Chapter 4

Time Domain Analysis of Control System

PART (1)

By

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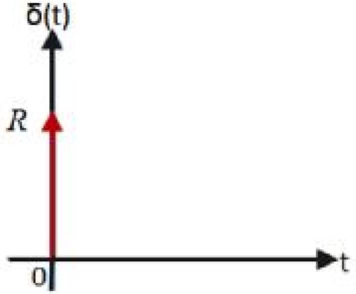
Typical Signals for the Time Response

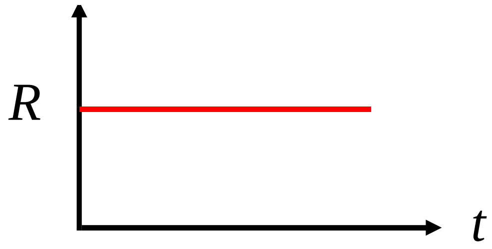
Some test signals

Impulse signal



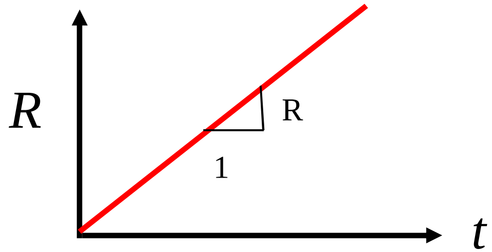
$$\boldsymbol{r(t) = R\mathrm{d}(t)} \qquad \boldsymbol{R(s) = R}$$

Step signal



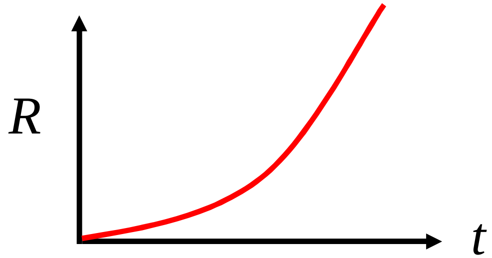
$$r(t) = Ru(t) \qquad \boldsymbol{R(s) = \frac{R}{s}}$$

Ramp signal



$$r(t) = Rtu(t) \qquad R(s) = \frac{R}{s^2}$$

Parabolic signal



$$r(t) = \frac{Rt^2}{2}u(t) \qquad R(s) = \frac{R}{s^3}$$

Relation between standard Test Signals

- Impulse

$$d(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Parabolic

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\int \qquad \int \qquad \int$$

$$\frac{d}{dt} \qquad \frac{d}{dt} \qquad \frac{d}{dt}$$

Control Systems Classifications

System Order

- The order of the system is the highest order of "s" in the denominator of the system T.F. C(s)/R(s). The procedure to determine the order of the system is:

1- Determine the equation which relates the system input and output.

2- Taking laplace transform of this equation and substituting zero initial conditions and calculate the T.F. of the system.

3- The highest order of s in the denominator of the system T.F. is the order of the system

$$\boxed{C(s)/R(s) = \frac{K}{s\,(s-1)}}$$

2nd order system

$$\boxed{C(s)/R(s) = \frac{24}{s(s+1)(s+6)}}$$

3rd order system

$$\boxed{C(s)/R(s) = \frac{K}{s^2(s+8)^2}}$$

4th order system

System Type

- The type of the system is the highest order of pole at the origin of the loop T.F. G(s)H(s). The procedure to determine the type of the system is:

1- Determine the forward and feedback path transfer functions G(s) and H(s) of the system.

2- Calculate the open loop T.F. G(s)H(s) of the system.

3- Factorize and simplify the expression for G(s) H(s) by algebric manipulations.

4- The highest order of pole at the origin of the loop T.F. G(s)H(s) is the type of the system

$$G(s) = \frac{K}{s^2(s+12)} \quad H(s) = 1$$

type 2 system

$$G(s) = \frac{K(s+3.15)}{s(s+1.5)(s+0.5)} \quad H(s) = 1$$

type 1 system

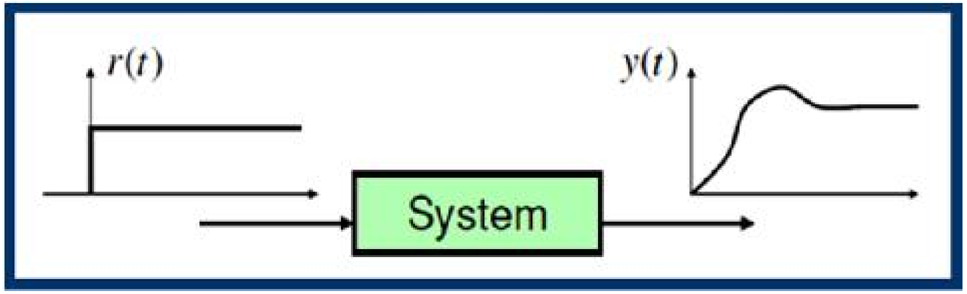
$$G(s) = \frac{5(s+1)}{(s+12)(s+5)} \quad H(s) = 1$$

type 0 system

Transient and Steady State Responses

System Response

- We would like to analyze a system property by applying a test input r(t) and observing a time response y(t).



- Time response can be divided as **Y(t) = C(t) = Output**

$$C(t) = C_t(t) + C_{ss}(t)$$

Transient **steady state**

Steady state response is the part of the total response that remains after the transient has died out.

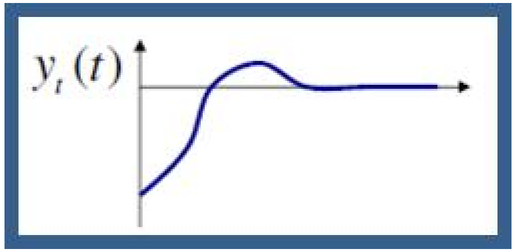
System Response

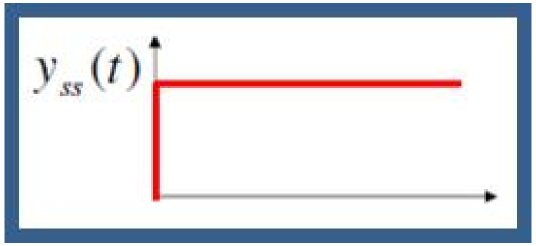
Time Responses – Input and Output

$$C(t) = C_t(t) + C_{ss}(t)$$

Transient response

Steady-state response





Transient response is the part of time response that goes to zero $t \longrightarrow \infty$

Steady state response is the part of the total response that remains after the transient has died out.

Poles and Zeros

- The output response of a system is the sum of two responses: the **Steady-state response** ***(forced response)*** and the **transient response** ***(natural response)***.

- The use of **poles** and **zeros** of the transfer function and their relationship to the time response of a system is such a technique.

- The concept of poles and zeros fundamental used in the analysis and design of control systems, to **simplifies** the evaluation of a system's response.

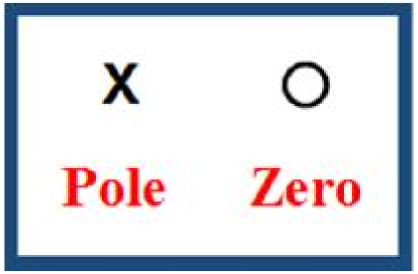
Poles, Zeros and S-plane

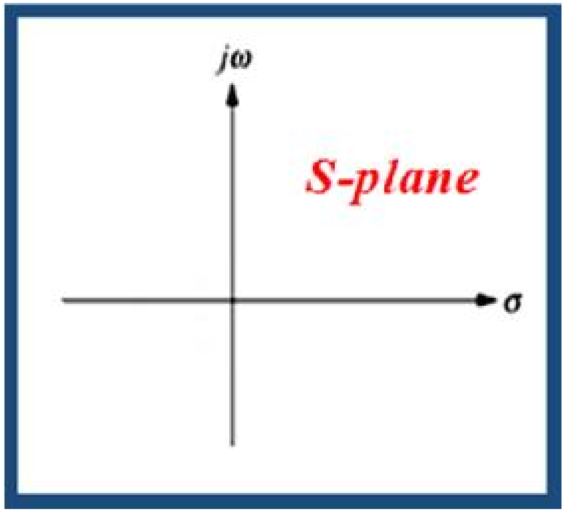
Poles of a Transfer Function

The poles of a transfer function are the roots of the characteristic polynomial in the denominator.

Zeros of a Transfer Function

The zeros of a transfer function are the roots of the characteristic polynomial in the nominator.





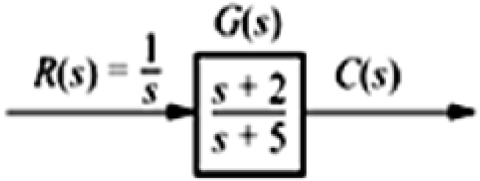
Poles and Zeros

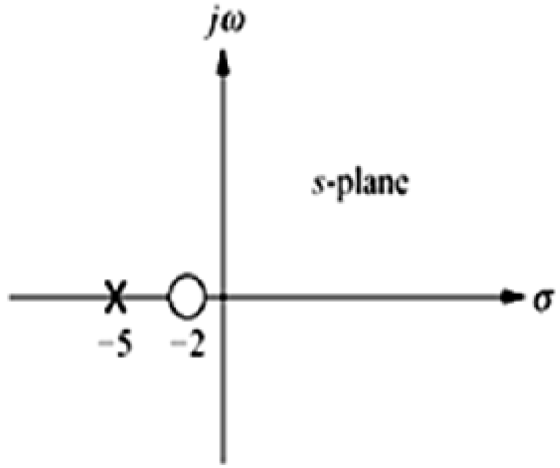
Poles and Zeros of a First-Order System

For the T.F. and input signal of the system given by

$$\frac{C(S)}{R(S)} = \frac{S+2}{S+5} \qquad R(S)=1/S$$

poles s = -5
Zeros s = -2





The output response is given by:

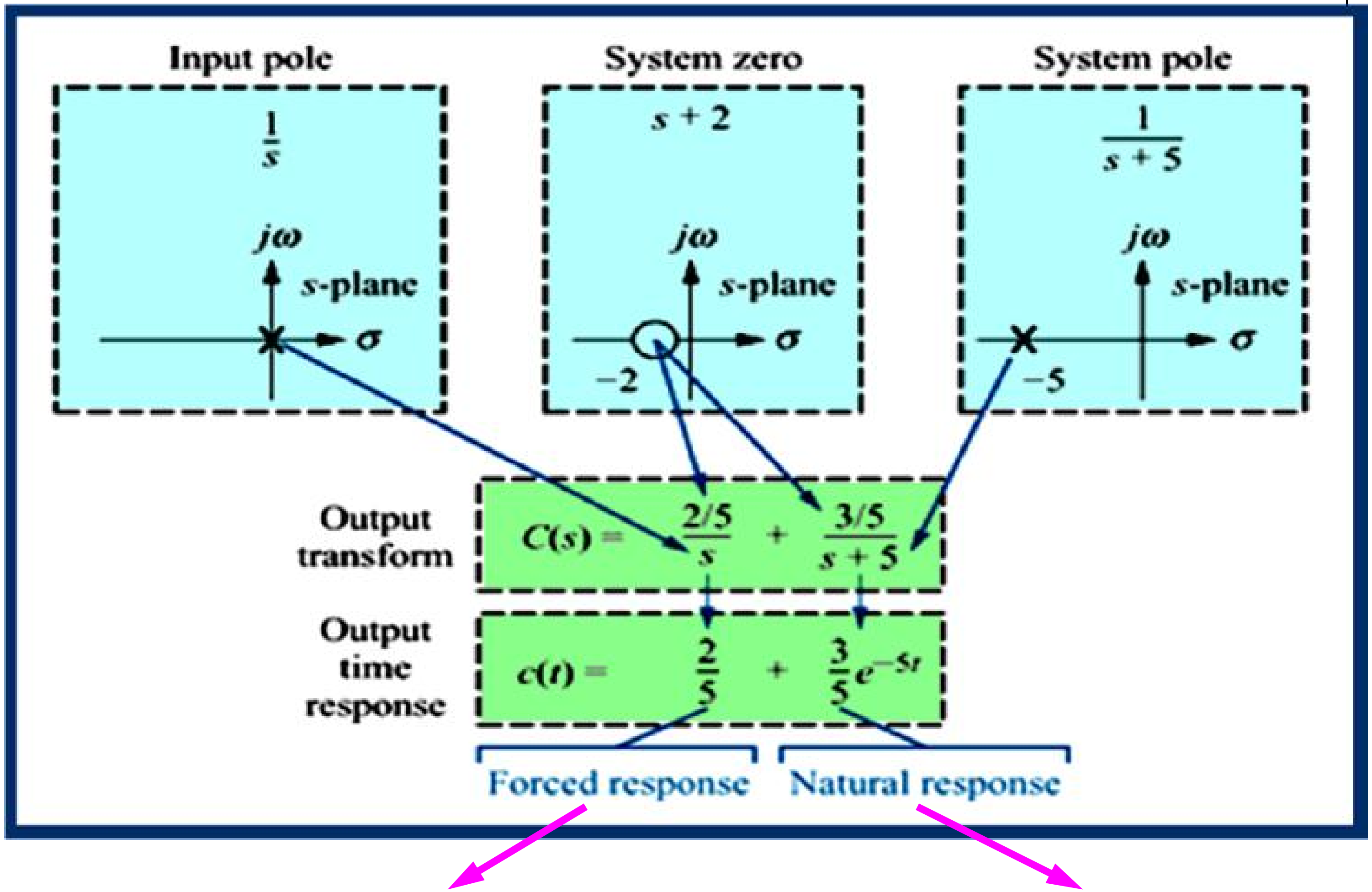
$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s\to 0} = \frac{2}{5} \qquad B = \left.\frac{(s+2)}{s}\right|_{s\to -5} = \frac{3}{5}$$

