CONTROLLERS

Sébastien Boisgérault, Mines ParisTech

PREAMBLE

```
from numpy import *
from numpy.linalg import *
from numpy.testing import *
from matplotlib.pyplot import *
from scipy.integrate import *
```

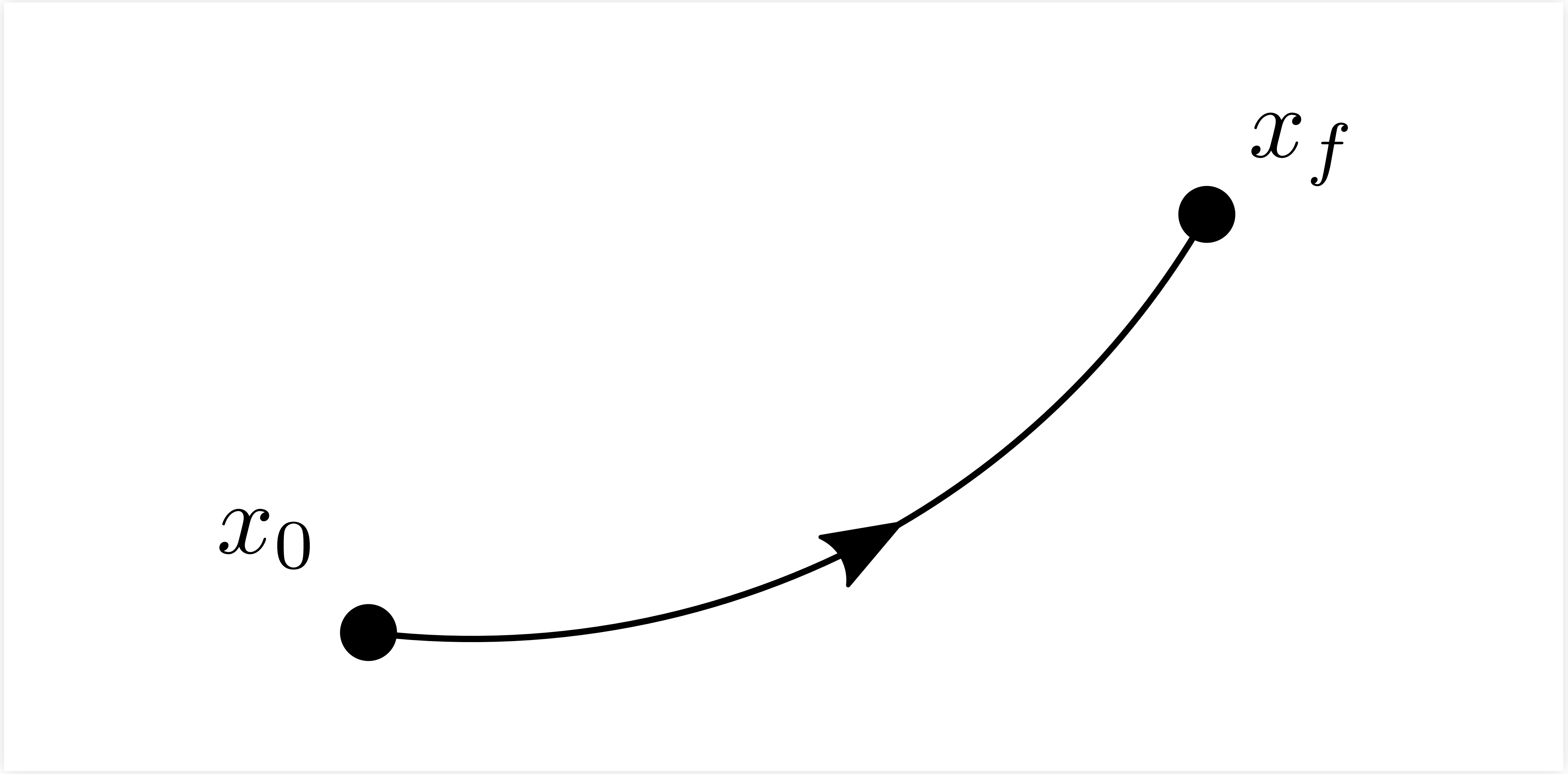
CONTROLLABILITY

DEFINITION

The system $\dot{x} = f(x, u)$ is **controllable** if

- for any $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n$ and $x_f \in \mathbb{R}^n$,
- there are $t_f > 0$ and $u : [t_0, t_f] \to \mathbb{R}^m$ such that
- the solution $x(t)$ such that $x(t_0) = x_0$ satisfies

$$x(t_f) = x_f.$$


x_0
x_f

👁 CONTROLLABILITY / CAR

The position x (in meters) of a car of mass m (in kg) on a straight road is governed by

$$m\ddot{x} = u$$

where u the force (in Newtons) generated by its motor.

The car is initially at the origin of a road and motionless. We would like to drive it to across the location $x_f > 0$ at speed v_f and at time $t_f > 0$.

Numerical values:

- $m = 1500 \, \mathrm{kg}$,
- $t_f = 10 \, \mathrm{s}$, $x_f = 100 \, \mathrm{m}$ and $v_f = 100 \, \mathrm{km/h}$.

STRATEGY

STEP 1 – TRAJECTORY PLANNING

We search for a **reference trajectory** for the state

$$X_r(t) = (x_r(t), \dot{x}_r(t))$$

such that:

- $x_r(0) = 0, \dot{x}_r(0) = 0,$
- $x_r(t_f) = x_f, \dot{x}_r(t_f) = v_f.$

STEP 2 – ADMISSIBILITY

We check that this reference trajectory is **admissible**, i.e. that we can find a control $u_r(t)$ such that the solution of the IVP is $X(t) = X_r(t)$ when $X(0) = X_r(t)$.

ADMISSIBLE TRAJECTORY

Here, if x_r is smooth and if we apply the control

$$u(t) = m\ddot{x}_r(t),$$

$$m\frac{d^2}{dt^2}(x - x_r) = 0,$$

$$(x - x_r)(0) = 0, \;\; \frac{d}{dt}(x - x_r)(0) = 0.$$

Thus, $x(t) = x_r(t)$ – and thus $\dot{x}(t) = \dot{x}_r(t)$ – for every $t \geq 0$.

REFERENCE TRAJECTORY

We can find x_r as a third-order polynomial in t

$$x_r(t) = at^3 + bt^2 + ct + d$$

with

$$a = \frac{v_f}{t_f^2} - 2\frac{x_f}{t_f^3},\ b = 3\frac{x_f}{t_f^2} - \frac{v_f}{tf},\ c = 0,\ d = 0.$$

