***Waves II*:**

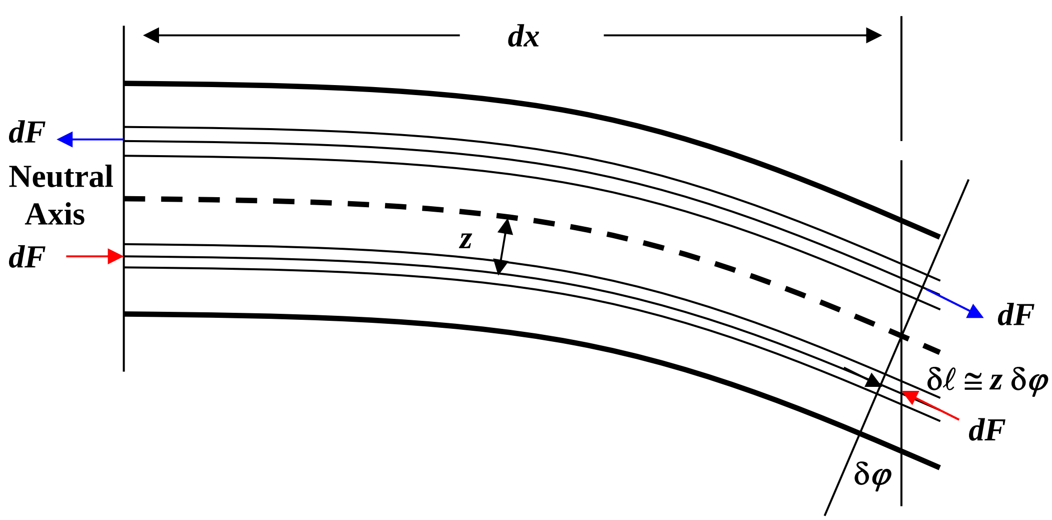
Vibrations of Real Strings

Thus far, our discussion(s) of wave propagation on strings has treated them in an idealized manner - *i.e*. that strings, while able to withstand a tension T, are perfectly elastic/flexible entities, and have no dissipative losses and/or non-linear properties associated with them. We made these simplifications in order to gain an understanding of the basic phenomena associated with wave propagation on strings. Now we wish to refine our treatment of string vibrations, in order to achieve a more physically realistic mathematical description of string vibrations.

Vibrations of Stiff Strings

Real vibrating strings on a guitar have a restoring force which is primarily due to the string tension, T. However, on heavier-gauge strings, the restoring force is also due to the stiffness of the string - it is not perfectly elastic. The wave equation we obtained in the previous lecture notes for vibrating strings did not take into account the possibility that the string could have a stiffness associated with it.

When a string of radius r_{string} bends in the process of vibrating (for small amplitude vibrations), the outer portion of the string is stretched slightly while the inner portion of the string is compressed slightly - the material of the string is in fact an elastic solid. Somewhere in between the outer and inner parts of the string exists a so-called neutral axis, whose length remains unchanged. A filament of the string located a distance z above (below) the neutral axis is stretched (compressed) by an amount $\delta\ell \cong z\,\delta\varphi$ as shown in the figure below, for a small segment (of infinitesimal length dx) of the string:



The strain, S is the fractional change in length of the string segment, *i.e*. the strain is the ratio $S = \delta\ell/\mathrm{d}x \cong z\,\delta\varphi/\mathrm{d}x$. The incremental amount of force, dF required to produce this strain is:

$$dF = Y_{string} S dA_{filament} \underset{1}{=} Yz\frac{\partial\varphi}{\partial x} dA_{filament}$$

where Y_{string} = Young's modulus (ratio of stress/strain) associated with the material of the string (stress is the force per unit area that creates the deformation of the string) and $dA_{filament}$ is the cross sectional area of the filament located a distance z from the neutral axis.

The moment of this force about the neutral axis is given by

$$dM = dF \ z = \left[Y_{string} \, z \frac{\partial \varphi}{\partial x} dA_{filament} \right] z$$

Thus, the total moment to compress/stretch all the filaments in the string is given by:

$$M = \int dM = Y_{string} \frac{\partial \varphi}{\partial x} \int z^2 dA_{filament}$$

The so-called radius of gyration, K of the cross section of the string is defined by:

$$K^2 \equiv \frac{1}{A_{string}} \int z^2 dA_{filament}$$

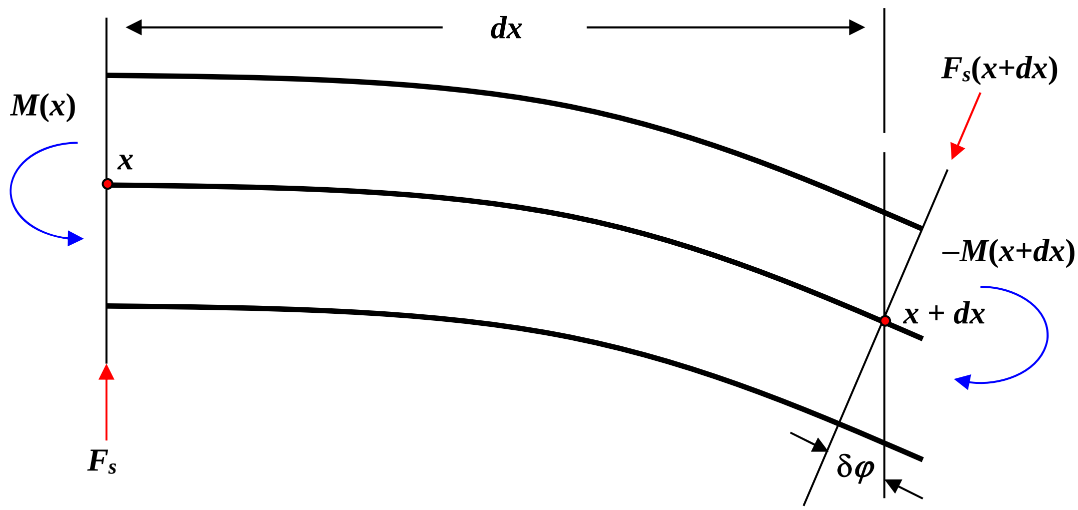
where $A_{string} = \pi r^2_{string}$ is the cross sectional area of the string. For a cylindrical string of radius r_{string}, the radius of gyration of the string is $K = r_{string}/2$.

The bending moment, $M(x,t)$ of the string segment dx at the point x and time t is thus:

$$M(x,t) = Y_{string} \frac{\partial \varphi(x,t)}{\partial x} A_{string} K^2 \cong -Y_{string} A_{string} K^2 \frac{\partial^2 y(x,t)}{\partial x^2}$$

since $\partial\varphi(x,t)/\partial x \cong -(\partial^2 y(x,t)/\partial x^2)$ for small angles $\delta\varphi$ (i.e. small amplitude vibrations of the string).

The bending moment, $M(x,t)$ is not the same for every portion of the string segment. In order to keep the string in equilibrium a shearing force, F_s must exist with bending moment $dM(x,t) = F_s(x,t)dx$ as shown for a "snapshot" of the string segment, dx in the figure below:



The shearing force, $F_s(x,t)$ is given by:

$$F_s(x,t) = \frac{\partial M(x,t)}{\partial x} = \frac{\partial}{\partial x}\left[-Y_{string} A_{string} K^2 \frac{\partial^2 y(x,t)}{\partial x^2}\right] = -Y_{string} A_{string} K^2 \frac{\partial^3 y(x,t)}{\partial x^3}$$

However, the shearing force, $F_s(x,t)$ acting on the string segment is not constant either. The net shearing force, $dF_s(x,t) = (\partial F_s(x,t)/\partial x)dx$ produces an acceleration perpendicular to the axis of the string. The equation of motion of the string segment, dx of mass, dm_{string} is given by Newton's second law, $dF_s(x,t) = dm_{string}\, a(x,t)$, were $a(x,t)$ is the acceleration of the string segment at the point x at time t:

$$\left(\frac{\partial F_s}{\partial x}\right)dx = dm_{string}\frac{\partial^2 y(x,t)}{\partial x^2} = \left(\rho_{string} A_{string}\, dx\right)\frac{\partial^2 y(x,t)}{\partial t^2}$$

$$-Y_{string} A_{string} K^2 \frac{\partial^4 y(x,t)}{\partial x^4} = \rho_{string} A_{string}\frac{\partial^2 y(x,t)}{\partial t^2}$$

$$-Y_{string} K^2 \frac{\partial^4 y(x,t)}{\partial x^4} = \rho_{string}\frac{\partial^2 y(x,t)}{\partial t^2}$$

This is a fourth-order differential equation, one which describes *bending* waves in a string (in the absence of tension in the string). The general solution of this differential equation has a complex transverse displacement, $y(x,t)$ of the form:

$$y(x,t) = \left[Ae^{kx} + Be^{-kx} + Ce^{ikx} + De^{-ikx}\right]e^{i\omega t}$$
$$= \left[A'\cosh(kx) + B'\sinh(kx) + C'\cos(kx) + D'\sin(kx)\right]\cos(\omega t + \varphi)$$

Thus, with four (apriori unknown) constants we must have four boundary conditions at the ends of the string in order to determine what these constants are. There are two boundary conditions at each end of the string. For fixed end supports, the boundary conditions at $x = 0$ and $x = L$ are $y(x,t) = 0$ and $a(x,t) = \partial^2 y(x,t)/\partial t^2 = 0$.

However, for transverse bending waves in a string, the magnitude of the longitudinal velocity of propagation of such waves, v_x^{bend} is not a simple relation, as in the case for transverse waves on an "ideal" string (where we found $v_x = \omega/k$). Here, we find that

$$\left(v_x^{bend}\right)^2 = \omega K \sqrt{Y_{string} / \rho_{string}} = \omega K c_L \qquad with \qquad c_L \equiv \sqrt{Y_{string} / \rho_{string}}$$

Thus, the longitudinal wave speed for bending waves on a string, v_x^{bend} is proportional to the square root of the frequency! For wave propagation in which the wave speed is not independent of the frequency, but instead has a frequency dependence, we say that such wave propagation has *dispersion*.

A modified wave equation for a string with tension T and which has an additional term for describing the restoring force associated with bending stiffness of the string is given by:

$$\mu \frac{\partial^2 y(x,t)}{\partial t^2} = T \frac{\partial^2 y(x,t)}{\partial x^2} - YA_{string} K^2 \frac{\partial^4 y(x,t)}{\partial x^4}$$

This differential equation can be solved, e.g. for standing waves on a string of length L with fixed ends at $x = 0$ and $x = L$. The modes of vibration are as before:

$$y_n(x,t) = y_{on} \sin(n\pi x / L) \sin(\omega_n t)$$

where $n = 1, 2, 3, 4, \ldots$ However, the frequencies of vibration of the various modes are given by:

$$f_n = n f_1^o \left[1 + \beta n^2\right]^{1/2}$$

where f_1^o is the fundamental frequency of the string without stiffness and the constant, β is given by:

$$\beta = \pi^2 YA_{string} K^2 \big/ TL^2$$

Finite stiffness in strings therefore raises the frequency of the string, slightly. Note that because of this, the harmonics of the fundamental will not be precisely integer multiples of the fundamental, for wavelengths that are integer fractions of the fundamental wavelength! Thus because the strings on a guitar have different thicknesses (gauges), there will be different amounts of stiffness associated with the strings. This is the primary reason for the need for separate adjustment of the bridge saddles on an electric guitar - in order for it to intonate properly when playing anywhere on the neck!

Effect of Motion of the End Supports on Vibrating Strings

On either an acoustic guitar or an electrical guitar, the strings contact the guitar at the nut (the zeroth fret) near the top of the neck of the guitar, and at the bridge of the guitar. On an acoustic guitar, the (wooden) bridge rests on the top plate (soundboard) of the guitar body, which is usually made of *e.g.* a thin sheet of spruce wood, supported by braces inside the guitar. The wooden bridge of an acoustic guitar may also have a thin strip of bone, ivory or synthetic material inserted into a length-wise slot in the bridge, which is where the strings make contact with the guitar. On an electric guitar, a “hard-tail” bridge assembly, usually consisting of a metal frame with individually adjustable bridge saddle is rigidly attached (via screws) to the body of the guitar, which often is a solid block of wood, such as alder, ash, mahogany, korina, poplar, *etc*. There also exist other kinds of bridge assemblies for electric guitars, *e.g*. vibrato bridges of various types, the design of which usually consists of a spring-loaded mechanical system that enables the guitar player to raise and/or lower the pitch (*i.e*. frequency) of the strings by pulling on a lever mechanism (or mechanisms). At the neck end of the guitar, the nut supporting the strings is again often made of bone, ivory or a synthetic material; sometimes even musical brass (also known as bell brass) is used for making the nut on a guitar. Some guitars, such as vintage (50’s and 60’s era) Gretsch guitars even used a “zeroth” fret as the nut of the guitar.

