CH4. Multi-Variable Distribution

Conditional pdf

Given X, Y's joint pdf is $f_{XY}(xy)$

$$f(x|y) = \frac{f(x,y)}{f(y)} \tag{1}$$

$$f(y|x) = \frac{f(x,y)}{f(x)} \tag{2}$$

Use $f(y|x)$ as examples:

1. for x, we don't need to concern the function relationship between x and y

2. for y, we need to concern the function relationship between x and y

If the condition is a point:

$$P(X < a|Y = y) = \int_{-\infty}^{a} f(x|y)dx = \int_{-\infty}^{a} \frac{f(x,y)}{f(y)}dx \tag{3}$$

If the condition is a range:

$$P(X < a|Y < b) = \frac{P(X < a \cap Y < b)}{P(Y < b)} \tag{4}$$

Moments

Given X and Y are bivariate random variables, their joint pdf is $f(x,y)$, then the expected value of $g(x,y)$ is:

$$E[g(X,Y)] = \int_y \int_x g(x,y)f(x,y)dxdy = \int_x \int_y g(x,y)f(x,y)dydx \tag{5}$$

Covariance and correlation coefficient

$$\sigma_{XY} = Cov(X,Y) = E\{[X - E(X)][Y - E(Y)]\} = E(XY) - E(X)E(Y) \tag{6}$$

$$\rho_{XY} = Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \tag{7}$$

Properties:

1. $E(aX + bY) = E(aX) + E(bY) = aE(x) + bE(y)$

2. $Cov(X, X) = Var(X)$

3. $Cov(X, c) = 0$

4. $Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$

5. $Cov(aX + b, cY + d) = acCov(X, Y)$

6. $Corr(aX + b, cY + d) = \frac{ac}{|ac|}Corr(X, Y) = sign(ac)Corr(X, Y)$

Given $X_1, X_2, \ldots, X_n$, then:

$$E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) \tag{8}$$

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n}\sum_{j\neq i} Cov(X_i, X_j) \tag{9}$$

$$E(\sum_{i=1}^{n} a_iX_i) = \sum_{i=1}^{n} a_iE(X_i) \tag{10}$$

$$Var(\sum_{i=1}^{n} a_iX_i) = \sum_{i=1}^{n} a_i^2Var(X_i) + \sum_{i=1}^{n}\sum_{j\neq i} a_ia_jCov(X_i, X_j) \tag{11}$$

Properties of individual r.v.

1. $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

2. $Var(X + Y) = Var(X - Y) = Var(X) + Var(Y)$

3. Let $Z = X + Y,\ M_Z(t) = E[e^{t(X+Y)}] = M_X(t) + M_Y(t)$

Given $X_1, X_2, \ldots, X_n$ are mutually independent

4. Let $Y = \sum_{i=1}^{n} X_i, M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$

i.i.d **random variables**

If a sequence of rnadom variables $X_1, X_2, \ldots X_n$ are mutually independent and identically distributed, they're called $i.i.d$ random variables.

1. $E(X_i) = \mu$
2. $Var(X_i) = \sigma^2$
3. $M_{X_i}(t) = M_X(t)$

Let $Y = \sum_{i=1}^{n} X_i$

1. $E(Y) = n\mu$
2. $Var(Y) = n\sigma^2$
3. $M_Y(t) = [M_X(t)]^n$

Let $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{Y}{n}$

1. $E(\bar{X}) = \mu$

2. $Var(\bar{X}) = \frac{\sigma^2}{n}$

3. $M_{\bar{X}}(t) = [M_X(\frac{t}{n})]^n$

Conditional Expectation and variance

conditional expectation

Given $X = x$, Y's conditional expectation is

$$E(Y|X = x) = \mu_{Y|X=x} = \int_y yf(y|x)dy \tag{12}$$

Given $Y = y$, X's conditional expectation is

$$E(X|Y = y) = \mu_{X|Y=y} = \int_x xf(x|y)dx \tag{13}$$

conditional variance

Given $X = x$, Y's conditional variance is

$$\begin{aligned} \sigma^2_{Y|X=x} &= Var(Y|X = x) = E\{[Y - E(Y|X = x)]^2|X = x\} \\ &= \int_y [y - E(Y|X = x)]^2 f(y|x)dy \end{aligned} \tag{14}$$

Given $Y = y$, X's conditional variance is

$$\begin{aligned} \sigma^2_{X|Y=y} &= Var(X|Y = y) = E\{[X - E(X|Y = y)]^2|Y = y\} \\ &= \int_x [x - E(X|Y = y)]^2 f(x|y)dx \end{aligned} \tag{15}$$

useful rule

$$E[h(X)|X] = h(X) \tag{16}$$

$$E[h(X)g(Y)|X] = h(X)E[g(Y)|X] \tag{17}$$