Examination	:	End Semester Examination – Nov/Dec 2024
Name of the Course	:	B.Tech. (IT & Mathematical Innovations)
Name of the Paper	:	Modelling Continuous Changes through Ordinary Differential Equations
Unique Paper Code	:	3122612301
Semester	:	III
Duration	:	3 hours
Maximum Marks	:	90

<u>*Instruction to students:*</u>

Attempt any ***<u>six</u>*** *questions.* ***<u>Question number 1 is compulsory.</u>*** *Each question carries equal marks.*

Q1) Fill in the blanks (1.5 x 10 = 15 marks)

a) The integrating factor of the differential equation $\frac{dy}{dx}+\frac{x}{1-x}y=1-x$ is ___________

b) Solution of the differential equation $\ln\left(\frac{dy}{dx}\right)=ax+by$ is ______________

c) The differential equation whose solution is given by $y(x) = (A + Bx + Cx^2)e^{3x}$ is ____________

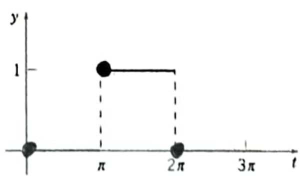
d) The Laplace inverse transform of $\frac{1}{s^2+2s}$ is ______________

e) A particle, initially at the origin moves along x-axis according to the rule $\frac{dy}{dt}=y+3$, where y is the displacement. The time taken by the particle to traverse a distance of 45 units is____________

f) The complimentary solution of the differential equation $\frac{d^2y}{dx^2}+a^2y=0$ is _________

g) The Laplace transformation of $f(t) = t^2 \sinh at$ is ______________

h) In terms of the step function, the following graph can be represented as _________



i) The value of a for which the equation $(ye^{2xy} + x) + bxe^{2xy}y' = 0$ is exact is _________

j) The solution of the differential equation $x^2y'' + xy' + y = 0$ is ______________

Q2) (a) Find the complete solution of the second order differential equation

$$x^2y'' + xy' - 25y = x^5 \, , y(1) = 3, y'(1) = 5$$

(b) Show that the equation $(y + x(x^2 + y^2)^2)dx + (y(x^2 + y^2)^2 + x)dy = 0$ is exact and hence solve it. (10 + 5 = 15 marks)

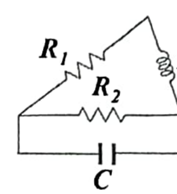
Q3) A shallow reservoir has a one square kilometer surface and an average water depth of 2 meters. Initially it is filled with fresh water but at time $t = 0$ water contaminated with a liquid begins flowing into the reservoir at the rate of 200 thousand cubic meters per month. The well mixed water in the reservoir flows out at the same rate.

a. Find the amount of pollutant in the reservoir at any time t.
b. If the incoming water has a pollutant concentration of 10 L/m^3, sketch a rough graph of pollutant concentration. From the graph determine the long term pollutant content in the reservoir.
c. Find the time that it takes the pollutant concentration in the reservoir to reach 5 L/m^3.

(15 marks)

Q4) Consider the circuit shown in the Figure. The current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$L\frac{dI}{dt} = -R_1 I - V, \quad C\frac{dV}{dt} = I - \frac{V}{R_2}$$



a. Write the system of equations as a single second order differential equation
b. Write the equation as X' = AX. Find a condition on R_1, R_2, C, and L that must be satisfied if the eigenvalues of the coefficient matrix are to be real and different. If the condition is satisfied, show that both eigenvalues are negative.
c. Find the general solution of the system if $R_1 = 1$, $R_2 = 3/5$, $L = 2$ and $C = 2/3$ units. Show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$, regardless of the initial values $I(0)$ and $V(0)$.

(15 marks)

Q5) A population of sterile rabbits $X(t)$ is preyed upon by a population of foxes $Y(t)$. A model for this population interaction is the pair of differential equations

$$\frac{dX}{dt} = -aXY, \quad \frac{dY}{dt} = bXY - cY$$

where a, b and c are positive constants.

(a) Eliminate t to obtain a relationship between the population of foxes and rabbits.
(b) Give a rough sketch of the phase plane trajectories, indicating the direction of movement along the trajectories. Justify your sketch.
(c) According to the model, is it possible for the foxes to completely wipe out the rabbit population? Give reasons. (15 marks)

Q6) Consider the harvesting model

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - hX$$

where r is the growth rate, K is the carrying capacity of the population and h is the rate at which the population is harvested.
Find the population at any time t. Find the non-zero equilibrium populations and the critical harvesting rate at which the population becomes extinct. (15 marks)

Q7) Use the method of series solution to the equation $xy'' + 2y' + xy = 0$. Find the indicial equation. Show that for the smaller root of the indicial equation the solution can be written as $y = c_1 \frac{\cos x}{x} + c_2 \frac{\sin x}{x}$ (15 marks)

Q8) a) Plot the graph for $u(t) + u(t-1) + u(t-2$. Show that $\mathcal{L}\{\sum_{n=0}^{\infty} u(t-n)\} = \frac{e}{s(e-1)}$, where $\mathcal{L}\{f(t)\}$ represents the Laplace transformation of the function and the function

$$u(t-n) = \begin{cases} 0 \text{ for } t < n \\ 1 \text{ for } t \geq n \end{cases}$$

is the unit step function.

b) Use Laplace transformations to solve the following differential equation
$y'' - 2y' + 2y = \cos t; \ y(0) = 1, y'(0) = 0$ (7½ + 7½ = 15 marks)

Table of Laplace Transforms

$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. $\mathbf{e}^{at}$	$\frac{1}{s-a}$
3. $t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4. $t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6. $t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t\sin(at)$	$\frac{2as}{\left(s^2+a^2\right)^2}$	10. $t\cos(at)$	$\frac{s^2-a^2}{\left(s^2+a^2\right)^2}$
11. $\sin(at)-at\cos(at)$	$\frac{2a^3}{\left(s^2+a^2\right)^2}$	12. $\sin(at)+at\cos(at)$	$\frac{2as^2}{\left(s^2+a^2\right)^2}$
13. $\cos(at)-at\sin(at)$	$\frac{s\left(s^2-a^2\right)}{\left(s^2+a^2\right)^2}$	14. $\cos(at)+at\sin(at)$	$\frac{s\left(s^2+3a^2\right)}{\left(s^2+a^2\right)^2}$
15. $\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17. $\sinh(at)$	$\frac{a}{s^2-a^2}$	18. $\cosh(at)$	$\frac{s}{s^2-a^2}$
19. $\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20. $\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21. $\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22. $\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23. $t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25. $u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26. $\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27. $u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28. $u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29. $\mathbf{e}^{ct}f(t)$	$F(s-c)$	30. $t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31. $\frac{1}{t}f(t)$	$\int_s^{\infty}F(u)\,du$	32. $\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34. $f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35. $f'(t)$	$sF(s)-f(0)$	36. $f''(t)$	$s^2F(s)-sf(0)-f'(0)$

37. $f^{(n)}(t)$ $\qquad s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$