When a string undergoes transverse vibrations, there are time-dependent forces that act on the end supports of the string, in addition to the static tension, T of the string.
If the end supports are perfectly rigid, then these forces are in fact of no consequence. However, on a real guitar, the end supports are *not* perfectly rigid. The bridge attached to the soundboard of an acoustic guitar can vibrate (differently) in three dimensions - because the wood of the soundboard, with its accompanying bracing is not perfectly rigid. Indeed, vibration of the soundboard of an acoustic guitar is *required* in order for it to produce its sound - via the acoustical cavity inside the body of the guitar!

Because of the grain structure of the spruce wood - usually aligned parallel to the strings of the acoustic guitar, the structural rigidity of the bridge-top plate assembly of an acoustic guitar is not the same when vibrating in three possible mutually-perpendicular directions. By conscious design, the bridge of an acoustic guitar vibrates most easily perpendicular to the plane of the soundboard of the guitar (= plane of the strings of the guitar), and to a lesser extent, along the string direction, and/or perpendicular to the strings, in the plane of the strings, enabling the efficient transfer of energy from the strings of an acoustic guitar to the (resonant) body cavity of the guitar, thereby enabling the sound of the strings to be amplified in a natural manner.

For an electric solid-body guitar, because of the fact that the wooden guitar body is so much thicker and thus much more massive than the soundboard of an acoustical guitar, the vibrations associated with the bridge of an electric guitar are considerably less than that associated with an acoustic guitar.

An electric guitar of necessity must use magnetically-permeable metal strings. As a consequence of this, the string tension, T associated with an electric guitar is ~ twice that of an acoustic guitar strung *e.g*. with nylon strings. The neck of an electric guitar is also usually considerably thinner, and therefore less rigid than the neck of an acoustic guitar. So much so, that almost without exception an adjustable truss rod (internal to the neck of an electric guitar) is used to counter the tensile forces of the strings in order to keep the neck from warping over a period of time. Thus, the neck of either an electric or acoustic guitar behaves as an *axially-loaded, cantilevered beam*. Therefore, the nut attached near the top of the neck of any guitar will not be perfectly rigid as the strings of the electric guitar vibrate. The nut on an electric guitar will in fact vibrate more than that of an acoustic guitar because of the thinner neck, and also because of the increased tension, due to the use of magnetically-permeable metal strings - we will shortly see that the forces acting on the end supports are proportional to the string tension.

The Impedance of a Driven String

A guitar string that is excited purely by the vibrations of one (or both) of end supports - *i.e.* the bridge or the nut, is known as a *driven* string. Mechanical energy is input to the string vibration(s) from the vibrations of one (or both) of the end supports.

In order to discuss this phenomenon, we first consider some simplified physical situations. We will also use complex notation to discuss the mathematics associated with this phenomena - see *e.g.* the preceding lecture notes on Fourier analysis for a discussion on complex notation.

Suppose we have an *infinitely* long string, with tension T, and apply a sinusoidally time-varying, complex transverse force of the form $F(t) = |F|e^{i\omega t} = |F|[\cos(\omega t)+i\sin(\omega t)]$ to the end of the string at x = 0, which is a *free* (i.e. not fixed) end, as an *approximation* to *e.g.* a vibrating bridge or nut of a guitar, driving the string. The magnitude of this complex driving force is |F|, a *real* quantity (i.e. a simple constant). Since the string is infinitely long, we do *not* create transverse *standing* waves by driving the string from one end. Instead, transverse *travelling* waves are created by driving the free end of the string in this manner. Thus, the transverse displacement at a point x along the driven string, at time, *t* is given by:

$$y(x,t) = y_o e^{i(\omega t - kx)}$$

Mathematically, this represents a travelling wave moving to the *right* (i.e. increasing x) as time, *t* increases. Note also that the transverse displacement amplitude, y_o is *complex*, i.e. $y_o = |y_o|e^{i\delta} = |y_o|[\cos\delta + i\sin\delta]$ where $|y_o|$ is the *magnitude* of the complex amplitude, y_o and δ is the phase angle, *relative* to the (complex) transverse driving force, F(t). Note also that we have the usual relations, the longitudinal wave speed, $v_x = \omega/k$, with $\omega = 2\pi f$ and $k = 2\pi/\lambda$.

At the driven end of the string, located at *x* = 0, since this end of the string is assumed to be ideally free, there can be no *net* force acting on the string. Therefore the transverse driving force, *F* must balance the transverse component of the tension, T_y, i.e. $F(t) = -T\sin\theta(t)$, which for small amplitude vibrations, corresponding to small-argument Taylor series expansions of $\sin\theta(t) \cong \tan\theta(t) \cong \theta(t) \cong \partial y(x=0,t)/\partial x$ at x = 0, thus the driving force, $F(t) \cong -T(\partial y(x=0,t)/\partial x)$ at x = 0. The slope, $\partial y(x=0,t)/\partial x$ at x = 0 is $\partial y(x=0,t)/\partial x = \partial/\partial x\{y(x,t)\}|_{x=0} = \partial/\partial x\{y_o e^{i(\omega t-kx)}\}|_{x=0} = -ik\, y_o e^{i\omega t} = -ik\, y(x=0,t)$, and hence $F(t) = |F|e^{i\omega t} \cong +ikT\, y(x=0,t) = ikT\, y_o e^{i\omega t} = ikT\, |y_o|e^{i\delta}e^{i\omega t} = ikT\, |y_o|e^{i(\omega t+\delta)}$.

Thus, we see that $|y_o| = |F|e^{-i\delta}/ikT = -i|F|e^{-i\delta}/kT$. However, note that *magnitudes* of *complex* quantities a.) *must* be purely real (i.e. they cannot have any i's in their mathematical expressions)

$$y(x,t) = y_o e^{i(\omega t-kx)} = |y_o|e^{i\delta}e^{i(\omega t-kx)} = \frac{-i|F|}{kT}e^{i(\omega t-kx)} = -i\frac{|F|}{kT}e^{i(\omega t-kx)}$$

and b.) non-zero magnitudes of complex quantities *must* be positive. Therefore, the only way that the magnitude, $|y_o|$ can be purely real *and* positive is if $e^{-i\delta}/i = -i\,e^{-i\delta} = -i[\cos\delta - i\sin\delta] = -i\cos\delta + (i*i)\sin\delta = -i\cos\delta - \sin\delta = +1$, or $e^{-i\delta} = e^{+i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$. This requires the phase angle, $\delta = -90^o = -\pi/2$. Thus, the complex transverse displacement:

Note that at x = 0, the complex displacement $y(x=0,t) = |F|/kT\, e^{i\omega t}$ is 90^o out of phase with the complex driving force, $F(t) = |F|e^{i\omega t}$.

The (complex) transverse velocity, $u_y(x, t) = \partial y(x, t)/\partial t$ of the string at the position, x and at time, *t* is therefore:

$$u_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \frac{\omega |F|}{kT} e^{i(\omega t - kx)} = \frac{v_x |F|}{T} e^{i(\omega t - kx)}$$

Note that the complex transverse velocity, $u_y(x=0, t) = i\, v_x |F|/T$ is in phase with the driving force, $F(t) = |F|e^{i\omega t}$.

We now *define* the complex mechanical input impedance, Z^{input} of the driven string (units of Newtons/(m/sec) = kg/sec) as the *ratio* of the complex driving force, F(t) to the complex transverse velocity, $u_y(x=0, t)$ at the driving point, $x = 0$:

$$Z^{input} \equiv \frac{F(t)}{u_y(x=0,t)} = \frac{T}{v_x}$$

In this situation, the complex mechanical input impedance of an infinitely long, driven string is a *purely real* quantity. This impedance is *purely "resistive"* - not "reactive" (i.e. it has no *imaginary* part), and is also known as the *characteristic* impedance, $Z_o = T/v_x = (\mu T)^{1/2} = \mu v_x$ of the infinitely long driven string.

The (complex) power, or time rate of energy transfer from the "free" end support at x = 0 to the string (units of Watts, or Joules/second) is defined by:

$$P(t) = \frac{dE(t)}{dt} \equiv F(t) u_y^*(x=0,t) = \frac{\omega |F|^2}{kT} = \frac{v_x |F|^2}{T} = \frac{|F|^2}{Z_o}$$

Where $u_y^*(x,y)$ is the complex conjugate of $u_y(x,t)$. In general, if $z = x + iy = |z|e^{i\varphi} = |z|[\cos\varphi + i\sin\varphi]$, then $z^* \equiv x - iy = |z|e^{-i\varphi} = |z|[\cos\varphi - i\sin\varphi]$. Here again, in this situation, the power is a *purely real* quantity. Note also that it has *no* time dependence. Thus, here in this situation the instantaneous power is *constant*. As a consequence of this the instantaneous power is *also* equal to the time-averaged power, here.

In general, the physical meaning of the *real* part of the complex power, P(t) is that Re(P(t)) is the rate of energy transferred from the power source to the string. We shall see below in the next example that the physical meaning of the *imaginary* part of the complex power, P(t) is that Im(P(t) is the rate of energy returned from the string back to the power source.

Next, we consider the (somewhat more physically realistic) case where the driven string has *finite* length (say, of length L). The string is again driven at x = 0 by the force $F(t) = |F|e^{i\omega t}$. We assume that the string is fixed at x = L (*i.e.* the end support at *x* = *L* is infinitely rigid = infinitely massive). For an acoustic guitar (and also for a hollow-body electric guitar), we imagine the bridge to be the driven, "free" end at x = 0, and the nut to be the fixed end at x = L, since the vibrations of the nut on an acoustic guitar are so much less than at the bridge. For a solid-body electric guitar, we imagine the nut to be the driven, "free" end at x = 0, and the bridge to be the fixed end at x = L, since the vibrations of the bridge on an electric solid-body guitar are so much less than at the nut. Then the right-moving travelling waves created by the driven, "free" end support at x = 0 are reflected and polarity-flipped at x = L and converted into left-moving travelling waves. Since the initial right-moving traveling wave is a continuous wave-train of sinusoidal, time-varying travelling waves, then when the reflected, left-moving travelling waves overlap with the right-moving traveling wave, a standing wave is generated on the driven string.

The most general form of the complex transverse displacement, y(x, t) of the standing wave is given by:

$$y(x,t) = y_{oR} e^{i(\omega t - kx)} + y_{oL} e^{i(\omega t + kx)}$$

where $y_{oR} = |y_{oR}|e^{i\delta}$ is the complex amplitude of the right-moving travelling wave and $y_{oL} = |y_{oL}|e^{i\delta}$ is the complex amplitude of the left-moving travelling wave.

