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Statistics and Probability (S2013)

Assignment 2

The content:

- Binomial Probability Distribution.
- Poisson Probability Distribution.
- Normal Probability Distribution.
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Computer Sciences

Binomial Probability Distribution

Definition:

It is the discrete probability distribution that explains the outcomes of a **binomial experiment** and the corresponding probabilities of these outcomes.

Binomial Experiment:

It is a probability experiment that satisfies the following requirements:

- There must be a fixed number of trials.
- Each trial can have only two outcomes or outcomes that can be reduced to two outcomes. These outcomes can be considered as either success or failure.
- The outcomes of each trial must be independent of one another.
- The probability of a success must remain the same for each trial.

Binomial Probability Formula:

$$\boldsymbol{P(X = x) = \binom{n}{x} p^x q^{(n-x)}}$$

Where:

x is the number of successes that is $0 \leq X \leq n$ and $X = 1,2,3, \dots., n$.

n is the number of trials.

$\mathrm{p} = \mathrm{P(S)}$ and $\mathrm{q} = \mathrm{P(F)}$; S = successes F = failures $\boldsymbol{p + q = 1}$

The Moments Generating Function for the Binomial Distribution:

$$\boldsymbol{M_x(t) = E(e^{tX})} = \sum e^{tx} P(x) = \sum e^{tx} \binom{n}{x} p^x q^{(n-x)} = \sum e^{tx} \binom{n}{x} p^x q^{(n-x)}$$

$$= \sum \binom{n}{x} (pe^t)^x (1-p)^{(n-x)} = \boldsymbol{[(1-p) + pe^t]^n}$$

$$\therefore \mathbf{M_x(t) = [(1-p) + pe^t]^n}$$

Expectation and Variance:

1. Using PMF of the Binomial Distribution:

a. $\boldsymbol{E\left(X^k\right) = \sum x^k . p(x)}$

$$= \sum_{x=0}^{n} x^k \binom{n}{x} p^x q^{(n-x)} = \sum_{x=0}^{n} x^k \binom{n}{x} p^x (1-p)^{(n-x)} = 0 + \sum_{x=1}^{n} x^k \binom{n}{x} p^x (1-p)^{(n-x)}$$

Using the identity: $x\binom{n}{x} = n\binom{n-1}{x-1}$

$$= np \sum_{x=1}^{n} x^{k-1} \binom{n-1}{x-1} p^{x-1} (1-p)^{(n-x)}$$

Make: $j = x - 1$

$$= np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} = np\, E[(Y+1)^{k-1}]; Y(n-1, p)$$

When $k = 1$: $\boldsymbol{E(X)} = np\, E[(Y+1)^0] = np(1) = \boldsymbol{np}$ $\quad\quad \therefore \boldsymbol{E(X) = n.p}$,

b. $\mathbf{Var(X)} = \boldsymbol{E(X^2) - [E(X)]^2 = (\sum x^2 . p(x)) - \mu^2}$

$E(X^2) = np\, E[(Y+1)^0] = np\, [(n-1)p + 1]$

$\boldsymbol{Var(X)} = np\, [(n-1)p + 1] - (np)^2 = np(1-p) = \boldsymbol{npq}.$

$$\therefore \boldsymbol{Var(X) = npq},$$

2. Using MGF of the Binomial Distribution:

a. $\boldsymbol{E(x) = M'(0)}$

$\frac{d}{dt} M(t) = \frac{d}{dt}([(1-p) + pe^t]^n) = n[(1-p) + pe^t]^{n-1}\, (pe^t)|_{\,t=0} = np(q+p) = np$

$$\therefore \boldsymbol{E(x) = M'(0) = np}$$

b. $\boldsymbol{\sigma^2 = E(x^2) - \left(E(x)\right)^2 = M''(0) - (\mu)^2}$

$M''(t) = n[(1-p) + pe^t]^{n-1}(pe^t) + n(n-1)[(1-p) + pe^t]^{n-2}(pe^t)^2$.

