ELE 509: Midterm 2

Duration of the examination is 72 hours

Please provide a comprehensive description of your work. Your grades depend not only on the final answers but also on the details and proofreading of your work. You may utilize your notes, textbooks, and the internet as resources. However, seeking assistance from others to solve the questions is not permitted. Please refrain from sharing your problem-solving approach with others, and if you have any questions, kindly direct them to me.

This exam has 4 questions.

1) **(25 Mark)** Let $X_k \in \{-\sqrt{k}, \sqrt{k}\}$ with distribution:

$$P(X_k = \sqrt{k}) = P(X_k = -\sqrt{k}) = \frac{1}{2}$$

and let

$$Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k$$

a) Find the characteristic functions of X_k and Y_n.
b) Using characteristic functions, find

$$\lim_{n \to \infty} F_n(x)$$

where F_n is the CDF of Y_n.

2) **(25 Mark)**
Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of iid random variables where all moments exist (and are finite). In particular, we put

$$\begin{aligned} m &= E[X_i] \\ \sigma^2 &= var[X_i] \end{aligned}$$

We are interested in evaluating the *weighted sum* and *average*

$$\begin{aligned} S_n &= \sum_{i=1}^{n} a_i X_i \\ M_n &= \frac{S_n}{\sum_{i=1}^{n} a_i} \end{aligned}$$

where $\{a_i\}_{i=1}^{\infty}$ is a sequence of positive constants. In this problem we put

$$\begin{aligned} a_1 &= k \\ a_n &= a_{n-1} + k, \quad n > 1 \end{aligned}$$

for some positive constant k.

a) Find the means and variances of S_n and M_n.
b) Show that

$$M_n \xrightarrow{P} m$$

c) Show that

$$\frac{S_n - E[S_n]}{\sqrt{var[S_n]}} \xrightarrow{D} Z$$

where Z is an $N(0,1)$ random variable.

3) (**25 Mark**) Let X_i be iid random variables with the pdf

$$f(x) = \frac{1}{2}e^{-|x|}$$

Define

$$S_n = \sum_{i=1}^{n} X_i$$

a) Find $\lim_{n\to\infty} P(S_n^2 > n)$
b) Find $\lim_{n\to\infty} P(S_n^2 > n^2)^{\frac{1}{n}}$

4) (**25 Mark**) Let

$$V_j = |\{n \geq 1 : X_n = j\}|$$

be the number of visits of the Markov chain X_n to j. Also, define

$$\mu_{ij} = P(V_j = \infty | X_0 = i)$$

Show that

a)

$$\mu_{ii} = \begin{cases} 1 & \text{if } i \text{ is persistent} \\ 0 & \text{if } i \text{ is transient} \end{cases}$$

b)

$$\mu_{ij} = \begin{cases} P(T_j < \infty | X_0 = i) & \text{if } j \text{ is persistent} \\ 0 & \text{if } j \text{ is transient} \end{cases}$$

where

$$T_j = \min\{n \geq 1 | X_n = j\}$$