A2023
Ordinary Differential Equations 1

Class Test

Date: 03 June 2024 **Student Number:** ____________________

Duration: 4 Days (From the 4th to the 7th of June) **Student Name:** ____________________

Instructions

1. The test consists of two sections: Section A is multiple choice questions (MCQs), and Section B is written questions.
2. Read all the questions carefully and attempt all the questions.
3. Section A has 10 multiple choice questions. Each question carries 1 mark. Copy the correct answer to your answer booklet.
4. Section B has 11 written questions. Each question requires detailed working and carries different marks as specified.
5. Show all your workings in the space provided for each question. Partial credit will be given for correct steps even if the final answer is incorrect.
6. Maintain accuracy to four decimal places unless otherwise specified.
7. Manage your time effectively. The total duration of the test is 4 days.
8. Once you have completed the test, ensure that your name, student number, and all answers are clearly written on the answer sheets.
9. Cheating or any form of academic dishonesty will not be tolerated and will result in immediate disqualification.

Question 1 — Multiple Choices (10 Marks)

Read each question carefully and select the best answer for each question. There is ONLY ONE correct answer per question

1. Which of the following differential equations is NOT separable? [1 Mark]

   (a) $\frac{dy}{dx} = xy + y$

   (b) $\frac{dy}{dx} = \frac{y}{x}$

   (c) $\frac{dy}{dx} = x + y$

   (d) $\frac{dy}{dx} = 0$

   (e) $\frac{dy}{dx} = e^x$

2. The general solution of the homogeneous differential equation $\frac{dy}{dx} = y - x$ is: [1 Mark]

   (a) $y = Ce^x$

   (b) $y = Ce^x + 1 + x$

   (c) $y = Cxe^x$

   (d) $y = C(x - y)$

   (e) $y = C(x + y)$

3. Which of the following is an example of a Bernoulli differential equation? [1 Mark]

   (a) $\frac{dy}{dx} + P(x)y = Q(x)y^n$

   (b) $y'' + p(x)y' + q(x)y = 0$

   (c) $\frac{dy}{dx} = f(x)g(y)$

   (d) $y' = y - x$

   (e) $\frac{d^2y}{dx^2} + P(x)y = Q(x)$

4. The differential equation $\frac{dy}{dx} + y = e^x$ is: [1 Mark]

   (a) Linear

   (b) Non-linear

   (c) Homogeneous

   (d) Exact

   (e) Bernoulli

5. For the exact differential equation $M(x,y)dx + N(x,y)dy = 0$, which of the following conditions must hold? [1 Mark]

   (a) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

   (b) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$

   (c) $M = N$

   (d) $\frac{\partial^2 M}{\partial x \partial y} = \frac{\partial^2 N}{\partial y \partial x}$

   (e) $M + N = 0$

6. A Riccati differential equation has the form: [1 Mark]

   (a) $\frac{dy}{dx} + P(x)y + Q(x)y^2 = R(x)$

   (b) $y'' + P(x)y' + Q(x)y = 0$

   (c) $\frac{dy}{dx} = f(x)g(y)$

   (d) $\frac{dy}{dx} + P(x)y = Q(x)y^n$

   (e) $\frac{d^2y}{dx^2} + Q(x)y = R(x)$

7. The differential equation $\frac{dy}{dx} = ky$ models: [1 Mark]

   (a) Newton's Law of Cooling

   (b) Exponential Growth/Decay

   (c) Logistic Growth

   (d) Population Dynamics

   (e) Chemical Reactions

8. Newton's Law of Cooling is given by which of the following differential equations? [1 Mark]

   (a) $\frac{dy}{dx} = ky$

   (b) $\frac{dT}{dt} = -k(T - T_{\text{env}})$

   (c) $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$

   (d) $\frac{dN}{dt} = rN$

   (e) $\frac{dS}{dt} = -kS$

9. The population growth model $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$ is known as: [1 Mark]

   (a) Exponential Growth

   (b) Logistic Growth

   (c) Newton's Law of Cooling

   (d) Malthusian Model

   (e) Decay Model

10. The particular solution to the differential equation $y' - \frac{y}{x} = x^2$ with the initial condition $y(1) = 2$ is: [1 Mark]

   (a) $y = \frac{x^3}{2} + 2x$

   (b) $y = \frac{x^3}{2} + x$

   (c) $y = \frac{x^3}{2} + \frac{2}{x}$

   (d) $y = \frac{x^3}{2} + 3x$

   (e) $y = \frac{x^3}{2} + \frac{3x}{2}$

Question 2 — Written Questions (54 Marks)

[2.1] Solve the following differential equation:

$$(x^2+1)\frac{dy}{dx} = y^2+1,$$

and then find the particular solution given the initial condition $y(0)=1$.

