1 Two periods Europian put option pricing

Risk neutral probability q

$$q = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05*\frac{1}{2}} - 0.8}{1.2 - 0.8} = \frac{0.2253}{0.4} = 0.5633$$

Find f_u and f_d

$$\begin{aligned} f_u &= e^{-rT}[q \times \max(K - S_0u^2, 0) + (1-q)\max(K - S_0ud, 0)] \\ &= e^{-0.05*\frac{1}{2}}[0.5633 \times 0 + 0.4367 \times 4] = 0.9753(0.4367 \times 4) = 1.704 \end{aligned}$$

$$\begin{aligned} f_d &= e^{-rT}[q \times \max(K - S_0du, 0) + (1-q)\max(K - S_0d^2, 0)] \\ &= e^{-0.05*\frac{1}{2}}[0.5633 \times 4 + 0.4367 \times 20] = 0.9753(0.5633 \times 4 + 0.4367 \times 20) = 10.72 \end{aligned}$$

Find f

$$\begin{aligned} f &= e^{-rT}[q \times \max(K - S_0u, 0) + (1-q)\max(K - S_0d, 0)] \\ &= e^{-0.05*\frac{1}{2}}[0.5633 \times 1.704 + 0.4367 \times 10.72] \\ &= 0.9753(0.5633 \times 1.704 + 0.4367 \times 10.72) = 5.5 \end{aligned}$$

2 Two periods American put option pricing

In American put option, we need to consider whether the investor will exercise the put option before the expire day.

1. When S_0u, the value of early exercise will be 0, smaller then the value of waiting for expire day which = 1.704
2. When S_od, the value of early exercise will be 12, larger then the value of waiting for expire day which = 10.72

So, the price of put option f is

$$f = 0.9753(0.5633 \times 1.704 + 0.4367 \times 12) = 6.05$$

3 The price of Europian call option when $\Delta t = \frac{T}{N}$

$$\begin{aligned} f_u &= e^{-r\Delta t}[q \times f_{uu} + (1-q) \times f_{ud}] \ , \ f_d = e^{-r\Delta t}[q \times f_{du} + (1-q) \times f_{dd}] \\ f &= e^{-r\Delta t}[q \times f_u + (1-q) \times f_d] \\ &= e^{-2r\Delta t}[q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}] \end{aligned}$$

From the above formula, we observed that it is following the Binominal Distribution. So the generalized formula when $\Delta t = \frac{T}{N}$ is

$$\begin{aligned} f &= e^{-Nr\Delta t}\Big[C_N^N q^N (1-q)^0 f_{C_N^N u^N d^0} + C_{N-1}^N q^{N-1}(1-q)^1 f_{C_{N-1}^N u^{N-1} d^1} + \cdots + C_0^N q^0 (1-q)^N f_{C_0^N u^0 d^N}\Big] \\ &= e^{-Nr\Delta t}\Big[\sum_{j=0}^{N} C_j^N p^j (1-p)^{N-j} \max\big[0, u^j d^{N-j} S - K\big]\Big] \end{aligned}$$

Let a stand for the minimum number of upward moves that the stock must make over the next n periods for the call to finish in-the-money.

$\forall\ j < a$,

$$\max\big[0, u^j d^{N-j} S - K\big] = 0$$

$\forall\ j \geq a$,

$$\max\big[0, u^j d^{N-j} S - K\big] = u^j d^{N-j} S - K$$

So,

$$f = e^{-Nr\Delta t}\left[\sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j} \big[u^j d^{N-j} S - K\big]\right]$$

4 Issues from *Option Pricing-A Simplified Approach*

$$
\begin{aligned}
f &= e^{-Nr\Delta t}\left[\sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j}\left[u^j d^{N-j} S - K\right]\right] \\
&= S\left\{\sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j}\left[\frac{u^j d^{N-j}}{e^{Nr\Delta t}}\right]\right\} - Ke^{-Nr\Delta t}\left[\sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j}\right]
\end{aligned}
$$

We know that

$$\sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j}$$

is a complementary Binominal Distribution. So we can denote it as

$$\phi(a; N, p)$$

We also define that

$$p' \equiv \frac{u}{e^{Nr\Delta t}} p \quad and \quad 1 - p' \equiv \frac{d}{e^{Nr\Delta t}}(1-p)$$

So

$$p^j (1-p)^{N-j}\left[\frac{u^j d^{N-j}}{e^{Nr\Delta t}}\right] = p'^j (1-p')^{N-j}$$

and

$$\sum_{j=a}^{N} C_j^N p'^j (1-p')^{N-j} = \phi(a; n, p')$$

Now, we can rewrite the full formula as

$$f = S\phi(a; n, p') - Ke^{-Nr\Delta t}\phi(a; n, p')$$

In the continuous time model, and $N \to \infty$, Binominal Distribution asymptotically approaches to the Normal Distribution. So the formula will be

$$f = SN(x) - Ke^{-Nr\Delta t}N(x)$$