$$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5} \Rightarrow c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

Poles and Zeros

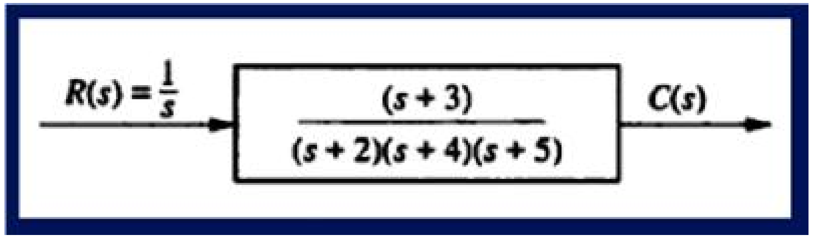
Poles and Zeros of a First-Order System



Steady-state response **Transient response**

Example 1

Given the system of Figure below, write the output, *c(t)*, in general terms. Specify the forced and natural parts of the solution.



Solution

By inspection, each system pole generates an exponential as part of the natural response. The input's pole generates the forced response. Thus,

$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\substack{\text{Forced}\\ \text{response}}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\substack{\text{Natural}\\ \text{response}}}$$

Taking the inverse Laplace transform, we get

$$C(t) = \underbrace{k_1}_{\text{Steady-state response}} + \underbrace{k_2 e^{-2t} + k_3 e^{-4t} + k_4 e^{-5t}}_{\text{Transient response}}$$

First Order Systems

First Order Systems

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

- Where ***K*** is the D.C gain and ***T*** is the time constant of the system.

- Time constant is a measure of how quickly a 1st order system responds to a unit step input.

- D.C Gain of the system is the ratio between the input signal and the steady state value of output.

First Order Systems

- For the first order system given below

$$\boxed{G(s) = \frac{10}{3s+1}}$$

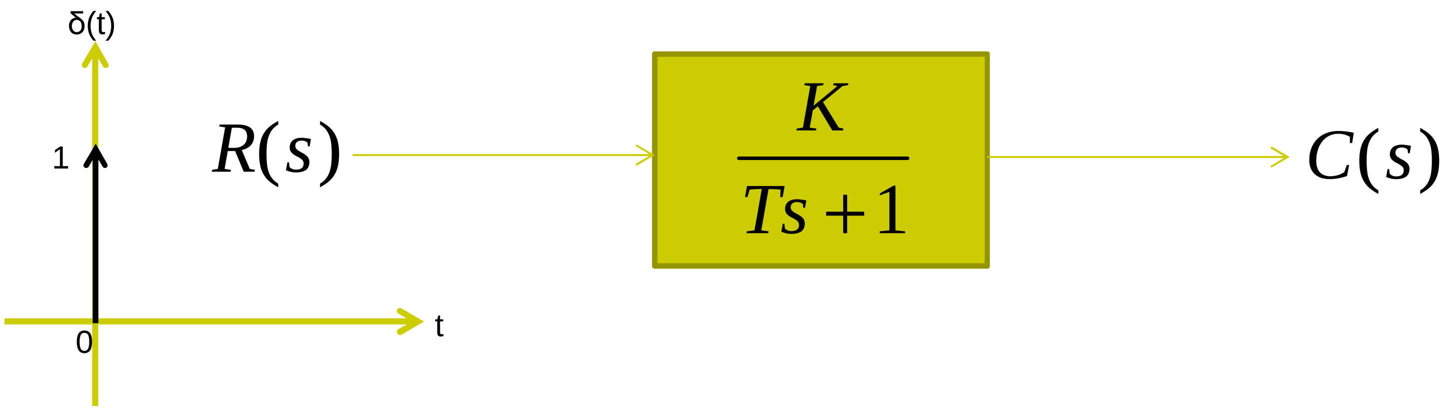
- D.C gain is 10 and time constant is 3 seconds.
- And for following system

$$G(s) = \frac{3}{s+5} = \frac{3/5}{1/5s+1}$$

- D.C Gain of the system is 3/5 and time constant is 1/5 seconds.

Impulse Response of 1st Order System

- Consider the following 1st order system



$$R(s) = d(s) = 1$$

$$C(s) = \frac{K}{Ts+1}$$

Impulse Response of 1st Order System

$$C(s) = \frac{K}{Ts+1}$$

- Re-arrange above equation as

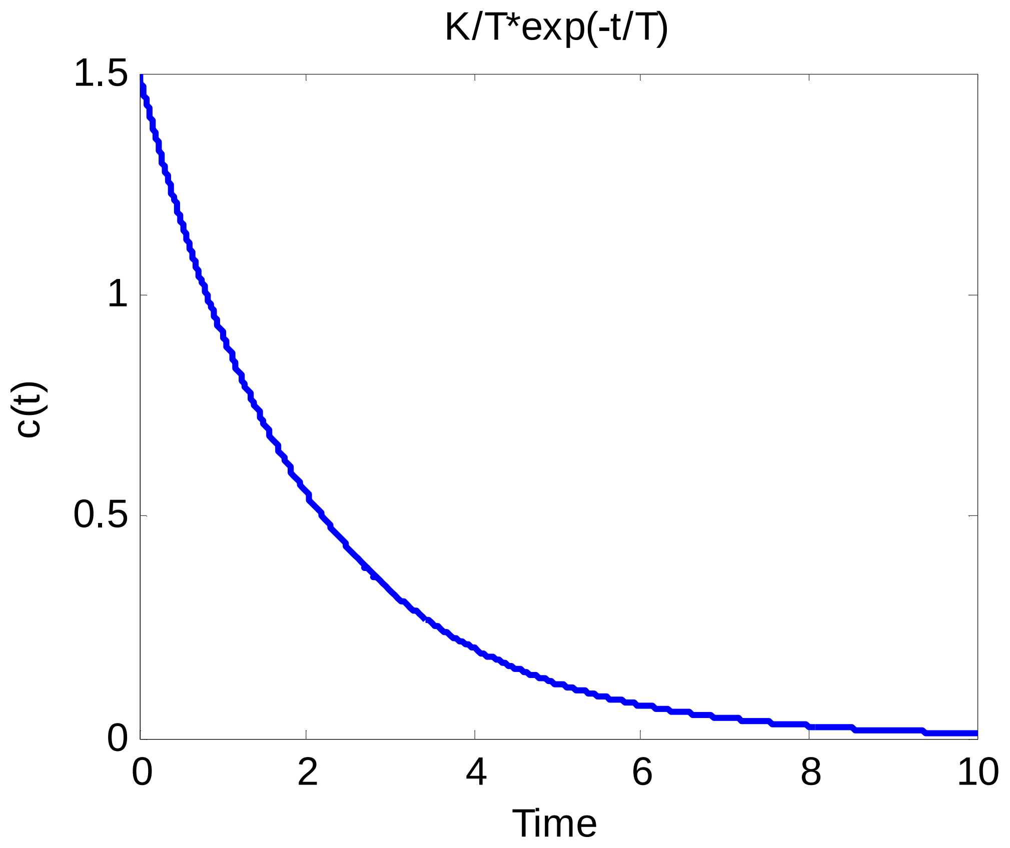
$$C(s) = \frac{K/T}{s+1/T}$$

- In order to represent the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

$$L^{-1}\left(\frac{A}{s+a}\right) = Ae^{-at} \quad \Rightarrow \quad c(t) = \frac{K}{T}e^{-t/T}$$

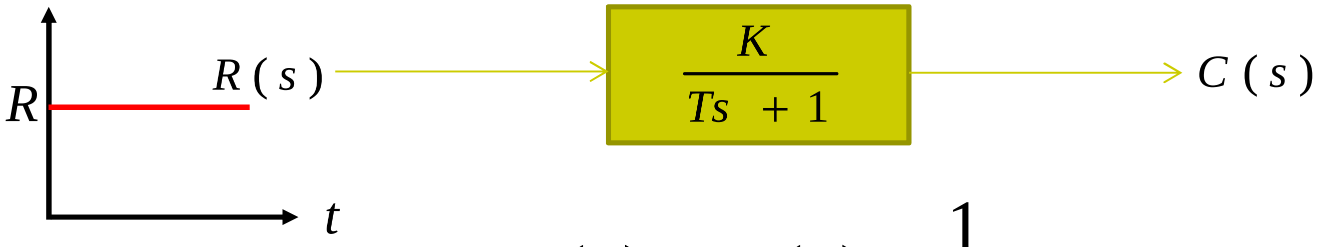
Impulse Response of 1st Order System

- If K=3 and T=2s then $$c(t) = \frac{K}{T}e^{-t/T} = \frac{3}{2}e^{-t/2} = 1.5e^{-0.5t}$$