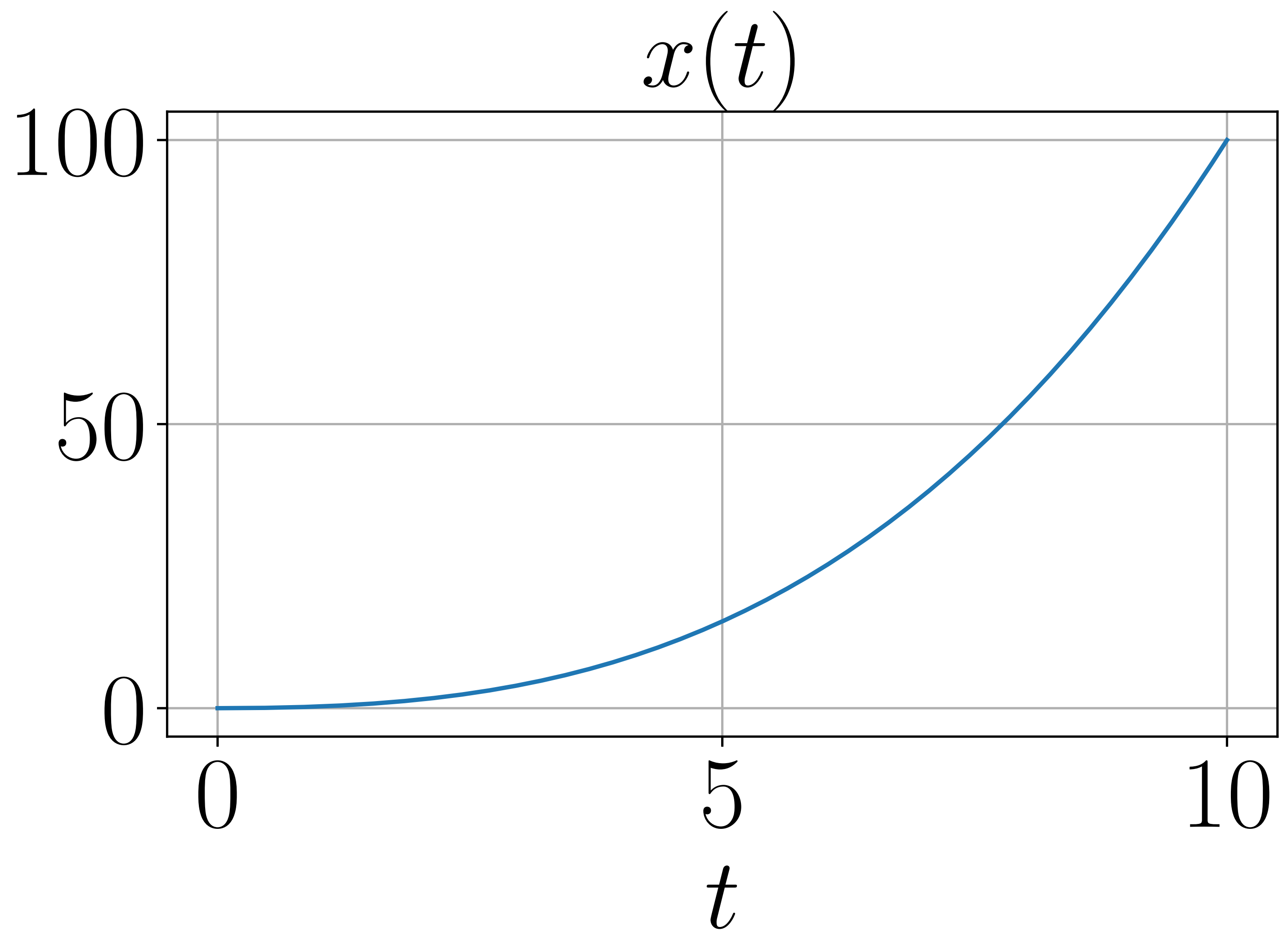
(equivalently, with $u(t)$ as an affine function of t).

```
m = 1500.0
xf = 100.0
vf = 100.0 * 1000 / 3600 # m/s
tf = 10.0
a = vf/tf**2 - 2*xf/tf**3
b = 3*xf/tf**2 - vf/tf
```

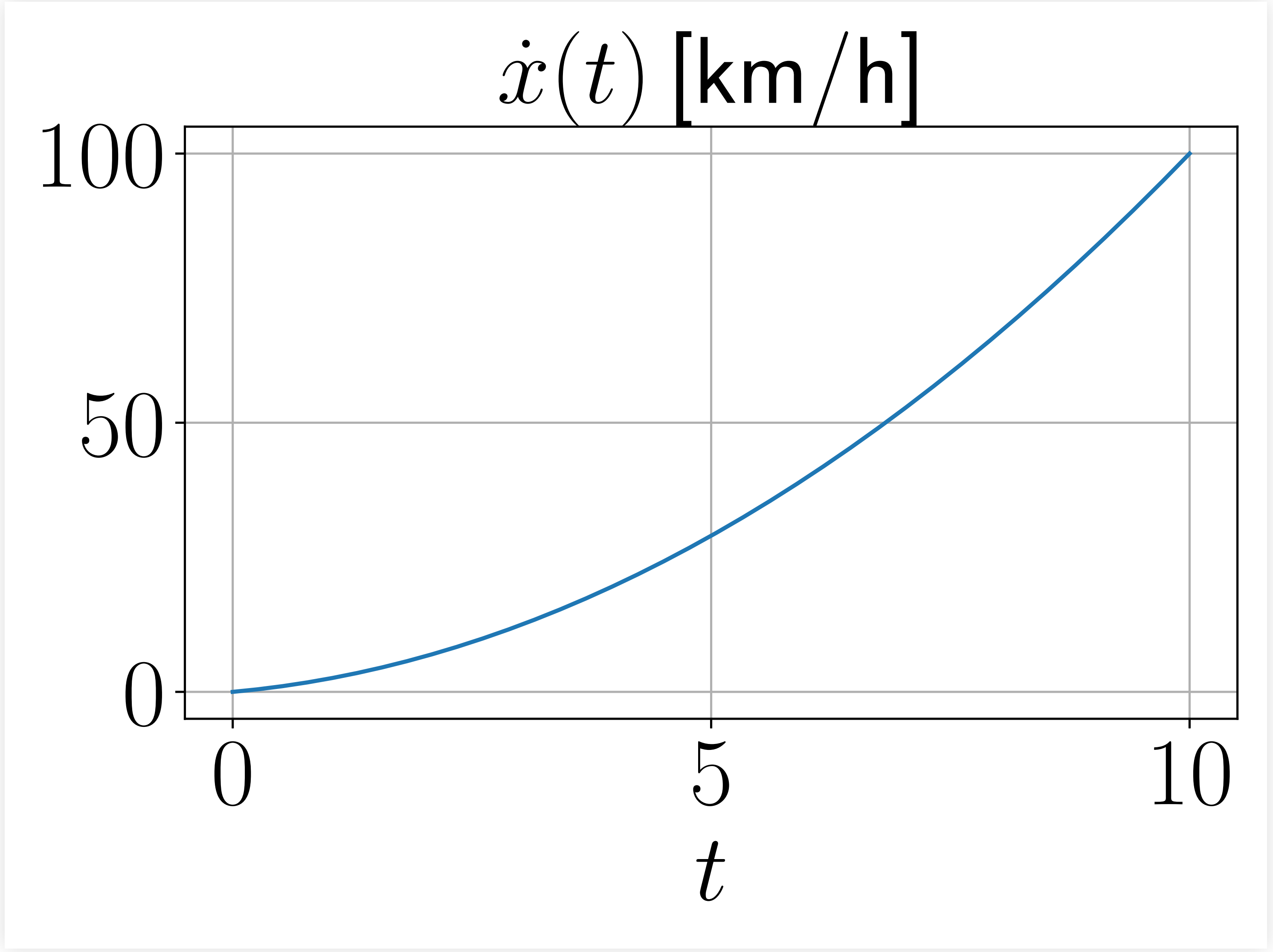
```
def x(t):
    return a * t**3 + b * t**2
def d2_x(t):
    return 6 * a * t + 2 * b
def u(t):
    return m * d2_x(t)
```

```
y0 = [0.0, 0.0]
def fun(t, y):
    x, d_x = y
    d2_x = u(t) / m
    return [d_x, d2_x]
result = solve_ivp(fun, [0.0, tf], y0,
dense_output=True)
```

```
figure()
t = linspace(0, tf, 1000)
xt = result["sol"](t)[0]
plot(t, xt)
grid(True); xlabel("$t$"); title("$x(t)$")
```


