TWO-COMPARTMENT MODEL TO SIMULATE THE FLOW OF WATER IN A CROP

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Aim

Our aim is to try to model and simulate a crop using the two-compartment model. The first compartment here represents the soil, while the second compartment represents the body of the plant being considered. The variables in this model are the time, and the amounts of water that are present in each of the constituent compartments at any given time.

Two-Compartment Model – Introduction

The **two-compartment model** is a general pharmacokinetic model that describes the distribution and elimination of a drug within the body. It consists of two interconnected compartments:

Central Compartment

- Represents the bloodstream and highly perfused organs, such as the heart, liver, and kidneys.
- Drug entry occurs via this compartment, either through intravenous or oral administration.
- Rapid equilibrium is established between the central and peripheral compartments.

Peripheral Compartment

- Represents tissues with lower blood perfusion, such as fat and muscle.
- Drug exchange between the central and peripheral compartments occurs at a slower rate.

Key Processes

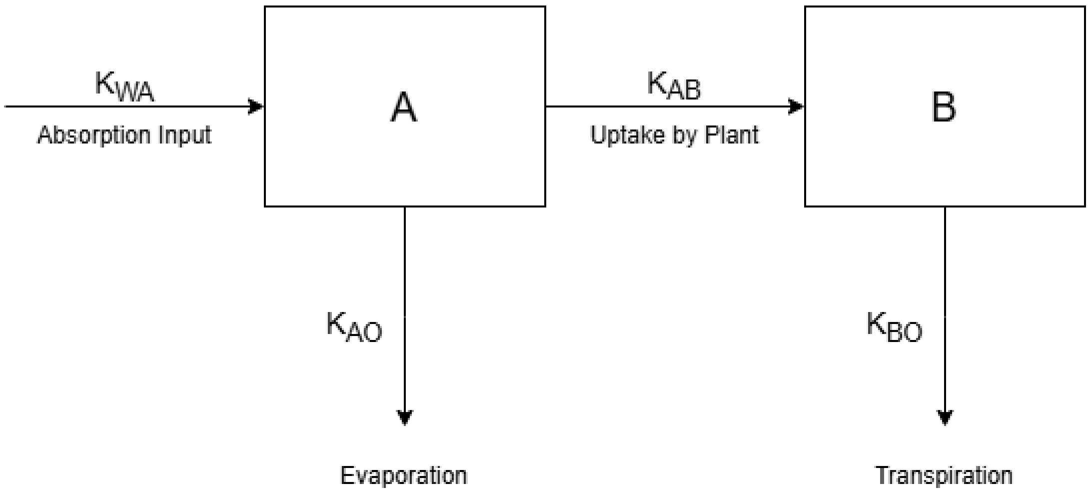
1. **Absorption**:
   Drug enters the central compartment unless administered directly via IV.
2. **Distribution**:
   Drug moves between the central and peripheral compartments.
3. **Elimination**:
   Drug is metabolized or excreted, primarily from the central compartment.

Uses

- Predicts drug concentration over time.
- Helps in determining appropriate dosing regimens.

This model is more complex than the single-compartment model and is used when the drug demonstrates a **biphasic concentration-time curve**.

Modelling the Crop



$$\frac{dC_A}{dt} = K_{WA}C_W - (K_{AB} + K_{AO})C_A$$

$$\frac{dC_B}{dt} = K_{AB}C_A - K_{BO}C_B$$

C_W: Amount of water given to the soil

$C_A(t)$: Amount of water in the soil

$C_B(t)$: Amount of water in the plant

K_{WA}: The absorption factor of water into the soil

K_{AB}: The transportation factor of water from the soil to the plant

K_{AO}: The evaporation factor of water from the soil

K_{BO}: The transpiration factor of water from the plant

At $t = 0$, $C_A = C_0$.

For dry soil, $C_0 = 0$.

Analysis of each term of our equation

Required dimension for each term: $[L^3T^{-1}]$

$K_{WA}C_W$

The water is introduced over the soil. We are assuming that there is no parallel flow.

$C_W = [L^3]$

K_W has to be $[T^{-1}]$.

We introduce a term called the **coefficient of permeability (k)** of soil, having the dimension $[LT^{-1}]$. The coefficient of permeability comes from Darcy's law, which happens to be the most fundamental mathematical equation for the discharge of water from the soil.

