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Faculty of Mathematical Sciences and Informatics

Statistics and Probability (S2013)

Assignment 5

(Some Probability Distributions)

Content:

- Gamma Distribution (Γ).
- Beta Distribution (β).
- Chi-Square Distribution (χ^2).
- Related Distributions.
- References.

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Computer Science

Gamma Distribution (Γ)

Definition

The definition of Gamma Distribution requires the gamma (Γ) function from calculus that exists for α > 0 and that the value of the integral is a positive number.

Properties

$$\Gamma'(1) = \int_0^\infty \ln t \, e^{-t} \, dt = -\gamma \tag{18.40}$$

Multiplication formula:

$$\Gamma(2z) = \pi^{-\frac{1}{2}} \, 2^{2z-1} \, \Gamma(z) \, \Gamma\left(z + \frac{1}{2}\right) \tag{18.41}$$

Reflection formulas:

$$\begin{aligned} &\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \\ &\Gamma\left(\frac{1}{2}+z\right)\,\Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos \pi z}, \\ &\Gamma(z-n) = (-1)^n \Gamma(z) \frac{\Gamma(1-z)}{\Gamma(n+1-z)} = \frac{(-1)^n \, \pi}{\sin \pi z \, \Gamma(n+1-z)} \end{aligned} \tag{18.42}$$

The gamma function has the recursion formula:

$$\Gamma(z+1) = z\,\Gamma(z) \tag{18.43}$$

The relation $\Gamma(z) = \Gamma(z+1)/z$ can be used to define the gamma function in the left half plane, $z \neq 0, -1, -2, \ldots$

The gamma function has simple poles at $z = -n$, (for $n = 0, 1, 2, \ldots$), with the respective residues $(-1)^n/n!$. That is,

$$\lim_{z \to -n} (z+n)\Gamma(z) = \frac{(-1)^n}{n!} \tag{18.44}$$

$$\Gamma(n+1) = n!$$

The Probability Density Function (PDF) of Gamma Distribution

A random variable X is said to have a gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ if its density function (pdf) is given by:

$$f(x) = \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} \qquad 0 < x < \infty$$

In the second reference page 174: ... with parameters (α, λ), $\lambda > 0$ *and* $\alpha > 0$

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \qquad x \geq 0$$

Where:

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y}\, dy$$

In which case, we often write that X has $\Gamma(\alpha, \beta)$ distribution.

The probability density function is skewed to the right. For fixed β the tail becomes heavier as α increases.

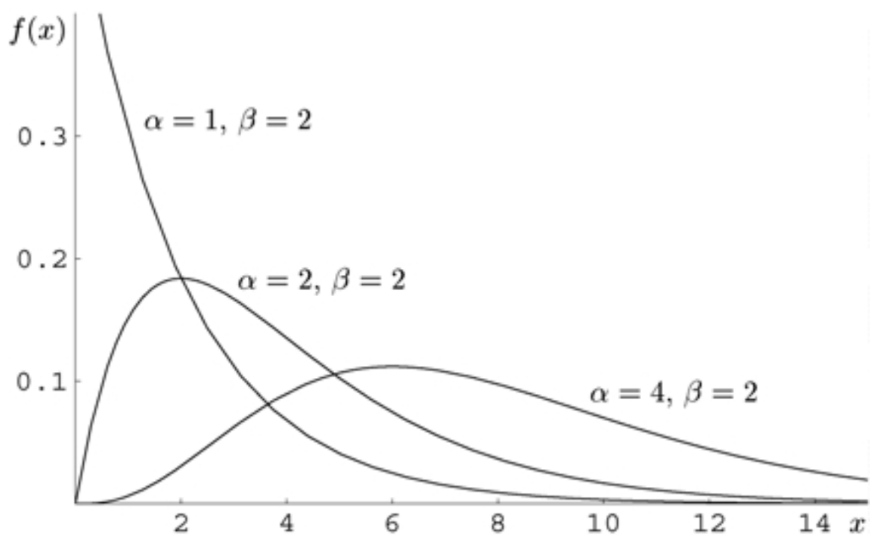

Figure 6.10: Probability density functions for a gamma random variable.