$M''(0) = np + n(n-1)p^2.$

$Var(X) = (np + n(n-1)p^2) - (np)^2 = np(1-p) = \boldsymbol{npq}.$

$\therefore \boldsymbol{\sigma^2 = np(1-p) = npq}$ **->** $\boldsymbol{\sigma = \sqrt{n\ p\ q}}$

Uses of the Binomial Distribution:

- The binomial distribution is used to characterize the number of successes in a sample size n.
- It is used to model some very common experiments in which a sample of size *n* is taken from an infinite population such that each element is selected independently and has the same probability, *p*, of having a specified attribute.
- Problems have only two outcomes or can be reduced to two outcomes.
- Examples:
    1. a medical treatment can be classified as effective or ineffective, depending on the results.
    2. A person can be classified as having normal or abnormal blood pressure, depending on the measure of the blood pressure gauge.
    3. A multiple-choice question, even though there are four or five answer choices, can be classified as correct or incorrect.

Example:

There are five flights daily from a city into the Airport. Suppose the probability that any flight arrives late is 0.20. What is the probability that none of the flights are late today? What is the mean, variance?

Solution:

In this case, a "success" is a flight that arrives late. Because there are no late arrivals, x=0;

n = 5;   p = 0.20;   q = 1 – p = 1 – 0.20 = 0.80;

$$\boldsymbol{P(X) = \binom{n}{x} p^x q^{(n-x)}}$$

$$P(0) = C_0^5 (0.20)^0 (0.80)^5 = (1)(1)(0.3277) = 0.3277$$

The probability that none of the flights are late today is 0.3277 and it's unlikely to occur.

$$\boldsymbol{\mu} = \boldsymbol{n.p} = 5 * 0.2 = \mathbf{1.}$$

The Expectation of how many flights arrives late today is one flight.

$\boldsymbol{\sigma^2} = \boldsymbol{n\ p\ q}$ = 5(*0.*20)(0.80) = *0.80.*

Poisson Probability Distribution

Definition:

It is a discrete probability distribution that is useful when n is large and p is small and when the independent variables occur over a period of time.

- In addition to being used for the stated conditions (i.e., n is large, p is small, and the variables occur over a period of time),
- The Poisson distribution can be used when a density of items is distributed over a given area or volume, such as the number of plants growing per acre or the number of defects in a given length of videotape.
- It is can be used with large and infinite samples "n is very big".
- The probability of an event in a short time or a small space is the ratio of that time or space to a specific range without counting the occurrence of the events out of that range.

Poisson Probability Formula:

$$P(x;\lambda) = \frac{e^{-\lambda} \quad \lambda^{x}}{x!}$$

Where:

X is a discrete random variable.

λ is the mean number of occurrences per unit (time, volume, etc.).

e is a constant approximately equal to 2.7183.

The Moments Generating Function for the Poisson Distribution:

$$M_x(t) = E(e^{tx}) = \sum e^{tx} P(x) = \sum e^{tX} * \frac{e^{-\lambda} \quad \lambda^{x}}{X!} = e^{-\lambda} \sum \frac{e^{tX} \quad \lambda^{X}}{X!} =$$

$$e^{-\lambda} \sum \frac{(e^{t} \quad \lambda)^{X}}{X!} = (e^{-\lambda})\left(e^{\lambda e^{t}}\right) = e^{\lambda(e^{t}-1)}$$

$$\therefore \; M_x(t) = e^{\lambda(e^{t}-1)}$$

Expectation and Variance:

1. Using PMF of the Poisson Distribution:

a. $\boldsymbol{E(X) = \sum x * \ P(x)}$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \ \ \lambda^{x}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{x \ \lambda^{x}}{x!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$= \lambda e^{-\lambda} e^{\lambda} \ = \boldsymbol{\lambda}$

$$\therefore \boldsymbol{E(X) = \mu = \lambda}$$

b. $\mathbf{Var(X)} = \ \boldsymbol{E(x^2) - \left(E(x)\right)^2}$

$\boldsymbol{E(x^2) = \sum x^2 * \ P(x)} = \sum x^2 \frac{e^{-\lambda} \ \ \lambda^{x}}{X!} = \ = \sum [x(x \ - \ 1) + \ x] \frac{e^{-\lambda} \ \ \lambda^{x}}{X!}$

$= \sum x(x - 1) \frac{e^{-\lambda} \ \ \lambda^{x}}{X!} + \sum x \frac{e^{-\lambda} \ \ \lambda^{x}}{X!} = \ \ \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} = \ \ \lambda^2 e^{-\lambda} e^{\lambda} = \ \boldsymbol{\lambda^2}$

$$\therefore \boldsymbol{Var(X) = \ \ \lambda^2 + \lambda - \ \ \lambda^2 \ = \lambda}$$

