1 Two periods Europian put option pricing

Risk neutral probability q

$$q = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05*\frac{1}{2}} - 0.8}{1.2 - 0.8} = \frac{0.2253}{0.4} = 0.5633$$

Find f_u and f_d

$$\begin{aligned} f_u &= e^{-rT}[q \times \max(K - S_0u^2, 0) + (1-q)\max(K - S_0ud, 0)] \\ &= e^{-0.05*\frac{1}{2}}[0.5633 \times 0 + 0.4367 \times 4] = 0.9753(0.4367 \times 4) = 1.704 \end{aligned}$$

$$\begin{aligned} f_d &= e^{-rT}[q \times \max(K - S_0du, 0) + (1-q)\max(K - S_0d^2, 0)] \\ &= e^{-0.05*\frac{1}{2}}[0.5633 \times 4 + 0.4367 \times 20] = 0.9753(0.5633 \times 4 + 0.4367 \times 20) = 10.72 \end{aligned}$$

Find f

$$\begin{aligned} f &= e^{-rT}[q \times \max(K - S_0u, 0) + (1-q)\max(K - S_0d, 0)] \\ &= e^{-0.05*\frac{1}{2}}[0.5633 \times 1.704 + 0.4367 \times 10.72] \\ &= 0.9753(0.5633 \times 1.704 + 0.4367 \times 10.72) = 5.5 \end{aligned}$$

2 Two periods American put option pricing

In American put option, we need to consider whether the investor will exercise the put option before the expire day.

1. When S_0u, the value of early exercise will be 0, smaller then the value of waiting for expire day which = 1.704
2. When S_od, the value of early exercise will be 12, larger then the value of waiting for expire day which = 10.72

So, the price of put option f is

$$f = 0.9753(0.5633 \times 1.704 + 0.4367 \times 12) = 6.05$$

3 The price of Europian call option when $\Delta t = \frac{T}{N}$

First, according to the two periods Europian call option pricing formula, we know

$$\begin{aligned} f_u &= e^{-r\Delta t}[q \times f_{uu} + (1-q) \times f_{ud}] \\ f_d &= e^{-r\Delta t}[q \times f_{du} + (1-q) \times f_{dd}] \\ f &= e^{-r\Delta t}[q \times f_u + (1-q) \times f_d] \\ &= e^{-2r\Delta t}[q^2 f_{uu} + 2q(1-q) f_{ud} + (1-q)^2 f_{dd}] \end{aligned}$$

So, when $N = 3$, we have

$$f = e^{-3r\Delta t}\Big[q^3 f_{C_3^3 u^3 d^0} + 3q^2(1-q) f_{C_2^3 u^2 d^1} + 3q(1-q)^2 f_{C_1^3 u^1 d^2} + (1-q)^3 f_{C_0^3 u^0 d^3}\Big]$$

From the above formula, we observed that the parameters are following the Binominal Distribution. So the generalized formula when $\Delta t = \frac{T}{N}$ is

$$\begin{aligned} f &= e^{-Nr\Delta t}\Big[C_N^N q^N (1-q)^0 f_{C_N^N u^N d^0} + C_{N-1}^N q^{N-1}(1-q)^1 f_{C_{N-1}^N u^{N-1} d^1} + \cdots + C_0^N q^0 (1-q)^N f_{C_0^N u^0 d^N}\Big] \\ &= e^{-Nr\Delta t}\Big[\sum_{j=0}^{N} C_j^N p^j (1-p)^{N-j} \max\big[0, u^j d^{N-j} S - K\big]\Big] \end{aligned}$$

Let a stand for the minimum number of upward moves that the stock must make over the next n periods for the call to finish in-the-money.

$\forall\ j < a$,

$$\max\big[0, u^j d^{N-j} S - K\big] = 0$$

$\forall\ j \geq a$,

$$\max\big[0, u^j d^{N-j} S - K\big] = u^j d^{N-j} S - K$$

So,

$$f = e^{-Nr\Delta t}\left[\sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j} \big[u^j d^{N-j} S - K\big]\right]$$

4 Issues from *Option Pricing-A Simplified Approach*

$$
\begin{aligned}
f &= e^{-Nr\Delta t}\left[\sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j}\left[u^j d^{N-j} S - K\right]\right] \\
&= S\left[\sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j}\left(\frac{u^j d^{N-j}}{e^{Nr\Delta t}}\right)\right] - Ke^{-Nr\Delta t}\left[\sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j}\right]
\end{aligned}
$$

We know that

$$\sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j}$$

is a complementary Binominal Distribution. So we can denote it as

$$\phi(a; N, p)$$

We also define that

$$p' \equiv \frac{u}{e^{Nr\Delta t}} p \quad \; and \quad \; 1-p' \equiv \frac{d}{e^{Nr\Delta t}}(1-p)$$

So

$$p^j (1-p)^{N-j}\left[\frac{u^j d^{N-j}}{e^{Nr\Delta t}}\right] = p'^j (1-p')^{N-j}$$

and

$$\sum_{j=a}^{N} C_j^N p'^j (1-p')^{N-j} = \phi(a; n, p')$$

Now, we can rewrite the full formula as

$$f = S\phi(a; n, p') - Ke^{-Nr\Delta t}\phi(a; n, p)$$

Assume that S^* is the stock price over N peiods, and there are j upwards. So

$$\log(\frac{S^*}{S}) = \log(u^j d^{n-j}) = j\log(\frac{u}{d}) + n\log d$$

As j is a random variable

$$E(\log(\frac{S^*}{S})) = E(j)\log(\frac{u}{d}) + n\log d$$

$$Var(\log(\frac{S^*}{S})) = Var(j)[\log(\frac{u}{d})]^2$$

From the above equations, we know that

$$\phi(a; n, p) = P(j \geq a)$$

So we can conclude

$$1 - \phi[a; n, p] = P(j \leq a - 1) = P(\frac{j - np}{\sqrt{np(1-p)}} \leq \frac{a - 1 - np}{\sqrt{np(1-p)}})$$

If we consider a stock which in each period will move to uS with probability p and to dS with probability $1 - p$, and $\log(\frac{S^*}{S}) = j\log(\frac{u}{d}) + n\log(d)$. The mean and variance of this stock are

$$\hat{\mu_p} = p\log(\frac{u}{d}) + \log d$$

$$\hat{\sigma_p^2} = p(1-p)[\log(\frac{u}{d})]^2$$

Using these equalities, we find that

$$\frac{j - np}{\sqrt{np(1-p)}} = \frac{\log(\frac{S^*}{S}) - \hat{\mu_p}n}{\hat{\sigma_p}\sqrt{n}}$$

and [1]

$$\frac{a - 1 - np}{\sqrt{np(1-p)}} = \frac{\log(\frac{K}{S}) - \hat{\mu_p}n - \varepsilon\log(\frac{u}{d})}{\hat{\sigma_p}\sqrt{n}}$$

In the continuous time model, and $N \to \infty$, Binominal Distribution asymptotically approaches to the Normal Distribution. So the formula will be

$$f = SN(x) - Ke^{-Nr\Delta t}N(x - \sigma\sqrt{t})$$

Where

$$x \equiv \frac{\log(S/Ke^{-Nr\Delta t})}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}$$

[1] I can't clearly know what actually appends below this equaiton, so I briefly write the solution.