$x(t)$
100
50
0
0
5
10
t

```python
figure()
vt = result["sol"](t)[1]
plot(t, 3.6 * vt)
grid(True); xlabel("$t$")
title(r"$\dot{x}(t) \, \mbox{[km/h]}$")
```



$\dot{x}(t)$ [km/h]
100
50
0
0
5
10
t

? NON-ADMISSIBLE TRAJECTORY

Let

$$\dot{x} = Ax \;\text{ with }\; x \in \mathbb{R}^2,\; A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Find a smooth reference trajectory

$$x_r(t),\; t \in [0, 1]$$

which is not admissible.

? PENDULUM

Consider the pendulum with dynamics:

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell\sin\theta = u$$

- [💡, $\mathbf{x}^2$] Find a smooth reference trajectory that leads the pendulum from $\theta(0) = 0$ and $\dot{\theta}(0) = 0$ to $\theta(t_f) = \pi$ and $\dot{\theta}(t_f) = 0$.
- [💡, $\mathbf{x}^2$] Show that the reference trajectory is admissible and compute the corresponding input $u(t)$ as a function of t and $\theta(t)$.

- [🧪, 💻] Simulate the result with standard and high-precision (small steps). What should happen theoretically after $t = t_f$ if $u(t) = 0$ is applied ? What does happen in practice ?

  Numerical Values:

  $$m = 1.0, \; l = 1.0, \; b = 0.1, \; g = 9.81, \; t_f = 10.$$

CONTROLLABILITY / LTI SYSTEM

For a LTI system, it is sufficient to check that

- from the origin $x_0 = 0$ at $t_0 = 0$,
- we can reach any state $x_f \in \mathbb{R}^n$.

KALMAN CRITERION

The system $\dot{x} = Ax + Bu$ is controllable iff:

$$\operatorname{rank}\ \left[B, AB, \ldots, A^{n-1}B\right] = n$$

$[B, \ldots, A^{n-1}B]$ is the **Kalman controllability matrix**.

👁 KALMAN CONTROLLABILITY MATRIX

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

COMPUTATION

```
def KCM(A, B):
    n = shape(A)[0]
    mp = matrix_power
    cs = column_stack
    return cs([mp(A, k) @ B for k in range(n)])
```

```
n = 3
A = zeros((n, n))
for i in range(0, n-1):
    A[i,i+1] = 1.0

B = zeros((n, 1))
B[n-1, 0] = 1.0
```

```
C = KCM(A, B)

C_expected = [[0, 0, 1], [0, 1, 0], [1, 0, 0]]
assert_almost_equal(C, C_expected)
```

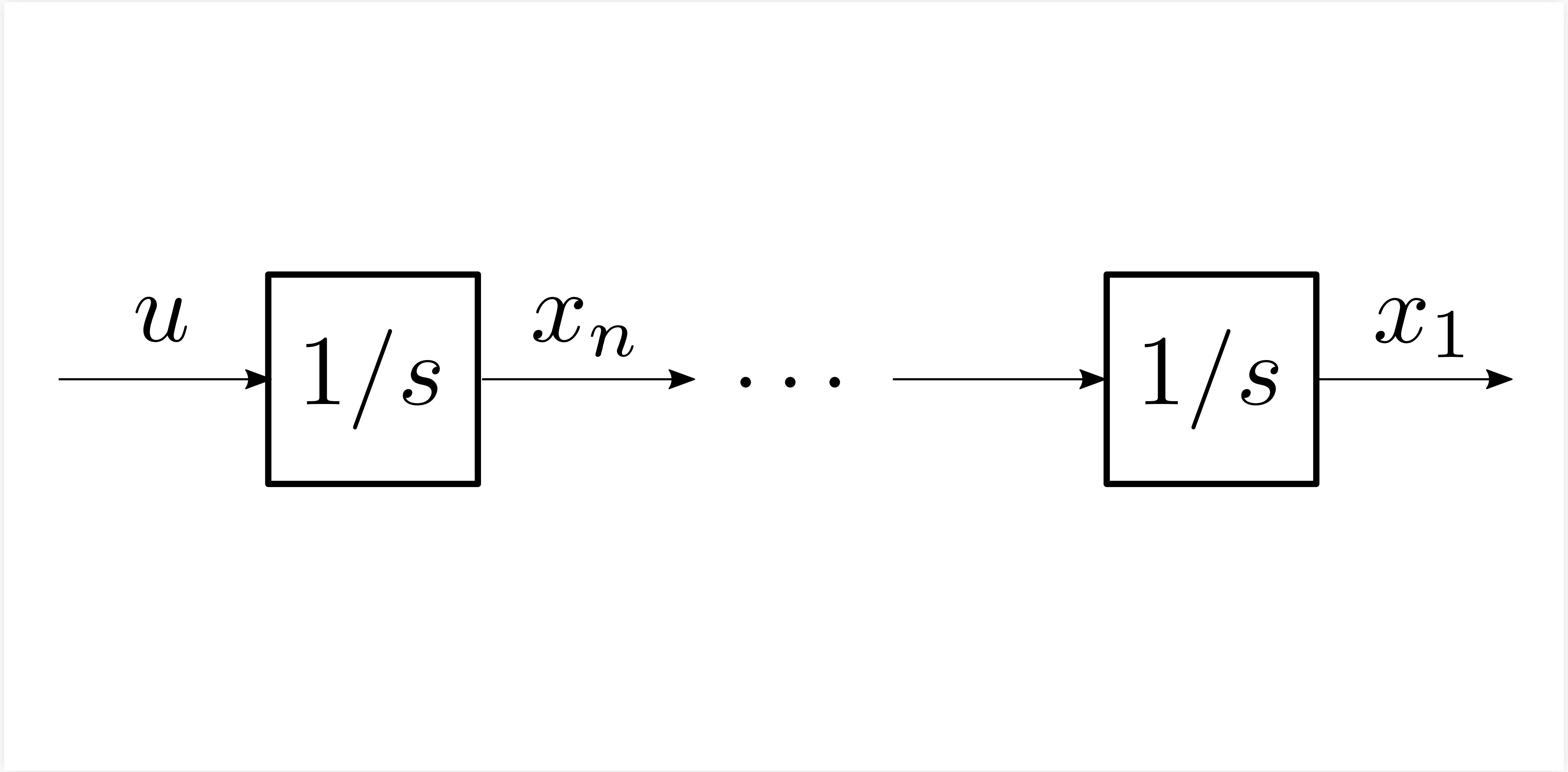
? FULLY ACTUATED SYSTEM

Consider $\dot{x} = Ax + Bu$ with $x \in \mathbb{R}^n, u \in \mathbb{R}^n$ and

$$\operatorname{rank} B = n.$$

- [💡, **x²**] Is the systems controllable ?
- [💡, **x²**] Given x_0, x_f and $t_f > 0$, show that any smooth trajectory that leads from x_0 to x_f in t_f seconds is admissible.