Darcy's law: $q = kiA$

The unit of the coefficient of permeability (k) is same as the units of velocity, like cm/s or mm/s. The value of k is a measure of the resistance offered by the soil to the flow of water. This coefficient of permeability further depends on the viscosity of water, average particle size, and void ratio of the soil present at the location.

This leaves us with the need to find another term with dimension [L] which will help balance the quantity K_W dimensionally.

Chosen quantity: **root zone depth (d)**

Root zone depth is the depth within the soil profile that commodity crop (cc) roots can effectively extract water and nutrients for growth.

$$K_{WA} = \frac{Coefficient\ of\ Permeability\ (k)}{Root\ Zone\ Depth\ (d)}$$

$K_{AB}C_A$ and $K_{BO}C_B$

$C_A = [L^3]$

$C_B = [L^3]$

The rate of water absorption from the soil compartment to the plant compartment, and the rate of water loss from the plant by transpiration, can be measured in terms of fractions of the whole compartment per hour. This gives us the dimension $[T^{-1}]$.

Thus:

$$K_{AB} = Fraction\ of\ water\ from\ compartment\ A\ transferred\ to\ compartment\ B\ per\ hour\ (F_{AB})$$

$$K_{BO} = Fraction\ of\ water\ in\ compartment\ B\ lost\ to\ transpiration\ per\ hour\ (F_{BO})$$

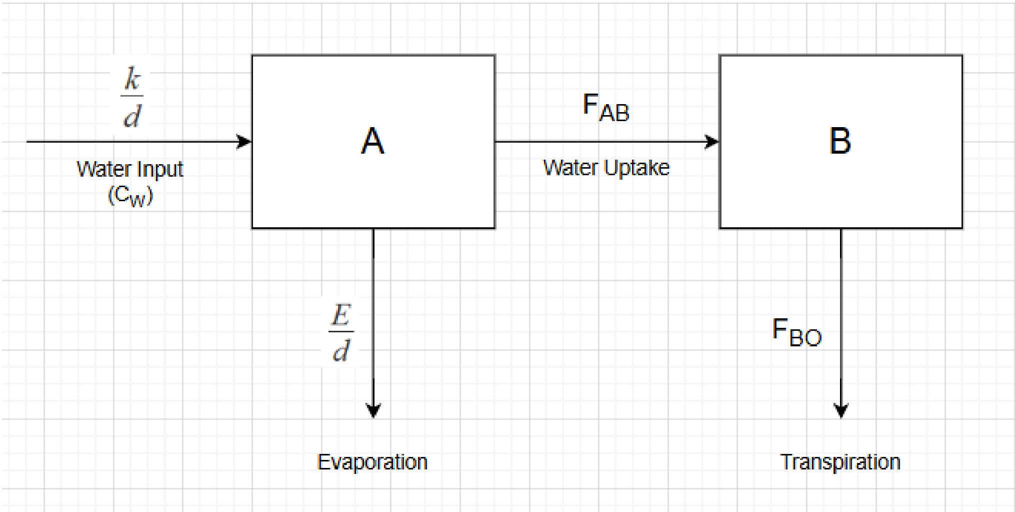
$K_{AO}C_A$

$C_A = [L^3]$

The dimension for evaporation (with respect to time) is $[LT^{-1}]$. We introduce the root zone depth here to balance the dimensions.

$$K_{AO} = \frac{Rate\ of\ Evaporation\ (E)}{Root\ Zone\ Depth\ (d)}$$

Final formula:



$$\frac{dC_A}{dt} \approx \frac{k}{d} C_W - \left(\frac{F_{AB}}{3600} + \frac{E}{d}\right) C_A$$

$$\frac{dC_B}{dt} \approx \frac{F_{AB}}{3600} C_A - \frac{F_{BO}}{3600} C_B$$

I have used the proportionality symbol here instead of the equality symbol because we never know the exact relationship between the quantities.

