1 Binomial Model

one-period standard Eurpoean Call option Binomial model

Risk Neutral probability q is derived as

$$q = \frac{e^{rT} - d}{u - d}$$

So, the price of the option is

$$f = e^{-rT}\Big[q \times \max(uS_0 - K, 0) + (1-q) \times \max(dS_0 - K, 0)\Big]$$

Generalized European Call option Binomial Model

n periods, where $\Delta t = \dfrac{T}{n}$

$$f_u = e^{-r\Delta t}\Big[q \times f_{uu} + (1-q) \times f_{ud}\Big],\ f_d = e^{-r\Delta t}\Big[q \times f_{du} + (1-q) \times f_{dd}\Big]$$
$$f = e^{-r\Delta t}\Big[q \times f_u + (1-q) \times f_d\Big] = e^{-2r\Delta t}\Big[q^2 \times f_{uu} + 2q(1-q) \times f_{ud} + (1-q)^2 \times f_{dd}\Big]$$

So, when $n = 3$, we have

$$f = e^{-3r\Delta t}\Big[q^3 f_{u^3d^0} + 3q^2(1-q) f_{u^2d^1} + 3q(1-q)^2 f_{u^1d^2} + (1-q)^3 f_{u^0d^3}\Big]$$

As we expand to n periods, we will have

$$\begin{aligned} f &= e^{-nr\Delta t}\Big[C_n^n q^n (1-q)^0 f_{u^nd^0} + C_{n-1}^n q^{n-1}(1-q)^1 f_{u^{n-1}d^1} + \cdots + C_0^n q^0 (1-q)^n f_{u^0d^n}\Big] \\ &= e^{-nr\Delta t}\Big[\sum_{i=0}^{n} C_i^n q^i (1-q)^{n-i} \max(u^i d^{n-i} S_0 - K, 0)\Big] \\ &= e^{-nr\Delta t}\Big[\sum_{i=0}^{n} C_i^n q^i (1-q)^{n-i} \max(u^i d^{n-i} S_0 - K, 0)\Big] \end{aligned}$$

Assume that a is the minimum number of upward moves that the stock must finish in-the-money.

$\forall i < a,\ \max(u^i d^{n-i} S_0 - K, 0) = 0$

$\forall i \geq a,\ \max(u^i d^{n-i} S_0 - K, 0) = u^i d^{n-i} S_0 - K$

So,

$$\begin{aligned} f &= e^{-nr\Delta t}\Big[\sum_{i=a}^{n} C_i^n q^i (1-q)^{n-i} (u^i d^{n-i} S_0 - K)\Big] \\ &= S\Big[\sum_{i=a}^{n} C_i^n (que^{-r\Delta t})^i [(1-q)de^{-r\Delta t}]^{n-i}\Big] - Ke^{-nr\Delta t}\sum_{i=a}^{n} C_i^n q^i (1-q)^{n-i} \end{aligned}$$

Let $q' = que^{-r\Delta t}$, $(1-q)' = (1-q)de^{-r\Delta t}$

$$\begin{aligned} f &= S\Big[\sum_{i=a}^{n} C_i^n q'^i [(1-q)']^{n-i}\Big] - Ke^{-nr\Delta t}\sum_{i=a}^{n} C_i^n q^i (1-q)^{n-i} \\ &= SP(X \geq a) - Ke^{-rT}P(Y \geq a) \end{aligned}$$

Where $X \sim Bin(n, q')$, $Y \sim Bin(n, q)$

CRR to BS

According to Geometric Brownian Motion (GBM), we have

$$\frac{S_t}{S} = rdt + \sigma dW_t \leftrightarrow \ln(\frac{S_T}{S}) = (r - \frac{\sigma^2}{2})T + \sigma W_T$$

