Discrete Structures CS2023

Tutorial II

Question 1

Determine the truth value of each of these statements, where the domain is the set of integer numbers.

1. $\exists n\ (n = -n)$
2. $\exists n\ \exists m\ (\ n + m = 4 \quad \wedge \quad n - m = 1)$
3. $\forall n\ \exists m\ (n^2 = m)$
4. $\forall n\ \forall \mathrm{m}\ \exists z\ \left(z = \frac{n+m}{2}\right)$

Question 1

Solution

1. $\exists n \ (n = -n)$

Consider the equation:

$$n = -n$$

Adding n to both sides yields:

$$2n = 0$$

Dividing both sides by 2 gives:

$$n = 0$$

Since the domain is the set of integers and zero belongs to it, we conclude that $\exists n \ (n = -n)$ is a true proposition.

Question 1

Solution

2. $\exists n\, \exists m\, (\, n + m = 4 \quad \land \quad n - m = 1)$

Consider the equations:

$$n + m = 4 \;\; (1)$$
$$n - m = 1 \;\; (2)$$

Adding the two equations gives:

$$2n = 5$$

Dividing both sides by 2 gives:

$$n = \frac{5}{2}$$

Question 1

Solution

Substituting $n = \frac{5}{2}$ in equation (1) and solving it for m gives:

$$m = 4 - \frac{5}{2} = \frac{8-5}{2} = \frac{3}{2}$$

Since the domain of m and n is the set of integers and the obtained values of n are not integers, $\exists n\, \exists m\, (\, n + m = 4 \wedge n - m = 1)$ is a false proposition.

Question 1

Solution

3. $\forall n\ \exists m\ (n^2 = m)$

The proposition states that the square of any integer number is an integer number as well. Since the set of integers is closed under multiplication, the proposition $\forall n\ \exists m\ (n^2 = m)$ is true.

Question 1

Solution

4. $\forall n \ \forall \mathrm{m} \ \exists z \ (z = \frac{n+m}{2})$

The proposition states that the average of any two integer numbers is an integer number as well. The statement is true when the summation $n + m$ is an even number so that $\frac{n+m}{2}$ is an integer number. But when the summation $n + m$ is an odd number the number $\frac{n+m}{2}$ is not an integer.

Therefore, $\forall n \ \forall \mathrm{m} \ \exists z \ \left(z = \frac{n+m}{2}\right)$ is a false proposition.

Question 2

a. Show that $\forall x\, P(x) \vee \forall x\, Q(x)$ and $\forall x\, \big(P(x) \vee Q(x)\big)$ are not logically equivalent

b. Show that $\exists x\, P(x) \wedge \exists x\, Q(x)$ and $\exists x\, \big(P(x) \wedge Q(x)\big)$ are not logically equivalent.

c. Express the quantification $\exists!\, x\, P(x)$ using universal quantifications, existential quantifications, and logical operators.

Question 2

Solution

a. Show that $\forall x\, P(x) \vee \forall x\, Q(x)$ and $\forall x\, \big(P(x) \vee Q(x)\big)$ are not logically equivalent.

$\forall x\, \big(P(x) \vee Q(x)\big)$ means that every x in the domain either satisfies $P(x)$ or $\mathrm{Q}(x)$, while $\forall x\, P(x) \vee \forall x\, Q(x)$ means that every x in the domain satisfies P, or every other element in the domain satisfies Q as well. Lets prove that they are not equivalent using a counter example.

let $P(x)$ be the statement that x is odd, and let $\mathrm{Q}(x)$ be the statement that x is even and the domain be the set of positive integers.

$\forall x\, \big(P(x) \vee Q(x)\big)$ states that each positive integer is either odd or even, and its true.

$\forall x\, P(x)$ states that all positive integers are odd and it's false, and $\forall x\, Q(x)$ states that all positive integers are even and it's also false. Therefore, the disjunction $\forall x\, P(x) \vee \forall x\, Q(x)$is false.

Since $\forall x\, \big(P(x) \vee Q(x)\big)$ is true, and $\forall x\, P(x) \vee \forall x\, Q(x)$ is false, they are not logically equivalent.

Question 2

Solution

b. Show that $\exists x\, P(x) \land \exists x\, Q(x)$ and $\exists x\, \big(P(x) \land Q(x)\big)$ are not logically equivalent.

To show these are not logically equivalent, let $P(x)$ be the statement "x is positive," and let $\mathrm{Q}(x)$ be the statement "x is negative" with domain the set of integers.