Step Response of 1st Order System

- Consider the following 1st order system



$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion

Steady state Response **Transient Response**

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

Step Response of 1st Order System

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

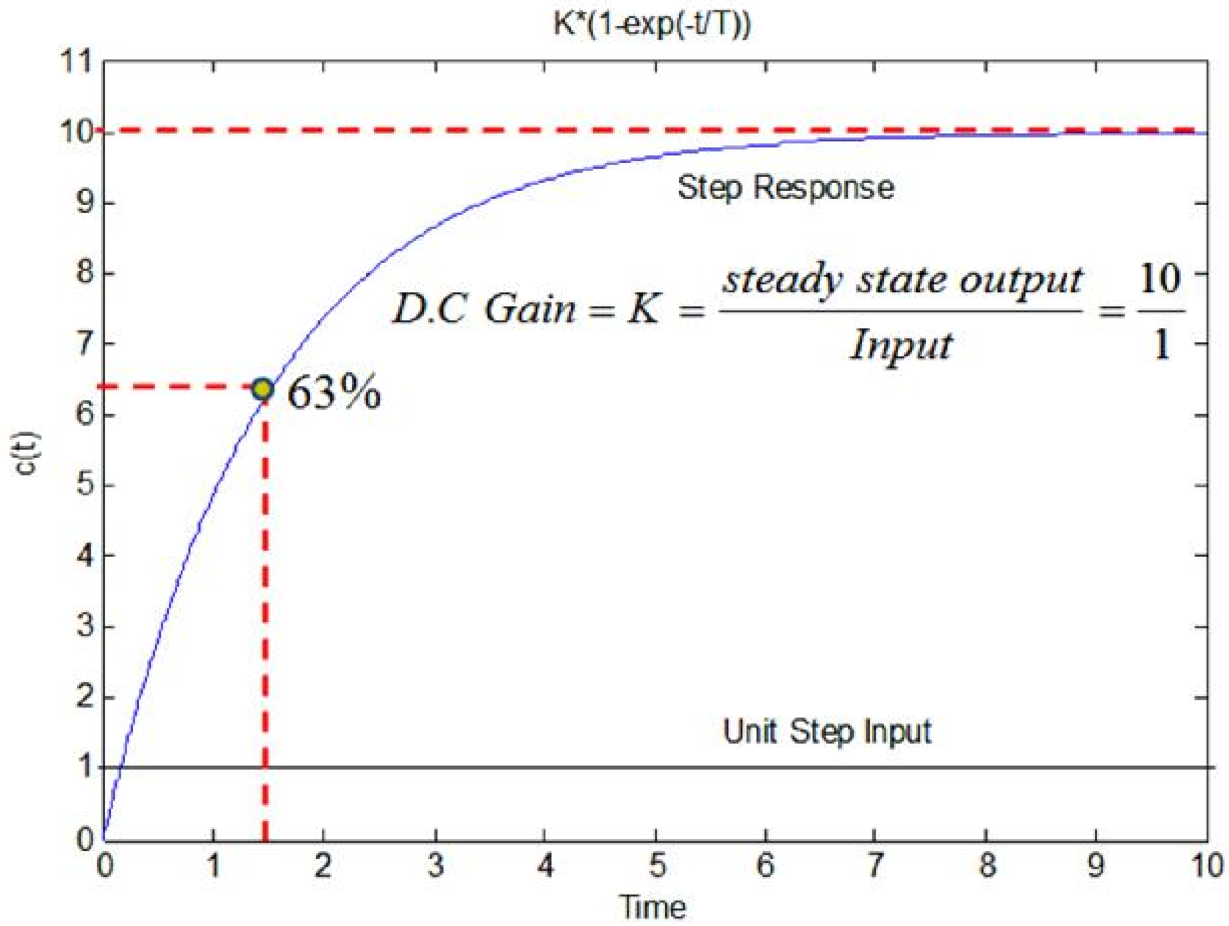
$$c(t) = K\left(1 - e^{-t/T}\right)$$

- When t=T

$$c(t) = K\left(1 - e^{-1}\right) = 0.632K$$

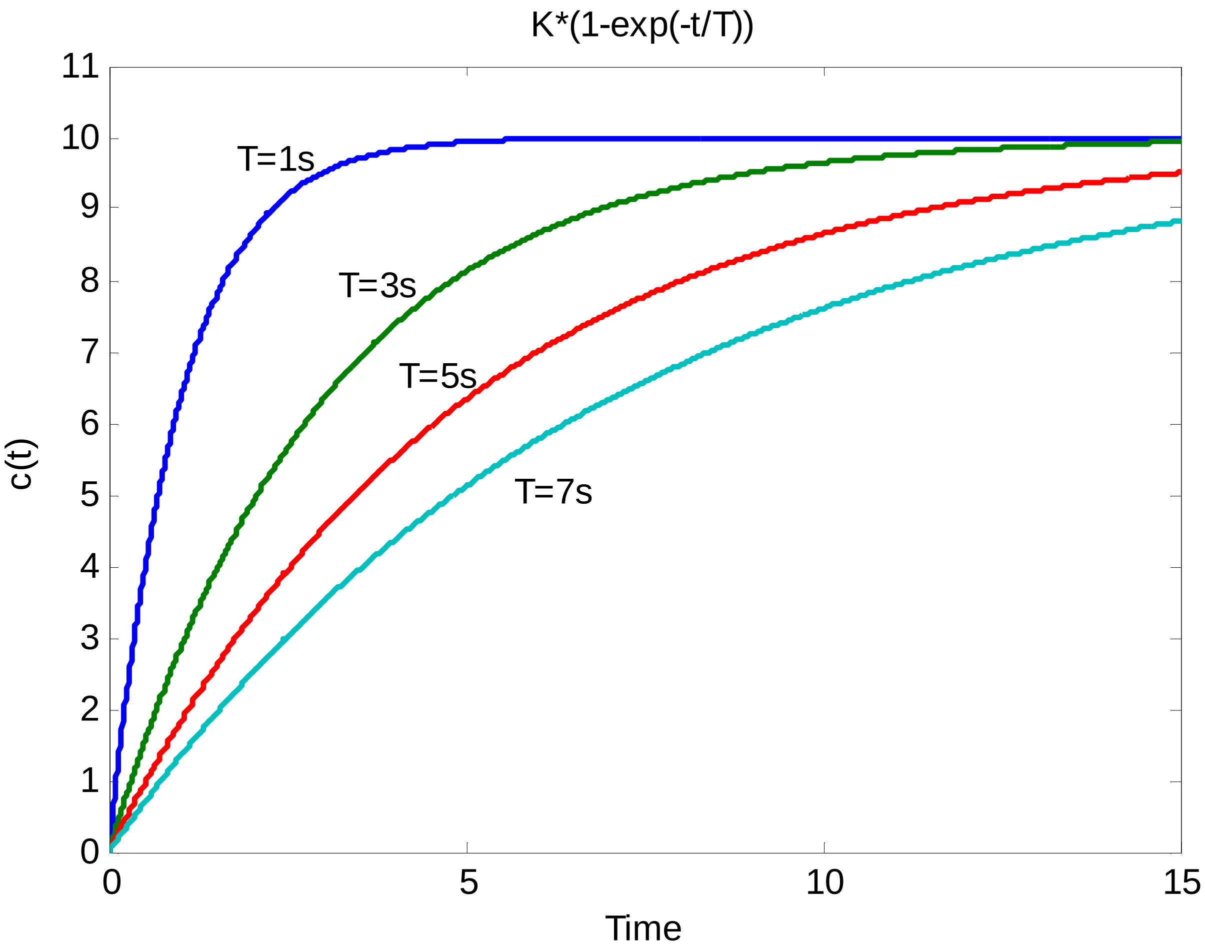
Step Response of 1st Order System

- If K=10 and T=1.5s then $c(t) = K\left(1 - e^{-t/T}\right) = \mathbf{10\left(1 - e^{-2t/3}\right)}$



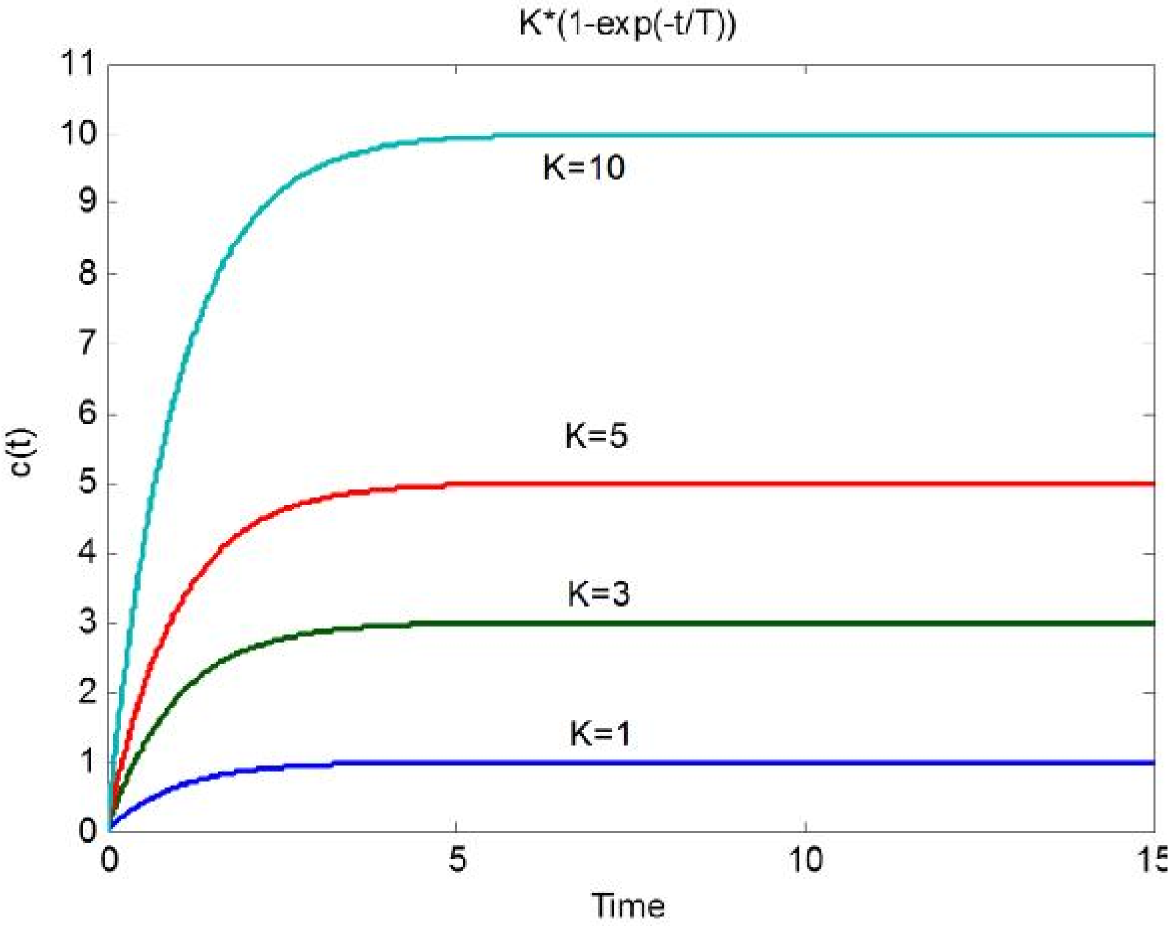
Step Response of 1st Order System

- If K=10 and T=1, 3, 5, 7 $\quad c(t) = K\left(1 - e^{-t/T}\right)$



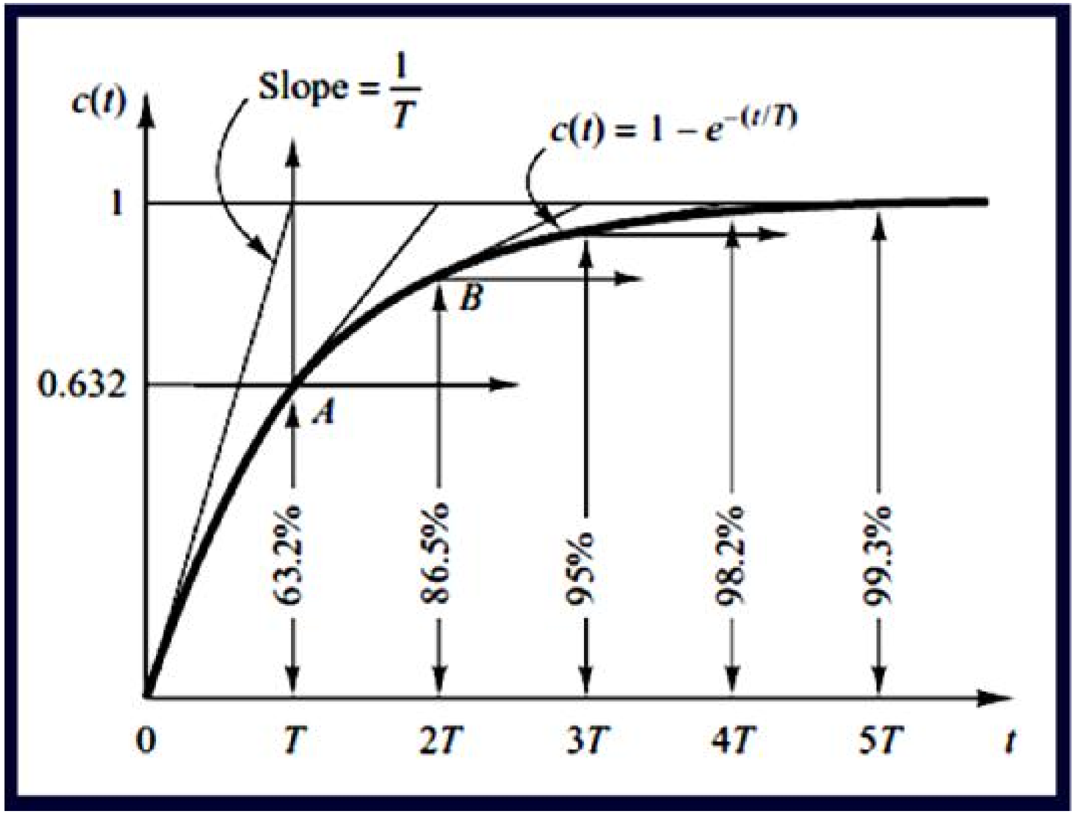
Step Response of 1st Order System

- If K=1, 3, 5, 10 and T=1 $$c(t) = K\left(1 - e^{-t/T}\right)$$



Step Response of 1st Order System

- System takes five time constants to reach its final value.