The Cumulative Distribution Function (CDF) of Gamma Distribution

$$\boldsymbol{F(x) = \int_{0}^{x} f(x)dx}$$

$$= \int_{0}^{x} \frac{x^{\alpha-1}e^{\frac{-x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}dx$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_{0}^{x} x^{\alpha-1}e^{\frac{-x}{\beta}}\,dx$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} * \beta^{\alpha}\gamma(\alpha, \beta x)$$

$$= \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}$$

$$\therefore \boldsymbol{F(x) = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}}$$

Where:

$$\boldsymbol{\gamma(s, x)} = \int_{0}^{x} t^{s-1}e^{-t}\,dt$$

$$\boldsymbol{\Gamma(\alpha)} = \int_{0}^{\infty} t^{\alpha-1}e^{-t}\,dt$$

The Moment Generating Function (MGF) of Gamma Distribution

Before starting finding the MGF we should remember that β must be greater than 0, and the MGF generates moments when $t = 0$, so t must be less than β.

$$\boldsymbol{M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} f(x) dx}$$

$$= \int_0^{\infty} e^{tx} \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\beta^\alpha \Gamma(\alpha)} dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-\left(\frac{1}{\beta}-t\right)x} dx$$

Assume $u = \left(\frac{1}{\beta} - t\right) x \;\; then \;\; x = \frac{u}{\frac{1}{\beta}-t}$. And Because of $t < \beta$:

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} \left(\frac{u}{\frac{1}{\beta} - t}\right)^{\alpha-1} e^{-u} \frac{du}{\frac{1}{\beta} - t}$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha) \left(\frac{1}{\beta} - t\right)^\alpha} \int_0^{\infty} (u)^{\alpha-1} e^{-u} du$$

$$= \frac{\Gamma(\alpha)}{\beta^\alpha \Gamma(\alpha) \left(\frac{1}{\beta} - t\right)^\alpha} = \left(\frac{1}{\beta\left(\frac{1}{\beta} - t\right)}\right)^\alpha = \boldsymbol{(1 - \beta t)^{-\alpha}}$$

$$\therefore \; \boldsymbol{M_X(t) = (1 - \beta t)^{-\alpha}}$$

Theorem 3.3.1. *Let $X_1, \ldots, X_n$ be independent random variables. Suppose, for $i = 1, \ldots, n$, that X_i has a $\Gamma(\alpha_i, \beta)$ distribution. Let $Y = \sum_{i=1}^n X_i$. Then Y has a $\Gamma(\sum_{i=1}^n \alpha_i, \beta)$ distribution.*

Proof: Using the assumed independence and the mgf of a gamma distribution, we have by Theorem 2.6.1 that for $t < 1/\beta$,

$$M_Y(t) = \prod_{i=1}^n (1 - \beta t)^{-\alpha_i} = (1 - \beta t)^{-\sum_{i=1}^n \alpha_i},$$

which is the mgf of a $\Gamma(\sum_{i=1}^n \alpha_i, \beta)$ distribution. ∎

Expectation and Variance of a Variable That Gamma Distributed

Let X be a random variable gamma distributed has $M_X(t) = (1-\beta t)^{-\alpha}$