Then the mean and variance of stock return can be present as

$$E\left[\ln(\frac{S_T}{S})\right] = \left(r - \frac{1}{2}\sigma^2\right)T$$

$$Var\left[\ln(\frac{S_T}{S})\right] = \sigma^2 T$$

Based on CRR, $S_T = Su^I d^{N-I} \Rightarrow \ln(\frac{S_T}{S}) = \ln(\frac{Su^I d^{N-I}}{S}) = N\ln d + I\ln(\frac{u}{d})$, Let $I = Nq$

$$E\left[\ln\left(\frac{S_T}{S}\right)\right] = N\ln d + Nq\ln(\frac{u}{d}) \equiv \hat{\mu}N$$

$$Var\left[\ln\left(\frac{S_T}{S}\right)\right] = Nq(1-q)\ln^2(\frac{u}{d}) \equiv \hat{\sigma}^2 N$$

To converge the pricing formula, it must satisfy the following properties

(i) $\hat{\mu}N \to \left(r - \frac{1}{2}\sigma^2\right)T$ as $n \to \infty$

(ii) $\hat{\sigma}^2 N \to \sigma^2 T$ as $n \to \infty$

Furthermore, we set $u = e^{\sigma\sqrt{\Delta t}}$, $\ d = e^{-\sigma\sqrt{\Delta t}}$. According to above, we devide the model into 2 parts, $P(X \geq a)$ and $P(Y \geq a)$

$$1 - P(Y \geq a) = P(Y \leq a) = P(\frac{Y - Nq}{\sqrt{Nq(1-q)}} \leq \frac{(a-1) - Nq}{\sqrt{Nq(1-q)}})$$

Because $u^a d^{N-a} S - K > 0$, so $a > \ln(\frac{K}{Sd^N})/\ln(\frac{u}{d})$. And there exists a $\varepsilon \in [0,1]$, such that $a - 1 = \ln(\frac{K}{Sd^N})/\ln(\frac{u}{d}) - \varepsilon$. Furthermore, from above, we know that

$$\ln d = \hat{\mu} - q\ln(\frac{u}{d})$$
$$\ln(\frac{u}{d}) = \frac{\hat{\sigma}}{\sqrt{q(1-q)}}$$

So we can replace $a-1$ by them and denote as

$$\begin{aligned}\frac{(a-1)-Nq}{\sqrt{Nq(1-q)}} &= \frac{(\ln(\frac{K}{Sd^N})/\ln(\frac{u}{d}) - \varepsilon) - Nq}{\sqrt{Nq(1-q)}} \\ &= \frac{\ln(\frac{K}{Sd^N})/\ln(\frac{u}{d})}{\sqrt{Nq(1-q)}} - \frac{\varepsilon + Nq}{\sqrt{Nq(1-q)}}\end{aligned}$$

From above, we can know that $\hat{\sigma} = \sqrt{q(1-q)}\ln(\frac{u}{d})$

$$\begin{aligned}\frac{(a-1)-Nq}{\sqrt{Nq(1-q)}} &= \frac{\ln(\frac{K}{S}) - N\ln d}{\sqrt{N}\hat{\sigma}} - \frac{\varepsilon + Nq}{\sqrt{Nq(1-q)}} = \frac{\ln(\frac{K}{S}) - N(\hat{\mu} - q\ln(\frac{u}{d}))}{\sqrt{N}\hat{\sigma}} - \frac{\varepsilon + Nq}{\sqrt{Nq(1-q)}} \\ &= \frac{\ln(\frac{K}{S}) - N(\hat{\mu} - q\ln(\frac{u}{d}))}{\sqrt{N}\hat{\sigma}} - \frac{\ln(\frac{u}{d})(\varepsilon + Nq)}{\sqrt{N}\hat{\sigma}} \\ &= \frac{\ln(\frac{K}{S}) - N\hat{\mu} - \ln(\frac{u}{d}\varepsilon)}{\sqrt{N}\hat{\sigma}}\end{aligned}$$

Follow the below properties

(i) $\hat{\mu}N \to (r - \frac{1}{2}\sigma^2)T$ as $N \to \infty$

(ii) $\hat{\sigma^2}N \to \sigma^2 T$ as $N \to \infty$

(iii) $\ln(\frac{u}{d}) = \ln(\frac{e^{\sigma\sqrt{\Delta t}}}{e^{-\sigma\sqrt{\Delta t}}}) = 2\sigma\sqrt{\Delta t} \to 0$ as $N \to \infty$

