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$\boxed{y_n = f_n(y_{n-1}w_n + b_n) \quad \text{第n层}}$

$f_{n,i}(t_1, t_2, \cdots t_m)$

① $\frac{\partial y_{n,i}}{\partial b_n} = f'_{n,i} \Rightarrow \left(\frac{\partial f_{n,i}}{\partial t_1}, \frac{\partial f_{n,i}}{\partial t_2}, \cdots, \frac{\partial f_{n,i}}{\partial t_m}\right)$

② $\frac{\partial y_{n,i}}{\partial w_n} = y_{n-1}^T f'_{n,i}$

$$f'_n = \begin{pmatrix} f'_{n,1} \\ f'_{n,2} \\ \vdots \\ f'_{n,m} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_{n,1}}{\partial t_1}, & \frac{\partial f_{n,1}}{\partial t_2}, & \cdots & \frac{\partial f_{n,1}}{\partial t_m} \\ \frac{\partial f_{n,2}}{\partial t_1}, & \frac{\partial f_{n,2}}{\partial t_2}, & \cdots & \frac{\partial f_{n,2}}{\partial t_m} \\ \vdots & & & \\ \frac{\partial f_{n,m}}{\partial t_1} & \frac{\partial f_{n,m}}{\partial t_2} & \cdots & \frac{\partial f_{n,m}}{\partial t_m} \end{pmatrix}$$

③ $\frac{\partial y_{n,i}}{\partial b_{n-1}} = f'_{n,i}\, w_n^T \cdot f'_{n-1}$

④ $\frac{\partial y_{n,i}}{\partial w_{n-1}} = y_{n-2}^T\, f'_{n,i}\, w_n^T \cdot f'_{n-1}$

⑤ $\frac{\partial y_{n,i}}{\partial b_{n-2}} = f'_{n,i}\, w_n^T f'_{n-1} \cdot w_{n-1}^T \cdot f'_{n-2}$ ☆☆☆

⑥ $\frac{\partial y_{n,i}}{\partial w_{n-2}} = y_{n-3}^T \cdot f'_{n,i}\, w_n^T f'_{n-1} \cdot w_{n-1}^T \cdot f'_{n-2}$

$\vdots$

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若 $J = \sum_{i=1}^{10}(y_{n,i} - y_i)^2$ 则 ① $\frac{\partial J}{\partial b_n} = \sum_{i=1}^{10} 2(y_{n,i} - y_i)\frac{\partial y_{n,i}}{\partial b_n} = \sum_{i=1}^{10} 2(y_{n,i} - y_i) \cdot f'_{n,i}$

$= 2(y_n - y) \cdot f'_n$

② $\frac{\partial J}{\partial w_n} = \sum_{i=1}^{10} 2(y_{n,i} - y_i)\frac{\partial y_{n,i}}{\partial w_n} = \sum_{i=1}^{10} 2(y_{n,i} - y_i) \cdot y_{n-1}^T \cdot f'_{n,i}$

$= y_{n-1}^T \cdot \sum_{i=1}^{10} 2(y_{n,i} - y_i) f'_{n,i} = y_{n-1}^T \cdot 2(y_n - y) \cdot f'_n$

类似的

③ $\frac{\partial J}{\partial b_{n-1}} = 2(y_n - y) \cdot f'_n \cdot w_n^T \cdot f'_{n-1}$

④ $\frac{\partial J}{\partial w_{n-1}} = y_{n-2}^T\, 2(y_n - y) \cdot f'_n w_n^T f'_{n-1}$

$\ddots$

$y_n = f_n(y_{n-1} \cdot w_n + b_n)$. 求 $\frac{\partial y_{n,i}}{\partial b_n}$. 假设 $f_{n,i}(t_1, t_2, \dots, t_m)$

① $$\frac{\partial y_{n,i}}{\partial b_{n,p}} = \sum_{k=1}^{m} \frac{\partial f_{n,i}}{\partial t_k} \frac{\partial b_{n,k}}{\partial b_{n,p}} = \frac{\partial f_{n,i}}{\partial t_p}$$

② $$\frac{\partial y_{n,i}}{\partial b_n} = \left( \frac{\partial f_{n,i}}{\partial t_1}, \frac{\partial f_{n,i}}{\partial t_2}, \dots, \frac{\partial f_{n,i}}{\partial t_m} \right) \overset{定义为}{=\!=} f_{n,i}' \quad (这是行向量).$$

$y_n = f_n(y_{n-1} \cdot w_n + b_n)$. 求 $\frac{\partial y_{n,i}}{\partial w_n}$ 假设 $f_{n,i}(t_1, t_2, \cdots, t_m)$

$w_n : r \times m$

$w_{n,k}$ 代表 $w_n$ 的第k列.