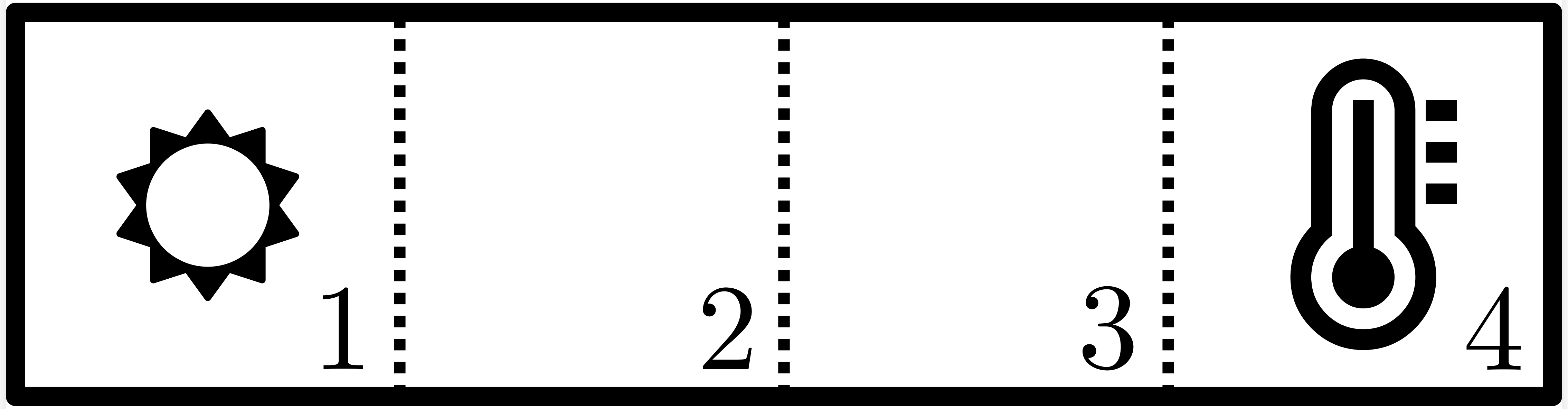
INTEGRATOR CHAIN



$$\dot{x}_n = u,\ \dot{x}_{n-1} = x_n,\ \cdots\ ,\dot{x}_1 = x_2.$$

- [💡, $\mathbf{x}^2$] Show that the system is controllable

? HEAT EQUATION



- $dT_1/dt = u + (T_2 - T_1)$
- $dT_2/dt = (T_1 - T_2) + (T_3 - T_2)$
- $dT_3/dt = (T_2 - T_3) + (T_4 - T_3)$
- $dT_4/dt = (T_3 - T_4)$

- [💡, $\mathbf{x}^2$] Show that the system is controllable.
- [💡, $\mathbf{x}^2$] Is it still true if the four cells are organized as a square and the heat sink/source is in any of the corners ? How many independent sources do you need to make the system controllable and where can you place them?

EXTRA EXERCICES

- Unreachable states
- Brunovsky form
- Controllability in prey-predator systems (via the invariant)
- etc.

ASYMPTOTIC STABILIZATION

STABILIZATION

When the system

$$\dot{x} = Ax, \;\; x \in \mathbb{R}^n$$

is not asymptotically stable at the origin,

maybe there are some inputs $u \in \mathbb{R}^m$ such that

$$\dot{x} = Ax + Bu$$

that we can use to stabilize asymptotically the system?

LINEAR FEEDBACK

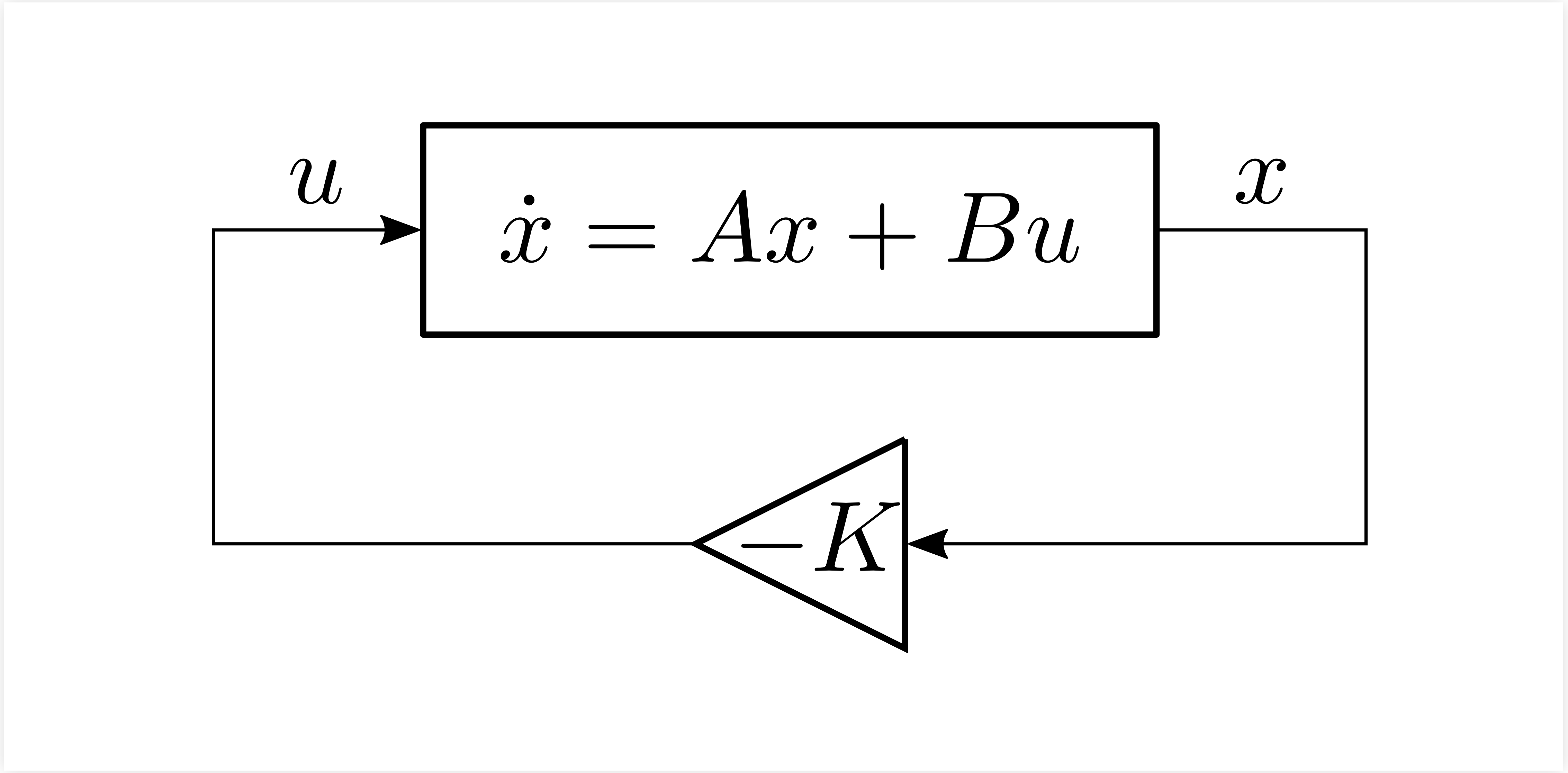
We can try to compute u as

$$u(t) = -Kx(t)$$

fro some $K \in \mathbb{R}^{m \times n}$

Note. This strategy requires the system state $x(t)$ to be known (measured); this information is then **fed back** into the system.