Then:

$$\boldsymbol{E(X) = M'_X(0)}$$

$$\frac{d}{dt}\left(M_X(t)\right) = \frac{d}{dt}(1-\beta t)^{-\alpha}\Big|_{t=0}$$

$$= (-\alpha)(1-\beta t)^{-\alpha-1}(-\beta)|_{t=0}$$

$$= \alpha\beta$$

$$\therefore \boldsymbol{E(x) = \alpha\beta}$$

$$\boldsymbol{var(X) = M''_X(0) - [M'_X(0)]^2}$$

$$\boldsymbol{M''_X(0) = \frac{d}{dt}\left(M'_X(t)\right)\Big|_{t=0}}$$

$$= \frac{d}{dt}(-\alpha)(1-\beta t)^{-\alpha-1}(-\beta)|_{t=0}$$

$$= (-\alpha)(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)^2|_{t=0}$$

$$= \alpha(\alpha+1)\beta^2$$

$$\therefore \boldsymbol{Var(X) =} \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 \boldsymbol{= \alpha\beta^2}$$

Examples

Example 1. Let the waiting time W have a gamma p.d.f. with $\alpha = k$ and $\beta = 1/\lambda$. Accordingly, $E(W) = k/\lambda$. If $k = 1$, then $E(W) = 1/\lambda$; that is, the expected waiting time for $k = 1$ changes is equal to the reciprocal of λ.

Example 2. Let X be a random variable such that

$$E(X^m) = \frac{(m+3)!}{3!} 3^m, \qquad m = 1, 2, 3, \ldots.$$

Then the moment-generating function of X is given by the series

$$M(t) = 1 + \frac{4!\ 3}{3!\ 1!} t + \frac{5!\ 3^2}{3!\ 2!} t^2 + \frac{6!\ 3^3}{3!\ 3!} t^3 + \cdots.$$

This, however, is the Maclaurin's series for $(1 - 3t)^{-4}$, provided that $-1 < 3t < 1$. Accordingly, X has a gamma distribution with $\alpha = 4$ and $\beta = 3$.

Example 6a

Let X be a gamma random variable with parameters α and λ. Calculate (a) $E[X]$ and (b) $\mathrm{Var}(X)$.

Solution (a)

$$\begin{aligned} E[X] &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda x e^{-\lambda x} (\lambda x)^{\alpha-1}\, dx \\ &= \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty \lambda e^{-\lambda x} (\lambda x)^{\alpha}\, dx \\ &= \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} \\ &= \frac{\alpha}{\lambda} \quad \text{by Equation (6.1)} \end{aligned}$$

(b) By first calculating $E[X^2]$, we can show that

$$\mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$$

The details are left as an exercise. ■

Beta Distribution (β)

Definition

It can be defined as a gamma distribution with parameter values of $\alpha + \beta$ and 1.

Properties

- The beta distribution can be used to model a random phenomenon whose set of possible values is some finite interval [c, d]—which, by letting c denote the origin and taking d - c as a unit measurement, can be transformed into the interval [0, 1].

- When a = b, the beta density is symmetric about 1 2, giving more and more weight to regions about 1 2 as the common value a increases.

- When a = b = 1, the beta distribution reduces to the uniform (0, 1) distribution.

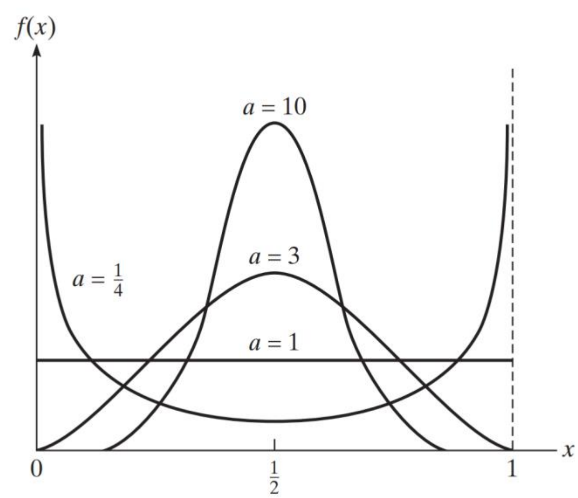

Figure 5.8 Beta densities with parameters (a, b) when $a = b$.

