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1- $y'' - 4y' + 8y = (2x^2 - 3x)e^{2x}\cos(2x) + (10x^2 - x - 1)e^{2x}\sin(2x)$

I) Find y_c by solve: $y'' - 4y' + 8y = 0$

Aux: $m^2 - 4m + 8 = 0$

$a = 1 \quad b = -4 \quad c = 8$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m_{1,2} = \frac{4 \pm \sqrt{16 - 32}}{2} = 2 \pm 2i$$

$\therefore y_c = e^{2x}\left[C_1 \cos(2x) + C_2 \sin(2x)\right]$

where C_1, C_2 are arbitrary constants $\in R$

2] Find y_p

$$y_p = e^{2x}\left[A\cos(2x) + B\sin(2x)\right] + x e^{2x}\left[E\cos(2x) + F\sin(2x)\right] + x^2 e^{2x}\left[G\cos(2x) + H\sin(2x)\right]$$

3] Check repetition :-

Rs of y_p , $m = 2 \pm 2i$ 3 times

$\therefore m = 2 \pm 2i$ repeted 4 times

$\therefore$ We have to multiply $y_p * x$

$$\therefore y_p = x e^{2x}\left[A\cos(2x) + B\sin(2x)\right] + x^2 e^{2x}\left[E\cos(2x) + F\sin(2x)\right] + x^3 e^{2x}\left[G\cos(2x) + H\sin(2x)\right]$$

$$y_p = e^{2x}\left[(Ax + Ex^2 + Gx^3)\cos(2x) + (Bx + Fx^2 + Hx^3)\sin(2x)\right]$$

$$y_p' = e^{2x}\Big[(Ax + Ex^2 + Gx^3)(-\sin(2x)) + (A + 2Ex + 3Gx^2)\cos(2x)$$
$$+ (Bx + Fx^2 + Hx^3)(\cos(2x)) + (B + 2Fx + 3Hx^2)\sin(2x)\Big]$$
$$+ (2e^{2x})\Big[(Ax + Ex^2 + Gx^3)\cos(2x) + (Bx + Fx^2 + Hx^3)\sin(2x)\Big]$$

$$= e^{2x}\big[A\cos(2x) + B\sin(2x)\big]$$
$$+ x e^{2x}\big[(2E + B + 2A)\cos(2x) + (-A + 2B + 2F)\sin(2x)\big]$$
$$+ x^2 e^{2x}\big[(3G + F + 2E)\cos(2x) + (-E + 3H + 2F)\sin(2x)\big]$$
$$+ x^3 e^{2x}\big[(H + 2G)\cos(2x) + (-G + 2H)\sin(2x)\big]$$

$$\therefore y_p' = e^{2x}\Big[\big[A + x(2E+B+2A) + x^2(3G+F+2E) + x^3(H+2G)\big]\cos(2x)$$
$$+ \big[B + x(-A+2B+2F) + x^2(-E+3H+2F) + x^3(-G+2H)\big]\sin(2x)\Big]$$

$$y_p'' = e^{2x}\Big[[A + x(2E + B + 2A) + x^2(3G + F + 2E) + x^3(H + 2G)]\,(-\sin(2x))$$

$$+ \cos(2x)\,[(2E + B + 2A) + x(6G + 2F + 4E) + x^2(3H + 6G)$$

$$+ [B + x(-A + 2B + 2E) + x^2(-E + 3H + 2F) + x^3(-G + 2H)]\,(\cos(2x))$$

$$+ \sin(2x)\,[(-A + 2B + 2F) + x(-2E + 6H + 4F) + x^2(-3G + 6H)]$$

$$+ (2e^{2x})\Big[(A + x(2E + B + 2A) + x^2(3G + F + 2E) + x^3(H + 2G))\cos(2x)$$

$$+ (B + x(-A + 2B + 2F) + x^2(-E + 3H + 2F) + x^3(-G + 2H))\sin(2x)\Big]$$

$$= e^{2x}\big[(2E + 2B + 4A)\cos(2x) + (-2A + 2\overset{F}{\cancel{B}} + 4B)\sin(2x)\big]$$

