1 Binary Options

Dynamics of two assets under R_1 and R_2 measure

Given the dynamics of two assets under P measure:

$$\begin{cases} dS_{1,t} = \mu_1 S_{1,t}dt + \sigma_1 S_{1,t}dW_{1,t}^P \\ \\ S_{1,T} = S_{1,t}e^{(\mu_1 - 0.5\sigma_1^2)(T-t)+\sigma_1 W_{1,T-t}^P} \end{cases} \quad \begin{cases} dS_{2,t} = \mu_2 S_{2,t}dt + \sigma_2 S_{2,t}dW_{2,t}^P \\ \\ S_{2,T} = S_{2,t}e^{(\mu_2 - 0.5\sigma_2^2)(T-t)+\sigma_2 W_{2,T-t}^P} \end{cases}$$

$$\frac{dS_{1,t}}{S_{1,t}} = (r+\sigma_1^2)dt + \sigma_1 dW_{1,t}^R \quad \frac{dS_{2,t}}{S_{2,t}} = (r+\rho\sigma_1\sigma_2)dt + \sigma_2 dW_{2,t}^R$$

$$S_{1,T} = S_{1,t}e^{(r+\frac{1}{2}\sigma_1^2)(T-t)+\sigma_1(W_{1,T}^R - W_{1,t}^R)} \quad S_{2,T} = S_{2,t}e^{(r-\frac{1}{2}\sigma_2^2+\rho\sigma_1\sigma_2)(T-t)+\sigma_2(W_{2,T}^R - W_{2,t}^R)}$$

Derive the pricing formula of the A-Brick option

$$payoff = \begin{cases} S_{1,T}+S_{2,T}, \text{if } K_1 < S_{1,T} < K_2 \text{ and } K_3 < S_{2,T} < K_4 \\ \\ 0 \qquad\qquad\quad , \text{otherwise} \end{cases}$$

$$\begin{aligned} A - Brick_t &= E^Q[e^{-r(T-t)}(S_{1,T} + S_{2,T} \cdot \mathbf{1}_{K_1<S_{1,T}<K_2,\ K_3<S_{2,T}<K_4}|\mathcal{F}_t)] \\ &= S_{1,t}P^R(K_1 < S_{1,T} < K_2,\ K_3 < S_{2,T} < K_4|\mathcal{F}_t) \\ &\quad + S_{2,t}P^R(K_1 < S_{1,T} < K_2,\ K_3 < S_{2,T} < K_4|\mathcal{F}_t) \end{aligned}$$

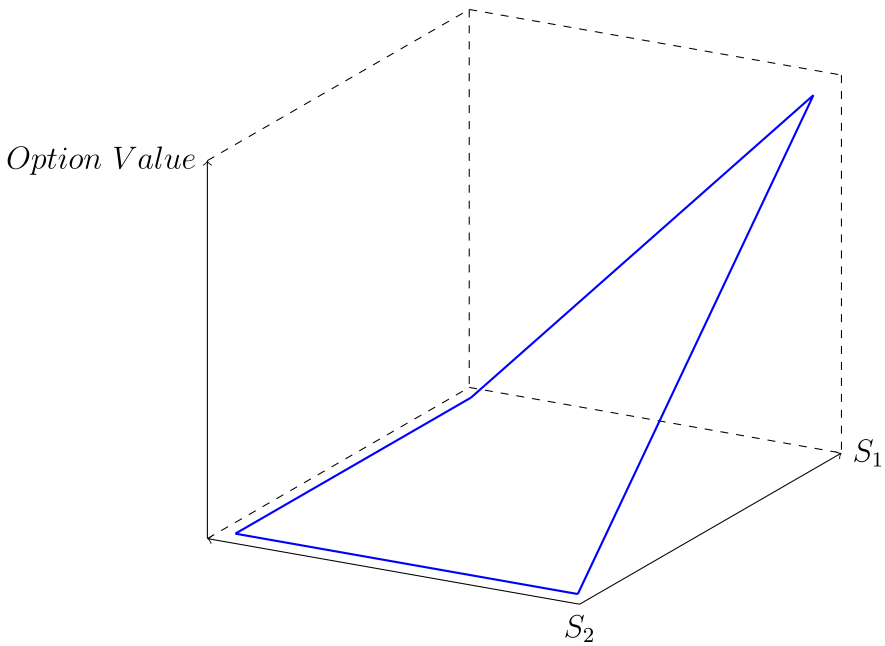
Let

$$f_{1,K_i} = \frac{\ln(\frac{S_{1,t}}{K_i}) + (r+0.5\sigma_1^2)(T-t)}{\sigma_1\sqrt{T-t}}, \quad g_{1,K_j} = \frac{\ln(\frac{S_{1,t}}{K_j}) + (r-0.5\sigma_2^2+\rho\sigma_1\sigma_2)(T-t)}{\sigma_2\sqrt{T-t}}$$

