1 Proof properties of Wiener process

Assume that $t < s$

$$X_s \sim N(0, s) \quad X_t \sim N(0, t)$$

1. By independent increments, we know that $E(X_s|X_t = B) = E(X_s - X_t + X_t|X_t = B) = E(X_s - X_t|X_t = B) + X_t = 0 + B = B$
2. $Var(X_s|X_t = B) = E\{[X_s - E(X_s|X_t = B)]^2|X_t = B\} = E[(X_s - B)^2|X_t = B]$
$= E[X_s^2 - 2X_sB + B^2|X_t = B] = E[X_s^2|X_t = B] - 2BE(X_s|X_t = B) + B^2$
$= E[X_s^2|X_t = B] - B^2 = Var(X_s) - Var(X_t) = s - t$
3. $X_t|X_s = B$ is a bivariate normal distribution which $\sim N(B \times \frac{t}{s}, (s - t) \times \frac{t}{s})$. So $E(X_t|X_s = B) = B \times \frac{t}{s}$
4. According to 3. $Var(X_t|X_s = B) = (s - t) \times \frac{t}{s}$
5. $Cov(X_s, X_t) = E(X_sX_t) - E(X_s)E(X_t) = E(X_sX_t) - 0 = E(X_sX_t)$
$= E(X_s - X_{s-1} + X_{s-1} - X_{s-2} + \cdots + X_{t+1} - X_t + X_t)X_t$
$= E(X_S - X_{s-1})X_t + \cdots + E(X_t)X_t = E(X_t^2) = t^2$

2 Proof the Quadratic Variation of Brownian Motion

According to standard Brownian Motion

$$dX_t = adt + bdW_t$$

By definition, $(dZ)^2 = \varepsilon^2 dt,\ \forall \varepsilon \sim N(0, 1)$. So

$$Var(\varepsilon) = 1 = E[\varepsilon^2] \Rightarrow E[(dZ)^2] = E[\varepsilon^2 dt] = 1dt = dt$$

In addition, $Var((dZ^2)) = Var(\varepsilon^2 dt) = (dt^2)Var(\varepsilon) \to 0$ (because $(dt^2 \to 0)$). So we say

$$(dZ)^2 \overset{a.s.}{=} dt$$

Based on the result of $(dZ)^2 = dt$, we can infer that the quadratic variation of Brownian Motion over $[0, T]$, i.e., $[W, W](t) = \lim_{n\to\infty} \sum_{i=1}^{n} |B(t_i^n) - B(t_{i-1}^n)|^2 = T$.

<Proof>
At first we know $(dZ)^2 \sim N(dt, 0)$ when $n \to \infty$ or $dt \to 0$, $(B(t_i^n) - B(t_{i-1}^n)) \overset{a.s.}{=} (t_i^n - t_{i-1}^n)$.
This is because $E[(B(t_i^n) - B(t_{i-1}^n))^2] = E[\varepsilon^2(t_i^n - t_{i-1}^n)] = t_i^n - t_{i-1}^n$, and
$Var((B(t_i^n) - B(t_{i-1}^n))^2) = Var(\varepsilon^2(t_i^n - t_{i-1}^n)) = (t_i^n - t_{i-1}^n)^2 Var(\varepsilon^2) \to 0$.

Thus, we can conclude that when $n \to \infty$, or $t_i^n - t_{i-1}^n \to 0$, $\lim_{n\to\infty} \sum_{i=1}^{n} |B(t_i^n) - B(t_{i-1}^n)|^2 = T$

3 Application of Itô's Lemma (I)

Assume the stock price change process is

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

$$f_t = \ln S_t$$

According to Taylor series

$$df_t = (\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}\mu S_t(S_t, t) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2(S_t, t))dt + \frac{\partial f}{\partial S}\sigma S_t(S_t, t)dz_t$$

Because $f_t = \ln S_t$

$$\begin{aligned} d\ln S_t &= (\frac{1}{S_t}\mu S_t + \frac{-1}{2S_t^2}\sigma^2 S_t^2)dt + \frac{1}{S_t}\sigma S_t dz_t \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz_t \end{aligned}$$

Then we inergrate it to get the close form of stock price.

$$\begin{aligned} \int_0^T d\ln S_t &= \int_0^T (\mu - \frac{1}{2}\sigma^2)dt + \int_0^T \sigma dz_t \\ \ln(S_T) - \ln(S_0) = \ln(\frac{S_T}{S_0}) &= (\mu - \frac{1}{2}\sigma^2)T + \sigma z_T \\ \frac{S_T}{S_0} &= \exp\left[(\mu - \frac{1}{2}\sigma^2)T + \sigma z_T\right] \\ S_T &= S_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)T + \sigma z_T\right] \end{aligned}$$

If S_T is Logarithm Normal Distributed, then $\ln(S_T)$ is Normal Distributed.