Adopt the Central Limit Theorem (CLT)

$$1 - P(Y \geq a) = P(\frac{Y - Nq}{\sqrt{Nq(1-q)}} \leq \frac{(a-1)-Nq}{\sqrt{Nq(1-q)}}) \to P(Z \leq \frac{\ln(K/S) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})$$

So

$$P(Y \geq a) \to P(Z \leq \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) = N(d_2)$$

We use the same method on $P(X \geq a) \to N(d_1)$

2 Stochastic Process and Itô's Lemma

Itô Process

$$\begin{aligned} df_t = \Big(&\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial X_{1,t}}\mu_1 X_{1,t} + \frac{\partial f_t}{\partial X_{2,t}}\mu_2 X_{2,t} + \frac{1}{2}\frac{\partial^2 f_t}{\partial X_{1,t}^2}\sigma_1^2 X_{1,t}^2 + \frac{1}{2}\frac{\partial^2 f_t}{\partial X_{2,t}^2}\sigma_2^2 X_{2,t}^2 \\ &+ \frac{\partial^2 f_t}{\partial X_{1,t}\partial X_{2,t}}\sigma_1\sigma_2 X_{1,t}X_{2,t}\rho_{12}\Big)dt + \frac{\partial f_t}{\partial X_{1,t}}\sigma_1 X_{1,t}dW_{1,t} + \frac{\partial f_t}{\partial X_{2,t}}\sigma_2 X_{2,t}dW_{2,t} \end{aligned}$$

Derive Itô's Lemma

$f_t = \dfrac{X_{1,t}}{X_{2,t}}, \quad dX_{1,t} = \mu_1 X_{1,t} + \sigma X_{1,t}dW_{1,t}, \;\; dX_{2,t} = \mu_2 X_{2,t} + \sigma X_{2,t}dW_{2,t}$

We can first write down

$$\frac{\partial f_t}{\partial t} = 0, \;\; \frac{\partial f_t}{\partial X_{1,t}} = \frac{1}{X_{2,t}}, \;\; \frac{\partial f_t}{\partial X_{2,t}} = -\frac{X_{1,t}}{X_{2,t}^2}, \;\; \frac{\partial^2 f_t}{\partial X_{1,t}^2} = 0, \;\; \frac{\partial^2 f_t}{\partial X_{2,t}^2} = \frac{2X_{1,t}}{X_{2,t}^3}, \;\; \frac{\partial^2 f_t}{\partial X_{1,t}\partial X_{2,t}} = -\frac{1}{X_{2,t}^2}$$

So

$$\begin{aligned} df_t &= \Big(\frac{1}{X_{2,t}}\mu_1 X_{1,t} - \frac{X_{1,t}}{X_{2,t}^2}\mu_2 X_{2,t} + \frac{1}{2}\frac{2X_{1,t}}{X_{2,t}^3}\sigma_2^2 X_{2,t}^2 - \frac{1}{X_{2,t}^2}\sigma_1\sigma_2 X_{1,t}X_{2,t}\rho_{12}\Big)dt \\ &\quad + \frac{1}{X_{2,t}}\sigma_1 X_{1,t}dW_{1,t} - \frac{X_{1,t}}{X_{2,t}^2}\sigma_2 X_{2,t}dW_{2,t} \\ &= \frac{X_{1,t}}{X_{2,t}}\Big[\Big(\mu_1 + \mu_2 + \sigma_2^2 - \sigma_1\sigma_2\rho_{12}\Big)dt + \sigma_1 dW_{1,t} - \sigma_2 dW_{2,t}\Big] \end{aligned}$$

Then

$$\frac{df_t}{f_t} = \Big(\mu_1 + \mu_2 + \sigma_2^2 - \sigma_1\sigma_2\rho_{12}\Big)dt + \sigma_1 dW_{1,t} - \sigma_2 dW_{2,t}$$

3 Black-Scholes Model and Merton Pricing Formula

Assumptions of Black-Scholes Model

1. Stock Price follows Itô process $dS_t = \mu S_t dt + \sigma S_t dW_t$, under this assumption, we can derive that the stock price follows logarithmic normal distribution, and both μ, σ are constant.
2. Market is frictioness. There are no transcation costs, no tax, no liquidity limitation, and no law restrictions.
3. Stock trading in the market is continuous.
4. We can short stocks without any restriction.
5. No underlying dividend.
6. Options are European.
7. risk free rate r exists.