At the driven end of the string, at x = 0, the complex driving force, F(t) must again balance against the transverse component of the tension, $T_y(x=0, t)$ at x = 0, which again for small amplitudes, is $T_y = T \sin\theta \cong T\,(\partial y(x=0, t)/\partial x)$, thus we must have

$$F(t) = |F|\, e^{i\omega t} = T \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0} = T\left[-iky_{oR} e^{i\omega t} + iky_{oL} e^{i\omega t}\right] = -ikT\left[y_{oR} - y_{oL}\right] e^{i\omega t}$$

Thus, we see that:

$$|F| = -ikT \;\; y_{oL}\left[e^{+2ikL} + 1\right]$$

At the fixed end, x = L, the boundary condition on the transverse displacement is y(x=L,t) = 0. Thus,

$$y(x = L, t) = y_{oR} e^{i(\omega t - kL)} + y_{oL} e^{i(\omega t + kL)} = \left[y_{oR} e^{-ikL} + y_{oL} e^{ikL}\right] e^{i\omega t} = 0$$

This relation *must* hold for any/all times, t. Thus, we must have:

$$\left[y_{oR} e^{-ikL} + y_{oL} e^{+ikL}\right] = 0$$

i.e. this implies that $y_{oR}\, e^{-ikL} = -y_{oL}\, e^{+ikL}$, or that $y_{oR} = -y_{oL}\, e^{+2ikL}$. Plugging this result into the above driving force - tension balancing result, we see that

$$|F| = -ikT\left[y_{oR} - y_{oL}\right]$$

or:

$$y_{oR} = -y_{oL} e^{+2ikL} = -i \frac{|F|\, e^{+ikL}}{2kT\cos(kL)}$$

since $\cos(x) = ½\,[e^{+ix} + e^{-ix}]$, note also that $\sin(x) = ½i\,[e^{+ix} - e^{-ix}]$. Then we find that

$$y_{oL} = i\frac{|F|}{kT\left[1 + e^{+2ikL}\right]} = i\frac{|F|\, e^{-ikL}}{kT\left[e^{-ikL} + e^{+ikL}\right]} = i\frac{|F|\, e^{-ikL}}{kT\left[e^{+ikL} + e^{-ikL}\right]} = i\frac{|F|\, e^{-ikL}}{2kT\cos(kL)}$$

Thus, the complex transverse displacement, *y*(*x*, *t*) of the driven string, at an arbitrary point, x, and time, *t* is given by:

$$\begin{aligned} y(x,t) &= y_{oR} e^{i(\omega t - kx)} + y_{oL} e^{i(\omega t + kx)} \\ &= -i\frac{|F|\, e^{i\omega t}}{2kT\cos(kL)} e^{ik(L-x)} + i\frac{|F|\, e^{i\omega t}}{2kT\cos(kL)} e^{-ik(L-x)} = \frac{|F|\sin[k(L-x)]}{kT\cos(kL)} e^{i\omega t} \end{aligned}$$

Thus, the complex transverse velocity, $u_y(x, t) = \partial y(x, t)/\partial t$ of the driven string, at an arbitrary point, x and time, t is given by:

$$u_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \frac{i\omega\,|\,F\,|\sin[k(L-x)]}{kT\cos(kL)}e^{i\omega t}$$

Again, the complex mechanical input impedance, Z^{input} of the driven string at the driving point, x = 0 is given by:

$$\begin{aligned} Z^{input} &\equiv \frac{F(t)}{u_y(x=0,t)} = \frac{kT\cos(kL)}{i\omega\sin(kL)} = -i\frac{kT}{\omega}\cot(kL) \\ &= -i\frac{T}{v_x}\cot(kL) = -i\sqrt{\mu T}\cot(kL) = -i\mu v_x\cot(kL) \\ &= -iZ_o\cot(kL) \end{aligned}$$

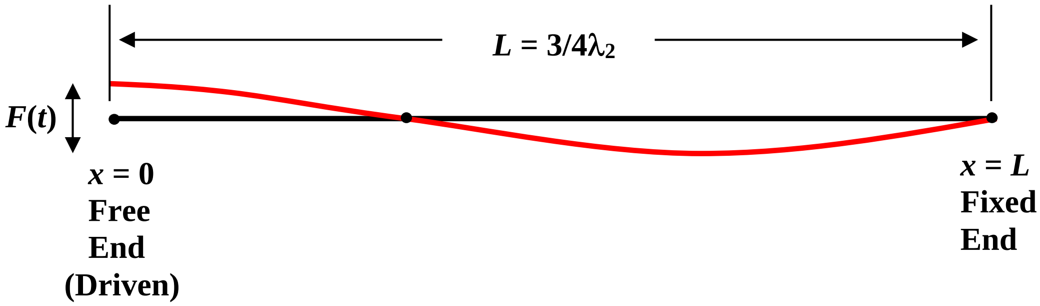
Note that here, for the driven free end-fixed end boundary conditions of this string, the complex mechanical input impedance, Z^{input} of the driven string at the driving point, x = 0 is *purely* *<u>reactive</u>* - i.e. it is purely imaginary!

The (complex) power, or time rate of energy transfer from the "free" end support at x = 0 to the string is given by:

$$\begin{aligned} P(t) &= \frac{\partial E(t)}{\partial t} = F(t)u_y^*(x=0,t) = -i\frac{\omega\,|\,F\,|^2\sin(kL)}{kT\cos(kL)} \\ &= -i\frac{v_x\,|\,F\,|^2}{T}\tan(kL) = -i\frac{v_x\,|\,F\,|^2}{T\cot(kL)} = \frac{v_x\,|\,F\,|^2}{iT\cot(kL)} \\ &= \frac{|\,F\,|^2}{(Z^{input})^*} = \frac{|\,F\,|^2}{iZ_o\cot(kL)} \end{aligned}$$

Here, the power is (also) purely reactive - i.e. purely imaginary, or 90^o out of phase with the driving force. The physical meaning of this is the following. The driving force inputs energy into the string at x = 0, creating a right-moving traveling wave. The right-moving traveling wave propagates on the string until it reaches the fixed end at x = L, whereupon it is reflected and polarity-flipped, converted into left-moving traveling wave. The left-moving traveling wave then propagates back to x = 0, where it leaves the string, returning its associated energy to the power source. Thus, a given amount of energy is put into the vibrating string at x = 0 by the complex driving force, F(t), however a little while later, this same energy comes back and is returned to the driving force. Thus, for the equilibrium situation, no *net* energy is transferred to/from this string!

The driving force, F(t) has angular frequency, $\omega = 2\pi f$. In principle, the free end support at x = 0 can be driven at any frequency, f. Because of the free end-fixed end boundary conditions at x = 0 and x = L, respectively, at certain driving frequencies *resonances* (maximum amplitudes) and *anti-resonances* (minimum amplitudes) in the transverse vibrations of the string of length, L will occur. However, because of the free end-fixed end boundary conditions, the modes of vibration in this situation will be different than those associated with fixed end-fixed end boundary conditions. For free end-fixed end boundary conditions, we must have a displacement *anti-node* at the x = 0 free end (because the slope of the string at x = 0, $\partial y(x=0, t)/\partial x = 0$). At the x = L fixed end, we must have a displacement *node* $y(x=0, t) = 0$. In the figure below, we show the second lowest mode of vibration of a standing wave on a string with free end-fixed end boundary conditions.



The transverse standing wave modes of vibration on a string with free end-fixed end boundary conditions are such that $L = {}^{1}/_{4}\lambda_1, {}^{3}/_{4}\lambda_2, {}^{5}/_{4}\lambda_3, {}^{7}/_{4}\lambda_4, {}^{9}/_{4}\lambda_5$, ... i.e. $L = {}^{(2n-1)}/_{4}\lambda_n$, or wavelength, $\lambda_n = {}^{4}/_{(2n-1)}L$, and wavenumber, $k_n = 2\pi/\lambda_n = {}^{(2n-1)\pi}/_{2L}$ and frequency, $f_n = v_x/\lambda_n = {}^{(2n-1)v_x}/_{4L}$, with n = 1, 2, 3, 4, ...

For driving frequencies, $f = f_n$ associated with the production of transverse standing wave modes of vibration on a string (i.e. *resonances*) with free end-fixed end boundary conditions, the complex mechanical input impedance, Z_n^{input} of the driven string at the driving point, x = 0 for the n^th mode of this type of transverse vibration is given by:

$$Z_n^{input} \equiv \frac{F(t)}{u_{ny}(x=0,t)} = \frac{k_n T\cos(k_n L)}{i\omega_n \sin(k_n L)}$$

$$= -i\frac{k_n T}{\omega_n}\cot(k_n L) = -i\frac{T}{v_x}\cot(k_n L) = -i\frac{T}{v_x}\cot\left(\tfrac{(2n-1)\pi}{2}\right) = -iZ_o\cot\left(\tfrac{(2n-1)\pi}{2}\right) = 0$$

because $\cot[(2n-1)\pi/2] = 0$ for n = 1, 2, 3, 4, ... Thus, at these resonant frequencies, the complex driving force, F(t) can very easily transfer energy from the transversely vibrating free end support to the string, generating a standing wave on the string at this same frequency. (This same energy will come back out of the string a little while later.)

One can also run this same process *backwards* in time (i.e. use the symmetry principle of *time-reversal invariance*): Conversely, if one plucks a string such that e.g. its fundamental, or one of the higher harmonics of the string vibrates at one of these resonant frequencies, the vibrating forces acting on this “free” end support will efficiently transfer energy from the string to the “free” end support at that vibrational frequency, causing it to vibrate, thereby draining/damping energy from the vibrating string. In other words, for a guitar with such a “free” end support, the *sustain* of this guitar will be very significantly degraded if the string has a vibrational mode at (or near) such a resonant frequency of a free end-fixed end support system - the “free” end being an approximation to the bridge on an acoustic (or hollow-body electric) guitar, or an approximation to the nut at the end of the neck on a solid body electric guitar, or e.g. a *massless* slide/bottleneck!

The (complex) power, or time rate of energy transfer from the "free" end support at $x = 0$ to the string (or vice versa, for the time-reversed situation), at the resonant frequency, $f_n = v_x/\lambda_n = {}^{(2n-1)V_x}/_{4L}$, n = 1, 2, 3, 4, ... is given by:

$$P_n(t) = \frac{dE_n(t)}{dt} \equiv F(t)u_{ny}^*(x=0,t) = -i\frac{\omega_n \,|\,F\,|^2 \sin(k_n L)}{k_n T\cos(k_n L)}$$

$$= -i\frac{v_x \,|\,F\,|^2}{T}\tan(k_n L) = -i\frac{v_x \,|\,F\,|^2}{T\cot(k_n L)} = \frac{v_x \,|\,F\,|^2}{iT\cot(k_n L)}$$

$$= \frac{|\,F\,|^2}{(Z_n^{input})^*} = \frac{|\,F\,|^2}{iZ_o\cot(k_n L)} = \frac{|\,F\,|^2}{iZ_o\cot\left[\frac{(2n-1)\pi}{2}\right]} = \infty$$

Of course, since we are driving the string with a finite force, we can't possibly expect to have an infinite amount of energy/power returned to the power source, if it wasn't put in, in the first place. The reason for this result is that we have implicitly assumed in this situation that the *width* of the resonances, f_n are infinitely narrow, which is in fact not the case for real strings on a guitar - the widths of the resonances are finite.

If the complex mechanical input impedance, $Z_n^{input} = 0$ at a succession of resonant driving frequencies, $f_n = v_x/\lambda_n = {}^{(2n-1)V_x}/_{4L}$, n = 1, 2, 3, 4, ... how does the complex mechanical input impedance, Z^{input} behave *in-between* such resonances, say in- between the resonant frequencies $f_n = {}^{(2n-1)V_x}/_{4L}$ and $f_{n+1} = = {}^{(2n+1)V_x}/_{4L}$?

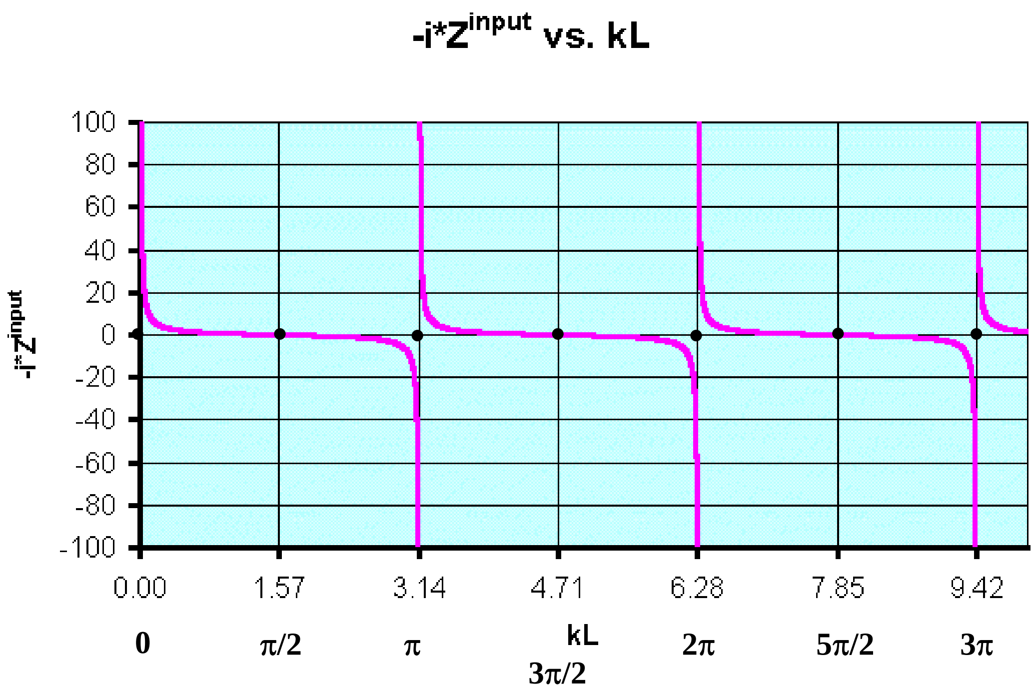
When the frequency, f of the complex driving force, F(t) is precisely at a frequency, $f_m = mv_x/2L$ for m = 1, 2, 3, 4, ... the complex mechanical input impedance, Z_m^{input} of the driven string at the driving point, x = 0 will be $Z_m^{input} = \pm\, i\, \infty$, because then $\cot(m\pi) = \pm\,\infty$ for m = 1, 2, 3, 4, ... The *magnitude* of the complex mechanical input impedance of the driven string, $|Z_m^{input}| = \infty$ when $f = f_m = mv_x/2L$ for m = 1, 2, 3, 4, ... These frequencies correspond to so-called *anti-resonances*. The free end support is vibrating up and down, transverse to the string at the frequency $f = f_m = mv_x/2L$ for m = 1, 2, 3, 4, ... but because the complex mechanical input impedance has magnitude, $|Z_m^{input}| = \infty$ at such frequencies, no energy can be transferred by the driving force, F(t) from the driven free end of the string to the string at these anti-resonant frequencies!