Example 2

- For the impulse response of a 1^{st} order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T
  - D.C Gain K
  - Transfer Function
  - Step Response

Example 2 (cont'd)

- Taking Laplace Transform of the impulse response to get the transfer function of the system.

$$L^{-1}\left(\frac{A}{s+a}\right)=Ae^{-at}$$

$$c(t)=3e^{-0.5t}$$

$$C(s)=\frac{3}{S+0.5}\times\delta(s)=\frac{3}{S+0.5}\times 1$$

$$\frac{C(s)}{d(s)}=\frac{C(s)}{R(s)}=\frac{3}{S+0.5}$$

$$\frac{C(s)}{R(s)}=\frac{6}{2S+1}$$

Example 2 (cont'd)

- Then for the Impulse response of a 1st order system is given by:

$$c(t) = 3e^{-0.5t}$$

- Transfer Function $\dfrac{C(s)}{R(s)} = \dfrac{6}{2S+1} \Rightarrow \dfrac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \dfrac{\boldsymbol{K}}{\boldsymbol{Ts+1}}$

- Time constant T=2

- D.C Gain K=6

Example 2 (cont'd)

- For step response integrate impulse response

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)dt = 3\int e^{-0.5t}dt$$

$$c_s(t) = -6e^{-0.5t} + C$$

- We can find out *C* if initial condition is known e.g. $c_s(0)=0$

$$0 = -6e^{-0.5\times 0} + C$$

1

$$C = 6$$

$$c_s(t) = 6 - 6e^{-0.5t}$$ ⇨ **Step Response**

Example 2 (cont'd)

- If initial Conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

since $R(s)$ **is a step input**, $R(s) = \frac{1}{s}$

$$C(s) = \frac{6}{s(2S+1)}$$

$$\frac{6}{s(2S+1)} = \frac{A}{s} + \frac{B}{2s+1}$$

$$\frac{6}{s(2S+1)} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$c(t) = 6 - 6e^{-0.5t}$$ ⇨ **Step Response**

Second Order Systems

Second Order Systems

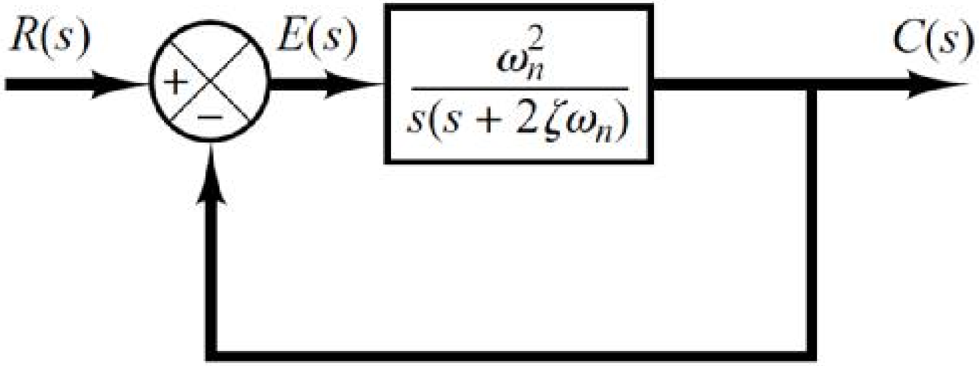
- For the first-order system parameters **(T, K)** simply changes the speed and offset of the response.

- Whereas, changes in the parameters of a second-order system can change the **form** of the response.

- A second-order system can display characteristics much like a first-order system or, depending on component values, display **damped** or **pure oscillations** for its ***transient response***.

Second Order Systems

- A general second-order system (without zeros) is characterized by the following transfer function.



$$G(s) = \frac{w_n^2}{s(s+2zw_n)}$$ Open-Loop Transfer Function

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s^2+2zw_ns+w_n^2}$$ Closed-Loop Transfer Function

Second Order Systems

Transient Response of second-order system

The general form of the T.F. for second-order system is:

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

$Z \longrightarrow$ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

$w_n \longrightarrow$ undamped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.

Example 3

- Determine the undamped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

Solution

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s^2 + 2zw_n s + w_n^2}$$

$$w_n^2 = 4 \qquad \Rightarrow w_n = 2\ rad/\sec$$

$$\cancel{s^2} + 2zw_n s + \cancel{w_n^2} = \cancel{s^2} + 2s + \cancel{4}$$

$$\Rightarrow 2zw_n s = 2s$$

$$\Rightarrow zw_n = 1$$

$$\Rightarrow z = 0.5$$

Second Order Systems

- For the 2nd order system

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

- The closed-loop poles of the system are

$$P_1 = -\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$P_2 = -\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- Depending upon the value of ζ , a second-order system can be set into one of the four categories

