1 Proof properties of Wiener process

Assume that $t < s$

$$X_s \sim N(0, s) \quad X_t \sim N(0, t)$$

1. By independent increments, we know that $E(X_s|X_t = B) = E(X_s - X_t + X_t|X_t = B) = E(X_s - X_t|X_t = B) + X_t = 0 + B = B$

2. $Var(X_s|X_t = B) = E\{[X_s - E(X_s|X_t = B)]^2|X_t = B\} = E[(X_s - B)^2|X_t = B]$
$= E[X_s^2 - 2X_sB + B^2|X_t = B] = E[X_s^2|X_t = B] - 2BE(X_s|X_t = B) + B^2$
$= E[X_s^2|X_t = B] - B^2 = Var(X_s) - Var(X_t) = s - t$

3. $X_t|X_s = B$ is a bivariate normal distribution which $\sim N(B \times \frac{t}{s}, (s - t) \times \frac{t}{s})$. So $E(X_t|X_s = B) = B \times \frac{t}{s}$

4. According to 3. $Var(X_t|X_s = B) = (s - t) \times \frac{t}{s}$

5. $Cov(X_s, X_t) = E(X_sX_t) - E(X_s)E(X_t) = E(X_sX_t) - 0 = E(X_sX_t)$
$= E(X_s - X_{s-1} + X_{s-1} - X_{s-2} + \cdots + X_{t+1} - X_t + X_t)X_t$
$= E(X_S - X_{s-1})X_t + \cdots + E(X_t)X_t = E(X_t^2) = t^2$

2 Proof the Quadratic Variation of Brownian Motion

Proof.

3 Application of Itô's Lemma (I)

Assume the stock price change process is

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

$$f_t = \ln S_t$$

According to Taylor series

$$df = (\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}\mu S_t(S_t, t) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2(S_t, t))dt + \frac{\partial f}{\partial S}\sigma S_t(S_t, t)dz_t$$

Because $f_t = \ln S_t$

$$\begin{aligned} d\ln S_t &= (\frac{1}{S_t}\mu S_t + \frac{-1}{2S_t^2}\sigma^2 S_t^2)dt + \frac{1}{S_t}\sigma S_t dz_t \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz_t \end{aligned}$$

Then we inergrate it to get the close form of stock price.

$$\begin{aligned} \int_0^T d\ln S_t = \int_0^T (\mu - \frac{1}{2}\sigma^2)dt + \int_0^T \sigma dz_t \\ \ln(S_T - S_0) = (\mu - \frac{1}{2}\sigma^2)T + \sigma z_T \\ (S_T - S_0) = \exp[(\mu - \frac{1}{2}\sigma^2)T + \sigma z_T] \\ S_T = S_0 + \exp[(\mu - \frac{1}{2}\sigma^2)T + \sigma z_T] \end{aligned}$$

Expectation value and variance

4 Application of Itô's Lemma (II)

Assume the stock price change process is

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

$$f_t = S_t - Ke^{-r(T-t)}$$

According to Taylor series

$$\begin{aligned} df &= (-rK^{-r(T-t)} + \mu S_t + \frac{(\sigma S_t)^2}{2} \times 0)dt + \sigma S_t dz_t \\ &= (\mu S_t - rK^{-r(T-t)})dt + \sigma S_t dz_t \end{aligned}$$

5 Application of Itô's Lemma (III)

Assume the stock price change process is

$$dr_t = \kappa$$

$$f_t = \ln S_t$$