

1 Proof properties of Wiener process

Assume that $t < s$

$$X_s \sim N(0, s) \quad X_t \sim N(0, t)$$

1. By independent increments, we know that $E(X_s|X_t = B) = E(X_s - X_t + X_t|X_t = B) = E(X_s - X_t|X_t = B) + X_t = 0 + B = B$
2. $Var(X_s|X_t = B) = E\{[X_s - E(X_s|X_t = B)]^2|X_t = B\} = E[(X_s - B)^2|X_t = B]$
 $= E[X_s^2 - 2X_sB + B^2|X_t = B] = E[X_s^2|X_t = B] - 2BE(X_s|X_t = B) + B^2$
 $= E[X_s^2|X_t = B] - B^2 = Var(X_s) - Var(X_t) = s - t$
3. $X_t|X_s = B$ is a bivariate normal distribution which $\sim N(B \times \frac{t}{s}, (s - t) \times \frac{t}{s})$. So $E(X_t|X_s = B) = B \times \frac{t}{s}$
4. According to 3. $Var(X_t|X_s = B) = (s - t) \times \frac{t}{s}$
5. $Cov(X_s, X_t) = E(X_s X_t) - E(X_s)E(X_t) = E(X_s X_t) - 0 = E(X_s X_t)$
 $= E(X_s - X_{s-1} + X_{s-1} - X_{s-2} + \dots + X_{t+1} - X_t + X_t)X_t$
 $= E(X_s - X_{s-1})X_t + \dots + E(X_t)X_t = E(X_t^2) = t^2$

2 Proof the Quadratic Variation of Brownian Motion

According to standard Brownian Motion

$$dX_t = adt + bdW_t$$

By definition, $(dZ)^2 = \varepsilon^2 dt$, $\forall \varepsilon \sim N(0, 1)$. So

$$Var(\varepsilon) = 1 = E[\varepsilon^2] \Rightarrow E[(dZ)^2] = E[\varepsilon^2 dt] = 1dt = dt$$

In addition, $Var((dZ)^2) = Var(\varepsilon^2 dt) = (dt^2)Var(\varepsilon) \rightarrow 0$ (because $(dt^2 \rightarrow 0)$). So we say

$$(dZ)^2 \stackrel{a.s.}{=} dt$$

Based on the result of $(dZ)^2 = dt$, we can infer that the quadratic variation of Brownian Motion over $[0, T]$, i.e., $[W, W](t) = \lim_{n \rightarrow \infty} \sum_{i=1}^n |B(t_i^n) - B(t_{i-1}^n)|^2 = T$.

<Proof>

At first we know $(dZ)^2 \sim N(dt, 0)$ when $n \rightarrow \infty$ or $dt \rightarrow 0$, $(B(t_i^n) - B(t_{i-1}^n)) \stackrel{a.s.}{=} (t_i^n - t_{i-1}^n)$. This is because $E[(B(t_i^n) - B(t_{i-1}^n))^2] = E[\varepsilon^2(t_i^n - t_{i-1}^n)] = t_i^n - t_{i-1}^n$, and $Var((B(t_i^n) - B(t_{i-1}^n))^2) = Var(\varepsilon^2(t_i^n - t_{i-1}^n)) = (t_i^n - t_{i-1}^n)^2 Var(\varepsilon^2) \rightarrow 0$.

Thus, we can conclude that when $n \rightarrow \infty$, or $t_i^n - t_{i-1}^n \rightarrow 0$, $\lim_{n \rightarrow \infty} \sum_{i=1}^n |B(t_i^n) - B(t_{i-1}^n)|^2 = T$

3 Application of Itô's Lemma (I)

Assume the stock price change process is

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

$$f_t = \ln S_t$$

According to Taylor series

$$df_t = \left(\frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} \mu S_t(S_t, t) + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2(S_t, t) \right) dt + \frac{\partial f}{\partial S} \sigma S_t(S_t, t) dz_t$$

Because $f_t = \ln S_t$

$$\begin{aligned} d \ln S_t &= \left(\frac{1}{S_t} \mu S_t + \frac{-1}{2 S_t^2} \sigma^2 S_t^2 \right) dt + \frac{1}{S_t} \sigma S_t dz_t \\ &= \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dz_t \end{aligned}$$

Then we integrate it to get the close form of stock price.

$$\begin{aligned} \int_0^T d \ln S_t &= \int_0^T \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \int_0^T \sigma dz_t \\ \ln(S_T - S_0) &= \left(\mu - \frac{1}{2} \sigma^2 \right) T + \sigma z_T \\ (S_T - S_0) &= \exp \left[\left(\mu - \frac{1}{2} \sigma^2 \right) T + \sigma z_T \right] \\ S_T &= S_0 + \exp \left[\left(\mu - \frac{1}{2} \sigma^2 \right) T + \sigma z_T \right] \end{aligned}$$