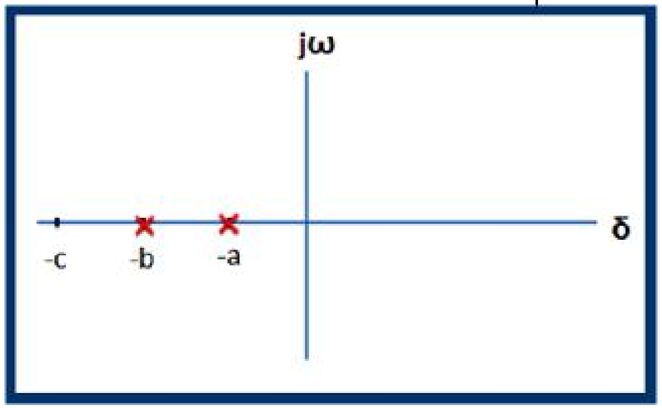
Second Order Systems

1- *Over damped system* $(Z > 1)$

Occurs when the system has two **Real distinct poles**

$$P_1 = -w_n z + w_n\sqrt{z^2 - 1}$$

$$P_2 = -w_n z - w_n\sqrt{z^2 - 1}$$

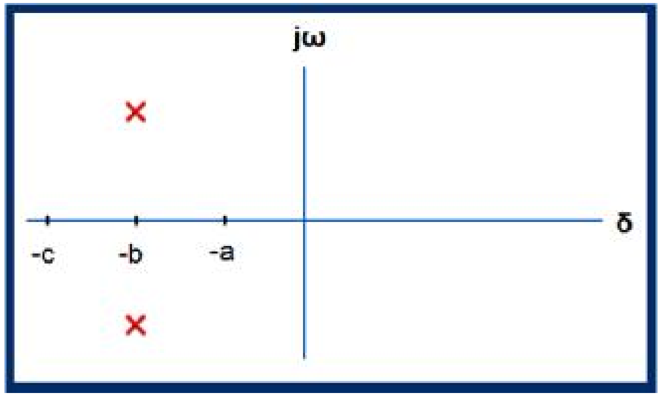


2- *Under damped system* $(0 < Z < 1)$

Occurs when the system has two **Complex conjugate poles**

$$P_1 = -w_n z + w_n\sqrt{z^2 - 1}$$

$$P_2 = -w_n z - w_n\sqrt{z^2 - 1}$$



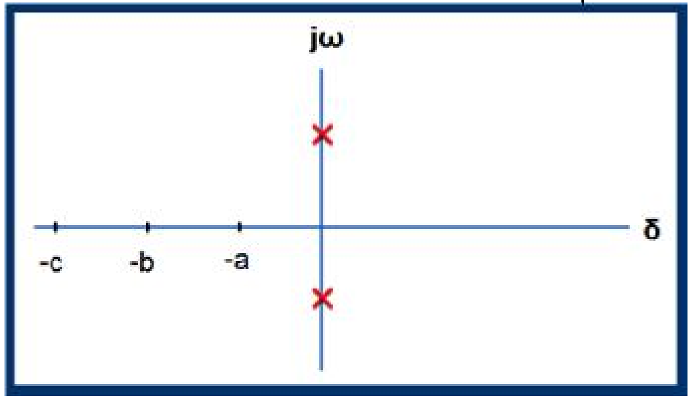
Second Order Systems

3- Undamped system $(Z = 0)$

Occurs when the system has two **Imaginary poles**

$$P_1 = -w_n z + w_n\sqrt{z^2 - 1}$$

$$P_2 = -w_n z - w_n\sqrt{z^2 - 1}$$

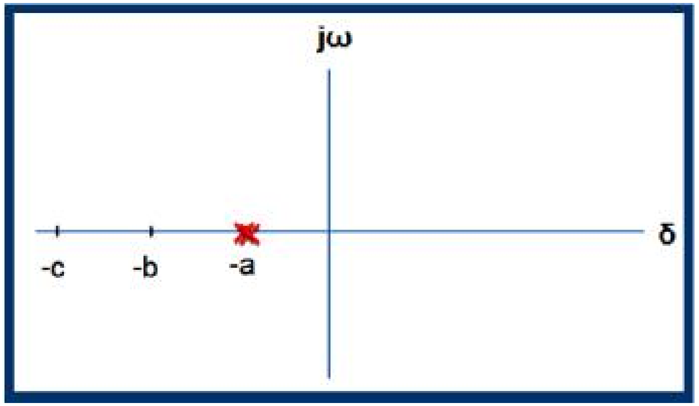


4- Critically damped system $(Z = 1)$

Occurs when the system has two **Real but equal poles**

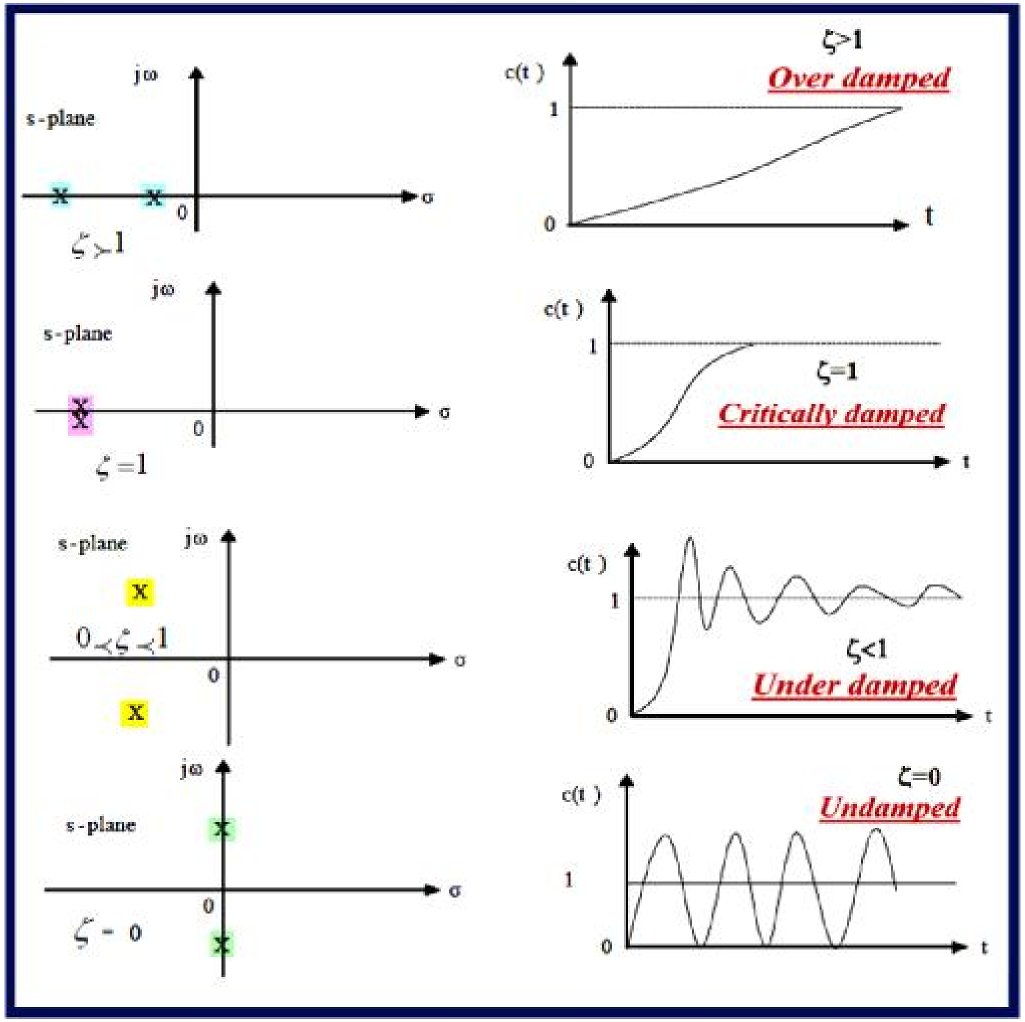
$$P_1 = -w_n z + w_n\sqrt{z^2 - 1}$$

$$P_2 = -w_n z - w_n\sqrt{z^2 - 1}$$



Second Order Systems

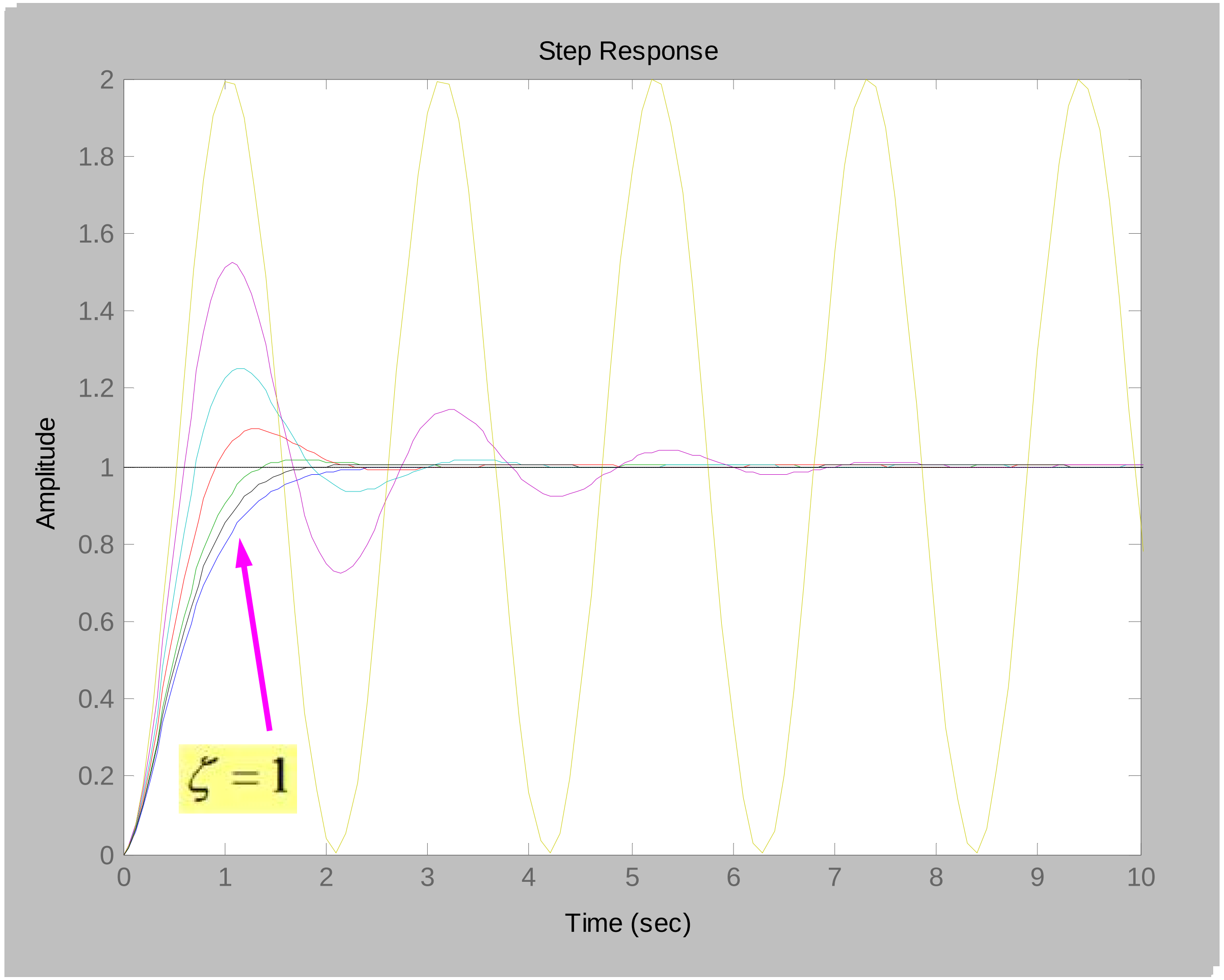
Step response comparison for various roots locations in S-plane



Second Order Systems

Step response

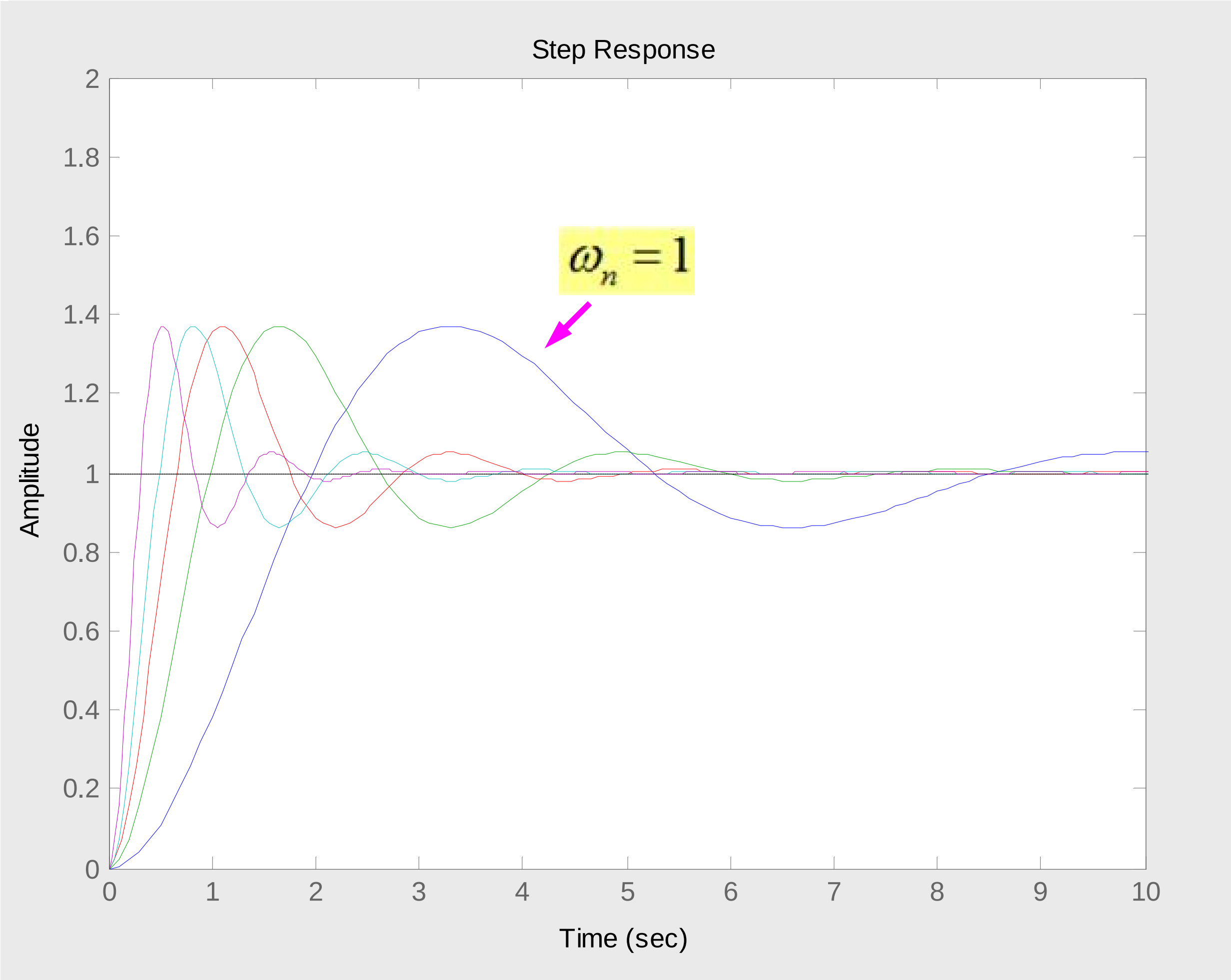
$$w_n = 3 \quad z = 1, 0.8, 0.6, 0.4, 0.2, 0$$



Second Order Systems

Step response

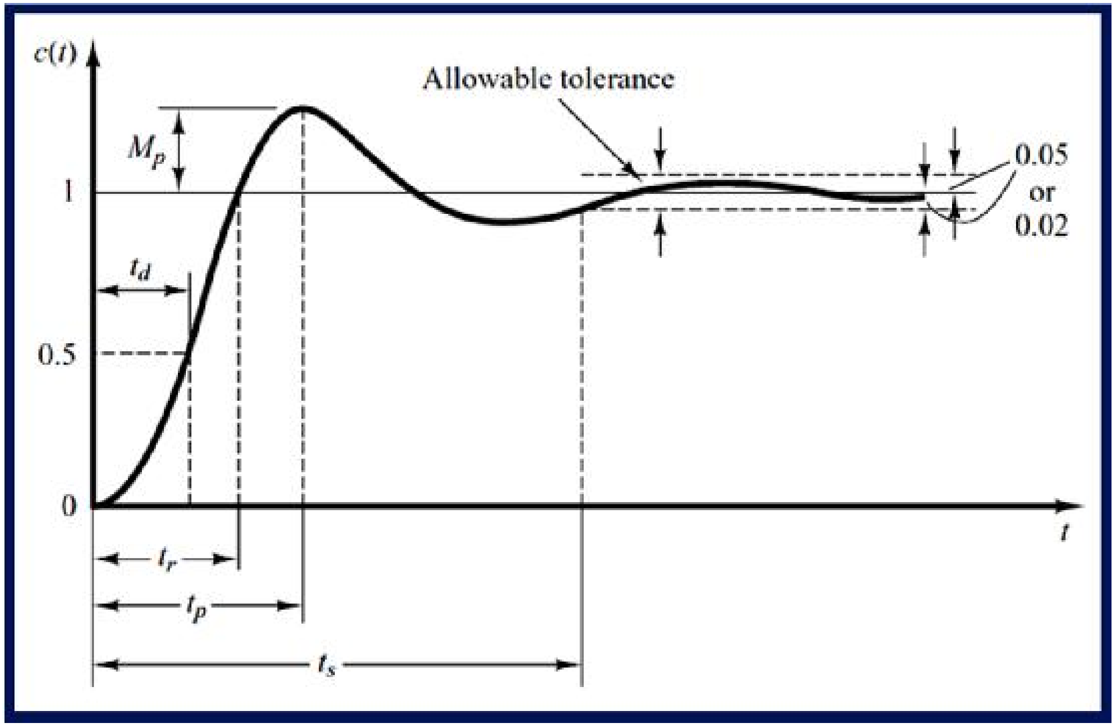
$$z = 0.3 \quad w_n = 1, 2, 3, 4, 6.28$$



