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① $X \sim U[-\theta, \theta]$, θ = Unknown

(i)- $\hat{\theta} = ?$

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \quad , \quad m_1 = E(X)$$

$$\because X \sim U[-\theta, \theta] \Rightarrow E(X) = \frac{\theta + (-\theta)}{2} = 0$$

$$M_1 = m_1 \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i = E(X) = 0$$

$$M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \quad , \quad m_2 = E(X^2)$$

$$\because V(X) = E(X^2) - [E(X)]^2 \Rightarrow Var(X) = E(X^2)$$

$$\because X \sim U[-\theta, \theta] \Rightarrow Var(X) = \frac{(\theta - (-\theta))^2}{12} = \frac{\theta^2}{3}$$

$$M_2 = m_2 \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\hat{\theta}^2}{3}$$

$$\therefore \hat{\theta} = \sqrt{\frac{3}{n}\sum_{i=1}^{n} X_i^2} \quad , \quad -\hat{\theta} = -\sqrt{\frac{3}{n}\sum_{i=1}^{n} X_i^2}$$

(ii) - $x_1 = -0{,}808$; $x_2 = 2{,}59$; $x_3 = 2{,}314$; $x_4 = -0{,}268$

$n = 4$, $\hat{\theta} = ?$

from (i)

$$\hat{\theta} = \sqrt{\frac{3}{4}\left[(-0{,}808)^2 + (2{,}59)^2 + (2{,}314)^2 + (-0{,}268)^2\right]}$$

$$= \frac{1}{2}\sqrt{38.362152}$$

$$\therefore \hat{\theta} \approx 3{,}0969$$

② $X \sim Geo(p)$ $\hat{p} = ?$

(i) $L_x(p) = \prod_{i=1}^{n} P(x_i) = \prod_{i=1}^{n} p(1-p)^{x_i-1}$

(ii) $\ell_x(p) = \ln \prod_{i=1}^{n} P(x_i) = \sum_{i=1}^{n} \ln\left[p(1-p)^{x_i-1}\right]$

$= \sum_{i=1}^{n} \ln(p) + \sum_{i=1}^{n} (x_i-1)\ln(1-p)$

$= n\ln(p) + \ln(1-p)\sum_{i=1}^{n} x_i - n\ln(1-p)$

(iii) $\frac{\partial \ell_x}{\partial p} = \frac{n}{p} - \frac{\sum x_i}{1-p} + \frac{n}{1-p} = \frac{n}{p} - \frac{(\sum x_i - n)}{1-p}$

(iv) $\frac{\partial \ell_x}{\partial p} = 0 \Rightarrow \frac{n}{\hat{p}} = \frac{\sum x_i - n}{1-\hat{p}}$

$\Rightarrow \frac{1-\hat{p}}{\hat{p}} = \frac{\sum x_i}{n} - 1 \Rightarrow \frac{1}{\hat{p}} \cancel{-1} = \bar{X} \cancel{-1}$

$\therefore \hat{p} = \frac{1}{\bar{X}}$ ⁎

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2.2) Show that $\hat{p}$ is unbiased.

$\because X \sim Geo(p) \Rightarrow \therefore E(X) = \frac{1}{p}$

but $E(x) = \mu$ and $\hat{\mu} = \bar{X}$

$\therefore \bar{X} = \frac{1}{\hat{p}} \Rightarrow \hat{p} = \frac{1}{\bar{X}}$ #

③ $X \sim Gamma(\alpha, \beta)$, $\hat{\alpha} = ?$, $\hat{\beta} = ?$

(i) $$L_X(\alpha,\beta) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1} e^{-\frac{x_i}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}$$

(ii) $$\ell_X(\alpha,\beta) = \ln \prod_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} \ln\left[\frac{x_i^{\alpha-1} e^{-\frac{x_i}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}\right]$$

$$= \sum_{i=1}^{n}(\alpha-1)\ln(x_i) + \sum_{i=1}^{1}\left(\frac{-x_i}{\beta}\right)\ln(e) - \sum_{i=1}^{n}\alpha\ln(\beta) + \sum_{i=1}^{n}\ln(\Gamma(\alpha))$$

$$= (\alpha-1)\cancel{\ln}\sum_{i=1}^{n}\ln(x_i) - \frac{1}{\beta}\sum_{i=1}^{n}x_i - n\alpha\ln(\beta) + n\ln(\Gamma(\alpha))$$

(iii) $$\frac{\partial \ell_X(\alpha,\beta)}{\partial\beta} \cancel{=0} \Rightarrow 0 + \frac{1}{\beta^2}\sum x_i - \frac{n\alpha}{\beta} + 0$$

(vi) $$\frac{\partial \ell_X}{\partial\beta} = 0 \Rightarrow \frac{1}{\hat{\beta}^2}\sum_{i=1}^{n}x_i = \frac{n\hat{\alpha}}{\hat{\beta}} \Rightarrow \hat{\beta} = \frac{\sum x_i}{n\hat{\alpha}} =$$

$$\therefore \hat{\beta} = \frac{\overline{X}}{\hat{\alpha}}$$

(v) $$\frac{\partial \ell_X(\alpha,\beta)}{\partial\alpha} = \sum_{i=1}^{n}\ln(x_i) - n\ln(\hat{\beta}) + n\left(\frac{\frac{\partial}{\partial\alpha}\Gamma(\alpha)}{\Gamma(\alpha)}\right)$$

(iv) $$\frac{\partial \ell_X(\alpha,\beta)}{\partial\alpha} = 0 \Rightarrow \frac{\frac{\partial}{\partial\alpha}(\Gamma(\hat{\alpha}))}{\Gamma(\alpha)} = \ln(\hat{\beta}) - \frac{\sum_{i=1}^{n}\ln(x_i)}{n}$$

$$\frac{\frac{\partial}{\partial \alpha}\left(\Gamma(\alpha)\right)}{\Gamma(\alpha)} + \ln(\hat{\alpha}) = \ln(\bar{x}) - \frac{\sum \ln(x_i)}{n}$$

$\vdots$

$$\hat{\alpha} = \frac{\bar{X}^2}{\frac{1}{n}\sum x_i^2 - \bar{X}^2}$$

$$\therefore \hat{\beta} = \frac{\bar{X}}{\hat{\alpha}} = \frac{\bar{X}}{\bar{X}^2} \cdot \left(\frac{1}{n}\sum x_i^2 - \bar{X}^2\right)$$

$$\therefore \hat{\beta} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{X}^2}{\bar{X}}$$

$$\therefore \hat{\beta} = \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i} - \bar{X}$$