Expectation value and variance

4 Application of Itô's Lemma (II)

Assume the stock price change process is

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

$$f_t = S_t - Ke^{-r(T-t)}$$

According to Taylor series

$$\begin{aligned} df_t &= (-rKe^{-r(T-t)} + \mu S_t + \frac{(\sigma S_t)^2}{2} \times 0)dt + \sigma S_t dz_t \\ &= (\mu S_t - rKe^{-r(T-t)})dt + \sigma S_t dz_t \end{aligned}$$

5 Application of Itô's Lemma (III)

Assume the interest rate change process is

$$dr_t = \kappa(\theta - r_t)dt + \sigma dz_t$$

$$f_t = r_t e^{\kappa t}$$

According to Taylor series

$$\begin{aligned} df_t &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} \kappa(\theta - r_t) + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \sigma^2(r_t, t) \right) dt + \frac{\partial f}{\partial r} \sigma(r_t, t) dz_t \\ &= (\kappa r_t e^{\kappa t} + e^{\kappa t} \kappa(\sigma - r_t))dt + e^{\kappa t} \sigma dz_t \end{aligned}$$

Because $f_t = r_t e^{\kappa t}$

$$\begin{aligned} dr_t e^{\kappa t} &= (\kappa r_t e^{\kappa t} + e^{\kappa t} \kappa(\sigma - r_t))dt + e^{\kappa t} \sigma dz_t \\ dr_t &= (\kappa r_t + \kappa(\sigma - r_t))dt + \sigma dz_t \\ &= \kappa \sigma dt + \sigma dz_t \end{aligned}$$

Then we integrate it to get the close form of interest rate.

$$\begin{aligned} \int_0^T dr_t &= \int_0^T \kappa \sigma dt + \int_0^T \sigma dz_t \\ r_T - r_0 &= \int_0^T \kappa \sigma dt + \int_0^T \sigma dz_t \\ r_T &= r_0 + \kappa \sigma T + \sigma z_T \end{aligned}$$

Expected Value and Variance

6 Application of Itô's Lemma (IV)

Assume the stock price change process and foreign exchange rate change process are

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dz_{S_t} \\ dX_t &= \mu X_t dt + \sigma X_t dz_{X_t} \\ Cov(dz_{S_t}, dz_{X_t}) &= \rho dt \end{aligned}$$

if $f_t = f(t, X_t, S_t) = S_t X_t$, According to Taylor series

$$\begin{aligned} df_t &= \left(\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial S_t} \mu_S S_t + \frac{\partial f_t}{\partial X_t} \mu_X X_t + \frac{1}{2} \frac{\partial^2 f_t}{\partial S_t^2} \sigma_S^2 S_t^2 + \frac{1}{2} \frac{\partial^2 f_t}{\partial X_t^2} \sigma_X^2 X_t^2 + \frac{\partial^2 f_t}{\partial S_t \partial X_t} \sigma_S S_t \sigma_X X_t \rho \right) dt \\ &\quad + \frac{\partial f_t}{\partial S_t} \sigma_S S_t dz_{S_t} + \frac{\partial f_t}{\partial X_t} \sigma_X X_t dz_{X_t} \\ &= (X_t \mu_S S_t + S_t \mu_X X_t + \sigma_S \sigma_X S_t X_t \rho) dt + X_t \sigma_S S_t dz_{S_t} + S_t \sigma_X X_t dz_{X_t} \end{aligned}$$

Because $f_t = S_t X_t$

$$dS_t X_t = (X_t \mu_S S_t + S_t \mu_X X_t + \sigma_S \sigma_X S_t X_t \rho) dt + X_t \sigma_S S_t dz_{S_t} + S_t \sigma_X X_t dz_{X_t}$$

Then we integrate it to get the close form of stock price with NTD dollars.

$$\begin{aligned} \int_0^T dS_t X_t &= \int_0^T (X_t \mu_S S_t + S_t \mu_X X_t + \sigma_S \sigma_X S_t X_t \rho) dt + \int_0^T X_t \sigma_S S_t dz_{S_t} + \int_0^T S_t \sigma_X X_t dz_{X_t} \\ (S_T - S_0)(X_T - X_0) &= \int_0^T (S_t X_t (\mu_S + \mu_X + \sigma_S \sigma_X \rho)) dt + \int_0^T S_t X_t (\sigma_S dz_{S_t} + \sigma_X dz_{X_t}) \\ &= [(S_T X_T - S_0 X_0)(\mu_S + \mu_X + \sigma_S \sigma_X \rho)] T + (S_T X_T - S_0 X_0)(\sigma_S dz_{S_T} + \sigma_X dz_{X_T}) \end{aligned}$$