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The beta function has the properties:

(a) $B(p,q) = B(q,p)$

(b) $B(p,q+1) = \frac{q}{p}\,B(p+1,q) = \frac{q}{p+q}\,B(p,q)$

-

(c) $B(p,q)\,B(p+q,r) = \frac{\Gamma(p)\,\Gamma(q)\,\Gamma(r)}{\Gamma(p+q+r)}$

-

The Probability Density Function (PDF) of Beta Distribution

A random variable X is said to have a beta distribution with parameters $\alpha > 0$ and $\beta > \mathbf{0}$ if its density function (pdf) is given by:

$$\boldsymbol{f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)} \qquad 0 \le x \le 1}$$

In the second reference page 207:

$$\boldsymbol{f(x) =}\ \dots\ \mathbf{0} < x < 1$$

Where:

$$\boldsymbol{\mathrm{B}(\alpha,\beta)} = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\, dy$$

We often denote it by: $\mathbf{B(X;\alpha,\beta)}$.

The Cumulative Distribution Function (CDF) of Beta Distribution

$$\boldsymbol{F(x) = \int_0^x f(x)dx}$$

$$= \int_0^x \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}\, dx$$

$$= \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^x x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

From the definition of the incomplete beta function

$$\mathrm{B}(X;\alpha,\beta) = \int_0^x x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$\therefore \boldsymbol{F(x) = \frac{\mathrm{B}(X;\alpha,\beta)}{\mathrm{B}(\alpha,\beta)}}$$

The Moment Generating Function (MGF) of Beta Distribution

$$\boldsymbol{M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} f(x) dx}$$

$$= \int_0^1 e^{tx} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)} dx = \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^1 e^{tx}\, x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$= \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(tx)^k}{k!} \right) x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$= \frac{1}{\mathrm{B}(\alpha,\beta)} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \right) \int_0^1 x^{\alpha+k-1}(1-x)^{\beta-1} dx$$

$$= \frac{1}{\mathrm{B}(\alpha,\beta)} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \right) \left(\mathrm{B}(\alpha+k,\beta) \right)$$

$$= \frac{\mathrm{B}(\alpha+k,\beta)}{\mathrm{B}(\alpha,\beta)} \left(\frac{t^0}{0!} \right) + \left(\sum_{k=1}^{\infty} \frac{t^k}{k!} \right) \left(\frac{\mathrm{B}(\alpha+k,\beta)}{\mathrm{B}(\alpha,\beta)} \right)$$

$$= 1 + \sum_{k=1}^{\infty} \left(\frac{\Gamma(\alpha) \prod_{r=0}^{k} (\alpha+r)}{\Gamma(\alpha)} * \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta) \prod_{r=0}^{k} (\alpha+\beta+r)} \right) \frac{t^k}{k!}$$

$$= 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right) \frac{t^k}{k!}$$

$$\boldsymbol{\therefore\ M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right) \frac{t^k}{k!}}$$

Expectation and Variance of a Variable That Beta Distributed

Let X be a random variable beta distributed has $M_X(t) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$.

Then:

$$\boldsymbol{E(X) = M_X'(0)}$$

$$\frac{d}{dt}\left(1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}\right) = \frac{d}{dt}1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}\bigg|_{t=0}$$

$$= \boldsymbol{\frac{\alpha}{\alpha+\beta}}$$

$$\therefore \boldsymbol{E(x) = \frac{\alpha}{\alpha+\beta}}$$

$$\boldsymbol{var(X) = M_X''(0) - [M_X'(0)]^2}$$

$$\boldsymbol{M_X''(0) = \frac{d}{dt}\left(M'_X(t)\right)\Big|_{t=0}}$$

$$= \frac{d^2}{dt^2}1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}\bigg|_{t=0}$$

$$\therefore \boldsymbol{Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

Example 3.3.6 (Dirichlet Distribution). Let $X_1, X_2, \ldots, X_{k+1}$ be independent random variables, each having a gamma distribution with $\beta = 1$. The joint pdf of these variables may be written as

$$h(x_1, x_2, \ldots, x_{k+1}) = \begin{cases} \prod_{i=1}^{k+1} \frac{1}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} e^{-x_i} & 0 < x_i < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Solution:

Let

$$Y_i = \frac{X_i}{X_1 + X_2 + \cdots + X_{k+1}}, \quad i = 1, 2, \ldots, k,$$

and $Y_{k+1} = X_1 + X_2 + \cdots + X_{k+1}$ denote $k+1$ new random variables. The associated transformation maps $\mathcal{A} = \{(x_1, \ldots, x_{k+1}) : 0 < x_i < \infty,\ i = 1, \ldots, k+1\}$ onto the space:

$$\mathcal{B} = \{(y_1, \ldots, y_k, y_{k+1}) : 0 < y_i,\ i = 1, \ldots, k,\ y_1 + \cdots + y_k < 1,\ 0 < y_{k+1} < \infty\}.$$

The single-valued inverse functions are $x_1 = y_1 y_{k+1}, \ldots, x_k = y_k y_{k+1}, x_{k+1} = y_{k+1}(1 - y_1 - \cdots - y_k)$, so that the Jacobian is

$$J = \begin{vmatrix} y_{k+1} & 0 & \cdots & 0 & y_1 \\ 0 & y_{k+1} & \cdots & 0 & y_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & y_{k+1} & y_k \\ -y_{k+1} & -y_{k+1} & \cdots & -y_{k+1} & (1 - y_1 - \cdots - y_k) \end{vmatrix} = y_{k+1}^k.$$

Hence the joint pdf of $Y_1, \ldots, Y_k, Y_{k+1}$ is given by

$$\frac{y_{k+1}^{\alpha_1 + \cdots + \alpha_{k+1} - 1} y_1^{\alpha_1 - 1} \cdots y_k^{\alpha_k - 1} (1 - y_1 - \cdots - y_k)^{\alpha_{k+1} - 1} e^{-y_{k+1}}}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k) \Gamma(\alpha_{k+1})},$$

provided that $(y_1, \ldots, y_k, y_{k+1}) \in \mathcal{B}$ and is equal to zero elsewhere. By integrating out y_{k+1}, the joint pdf of $Y_1, \ldots, Y_k$ is seen to be

$$g(y_1, \ldots, y_k) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_{k+1})}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{k+1})} y_1^{\alpha_1 - 1} \cdots y_k^{\alpha_k - 1} (1 - y_1 - \cdots - y_k)^{\alpha_{k+1} - 1}, \quad (3.3.10)$$

when $0 < y_i,\ i = 1, \ldots, k,\ y_1 + \cdots + y_k < 1$, while the function g is equal to zero elsewhere. Random variables $Y_1, \ldots, Y_k$ that have a joint pdf of this form are said to have a **Dirichlet pdf**. It is seen, in the special case of $k = 1$, that the Dirichlet pdf becomes a beta pdf. Moreover, it is also clear from the joint pdf of $Y_1, \ldots, Y_k, Y_{k+1}$ that Y_{k+1} has a gamma distribution with parameters $\alpha_1 + \cdots + \alpha_k + \alpha_{k+1}$ and $\beta = 1$ and that Y_{k+1} is independent of $Y_1, Y_2, \ldots, Y_k$. ■

Chi-Square Distribution (χ^2)

Definition

Let us now consider a special case of the gamma distribution in which $\alpha = r/2$, where r is a positive integer, and $\beta = 2$. A random variable X of the continuous type.

The Probability Density Function (PDF) of Chi-Square Distribution

It has the same PDF of Gamma Distribution with parameters $({}^{r}/_{2}\ , 2)$, so it is given by:

$$\boldsymbol{f(x) = \frac{x^{(r/2)-1} e^{-x/2}}{2^{r/2}\Gamma(r/2\ )}} \qquad \boldsymbol{0 < x < \infty\ ,} \qquad \boldsymbol{r > 0}$$

Where:

$\boldsymbol{r} \equiv$ A positive integer

$$\boldsymbol{\Gamma(r/2)} = \int_0^\infty y^{(r/2)-1} e^{-y}\, dy$$

In which case, we often write that X has $\boldsymbol{\Gamma(\alpha, \beta)}$ distribution.