$$+ xe^{2x}\big[(6G + 8E + 4F + 4B + 3A)\cos(2x) + (-4E + 8F + 3B + 6H - 4A)\sin(2x)\big]$$

$$+ x^2e^{2x}\big[(6H + 12G + 4F + 3E)\cos(2x) + (-6G + 12H + 3F - 4E)\sin(2x)\big]$$

$$+ x^3e^{2x}\big[(4H + 3G)\cos(2x) + (-4G + 3H)\sin(2x)\big]$$

$$\therefore y_p'' = e^{2x}\Big[\big[(2E + 2B + 4A) + x(6G + 8E + 4F + 4B + 3A) + x^2(6H + 12G + 4F + 3E)$$

$$+ x^3(4H + 3G)\big]\cos(2x)$$

$$+ \big[(-2A + 2F + 4B) + x(-4E + 8F + 3B + 6H - 4A)$$

$$+ x^2(-6G + 12H + 3F - 4E) + x^3(-4G + 3H)\big]\sin(2x)\Big]$$

$$y_p'' - 4y_p' + 8y_p = (2x^2-3x)e^{2x}\cos(2x) + (10x^2-x-1)e^{2x}\overset{\sin}{\cancel{\cos}}(2x)$$

y_p''

$$e^{2x}\Big[\big[(2E+2B+\cancel{4A}) + x(6G+\cancel{8E}+4F+\cancel{4B}+3A) + x^2(6H+\cancel{12G}+\cancel{4F}+3E) + x^3(\cancel{4H}+3G)\big]\cos(2x)$$
$$+\big[(-2A+2F+\cancel{4B}) + x(-4E+\cancel{8F}+3B+6H-\cancel{4A}) + x^2(-6G+\cancel{12H}+3F-\cancel{4E}) + x^3(-\overset{G}{\cancel{4}}+3H)\big]\sin(2x)\Big]$$

$-4y_p'$

$$+e^{2x}\Big[\big[\cancel{-4A} + x(\cancel{-8E}-\cancel{4B}-\cancel{8A}) + x^2(\cancel{-12G}-\cancel{4F}-\cancel{8E}) + x^3(\cancel{-4H}-\cancel{8G})\big]\cos(2x)$$
$$+\big[\cancel{-4B} + x(\cancel{4A}-\cancel{8B}-\cancel{8F}) + x^2(\cancel{4E}-\cancel{12H}-\cancel{8F}) + x^3(\cancel{4G}-\cancel{8H})\big]\sin(2x)\Big]$$

$8y_p$

$$+e^{2x}\Big[\big[0 + x(\cancel{8A}) + x^2(\cancel{8E}) + x^3(\cancel{8G})\big]\cos(2x)$$
$$+\big[0 + x(8B) + x^2(8F) + x^3(8H)\big]\sin(2x)\Big]$$

$$= e^{2x}\big[(2x^2-3x)\cos(2x) + (-1-x+10x^2)\sin(2x)\big]$$

$$\therefore e^{2x}\Big[\big[(2E+2B) + x(6G+4F+3A) + x^2(6H+3E) + x^3(3G)\big]\cos(2x)$$
$$+\big[(-2A+2F) + x(-4E+3B+6H) + x^2(-6G+3F) + x^3(3H)\big]\sin(2x)\Big]$$

$$\cancel{= e^{2x}\big[[0+x(8B)+x^2}$$

$$= e^{2x}\Big[\big[0 + x(-3) + x^2(2) + x^3(0)\big]\cos(2x) + \big[-1 + x(-1) + x^2(10) + x^3(0)\big]\sin(2x)\Big]$$

بمقارنة المعاملات

$x^0 e^{2x}\cos(2x)$: $2E+2B=0 \Rightarrow B=-E \Rightarrow \boxed{B=\frac{-2}{3}}$