C_W: Amount of water given to the soil

$C_A(t)$: Amount of water in the soil

$C_B(t)$: Amount of water in the plant

k: Coefficient of Permeability of the soil

d: Root Zone Depth of the chosen plant

E: Rate of Evaporation

F_{AB}: Fraction of the water absorbed from the soil compartment to the plant compartment in 1 hour

F_{BO}: Fraction of the water transpired from the plant compartment in 1 hour

Determination of values for testing our model

1. Coefficient of Permeability (k)

We will consider the study area to be that of New Delhi for this project. From [1], we get that temperature is one of the parameters that influences the value of k. For New Delhi, the following values were calculated by them (for the year 2019).

$$k\ for\ T_{max} = 20.711\ cm\ s^{-1}$$

$$k\ for\ T_{min} = 7.325\ cm\ s^{-1}$$

For the current work, we will consider the minimum value, which was recorded in December 2019.

$$k\ = 14.018\ cm\ s^{-1}$$

OR

$$k\ = 14.018\ \times 10^{-2}\ m\ s^{-1}$$

2. Rate of Evaporation (E)

According to a report published by the Indian Meteorological Department on 29th December 2023 [2], the weekly normal evaporation rate for New Delhi for the week running from 23rd to 29th December (upto 2023) was 2.2 mm/day. We will use that value for our simulation.

$$E\ =\ 2.2\ mm\ day^{-1}$$

OR

$$E\ \approx\ 2.546 \times 10^{-8}\ m\ s^{-1}$$

3. Root Zone Depth (d)

Database: West Bengal Accelerated Development of Minor Irrigation Project [3].

Considering a banana plant, the following is the value for the root zone depth.

$$d\ =\ 0.7\ m$$

4. Fraction of water absorbed by the plant (per hour)

Database*: West Bengal Accelerated Development of Minor Irrigation Project [3].*

Soil: Silt Loam (upto 2 m)

Plant considered: Banana

$$Water\ Holding\ Capacity\ (from\ database)\ =\ 185\ mm\ m^{-1}$$

$$Soil\ Depth\ (let)\ =\ 2\ m$$

$$\therefore\ Water\ held\ by\ the\ soil\ =\ Water\ Holding\ Capacity \times Soil\ Depth$$

$$=\ 185\ mm\ m^{-1} \times 2\ m$$

$$=\ 370\ mm$$

$$=\ 370 \times 10^{-3}\ m$$

In general, a banana plant needs 1 to 1.5 inches (0.0254 m and 0.0381 m) of water per week [4]. We will consider the average of these edge values.

$$Water\ absorbed\ by\ the\ plant\ =\ 0.0318\ m\ week^{-1}$$

$$=\ \frac{0.0318}{7 \times 24}\ m\ hr^{-1}$$

$$=\ 0.000189\ m\ hr^{-1}$$

$$\therefore\ F_{AB}\ =\ \frac{0.000189\ m\ hr^{-1}}{370 \times 10^{-3}\ m}$$

$$=\ 5.108 \times 10^{-4}\ hr^{-1}$$

5. Fraction of water transpired (per hour)

Database: *West Bengal Accelerated Development of Minor Irrigation Project [3].*

In general, 90% of a plant's weight consists of water [5], [6].

$$Weight\ of\ a\ banana\ plant\ (let)\ =\ 65\ kg$$

$$Water\ lost\ by\ an\ irrigated\ banana\ plant\ =\ 7\ kg\ day^{-1}$$ [7]

$$=\ 0.292\ kg\ hr^{-1}$$

$$\therefore\ F_{BO}\ =\ \frac{0.292\ kg\ hr^{-1}}{65\ kg}$$

$$=\ 4.492 \times 10^{-3}\ hr^{-1}$$

Simulation

The pseudostem of a banana plant can hold about 15 to 25 L of water, we'll take it to be around $20 \times 10^{-3}\ m^3$.

For each of our simulations, the quantities taken are in SI units. The simulations have been performed over a very long period of time, that is $(t = 0)$ *to* $(t = 7.2 \times 10^7)$ *s, or 833 days. By this point, the graphs* ***almost*** *become asymptotic.*

Disclaimer: *The results achieved here are very much theoretical in nature, as per our observations and thought process.*

Enhancing the Equations

Four scaling factors, A, B, C and D have been introduced to the two equations. Our enhanced equations are:

$$\frac{dC_A}{dt} = A\frac{k}{d}C_W - B\left(\frac{F_{AB}}{3600} + \frac{E}{d}\right)C_A$$

$$\frac{dC_B}{dt} = C\frac{F_{AB}}{3600}C_A - D\frac{F_{BO}}{3600}C_B$$

Simulation 1

The soil compartment initially has 50 L of water, and the plant body is saturated with 20 L of water in it. 50 L of water is poured over the soil. The plant being already full, should not be able to take in any additional water in it, so the graph for $C_B(t)$ should peak at 20 L.

In the simulation below, the following scaling values have been taken.