Time-Domain Specification Of 2nd Order Systems

Time-Domain Specification Of 2^{nd} Order Systems

For $0 < z < 1$ and $\omega_n > 0$, the 2^{nd} order system's response due to a unit step input as shown

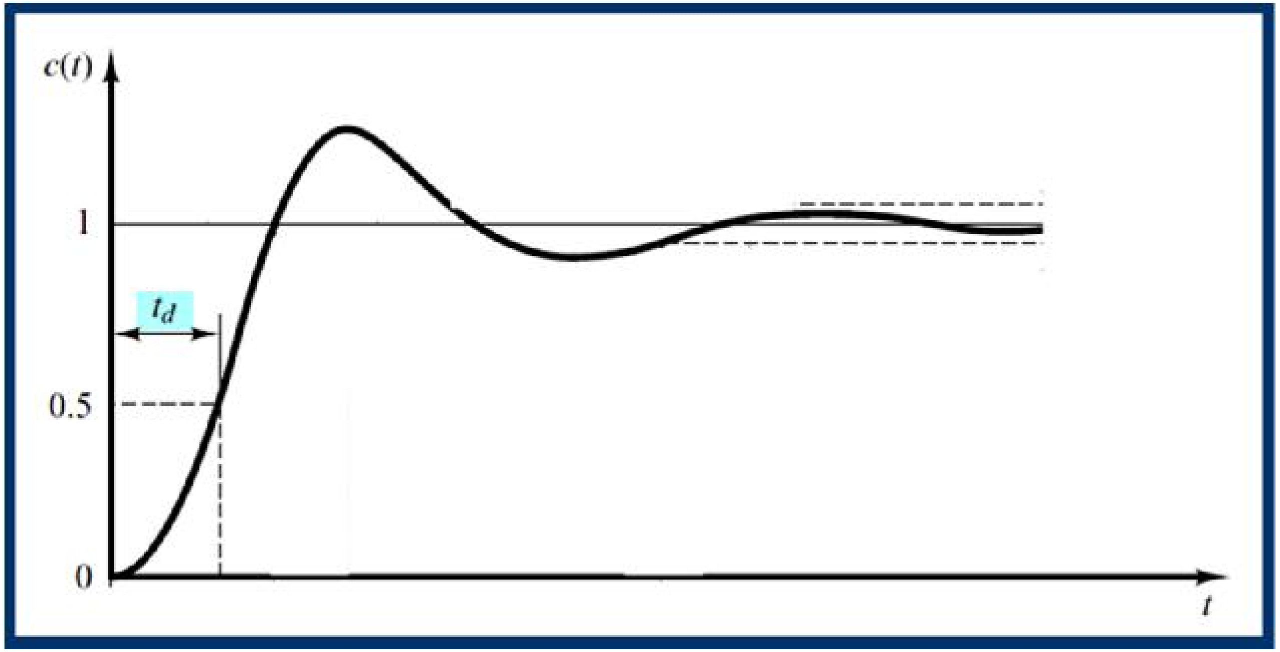


Performance Characteristics

Time-Domain Specification

Delay time (t_d)

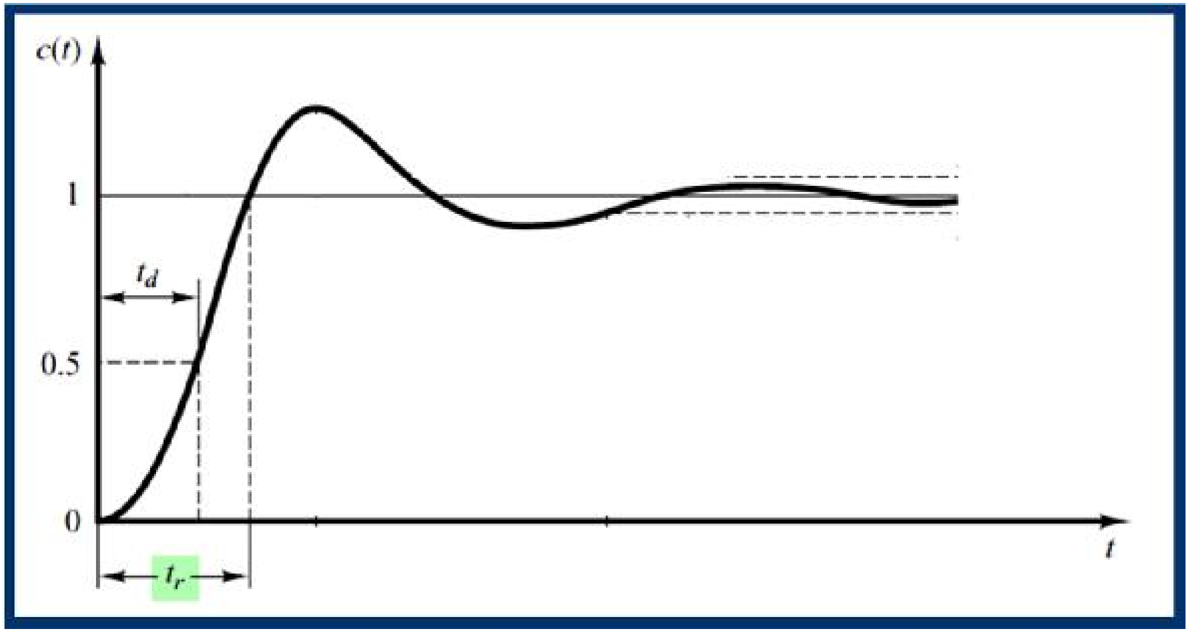
- The delay time is the time required for the response to reach half the final value the very first time.



Time-Domain Specification

Rise time (t_r)

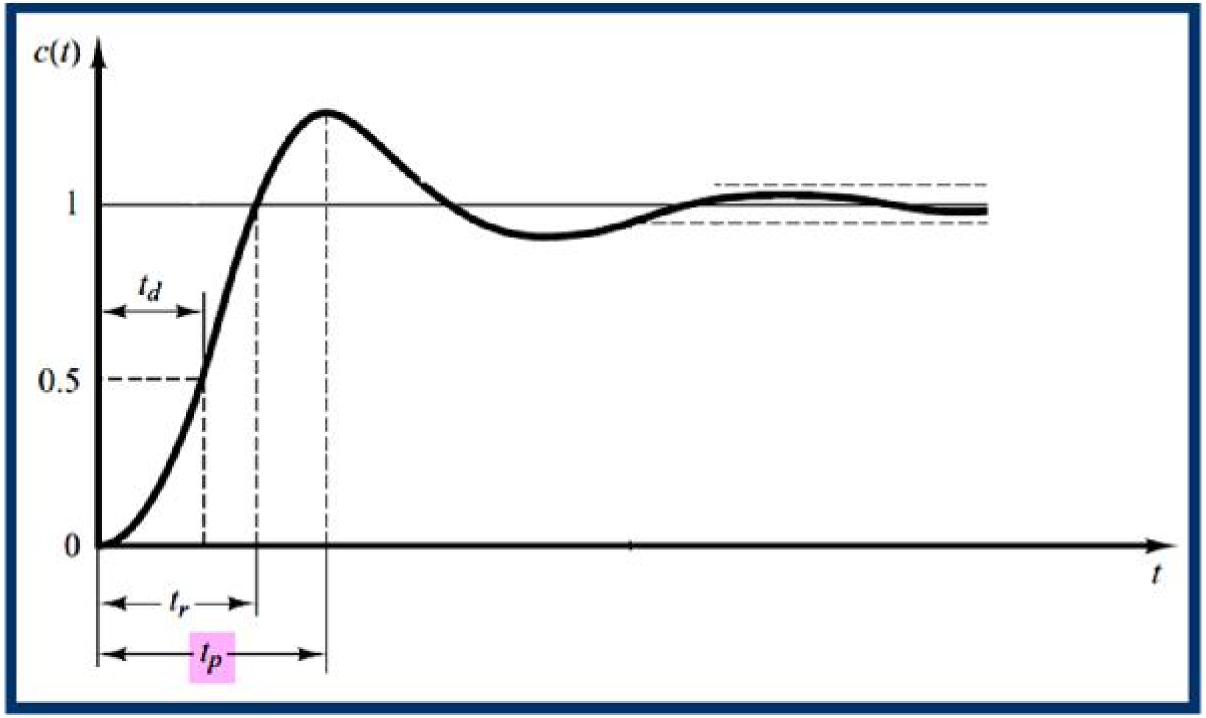
- The rise time is the time required for the response to rise from 0% to 100% of its final value (under damped second order systems), or from 10% to 90% of its final value (for the over damped systems).



Time-Domain Specification

Peak time (t_p)

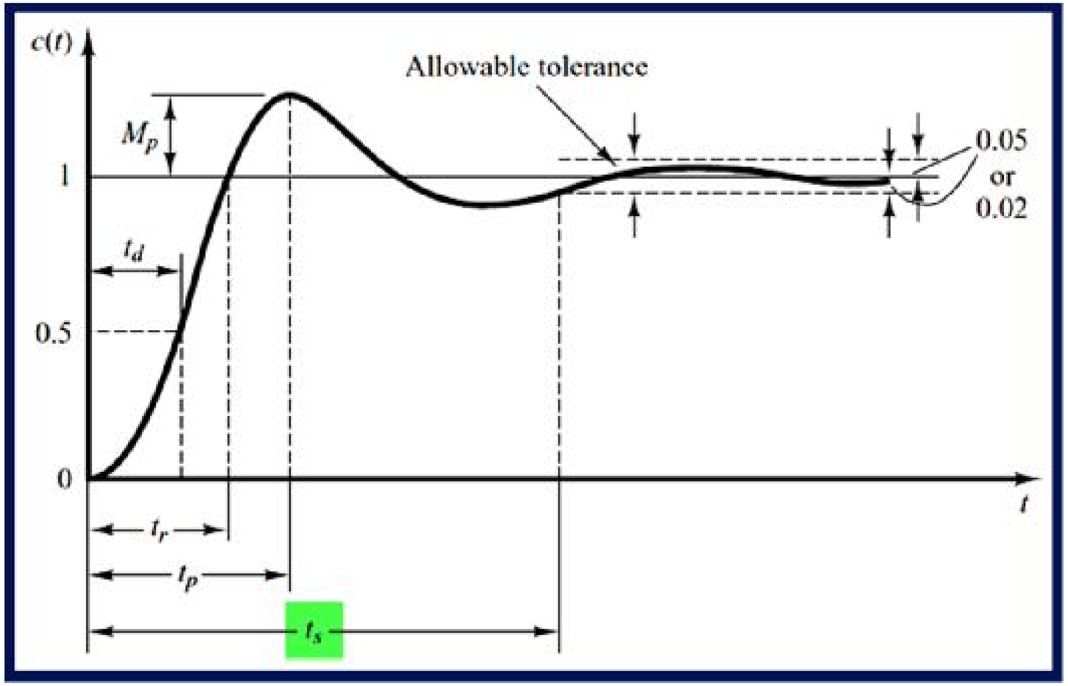
- The peak time is the time required for the response to reach the first peak of the overshoot.



