بسم الله الرحمن الرحيم

University of Khartoum

Faculty of Mathematical Sciences and Informatics

Statistics and Probability (S2013)

Assignment 1

The content:

- Moments Generating Function.
- Cumulant Generating Function.
- Factorial Moment Generating Function.
- References.

Written by:

Abobaker Ahmed Khidir Hassan. 21-304

Computer Sciences

Moments Generating Function

Definition:

Let X be a random variable such that for some h > 0, the expectation of $\boldsymbol{e^{tx}}$ is exists for -h < t < h.

The moment generating function of X is defined to be the function

$M(t) = E[e^{tX}]$ $\forall - \text{h} < \text{t} < \text{h}$. and we use the abbreviation MGF to denote the moment generating function of a random variable.

The moment generating function $M(t)$ of the random variable X is defined for all real values of t by:

$$M(t) = E(e^{tX}) = \begin{cases} \displaystyle\sum_x e^{tx}\, P(x)\ , if\ X\ is\ discrete\ with\ mass\ function\ p(x) \\ \displaystyle\int_{-\infty}^{\infty} e^{tx}\, f(x)\, dx\,, if\ X\ is\ continuous\ with\ density\ f(x) \end{cases}$$

- We call $M(t)$ the moment generating function because all of the moments of X can be obtained by successively differentiating $M(x)$ and then evaluating the result at $t = 0$. $\{\ M^{(n)}(0) = E(x^n)\ \}$.
- If we are discussing several random variables, it is often useful to subscript $M(t)$ as $M_x(t)$ to denote that this is the MGF of X.
- If a random variable X has f(X) - pmf ⊕ pdf - and we want to compute E[g(X)] where (g(X) is a some function of X); this function has a MGF if and only if the its (series sum ⊕ integration) of E(g(X)) is **convergent**, else that isn't exists. Note: $A \oplus \boldsymbol{B} \equiv \boldsymbol{A\ or\ B\ but\ not\ both}$.

- Important formulas:
  - $\boldsymbol{M(0) = 1}$
  - $\boldsymbol{M'(0) = E(x)} = \frac{d}{dt}M(t)\ \ when\ t = 0;$
  - $\boldsymbol{M''(0) - (M'(0))^2 = var(X)} = E(x^2) - \left(E(x)\right)^2 = \frac{d^2}{dt^2}M(t) - (\frac{d}{dt}M(t)\ )^2\ \ when\ t = 0;$

Example:

If x a random variable and its probabilities is distributed by:

$$f(x) = \begin{cases} \frac{x}{10}, & x = 1,2,3,4 \\ 0, & otherwise \end{cases}$$

1- Find the moment generating function of x.

2- Find the expected value and the variance of X using the old method and using the MGF.

Solution:

First: Getting the MGF.

$$M(t) = E(e^{tx}) = \sum_x e^{tx} P(x)$$

$$= \sum_x e^{tx} * \frac{x}{10} = e^t * \frac{1}{10} + e^{2t} * \frac{2}{10} + e^{3t} * \frac{3}{10} + e^{4t} * \frac{4}{10} = \frac{e^t + 2e^{2t} + 3e^{3t} + 4e^{4t}}{10}$$

$$\therefore M(t) = \frac{1}{10}\left(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t}\right)$$

Second:

a. Using the old method:

$$E(X) = \sum x * p(x) \quad = \sum \frac{x^2}{10} = \frac{1^2}{10} + \frac{2^2}{10} + \frac{3^2}{10} + \frac{4^2}{10} = \frac{30}{10} = 3$$

$$var(X) = E(x^2) - \left(E(x)\right)^2 = \sum x^2 \frac{x}{10} - (3)^2;$$

$$\left(\sum \frac{x^3}{10} = \frac{1}{10} + \frac{2^3}{10} + \frac{3^3}{10} + \frac{4^3}{10} = 10\right)$$

$$\therefore var(X) = 10 - 9 = 1$$

b. Using MGF:

$$E(x) = M'(0) = \frac{d}{dt} M(t) = 0.1 * e^t + 0.2 * 2e^{2t} + 0.3 * 3\, e^{3t} + 4e^{4t}\big|_{t=0}$$

$$E(x) = 0.1 + 0.2 * 2 + 0.3 * 3 + 0.4 * 4 = 3.$$

$$Var(X) = M''(0) - (M'(0))^2$$

$$M''(0) = 0.1 * e^t + 0.4 * 2 * e^{2t} + 0.9 * 3 * e^{3t} + 1.6 * 4 * e^{4t}\big|_{t=0}$$

$$M''(0) = 0.1 + 0.8 + 2.7 + 6.4 = 10 \quad \sigma^2 = 10 - (3)^2 = 1$$

Theorems:

Let X be a random variable has MGF $M_X(t)$ and $\boldsymbol{Y = aX + b}$; $a, b \in \mathbb{R}$

Then:

1. $\boldsymbol{M_{aX}(t) = M_X(at)}$.

2. $\boldsymbol{M_{X+b}(t) = e^{bt} M_X(t)}$

 Based on it: $\boldsymbol{M_{X-\mu}(t) = e^{-\mu t} M_X(t)}$

$$\boldsymbol{\mu_r = M_{X-\mu}^{(r)}(0) = \left.\frac{d^r\left(e^{-\mu t} M_X(t)\right)}{dt^r}\right|_{t=0}}$$

3. $\boldsymbol{M_Y(t) = e^{bt} M_{aX}(t) = e^{bt} M_X(at)}$

4. $\boldsymbol{M_{\frac{X+b}{a}}(t) = e^{\frac{bt}{a}} * M_X\left(\frac{t}{a}\right)}$

5. $\boldsymbol{M_{Y+X}(t) = M_x(t) M_y(t)}$; $\forall -h < t < h$.

6. If $X_1, X_2, X_3, \ldots.., X_n$ are independent random variables and $S = X_1 + X_2 \ldots + X_n$ Then:

$$\boldsymbol{M_S(t) = [M_X(t)]^n}$$

Cumulant Generating Function

Definition:

If we assume $M_X(t)$ to be the moment generating function, then its Cumulant Generating Function $C_x(t)$ is defined as follows:

$$C_X(t) = \ln[M_X(t)] = \sum_{i=1}^{r} \frac{k_i t}{i!} \qquad \text{Or} \qquad M_X(t) = e^{[C_x(t)]}$$

- $C_X(t)$ is the cumulant generating function. The constants $k_1, k_2, \ldots k_r$ are the cumulants (or semi–invariants) of the distribution.
- The r^{th} derivative of C_X with respect to t, evaluated at 0 is the r^{th} cumulant. And The function C_X generates the cumulants:

$$\boldsymbol{k_n = C_X^{(n)}(0) = \left.\frac{d^n C_X}{dt^n}\right|_{t=0}} \tag{1}$$

- The moment generating functions are all positive so that the cumulant generating functions are defined wherever the moment generating functions are.
- It also called semi–invariant generating function and it also has CGF and $\psi_x(t)$ symbols.
- Important formulas:
  - ❖ $\boldsymbol{C_X(0) = 0}$
  - ❖ $\boldsymbol{C_X'(0) = E(X)} = \frac{d}{dt}\ln[M_x(0)] = \frac{M_x'(0)}{M_x(0)} = \frac{E(X)}{(1)}.$
  - ❖ $\boldsymbol{C_X''(0) = Var(X)} = \frac{M_x(0)M_x''(0) - [M'_x(0)]^2}{[M_x(0)]^2} = \frac{(1)\left(E\left(X^2\right)\right) - [E(X)]^2}{(1)^2}$

Example:

Find the CGF of the Poisson distribution and use it to find the expected value and variance.