$x e^{2x}\cos(2x)$: $6G+4F+3A=-3 \Rightarrow 6G = -3-4(\frac{10}{3})-3(\frac{23}{6}) \Rightarrow G$

$x^2 e^{2x}\cos(2x)$: $6H+3E=2 \Rightarrow 3E=2 \Rightarrow \boxed{E=\frac{2}{3}}$

$x^3 e^{2x}\cos(2x)$: $3G=0 \Rightarrow \boxed{G=0}$

$x^0 e^{2x}\sin(2x)$: $-2A+2F=-1 \Rightarrow -2A=-1-\frac{10}{3}(2) \Rightarrow \boxed{A=\frac{23}{6}}$

$x e^{2x}\sin(2x)$: $-4E+3B+6H=-1$

$x^2 e^{2x}\sin(2x)$: $-6G+3F=10 \Rightarrow 3F=10 \Rightarrow \boxed{F=\frac{10}{3}}$

$x^3 e^{2x}\sin(2x)$: $3H=0 \Rightarrow \boxed{H=0}$

$$\therefore y_p = e^{2x}\left[\left(\frac{23}{6}x+\frac{2}{3}x^2\right)\cos(2x)+\left(\frac{-2}{3}x+\frac{10}{3}x^2\right)\sin(2x)\right]$$

Find the general solution

$\therefore y = y_c + y_p$

$\cancel{\therefore y = e^{2x}[c_1 + c_2}$

$$\therefore y = e^{2x}\left[c_1\cos(2x)+c_2\sin(2x)\right] + \cancel{e^{2x}}$$
$$+ e^{2x}\left[\left(\frac{23}{6}x+\frac{2}{3}x^2\right)\cos(2x)+\left(\frac{-2}{3}x+\frac{10}{3}x^2\right)\sin(2x)\right]$$

$$\therefore y = e^{2x}\left[\left(c_1+\frac{23}{6}x+\frac{2}{3}x^2\right)\cos(2x)+\left(c_2-\frac{2}{3}x+\frac{10}{3}x^2\right)\sin(2x)\right]$$

where c_1, c_2 are arbitrary constants $\in \mathbb{R}$

2. $m = 4kg$, $k = 100 N/m$, $\beta \frac{dx}{dt} = 10 \frac{dx}{dt}$

$x(0) = -0{,}1$

because it is in the opposite side.

$\dot{x}(0) = 2 m/s$

a) $\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$

$\therefore \ddot{x} + \frac{5}{2}\dot{x} + 25x = 0$

b) Aux : $m^2 + \frac{5}{2}m + 25 = 0$

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\frac{5}{2} \pm \sqrt{\frac{25}{4} - (4)(25)}}{2}$$

$$m_{1,2} = -\frac{5}{4} \pm \frac{5\sqrt{15}}{4} i$$ Undamed

$$\therefore x = e^{-\frac{5}{4}t}\left[A\cos\left(\frac{5\sqrt{15}}{4}t\right) + B\sin\left(\frac{5\sqrt{15}}{4}t\right)\right]$$

where A, B are arbitrary constants $\in R$

c) $x(0) = -0.1$ $\quad \dot{x}(0) = 2$

Let $s = \frac{5\sqrt{15}}{4}$

$$x(0) = -0.1 = \cancel{e^{-\frac{5}{4}(0)}} \left[A \cancel{\cos}(s(0)) + \cancel{B \sin(s(0))}\right]$$

$\therefore A = -0.1$

$$\dot{x}(t) = \left(-\frac{5}{4} e^{-\frac{5}{4}t}\right)\left[A\cos(st) + B\sin(st)\right] + e^{-\frac{5}{4}t}\left[(As)(-\sin(st)) + (Bs)\cos(st)\right]$$

$$\therefore \dot{x}(t) = e^{-\frac{5}{4}t}\left[(Bs - \tfrac{5}{4}A)\cos(st) + (-As - \tfrac{5}{4}B)\sin(st)\right]$$

$$\because \dot{x}(0) = 2 = \cancel{e^{-\frac{5}{4}(0)}}\left[(Bs - \tfrac{5}{4}A)\cancel{\cos}(s(0)) + \cancel{(-As - \tfrac{5}{4}B)\sin(0)}\right]$$