Black-Scholes PDE

Assume the stock change process is $dS_t = \mu S_t dt + \sigma S_t dW_t$

Let f_t be the price of a European call option

Then

$$df_t = (\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma^2 S_t^2)dt + \frac{\partial f_t}{\partial S_t}\sigma S_t dW_t$$

Now, construct a risk-free investment portfolio π_t, where $\pi_t = -f_t + \frac{\partial f_t}{\partial S_t}S_t$, So $d\pi_t = -df_t + \frac{\partial f_t}{\partial S_t}dS_t$

$$\begin{aligned}
d\pi_t &= -(\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma^2 S_t^2)dt - \frac{\partial f_t}{\partial S_t}\sigma S_t dW_t + \frac{\partial f_t}{\partial S_t}dS_t \\
&= -(\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma^2 S_t^2)dt - \frac{\partial f_t}{\partial S_t}\sigma S_t dW_t + \frac{\partial f_t}{\partial S_t}(\mu S_t dt + \sigma S_t dW_t) \\
&= -(\frac{\partial f_t}{\partial t} + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma^2 S_t^2)dt = r\pi_t dt = r(-f_t + \frac{\partial f_t}{\partial S_t}S_t)dt
\end{aligned}$$

Both devided by dt

$$-(\frac{\partial f_t}{\partial t} + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma^2 S_t^2) = r(-f_t + \frac{\partial f_t}{\partial S_t}S_t)$$

$$rf_t = \frac{\partial f_t}{\partial t} + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma^2 S_t^2 + r\frac{\partial f_t}{\partial S_t}S_t$$

Feynman-Kac Thoerem

According to the above, we know that $rf_t = \dfrac{\partial f_t}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 f_t}{\partial S_t^2}\sigma^2 S_t^2 + r\dfrac{\partial f_t}{\partial S_t}S_t$

Let $f_T = \max(S_T - K, 0) = (S_T - K)\mathbb{1}_{S_T>K} \leftrightarrow f_t = E_Q[e^{-r(T-t)}f_T|\mathcal{F}_t]$

$$f_t = e^{-r(T-t)}E_Q[f_T|\mathcal{F}_t] = e^{-r(T-t)}E_Q[(S_T - K)\mathbb{1}_{S_T>K}|\mathcal{F}_t]$$

$$= e^{-r(T-t)}E_Q[S_T\mathbb{1}_{S_T>K}] - Ke^{-r(T-t)}E_Q[\mathbb{1}_{S_T>K}|\mathcal{F}_t]$$

Assume we set $E_1 = e^{-r(T-t)}E_Q[S_T\mathbb{1}_{S_T>K}]$ and $E_2 = Ke^{-r(T-t)}E_Q[\mathbb{1}_{S_T>K}|\mathcal{F}_t]$

And we know $S_T = S_t\exp[(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\Delta W_{T-t}]$, $\Delta W_{T-t} = W_T - W_t \sim N(0, T-t)$

Thus, $S_T \sim \log N(\ln S_t + (r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))$, Let $\ln S_t + (r - \frac{1}{2}\sigma^2)(T-t) = a$

, and $\sigma^2(T-t) = b^2$

Last, let $S_T = Y = e^x$. That is, $Y = e^x \sim \log N(a, b^2)$, $x = \log Y \sim N(a, b^2)$

$$\begin{aligned}
E_Q[e^x\mathbb{1}_{x>\ln K}|\mathcal{F}_t] &= \int_{\ln K}^{\infty} e^x f(x)dx = \int_{\ln K}^{\infty} e^x \frac{1}{\sqrt{2\pi b^2}}\exp(-\frac{(x-a)^2}{2b^2})dx \\
&= \int_{\ln K}^{\infty} e^x \frac{1}{\sqrt{2\pi b^2}}\exp(-\frac{1}{2b^2}(x^2 - 2ax + a^2))dx \\
&= \int_{\ln K}^{\infty} \frac{1}{\sqrt{2\pi b^2}}\exp(-\frac{1}{2b^2}(x^2 - 2(a+b^2)x + a^2))dx \\
&= \int_{\ln K}^{\infty} \frac{1}{\sqrt{2\pi b^2}}\exp(-\frac{1}{2b^2}(x - (a+b^2)^2)dx \times \exp[\frac{1}{2b^2}(2ab^2 + b^4)]
\end{aligned}$$

