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Faculty of Mathematical Sciences and Informatics

Statistics and Probability (S2013)

Assignment 3

The content:

- Uniform Continuous Probability Distribution.
    - Its PDF, CDF, and MGF.
    - Its Expectation and Variance.
    - Uses and Example.
- References.

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Computer Sciences

Uniform Continuous Probability Distribution

Definition:

A random variable is said to be *uniformly* distributed over the interval (0, 1) if its probability density function is given by:

$$f(x) = \begin{cases} 1, & 0 < x < 0 \\ 0, & other\ wise\ x \geq 0 \end{cases}$$

Note that Equation is a density function, since $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = \mathbf{1}$. Because $f(x) > 0$ only when $x \in (0, 1)$, it follows that X must assume a value in interval (0, 1). Also, since $f(x)$ is constant for $x \in (0, 1)$, X is just as likely to be near any value in (0, 1) as it is to be near any other value. To verify this statement, note that for any $0 < a < b < 1$,

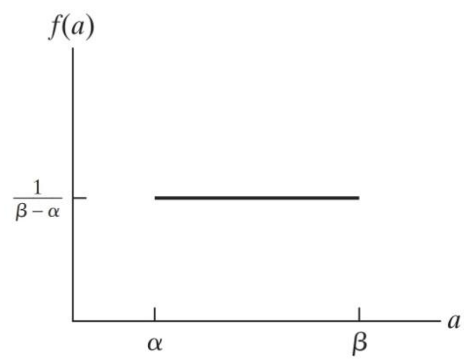
$$p(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx = b - a$$

In other words, the probability that $\boldsymbol{X}$ is in any particular subinterval of $(\mathbf{0}, \mathbf{1})$ equals the length of that subinterval.

PDF of the Uniform Continuous Distribution:

In general, we say that X is a uniform continuous random variable on the interval (α, β) if the probability density function of X is given by:

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & if\ \alpha \leq x \leq \beta \\ 0, & otherwise \end{cases}$$



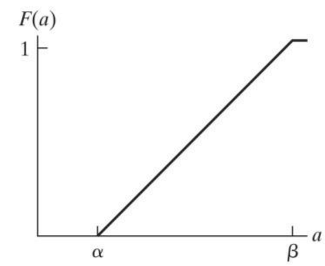
Where:

$\beta \equiv The\ maximum\ value\ that\ X\ can\ assume$

$\alpha \equiv The\ minimum\ value\ that\ X\ can\ assume$

CDF of the Uniform Continuous Distribution:

$$F(x) = \begin{cases} 0 & x < \alpha \\ \frac{x - \alpha}{\beta - \alpha} & \alpha \leq x \leq \beta \\ 1 & x > \beta \end{cases}$$



MGF of the Uniform Continuous Distribution:

$$\boldsymbol{M_x(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} f(x)\, dx}$$

$$= \int_a^b \frac{e^{tx}}{b-a} dx = \frac{1}{b-a}[\frac{1}{t} e^{tx}]_a^b = \frac{1}{b-a}\left(\frac{1}{t}\right)(e^{tb} - e^{ta}) = \frac{e^{tb} - e^{ta}}{t(b- \ a)}$$

$$\therefore \ \mathbf{M_x\,(t)} = \boldsymbol{\frac{e^{tb} - e^{ta}}{t(b- \ a)}}$$

Expectation and Variance:

Let X be a continues random variable uniformly distributed over (a, b); then:

$$\boldsymbol{E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx}$$

$$= \int_a^b \frac{x}{b-a} dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\left(\frac{1}{2}\right)(b^2 - a^2) = \frac{1}{b-a}\left(\frac{1}{2}\right)(b-a)(b+a)$$

$$\therefore \boldsymbol{E(x) = \frac{b+a}{2}}$$

$$\boldsymbol{var(X) = \ E(X^2) - [E(X)]^2}$$

$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx$

$$= \int_a^b \frac{x^2}{b-a} dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a}\left(\frac{1}{3}\right)(b^3 - a^3) = \frac{(b^2 + ab + a^2)(b-a)}{3(b-a)}$$

$$\therefore E(x^2) = \frac{b^2 + ab + a^2}{3}$$

$$\therefore var(X) = \frac{b^2 + ab + a^2}{3} - (\frac{b+a}{2})^2 = \frac{4(b^2 + ab + a^2) - 3(b^2 + 2ab + a^2)}{12}$$

$$\therefore \boldsymbol{Var(X) = \frac{(b-a)^2}{12}}$$

Expectation and Variance using MGF:

The expected value and variance **cannot** be calculated using MGF.