Again, by time reversal invariance, if a plucked string has a fundamental, or one of the higher harmonics vibrates at one of these *anti-resonant* frequencies, $f_m = mv_x/2L$ for m = 1, 2, 3, 4, ... of the "free" end support, there will be *no* energy transferred from the vibrating string to the "free" end support at such a frequency, thus the *sustain* of the string will be as long as in the case of fixed, perfectly rigid end supports at both ends of the string!

The complex power, P(t) for the anti-resonant frequencies, $f_m = mv_x/2L$ for m = 1, 2, 3, 4, ... is given by:

$$P(t) = \frac{\partial E(t)}{\partial t} = F(t)u_y^*(x=0,t) = \frac{|\,F\,|^2}{(Z^{input})^*} = \frac{|\,F\,|^2}{iZ_o\cot(k_m L)} = \frac{|\,F\,|^2}{iZ_o\cot(m\pi)} = 0$$

The following figure shows a plot of the quantity $-i\, Z^{input} = (T/v_x)\cot(kL)$ vs. kL for the case of free end-fixed end supports at x = 0 and x = L, respectively, with $(T/v_x) = 1.0$.



It can be seen that from this curve that $|Z^{input}| \sim 0$ for much of the range of kL values, except near $kL \sim m\pi$, m = 0, 1, 2, 3, 4, ... and that $|Z^{input}| = 0$ for $kL = (2n-1)\pi/2$, n = 1, 2, 3, 4, ...

<u>A More Realistic Motion of the End Supports</u>

If one of the end supports on a guitar - either the bridge or the nut at the end of the neck of a guitar is not completely rigid, or a slide/bottleneck of finite mass, M is used on the strings of the guitar, then this quasi-fixed/quasi-free end support will have associated with it a *finite*, complex impedance, $Z = Z_r + iZ_i$,, with $|Z| = (Z^*Z)^{½} = [(Z_r - iZ_i)(Z_r + iZ_i)]^{½} = (Z_r^2 + Z_i^2)^{½}$. If the so-called *imaginary* part, Z_i (i.e. the out-of-phase component) of the complex impedance, Z is *positive*, the motion of this not-completely-rigid end support relative to the string at this point is “mass-like”, and the fixed end-fixed end resonances - transverse vibrational modes of the string, $f_n = v_x/\lambda_n = nv_x/2L$, n = 1, 2, 3, 4, ... will be slightly *raised*. If the so-called imaginary part, Z_i of the complex impedance, Z is *negative*, the motion of the not-completely-rigid end support is “spring-like”, and the fixed end-fixed end resonances/vibrational modes of the string, $f_n = v_x/\lambda_n = nv_x/2L$, n = 1, 2, 3, 4, ... will be slightly *lowered* as a result. The so-called real part, Z_r (i.e. the in-phase component) of the complex impedance, Z is proportional to the rate of energy transfer from the string to the not-completely-rigid end support.

Suppose the string with tension, *T* has a rigid, fixed end support at $x = 0$, but at $x = L$, this end support is not completely rigidly fixed, but instead, this end support behaves as if it is quasi-free, with a finite mass, *M*. Note that a perfectly rigid, fixed end support can be thought of as having infinite mass, $M = \infty$. Then as the string vibrates transversely, for small-amplitude vibrations, it will exert a transverse force, $F_y(t) \cong -T\,(\partial y(x{=}L,t)/\partial x)$ on this quasi-free end support at x = L, of mass, M. By Newton's second law (F = Ma), we have:
Note that the acceleration, a of the mass, M at the end support located at x = L is the same as the

$$F_y(t) = -T\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=L} = Ma = M\left.\frac{\partial^2 y(x,t)}{\partial t^2}\right|_{x=L}$$

transverse acceleration of the string, $a_y(x{=}L) = \partial^2 y(x{=}L,t)/\partial t^2$ at this end support. This force will in general be complex.

The (complex) transverse displacement of the string at an arbitrary point, x along its length, L at a given time, t is again given by:

$$y(x,t) = y_{oR}e^{i(\omega t - kx)} + y_{oL}e^{i(\omega t + kx)}$$

with $y_{oR} = |y_{oR}|e^{i\delta}$ and $y_{oL} = |y_{oL}|e^{i\delta}$.We apply the fixed-end boundary condition at x = 0, namely that y(x=0,t) = 0, where:

$$y(x=0,t) = y_{oR}e^{i(\omega t - kx)}\Big|_{x=0} + y_{oL}e^{i(\omega t + kx)}\Big|_{x=0} = y_{oR}e^{i\omega t} + y_{oL}e^{i\omega t} = \left(y_{oR} + y_{oL}\right)e^{i\omega t} = 0$$

This can only be satisfied for any/all time(s) t, if $y_{oR} = -y_{oL}$. Then the transverse displacement, y(x,t) of the string, for an arbitrary point, *x* and time, *t* is:

$$y(x,t) = y_{oR}\left(e^{-ikx} - e^{+ikx}\right)e^{i\omega t} = -y_{oR}\left(e^{+ikx} - e^{-ikx}\right)e^{i\omega t} = -2i\ y_{oR}\sin(kx)e^{i\omega t}$$

Then:

$$u_y(x,t) = \frac{\partial y(x,t)}{\partial t} = 2\omega y_{oR}\sin(kx)e^{i\omega t}$$

and:

$$\frac{\partial y(x,t)}{\partial x} = -2iky_{oR}\cos(kx)e^{i\omega t}$$

and:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = +2i\omega^2 y_{oR}\sin(kx)e^{i\omega t}$$

Inserting these relations into the above force relation at $x = L$, we have:

$$F(t) = +2iky_{oR}T\cos(kL)e^{i\omega t} = 2iM\omega^2 y_{oR}\sin(kL)e^{i\omega t}$$

or:

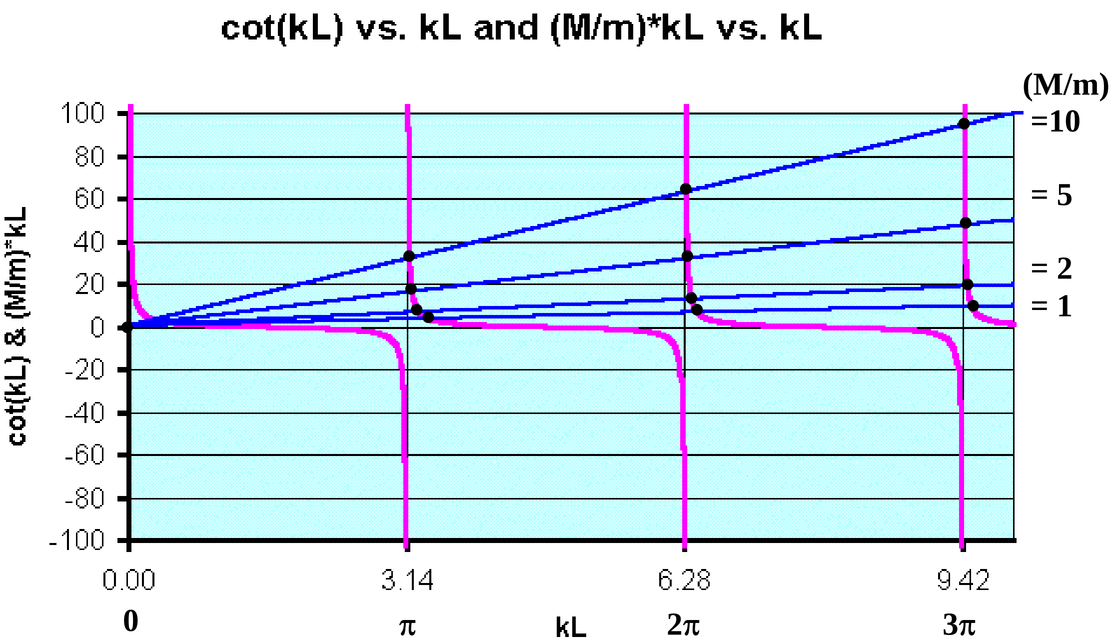
$$kT\cos(kL) = M\omega^2 \sin(kL)$$

or:

$$\cot(kL) = \frac{\cos(kL)}{\sin(kL)} = \frac{M\omega^2}{kT} = \frac{kM}{T}\left(\frac{\omega}{k}\right)^2 = \frac{kM}{T}v_x^2 = \frac{kM}{\mu} = \left(\frac{M}{m}\right)kL$$

where we used the relations $v_x = \omega/k$, $v_x^2 = T/\mu$ and $m = \mu L$ = total mass of the string of length, L. The relation cot(kL) = (M/m)kL is a non-linear equation, known as a so-called *transcendental equation* - because it *transcends* known *analytic* mathematical methods for solving this equation. Before the age of computers, solving such equations could only be done in a reasonable amount of time by using graphical techniques!

In the figure below, the magenta curves are the graphs of cot(kL) vs. kL. The dark blue straight lines are the (M/m)kL vs. kL relations, for values of the slope of each straight line, (M/m) = 1, 2, 5 and 10, respectively.



The kL values where each of the dark blue straight-line relations (M/m)kL vs. kL *intersect* with each of the magenta cot(kL) vs. kL curves (shown by black dots) thus determine the vibrational modes of the string with one fixed end support at x = 0 and one quasi-fixed/quasi-free end support of mass, M located at x = L. For example, if the ratio of end support mass to the total string mass is (M/m) = 10, then $k_1L \cong \pi$, $k_2L \cong 2\pi$, $k_3L \cong 3\pi$, etc., or $k_nL \cong n\pi$, hence $k_n \cong n\pi/L$, n = 1, 2, 3, 4, ... (In reality, M >> m on a real guitar), except for the case of slide/bottleneck guitar, if the slide/bottleneck is *e.g.* made of thin glass or plastic. Recall that for fixed end-fixed end boundary conditions on the string (equivalent to M = ∞, and hence (M/m) = ∞), that $k_n = n\pi/L$, n = 1, 2, 3, 4, ... Thus, we see from the above figure, that for finite mass, M of the end support located at x = L, that in fact $k_n > n\pi/L$, and therefore since $k_n = 2\pi/\lambda_n$, then $2\pi/\lambda_n > n\pi/L$, or $\lambda_n < 2L/n$ for a quasi-fixed/quasi-free end support of mass M located at x = L. A *shorter* wavelength, λ_n implies a *higher* frequency, f_n associated with vibrational modes of a string with one quasi-fixed/quasi-free end support of mass M, in comparison to the vibrational modes associated with rigid/fixed ($M = \infty$) end supports at both ends of the string.

Note also that because of this non-linear relationship (cot(kL) = (M/m)kL), the frequency of the lowest mode (n = 1) is raised *more* from its ($M = \infty$) fixed end-fixed end support value than that of the higher modes (n > 1). Because of this, the frequencies of the higher (n > 1) harmonics will not be precisely at integer multiples of the fundamental (n = 1) - i.e. $f_n \neq nf_1$ for n = 2, 3, 4, ...!!

The complex input impedance of the string at this quasi-fixed/quasi-free end support located at x = L is given by:

$$Z^{input} \equiv \frac{F(t)}{u_y(x=L,t)} = \frac{2ikTy_{oR}\cos(kL)e^{i\omega t}}{+2\omega y_{oR}\sin(kL)e^{i\omega t}} = i\left(\frac{T}{v_x}\right)\cot(kL) = iM\omega$$

From the above transcendental equation, cot(kL) = (M/m)kL. Thus, the complex input impedance is

$$Z^{input} = i\left(\frac{T}{v_x}\right)\cot(kL) = i\left(\frac{T}{v_x}\right)\left(\frac{M}{m}\right)kL = iM\omega$$

Note that this "mass-like" impedance associated with the quasi-free end support at x = L is again purely imaginary, and as stated earlier, positive imaginary. The last term(s) on the right hand side of this relation may seem odd at first, but they aren't. Removing common factors, we have $[T/(mv_x)]kL = \omega$, but since $m = \mu L$, this is $[(T/\mu)/v_x]k = \omega$, and since $(T/\mu) = v_x^2$, this is $v_x k = \omega$, and since $\omega/k = v_x$, then we simply find that 1 = 1!