Because $\dfrac{1}{\sqrt{2\pi b^2}}\exp(-\dfrac{1}{2b^2}(x - (a+b^2)^2)$ means that $x \sim N(a+b^2, b^2)$

$$\begin{aligned}
E_Q[e^x\mathbb{1}_{x>\ln K}|\mathcal{F}_t] &= p(x > \ln K) \times \exp[a + \frac{1}{2b^2}] = P(\frac{x-(a+b^2)}{b} > \frac{\ln K - (a+b^2)}{b}) \times \exp[a + \frac{1}{2}b^2] \\
&= P(Z > \frac{\ln K - (a+b^2)}{b}) \times \exp[a + \frac{1}{2}b^2] = P(Z < \frac{(a+b^2) - \ln K}{b}) \times \exp[a + \frac{1}{2}b^2] \\
&= \Phi\left(\frac{(a+b^2) - \ln K}{b}\right) \times \exp[a + \frac{1}{2}b^2]
\end{aligned}$$

We know that $a = \ln S_t + (r - \frac{1}{2}\sigma^2)(T-t)$, and $b^2 = \sigma^2(T-t)$

$$\Phi\left(\frac{(a+b^2) - \ln K}{b}\right) = \Phi\left(\frac{\ln S_t + (r - \frac{1}{2}\sigma^2)(T-t) + \sigma^2(T-t) - \ln K}{\sigma\sqrt{T-t}}\right)$$
$$= \Phi\left(\frac{\ln \frac{S_t}{K_t} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) = N(d_1)$$

$$\exp[a + \frac{1}{2}b^2] = \exp[\ln S_t + (r - \frac{1}{2}\sigma^2)(T-t) + \frac{1}{2}\sigma^2(T-t)] = S_t e^{r(T-t)}$$

So,

$$e^{-r(T-t)} E_Q[S_T \mathbb{1}_{S_T > K}] = S_t N(d_1)$$

E_2 has the same way to get it.

$$E_Q[\mathbb{1}x > \ln K | \mathcal{F}_t] = \int_{\ln K}^{\infty} \frac{1}{\sqrt{2\pi b^2}} \exp(-\frac{(x-a)^2}{2b^2}) dx$$

We know that $\frac{1}{\sqrt{2\pi b^2}} \exp(-\frac{(x-a)^2}{2b^2})$ means that $x \sim N(a, b^2)$

$$E_Q[\mathbb{1}x > \ln K | \mathcal{F}_t] = P(x > \ln K) = P(\frac{x-a}{b} > \frac{\ln K - a}{b}) = P(Z > \frac{\ln K - a}{b}) = P(Z < \frac{a - \ln K}{b})$$
$$= \Phi\left(\frac{a - \ln K}{b}\right) = \Phi\left(\frac{\ln S_t + (r - \frac{1}{2}\sigma^2)(T-t) - \ln K}{\sigma\sqrt{T-t}}\right) = N(d_2)$$

$$E_2 = K e^{-r(T-t)} E_Q[\mathbb{1}x > \ln K | \mathcal{F}_t] = K e^{-r(T-t)} N(d_2)$$

Hedge parameters for Delta

Delta is the hedge ratio of the option. $\delta = \frac{\partial C}{\partial S}$

$$C_T = S N(d_1) - K e^{-rT} N(d_2)$$

$$\frac{\partial C}{\partial S} = N(d_1) + S \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial S} - K e^{-rT} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial S}$$
$$= N(d_1) + S \frac{1}{\sqrt{2\pi}} \exp(-\frac{d_1^2}{2}) \frac{\partial d_1}{\partial S} - K e^{-rT} \frac{1}{\sqrt{2\pi}} \exp(-\frac{d_2^2}{2}) \frac{\partial d_2}{\partial S}$$

