Assignment 8

Question 1

Given the estimators of the mean $T_1 = (X_1 + 2X_2 + X_3)/4$ and $T_2 = (X_1 + X_2 + X_3)/3$, where X_1, X_2, X_3 is a random sample from a $N(\mu, \sigma^2)$ distribution, prove that T_2 is more efficient than T_1.

Question 2

Let X be a $Bin(n, p)$ random variable.

(a) Find the mean squared error of the p parameter estimators $T_1 = X/n$ and $T_2 = (X + 2)/(n + 4)$.

(b) When $n = 20$ and $p = 0.4$, which estimator, T_1 or T_2, has the smaller MSE?

(c) Plot the efficiency of T_2 relative to T_1 versus p values in $(0, 1)$ for n values from 1 to 10.

Question 3

Consider a random sample of size n from a geometric distribution.

(a) Find the method of moments estimator of p.

(b) Find the maximum likelihood estimator of p.

(c) Use the results from (a) and (b) to compute the method of moments and maximum likelihood estimates from the sample $\{8, 1, 2, 0, 0, 0, 2, 1, 3, 3\}$, which represents the number of Bernoulli trials that resulted in failure before the first success in 10 experiments.

Question 4

Consider the density function

$$f(x) = (\theta + 1)(1 - x)^\theta, 0 \leq x \leq 1, \theta > 0.$$

(a) Find the maximum likelihood estimator of θ for a random sample of size n.

(b) Set the seed equal to 3, and generate 20,000 values from $f(x)$ when $\theta = 5$. Calculate the maximum likelihood estimate of θ from the generated values.

(c) How close is the maximum likelihood estimate in (b) to $\theta = 5$?

Question 5

Consider an exponential distribution with mean θ and the following estimators of θ:

$$\hat{\theta}_1 = X_1, \hat{\theta}_2 = \frac{1}{2}(X_1 + X_2), \hat{\theta}_3 = \min(X_1, X_2, X_3).$$

(a) Find the mean and variance of each estimator.

(b) Are any of the estimators efficient?

(c) Which estimator is the MLE?

(d) Let X be an exponential random variable with mean $\theta + 2$. Which estimator is an unbiased estimator of θ?

Question 6

Consider the density function

$$f(x) = 3\pi\theta x^2 e^{-\theta\pi x^3}, x \geq 0.$$

(a) Set the seed equal to 102, and generate a random sample of size $n = 20,000$ with $\theta = 5$.

(b) Find the sample mean and the sample variance of the random values generated in (a).

(c) Create a density histogram of the simulated values from (a) and superimpose the density function over the density histogram.

(d) Find the maximum likelihood estimate of θ.

(e) Plot the logarithm of the likelihood function versus θ. Use values for θ from 0 to 15.