$A = 1.16 \times 10^{-6}$

$B = 0.5$

$C = 0.6728$

$D = 0.498$

$C_A(t)$ peaks at around 130 L and $C_B(t)$ peaks at 20 L (approximately). Despite there being this much supply of water, the amount of water in the plant temporarily falls. If the soil compartment can hold this much water, then this is a valid model. *There will always be other combinations of the scaling factors which will satisfy this model with the same initial conditions.*

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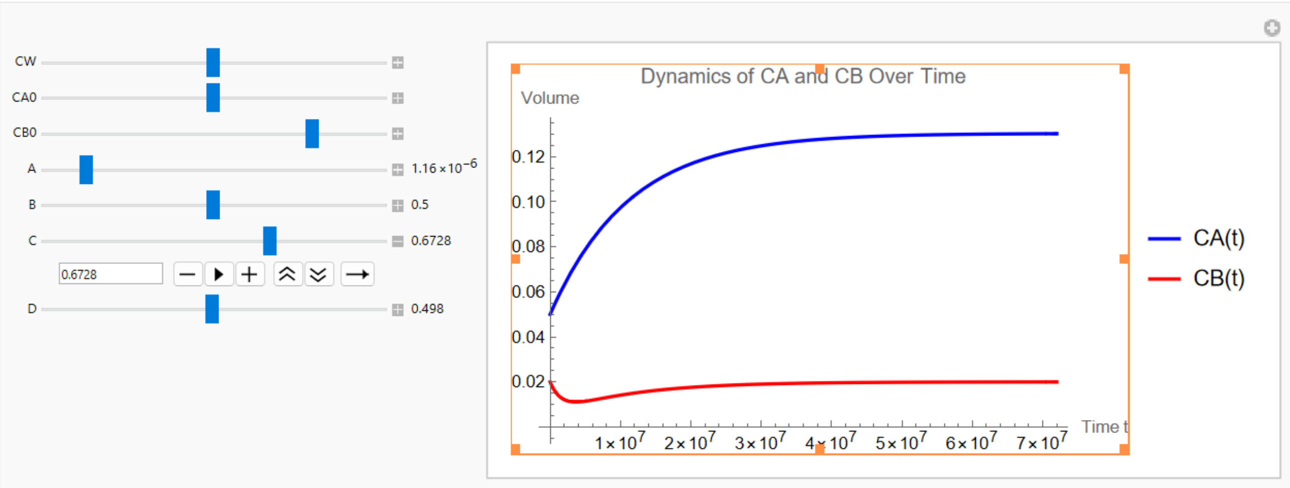

Simulation 2

The soil compartment initially has 0 L of water, that is, it is dry, while the plant body is saturated with 20 L of water in it. 50 L of water is poured over the soil. The plant being already full, should not be able to take in any additional water in it, so the graph for $C_B(t)$ should peak at 20 L.

In the simulation below, the following scaling values have been taken.