2. Using MGF of the Poisson Distribution:

a. $\boldsymbol{E(x) = M'(0)}$

$\boldsymbol{\frac{d}{dt} M_x(t) = \frac{d}{dt} \ e^{\lambda(e^t - 1)} = (\lambda e^t)(e^{\lambda(e^t-1)})}.$

$$\therefore \boldsymbol{E(x) = M'(0) = \ \lambda}$$

b. $\boldsymbol{Var(X) = \ E(x^2) - \left(E(x)\right)^2} \quad = \ M''(0) - \ (\mu)^2 = (\boldsymbol{\lambda^2} + \ \boldsymbol{\lambda}) - \boldsymbol{\lambda^2} = \boldsymbol{\lambda}.$

$$\therefore \boldsymbol{Var(X) = \ \lambda}$$

Uses of the Poisson Distribution:

- It is used to describe the number of events occurring in a time interval or region of opportunity.
- Examples:
    1. The number of misprints on a page (or a group of pages) of a book.
    2. The number of customers entering a post office on a given day.
    3. The number of customers waiting to be served at a restaurant.
    4. The number of vacancies occurring during a year in the federal judicial system.
    5. The number of defective parts in outgoing shipments.

Example:

If there are 200 typographical errors randomly distributed in a 500 pages manuscript, find the probability that a given page contains exactly 3 errors, and then find the expected value and the standard deviation.

Solution:

First, find the mean number λ of errors. $\boldsymbol{\lambda = \frac{200}{500} = 0.4}$

Since there are 200 errors distributed over 500 pages, each page has an average 0.4 error per page.

Second, X =3, substituting into the formula yields:

$$\boldsymbol{P(X;\lambda) = \frac{e^{-\lambda}\ \lambda^{x}}{X!}} \qquad \boldsymbol{P(3;0.4)} = \frac{(2.7183)^{-0.4}\ \ (0.4)^{3}}{3!} = \boldsymbol{0.0072}$$

The probability of selecting a page has 3 errors is 0.0072 and it's unlikely to occur because the average of errors is 0.4/p.

$$\boldsymbol{E(x) = \lambda = 0.4} \qquad \boldsymbol{\sigma = \sqrt{\lambda}} = \sqrt{0.4} = \boldsymbol{0.2}$$

Relationship between Poisson and Binomial Distributions:

The Poisson distribute on can also be used to approximate the binomial distribution when the expected value $\boldsymbol{\lambda} = \boldsymbol{n}.\boldsymbol{p}$ is less than **5**.

Example:

If approximately 2% of the people in a room of 200 people are left-handed,

Find the probability that exactly 5 people there are left-handed.

<u>Solution:</u>

$\boldsymbol{\lambda} = \boldsymbol{n}\,.\boldsymbol{p}$, then $\lambda = (200)(0.02) = \mathbf{4}$.

$$\boldsymbol{P}(\boldsymbol{x};\boldsymbol{\lambda}) = \frac{\boldsymbol{e}^{-\boldsymbol{\lambda}}\ \boldsymbol{\lambda}^{\boldsymbol{x}}}{\boldsymbol{x}!} \qquad \boldsymbol{P}\big(\mathbf{5};\mathbf{4}\big) = \frac{(2.7183)^{-4}\ 4^{5}}{5!} = \mathbf{0.1563}$$

This is verified by the formula:

$$\boldsymbol{P}(\boldsymbol{X}) = \binom{\boldsymbol{n}}{\boldsymbol{x}}\boldsymbol{p}^{\boldsymbol{x}}\boldsymbol{q}^{(\boldsymbol{n}-\boldsymbol{x})} \qquad \boldsymbol{P}(\mathbf{5}) = \binom{200}{5}(0.02)5(0.98)195 = \mathbf{0.1579}.$$

The difference between the two answers is based on the fact that the Poisson distribution is an approximation and rounding has been used.