The power transferred from the quasi-free end mass-like end support located at x = L to the string (or from the string to the quasi-free, mass-like end support, if time is reversed), is given by:

$$P(t) = \frac{\partial E(t)}{\partial t} = F(t)u_y^*(x=L,t) = \frac{|F|^2}{(Z^{input})^*} = 4iM\omega^3\,|\,y_{oR}\,|^2\,\sin^2(kL)$$

Again, the power here is purely imaginary - the energy transferred from the vibrating string to this mass-like end support (or vice versa) is (eventually) returned back from the end support of mass, M to the string (or vice versa). We have not (yet) put in the physics of energy flowing from the vibrating string, passing thru such and exciting a quasi-free end support, and then flowing on to other portions of the guitar (or vice versa, for the time-reversed situation)!

Damping Effects in Vibrating Strings

As we learned in the previous lecture notes on waves, a vibrating string has energy associated with it. This energy is imparted to the string at the instant it is e.g. plucked, say at time t = 0 sec. We all know that if we wait long enough, the string vibrations will slowly die out; eventually the string ceases to vibrate altogether. Thus, the energy that was initally imparted to the string by plucking it is *dissipated*. In fact, all of the initial energy associated with the vibrating string ultimately all winds up as heat, somewhere. Dissipation of the string's energy is also known as a form of *damping* of the vibrating string. In general, whenever damping processes exist, independent of the physical origin of the damping process, they will have an impact on the frequenc(ies) associated with the string vibrations. There are various physical mechanisms associated with dissipative losses of the string's initial energy/damping of the string's vibrations which we will discuss in turn, below. However, first we wish to discuss damping of the string vibrations in a general manner.

Consider a stretched string of length L and mass per unit length, μ completely immersed in a viscous medium that provides a damping force, F on the string that is propotional to the transverse velocity of the string, $u_y(x,t)$.

The (generic) damping force, $dF_{damp}(x,t)$ acting on an infinitesimal element, or segment, dx of the string at the point, x along the string at time, t is given by:

$$dF_{damp}(x,t) = -b\ dx\ u_y(x,t) = -b\ dx \frac{\partial y(x,t)}{\partial t}$$

where b is the constant of proportionality for the damping force. The minus sign indicates that the damping force always acts in such a way as to oppose the motion of the vibrating string.

The wave equation describing an infinitesimal element, *dx* of vibrating string in the *absence* of viscous damping, for small amplitude vibrations of the string is given by:

$$\mu\ dx \frac{\partial^2 y(x,t)}{\partial t^2} = T\{\sin(\theta + d\theta) - \sin\theta\} = T\ \ d\sin\theta = T\cos\theta\ \ d\theta \cong T\frac{\partial\theta}{\partial x}dx \cong T\frac{\partial^2 y(x,t)}{\partial x^2}dx$$

or:

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{\mu}{T}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

or:

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v_x^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

Mathematically, this wave equation, which describes the undamped vibrations of transverse waves on a stretched string is a so-called second-order, linear, *homogeneous* differential equation. It is a *second-order* differential equation because it has *double derivatives* of y(x,t), one of position, x ($\partial/\partial x^2$) and the other, of time, t ($\partial/\partial t^2$). It is a *linear* differential equation, because all terms in this equation have a *linear* dependence on y(x,t). It is a *homogeneous* differential equation because there are *only* the double-derivatives in this 2nd order linear differential equation - not single, or triple, quadruple etc. derivatives.

In the presence of viscous damping of the string, the wave equation is modified:

$$\mu\ dx \frac{\partial^2 y(x,t)}{\partial t^2} = T\frac{\partial^2 y(x,t)}{\partial x^2}dx - bdx\ u_y(x,t) = T\frac{\partial^2 y(x,t)}{\partial x^2}dx - bdx\frac{\partial y(x,t)}{\partial t}$$

or:

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{\mu}{T}\frac{\partial^2 y(x,t)}{\partial t^2} = +\frac{b}{T}\frac{\partial y(x,t)}{\partial t}$$

or:

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v_x^2}\frac{\partial^2 y(x,t)}{\partial t^2} = +\frac{b}{T}\frac{\partial y(x,t)}{\partial t}$$

Mathematically, this wave equation, which describes the (viscously) damped vibrations of transverse waves on a stretched string is a so-called second-order, linear, but (now) *inhomogeneous* differential equation. This 2^{nd} order linear differential equation is inhomogeneous because it has a term which is a single derivative of y(x,t) - the velocity-dependent damping term on the right-hand side of the above equation.

The solution, y(x,t) of this 2^{nd} order linear inhomogeneous wave equation that describes the behavior of e.g. viscously-damped right-moving traveling waves on the string is of the form: Plugging this solution in to the above wave equation, we obtain the so-called *characteristic*

$$y(x,t) = y_o e^{i(\omega t - k'x)}$$

equation:

$$k'^2 = \omega^2\left(\frac{\mu}{T}\right)\left[1 - i\frac{b}{\mu\omega}\right]$$

Thus, it can be seen that the wavenumber, k' associated with the viscously damped, right-moving traveling wave is complex - i.e. it has a real part, k'_R and an imaginary part, k'_I:

$$k' = k'_R - ik'_I$$

Note that the minus sign in the above formula is extremely important - for a right-moving traveling wave, the amplitude of this wave is exponentially damped as x increases (for a left-moving traveling wave, the amplitude is exponentially damped as x decreases).

Then $k'^2 = k'^2_R - 2ik'_R k'_I - k'^2_I$, we plug this into the above characteristic equation, identify the resulting terms that are purely real, and those that are purely imaginary:

$$k'^2_R - k'^2_I = \omega^2\left(\frac{\mu}{T}\right) \qquad \textit{and} \qquad -2ik'_R\, k'_I = i\omega^2\left(\frac{\mu}{T}\right)\left[\frac{b}{\mu\omega}\right] = i\omega\frac{b}{T}$$

We can then solve the two resulting quadratic equations for k'_R and k'_I with the physical constraint(s) that both k'_R and k'_I must be positive, and after some algebra, we obtain:

$$k'_R = \omega\left(\frac{\mu}{2T}\right)^{1/2}\left[1 + \sqrt{1 + \left(\frac{b}{\mu\omega}\right)^2}\right]^{1/2}$$

and:

$$k'_I = \omega\left(\frac{\mu}{2T}\right)^{1/2}\left[-1+\sqrt{1+\left(\frac{b}{\mu\omega}\right)^2}\right]^{1/2}$$

Now since:

$$y(x,t) = y_o e^{i(\omega t - k'x)} = y_o e^{i\omega t} e^{-ik'x} = y_o e^{i\omega t} e^{-i(k'_R - ik'_I)x} = y_o e^{i\omega t} e^{-ik'_R} e^{-k'_I x} = y_o e^{i(\omega t - k'_R x)} e^{-k'_I x}$$

we see that the amplitude of the transverse displacement, y(x,t) associated with a right-moving transverse traveling wave on the string, vibrating in a viscous, dissipative medium will be attenuated to 1/e of its initial value (here assuming the wave originates at x = 0) in a characteristic distance $x = 1/k'_I$. We therefore define this characteristic damping distance as the so-called attenuation length, $\lambda_{atten} \equiv 1/k'_I$:

$$\lambda_{atten} \equiv \frac{1}{k'_I} = \frac{\frac{1}{\omega}\left(\frac{2T}{\mu}\right)^{1/2}}{\left[-1+\sqrt{1+\left(\frac{b}{\mu\omega}\right)^2}\right]}$$

The *phase speed*, v'_x (= longitudinal propagation speed) of this viscously damped right-moving traveling wave is no longer $v'_x = T/\mu$, because of the presence of damping. It is now $v'_x = \omega/k'_R$. Note that when b = 0 (i.e. no viscous damping of the string) then $k = k'_R = \omega/v_x$ (since then $v'_x = v_x = (T/\mu)^{½}$), and $k'_I = 0$. Thus,

$$v'_x = \frac{\omega}{k'_R} = \frac{\left(\frac{2T}{\mu}\right)^{1/2}}{\left[1+\sqrt{1+\left(\frac{b}{\mu\omega}\right)^2}\right]^{1/2}}$$

Thus, we see that in the presence of viscous damping ($b \neq 0$) the phase speed/longitudinal wave speed, $v'_x < v_x = (T/\mu)^{½}$.

The characterstic damping time, $t = \tau_{damp}$ in which the amplitude of the transverse wave is exponentially damped to $1/e = 1/e^{+1} = e^{-1} = 1/2.71828 = 0.36788$ of its initial value is given by:

$$\tau_{damp} = \frac{\lambda_{atten}}{v'_x} = \frac{1/k'_I}{\omega/k'_R} = \frac{1}{\omega}\left(\frac{k'_R}{k'_I}\right) = \frac{1}{\omega}\frac{\left[1+\sqrt{1+\left(\frac{b}{\mu\omega}\right)^2}\right]^{1/2}}{\left[-1+\sqrt{1+\left(\frac{b}{\mu\omega}\right)^2}\right]^{1/2}}$$

Note that this damping time is valid not only for transverse traveling waves, but also for transverse standing waves. If the amount of viscous damping is small, i.e. $b \ll \mu\omega$, then using the Taylor series expansion $(1+\varepsilon)^{½} \cong 1+ ½\varepsilon$ for $\varepsilon \ll 1$, the damping time, τ_{damp} is:

$$\tau_{damp} \cong \frac{1}{\omega}\frac{\left[1+1+\frac{1}{2}\left(\frac{b}{\mu\omega}\right)^2\right]^{1/2}}{\left[-1+1+\frac{1}{2}\left(\frac{b}{\mu\omega}\right)^2\right]^{1/2}} = \frac{1}{\omega}\frac{\left[2+\frac{1}{2}\left(\frac{b}{\mu\omega}\right)^2\right]^{1/2}}{\left[\frac{1}{2}\left(\frac{b}{\mu\omega}\right)^2\right]^{1/2}} = \frac{2}{\omega}\frac{\left[1+\left(\frac{b}{2\mu\omega}\right)^2\right]^{1/2}}{\left(\frac{b}{\mu\omega}\right)} = \frac{2\mu}{b}\left[1+\left(\frac{b}{2\mu\omega}\right)^2\right]^{1/2}$$

The frequency of vibration of transverse waves propagating in a viscous medium is also changed from its b = 0 value. If the amount of viscous damping is small, i.e. $b \ll \mu\omega$, then using the Taylor series expansion $(1+\varepsilon)^{1/2} \cong 1+ \frac{1}{2}\varepsilon$ for $\varepsilon \ll 1$, it can be shown that:

$$\omega' \cong \omega\sqrt{1-\left(\frac{b}{2\mu\omega}\right)^2}$$

and:

$$f' \cong f\sqrt{1-\left(\frac{b}{4\pi\mu f}\right)^2}$$

Air Damping Effects of Vibrating Strings

In an electric guitar, or any stringed instrument there are in fact several dissipative mechanisms that are responsible for damping of the string vibrations. One such mechanism is due to viscous air-drag effects - the vibrating string(s) of a guitar do not occur in a vacuum - they occur in air, usually at atmospheric pressure (if at sea level). Microscopically, energy from the vibrating string is transferred to individual air molecules by their many collisions with the string as it vibrates. This is a statistical process. The vibrating string pushes air molecules out of the way as it vibrates - the air is essentially a viscous fluid, and has a viscous drag associated with it.

A string with either a transverse travelling wave or a transverse standing wave vibrates (transversely) back and forth in a plane. As it does so, it produces a compression wave in the air in front of it, and a rarefaction of the air behind it. From this, one might think that this would enable the vibrating string to couple extremely well to the air, and therefore act as a good radiator of sound waves. However, this is not the case, because the diameter of the string is so small in comparison to the wavelength(s) of sound in air.