(4 Marks)

[2.2] Solve the following differential equation:

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy},$$

and then find the particular solution given the initial condition $y(1)=2$.

(4 Marks)

[2.3] Solve the differential equation:

$$\frac{dy}{dx} = \frac{2x+y}{x+2y}.$$

(4 Marks)

[2.4] Test for exactness and solve the following differential equation:

$$(2xy+y^2)dx+(x^2+2xy)dy = 0,$$

and if the equation is exact, then solve it.

(4 Marks)

[2.5] Solve the following differential equation:

$$\frac{dy}{dx}+3y = 6x,$$

and then find the particular solution given the initial condition $y(0)=2$.

(4 Marks)

[2.6] Solve the following Bernoulli differential equation:

$$\frac{dy}{dx}+y = xy^2$$

(4 Marks)

[**2.7**] Consider a population $P(t)$ of a species in a habitat with carrying capacity $K = 1000$ individuals. The population grows according to the logistic growth model with a growth rate $r = 0.05$ per year:

$$\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{1000}\right)$$

1. Derive the general solution of this differential equation. [2 Marks]
2. If the initial population is $P(0) = 50$, find the particular solution for $P(t)$. [2 Marks]
3. How long will it take for the population to reach 500 individuals? [2 Marks]

(6 Marks)

[**2.8**] A cup of coffee with initial temperature 90°C is placed in a room with ambient temperature 20°C. According to Newton's Law of Cooling, the temperature $T(t)$ of the coffee changes at a rate proportional to the difference between the temperature of the coffee and the ambient temperature:

$$\frac{dT}{dt} = -0.1(T - 20)$$

1. Derive the general solution of this differential equation. [2 Marks]
2. Find the particular solution for $T(t)$ given the initial temperature. [2 Marks]
3. What will be the temperature of the coffee after 30 minutes? [2 Marks]

(6 Marks)

[**2.9**] A chemical substance decomposes at a rate proportional to the square of the amount $x(t)$ of the substance remaining. Initially, there are 10 grams of the substance, and after 2 hours, 5 grams remain. The differential equation governing the decomposition is:

$$\frac{dx}{dt} = -kx^2$$

1. Solve this differential equation for the general solution. [2 Marks]
2. Find the particular solution for $x(t)$ given the initial condition. [2 Marks]
3. How long will it take for 90% of the substance to decompose? [2 Marks]

(6 Marks)

[2.10] A tank initially contains 100 liters of pure water. A salt solution with concentration of 2 grams of salt per liter is added to the tank at a rate of 3 liters per minute. The well-mixed solution is drained from the tank at the same rate. Let $Q(t)$ represent the amount of salt in the tank at time t. The differential equation governing this process is:

$$\frac{dQ}{dt} = 6 - \frac{3Q}{100}$$

1. Solve this differential equation for the general solution. [2 Marks]

2. If the initial amount of salt in the tank is $Q(0) = 0$, find the particular solution for $Q(t)$. [2 Marks]

3. Determine the amount of salt in the tank after 20 minutes. [2 Marks]

(6 Marks)

[2.11] A certain radioactive substance decays at a rate proportional to the amount present. Initially, there are 50 grams of the substance. After 4 hours, 25 grams remain. The differential equation describing the amount $A(t)$ of the substance at time t is:

$$\frac{dA}{dt} = -kA$$

1. Solve this differential equation for the general solution. [2 Marks]

2. Find the particular solution for $A(t)$ given the initial condition. [2 Marks]

3. Determine the half-life of the substance. [2 Marks]

(6 Marks)

Good luck and do your best!