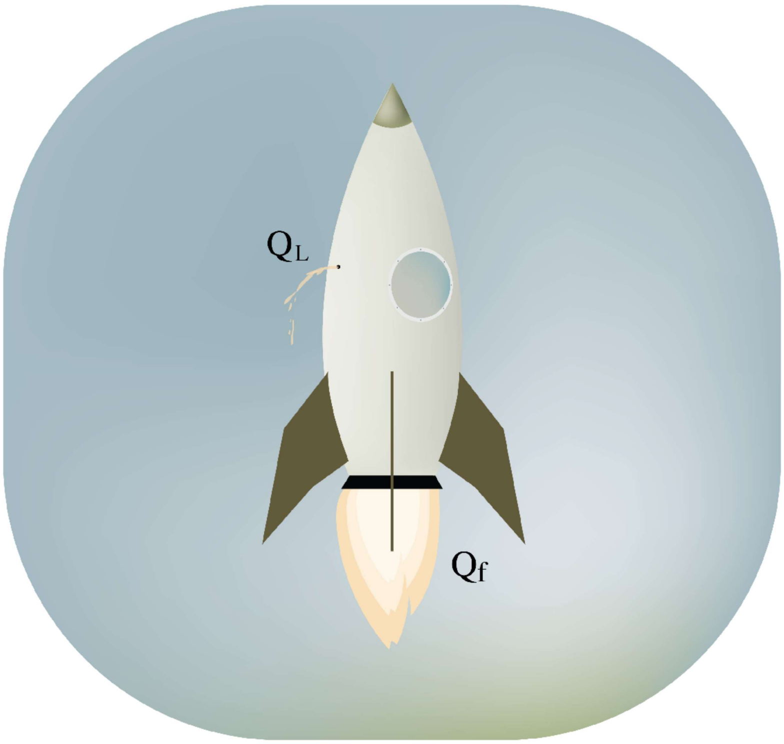
20-P-MOM-DY-41

A rocket with mass (including fuel) $m_0\ kg$ has $m_f\ kg$ of its mass dedicated to fuel. During flight, it is discovered that there is a puncture that allows $Q_L\ kg/s$ of fuel to leak from the system at $v_L\ m/s$ relative to the rocket. The rocket engine ejects fuel at $Q_f\ kg/s$ with velocity $v_f\ m/s$ relative to the rocket. Assuming up to be positive, determine the magnitude of the maximum velocity of the rocket if the rocket begins at rest. Ignore the effect of altitude on the weight of the rocket, and drag forces. Assume the propulsion of the leak is in the same direction as the primary fuel ejection.



Solution:

Recall: $\Sigma F_{cv} = m\frac{dv}{dt} - v_{D/e}\frac{dm_i}{dt}$

$\rightarrow -W = m\frac{dv}{dt} - Q_f v_f - Q_L v_L$

In order to solve for v_{max}, we rearrange the equation and integrate, using the information that v(0) = 0:
$\int_0^v dv' = \int_0^t (\frac{Q_f v_f + Q_L v_L}{m} - g)d\tau$

Since $m = (m_0 - Q_t t)$, where $Q_t = Q_f + Q_L$, we get $\int_0^v dv' = \int_0^t (\frac{Q_f v_f + Q_L v_L}{m_0 - Q_t t} - g)d\tau$

$\rightarrow v(t) = [-\frac{Q_f v_f + Q_L v_L}{Q_t}\ln(m_0 - Q_t \tau) - g\tau]_0^t$

$\rightarrow v(t) = \frac{Q_f v_f + Q_L v_L}{Q_t}\ln(\frac{m_0}{m_0 - Q_t t}) - gt$

The time t' needed to consume all the fuel is $t' = \frac{m_f}{Q_t}$, so:

$$v_{max} = v(t') = \frac{Q_f v_f + Q_L v_L}{Q_t} \ln\left(\frac{m_0}{m_0 - m_f}\right) - \frac{g m_f}{Q_t} = \frac{Q_f v_f + Q_L v_L}{Q_f + Q_L} \ln\left(\frac{m_0}{m_0 - m_f}\right) - \frac{g m_f}{Q_f + Q_L}$$