CLOSED-LOOP DIAGRAM



CLOSED-LOOP DYNAMICS

When

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

the state $x \in \mathbb{R}^n$ evolves according to:

$$\dot{x} = (A - BK)x$$

The closed-loop system is asymptotically stable iff every eigenvalue of the matrix

$$A - BK$$

is in the open left-hand plane.

POLE ASSIGNMENT

Assume that:

- The systems $\dot{x} = Ax + Bu$ is controllable.
- Let $\Lambda = \{\lambda_1, \dots, \lambda_n\} \in \mathbb{C}^n$ be a (multi-)set of complex numbers (a value may appear several times) which is symmetric:

  if $\lambda \in \Lambda$, then $\overline{\lambda} \in \Lambda$ (with the same multiplicity)

- Let $\sigma(A - BK)$ denote the (multi-)set of eigenvalues of $A - BK$ (eigenvalues are counted with their multiplicity).
- Then there is a matrix K such that

$$\sigma(A - BK) = \Lambda.$$

👁 STABILIZATION/POLE ASSIGNMENT

Consider the double integrator $\ddot{x} = u$

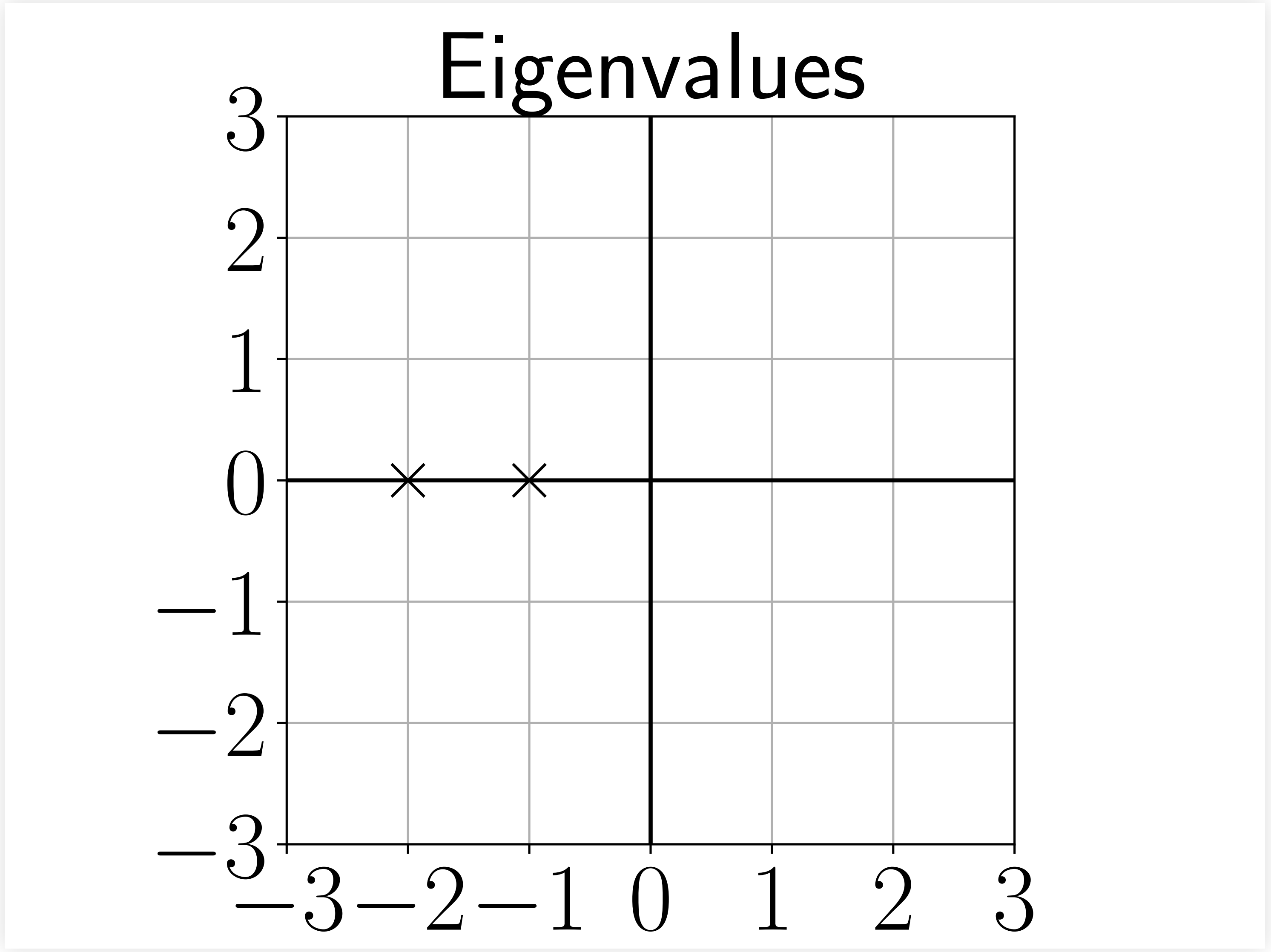
$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(in standard form)

```
from scipy.signal import place_poles
A = array([[0, 1], [0, 0]])
B = array([[0], [1]])
poles = [-1, -2]
K = place_poles(A, B, poles).gain_matrix
```

```
assert_almost_equal(K, [[2.0, 3.0]])
eigenvalues, _ = eig(A - B @ K)
assert_almost_equal(eigenvalues, [-1, -2])
```

```
figure()
x = [real(s) for s in eigenvalues]
y = [imag(s) for s in eigenvalues]
plot(x, y, "kx", ms=12.0)
xticks([-3, -2,-1, 0,1, 2,3])
yticks([-3, -2,-1, 0,1, 2,3])
plot([0, 0], [-3, 3], "k")
plot([-3, 3], [0, 0], "k")
```


Eigenvalues
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3

⚠ IMPLEMENTATION DETAIL

- The `place_poles` function will not accept eigenvalues whose multiplicity is higher than the rank of B.
- So here `poles = [-1, -1]` won't work.
- But `poles = [-1, -1.001]` may work.

POLE ASSIGNMENT / DEFAULT

Consider system with dynamics

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2 + u \\ \dot{x}_2 &= -x_1 + x_2 + u\end{aligned}$$

- [💡,**x²**]. We apply the control law
  $$u = -k_1 x_1 - k_2 x_2;$$
  can we move the poles of the system where we want by a suitable choice of k_1 and k_2?
- [💡] Explain this result.

PENDULUM

Consider the pendulum with dynamics:

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell \sin\theta = u$$

- [💡, $\mathbf{x^2}$] Compute the linearized dynamics of the system around the equilibrium $\theta = \pi$ and $\dot{\theta} = 0$.

- [💡, $\mathbf{x^2}$] Design a control law

$$u = -k_1(\theta - \pi) - k_2\dot{\theta}$$

such that the closed-loop linear system is asymptotically stable, with a time constant smaller than 10 sec.

Numerical Values:

$$m = 1.0,\ l = 1.0,\ b = 0.1,\ g = 9.81$$

- [🧪, 💻] Simulate this control law on the nonlinear systems when $\theta(0) = 0$ and $\dot{\theta}(0) = 0$; compare with the open-loop strategy that we have already considered.