$$\begin{aligned} A - Brick_t &= S_{1,t}P^R\left(f_{1,K_2} < -\frac{\Delta W_{1,T-t}^R}{\sqrt{T-t}} < f_{1,K_1},\ g_{1,K_4} < -\frac{\Delta W_{2,T-t}^R}{\sqrt{T-t}} < g_{1,K_3}\right) \\ &\quad + S_{2,t}P^R\left(f_{1,K_2} < -\frac{\Delta W_{1,T-t}^R}{\sqrt{T-t}} < f_{1,K_1},\ g_{1,K_4} < -\frac{\Delta W_{2,T-t}^R}{\sqrt{T-t}} < g_{1,K_3}\right) \\ &= (S_{1,t}+S_{2,t})\int_{f_{1,K_2}}^{f_{1,K_1}}\int_{g_{1,K_4}}^{g_{1,K_3}} f_E(\varepsilon_1,\varepsilon_2)d\varepsilon_2 d\varepsilon_1 \\ &= (S_{1,t}+S_{2,t})\Bigg[\int_{-\infty}^{f_{1,K_1}}\int_{-\infty}^{g_{1,K_3}} f_E(\varepsilon_1,\varepsilon_2)d\varepsilon_2 d\varepsilon_1 - \int_{-\infty}^{f_{1,K_1}}\int_{-\infty}^{g_{1,K_4}} f_E(\varepsilon_1,\varepsilon_2)d\varepsilon_2 d\varepsilon_1 \\ &\quad - \int_{-\infty}^{f_{1,K_2}}\int_{-\infty}^{g_{1,K_3}} f_E(\varepsilon_1,\varepsilon_2)d\varepsilon_2 d\varepsilon_1 + \int_{-\infty}^{f_{1,K_2}}\int_{-\infty}^{g_{1,K_4}} f_E(\varepsilon_1,\varepsilon_2)d\varepsilon_2 d\varepsilon_1\Bigg] \\ &= (S_{1,t}+S_{2,t})\Big[N_2(f_{1,K_1},g_{1,K_3};\rho) - N_2(f_{1,K_1},g_{1,K_4};\rho) - N_2(f_{1,K_2},g_{1,K_3};\rho) + N_2(f_{1,K_2},g_{1,K_4};\rho)\Big] \end{aligned}$$

2 Exchange Option

Graph the exchange call option



Linear homogeneous property of an exchange option

$$\lambda \cdot \omega(S_{1,t}, S_{2,t}; T-t) = \omega(\lambda \cdot S_{1,t}, \lambda \cdot S_{2,t}; T-t)$$

Application of Euler theorem

According to Euler's Theorem, we can get

$$\omega(S_{1,t}, S_{2,t}; T-t) = \frac{\partial \omega(S_{1,t}, S_{2,t}; T-t)}{\partial S_{1,t}} S_{1,t} + \frac{\partial \omega(S_{1,t}, S_{2,t}; T-t)}{\partial S_{2,t}} S_{2,t}$$

$$-\omega(S_{1,t}, S_{2,t}; T-t) + \frac{\partial \omega(S_{1,t}, S_{2,t}; T-t)}{\partial S_{1,t}} S_{1,t} + \frac{\partial \omega(S_{1,t}, S_{2,t}; T-t)}{\partial S_{2,t}} S_{2,t} = 0$$

And $\frac{\partial \omega(S_{i,t}, S_{j,t}; T-t)}{\partial S_{i,t}}$ is the holding units of Assets i.
So, under instant time,

$$d\omega(S_{1,t}, S_{2,t}; T-t) - \frac{\partial \omega(S_{1,t}, S_{2,t}; T-t)}{\partial S_{1,t}} dS_{1,t} - \frac{\partial \omega(S_{1,t}, S_{2,t}; T-t)}{\partial S_{2,t}} dS_{2,t} = 0$$