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① $$\frac{\partial y_{n,i}}{\partial w_{n,p,q}} = \sum_{k=1}^{m} \frac{\partial f_{n,i}}{\partial t_k} \cdot \frac{\partial (y_{n-1} \cdot w_n)_k}{\partial w_{n,p,q}} = \sum_{k=1}^{m} \frac{\partial f_{n,i}}{\partial t_k} \cdot y_{n-1} \cdot \frac{\partial w_{n,k}}{\partial w_{n,p,q}}$$

$$= \frac{\partial f_{n,i}}{\partial t_q} \cdot y_{n-1,p}$$

② $$\frac{\partial y_{n,i}}{\partial w_n} = \begin{pmatrix} \frac{\partial f_{n,i}}{\partial t_1} y_{n-1,1} & \frac{\partial f_{n,i}}{\partial t_2} \cdot y_{n-1,1} & \cdots & \frac{\partial f_{n,i}}{\partial t_m} y_{n-1,1} \\ \frac{\partial f_{n,i}}{\partial t_1} y_{n-1,2} & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_{n,i}}{\partial t_1} y_{n-1,r} & \cdots & \cdots & \frac{\partial f_{n,i}}{\partial t_m} \cdot y_{n-1,r} \end{pmatrix}$$

$$= \begin{pmatrix} y_{n-1,1} \\ y_{n-1,2} \\ \vdots \\ y_{n-1,r} \end{pmatrix} \left( \frac{\partial f_{n,i}}{\partial t_1}, \frac{\partial f_{n,i}}{\partial t_2}, \quad , \quad \frac{\partial f_{n,i}}{\partial t_m} \right)$$

$$= y_{n-1}^T \cdot f_{n,i}'$$

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默认 行向量求导是行向量

列向量求导是列向量

$y_n$, $y_{n-1}$, $b_n$ 都是行向量

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$y_n = f_n(y_{n-1}\cdot w_n + b_n)$ 　　求 $\dfrac{\partial y_{n,i}}{\partial b_{n-1}}$ 　　假设 $f_{n,i}(t_1, t_2, \cdots, t_m)$

$f_{n-1,i}(z_1, z_2, \cdots, z_r)$

---

列向量. $w_n$ 的第k列

$$\frac{\partial y_{n,i}}{\partial b_{n-1,p}} = \sum_{k=1}^{m} \frac{\partial f_{n,i}}{\partial t_k}\left(\frac{\partial y_{n-1}}{\partial b_{n-1,p}}\right)\cdot w_{n,k}$$

$$= \sum_{k=1}^{m} \frac{\partial f_{n,i}}{\partial t_k}\cdot \underbrace{\left(\frac{\partial y_{n-1,1}}{\partial b_{n-1,p}}, \frac{\partial y_{n-1,2}}{\partial b_{n-1,p}}, \cdots \frac{\partial y_{n-1,r}}{\partial b_{n-1,p}}\right)}\cdot w_{n,k}$$

$$\left(\frac{\partial f_{n-1,1}}{\partial z_p}, \frac{\partial f_{n-1,2}}{\partial z_p}, \cdots, \frac{\partial f_{n-1,r}}{\partial z_p}\right) \overset{\text{定义为}}{:=} f_{n-1}^{[p]}$$

$$= \sum_{k=1}^{m} \frac{\partial f_{n,i}}{\partial t_k}\cdot\left(f_{n-1}^{[p]}\cdot w_{n,k}\right)$$

$$= f_{n-1}^{[p]}\cdot w_n \cdot f_{n,i}'^{T} \quad \overset{\text{或者}}{=} \quad f_{n,i}' w_n^T \cdot f_{n-1}^{[p]^T}$$