Because MGF derivatives any order when $t = 0$ are **not** exist.

For example:

$\boldsymbol{E(X) = M'(0)}$

$$\frac{d}{dt}M(t) = \frac{d}{dt}\left(\frac{\boldsymbol{e^{tb} - e^{ta}}}{\boldsymbol{t(b - a)}}\right) = \frac{1}{\boldsymbol{(b - a)}} * \frac{\boldsymbol{t\left(\mathbf{b}e^{tb} - \mathbf{a}e^{ta}\right) - \left[\left(e^{tb} - e^{ta}\right)(1)\right]}}{(\boldsymbol{t})^2}$$

$$\therefore \boldsymbol{E(x) = M'(0) =} \textbf{\textit{is not exist}}!$$

Uses of the Uniform Continuous Distribution:

- It is used when the probabilities of all possible outcomes of a continuous random variable are equals and equals same constant.
- In other words, it is used when the probability of any possible outcome of a continuous random variable has the same chance to be got.
- Real Examples:
    1. If a person arrived to the bus stop and he found that the bus has left, and the next bus will arrive after 20m after the first one.
       Let a random variable X represents the waiting time per minutes, then it will be uniformly distributed because he will wait a time from 0 to 20 minutes and they have the same probability to be waited.

    2. If a family went on a trip by car and they made a room reservation before their leaving. But they don't arrive and it seems to be they have a problem in somewhere in the road and a team will go to find them.
       Let Y represents the distance that the team will find them after it, then Y is uniformly distributed because they can find them after any distance between their home and their reservation.

Example:

If a random variable X is uniformly distributed over (0, 10), calculate that:

$a)\ f(x),\quad b)\ E(x),\quad c)\ Var(x),\quad d)\ F(x),\quad$ e) $P(X < 3),\quad$ g) $P(X > 6)$,

Solution:

Because of X is uniformly distributed over $(a = 0, b = 10)$ then:

$$a)\quad f(x) = \begin{cases} \frac{1}{b-a}, & if\ a \leq x \leq b \\ 0, & otherwise \end{cases} = \begin{cases} \frac{\mathbf{1}}{\mathbf{10}}, & \boldsymbol{if}\ \mathbf{0} \leq x \leq 10 \\ \mathbf{0}, & \boldsymbol{otherwise} \end{cases}$$

$$b)\quad E(X) = \frac{b+a}{2} = \frac{10+0}{2} = \mathbf{5}$$

$$c)\quad Var(X) = \frac{(b-a)^2}{12} = \frac{(10-0)^2}{12} = \frac{100}{12} \approx \mathbf{8.3333}$$

$$d)\quad F(x) = \begin{cases} 0 & x < \alpha \\ \frac{x-a}{b-a} & \alpha \leq x \leq b \\ 1 & x > \alpha \end{cases} = \begin{cases} \mathbf{0} & \boldsymbol{x} < 0 \\ \frac{\boldsymbol{x}}{\mathbf{10}} & \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{10} \\ \mathbf{1} & \boldsymbol{x} > 10 \end{cases}$$

$$e)\quad P(X < 3) = \int_{-\infty}^{3} f(x)\,dx = \int_{0}^{3} \frac{1}{10}\,dx = \left[\frac{x}{10}\right]_0^3 = \frac{\mathbf{3}}{\mathbf{10}} = \mathbf{0.3}$$

$$g)\quad P(X > 6) = \int_{6}^{\infty} f(x)\,dx = \int_{6}^{10} \frac{1}{10}\,dx = \left[\frac{x}{10}\right]_6^{10} = \frac{10}{10} - \frac{6}{10} = \frac{4}{10} = \mathbf{0.4}$$

References:

- Sheldon Ross. A First Course in Probability 9th Edition.
- Robert V. Hogg, Joseph W. McKean, Allen T. Craig. Introduction to Mathematical Statistics 8th Edition.
- Daniel Zwillinger. CRC Standard Probability and Statistics Tables and Formulae.

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The End

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