Derive the pricing formula of exchange option

$$\begin{aligned}\omega(t) &= e^{-r(T-t)}E_t^Q(S_{1,T} - S_{2,T}\mathbf{1}_{\{S_{1,T}>S_{2,T}\}}) \\ &= e^{-r(T-t)}E_t^Q(S_{1,T}\mathbf{1}_{\{S_{1,T}>S_{2,T}\}}) - e^{-r(T-t)}E_t^Q(S_{2,T}\mathbf{1}_{\{S_{1,T}>S_{2,T}\}})\end{aligned}$$

Assume we let $E_1 = E_t^Q(S_{1,T}\mathbf{1}_{\{S_{1,T}>S_{2,T}\}})$, and $E_2 = E_t^Q(S_{2,T}\mathbf{1}_{\{S_{1,T}>S_{2,T}\}})$

$$\begin{aligned}e^{-r(T-t)}E_1 &= S_{1,t}P_t^{R_1}(S_{1,t}e^{(r+0.5\sigma_1^2)(T-t)+\sigma_1\Delta W_{1,T-t}^{R_1}} > S_{2,t}e^{(r-0.5\sigma_2^2+\rho\sigma_1\sigma_2)(T-t)+\sigma_2\Delta W_{2,T-t}^{R_2}}) \\ &= S_{1,t}P_t^{R_1}\left(\frac{\ln(\frac{S_{1,t}}{S_{2,t}}) + \frac{1}{2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)(T-t)}{\sqrt{(\sigma_1^2 - 2\rho\sigma_1\sigma_2+\sigma_2^2)(T-t)}} > -\frac{\sigma_1\Delta W_{1,T-t}^{R_1} - \sigma_2\Delta W_{2,T-t}^{R_2}}{\sqrt{(\sigma_1^2 - 2\rho\sigma_1\sigma_2+\sigma_2^2)(T-t)}}\right) \\ &= S_{1,t}N\left(\frac{\ln(\frac{S_{1,t}}{S_{2,t}}) + \frac{1}{2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)(T-t)}{\sqrt{(\sigma_1^2 - 2\rho\sigma_1\sigma_2+\sigma_2^2)(T-t)}}\right) \\ &= S_{1,t}N\left(\frac{\ln(\frac{S_{1,t}}{S_{2,t}}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\ &= S_{1,t}N(d_1)\end{aligned}$$

Same method will be used at $e^{-r(T-t)}E_2$, and the outcome will be $S_{2,t}N(d2)$

Pricing formula of an exchange call option if $S_{2,t} = Ke^{-r(T-t)}$

$$\omega(t) = S_{1,t}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

Pricing formula of an exchange put option if $S_{2,t} = Ke^{-\int_t^T r(s)ds}$

$$\omega(t) = S_{1,t}N(d_1) - Ke^{-\int_t^T r(s)ds}N(d_2)$$

3 Chooser's option

State and prove the put-call parity

Assume we have portfolio A and portfolio B. A includes buying a call option C and save K in the bank. B includes buying a put option P and buy stock S to hedge.

$$\begin{aligned}A &= \max(S_T - K, 0) + K = \max(S_T, K) \\ B &= \max(K - S_T, 0) + S_T = \max(K, S_T)\end{aligned}$$

We can know that the future value of both portfolios are the same, it also implies that the initial value of both portfolios are the same. That is,

$$C + Ke^{-r(T-t)} = P + S_t$$

Decompose the chooser's call option

$$\begin{aligned}\omega(t) &= \max\left\{C(K,T-t,T), C(K,T-t,T) - S_t + Ke^{-r(T-t)}\right\}\\ &= C(K,T-t,T) + \max\left\{0, Ke^{-r(T-t)} - S_t\right\}\\ &= C(K,T-t,T) + P(Ke^{-r(T-t)},0,t)\end{aligned}$$

Pricing formula of chooser's call option

By put-call parity, we can know that

$$\begin{aligned}\omega(0) &= C(K,T,T) + P(Ke^{-r(T-t)},t,t)\\ &= S_0N(d_{1,K,T}) - Ke^{-rT}N(d_{2,K,T}) + Ke^{-r(T-t)}e^{-rt}N(-d_{2,Ke^{-r(T-t)},t}) - S_0N(-d_{1,Ke^{-r(T-t)},t})\\ &= S_0\left[N(d_{1,K,T}) - N(-d_{1,Ke^{-r(T-t)},t})\right] - Ke^{-rT}\left[N(d_{2,K,T}) - N(d_{2,Ke^{-r(T-t)},t})\right]\end{aligned}$$

