CH3. Single Variable Distribution

Probability mass function (pmf)

Given the range of *discrete* random variable X $supp(X) \in R$, function $f(x) : R \to [0,1]$ is defined as

$$f(x) = \begin{cases} P(X = x), & x \in supp(X) \\ 0, & x \notin supp(X) \end{cases} \tag{1}$$

$f(x)$ satisfy:

1. $f(x) > 0, \forall x \in supp(X)$
2. $\sum_{x \in supp(X)} f(x) = 1$
3. $P(X \in A) = \sum_{x \in A} f(x)$, A$\in supp(X)$

And we can say that $f(x)$ is the pmf of X.

Cumulative distribution function (CDF)

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x) \tag{2}$$

We can said that $F(x)$ is CDF.
CDF is 'Cumulative' distribution function, so it's a nondecreasing function.

Properties of discrete random variables' CDF

1. $P(X > a) = 1 - P(X \leq a) = 1 - F(a)$
2. $P(X \geq a) = P(X = a) + P(X > a) = f(a) + 1 - F(a) = 1 - F(a) + f(a)$
3. $P(a < X \leq b) = F(b) - F(a)$
4. $P(a < X < b) = F(b) - f(b) - F(a)$
5. $P(a \leq X \leq b) = F(b) - F(a) + f(a)$
6. $P(a \leq X < b) = F(b) - F(a) + f(a) - f(b)$
7. $\lim_{x \to -\infty} F(x) = 0,\ \lim_{x \to \infty} F(x) = 1$

Probability density function

$$P(a < x < b) = \int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \tag{3}$$

$f(x)$ must satisfy

1. $f(x) > 0, \forall x \in supp(X)$
2. $\int_{x \in supp(X)} f(x)dx = 1$

Something important:

- You need to write down the range of x to get full score
- Point probability = 0
- $f(x)$ is height, not probability, so it may > 1

Because

$$\int_a^b f(x)dx = [F(x)]_a^b \tag{4}$$

We can see that

$$f(x) = \frac{\partial F(x)}{\partial x}, \qquad F(x) = \int_{-\infty}^{x} f(u)du \tag{5}$$

Moments of random variables

Expected Value

$$E(x) = \int_a^b xf(x)dx \tag{6}$$

Variance

$$\begin{aligned} Var(x) &= E\{[x - E(x)]^2\} = E\{x^2 - 2xE(x) - [E(x)]^2\} \\ &= E(x^2) - 2[E(x)]^2 + [E(x)]^2 = E(x^2) - [E(x)]^2 \end{aligned} \tag{7}$$

Quantile

Given $0 < p < 1$, and $F(\cdot)$ is strictly increasing function, if

$$\pi_p = F^{-1}(p) \tag{8}$$

$F(\pi_p) = P(X \leq \pi_p) = p$, then we can say π_p is the $p-th$ quantile of r.v. X

Moment, central moment and standard moment

Given $E(X)=\mu$, $Var(X)=\sigma^2$

1. $k-th$ moment: $E(X^k)$
2. $k-th$ central moment: $\mu_k = E\Big[(X-\mu)^k\Big]$
3. $k-th$ standard moment: $\gamma_k = E(Z_X^k) = E\Bigg[(\frac{X-\mu}{\sigma})^k\Bigg] = \frac{E[(X-\mu)^k]}{\sigma^k}$

Moment generating function

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} f(x) = \sum_x e^{tx} P(X=x) \\ \int_x e^{tx} f(x)dx \end{cases} \tag{9}$$

To calculate moment by MGF

$$E(X^k) = M_X^{(k)}(t)|_{t=0} \tag{10}$$

Linear combination of MGF
Let $Y = aX + b$

$$M_Y(t) = E(e^{tY}) = E[e^{t(aX+b)}] = E[e^{tb}e^{atX}] = e^{bt}M_X(at) \tag{11}$$

Cumulant generating function
Let $K_X(t) = \ln M_X(t)$

1. $K'_X(t)|_{t=0} = E(X)$
2. $K''_X(t)|_{t=0} = Var(X)$
3. $K'''_X(t)|_{t=0} = E([X-E(X)]^3)$
4. $K''''_X(t)|_{t=0} = E([X-E(X)^4]) - 3[Var(X)]^2$

Jacobian transformation method

Step1. Inverse function: get $x = g^{-1}(y)$, and check if it's 1 to 1.
Step2. Jacobian: calculate $|J| = |\frac{\partial g^{-1}(y)}{\partial y}|$
Step3. Range: from $supp(X)$ and $g(y)$ get the range of new variable
Step4. Substitute: get $f_Y(y) = f_X[g^{-1}(y)] \times |J|$

Truncated distribution

$$f(x|X \leq a) = \frac{f(x)}{F(a)},\ x \leq a \tag{12}$$

$$f(x|X \geq a) = \frac{f(x)}{1 - F(a)},\ x \geq a \tag{13}$$

$$f(x|a \leq X \leq b) = \frac{f(x)}{F(b) - F(a)},\ a \leq X \leq b \tag{14}$$

Jensen's inequality

if $g(\cdot)$ satisfy $g[(1-\alpha)x + \alpha y] > (1-\alpha)g(x) + \alpha g(y)$, it's a strictly concave function. Then

$$E[g(X)] < g[E(X)] \tag{15}$$

if $g(\cdot)$ satisfy $g[(1-\alpha)x + \alpha y] < (1-\alpha)g(x) + \alpha g(y)$, it's a strictly conves function. Then

$$E[g(X)] > g[E(X)] \tag{16}$$

if $g(\cdot)$ satisfy $g[(1-\alpha)x + \alpha y] = (1-\alpha)g(x) + \alpha g(y)$, it's a linear function. Then

$$E[g(X)] = g[E(X)] \tag{17}$$

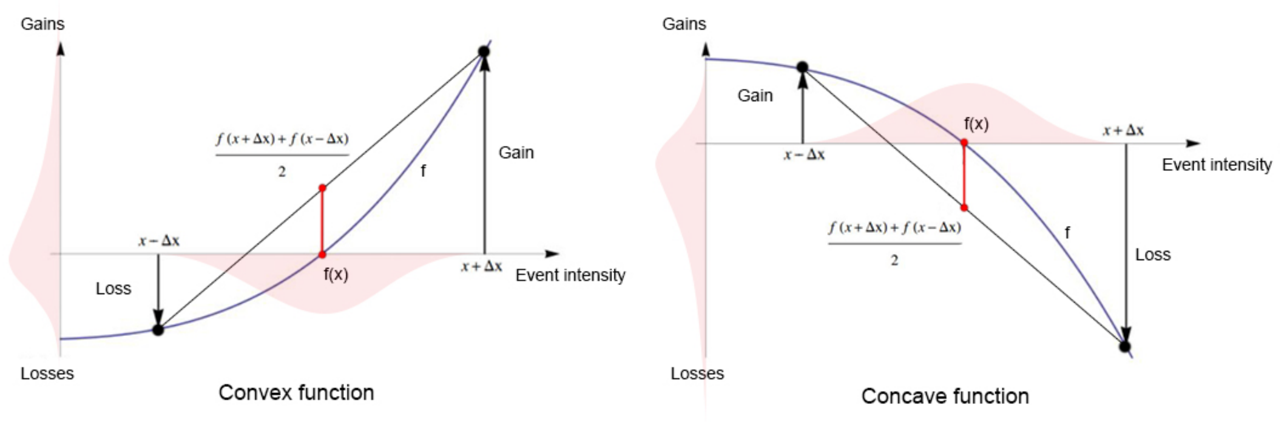

Figure 1: Jensen's Inequality