For example, electric guitar strings have diameters in the range *e.g.* ~ 0.009”–0.046” (*i.e.* 0.2286–1.1684 mm). The speed of sound in air, at sea level is v_s ~ 330 m/s. The high-E string of a guitar has frequency $f_{hi\text{-}E} = 330$ Hz. Thus, the wavelength in air associated with playing an open high-E string on a guitar is $\lambda_{hi\text{-}E} = v_s/f_{hi\text{-}E} = 1\ m$, which is much much greater than the diameter of the high-E string on a guitar. Since sound waves do indeed consist of compression & rarefaction of the air (i.e. local density perturbations in the air), the compression & rarefaction of air on the distance scale of the diameter of strings of a guitar is very inefficient at transferring sound energy from the string to the surrounding air - essentially at this distance scale, the air in the compressed region immediately ahead of the string, being pushed (*i.e.* compressed) by the vibrating string simply flows around the string to the backside of the string, where the air pressure is lower, thus cancelling out the sound even before it has time to radiate away!

However, this does not mean that the vibrating string is not affected by the presence of the air around it. Indeed it is!

Nearly 150 years ago (in 1851), George Stokes [1] solved the problem of viscous air drag on a vibrating string. The viscous force on the string has two components. One is a mass-like load that *lowers* the vibrational mode frequencies of the string very slightly, the other force component produces an exponential decay of the amplitude with time. The frequency of vibration of the string in air is given by:

$$f_{air} = f_o\left[1-\left(\frac{\gamma}{f_o}\right)^2\right]^{1/2}$$

where f_o = frequency of vibration of the string in a vacuum; the parameter, γ is given by:

$$\gamma = f_o\rho_{air}\pi r_{string}^2\left(\frac{\sqrt{2}}{M}+\frac{1}{2M^2}\right) = f_o\rho_{air}A_{string}\left(\frac{\sqrt{2}}{M}+\frac{1}{2M^2}\right)$$

where ρ_{air} = density of air $\cong$ 1.205 kg/m^3, (at T = 20^o C, P = 1 Atm) $u_y(x,t)$ = transverse velocity of the string at the point x, at time, t; r_{string} = radius of string, $A_{string} = \pi r^2_{string}$, and:

$$M = \tfrac{1}{2} r_{string}^2 \sqrt{2\pi f_o / \eta^*_{air}}$$

where $\eta^*_{air} \cong 1.52 \times 10^{-5}$ m^2/sec is the kinematic viscosity of air, i.e. $\eta^*_{air} = \eta_{air}/\rho_{air}$, where $\eta_{air} \cong 1.832 \times 10^{-5}$ kg/(m-sec) is the coefficient of viscosity of air. For a 0.010" diameter high-E string on a guitar, where $f_{hi\text{-}E}$ = 330 Hz, $M_{hi\text{-}E} \cong 0.44$. For a 0.046" diameter low-E string on a guitar, where $f_{lo\text{-}E}$ = 82 Hz, $M_{lo\text{-}E} \cong 2.35$. The value(s) of γ are extremely small, ranging from ~ 7.4 × 10^{-5} for the high-E string to ~ 4.8 × 10^{-5} for the low-E string. Thus, the frequency shifts due to viscous air drag effects on the vibrating strings of a guitar are exceedingly small, on the order of ~ 10^{-11} Hz.

For the range of string diameters and string vibration frequencies associated with stringed instruments, such as guitars, the viscous drag force, $dF_{drag}(x,t)$ acting on a infinitesimal segment of string of length dx, at the position, x along the string at time, t does so in such a way as to <u>always</u> <u>*oppose*</u> the motion of the string, i.e. <u>*retarding*</u> its motion, as the string passes through the air. This drag force, $dF_{drag}(x,t)$ acting at a point x on a vibrating string of length, L is given by:

The instantaneous power lost due to viscous damping of the vibrating string in air, $dP_{drag}(x,t)$

$$dF_{drag}(x,t) = 2\pi^2 f_o\rho_{air}u_y(x,t)r_{string}^2\left(\frac{\sqrt{2}}{M}+\frac{1}{2M^2}\right)dx = 2\pi f_o\rho_{air}\ u_y(x,t)A_{string}\left(\frac{\sqrt{2}}{M}+\frac{1}{2M^2}\right)dx$$

associated with the string segment, of length, dx at the point, x, at time t is given by:

The power loss, $dP_{drag}(x,t)$ is by definition the (time) rate of change of the total energy due to

$$dP_{drag}(x,t) = \frac{dE_{drag}(x,t)}{dt} = dF_{drag}(x,t) * u_y(x,t) = 2\pi f\rho_{air}u_y^2(x,t)A_{string}\left(\frac{\sqrt{2}}{M}+\frac{1}{2M^2}\right)$$

viscous air drag, $dE_{drag}(x,t)/dt$ associated with the string segment of infinitesimal length, dx, at the point x, at time, t. Note that since the viscous drag force, $dF_{drag}(x,t)$ acting on the string segment, dx is proportional to the transverse velocity of the string, $u_y(x,t)$ at the point x, at time, t, then the power loss and/or the rate of energy loss is proportional to the square of the transverse velocity, $u_y(x,t)$ at the point, *x* at time, *t*.

The *net* viscous drag force, $F_{drag}(t)$ acting on the *entire* length, L of vibrating string at a given instant in time, *t* is obtained by summing up all the force contributions, $dF_{drag}(x,t)$ from each of the infinitesimal string segments, dx along the vibrating string, from $0 \le x \le L$:

$$F_{drag}(t) = \sum_{n=1}^{n=N} dF_{drag}(x,t) = \sum_{n=1}^{n=N} \frac{dF_{drag}(x,t)}{dx} dx$$

Going to the limit of a true infinitesimal, when $dx \to 0$, this sum converts to an integral representation:

$$F_{drag}(t) = \int dF_{drag}(x,t) = 2\pi f \rho_{air} A_{string} \left(\frac{\sqrt{2}}{M} + \frac{1}{2M^2} \right) \int_{x=0}^{x=L} u_y(x,t) dx$$

Similarly, the instantaneous power loss, $P_{drag}(t)$ for the entire length, L of vibrating string at time, t is given by:

$$P_{drag}(t) = \frac{dE_{drag}(t)}{dt} = \int dP_{drag}(x,t) = \int dF_{drag}(x,t) u_y(x,t) = 2\pi f \rho_{air} A_{string} \left(\frac{\sqrt{2}}{M} + \frac{1}{2M^2} \right) \int_{x=0}^{x=L} u_y^2(x,t) dx$$

For a given mode of vibration of the string, of frequency, f_n the transverse displacement amplitude will decay exponentially with time, *i.e.*

$$y_n(x,t) = y_{on}\ e^{-t/\tau_n} \sin(k_n x) \sin(\omega_n t)$$

The decay time constant, τ_n^{air} = the time for the transverse displacement amplitude, $y_{on}\, e^{-t/\tau n}$ associated with the harmonic, n of frequency, f_n to fall to $e^{-\tau n/\tau n} = e^{-1} = 1/e = 1/2.71828 = 0.36788$ of its initial value, y_{on} at t = 0.

The decay time constant associated with air damping associated with a given mode, n of vibration of the string is given by:

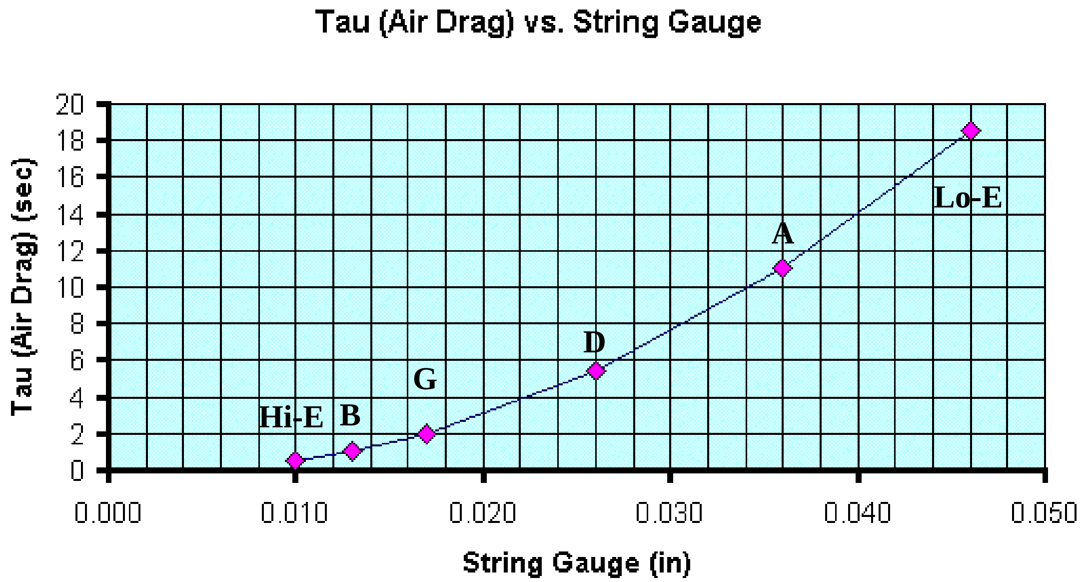
$$\tau_n^{air} = \frac{\rho_{string}}{2\pi \rho_{air} f_n} \left(\frac{2M_n^2}{2\sqrt{2} M_n + 1} \right)$$

where ρ_{string} is the *volume* mass density (kg/m^3) of the string, and

$$M_n = \tfrac{1}{2} r_{string}^2 \sqrt{2\pi}\, f_{on} / \eta^*_{air}$$

This decay time constant, τ_n^{air} is linearly proportional to ρ_{string}, but depends in a more complicated manner on the string radius and frequency. The decay time constant, $\tau_n^{air} \propto \rho_{string}\, r^2_{string}$ at low frequencies - independent of f_n, but $\tau_n^{air} \propto \rho_{string}\, r_{string} / \sqrt{f_n}$ at high frequencies. Thus, the decay time due to viscous air-damping of the strings will be shortest at high frequencies, affecting the higher harmonics first, and then successively lower harmonics from the time the strings are initially plucked.

In the figure below, we show the calculated decay time constant associated with air damping/air drag effects on vibrating steel strings of the electric guitar vs. string gauge, using the above formula, for air at 1 atmosphere of pressure (i.e. sea level) and temperature, T = 20° C.



The theory predicts a decay time in reasonable agreement with actual electric guitars for the low-frequency strings, but disagrees significantly for the higher frequency strings - the decay time here is *not* as short as the prediction would have you believe. Thus, for the smaller-gauge strings, the effect of viscous air damping is not as large as predicted!

Internal Damping Effects of Vibrating Strings

The material used in making e.g. electric guitar strings - steel, stainless steel for plain strings, and steel, stainless steel wrapped with pure nickel or a nickel alloy for the wound strings - behave as elastic materials. The strings of a guitar, of scale length, L_{scale} each have their own radius, r_{string} (m), (volume) mass density, ρ_{string} (kg/m^3). The ratio of stress (force per unit area, F/A) per unit strain (change in length per unit length, ΔL/L) in the material is known as Young's modulus, $Y = (F/A)/(\Delta L/L)$ (units: Newtons/m^2).

When a stress is applied to a material, an instantaneous strain occurs, but over a characteristic time scale, τ_{strain}, the strain increases slightly. Depending on the type of string material, this second strain can be moderately large or extremely small - the associated time scale, τ_{strain} can range from less than a millisecond to many seconds.

This relaxation behavior in strings can be represented mathematically by making Young's modulus, Y complex - i.e. an in-phase, or real component, Y_1 and a 90° out-of-phase, or imaginary component, Y_2, relative to the phase of a vibrating string. Then $Y = Y_1 + iY_2$, where $i \equiv \sqrt{-1}$, and $i \times i = -1$, $i \times -i = +1$. From the relaxation theory worked out by the Dutch physicist Peter Debye, the value of the out-of-phase/imaginary component of Young's modulus, Y_2 has a peak at the relaxation frequency $f_{strain} = 1/\tau_{strain}$. However, in a real material, there may in fact be several such relaxation times, associated with normal elastic bond distortions between atoms making up the material, as well as relaxation times associated with motion of dislocations in the material and/or kinks in polymer chains (in the case of e.g. nylon strings, used in acoustic guitars). Typically, the ratio Y_2/Y_1 is often less than 10^{-4} in hard crystalline materials, such as metals - steel, etc, but can be as large as $10^{-1} = 0.1$ in some polymer materials. In general, this ratio is also temperature dependent.

Internal damping of string vibrations arising from these microscopic physical processes leads to an exponential decay time, $\tau_n^{internal}$ of the transverse displacement amplitude(s) for the different modes of vibrations of the string, $y_n(x,t) = y_o \exp(-t/\tau_n^{internal}) \sin(k_n x) \sin(\omega_n t)$:

This kind of damping is negligible in comparison to air damping for solid metal strings, but

$$\tau_n^{internal} = \frac{1}{\pi f_n} \frac{Y_1}{Y_2}$$

can be the dominant damping mechanism in gut or nylon strings used on acoustic and classical guitars. Note that this decay time is shortest at high frequencies.