$$\frac{\partial y_{n,i}}{\partial b_{n-1}} = \left[f_{n,i}' w_n^T f_{n-1}^{[1]^T},\ f_{n,i}' w_n^T \cdot f_{n-1}^{[2]^T},\ \cdots,\ f_{n,i}' w_n^T \cdot f_{n-1}^{[r]^T}\right]$$

$$= f_{n,i}' w_n^T \cdot \left[f_{n-1}^{[1]^T}, f_{n-1}^{[2]^T}, \cdots f_{n-1}^{[r]^T}\right]$$

$$= f_{n,i}' w_n^T \cdot f_{n-1}'$$

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$y_n = f_n(y_{n-1} \cdot w_n + b_n)$. 求 $\frac{\partial y_{n,i}}{\partial w_{n-1}}$. 假设 $f_{n,i}(t_1, t_2, \cdots, t_m)$

$f_{n-1,i}(z_1, z_2, \cdots, z_r)$

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$$\frac{\partial y_{n,i}}{\partial w_{n-1,p,q}} = \sum_{k=1}^{m} \frac{\partial f_{n,i}}{\partial t_k} \underbrace{\left(\frac{\partial y_{n-1}}{\partial w_{n-1,p,q}}\right) \cdot w_{n,k}}$$ ↙ 列向量

$$= \frac{\partial y_{n-1}}{\partial w_{n-1,p,q}} \cdot w_n \cdot f_{n,i}'^{T} = f_{n,i}' \cdot w_n^T \cdot \left(\frac{\partial y_{n-1}}{\partial w_{n-1,p,q}}\right)^T$$

$$= f_{n,i}' w_n^T \cdot \begin{pmatrix} \frac{\partial y_{n-1,1}}{\partial w_{n-1,p,q}} \\ \frac{\partial y_{n-1,2}}{\partial w_{n-1,p,q}} \\ \vdots \\ \frac{\partial y_{n-1,r}}{\partial w_{n-1,p,q}} \end{pmatrix} = f_{n,i}' \cdot w_n^T \cdot \begin{pmatrix} \frac{\partial f_{n-1,1}}{\partial z_q} \cdot y_{n-2,p} \\ \frac{\partial f_{n-1,2}}{\partial z_q} \cdot y_{n-2,p} \\ \vdots \\ \frac{\partial f_{n-1,r}}{\partial z_q} \cdot y_{n-2,p} \end{pmatrix}$$

$$= \left(f_{n,i}' w_n^T \cdot f_{n-1}^{[q]T}\right) \cdot y_{n-2,p}$$

$$\frac{\partial y_{n,i}}{\partial w_{n-1}} = y_{n-2}^T \left[ f_{n,i}' w_n^T \cdot f_{n-1}^{[1]T},\ f_{n,i}' \cdot w_n^T \cdot f_{n-1}^{[2]T},\ \cdots,\ f_{n,i}' \cdot w_n^T \cdot f_{n-1}^{[r]T} \right]$$

$$= y_{n-2}^T f_{n,i}' w_n^T \cdot f_{n-1}'$$

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$y_n = f_n(y_{n-1} \cdot w_n + b_n)$. 求 $\dfrac{\partial y_{n,i}}{\partial b_{n-2}}$. 假设 $f_{n,i}(t_1, t_2, \cdots, t_m)$

$f_{n-1,i}(z_1, z_2, \cdots, z_r)$

$f_{n-2,i}(s_1, s_2, \cdots, s_d)$.

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列向量

$$\frac{\partial y_{n,i}}{\partial b_{n-2,p}} = \sum_{k=1}^{m} \frac{\partial f_{n,i}}{\partial t_k} \cdot \left(\frac{\partial y_{n-1}}{\partial b_{n-2,p}}\right) \cdot w_{n,k}.$$

$$= \frac{\partial y_{n-1}}{\partial b_{n-2,p}} \cdot w_n \cdot f_{n,i}'^{T} = f_{n,i}' \cdot w_n^T \left(\frac{\partial y_{n-1}}{\partial b_{n-2,p}}\right)^T$$