Because $d_2 = d_1 - \sigma\sqrt{T}$, and $\dfrac{\partial d_1}{\partial S} = \dfrac{\partial d_2}{\partial S}$

$$\frac{\partial C}{\partial S} = N(d_1) + S\frac{1}{\sqrt{2\pi}}\exp(-\frac{d_1^2}{2})\frac{\partial d_1}{\partial S} - Ke^{-rT}\frac{1}{\sqrt{2\pi}}\exp(-\frac{(d_1^2 - 2d_1\sigma\sqrt{T} + \sigma^2 T)}{2})\frac{\partial d_1}{\partial S}$$

As $d_1\sigma\sqrt{T} = \ln(\dfrac{S}{K}) + (r + \dfrac{1}{2}\sigma^2)T$

$$\begin{aligned} S\exp(-\frac{1}{2}d_1^2) &= Ke^{-rT}\exp(-\frac{1}{2}d_1^2 + \ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)T - \frac{1}{2}\sigma^2 T) \\ &= Ke^{-rT}\exp(-\frac{1}{2}d_1^2 + \ln(\frac{S}{K}) + rT) = S\exp(-\frac{1}{2}d_1^2) \end{aligned}$$

So, at last

$$\frac{\partial C}{\partial S} = N(d_1)$$

Derive Merton's pricing PDE

Same as 3-1, we first construct the risk-free investment portfolio $\pi_t = -f_t + \dfrac{\partial f_t}{\partial S_t}S_t$
In minimum time dt, change of the portfolio is

$$\begin{aligned} d\pi_t &= -df_t + \frac{\partial f_t}{\partial S_t}(dS_t + qS_t dt) \\ &= -\left(\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma^2 S_t^2\right)dt - \frac{\partial f_t}{\partial S_t}\sigma S_t dW_t + \frac{\partial f_t}{\partial S_t}(\mu S_t dt + \sigma S_t dW_t + qS_t dt) \\ &= -\left(\frac{\partial f_t}{\partial t} + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma^2 S_t^2 - \frac{\partial f_t}{\partial S_t}qS_t\right)dt = r\pi_t dt = r\left(-f_t + \frac{\partial f_t}{\partial S_t}S_t\right)dt \end{aligned}$$

So

$$rf_t = \frac{\partial f_t}{\partial t} + \frac{1}{2}\frac{\partial^2 f_t}{\partial S_t^2}\sigma^2 S_t^2 + \frac{\partial f_t}{\partial S_t}(r - q)S_t$$

Merton's pricing formula

$$C_t = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

Where q is the dividend yield.

4 Heat Equation

Derive Black-Scholes PDE

Solution of Heat Equation

$$U(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi c^2 t}} e^{-\dfrac{(x-y)^2}{4c^2 t}} g(y) dy$$

Where $U_t(x,t) = c^2 U_x x(x,t)$ subject to $U(x,0) = g(x)$

$$V_\tau(x,\tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi c^2 \tau}} e^{-\dfrac{(y-x)^2}{4c^2 \tau}} K(e^y - 1)^+ dy$$

Because $c^2 = \dfrac{1}{2}\sigma^2$

$$\begin{aligned}
V_\tau(x,\tau) &= K \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\dfrac{(y-x)^2}{2\sigma^2\tau}} (e^y - 1)^+ dy \\
&= K \int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\dfrac{(y-x)^2}{2\sigma^2\tau}} (e^y - 1) dy,\ y > 0 \leftrightarrow e^y > 1 \\
&= K \left[\int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\dfrac{(y-x)^2}{2\sigma^2\tau}} e^y dy - \int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\dfrac{(y-x)^2}{2\sigma^2\tau}} dy \right]
\end{aligned}$$

Let $z = \dfrac{y-x}{\sigma\sqrt{\tau}}$, $y = \sigma\sqrt{\tau} \cdot z + x$, $dz = \dfrac{dy}{\sigma\sqrt{\tau}}$, $z^* = -\dfrac{x}{\sigma\sqrt{\tau}} = -d_2$

$$\begin{aligned}
V_\tau(x,\tau) &= K \left[\int_{-d2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} e^{\sigma\sqrt{\tau}\cdot z + x} dz - \int_{-d2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \right] \\
&= K \left[\int_{-d2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma\sqrt{\tau})^2}{2}} e^{\frac{\sigma^2}{2}\tau + x} dz - \int_{-d2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \right] \\
&= K e^{\frac{\sigma^2}{2}\tau + x} \int_{-d2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma\sqrt{\tau})^2}{2}} dz - K \int_{-\infty}^{d2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz
\end{aligned}$$