DOUBLE SPRING SYSTEM

Consider the dynamics:

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - b_1\dot{x}_1 \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1) - b_2\dot{x}_2 + u \end{aligned}$$

Numerical values:

$$m_1 = m_2 = 1,\ k_1 = 1, k_2 = 100,\ b_1 = 0,\ b_2 = 20$$

- [🧪, 💻] Compute the poles of the system. Is it asymptotically stable?
- [🧪, 💻] Use a linear feedback to kill the oscillatory behavior of the solutions and “speed up” the eigenvalues associated to a slow behavior.

OPTIMAL CONTROL

WHY ?

Limitations of Pole Assignment

- it is not always obvious what set of poles we should target (especially for large systems),
- we do not control explicitly the trade-off between “speed of convergence” and “intensity of the control” (large input values maybe costly or impossible).

Let

$$\dot{x} = Ax + Bu$$

where

- $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}$ and
- $x(0) = x_0 \in \mathbb{R}^n$ is given.

Find $u(t)$ that minimizes

$$J = \int_0^{+\infty} x(t)^t Q x(t) + u(t)^t R u(t) \, dt$$

where:

- $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$,
- (to be continued …)

- Q and R are **symmetric** ($R^t = R$ and $Q^t = Q$),
- Q and R are **positive definite** (denoted “> 0”)

$$x^t Q x \geq 0 \text{ and } x^t Q x = 0 \text{ iff } x = 0$$

and

$$u^t R u \geq 0 \text{ and } u^t R u = 0 \text{ iff } u = 0.$$

HEURISTICS / SCALAR CASE

If $x \in \mathbb{R}$ and $u \in \mathbb{R}$,

$$J = \int_0^{+\infty} qx(t)^2 + ru(t)^2 \, dt$$

with $q > 0$ and $r > 0$.

When we minimize J:

- Only the relative values of q and r matters.
- Large values of q penalize strongly non-zero states:

  $\Rightarrow$ fast convergence.
- Large values of r penalize strongly non-zero inputs:

  $\Rightarrow$ small input values.

HEURISTICS / VECTOR CASE

If $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ and Q and R are diagonal,

$$Q = \mathrm{diag}(q_1, \cdots, q_n), \;\; R = \mathrm{diag}(r_1, \cdots, r_m),$$

$$J = \int_0^{+\infty} \sum_i q_i x_i(t)^2 + \sum_j r_j u_j(t)^2 \, dt$$

with $q_i > 0$ and $r_j > 0$.

Thus we can control the cost of each component of x and u independently.

OPTIMAL SOLUTION

Assume that $\dot{x} = Ax + Bu$ is controllable.

There is an optimal solution; it is a linear feedback

$$u = -Kx$$

The corresponding closed-loop dynamics is asymptotically stable.

ALGEBRAIC RICCATI EQUATION

The gain matrix K is given by

$$K = R^{-1} B^t \Pi,$$

where $\Pi \in \mathbb{R}^{n \times n}$ is the unique matrix such that $\Pi^t = \Pi, \Pi > 0$ and

$$\Pi B R^{-1} B^t \Pi - \Pi A - A^t \Pi - Q = 0.$$

? VALUE OF J

Consider the dynamics $\dot{x} = Ax + Bu$ where $u = -Kx$ is the optimal control associated to

$$J = \int_0^{+\infty} j(x(t), u(t)) \, dt$$

where

$$j(x, u) = x^t Q x + u^t R u.$$

- [💡, $\mathbf{x^2}$] Show that

$$j(x(t), u(t)) = -\frac{d}{dt} x(t)^t \Pi x(t)$$

- [💡, $\mathbf{x^2}$] What is the value of J in the optimal case?

👁 STABILIZATION/OPTIMAL CONTROL

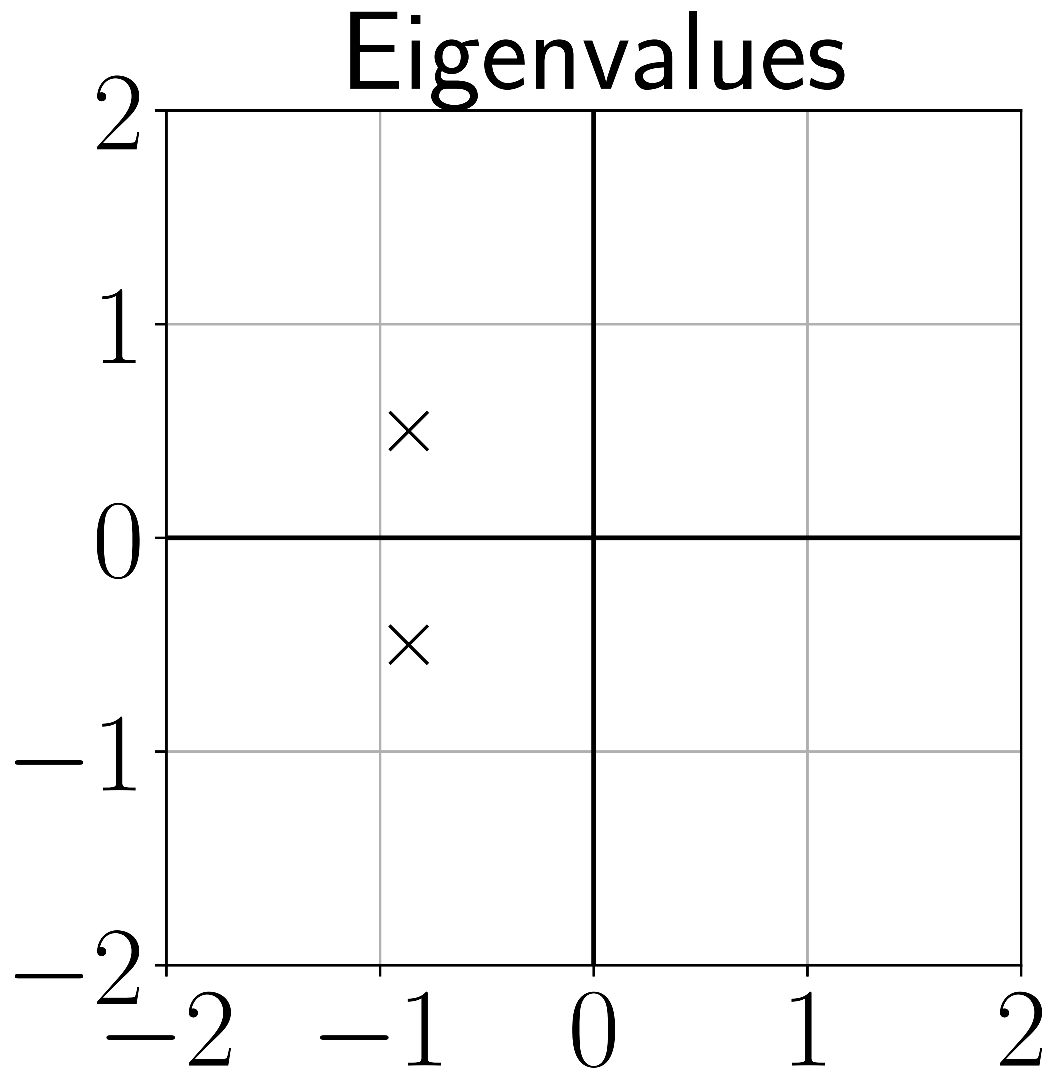
Consider the double integrator $\ddot{x} = u$