Solution:

We know that the Poisson distribution PMF is: $\boldsymbol{P(x;\lambda) = \frac{e^{-\lambda} \ \lambda^{x}}{x!}}$

First: Getting the MGF:

$$\boldsymbol{M_x(t) = E(e^{tx})} = \sum e^{tx} \, P(x) \ = \sum e^{tX} * \frac{e^{-\lambda} \ \lambda^{x}}{X!} = \ e^{-\lambda} \sum \frac{e^{tX} \ \lambda^{X}}{X!} =$$

$$e^{-\lambda} \sum \frac{(e^{t} \ \lambda)^{X}}{X!} = \ (e^{-\lambda})\left(e^{\lambda e^{t}}\right) = \ \boldsymbol{e^{\lambda(e^{t}-1)}}$$

Second: Getting the CGF:

$$\boldsymbol{C_x(t) = \ \ln[M_x(t)]} \qquad\qquad C_x(t) = \ln\left[\boldsymbol{e^{\lambda(e^{t}-1)}}\right] = \boldsymbol{\lambda(e^{\lambda} - 1)}$$

Finally: Getting the expected value and variance:

$$\mathbf{E(X) = \ C_x'(0)} = \frac{\mathrm{d}}{\mathrm{dt}}\lambda(e^{t} - 1) \quad = \lambda e^{t}|_{\mathrm{t=0}} = \ \boldsymbol{\lambda}$$

$$\boldsymbol{var(X) = C_X''(0)} = \frac{d}{dt}[\lambda e^{t}] = \boldsymbol{\lambda}$$

Theorems:

- From 1: Marcienkiewicz's theorem states that either all but the first two cumulants vanish (i.e., it is a normal distribution) or there are an infinite number of non-vanishing cumulants.
- Let $X_1, X_2, X_3, \ldots, X_n$ are independent random variables and $S_n = \sum_{i=1}^{\infty} X_i$ and M_{X_i} is exists. Then:

$$\boldsymbol{C_{S_n}(t) = n.C_X(t)}$$

Factorial Moment Generating Function

Definition:

Let X denote a random variable, the factorial moment generating function $P_X(t)$ is defined as:

$$P_X(t) = E(t^X) = \begin{cases} \sum_x t^x \, p(x) \, , & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} t^x \, f(x) \, dx \, , & if\ X\ is\ continuous \end{cases}$$

- We call it the factorial generating function because the fact that it satisfies:

$$\frac{\mathbf{d^n}}{\mathbf{dt^n}}[\mathbf{E}(\boldsymbol{t}^{\boldsymbol{X}})]_{\mathbf{t=1}} = \mathbf{E}[\mathbf{X}(\mathbf{X}-\mathbf{1})(\mathbf{X}-\mathbf{2}) \ldots \ldots (\mathbf{X}-\mathbf{n}+\mathbf{1})]$$

  And we note that the right-hand side is a factorial moment.

- The function P generates the factorial moments by:

$$\boldsymbol{\mu}_{[r]} = \boldsymbol{P}^{(r)}(\mathbf{1}) = \left.\frac{\boldsymbol{d}^{r}\boldsymbol{P}(\boldsymbol{t})}{\boldsymbol{d}\boldsymbol{t}^{r}}\right|_{\boldsymbol{t}=\mathbf{1}}$$

- It's also has **FMGF,** $\boldsymbol{K_X}(\boldsymbol{t})$, and $\boldsymbol{F_X}(\boldsymbol{t})$ symbols.

  If X is a discrete random variable, then we can write:

  $$\boldsymbol{F_X}(\boldsymbol{t}) = \sum_{\boldsymbol{x}} \boldsymbol{t}^{\boldsymbol{x}} \, \boldsymbol{P}(\boldsymbol{X}=\boldsymbol{x})$$

  And the factorial moment generating function is called the **probability-generating function**, since the coefficients of the power series give the probabilities.

