i=1}^{n} (ik)^2 \sigma^2 = \sigma^2 k^2 \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$

standardization: $\Rightarrow Z_n = \frac{S_n - E[S_n]}{\sqrt{Var(S_n)}}$ $\xrightarrow[\text{theorem}]{\text{lindeberg-Feller central limit}}$

for sequence of r.v: $(X_1, X_2, \dots, X_n)$ with means μ_i and variances σ_i^2, and for a sequeuse of constants: $(a_1, a_2, \dots, a_n)$ the

$$\frac{\sum_{i=1}^{n} a_i (X_i - \mu_i)}{\sqrt{\sum_{i=1}^{n} a_i^2 \sigma_i^2}} \xrightarrow{D} N(0,1)$$ if the

lideuberg condition satisfied.

for each $\epsilon > 0$

$$\lim \frac{1}{S_n^2} \sum_{i=1}^{n} E\left[(X_i - \mu_i)^2 \cdot 1\{|X_i - \mu_i| > \epsilon S_n\}\right] = 0$$

indicator fonction

$S_n^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2$

$\hookrightarrow$ represents the growing variance term and is $O(n^3)$.

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2) ⓒ given that ai grows linearly with "i"

and assuming that the X_i are from a distribution with tails that decrease sufficiently fast (for example. faster than the tails of any normal distribution) the lindenberg condition should be satisfied and the normalized sum should converge in distribution to a standard Normal distribution by the lindenberg-Feller central Limit Theorem.

define:

$$H_n = \sum_{i=1}^{n} (X_i - m)^2, \quad h^2 = \sum_{i=1}^{n} \sigma^2 = n\sigma^2$$

$$B_n = H_n / h_n$$

considering the limit involving higher moments of x_i for some $d>0$:

$$\text{if } \lim_{n\to\infty} \frac{\sum E[(X_i-m)^{2+d}]}{h_n^{2+d}} = 0 \Rightarrow B_n \xrightarrow{D} N(0,1) \quad d>0$$

Since $E[X_i - m] = 0$, the $\dfrac{S_n - E[S_n]}{\sqrt{Var(S_n)}} \xrightarrow{D} N(0,1)$

This proof relies on the fact that the higher moment of X_i are controlled (the limit involving the $(2+d)$-th moment is "o"), which is a stronger condition then required by the classical central limit theorem. this additional moment condition ensures that the weighted sum of the r.v.s normalized by its mean and standard deviation → standard Normal distribution as $n \to \infty$

⑦

3/

pdf X_i is iid

$f(x) = \frac{1}{2} e^{-|x|}$ $\longrightarrow X \rightarrow \mu = 0$, $\sigma^2 = 2$

$S_n = \sum_{i=1}^{n} X_i$

$\lim_{n\to\infty} P(S_n^2 > n) = ?$

$X_i \rightarrow$ iid $\Rightarrow$ Centeral Limit theorem

a)

$\Rightarrow \lim_{n\to\infty} S_n \rightarrow$ the distribution of S_n will approach a normal distribution with mean $= n\mu = 0 \rightarrow$

$\downarrow$ 0

* mean S_n $= 0$

$\frac{S_n}{\sqrt{2n}} \xrightarrow{D} N(0,1)$, $n \to \infty$

variance: $n\sigma^2 = n(2) = 2n$

$\Rightarrow S_n \rightarrow$ Normal distributed $N(0,n)$

$S_n^2 > n \Leftrightarrow |S_n| > \sqrt{n}$

$\Rightarrow \lim_{n\to\infty} P(S_n^2 > n) \longleftrightarrow P(|S_n| > \sqrt{n})$

$\Rightarrow P(|S_n| > \sqrt{n}) \rightarrow 0$ as: The probability that a standard normal variable is outside of the $\left[-\sqrt{\frac{n}{2n}}, \sqrt{\frac{n}{2n}}\right]$ is "0" $n \to \infty$ *

S_n is normally distributed as: $N(0,n)$

$\Rightarrow \frac{S_n}{\sqrt{2n}}$ will be a standard normal random variable as $n \to \infty$.

$\lim_{n\to\infty} P(S_n^2 > n)$: The probability that a standard normal random variable is greater than "1" or less than "-1". $P(Z>1) + P(Z<-1)$

standard normal r.v.

$P(Z>1) = 1 - P(Z\leq 1)$. as the standard normal distribution is symmetric, $P(Z<-1) = P(Z>1) \Rightarrow P(|Z|>1) = 2\,P(Z>1)$

$\Rightarrow$ Python code:

$$P(|Z|>1) = 1 - \underbrace{P(Z<1)}_{\text{norm.cdf(1)}} \rightarrow = 0.317$$

Also:

Markov $\rightarrow$ $\underbrace{P(|S_n^2| \geq n) \leq \frac{E[S_n^2]}{n} = 0 \quad n>0}_{\text{non-negative}} \Rightarrow$

$$P(|S_n^2| \geq n) = 0 \Rightarrow$$

$$\lim_{n\to\infty} P(|S_n^2| \geq n) = 0$$

Chebyshev $\rightarrow$ $P(|\pm S_n| > \sqrt{n}) \leq 1/n \Rightarrow$

$$\lim P(S_n^2 > n) \leq \lim_{n\to\infty} (1/n + 1/n) = \lim_{n\to\infty} (2/n) = 0$$

b) $\lim_{n\to\infty} P(S_n^2 > n^2)^{\frac{1}{n}} = ?$

$X \to (0,1) \xrightarrow[\text{limit theorem}]{\text{central}} S_n$ (the sum of n iid r.v. X_i) is normally distributed with $(0,n)$ for large "$\underline{n}$"