The Cumulative Distribution Function (CDF) of Chi-Square Distribution

$$\boldsymbol{F(x) = \int_{0}^{x} f(x)dx}$$

It has the same CDF of Gamma Distribution with parameters $(r/2, 2)$

$$\because \boldsymbol{CDF\ of\ Gamma\ Distribution\ F(x) = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}}$$

We can find the CDF of Chi-Square Distribution by assign its parameters $(r/2, 2)$

$$\therefore \boldsymbol{F(x) = \frac{\gamma(r/2, 2x)}{\Gamma(r/2)}}$$

Where:

$$\boldsymbol{\gamma(r/2, 2x)} = \int_{0}^{2x} t^{(r/2)-1} e^{-t}\, dt$$

$$\boldsymbol{\Gamma(r/2)} = \int_{0}^{\infty} t^{(r/2)-1} e^{-t}\, dt$$

The Moments Generating Function (MGF) of Chi-Square Distribution

It has the same MGF of Gamma Distribution with parameters $(r/2, 2)$

$$\because \boldsymbol{MGF\ of\ Gamma\ Distribution\ M_X(t) = (1 - \beta t)^{-\alpha}}$$

We can find the MGF of Chi-Square Distribution by assign its parameters $(r/2, 2)$

$$\therefore \boldsymbol{M_X(t) = (1 - 2t)^{-r/2}}$$

Expectation and Variance of a Variable That Chi-Square Distributed

Let X be a random variable chi-square distributed which is a spatial case of Gamma distribution when $\alpha = r/2$, and $\beta = 2$.

Then it will have the same expected value and variance so:

$$\boldsymbol{E(X) = M'_X(0)}$$

$\because$ *The expected value of a variable X that gamma distributed* $\boldsymbol{E(x) = \alpha\beta}$

We can find the expected value of a variable Y that chi-square Distributed by assign its parameters $(r/2, 2)$

$$\therefore \boldsymbol{E(Y)} = \left(\frac{r}{2}\right)2 = \boldsymbol{r}$$

$$\boldsymbol{var(X) = M''_X(0) - [M'_X(0)]^2}$$

$\because$ *The variance of a variable X that gamma distributed* $\boldsymbol{Var(X) = \alpha\beta^2}$

We can find the expected value of a variable Y that chi-square Distributed by assign its parameters $(r/2, 2)$

$$\therefore \boldsymbol{Var(Y)} = \left(\frac{r}{2}\right)2^2 = \boldsymbol{2r}$$

Examples

Example 3. If X has the p.d.f.

$$f(x) = \tfrac{1}{4}xe^{-x/2}, \qquad 0 < x < \infty,$$
$$= 0 \text{ elsewhere,}$$

then X is $\chi^2(4)$. Hence $\mu = 4$, $\sigma^2 = 8$, and $M(t) = (1 - 2t)^{-2}$, $t < \frac{1}{2}$.

Example 4. If X has the moment-generating function $M(t) = (1 - 2t)^{-8}$, $t < \frac{1}{2}$, then X is $\chi^2(16)$.

If the random variable X is $\chi^2(r)$, then, with $c_1 < c_2$, we have

$$\Pr(c_1 \le X \le c_2) = \Pr(X \le c_2) - \Pr(X \le c_1),$$

since $\Pr(X = c_1) = 0$. To compute such a probability, we need the value of an integral like

$$\Pr(X \le x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2}\, dw.$$

Tables of this integral for selected values of r and x have been prepared and are partially reproduced in Table II in Appendix B.