Then $\exists x\, P(x) \land \exists x\, Q(x)$ is true, but $\exists x\, \big(P(x) \land Q(x)\big)$ is false. therefore, they are not logically equivalent.

Question 2

Solution

c. Express the quantification $\exists!\, x\ P(x)$ using universal quantifications, existential quantifications, and logical operators.

Remember that $\exists!\, x\ P(x)$ means that there exist a unique x that satisfies $P(x)$, and no other element in the domain besides x satisfies the predicate. This is equivalent to:

$$\exists x \left(P(x) \land \forall y(P(y) \rightarrow x = y)\right)$$

Question 3

a. What are the truth values of these statements?

1. $\exists!\, x\ P(x) \rightarrow \exists x P(x)$
2. $\forall x\ P(x) \rightarrow \exists!\, x P(x)$
3. $\exists!\, x \neg P(x) \rightarrow \neg \forall x P(x)$

Question 3

Solution

1. $\exists!\, x\, P(x) \rightarrow \exists x P(x)$

The proposition is true since if there is a unique x that satisfies $P(x)$, then $\exists x P(x)$

2. $\forall x\, P(x) \rightarrow \exists!\, x P(x)$

The proposition is false since $\forall x\, P(x)$ means that all members of the domain satisfy $P(x)$, while $\exists!\, x P(x)$ means that only one unique member in the domain satisfies $P(x)$.

3. $\exists!\, x \neg P(x) \rightarrow \neg \forall x P(x)$

$\exists!\, x \neg P(x)$ means that there is only one unique member in the domain that does not satisfy $P(x)$, thus not all the members of the domain satisfy $P(x)$, which is $\neg \forall x P(x)$.

Therefore $\exists!\, x \neg P(x) \rightarrow \neg \forall x P(x)$ is true.

Question 4

Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators.

1. The average of two positive integers is positive.

2. The square of the sum of two integers is less than or equal to the sum of the squares of the two integers.

3. The quadratic equation $ax^2 + bx + c = 0$ has either two distinct real roots, one repeated real root, or no real roots at all depending on value of $b^2 - 4ac$. Express the case on which the quadratic equation has two distinct roots.

Question 4

Solution

1. The average of two positive integers is positive.

$$\forall m\, \forall n\, \big((m > 0 \land n > 0)\big) \rightarrow \left(\frac{m+n}{2} > 0\right), m, n \in \mathbb{Z}$$

Alternate way:

$$\forall m\, \forall n\, \left(\frac{m+n}{2} > 0\right), m, n \in \mathbb{Z}^{+}$$

2. The square of the sum of two integers is less than or equal to the sum of the squares of the two integers.

$$\forall m \forall n\big((m+n)^2 \leq m^2 + n^2\big), m, n \in \mathbb{Z}$$

3. The quadratic equation $ax^2 + bx + c = 0$ has either two distinct real roots, one repeated real root, or no real roots at all depending on value of $b^2 - 4ac$. Express the case on which the quadratic equation has two distinct roots.

$\forall a\, \forall b\, \forall c\, ((b^2 - 4ac) > 0 \rightarrow \exists x\, \exists y\, ((ax^2 + bx + c = 0) \land (ay^2 + by + c = 0) \land x \neq y)\; a, b, c, x, y \in R$

Assignment II

1. Determine the truth value of each of these statements, where the domain is the set of real numbers.

1. $\exists x\ (x^2 + 1 = 0)$
2. $\forall x\ \forall y\ (x + y \neq y + x)$
3. $\forall x\ \forall y\ \exists z\ (z = \frac{x+y}{2}\ )$

Assignment II

2. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators.

1. The set of real numbers has the density property. There is a real number between any two real numbers.
2. Express the other two cases of the $ax^2 + bx + c = 0$ where it has one repeated root, and no real roots at all.

Assignment II

3. Let F(x, y) be the statement “x can fool y,” where the domain consists of all people in the world. Use quantifiers to express each of these statements.

1. Everybody can fool Yassir.
2. There’s somebody who can fool exactly two people.
3. There is exactly one person whom everybody can fool.

Assignment II

- The submission deadline is: **Saturday, June 8th 2024, 23:59:59 GMT+2.**

- Upload a clearly captured photocopy of your answer-sheet to: https://forms.gle/zKwBG7oeinWdncw4A

- In cases of cheating, the student will suspect themselves to **strict** cheating penalties.

END