Fourier Transformation I

Proof $-i\lambda F(\lambda)$ is $f'(x)$'s Fourier Transformation

Given $F(\lambda) = F_x[f(x)](\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} f(x)\, dx$

$$F_x[f'(x)](\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} f'(x)\, dx$$

According to integral by parts

$$\int u\, dv = uv - \int v\, du$$

Let $dv = f'(x)dx$, $u = e^{i\lambda x}$. So $v = f(x)$, $du = i\lambda e^{i\lambda x}\, dx$

$$F_x[f'(x)](\lambda) = f(x)e^{i\lambda x}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(i\lambda e^{i\lambda x})\, dx$$

Because $\lim_{x\to\pm\infty} f(x) = 0$

$$F_x[f'(x)](\lambda) = -i\lambda \int_{-\infty}^{\infty} f(x)e^{i\lambda x}\, dx = -i\lambda F(\lambda)$$

Fourier Transformation II

Proof $(-i\lambda)^n F(\lambda)$ is $f^{(n)}(x)$'s Fourier Transformation

Given $F(\lambda) = F_x[f(x)](\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} f(x)\, dx$

$$F_x[f^{(n)}(x)](\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} f^{(n)}(x)\, dx$$

According to integral by parts

$$\int u\, dv = uv - \int v\, du$$

Let $dv = f^{(n)}(x)dx$, $u = e^{i\lambda x}$. So $v = f^{(n-1)}(x)$, $du = i\lambda e^{i\lambda x}\, dx$

$$F_x[f^{(n)}(x)](\lambda) = f^{(n-1)}(x)e^{i\lambda x}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f^{(n-1)}(x)(i\lambda e^{i\lambda x})\,dx$$

Because $\lim_{x\to\pm\infty} f^{(n-1)}(x) = 0$

$$F_x[f^{(n)}(x)](\lambda) = -i\lambda \int_{-\infty}^{\infty} f^{(n-1)}(x)e^{i\lambda x}\,dx = -i\lambda F_x[f^{(n-1)}(x)](\lambda)$$

$$= (-i\lambda)^2 F_x[f^{(n-2)}(x)](\lambda)$$

$$= \cdots = (-i\lambda)^n F_x[f(x)](\lambda) = (-i\lambda)^n F(\lambda)$$

Fourier Inverse Transformation I

Given $F(\lambda) = e^{i\mu\lambda + \frac{i^2\lambda^2\sigma^2}{2}}$, get $f(x)$ by Fourier Inverse Transformation

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-i\lambda x} F(\lambda) d\lambda \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} e^{i\mu\lambda + \frac{i^2\lambda^2\sigma^2}{2}} d\lambda \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda(\mu - x) + \frac{i^2\lambda^2\sigma^2}{2}} d\lambda \\
&= \frac{1}{2\pi} \times \left[\frac{1}{i(\mu - x) + i^2\sigma^2\lambda} e^{i\lambda(\mu - x) + \frac{i^2\lambda^2\sigma^2}{2}} \right]_{-\infty}^{\infty} \\
&= \frac{1}{2\pi} \times [0 - 0] = 0
\end{aligned}$$