There exist additional internal damping mechanisms. In metal strings, thermal conduction (i.e. conduction of heat) also results in damping of the string vibrations. However, again, this is a small effect. In compound strings - consisting of twisted fibers, or wound strings, there also exists internal friction due to the relative motion of the component parts. As strings age on the guitar, build-up of “grunge” on the strings from playing the guitar occurs - a combination of skin cells, finger grease, dirt and sweat, all of which can (and does) lead to significant damping of the string vibrations. These loss mechanisms are also frequency-dependent, in the same manner as the above-discussed loss mechanisms.

Energy Loss Through the End Supports of the Strings

As we have discussed above, if the bridge or nut on a guitar is not perfectly rigid, then at certain resonant frequencies, $f_n = v_x/\lambda_n = {}^{(2n-1)V_x}/_{4L}$, n = 1, 2, 3, 4, ... energy can be transferred from the mode, n of the vibrating string to the quasi-free end support. In a real guitar, this energy can then be transferred to places elsewhere in the guitar, with a commensurate loss in sustain of the vibrating guitar string at this frequency.

In considering the energy of a quasi-free end support, it is easier to work with the complex mechanical *admittance*, $Y \equiv 1/Z$ rather than complex mechanical *impedance*, Z. The complex mechanical admittance, $Y = G + iB$, where the real part of the complex mechanical admittance, $\mathrm{Re}(Y) = G$, is known as the mechanical *conductance*. The imaginary part of the complex mechanical admittance, $\mathrm{Im}(Y) = B$ is known as the mechanical *susceptance*.

If the complex mechanical impedance, $Z = R + iX$, where $\mathrm{Re}(Z) = R$, known as the mechanical *resistance* and $\mathrm{Im}(Z) = X$, known as the mechanical *reactance*, then it can be shown that:

$$G = \frac{R}{R^2 + X^2} = \frac{R}{|Z|^2} \qquad and \qquad B = \frac{-X}{R^2 + X^2} = \frac{-X}{|Z|^2}$$

and conversely, that:

$$R = \frac{G}{G^2 + B^2} = \frac{G}{|Y|^2} \qquad and \qquad X = \frac{-B}{G^2 + B^2} = \frac{-B}{|Y|^2}$$

Note the negative sign between the reactance, X and the susceptance, B formulae. Note also that the units of Z, R and X are kg/sec. The units of Y, G and B are therefore sec/kg.

For a given mode, n of a vibrating string on a guitar, which has a quasi-free end support located, say at x = L, which is now mechanically connected to other portion(s) of the guitar, then it can be shown that the transverse velocity of the quasi-free end support, located at x = L, for the mode, n of the vibrating string is given by:

$$u_{ny}(x=L,t) = \alpha G_n F_{ny}(t)$$

where $F_{ny}(t)$ is the vertical component of the force acting on the quasi-free end support located at x = L, associated with the vibrational mode, n. The dimensionless parameter, α is a constant of proportionality, and G_n is the conductance associated with that mode, n.

In general, it is very difficult to *apriori* reliably predict, in a "first-principles" manner, what the conductance, G_n, the transverse velocity $u_{ny}(x{=}L,t)$ and transverse force, $F_{ny}(t)$ will be for a given vibrational mode, n for an actual guitar, be it an acoustic or and electric guitar. These things will depend on the details of the design and construction of the guitar, the totality of the materials used to build it - different kinds of wood, how they are oriented, various types of glue(s), the finishes used on the surfaces of the guitar, etc. It is easier, and more reliable to experimentally measure e.g. $u_{ny}(x{=}L,t)$ and $F_{ny}(t)$, and then compute G_n (assuming α is known).

The decay time constant, $\tau_n^{support}$ associated with energy loss through a quasi-free end support physically attached to other portions of the guitar, for a given mode of vibration, *n* is given by:

$$\tau_n^{\sup port} = \frac{1}{8 m_{string} f_n^2 G_n}$$

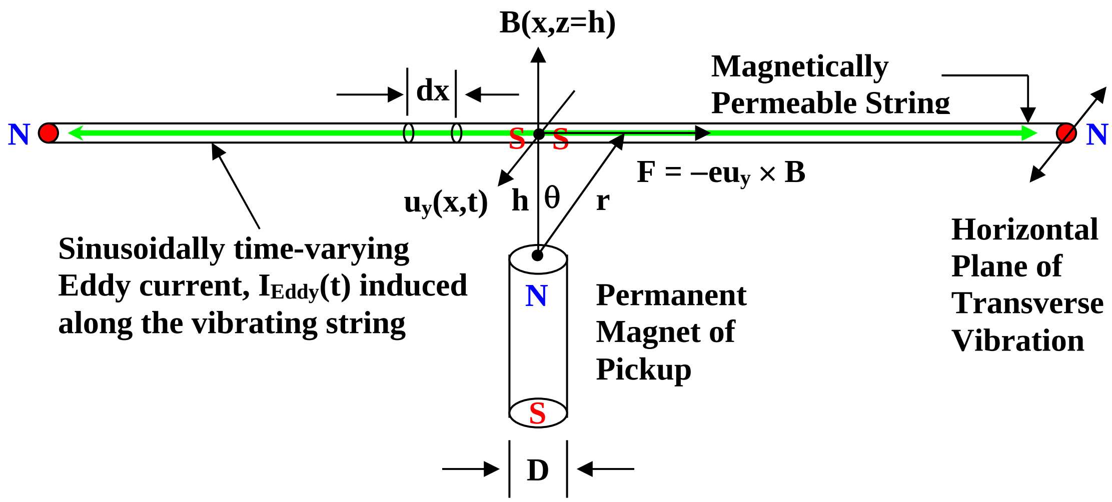
If the conductance, G_n of the quasi-free end support is large for the vibrational mode, n of the string, then the decay time, $\tau_n^{support}$ is short for this mode. Note that this decay time varies inversely as f_n^2, i.e. it has quite a strong frequency dependence, compared, e.g. to viscous air damping of the vibrating string.

Magnetic Damping Effects of Vibrating Strings on Electric Guitars

The pickups on electric guitars have permanent magnets which magnetize the strings of the guitar. The material used for electric guitar strings therefore must be magnetically permeable - typically steel, stainless steel on the three higher, small-gauge plain strings, and steel or stainless steel core wrapped with either steel or stainless steel, or pure nickel or a nickel alloy on the three lower, larger gauge wound strings.

Interestingly enough, the magnetic field of the pole piece of the pickup induces a *magnetic quadrupole field* (i.e. a linear {N-S} + {S-N} configuration) in the magnetically permeable strings of an electric guitar! As shown in the figure below, there are *two* South poles induced in the string in proximity to the North pole of the rod magnet. Due to the magnetic permeability (μ) properties of the electric guitar string, the string becomes a *magnetic flux tube*, *confining* the (majority of) magnetic field lines *inside* the string, thus connecting each South pole to its companion North pole, one at each end of the string!

Using the approximation that the magnetic field of the permanent rod magnet of the electric guitar pickup is equivalent to that of a point magnetic dipole (which it is not, due to its spatial extent), the magnitude of the magnetic field intensity, $|\mathbf{B}(\mathbf{r})|$ associated with a permanent magnetic pole of the guitar pickup decreases ~ as the *cube* of the perpendicular distance, z from the end of the permanent rod magnet, i.e. $|\mathbf{B}(\mathbf{r})| \sim 1/z^3$. {Permanent magnets, due to their finite physical size (and H x D aspect ratio), do indeed have higher-order magnetic field multipole moments (magnetic quadrupole, octupole, etc.), however the B-fields associated with each of these higher-order moments decrease (significantly) faster than $1/z^3$.}



As an electric guitar string vibrates in the magnetic field of a pickup, because the string is made of a electrically conducting metal, the free electrons in the metal of the strings in proximity to the magnetic poles of the guitar pickups experience a force, known as the Lorentz force, due to the fact the string (and thus the free electrons in the metal of the string) is moving with transverse velocity, $u_y(x,t)$ in the magnetic field $B(x,z=h)$ provided by the magnetic pole(s) of the guitar pickups, as shown in the figure below.

The Lorentz force is a vector force - it has both a magnitude and a direction, and is given by $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, where $q = -e$ is the electric charge of the electron, $\mathbf{v}$ is the velocity vector, $\mathbf{v}(x,t) = u_y(x,t)\mathbf{y}$, where $\mathbf{y}$ is a unit vector in the +y direction, and $\mathbf{B}(x,y,z) = B(x,y,z=h)\mathbf{z}$ near the magnetic pole of the pickup, and $\mathbf{z}$ is a unit vector in the +z direction (up). The direction of the Lorentz force acting on the free electrons is along the axis of the string (in the x direction), from application of the so-called “right-hand rule” in taking the cross product $\mathbf{u_y} \times \mathbf{B}$. Thus, the free electrons will collectively move along the axis of the string as a consequence of the Lorentz force acting on them in over the poles of the magnetic pickup(s). The “gas” of free electrons moving along the axis of the string due to the Lorentz force acting on them is an electrical flow of current!

Because the metal of the string has resistance, the “gas” of free electrons drift through the metal of the string along (i.e. parallel to) the axis of the string with a *terminal* velocity, $\mathbf{v_d} = v_d\mathbf{x}$. The resulting macroscopic current density, $\mathbf{J} = \sigma\mathbf{E} = (1/\rho)\mathbf{E}$ (units = Amperes/m^2) where σ is the conductivity of the metal (units = Ohm^{-1} m^{-1}), $\rho = 1/\sigma$ is the resistivity of the metal (units = Ohm-m) and $\mathbf{E}$ is the electric field in the metal, which arises from the Lorentz force acting on the free electrons, since $\mathbf{F} = q\mathbf{E} = q\mathbf{v} \times \mathbf{B}$. Thus, in our situation here, the electric field in the metal is given by

$$\mathbf{E}(x,y,z,t) = \mathbf{F}(x,y,z,t)/q = \mathbf{u_y}(x,t) \times \mathbf{B}(x,y,z)$$

Note that the electric field, **E** is anti-parallel to the Lorentz force, **F** for $q = -e$. The current density, **J** is

$$\mathbf{J}(x,y,z,t) = \sigma\mathbf{E}(x,y,z,t) = \sigma\mathbf{F}(x,y,z,t)/q = \sigma\{\mathbf{u_y}(x,t) \times \mathbf{B}(x,y,z)\}$$

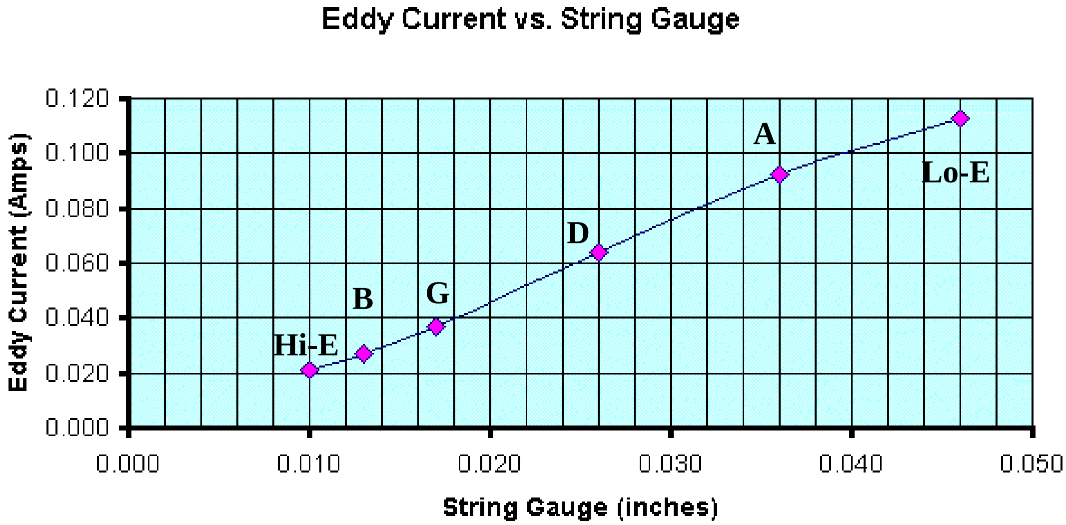
The macroscopic current $I = \mathbf{J} \cdot \mathbf{A_{string}}$, where $\mathbf{A_{string}} = \pi r^2_{string}\,\mathbf{n}$ = cross sectional area of the string, and **n** is a unit vector perpendicular to the cross sectional area of the string. This motionally-induced macroscopic current, arising from the Lorentz force acting on the free electrons in the metal of the vibrating strings of the guitar near the poles of the magnetic pickups of the guitar is in fact a type of Eddy current, I_{Eddy}!