$$\begin{aligned} \ln(S_T) &= \ln(S_0) + (\mu - \frac{1}{2}\sigma^2)T + \sigma z_T \sim N(\ln(S_0) + (\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T) \\ E(S_T) &= \exp[\ln(S_0) + (\mu - \frac{1}{2}\sigma^2)T + \frac{1}{2}\sigma^2 T] = S_0 e^{\mu T} \\ E(S_T^2) &= \exp[2\ln(S_0) + 2(\mu - \frac{1}{2}\sigma^2)T + 2\frac{1}{2}\sigma^2 T] = S_0^2 e^{2\mu T + \sigma^2 T} \\ Var(S_T) &= E(S_T^2) - [E(S_T)]^2 = S_0^2(e^{2\mu T + \sigma^2 T} - e^{2\mu T}) \end{aligned}$$

4 Application of Itô's Lemma (II)

Assume the stock price change process is

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$
$$f_t = S_t - Ke^{-r(T-t)}$$

According to Taylor series

$$\begin{aligned} df_t &= (-rK^{-r(T-t)} + \mu S_t + \frac{(\sigma S_t)^2}{2} \times 0)dt + \sigma S_t dz_t \\ &= (\mu S_t - rK^{-r(T-t)})dt + \sigma S_t dz_t \end{aligned}$$

5 Application of Itô's Lemma (III)

Assume the interest rate change process is

$$dr_t = \kappa(\theta - r_t)dt + \sigma dz_t$$
$$f_t = r_t e^{\kappa t}$$

According to Taylor series

$$\begin{aligned} df_t &= (\frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\kappa(\theta - r_t) + \frac{1}{2}\frac{\partial^2 f}{\partial r^2}\sigma^2(r_t, t))dt + \frac{\partial f}{\partial r}\sigma(r_t, t)dz_t \\ &= (\kappa r_t e^{\kappa t} + e^{\kappa t}\kappa(\sigma - r_t))dt + e^{\kappa t}\sigma dz_t \end{aligned}$$

Because $f_t = r_t e^{\kappa t}$

$$\begin{aligned} dr_t e^{\kappa t} &= (\kappa r_t e^{\kappa t} + e^{\kappa t}\kappa(\sigma - r_t))dt + e^{\kappa t}\sigma dz_t \\ dr_t &= (\kappa r_t + \kappa(\sigma - r_t))dt + \sigma dz_t \\ &= \kappa\sigma dt + \sigma dz_t \end{aligned}$$

Then we inergrate it to get the close form of interest rate.

$$\begin{aligned} \int_0^T dr_t &= \int_0^T \kappa\sigma dt + \int_0^T \sigma dz_t \\ r_T - r_0 &= \int_0^T \kappa\sigma dt + \int_0^T \sigma dz_t \\ r_T &= r_0 + \kappa\sigma T + \sigma z_T \end{aligned}$$

If r_T is Normal Distributed,

$$\begin{aligned} r_T &= r_0 + \kappa\sigma T + \sigma z_T \sim N(r_0 + \kappa\sigma T, \sigma^2 T) \\ E(r_T) &= r_0 + \kappa\sigma T \\ E(r_T^2) &= r_0^2 + 2r_0\kappa\sigma T + \kappa^2\sigma^2 T \\ Var(r_T) &= E(r_T^2) - [E(r_T)]^2 = r_0^2 + 2r_0\kappa\sigma T + \kappa^2\sigma^2 T - (r_0 + \kappa\sigma T)^2 = 0 \end{aligned}$$

6 Application of Itô's Lemma (IV)

Assume the stock price change process and foriegn exchange rate change process are

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dz_{St} \\
dX_t &= \mu X_t dt + \sigma X_t dz_{Xt} \\
Cov(dz_{St}, dz_{Xt}) &= \rho dt
\end{aligned}
$$

6.1 $f_t = S_t X_t$

if $f_t = f(t, X_t, S_t) = S_t X_t$, According to Taylor series

$$
\begin{aligned}
df_t = (&\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial S_t}\mu_S S_t + \frac{\partial f_t}{\partial X_t}\mu_X X_t + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma_S^2 S_t^2 + \frac{1}{2}\frac{\partial^2 f_t}{\partial X_t^2}\sigma_X^2 X_t^2 + \frac{\partial^2 f_t}{\partial S_t \partial X_t}\sigma_S S_t \sigma_X X_t \rho)dt \\
&+ \frac{\partial f_t}{\partial S_t}\sigma_S S_t dz_{St} + \frac{\partial f_t}{\partial X_t}\sigma_X X_t dz_{Xt} \\
&= (X_t \mu_S S_t + S_t \mu_X X_t + \sigma_S \sigma_X S_t X_t \rho)dt + X_t \sigma_S S_t dz_{St} + S_t \sigma_X X_t dz_{Xt}
\end{aligned}
$$