$\therefore Bs - \frac{5}{4}A = 2$ $\quad \cancel{\Rightarrow B = \frac{2 + \frac{5}{4}A}{s}}$

$s = \frac{5\sqrt{15}}{4}$, $A = -0.1$

$$\cancel{\therefore B = \frac{2\times 4}{5\sqrt{15}} + \frac{5}{4}\times\frac{4}{5\sqrt{15}} = \frac{13}{5\sqrt{15}}}$$

$$B = \left(2 + \frac{5}{4}A\right)/s = \left(2 + \frac{5(0.1)}{4}\right) \Big/ \frac{5\sqrt{15}}{4}$$

$$= \frac{8 + 5(0.1)}{4} \times \frac{4}{5\sqrt{15}} = \frac{8.5}{5\sqrt{15}} = \frac{8}{\frac{50\sqrt{15}}{25}}$$

$$\boxed{\therefore B = \frac{4}{25\sqrt{15}}}$$

$\therefore x = e^{-\frac{5}{4}t}\left[\frac{-1}{10}\cos\left(\frac{5\sqrt{15}}{4}t\right) + \frac{4}{25\sqrt{15}}\sin\left(\frac{5\sqrt{15}}{4}t\right)\right]$

d) $x(t) = 0 \quad t = ?$ for the 1st time

$0 = e^{-\frac{5}{4}t}\left[\frac{-1}{10}\cos\left(\frac{5\sqrt{15}}{4}t\right) + \frac{4}{25\sqrt{15}}\sin\left(\frac{5\sqrt{15}}{4}t\right)\right]$

$\frac{1}{10}\cos\left(\frac{5\sqrt{15}}{4}t\right) = \frac{4}{25\sqrt{15}}\sin\left(\frac{5\sqrt{15}}{4}t\right)$

mutiply it by $\frac{25\sqrt{15}}{4\cos\left(\frac{5\sqrt{15}}{4}t\right)}$

$\therefore \frac{1}{10} \times \frac{25\sqrt{15}}{4} = \frac{\sin\left(\frac{5\sqrt{15}}{4}t\right)}{\cos\left(\frac{5\sqrt{15}}{4}t\right)}$

$\rightarrow \tan\left(\frac{5\sqrt{15}}{4}t\right) = \frac{5\sqrt{15}}{4} \times \frac{1}{2}$

$\therefore t = \frac{4}{5\sqrt{15}}\tan^{-1}\left(\frac{5\sqrt{15}}{8}\right)$

$t \cong 0{,}2435$ second

f) $\dot{x}(t) = 0$ $t = ?$ for the 1st time

From number (c) but

$B_s - \frac{5}{4}A = \frac{\cancel{4}}{\cancel{25}\sqrt{15}_{5}} \cdot \frac{\cancel{5}\sqrt{15}}{\cancel{4}} + \frac{5}{4} \cdot \frac{-1}{10} = \frac{8+5}{40} = \frac{13}{40}$

$-A_s - \frac{5}{4}B = \frac{1}{\cancel{10}_{2}} * \frac{\cancel{5}\sqrt{15}}{4} - \frac{\cancel{5}}{\cancel{4}} * \frac{\cancel{4}}{\cancel{25}\sqrt{15}_{5}} = \frac{\sqrt{15}}{8} - \frac{1}{5\sqrt{15}} = \frac{5(15)-8}{40\sqrt{15}}$

$= \frac{67}{40\sqrt{15}}$

$\therefore \dot{x}(t) = e^{-\frac{5}{4}t}\left[\frac{13}{\cancel{40}} \cos\left(\frac{5\sqrt{15}}{4}t\right) + \frac{67}{\cancel{40}\sqrt{15}} \sin\left(\frac{5\sqrt{15}}{4}t\right)\right]$