$A = 1.57 \times 10^{-6}$

$B = 0.5$

$C = 0.5$

$D = 0.5$

$C_A(t)$ peaks at around 176 L and $C_B(t)$ peaks at 20 L. Initially, because of the lack of water in the soil, the amount of water in the plant drops, before picking up again. If the soil compartment can hold this much water, then this is a valid model.

```
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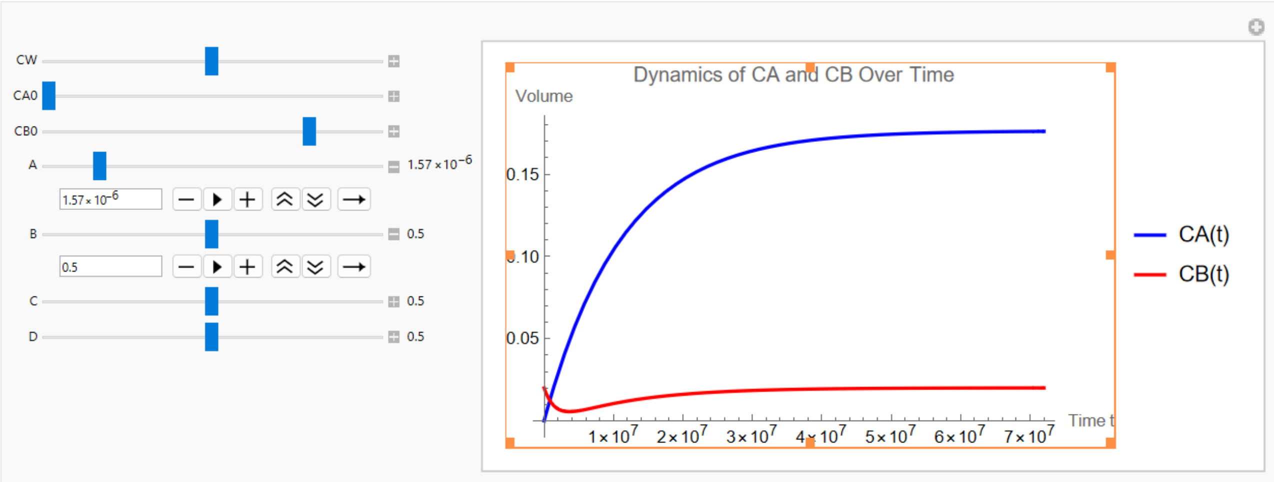

Simulation 3

The soil compartment initially has 50 L of water, and the plant body has 0 L of water in it. *Is it a dead plant? Let's assume it's not.* 50 L of water is poured over the soil. The graph for $C_B(t)$ should peak at 20 L.

For this one, let's also play with C and D.

$A = 9.8 \times 10^{-7}$

$B = 0.5$

$C = 1$

$D = 0.624$

$C_A(t)$ peaks at around 110 L and $C_B(t)$ peaks at 20 L. *The plant is not dead after all.* If the soil compartment can hold this much water, then this is a valid model.

```
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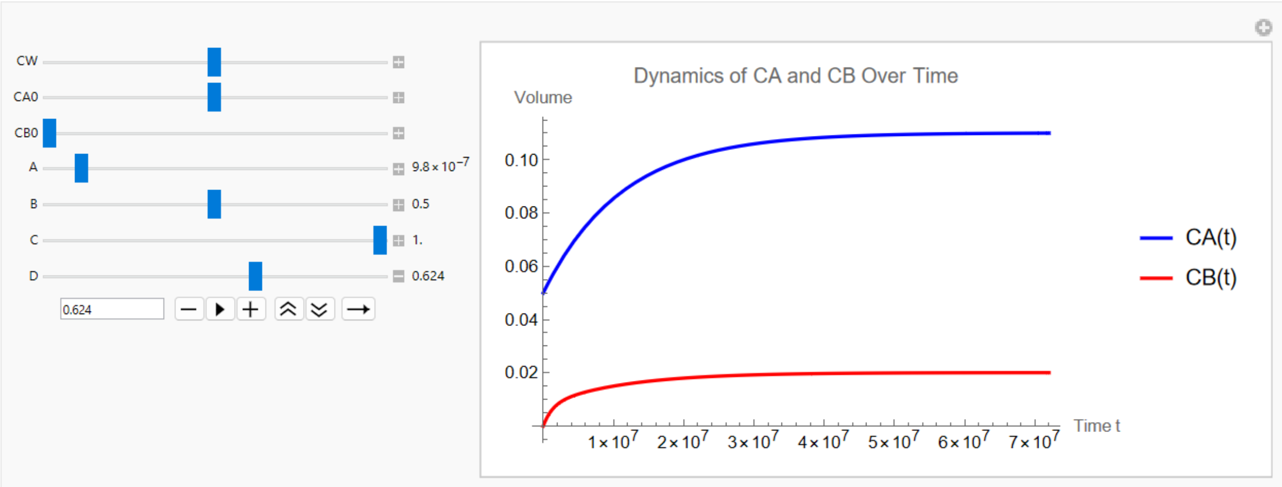

Simulation 4

The soil compartment initially has 0 L of water, and the plant body has 0 L of water in it. 50 L of water is poured over the soil. The graph for $C_B(t)$ should peak at 20 L, if it, the dry plant, somehow manages to survive in dry soil during a drought.

In the simulation below, the following scaling values have been taken.