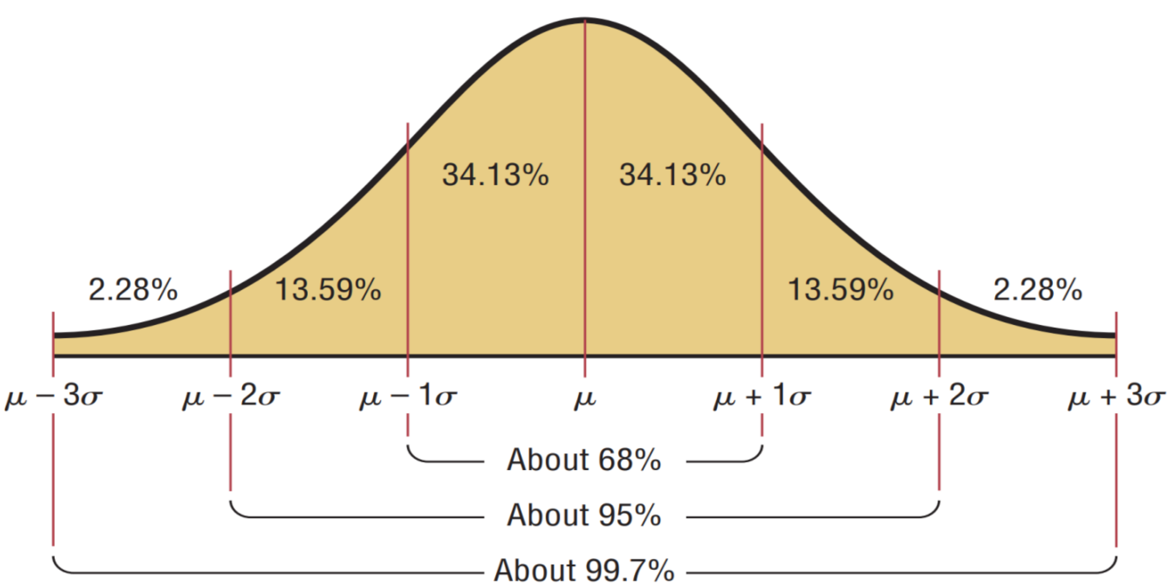
Normal Probability Distribution

Definition:

It is a continuous, symmetric, bell shaped distribution of a random variable.

Properties:

- We call X a normal random variable.
- The mean, median, and mode are equal and are located at the center of the distribution.
- A normal distribution curve is bell-shaped, unimodal, symmetric about the mean, and the curve is continuous. For each value o f X, there is a corresponding value of Y.
- The curve never touches the x axis but it gets increasingly closer.
- The area under the curve is classified by:



- Especial case: When $\boldsymbol{\mu} = \mathbf{1}$ and $\boldsymbol{\sigma} = 0$ we call it Standard Normal Distribution.

Normal Distribution Formula:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad for\ -\infty < x < \infty$$

Where:

μ is the mean of X.

σ^2 is the variance of X.

$e \approx 2.718$ $\quad\quad$ $\pi \approx 3.14$

Standard Normal Distribution Formula:

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{\frac{-z^2}{2}} \quad for\ -\infty < x < \infty$$

Where:

$z = \frac{X-\mu}{\sigma}$

The Moments Generating Function of the Normal Distribution:

To explain it, I will find the MGF for the standard normal distribution then generalize the concept.

$$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} (e^{tz})\left(\frac{1}{\sqrt{2\pi}}\right)\left(e^{\frac{-z^2}{2}}\right)dz = e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right) e^{\frac{-(z-t)^2}{2}} dz$$

Make $w = z - t$ and then $dw = dz$

$$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right) e^{\frac{-w^2}{2}} dw = e^{\frac{t^2}{2}}(1) = e^{\frac{t^2}{2}} \quad \therefore\ M_Z(t) = e^{\frac{t^2}{2}} \ \ \text{for } -\infty < t < \infty$$

To generalize the concept make $X = Z\sigma + \mu$

$$M_X = E\left[e^{t(Z\sigma+\mu)}\right] = e^{\mu t} * E[e^{t\sigma Z}] = e^{\mu t} * e^{\frac{\sigma^2 t^2}{2}} = e^{[\mu t + \frac{1}{2}\sigma^2 t^2]}$$

$$\therefore\ M_x(t) = e^{[\mu t + \frac{1}{2}\sigma^2 t^2]}$$

The Expectation and Variance:

1. <u>Using the PDF of the Normal Distribution</u>**:**

The normal distribution is a spatial case that is its **Properties** must be explained to determine its function.