Time-Domain Specification

Settling time (t_s)

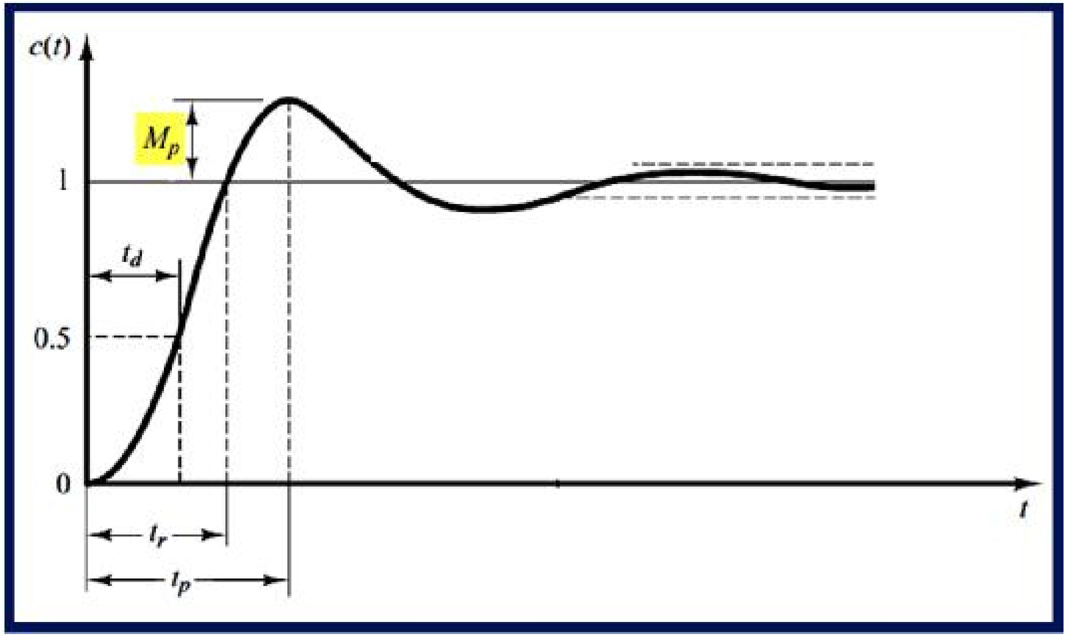
- The settling time is the time required for the response curve to reach and stay within a range usually 2% or 5% of the final value (steady-state value)



Time-Domain Specification

Maximum overshoot (M_p)

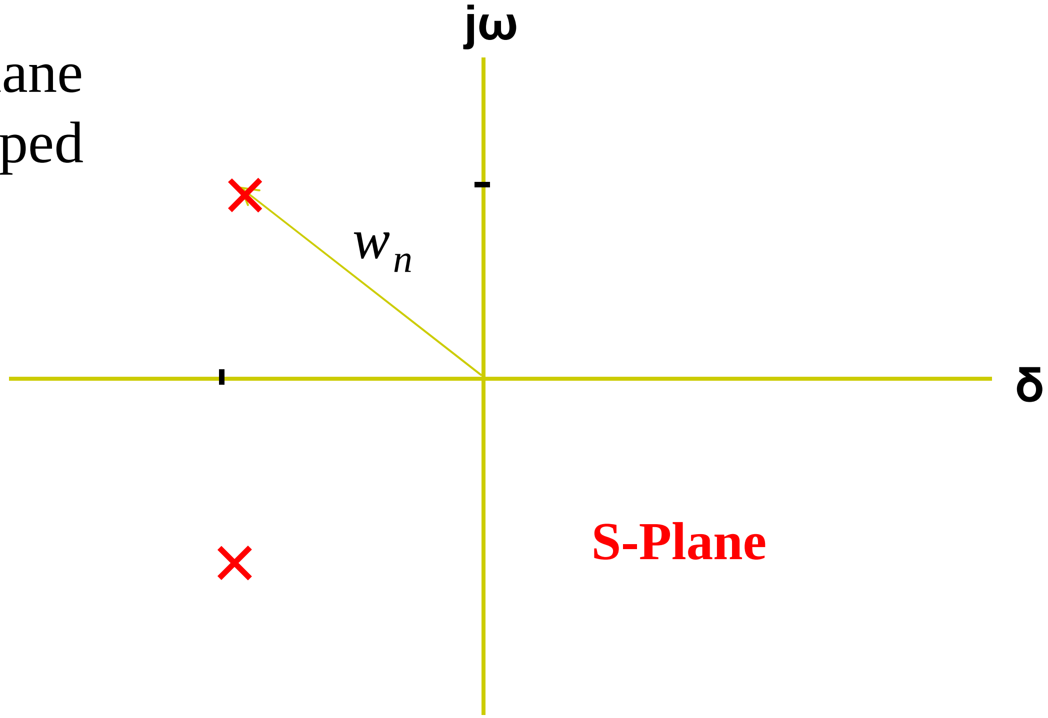
- The maximum overshoot is the maximum peak value of the response curve measured from unity.
- The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.



Time-Domain Specification

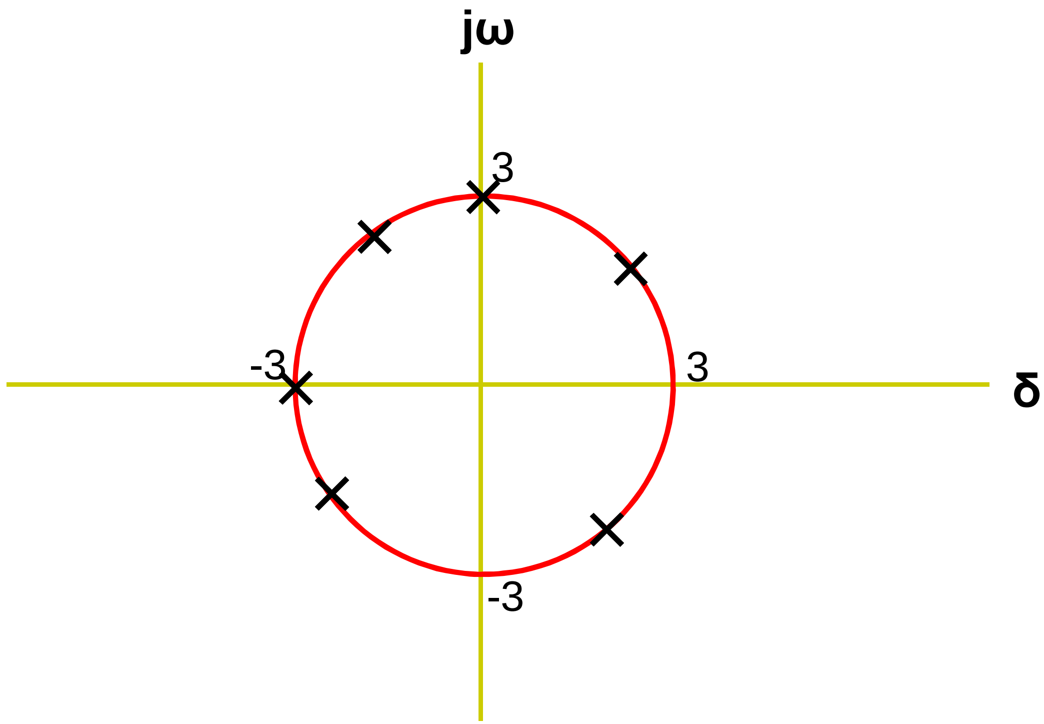
Natural undamped frequency (ω_n)

- Distance from the origin of s-plane to pole is natural undamped frequency ω_n in rad/sec.



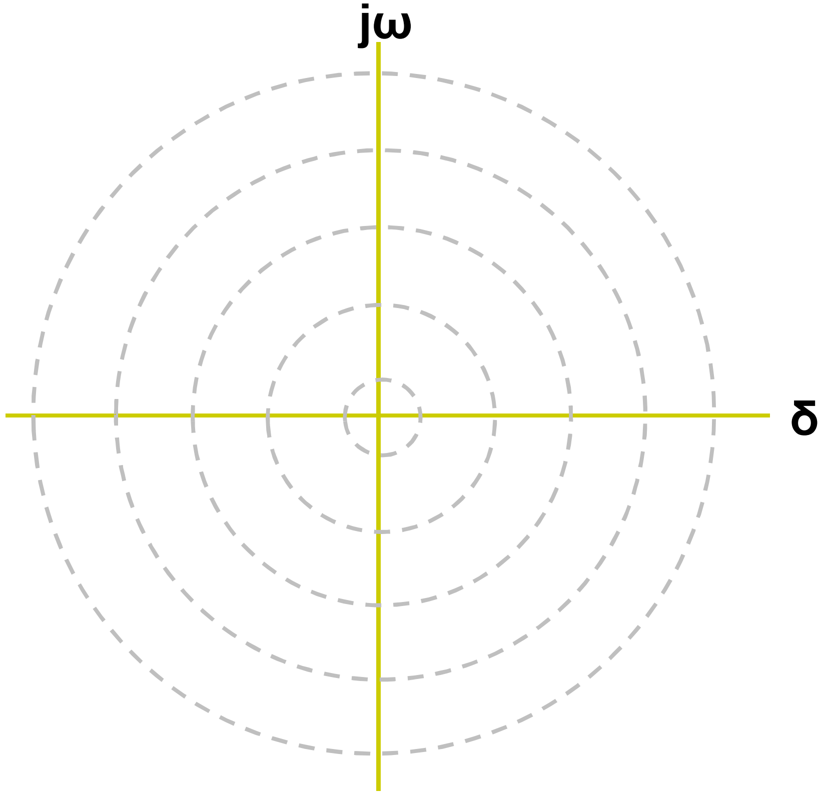
Time-Domain Specification

- Let us draw a circle of radius 3 in s-plane.
- If a pole is located anywhere on the circumference of the circle the natural undamped frequency would be *3 rad/sec*.



Time-Domain Specification

- Therefore the s-plane is divided into constant Natural undamped frequency (ω_n) Circles.