Related Distributions

Gamma Distribution

Let X be a gamma random variable with parameters α and β.

(1) The random variable X has a **standard gamma distribution** if $\alpha = 1$.

(2) If $\alpha = 1$ and $\beta = 1/\lambda$, then X has an **exponential distribution** with parameter λ.

(3) If $\alpha = \nu/2$ and $\beta = 2$, then X has a **chi–square distribution** with ν degrees of freedom.

(4) If $\alpha = n$ is an integer, then X has an **Erlang distribution** with parameters β and n.

(5) If $\alpha = \nu/2$ and $\beta = 1$, then the random variable $Y = 2X$ has a **chi–square distribution** with ν degrees of freedom.

(6) As $\alpha \to \infty$, X tends to a **normal distribution** with parameters $\mu = \alpha\beta$ and $\sigma^2 = \alpha\beta^2$.

(7) Suppose X_1 is a gamma random variable with parameters $\alpha = 1$ and $\beta = \beta_1$, X_2 is a gamma random variable with parameters $\alpha = 1$ and $\beta = \beta_2$, and X_1 and X_2 are independent. The random variable $Y = X_1/(X_1 + X_2)$ has a **beta distribution** with parameters β_2 and β_2.

(8) Let $X_1, X_2, \ldots, X_n$ be independent gamma random variables with parameters α and $\beta_i \; for \; i = 1, 2, \ldots, n$. The random variable $Y = X_1 + X_2 + \cdots + X_n$ has a **gamma distribution** with parameters α and $\beta = \beta_1 + \beta_2 + \cdots + \beta_n$.

Beta Distribution

Let X be a beta random variable with parameters α and β.

(1) If $\alpha = \beta = 1/2$, then X is an **arcsin** random variable.

(2) If $\alpha = \beta = 1$, then X is a **uniform** random variable with parameters $a = 0$ and $b = 1$.

(3) If $\beta = 1$, then X is a **power function** random variable with parameters $b = 1$ and $c = \alpha$

(4) As α and β tend to infinity such that α/β is constant, X tends to a **standard normal** random variable.

Chi-Square Distribution

(1) If X is a chi–square random variable with $r = 2$, then X is an **exponential** random variable with $\lambda = 1/2$.

(2) If X_1 and X_2 are independent chi–square random variables with parameters r_1 and r_2, then the random variable $(X_1/r_1)/(X_2/r_2)$ has an ***F*** **distribution** with r_1 and r_2 degrees of freedom.

(3) If X_1 and X_2 are independent chi–square random variables with parameters $r_1 = r_2 = r$, the random variable

$$\boldsymbol{Y = \frac{\sqrt{r}}{2}\frac{(X_1 - X_2)}{\sqrt{X_1 X_2}}}$$

has a ***t*** **distribution** with r degrees of freedom.

(4) Let X_i $(for\ i = 1, 2, \ldots, n)$ be independent chi–square random variables with parameters v_i . The random variable $Y = X_1 + X_2 + \cdots + X_n$ has a **chi–square distribution** with $v = v_1 + v_2 + \cdots + v_n$ degrees of freedom.

(5) If X is a chi–square random variable with v degrees of freedom, the random variable $\sqrt{X}$ has a **chi distribution** with parameter v. Properties of a chi random variable:

$$\boldsymbol{f(x) = \frac{x^{v-1} e^{-x^2/2}}{2^{(v/2)-1}\,\Gamma(v/2)}}$$

References

- ❖ Robert V. Hogg, Joseph W. McKean, Allen T. Craig. Introduction to Mathematical Statistics 8th Edition.
- ❖ Sheldon Ross. A First Course in Probability 9th Edition.
- ❖ Hogg Craig. Introduction to Mathematical Statistics 4th edition.
- ❖ Daniel Zwillinger. CRC Standard Probability and Statistics Tables and Formulae.
- ❖ Websites: proofwiki.org and statproofbook.github.io

The End

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