Because $f_t = S_t X_t$

$$
dS_t X_t = (X_t \mu_S S_t + S_t \mu_X X_t + \sigma_S \sigma_X S_t X_t \rho)dt + X_t \sigma_S S_t dz_{St} + S_t \sigma_X X_t dz_{Xt}
$$

Then we intergrate it to get the close form of stock price with NTD dollars.

$$
\begin{aligned}
\int_o^T dS_t X_t &= \int_0^T (X_t \mu_S S_t + S_t \mu_X X_t + \sigma_S \sigma_X S_t X_t \rho)dt + \int_o^T X_t \sigma_S S_t dz_{St} + \int_o^T S_t \sigma_X X_t dz_{Xt} \\
S_T X_T - S_0 X_0 &= \int_0^T (S_t X_t(\mu_S + \mu_X + \sigma_S \sigma_X \rho))dt + \int_0^T S_t X_t(\sigma_S dz_{St} + \sigma_X dz_{Xt}) \\
&= [(S_T X_T - S_0 X_0)(\mu_S + \mu_X + \sigma_S \sigma_X \rho)]T + S_T X_T(\sigma_S dz_{ST} + \sigma_X dz_{XT})
\end{aligned}
$$

$$
S_T X_T = S_0 X_0 + [(S_T X_T - S_0 X_0)(\mu_S + \mu_X + \sigma_S \sigma_X \rho)]T + S_T X_T(\sigma_S dz_{ST} + \sigma_X dz_{XT})
$$

6.2 $f_t = \frac{S_t}{X_t}$

if $f_t = f(t, X_t, S_t) = \frac{S_t}{X_t}$, According to Taylor series

$$
\begin{aligned}
df_t = (&\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial S_t}\mu_S S_t + \frac{\partial f_t}{\partial X_t}\mu_X X_t + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma_S^2 S_t^2 + \frac{1}{2}\frac{\partial^2 f_t}{\partial X_t^2}\sigma_X^2 X_t^2 + \frac{\partial^2 f_t}{\partial S_t \partial X_t}\sigma_S S_t \sigma_X X_t \rho)dt \\
&+ \frac{\partial f_t}{\partial S_t}\sigma_S S_t dz_{St} + \frac{\partial f_t}{\partial X_t}\sigma_X X_t dz_{Xt} \\
&= (\frac{1}{X_t}\mu_S S_t - \frac{S_t}{X_t}\mu_X + 2\frac{S_t}{X_t}\sigma_X^2 - \frac{1}{X_t}\sigma_S S_t \sigma_X \rho)dt + \frac{1}{X_t}\sigma_S S_t dz_{St} - \frac{S_t}{X_t}\sigma_X dz_{Xt}
\end{aligned}
$$

Because $f_t = \dfrac{S_t}{X_t}$

$$d\frac{S_t}{X_t} = (\frac{1}{X_t}\mu_S S_t - \frac{S_t}{X_t}\mu_X + 2\frac{S_t}{X_t}\sigma_X^2 - \frac{1}{X_t}\sigma_S S_t \sigma_X \rho)dt + \frac{1}{X_t}\sigma_S S_t dz_{St} - \frac{S_t}{X_t}\sigma_X dz_{Xt}$$

Then we intregrate it to get the close form of stock price with NTD dollars.

$$\begin{aligned}
\int_o^T d\frac{S_t}{X_t} &= \int_0^T (\frac{1}{X_t}\mu_S S_t - \frac{S_t}{X_t}\mu_X + 2\frac{S_t}{X_t}\sigma_X^2 - \frac{1}{X_t}\sigma_S S_t \sigma_X \rho)dt + \int_o^T \frac{1}{X_t}\sigma_S S_t dz_{St} - \int_o^T \frac{S_t}{X_t}\sigma_X dz_{Xt} \\
\frac{S_T}{X_T} - \frac{S_0}{X_0} &= \int_0^T (\frac{1}{X_t}\mu_S S_t - \frac{S_t}{X_t}\mu_X + 2\frac{S_t}{X_t}\sigma_X^2 - \frac{1}{X_t}\sigma_S S_t \sigma_X \rho)dt + \int_0^T \frac{1}{X_t}\sigma_S S_t dz_{St} - \int_0^T \frac{S_t}{X_t}\sigma_X dz_{Xt} \\
&= [(\frac{S_T}{X_T} - \frac{S_0}{X_0})(\mu_S - \mu_X + 2\sigma_X^2 - \sigma_S\sigma_X\rho)]T + \frac{S_T}{X_T}(\sigma_S dz_{ST} - \sigma_X dz_{XT})
\end{aligned}$$

$$\frac{S_T}{X_T} = \frac{S_0}{X_0} + [(\frac{S_T}{X_T} - \frac{S_0}{X_0})(\mu_S - \mu_X + 2\sigma_X^2 - \sigma_S\sigma_X\rho)]T + \frac{S_T}{X_T}(\sigma_S dz_{ST} - \sigma_X dz_{XT})$$