$= 0$

$\therefore -13\cos\left(\frac{5\sqrt{15}}{4}t\right) = \frac{67}{\sqrt{15}} \sin\left(\frac{5\sqrt{15}}{4}t\right)$

multiply it by $\frac{\sqrt{15}}{67\cos\left(\frac{5\sqrt{15}}{4}t\right)}$

$\therefore \frac{-13\sqrt{15}}{67} = \tan\left(\frac{5\sqrt{15}}{4}t\right)$

$\therefore t = \frac{4}{5\sqrt{15}} \tan^{-1}\left(\frac{-13\sqrt{15}}{67}\right) \approx 7,6269$ seconds

e) I have ~~find~~ found the time in the next section at first such $t \approx 7{,}6269$

Now I have to find x ~~the~~ in that time.

$$x(7{,}6269) = e^{-\frac{5}{4}(7{,}6269)}\left[-\frac{1}{10}\cos\left(\frac{5\sqrt{15}}{4}(7{,}6269)\right) + \frac{4}{25\sqrt{15}}\sin\left(\frac{5\sqrt{15}}{4}(7{,}6269)\right)\right]$$

$\approx$ meters under the equilibrium.

g) ~~$\dot{x}(2)$~~ $\dot{x}(2) = ?$

$$\dot{x}(2) = e^{-\frac{5}{4_2}(2)}\left[\frac{13}{40}\cos\left(\frac{5\sqrt{15}}{4_2}(2)\right) + \frac{67}{40\sqrt{15}}\sin\left(\frac{5\sqrt{15}}{4_2}(2)\right)\right]$$

$$= e^{-\frac{5}{2}}\left[\frac{13}{40}\cos\left(\frac{5\sqrt{15}}{2}\right) + \frac{67}{40\sqrt{15}}\sin\left(\frac{5\sqrt{15}}{2}\right)\right]$$

$\dot{x}(2)$ ~~is~~ m/s

h) $\ddot{x}(2) = ?$

$$\ddot{x}(t) = \left(\frac{-5}{4} e^{-\frac{5}{4}t}\right)\left[\frac{13}{40}\cos\left(\frac{5\sqrt{15}}{4}t\right) + \frac{67}{40\sqrt{15}}\sin\left(\frac{5\sqrt{15}}{4}t\right)\right]$$

$$+ e^{-\frac{5}{4}t}\left[\frac{13}{\cancel{40}_{8}}\cdot\frac{\cancel{5}\sqrt{15}}{4}\left(-\sin\left(\frac{5\sqrt{15}}{4}t\right)\right) + \frac{67}{\cancel{40}\sqrt{15}_{8}}\cdot\frac{\cancel{5}\sqrt{15}}{4}\cos\left(\frac{5\sqrt{15}}{4}t\right)\right]$$

$$\therefore \ddot{x}(t) = e^{-\frac{5}{4}t}\left[\left(-\frac{\cancel{5}}{4} * \frac{13}{\cancel{40}8} + \frac{67}{8*4}\right)\cos\left(\frac{5\sqrt{15}}{4}t\right) + \left(-\frac{\cancel{5}}{4} * \frac{67}{\cancel{40}\sqrt{15}_{8}} - \frac{13\sqrt{15}}{8*4}\right)\sin\left(\frac{5\sqrt{15}}{4}t\right)\right]$$

$$\therefore \ddot{x}(t) = e^{-\frac{5}{4}t}\left[\frac{27}{16}\cos\left(\frac{5\sqrt{15}}{4}t\right) + \left(\frac{-131}{16\sqrt{15}}\right)\sin\left(\frac{5\sqrt{15}}{4}t\right)\right]$$

$$\ddot{x}(2) = e^{-\frac{5}{2}}\left[\frac{27}{16}\cos\left(\frac{5\sqrt{15}}{2}\right) + \left(\frac{-131}{16\sqrt{15}}\right)\sin\left(\frac{5\sqrt{15}}{2}\right)\right]$$

$\simeq$ m/s^2