4 Option on the maximum and minimum assets

Two assets of maximum put option

$$Let\ A_1 = K > S_{1,T},\ S_{1,T} > S_{2,T} \quad and\ A_2 = K > S_{2,T},\ S_{2,T} > S_{1,T}$$

$$P_t^{\max} = Ke^{-r(T-t)}\left[E_t^Q(\mathbf{1}_{A_1}) + E_t^Q(\mathbf{1}_{A_2})\right] - e^{-r(T-t)}\left[E_t^Q(S_{1,T}\mathbf{1}_{A_1}) + E_t^Q(S_{2,T}\mathbf{1}_{A_2})\right]$$

Let $E_1 = \left[E_t^Q(\mathbf{1}_{A_1}) + E_t^Q(\mathbf{1}_{A_2})\right]$, $E_2 = e^{-r(T-t)}\left[E_t^Q(S_{1,T}\mathbf{1}_{A_1})\right]$ $E_3 = e^{-r(T-t)}\left[E_t^Q(S_{2,T}\mathbf{1}_{A_2})\right]$

$$\begin{aligned}E_2 &= e^{-r(T-t)}E_t^Q\left(S_{1,T}\mathbf{1}_{A_1}\right)\\ &= S_{1,t}P_t^{R_1}(K > S_{1,T},\ S_{1,T} > S_{2,T})\\ &= S_{1,t}P_t^{R_1}(K > S_{1,t}e^{(r+0.5\sigma_1^2)(T-t)+\sigma_1\Delta W_{1,T-t}^{R_1}}\\ &\qquad ,S_{1,t}e^{(r+0.5\sigma_1^2)(T-t)+\sigma_1\Delta W_{1,T-t}^{R_1}} > S_{2,t}e^{(r-0.5\sigma_2^2+\rho\sigma_1\sigma_2)(T-t)+\sigma_2\Delta W_{2,T-t}^{R_1}})\\ &= S_{1,t}N_2(-d_1^1, d_{1,2}^1, \Sigma_1^1)\end{aligned}$$

Using the same step from above, we can get:

$$\begin{aligned}E_3 &= S_{2,t}N_2(-d_1^2, d_{1,2}^2, \Sigma_1^2)\\ E_1 &= 1 - N_2(d_2^1, d_2^2, \Sigma_2)\end{aligned}$$

So, $P_t^{\max} = Ke^{-r(T-t)}\left[1 - N_2(d_2^1, d_2^2, \Sigma_2)\right] - S_{1,t}N_2(-d_1^1, d_{1,2}^1, \Sigma_1^1) - S_{2,t}N_2(-d_1^2, d_{1,2}^2, \Sigma_1^2)$

Portfolio

Show that the portfolio of minimum put option and the maximum put option equals the portfolio of the put option with the first underlying asset and the other put option with the second underlying asset.

$$\begin{aligned} P_T^{\max} + P_T^{\min} &= \max\left\{K - \max(S_{1,T}, S_{2,T}), 0\right\} + \max\left\{K - \min(S_{1,T}, S_{2,T}), 0\right\} \\ &= \begin{cases} \max\{K - S_{1,T}, 0\} + \max\{K - S_{2,T}, 0\},\ S_{1,T} > S_{2,T} \\ \max\{K - S_{2,T}, 0\} + \max\{K - S_{1,T}, 0\},\ S_{2,T} > S_{1,T} \end{cases} \\ &= P(S_{1,T}, K, T) + P(S_{2,T}, K, T) \end{aligned}$$

Hedge ratio

Show that how to hedge the maximum option with $n = 2$

$$C_t^{\max} = \sum_{i=1}^{2} \left(\frac{\partial C_t^{\max}}{\partial S_{i,t}}\right) S_{i,t} + \left(\frac{\partial C_t^{\max}}{\partial K}\right) K = \sum_{i=1}^{2} \Delta_{i,t}^{\max} S_{i,t} + \Delta_{i,t}^{\max} K$$

$\Delta_{i,t} > 0$, $i = 1, 2$ is the asset's hedge ratio.

5 Application of premium reduced option

Initial value of bond

$$10,000 \times 0.9 \times e^{-0.0045} \simeq 8,990$$

Participate rate of PGN

$$\frac{10,000 - 8,990}{2,020} = 0.5$$

Influence of the premium reduced option

If the market interest rate goes down, the initial bond value will be more higher, thus the remaining value of call option will be smaller, that is, the participate rate will be lower.