$A = 6.6 \times 10^{-7}$

$B = 0.5$

$C = 1$

$D = 0.42$

$C_A(t)$ peaks at around 74 L and $C_B(t)$ peaks at 20 L (approximately). Congratulations! Our plant survived.

```
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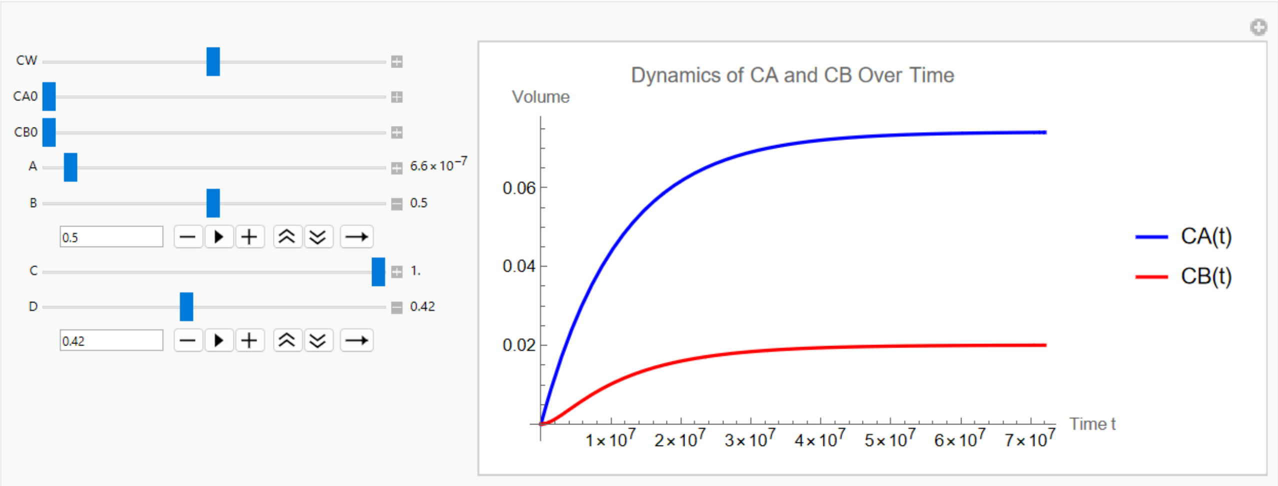

Simulation 5

Will our plant survive in a drought with no irrigation, nothing? NAH!

```
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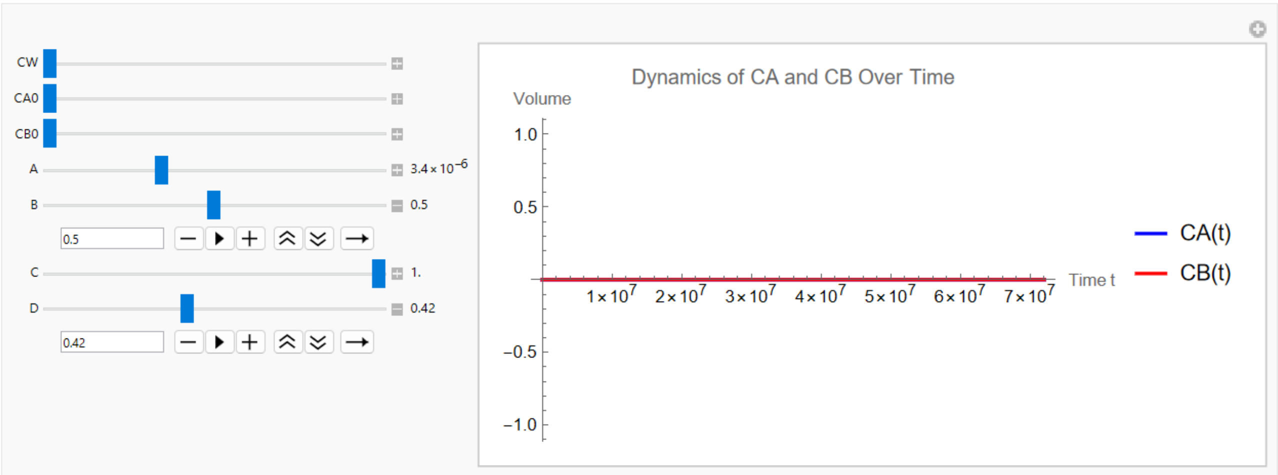

Future Steps

You see, for now, we do not see this exact model fit into any real-life scenario, nor do I have any data to test the same. However, this effort might lead to some applications in the future under two circumstances:

1. Proper guidance is received.
2. If this is actually supposed to work.

Until then, thanks for reading. *Signing off*.

Acknowledgement

We would like to thank our professor, Prof. Shobha Bagai for all the support, encouragement and guidance we have received from her during this project.

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