quantum information theory
problem sheet 13

to be submitted on Moodle before 09:00 on 02.02.2026

Note: Quantum error correction is examinable. However, submission of this problem sheet is optional, and the points will taken into consideration only if it improves your overall average grade.

Problem 1: Introduction to quantum error correction (2+1+1+2+2+1+2+2+2 points)

1. The circuit for encoding the three-qubit bit-flip code is shown in Figure 7.3 in the lecture notes (quantum error correction part 1). Verify that this circuit indeed implements such an encoding.

2. Argue why the three-qubit bit-flip code does *not* protect against phase-flip errors.

3. We can analyse the performance of an error-correcting code by using the fidelity as a figure of merit. Consider a qubit in a pure state $|\psi\rangle$ undergoing the bit-flip error channel given by

$$\mathcal{E}_1(|\psi\rangle) = (1-p)\,|\psi\rangle\langle\psi| + pX\,|\psi\rangle\langle\psi|\,X, \tag{1}$$

where p is the probability of a single bit-flip error.

(a) What is the minimum fidelity of the error-prone qubit to the initial state, $F_{\min}(|\psi\rangle, \mathcal{E}_1(|\psi\rangle))$?

Suppose that the three-qubit bit-flip code is now implemented. Since this code can only correct up to single-qubit errors, the state after the error channel and the error-correction procedure will be

$$\mathcal{E}_3(|\psi\rangle) = [(1-p)^3 + 3p(1-p)^2]\,|\psi\rangle\langle\psi| + \dots, \tag{2}$$

where we omit terms with two or three bit-flip errors (since these cannot be corrected to recover the original state $|\psi\rangle$).

(b) Derive a lower bound on the fidelity of the error-corrected state $F(|\psi\rangle, \mathcal{E}_3(|\psi\rangle))$. *Hint: make use of the positivity of quantum operations: for $\mathcal{E}(\rho) = \sum_i E_i\rho E_i^\dagger$, we have that $\langle\psi|\sum_i E_i\rho E_i^\dagger|\psi\rangle \geq \langle\psi|E_j\rho E_j^\dagger|\psi\rangle$ for all j.*

(c) What bound must the single-qubit error probability p satisfy in order to ensure that error correction improves the overall fidelity of the state?

4. The Shor code is a code that can correct both bit- and phase-flip errors.

   (a) What is the number of physical qubits, logical qubits and the distance of the Shor code, $[[n, k, d]]$?

   (b) Show that the syndrome measurement for phase-flip errors corresponds to the following observables:

$$X_1X_2X_3X_4X_5X_6 \quad \& \quad X_4X_5X_6X_7X_8X_9. \tag{3}$$

   (c) Suppose qubit i undergoes an error Y_i. Describe briefly (in a few words or lines of equation) how this error could be detected and corrected.

   (d) Suppose the physical qubits are prone to errors described by the depolarizing channel given by

$$\mathcal{E}(\rho) = (1-p)\rho + p\frac{\mathbb{1}}{2}, \tag{4}$$

   where p is the probability that the qubit is mapped to the maximally mixed state. Show that the Shor code can still protect against this type of error by discretizing the error channel. Assume that the error probability for two or more qubits can be neglected.