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(in standard form)

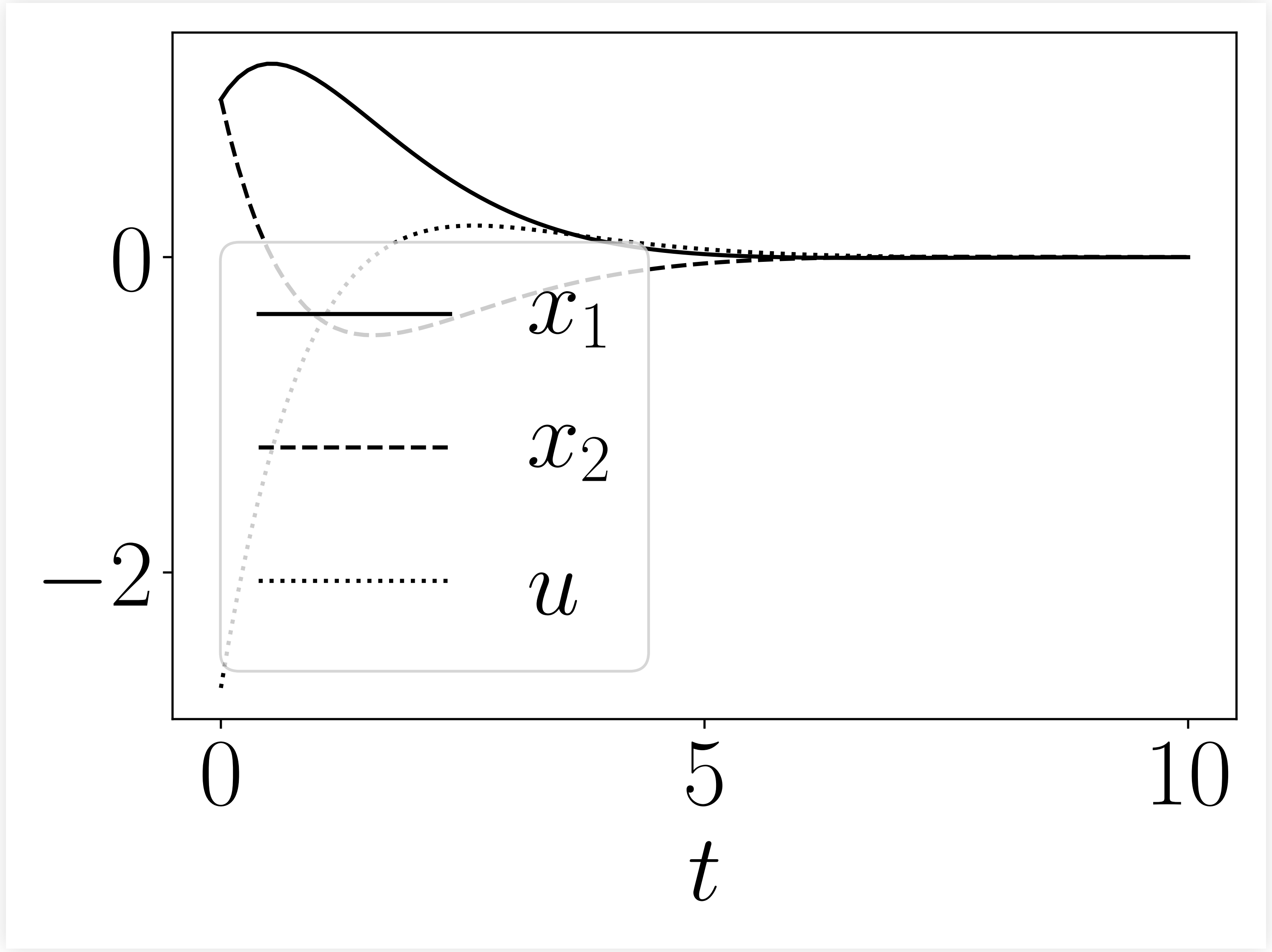
```
from scipy.linalg import solve_continuous_are
A = array([[0, 1], [0, 0]])
B = array([[0], [1]])
Q = array([[1, 0], [0, 1]]); R = array([[1]])
Pi = solve_continuous_are(A, B, Q, R)
K = inv(R) @ B.T @ Pi
eigenvalues, _ = eig(A - B @ K)
assert all([real(s) < 0 for s in eigenvalues])
```

```
figure()
x = [real(s) for s in eigenvalues]
y = [imag(s) for s in eigenvalues]
plot(x, y, "kx", ms=12.0)
xticks([-2, -1, 0, 1, 2])
yticks([-2, -1, 0, 1, 2])
plot([0, 0], [-2, 2], "k")
plot([-2, 2], [0, 0], "k")
```


Eigenvalues
2
1
0
−1
−2
−2
−1
0
1
2

```
y0 = [1.0, 1.0]
def f(t, x):
    return (A - B.dot(K)).dot(x)
result = solve_ivp(f, t_span=[0, 10], y0=y0,
max_step=0.1)
t = result["t"]
x1 = result["y"][0]
x2 = result["y"][1]
```

```
figure()
plot(t, x1, "k-", label="$x_1$")
plot(t, x2, "k--", label="$x_2$")
plot(t, u, "k:", label="$u$")
xlabel("$t$")
legend()
```