$S_n^2 > n^2 \leftrightarrow |S_n| > n \rightarrow \begin{matrix} S_n > n \\ -S_n > n \end{matrix}$

for large "$\underline{n}$" we can assume that S_n is approximately normally distributed as $N(0,n) \longrightarrow$ standardizing $\underline{S_n}$

$\Rightarrow \dfrac{S_n}{\sqrt{n}} \longrightarrow$ which is a standard normal r.v. as $n \to \infty$

$P(|S_n| > n)$ can be thought of as

$P\left(\left|\frac{S_n}{\sqrt{n}}\right| > \sqrt{n}\right)$, which is the probability that the absolute value of a standand normal r.v $> \sqrt{n} \underset{n\to\infty}{\longrightarrow} \underline{0}$

↳ because the tails of the normal distribution approach "$\underline{0}$".

$\Rightarrow P(|S_n| > n) \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow \underbrace{\lim_{n\to\infty} P((|S_n|)^2 > n^2)^{\frac{1}{n}} \to 0}$

based on the properties of limits and the behavior of the normal distribution's taib.

(10)

markov $\Rightarrow$

$$p(|s_n| \geq n) \leq \frac{E[s_n]}{n} = 0$$

$$\Rightarrow \lim p(s_n^2 > n^2)^{1/n} = \lim 0 = 0$$

chebyshev's $\Rightarrow$

$$p(|s_n| > n) \leq 1/n^2 \Rightarrow \lim p(s_n^2 > n^2)^{1/n} \leq \lim_{n\to\infty} (1/n^2 + 1/n^2)^{1/n}$$

$$= \lim_{n\to\infty} (2/n^2)^{\frac{1}{n}} = 0$$

(11)

4)

$V_j = |\{n \geq 1 : X_n = j\}|$

$\downarrow$

The number of visits of markov chain X_n to j.

$r_{ij} = P(V_j = \infty | X_0 = i)$

a) $r_{ii} = ?$

A state "i" is persistent (or recurrent) if, once visited, the probability of returning to it infinity often is $\underline{1}$. $\rightarrow$ $\underbrace{E(V_i | X_0 = i) \text{ is infinite}}_{\text{mathematically.}}$

and it is transient if the total expected number of visits is finite.

if "i" is persistent $\Rightarrow$ if start from state "$\underline{i}$" the probability that we will return to state "$\underline{i}$" infinitly often is "$\underline{1}$". $\Rightarrow$ $r_{ii} = 1$: The probability that we will return to state "$\underline{i}$" at least once is "$\underline{1}$". (since the state is recurrent) and due to the Markov property, each time we return to state "$\underline{i}$", the chain "restarts", and we have the same probability of returning to it again. since it can happen an infinite number of times, $r_{ii} = 1$

(4)

if "i" is transient;

the expected number of visits is finite, which means there is a probability of not returning to state "i" that is greater than "0".

As we keep visiting state "i", the probability of not returning to it again increases until it becomes certain that we wont return.

In this case, the probability that we will never return to state "i" after some finite number "n" of visits is > 0, which implies that the probability of returning to state "i" infinitely often is less than "1". Since the state is transient, this probability is actually 0, $\Rightarrow \rho_{ii} = 0$

$$\Rightarrow \rho_{ii} = \begin{cases} 1 & \text{if } i \text{ is persistent} \\ 0 & \text{if } i \text{ is transient} \end{cases}$$

(4)

b) $\mu_{ij} = ?$ $\Rightarrow$ $\mu_{ij} = \begin{cases} P(T_j < \infty \mid X_0 = i) & \text{if } j \text{ is persistent} \\ 0 & \text{if } j \text{ is transient} \end{cases}$

$\mu_{ij} \rightarrow$ The probability that starting from state "i" the markov chain will visit state "j" at least once.

$\underbrace{T_j = \min\{ n \geq 1 \mid X_n = j \}}$ is the hitting time for state "j"

if j is persistent (recurrent)

$\Rightarrow$ Probability that the markov chain, starting from any state i, will eventually hit state "j" is 1.

$\Rightarrow P(T_j < \infty \mid X_0 = i) = 1$ for all i.

Since the state is persistent, it will be visited eventually, and the hitting time T_j will be finite with the probability of "1".

if j is transient: $\Rightarrow$ there is a passitive probability that starting from state "i" the chain will never visit state "j" $\Rightarrow$ meaning the hitting time T_j would be infinite with possitive probability.

consequently, $P(T_j < \infty \mid X_0 = i) < 1$, However since we are looking at the probability that T_j is finite, this probability is "0". because a transient state, by definition, is not expected to be hit an infinite number of times.

$\Rightarrow \mu_{ij} = P(T_j < \infty \mid X_0 = i)$, when j is persistent. and

$\mu_{ij} = 0$, when j is transient

(14)