$$= f_{n,i}' \cdot w_n^T \cdot \begin{pmatrix} \frac{\partial y_{n-1,1}}{\partial b_{n-2,p}} \\ \frac{\partial y_{n-1,2}}{\partial b_{n-2,p}} \\ \vdots \\ \frac{\partial y_{n-1,r}}{\partial b_{n-2,p}} \end{pmatrix} = f_{n,i}' \, w_n^T \begin{pmatrix} f_{n-1,1}' \, w_{n-1}^T \cdot f_{n-2}^{[p]T} \\ f_{n-1,2}' \, w_{n-1}^T \, f_{n-2}^{[p]T} \\ \vdots \\ f_{n-1,r}' \, w_{n-1}^T \cdot f_{n-2}^{[p]T} \end{pmatrix}$$

$$= f_{n,i}' \, w_n^T \cdot f_{n-1}' \, w_{n-1}^T \cdot f_{n-2}^{[p]T}.$$

$$\frac{\partial y_{n,i}}{\partial b_{n-2}} = f_{n,i}' \, w_n^T \cdot f_{n-1}' \cdot w_{n-1}^T \left[ f_{n-2}^{[1]T}, f_{n-2}^{[2]T}, \cdots, f_{n-2}^{[d]T} \right]$$

$$= f_{n,i}' \, w_n^T \cdot f_{n-1}' \, w_{n-1}^T \cdot f_{n-2}'.$$

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由此递推易知

$$\frac{\partial y_{n,i}}{\partial b_{n-3}} = \frac{\partial y_{n,i}}{\partial b_{n-2}} \cdot w_{n-2}^T \cdot f_{n-3}'.$$

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$y_n = f_n(y_{n-1} \cdot w_n + b_n)$ 求 $\frac{\partial y_{n,i}}{\partial w_{n-2}}$ 假设 $f_{n,i}(t_1, t_2, \ldots, t_m)$
$f_{n-1,i}(z_1, z_2, \ldots, z_r)$
$f_{n-2,i}(s_1, s_2, \ldots, s_d)$

$$\frac{\partial y_{n,i}}{\partial w_{n-2,p,q}} = \sum_{k=1}^{m} \frac{\partial f_{n,i}}{\partial t_k} \left(\frac{\partial y_{n-1}}{\partial w_{n-2,p,q}}\right) \cdot w_{n,k}.$$

$$= \frac{\partial y_{n-1}}{\partial w_{n-2,p,q}} \cdot w_n \cdot f_{n,i}'^{T} = f_{n,i}' \cdot w_n^T \cdot \left(\frac{\partial y_{n-1}}{\partial w_{n-2,p,q}}\right)^T$$

$$= f_{n,i}' \cdot w_n^T \cdot \begin{pmatrix} \frac{\partial y_{n-1,1}}{\partial w_{n-2,p,q}} \\ \frac{\partial y_{n-1,2}}{\partial w_{n-2,p,q}} \\ \vdots \\ \frac{\partial y_{n-1,r}}{\partial w_{n-2,p,q}} \end{pmatrix} = f_{n,i}' \cdot w_n^T \cdot \begin{pmatrix} (f_{n-1,1}' \cdot w_{n-1}^T \cdot f_{n-2}^{[q]T}) \cdot y_{n-3,p} \\ (f_{n-1,2}' \cdot w_{n-1}^T \cdot f_{n-2}^{[q]T}) \cdot y_{n-3,p} \\ \vdots \\ (f_{n-1,r}' \cdot w_{n-1}^T \cdot f_{n-2}^{[q]T}) \cdot y_{n-3,p} \end{pmatrix}$$

$$= \left(f_{n,i}' w_n^T \cdot f_{n-1}' \cdot w_{n-1}^T \cdot f_{n-2}^{[q]T}\right) \cdot y_{n-3,p}.$$

$$\frac{\partial y_{n,i}}{\partial w_{n-2}} = y_{n-3}^T \cdot f_{n,i}' \cdot w_n^T \cdot f_{n-1}' \cdot w_{n-1}^T \cdot f_{n-2}'.$$

故由递推关系易知 $\frac{\partial y_{n,i}}{\partial w_{n-3}} = y_{n-4}^T \cdot \frac{\partial y_{n,i}}{\partial b_{n-3}}.$