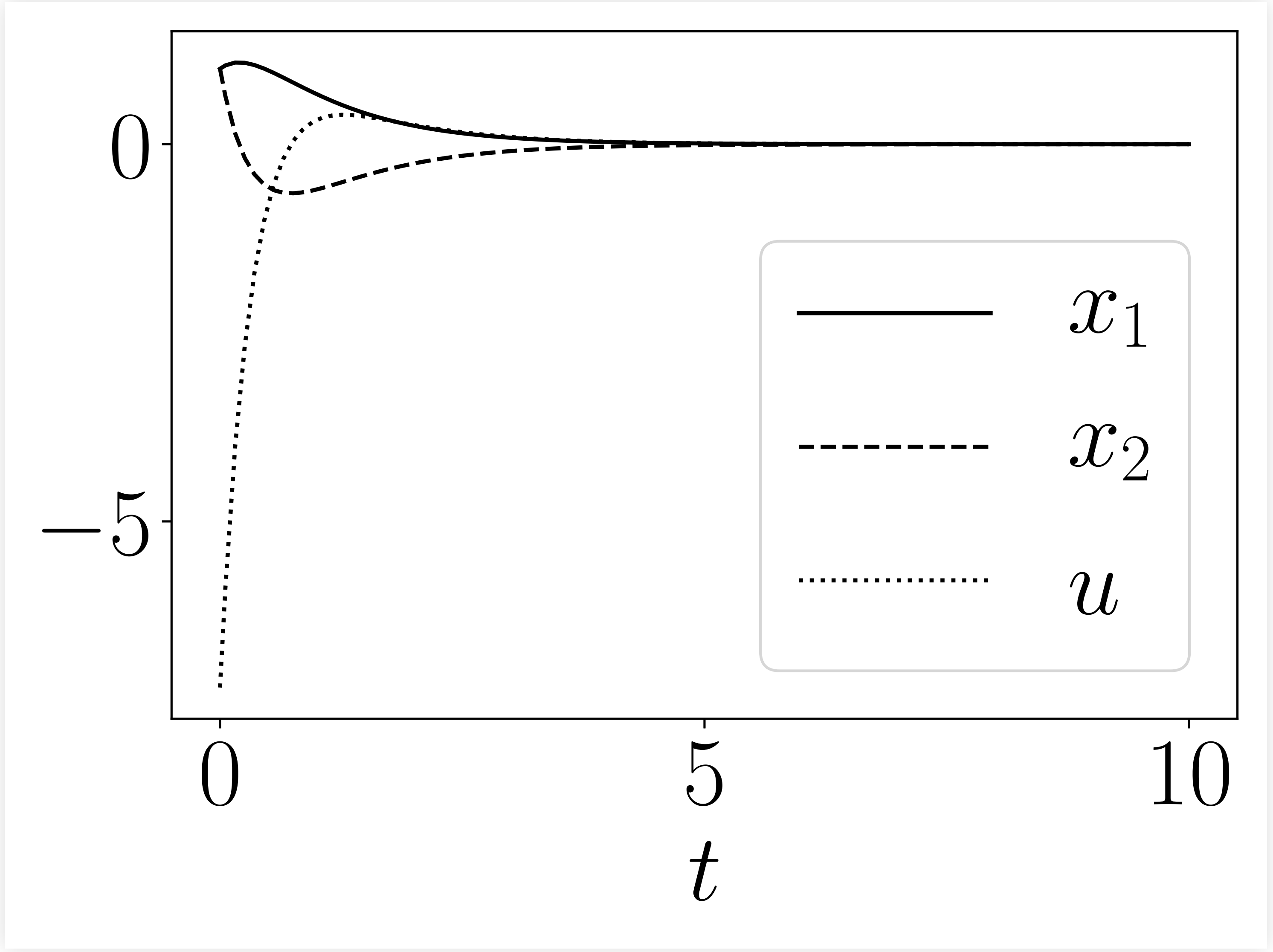


0
−2
0
5
10
t
x_1
x_2
u

```
Q = array([[10, 0], [0, 10]]); R = array([[1]])
Pi = solve_continuous_are(A, B, Q, R)
K = inv(R) @ B.T @ Pi
```

```
result = solve_ivp(f, t_span=[0, 10], y0=y0,
max_step=0.1)
t = result["t"]
x1 = result["y"][0]
x2 = result["y"][1]
u = -K.dot(result["y"]).flatten()
```

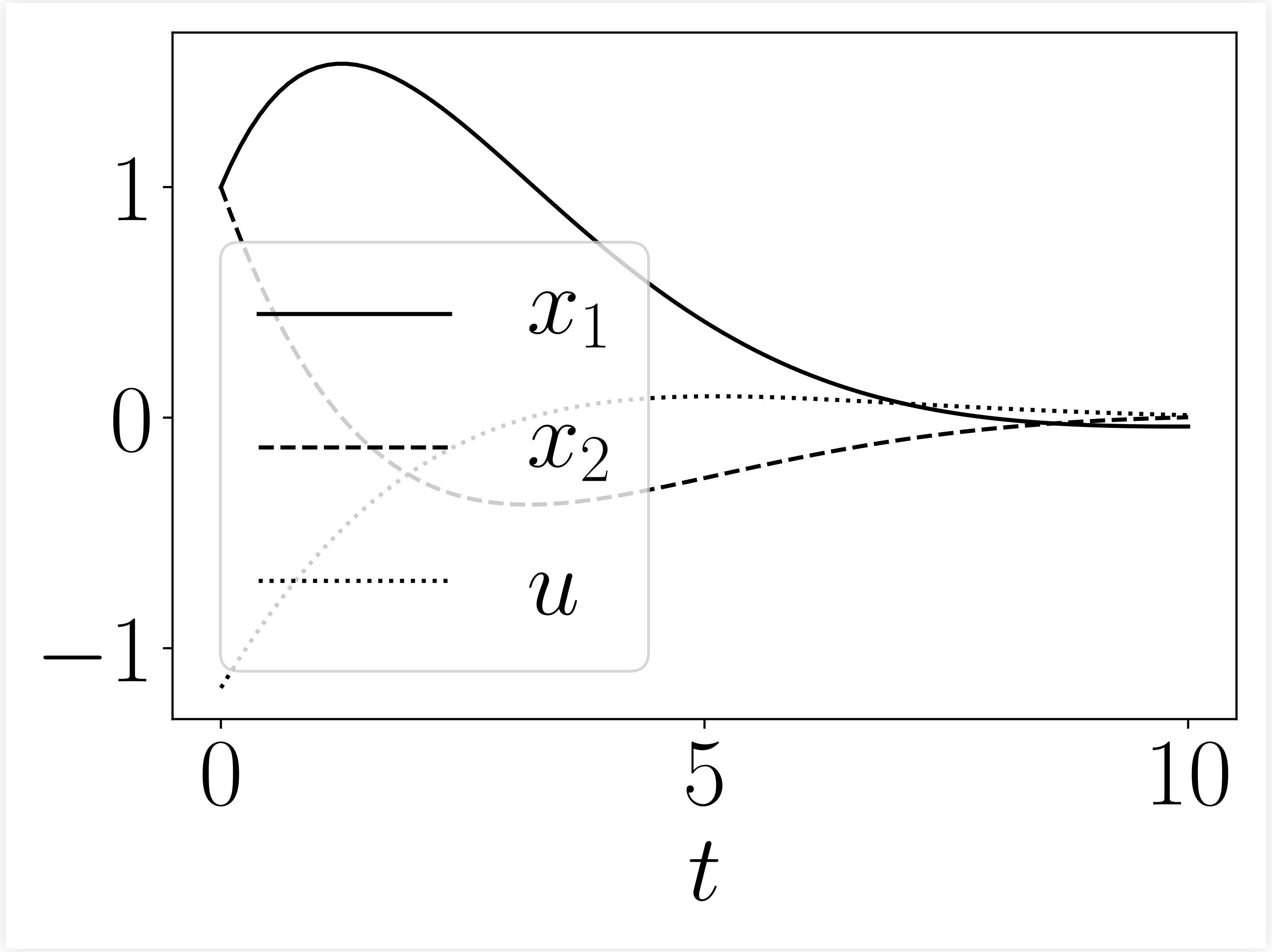
```
figure()
plot(t, x1, "k-", label="$x_1$")
plot(t, x2, "k--", label="$x_2$")
plot(t, u, "k:", label="$u$")
xlabel("$t$")
legend()
```


0
-5
0
5
10
t
x_1
x_2
u

```python
Q = array([[1, 0], [0, 1]]); R = array([[10]])
Pi = solve_continuous_are(A, B, Q, R)
K = inv(R) @ B.T @ Pi
```

```
result = solve_ivp(f, t_span=[0, 10], y0=y0,
max_step=0.1)
t = result["t"]
x1 = result["y"][0]
x2 = result["y"][1]
u = -K.dot(result["y"]).flatten()
```

```
figure()
plot(t, x1, "k-", label="$x_1$")
plot(t, x2, "k--", label="$x_2$")
plot(t, u, "k:", label="$u$")
xlabel("$t$")
legend()
```


1
0
−1
0
5
10
t
x_1
x_2
u

EXTRA EXERCISE

- “Lunar lander” for “rendez-vous” with limited fuel?