Useful sets

$\mathcal{F}_0 = \{\phi, \Omega\} \quad \mathcal{F}_1 = \{\{\omega_1, \omega_2, \omega_3\}, \cdots, \Omega\} \quad \mathcal{F}_2 = \{\{\omega_1\}, \cdots, \Omega\} \quad \mathcal{F}_3 \; with \; 64 \; items$
$A = \{\omega_1, \omega_4\} \quad B = \{\omega_1, \omega_2, \omega_3\} \quad C = \{\omega_3, \omega_4, \omega_5\}$

$$X_1(\omega_i) = \begin{cases} 1, & i = 1,2,3,4,5,6 \end{cases} \qquad X_2(\omega_i) = \begin{cases} 2, & i = 1,4 \\ 4, & i = 2,3,5,6 \end{cases}$$
$$X_4(\omega_i) = \begin{cases} 1, & i = 1,2,3 \\ 3, & i = 4,5,6 \end{cases} \qquad X_5(\omega_i) = \begin{cases} i, & i = 1,2,3,4,5,6 \end{cases}$$

1 Find Conditional Probability

Conidtional Probability

1. $P(A|\mathcal{F}_0) = P(A|\{\phi, \Omega\}) = P(\{\omega_1, \omega_4\}|\{\phi, \Omega\}) = \frac{1}{3}$
2. $P(C|\mathcal{F}_1) = P(C|\{\{\omega_1, \omega_2, \omega_3\}, \cdots, \Omega\}) = P(\{\omega_3, \omega_4, \omega_5\}|\{\{\omega_1, \omega_2, \omega_3\}, \cdots, \Omega\}) = \begin{cases} \frac{1}{2}, & i = 1,2,3 \\ \frac{2}{3}, & i = 4,5,6 \end{cases}$
3. $P(A|\mathcal{F}_2) = P(A|\{\{\omega_1\}, \cdots, \Omega\}) = P(\{\omega_1, \omega_4\}|\{\{\omega_1\}, \cdots, \Omega\}) = \begin{cases} 1, & i = 1,4 \\ 0, & i = 2,3,5,6 \end{cases}$

Proof. $A \in \mathcal{F}_2 \Rightarrow P(A|\mathcal{F}_2) = \mathbb{1}_A$ □

$$\begin{aligned} & P(A|\mathcal{F}_2)(\omega) = \frac{P(A \cap G_\omega)}{P(G_\omega)} = 1 \\ & \forall k \notin A \; then \; A \cap G_k = 0 \Rightarrow P(A \cap G_k) = 0 \Rightarrow P(A|\mathcal{F}_2)(k) = 0 \\ & Hence, \; P(A|\mathcal{F}) = \mathbb{1}_A \end{aligned}$$

2 Find conditional expectation

Proof. $E(X_2|\mathcal{F}_1)$ is an expectation of X_4 on $\mathcal{F}_1$ □

$$E(X_2|\mathcal{F}_1)(\omega_1) = E(X_2|\mathcal{F}_1)(\omega_4) = 2$$
$$E(X_2|\mathcal{F}_1)(\omega_3) = \cdots = E(X_2|\mathcal{F}_1)(\omega_6) = 4$$
$$E(X_2|\mathcal{F}_1) = 2 \times \frac{1}{3} + 4 \times \frac{2}{3} = \frac{10}{3}$$

Find $E(X_4)$

According to Law of Iterated Expectation (L.I.E), $E(X_4) = E[E(X_4|\mathcal{F}_1)]$. We know that X_4 is measurable in $\mathcal{F}_1$, so $E(X_4|\mathcal{F}_1) = X_4 = 2$. So $E(X_4) = E[E(X_4|\mathcal{F}_1)] = E(2) = 2$

Conditional expectation

1. $E(X_5|\mathcal{F}_2) = \begin{cases} 1, & i=1 \\ \frac{13}{5}, & i=2,3 \\ 4, & i=4 \\ \frac{16}{5}, & i=5,6 \end{cases}$

2. $E(X_0|\mathcal{F}_2) = 1, i = 1,2,3,4,5,6$

3. $E(X_4|\mathcal{F}_0) = 2$

4. $E(X_4|\mathcal{F}_2) = \begin{cases} 1, & i=1 \\ 1, & i=2,3 \\ 3, & i=4 \\ 3, & i=5,6 \end{cases}$

5. $E(X_4|\mathcal{F}_3) = \begin{cases} 1, & =1,2,3 \\ 3, & =4,5,6 \end{cases}$

6. $E(X_4|X_2 = 4) = E(X_4|X_2(\omega_i), i = 2,3,5,6) = \dfrac{1 \times 5 + 3 \times 3}{8} = \dfrac{7}{4}$

Varify $E(X_2X_4|\mathcal{F}_1) = X_4E(X_2|\mathcal{F}_1)$

X is $\mathcal{G}$ measurable $\Rightarrow X(\omega) = \sum C_i \mathbb{1}_{G_i}$
Suppose X_4 is $\mathcal{F}$ measurable, then $X_4 = \sum C_i \mathbb{1}_{G_i}$

$$\int_G X_4E(X_2|\mathcal{F}_1)dP = \int_{G\cap B} E(X_2|\mathcal{F}_1)dP = \int_{G\cap B} X_2 dP = \int_G X_4X_2 dP = \int_G E(X_4X_2|\mathcal{F}_1)dP$$

Proof. $E(X|\mathcal{F}_0) = E(X)$ □

If X is $\mathcal{F}_0$ measurable, that X must be $\mathcal{G}$ measurable. According to Law of Iterated Expectation (L.I.E)

$$E(X|\mathcal{F}_0) = E[E(X|\mathcal{F}_0)|\mathcal{G}] = E[E(X|\mathcal{G})|\mathcal{F}_0] = E[E(X|\mathcal{G})] = E(X)$$