1 Multiple Linear Equation

Proof

$$\min_{\alpha,\ \beta_i} E[(Y-\alpha-\beta_1X_1-\cdots-\beta_nX_n)^2] \Rightarrow \begin{cases} \hat{\alpha} & = E(Y)-\sum_{i=1}^n \beta_i X_i \\ \hat{\beta} & = Var(X)^{-1}Cov(X,Y) \end{cases}$$

To minimize $E[(Y-\alpha-\beta_1X_1-\cdots-\beta_nX_n)^2]$, we need to get the first order condition of α

$$\begin{aligned} &\hat{\alpha} : E(Y-\hat{\alpha}-\hat{\beta}_1X_1-\cdots-\hat{\beta}_nX_n)=0 \\ &\hat{\alpha} = E(Y)-E(\hat{\beta}_1X_1-\cdots-\hat{\beta}_nX_n) \\ &\hat{\alpha} = E(Y)-\sum_{i=1}^n \hat{\beta}_i E(X_i) \end{aligned}$$

$$\begin{aligned} Cov(\mathbb{X},Y) &= Cov(\mathbb{X},\alpha+\beta\mathbb{X}) = Cov(\mathbb{X},\beta\mathbb{X}) = \beta Cov(\mathbb{X},\mathbb{X}) = \beta Var(\mathbb{X}) \\ &\Rightarrow \hat{\beta} = Var(\mathbb{X})^{-1}Cov(\mathbb{X},Y) \end{aligned}$$

2 Radon Nikodym Derivative

Proof

$$\frac{dQ}{dP}\ exists \Rightarrow P >> Q$$

$P >> Q$ means that P dominates Q, if and only if Q is absolutely continuous $w.r.t$ P.

$$Q(B) = \int_B \frac{dQ}{dP}dP$$

If we assume $P(B)=0$, we know that $dP=0$

$$Q(B) = \int_B \frac{dQ}{dP}dP = \int_B 0 = 0$$

So we can conclude that if $\dfrac{dQ}{dP}$ exists, then $P(B)=0 \Rightarrow Q(B)=0(P>>Q)$

3 Explain Equation

Because $\xi = \dfrac{dQ}{dP}$, $dQ = \xi dP$

$$\int_G \frac{E_p(\xi X|\mathcal{G})}{E_p(\xi|\mathcal{G})}dQ = \int_G \frac{E_p(\xi X|\mathcal{G})}{E_p(\xi|\mathcal{G})}\xi dP$$

According to Law of Iterated Expectation (L.I.E), we know that

$$\int_G \frac{E_p(\xi X|\mathcal{G})}{E_p(\xi|\mathcal{G})}\xi dP = \int_G E_p\left[\frac{E_p(\xi X|\mathcal{G})}{E_p(\xi|\mathcal{G})}\xi\middle|\mathcal{G}\right]dP$$

Because $\dfrac{E_p(\xi X|\mathcal{G})}{E_p(\xi|\mathcal{G})}$ is a constant, we can conclude that

$$\int_G E_p\left[\frac{E_p(\xi X|\mathcal{G})}{E_p(\xi|\mathcal{G})}\xi\middle|\mathcal{G}\right]dP = \int_G \frac{E_p(\xi X|\mathcal{G})}{E_p(\xi|\mathcal{G})}E_p(\xi|\mathcal{G})dP = \int_G E_p(\xi X|\mathcal{G})dP$$

Thus, we know that X is $\mathcal{G}$ measurable and intergral random variable, So

$$\int_G E_p(\xi X|\mathcal{G})dP = \int_G X\xi dP = \int_G XdQ$$