Let $z_1 = z - \sigma\sqrt{\tau}$, the lower bound of $z_1 = -d_1 = -d_2 - \sigma\sqrt{\tau}$

$$\begin{aligned}
V_\tau(x,\tau) &= K e^{\frac{\sigma^2}{2}\tau + x} \int_{-d2-\sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_1^2}{2}} dz_1 - K N(d_2) \\
&= K e^{\frac{\sigma^2}{2}\tau + x} \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_1^2}{2}} dz_1 - K N(d_2) \\
&= K e^{\frac{\sigma^2}{2}\tau + x} N(d_1) - K N(d_2),\ x = \ln\frac{S}{K} + (r - \frac{1}{2}\sigma^2)\tau \\
&= S e^{r\tau} N(d_1) - K N(d_2)
\end{aligned}$$

Last

$$C(S_t, t) = e^{-r\tau} V(x, \tau) = SN(d_1) - Ke^{-r\tau} N(d_2)$$

5 Martingale Pricing Formula

Describe the Fundamental Theorem of Arbitrage-Free

Under the Arbitrage free condition, T exists an only risk neutral measurement Q which let the relative price of asset follows Martingale.

That is,

$$\frac{H_t}{\beta_t} = E_Q\left[\frac{H_T}{\beta_T}\middle|\mathcal{F}_t\right]$$

Describe the Girsanov's Theorem

If $E(e^{\int_0^t \beta_t dt}) < \infty$. Let P to $Q's$ Radon-Nikodym derivative $\xi_t = \dfrac{dP}{dQ}|\mathcal{F}_t = \exp\{\int_t^T X_s dW_s - \dfrac{1}{2}\int_t^T X_s^2 ds\}$

Derive Black-Scholes pricing formula by Martingale Pricing Method

First, transfer dynamic stock price process of P measure to Q measure.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P \to \frac{dS_t}{S_t} = rdt + \sigma dW_t^Q$$

Second, We let the relative price of MMA follows Martingale under Q measure.

$$\frac{C_t}{\beta_t} = E\left[\frac{C_T}{\beta_T}|\mathcal{F}_t\right]$$

We assume interest rate is constant

$$\begin{aligned}
C_t &= \frac{\beta_t}{\beta_T} E[C_T|\mathcal{F}_t] \\
&= e^{-r(T-t)} E[C_T|\mathcal{F}_t] \\
&= e^{-r(T-t)} E^Q(S_T \mathbb{1}_{S_T>K}|\mathcal{F}_t) - Ke^{-r(T-t)} E^Q(\mathbb{1}_{S_T>K}|\mathcal{F}_t)
\end{aligned}$$

Expand the close form of Stock Price

$$\begin{aligned}
E^Q(S_T \mathbb{1}_{S_T>K}|\mathcal{F}_t) &= E^Q(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T^Q - W_t^Q)} \mathbb{1}_{S_T>K}|\mathcal{F}_t) \\
&= S_t e^{r(T-t)} E^Q(e^{(\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T^Q - W_t^Q)} \mathbb{1}_{S_T>K}|\mathcal{F}_t)
\end{aligned}$$

Under Girsanov theorem

$$\frac{dQ}{dP}|\mathcal{F}_t = \exp\left\{\int_t^T \sigma dW_s - \frac{1}{2}\int_t^T \sigma^2 ds\right\}$$

So, we can derive the close form of stock price as

$$\begin{aligned} E^Q(e^{(\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T^Q-W_t^Q)}\mathbb{1}_{S_T>K}|\mathcal{F}_t) &= E^Q(\frac{dP}{dQ}\cdot\mathbb{1}_{S_T>K}|\mathcal{F}_t) \\ &= E^P(\mathbb{1}_{S_T>K}|\mathcal{F}_t) \\ &= P^P(S_T>K|\mathcal{F}_t) \end{aligned}$$