  That is, to obtain the probability that $X = \mathrm{x}$, calculate:

$$\boldsymbol{P}(\boldsymbol{X}=\boldsymbol{x}) = \frac{\mathbf{1}}{\boldsymbol{x}!} * \frac{\boldsymbol{d}}{\boldsymbol{dt}}(\boldsymbol{E}(\boldsymbol{t}^{\boldsymbol{X}}))\,|_{\boldsymbol{t}=\mathbf{1}}$$

- **Getting the expected value and standard deviation from FMGF:**
  - $\boldsymbol{F_X(1) = 1}$.
  - $\boldsymbol{F'_X(1) = E(X)} = \frac{d}{dt}[E(t^x)] = E(Xt^{X-1}) = E[X.(1)^{X-1}] = E(X)$.
  - $\boldsymbol{F'_X(1)[1 - F'_X(1)] + F''_X(1) = var(X)}$

$$F'_X(1)[1 - F'_X(1)] + F''_X(1) = [E(X)][1 - E(X)] + \frac{d}{dt}[E(X.t^x)]$$

$$= E(X) - E(X)^2 + \frac{1}{2} * E\big(X(X-1)(t^{X-1})\big)$$

$$= E(X) - E(X)^2 + E(X^2) - E(X)(1) = E(X^2) - [E(X)]^2$$

Example:

If a random variable X has pmf as following:

$$P(X) = \begin{cases} \frac{x}{6}, & x = 1,2,3 \\ 0, & otherwise \end{cases}$$

Find its expected value and standard deviation using the factorial moments generating function.

Solution:

First: Getting the FMGF:

$$P_x(t) = E(t^X) = \sum t^x * p(x) = \frac{t + 2t^2 + 3t^3}{6} = \boldsymbol{\frac{1}{6}(t + 2t^2 + 3t^3)}$$

Second: Getting the expected value and variance:

$$\boldsymbol{E(X) = F'_X(1)} = \frac{d}{dx}\left(\frac{1}{6}(t + 2t^2 + 3t^3)\right) = \frac{1}{6}(1 + 4t + 9t^2)_{t=1} = \frac{14}{6} \approx \boldsymbol{2.3}$$

$$\boldsymbol{Var(X) = F''_X(1) + F'_X(1) - [F'_X(1)]^2} = F''_X(1) + E(X) - [E(X)]^2$$

$$F''_X(1) = \frac{1}{6}(4 + 18t)_{t=1} = \frac{22}{6}$$

$$\therefore \boldsymbol{Var(X)} = \frac{22}{6} + \frac{14}{6} - \left[\frac{14}{6}\right]^2 = \frac{20}{36} \approx \boldsymbol{0.5555}$$

$$\boldsymbol{\sigma = \sqrt{var(X)}} = \sqrt{0.5555} \approx \boldsymbol{0,7453}$$

Theorems:

Suppose $P_X(t)$ is the factorial moment generating function for the random variable X and a, b are constants.

Then:

1. $P_{aX}(t) = P_X(t^a)$
2. $P_{X+b}(t) = t^b . P_X(t)$
3. $P_{\frac{X+b}{a}}(t) = t^{b/a} . P_X\left(t^{1/a}\right)$
4. $P_X(t) = M_X(lnt), where\ M_X(t)\ is\ the\ moment\ generating\ function\ for\ X.$
5. If $X_1 X_2, \ldots \ldots, X_n$ are **independent** random variables with factorial moment generating function $P_X(t)$ and $S = X_1 + X_2 + \cdots . + X_n$, then:

$$P_S(t) = [P_X(t)]^n.$$

References:

- Sheldon Ross. A First Course in Probability 9th Edition.
- Robert V. Hogg, Joseph W. McKean, Allen T. Craig. Introduction to Mathematical Statistics 8th Edition.
- Hogg Craig. Introduction to Mathematical Statistics 4th edition.
- Daniel Zwillinger. CRC Standard Probability and Statistics Tables and Formulae.
- Probability and Statistics (S2013) Lectures Summary by Mr. Elamin Hassan Salih.
- Old Researches.

The End

19th - May - 2024