a. $\boldsymbol{E(x) = M'(0)}$
b. $\boldsymbol{Var(X) = \sigma^2}$

2. <u>Using the MGF of the Normal Distribution:</u>

a. $\boldsymbol{E(x) = M'(0)}$

$$\frac{d}{dt}M_Z(t) = \frac{d}{dt}e^{\mu t} * e^{\left[\frac{1}{2}\sigma^2t^2\right]}\Big|_{t=0} = e^{\left[\frac{1}{2}\sigma^2t^2\right]} * \mu e^{\mu t} + e^{\mu t} * \sigma^2 t e^{\left[\frac{1}{2}\sigma^2t^2\right]}\Big|_{t=0} = \boldsymbol{\mu}$$

$$\therefore \boldsymbol{E(X) = \mu}$$

b. $\boldsymbol{Var(X) = E(x^2) - \left(E(x)\right)^2 = M''(0) - (\mu)^2}$

$$M''(0) = \frac{d}{dx}(\mu + \sigma^2 t).e^{t\mu + \frac{\sigma^2t^2}{2}}\Big|_{t=0} = e^{t\mu + \frac{\sigma^2t^2}{2}}(\sigma^2 + (\mu + \sigma^2 t)^2)\Big|_{t=0} = \sigma^2 + \mu^2$$

$$Var(X) = \sigma^2 + \mu^2 - \mu^2 = \boldsymbol{\sigma^2} \qquad \therefore \boldsymbol{Var(X) = \sigma^2}$$

Uses of the normal distribution:

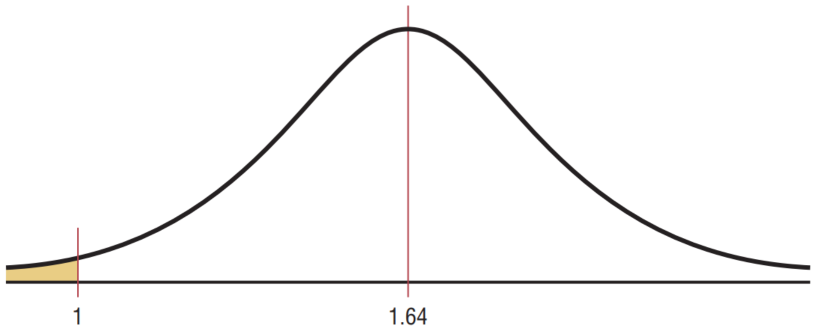
- It is used to inferential statistics to compare different groups and make estimates about populations using samples.
- Examples:
    1. People's height.
    2. Standard tests scores.
    3. IQ tests.
    4. Incomes.

Example:

Americans consume an average of 1.64 cups of coffee per day. Assume the variable is approximately normally distributed with a standard deviation of 0.24 cup. If 500 individuals are selected, approximately how many will drink less than 1 cup of coffee per day?

Solution:

First, Draw a figure and represent the area as shown in Figure:



Second, Find the *z* value for 1.

$$z = \frac{X - \mu}{\sigma} = \frac{1 - 1.64}{0.24} = -2.67$$

Third, Find the area to the left of z = - 2.67. It is 0.0038

Finally, To find how many people drank less than 1 cup of coffee, multiply the sample size 500 by 0.0038 to get 1.9,

Approximately 2 people will drink less than 1 cup of coffee a day.

Relationship between Normal and Binomial Distributions:

An important result in probability theory known as the DeMoivre-Laplace limit theorem states that when n is large, a binomial random variable with parameters n and p will have approximately the same distribution as normal random variable with the same mean and variance as the binomial.

It formally states that if we “standardize” the binomial by:

first subtracting its mean $(\boldsymbol{\mu = n.p})$ and then dividing the result by its standard deviation $(\sqrt{\boldsymbol{np(1-p)}})$, then the distribution function of this standardized random variable (which has mean 0 and variance 1) will converge to the standard normal distribution function as $\boldsymbol{n} \to \infty$.

The DeMoivre–Laplace limit theorem

If S_n denotes the number of successes that occur when n independent trials, each resulting in a success with probability p, are performed, then, for any $a < b$,

$$P\left\{a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right\} \to \Phi(b) - \Phi(a)$$

as $n \to \infty$.

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The End

20th - May - 2024