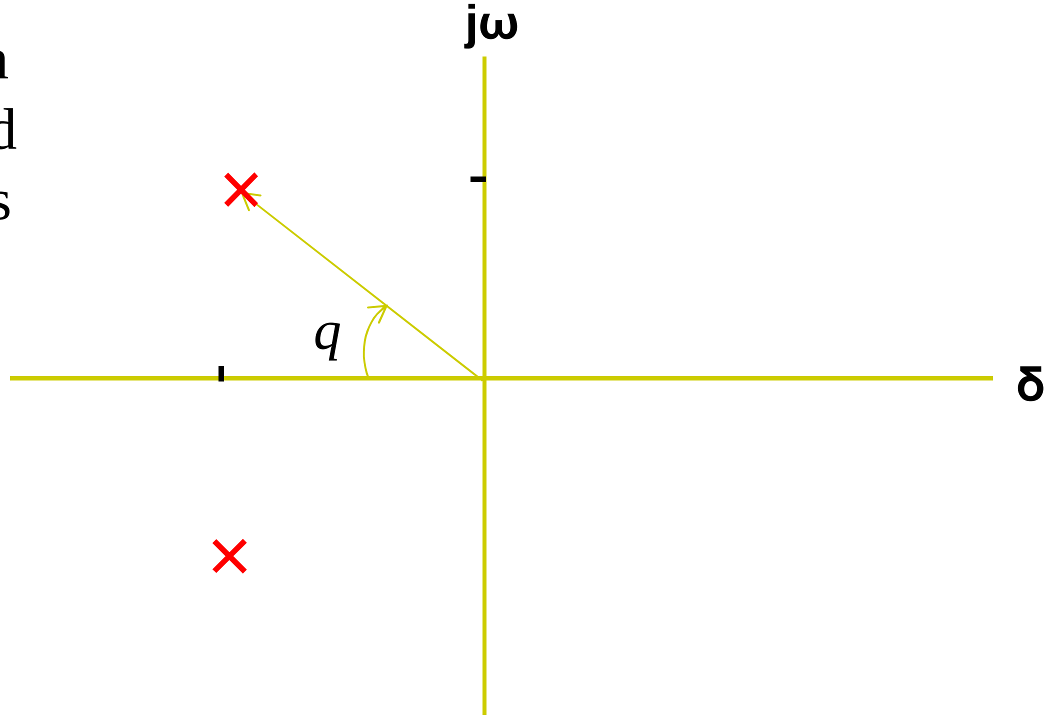


Time-Domain Specification

- **Damping ratio** (z)

- Cosine of the angle between vector connecting origin and pole and –ve real axis yields damping ratio.

$$\boxed{\zeta = \cos\theta}$$



Time-Domain Specification

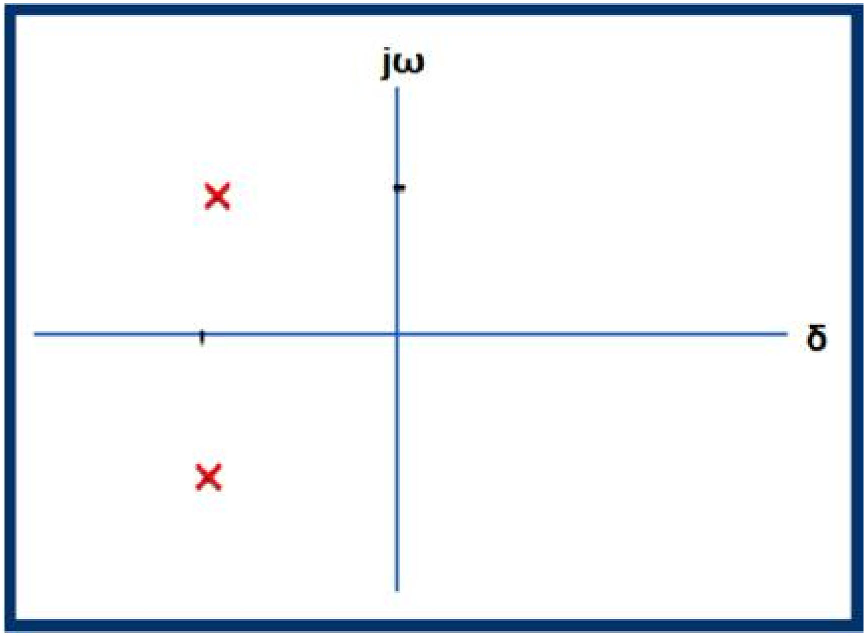
- For **Underdamped** system

$$0^0 < q < 90^0$$

therefore,

$$\boxed{\zeta = \cos\theta}$$

$$\boxed{0 < \zeta < 1}$$



Time-Domain Specification

- For **Undamped** system

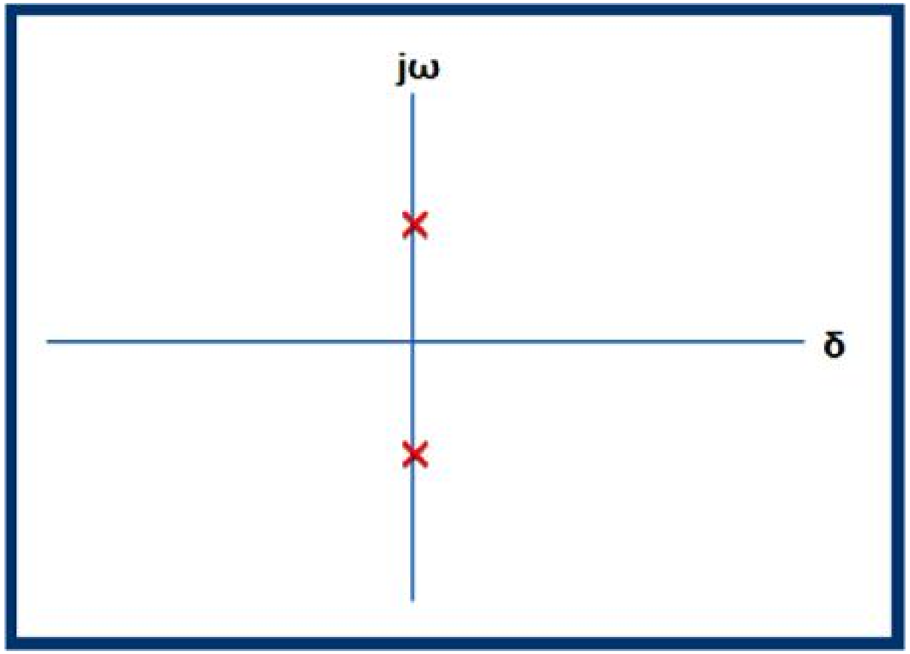
$$q = 90^{\mathbf{0}}$$

therefore,

$$\boxed{\zeta = \cos\theta}$$

$$\Downarrow$$

$$\boxed{\zeta = 0}$$



Time-Domain Specification

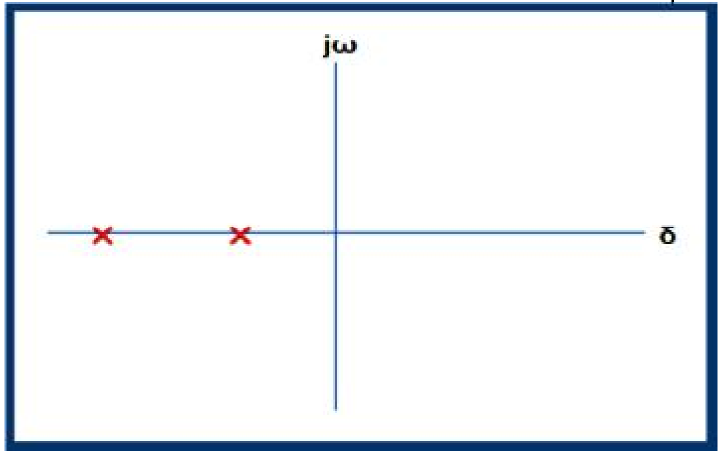
- For **over damped** and **critically damped** systems

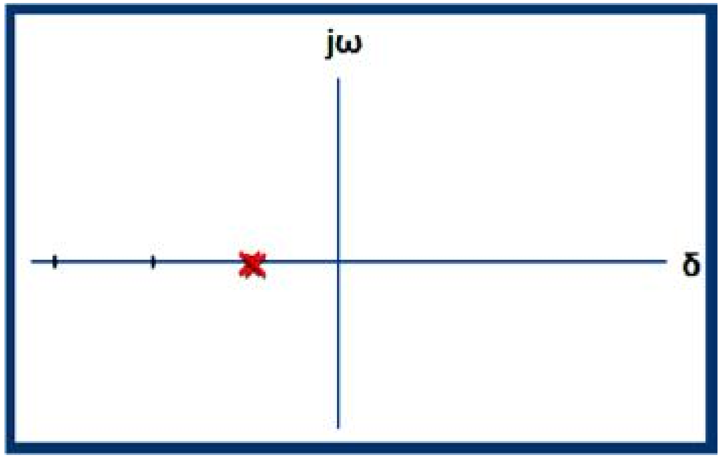
$$q = 0^{0}$$

therefore,

$$\boxed{\zeta = \cos\theta}$$

$$\boxed{\zeta \geq 1}$$





Time-Domain Specification

Damping factor (α)

$$\boxed{\alpha = \xi\omega_n}$$

Damping natural frequency (ω_d)

$$\omega_d = \sqrt{\omega_n^2 - \alpha^2} \qquad \alpha = \omega_n \zeta$$

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

ω_n: undamping natural frequency
ξ : damping ratio

Time-Domain Specification

Rise time (t_r)

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$\theta = \cos^{-1}\zeta$$

Peak time (T_p)

$$t_p = \frac{\pi}{\omega_d}$$

$$t_p = \frac{\pi}{\sqrt{\omega_n^2 - \alpha^2}}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\alpha = \xi\omega_n$$

ω_d: damping natural frequency
ω_n: undamping natural frequency
θ : phase angle
ξ : damping ratio

Time-Domain Specification

Settling time (t_s)

$$t_s = 3T$$

$$t_s = \frac{4}{\zeta\omega_n} \qquad (2\%\ \text{Criterion})$$

$$t_s = 4T$$

$$t_s = \frac{3}{\zeta\omega_n} \qquad (5\%\ \text{Criterion})$$

Max Overshoot (M_p)

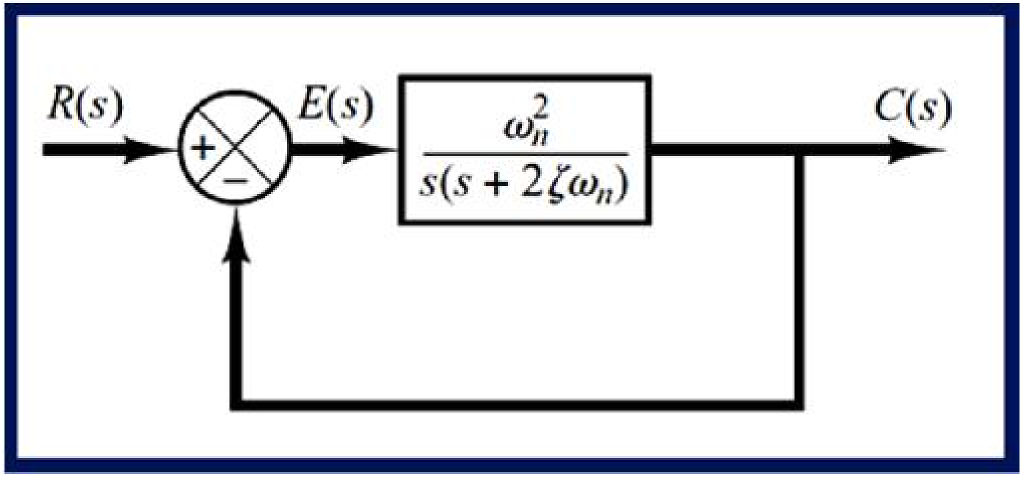
$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

ω_n: undamping natural frequency
α : damping factor
ξ : damping ratio

Example 4

Consider the system shown in following figure, where damping ratio is 0.6 and natural undamped frequency is 5 rad/sec. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time 2% and 5% criterion t_s when the system is subjected to a unit-step input.



Example 4 (Cont'd)

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$\theta = \cos^{-1}\zeta$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$q = Cos^{-1}x = \cos^{-1}0.6 = 53.13^{\circ}$$

$$q = 53.13 ´ \frac{p}{180} = 0.93 \text{rad}$$

$$t_r = \frac{3.141 - q}{w_n\sqrt{1 - z^2}}$$

$$t_r = \frac{3.141 - 0.93}{5\sqrt{1-0.6^2}} = 0.55s$$

Example 4 (Cont'd)

Peak Time

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

$$t_p = \frac{3.141}{4} = 0.785s$$

Settling Time (4%)

$$\boxed{t_s = \frac{3}{\zeta\omega_n}}$$

$$t_s = \frac{3}{0.6\times5} = 1s$$

Settling Time (2%)

$$\boxed{t_s = \frac{4}{\zeta\omega_n}}$$

$$t_s = \frac{4}{0.6\times5} = 1.33s$$

Example 4 (Cont'd)

Maximum Overshoot

$$\boxed{M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100}$$

$$M_p = e^{-\frac{3.141 \times 0.6}{\sqrt{1-0.6^2}}} \times 100 = 9.5\%$$

Example 4 (Cont'd)

Performance Characteristics