Problem 1.

$(X_t)_{t=0}^T$ is a martingale under $\mathcal{Q}$ measure $\Leftrightarrow$ $(\xi_t X_t)_{t=0}^T$ is a martingale under $\mathcal{P}$ measure.

First, we must prove $E^Q(X_t|\mathcal{F}_s) = X_s$

$$E^P(\xi_t X_t|\mathcal{F}_s) = \xi_s X_s, \quad \forall\ 0 \le s \le t$$

$$\begin{aligned} E^Q(X_t|\mathcal{F}_s) &= \frac{E^P(\xi X_t|\mathcal{F}_s)}{E^P(\xi|\mathcal{F}_s)} = \frac{E^P[E^P(\xi X_t|\mathcal{F}_t)|\mathcal{F}_s]}{\xi_s} = \frac{E^P[X_t E^P(\xi|\mathcal{F}_t)|\mathcal{F}_s]}{\xi_s} = \frac{E^P(X_t\xi_t|\mathcal{F}_s)}{\xi_s} \\ &= \frac{X_s E^P(\xi_t|\mathcal{F}_s)}{\xi_s} = \frac{X_S \xi_s}{\xi_s} = X_s \end{aligned}$$

Problem 2.

$P(W_1 \le X_1, W_2 \le X_2, W_3 \le X_3) = P(W_1 \le X_1, W_2 - W_1 \le X_2 - X_1, W_3 - W_2 \le X_3 - X_2)$

Based on Markov property

$$P(W_1 \le X_1)P(W_2 \le X_2|W_1 \le X_1)P(W_3 \le X_3|W_2 \le X_2)$$

Because $W_2 \le X_2 \Rightarrow W_1 + W_2 - W_1 \le X_2 \Rightarrow W_2 - W_1 \le X_2 - W_1$

$$P(W_2 - W_1 \le X_2 - W_1|W_1 \le X_1) = P(W_2 - W_1 \le X_2 - X_1)$$

And we do the same thing for $P(W_3 \le X_3|W_2 \le X_2)$, then we can get

$$P(W_3 \le X_3|W_2 \le X_2) = P(W_3 - W_2 \le X_3 - X_2)$$

Problem 3.

Use *Jacobian* proof the joint distribution of $W_{t_1}, \cdots, W_{t_k}$ is the same as the joint distribution $W_{t_1}, W_{t_2} - W_{t_1}, \cdots, W_{t_k} - W_{t_{k-1}}$

Problem 4.

$B(t) = \dfrac{1}{c}W(c^2t)$ is a Wiener Process

Problem 5.

$P(t,x,y) = \dfrac{1}{\sqrt{2\pi t}}\exp\left\{ -\dfrac{(x-y)^2}{2t}\right\}$ satisfied heat equation $\dfrac{1}{2}P_{xx} - P_t = 0$

Problem 6.

$W^2(t) - t$ is a martingale