Note that because of the nature of the cross product $\mathbf{v} \times \mathbf{B}$, in the Lorentz force equation, when **v** is *perpendicular* to **B**, this force is maximal. However, the Lorentz force *vanishes* when **v** and **B** are *parallel* (or anti-parallel) to each other. Thus, the magnitude of the Lorentz force $F = |\mathbf{F}| = q|\mathbf{v} \times \mathbf{B}| = qvB\sin\varphi$ where φ is the opening angle between **v** and **B**.

The amplitude of the Eddy current, $|I_{Eddy}|$ induced in the vibrating string of a guitar, for a pickup located at x_{pu}, is

$$|I_{Eddy}| = \sigma\omega|y_o||\sin(kx)|B(z=h)A_{string} = 4\sigma\pi^2 r^2_{string} f\,|y_o||\sin(kx_{pu})|B(z=h)$$

In the figure below, we show the magnitude of the Eddy current, $|I_{Eddy}|$ induced in each of the vibrating strings of a guitar, for open strings vibrating transversely in the plane of the strings, parallel to the body of the guitar, and perpendicular to the magnetic field(s) of the permanent magnets in the guitar pickups. As we have discussed in the 4th set of lecture notes on Fourier analysis, the neck pickup of an electric guitar is typically located at the *anti-node* of the 2nd harmonic, thus $x_{pu} = {}^5/_8 L_{scale}$. For steel strings on an electric guitar, steel has a conductivity, $\sigma = 1.1 \times 10^7\ \text{Ohm}^{-1}\ \text{m}^{-1}$. A typical magnetic field strength at the strings of a guitar, located ~ 4-5 mm above the poles of a guitar pickup is B(z = 4-5 mm) ~ 200 Gauss = 0.2 kilo-Gauss = 0.02 Tesla.



It can be seen that the magnitude of the Eddy current induced in the strings of an electric guitar ranges from I_{Eddy} ~ 20 mA for the high-E string to I_{Eddy} ~ 115 mA for the low-E string. The increase in I_{Eddy} with string gauge is due to the cross sectional area term, Astring = πr^2_{string} term, which is quadratic in the string gauge (= string diameter). However, I_{Eddy} also depends linearly on the string vibration frequency, f; thus the induced Eddy current in each of the strings is ~ linear with string gauge.

As a consequence of creating an induced time-dependent current, I_{Eddy} (t) flowing in the string, a time-dependent EMF (i.e. a voltage, or potential difference), ε(t) (units: volts) is also induced across the ends of the string, which is given by:

$$\varepsilon(x,y,z,t) = \int_{x=0}^{x=L} |\,E(x,y,x,t)\,|\,dx = \frac{1}{q}\int_{x=0}^{x=L} |\,F(x,y,x,t)\,|\,dx$$
$$= \int_{x=0}^{x=L} |\,u_y(x,t)\,||\,B(x,y,z=h)\,|\sin\varphi dx$$

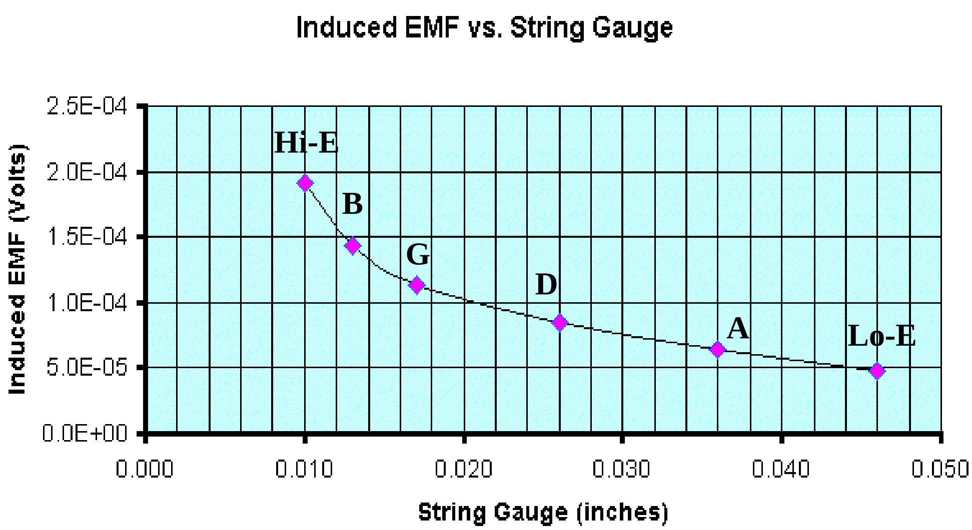
where $|u_y(x, t)|$ is the magnitude of the transverse velocity of the string at the point, x at time, t, and |B(x)| is the magnitude of the magnetic field intensity at the point, x on the string and sinφ is the opening angle between the plane of the transversely vibrating string and the direction of the magnetic field intensity, B(x) of the permanent magnet at the point, x on the string. Because the magnetic field intensity decreases so rapidly in moving away in any direction from the immediate vicinity of the pole of the permanent magnet of the pickup, located directly underneath the guitar string, the bulk of the contribution of the *integrand*, $|u_y(x,t)||B(x)|\sin\varphi dx$ of the above integral occurs in the x-region of the string in the vicinity of the pole of the permanent magnet.

Since the permanent magnets of a e.g. a single-coil guitar pickup typically have a diameter, D ~ 5 mm = 0.005 m = 5 × 10^{-3} m, the time-dependent EMF, ε(t) induced across the ends of the vibrating string is approximately $\varepsilon(t) \cong |u_y(x_{pu}, t)||B(z=h)|D$, for strings vibrating parallel to the plane of the guitar body. If the typical transverse displacement, y(x, t) of a guitar string has an amplitude, $|y_o| \sim 1.0$ mm = 0.001 m = 1×10^{-3} m, then e.g. for the fundamental mode of vibration of the high-E string, with $f_{hi\text{-}E} = 330$ Hz, then $y(x, t)|_{hi\text{-}E} = y_o \sin(k_{hi\text{-}E}\, x)\sin(\omega_{hi\text{-}E}\, t)$. The transverse velocity of the vibrating string, $u_y(x,t)|_{hi\text{-}E} = \partial/\partial t[y(x,t)|_{hi\text{-}E}] = \omega_{hi\text{-}E}\, y_o \sin(k_{hi\text{-}E}\, x)\cos(\omega_{hi\text{-}E}\, t)$. The neck pickup of an electric guitar is typically located at the *anti-node* of the 2nd harmonic, thus $x_{pu} = {}^5/_8 L_{scale}$, and since $\lambda_{hi\text{-}E} = 2L_{scale}$, $k_{hi\text{-}E} = 2\pi/\lambda_{hi\text{-}E} = \pi/L_{scale}$ and $\omega_{hi\text{-}E} = 2\pi f_{hi\text{-}E}$. The typical strength of the magnetic field intensity, B of a permanent magnet *at the pole of the magnet* is B(z=0) ~ 1.0 kilo-Gauss = 0.1 Tesla. The string of the guitar is typically located a few mm above the pole of the permanent magnet; the magnetic field strength |B(z = 4-5 mm)| at this point is typically ~ 20% of that at the pole of the permanent magnet, i.e. |B(z = 4-5 mm)| ~ 0.2|B(z=0)| ~ 200 Gauss = 0.2 kilo-Gauss = 0.02 Tesla.

Putting this all together, the typical *amplitude* of the time-dependent EMF, $|\varepsilon|$ induced across the ends of the vibrating string of an electric guitar, for an electric guitar pickup located at $x = {}^5/_8 L_{scale}$ from the nut of the guitar, with the string vibrating in the plane of the strings of the guitar (parallel to the body of the guitar, and perpendicular to the magnetic field of the poles of the guitar pickup) is given by:

$$
\begin{aligned}
|\varepsilon|_{hi-E} &\cong u_y(x = \tfrac{5}{8} L_{scale})\Big|_{hi-E} B(x = \tfrac{5}{8} L_{scale}) D \\
&= 2\pi f_{hi-E}\, |y_o| \sin(\tfrac{5}{8}\pi) B(x = \tfrac{5}{8} L_{scale}) D \\
&= 2\pi * 330Hz * 0.001m * \sin(\tfrac{5}{8}\pi) * 0.02Tesla * 0.005m \sim 0.2mV
\end{aligned}
$$

The EMF, $|\varepsilon|$ induced across the ends of each string of an electric guitar is shown below.



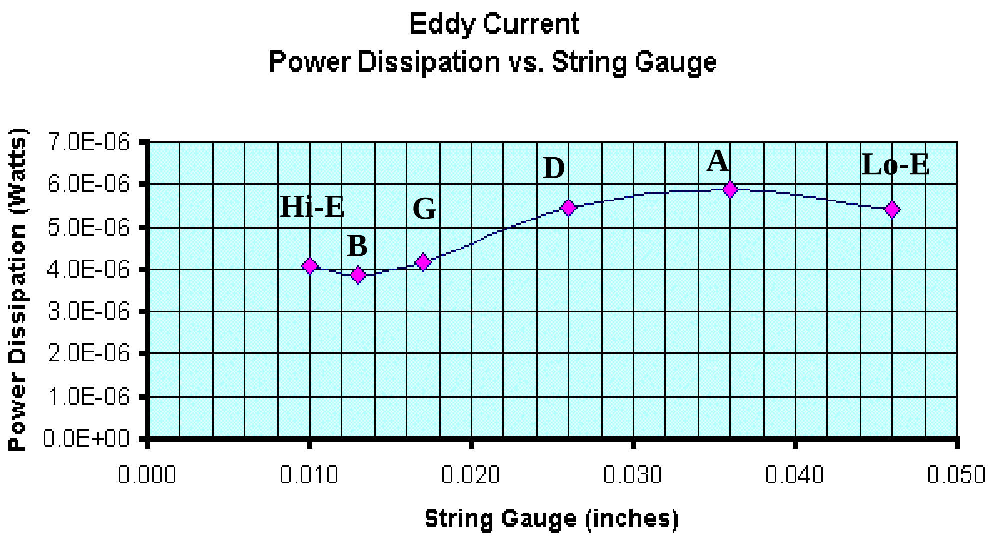
If the strings vibrate *perpendicular* to the plane of the guitar body - parallel to the magnetic field of the poles of the magnetic pickups of the guitar, *no* EMF is induced across the ends of the vibrating strings of a guitar - because $\sin\varphi = 0$ in this situation.

It can be seen from the above figure that the magnitude of the induced EMF, $|\varepsilon|$ is linear with frequency, f; thus the magnitude of the induced EMF, $|\varepsilon|$ has a quadratic dependence on string gauge/string diameter.

Because the metal of the vibrating string has a finite (but very small) electrical resistance, R_{string} associated with it, the electrical energy associated with the induced Eddy current(s) flowing in the string is dissipated - ultimately, it is converted into heat energy. The instantaneous power loss, $P_{Eddy}(t)$ due to the induced Eddy current(s) flowing in the string is given by:
The (peak) power dissipation, or power loss due to Eddy currents in each of the strings of the

$$P_{Eddy}(t) = \frac{dE_{Eddy}(t)}{dt} = \varepsilon(t) * I^*_{Eddy}(t) = \frac{|\varepsilon(t)|^2}{R_{string}}$$

guitar is shown in the figure below, for the same values of parameters as used in creating the above two figures.



Note that the (peak) power dissipation is approximately constant with string gauge/fundamental frequency, P_{Eddy} ~ 4-6 μW, and depends quadratically on B(z = h), since both I_{Eddy} and $|\varepsilon|$. Note also that the time-averaged, or root-mean-square (rms) power dissipation, <P_{Eddy}> is half of the peak power dissipation value.

The vibrational energy of the guitar strings is slowly dissipated via magnetically-induced Eddy current power losses in each of the strings. The vibrations of the guitar string are thus *magnetically damped*. The amplitude of the string vibrations slowly decays (exponentially) with time due to magnetic damping of the strings by the strings vibrating in the magnetic field(s) of the guitar pickups, with a characteristic time constant, τ_{Eddy} which is given by:

$$\tau_{Eddy} = \frac{2\mu}{b}\left[1 + \left(\tfrac{b}{2\mu\omega}\right)^2\right]^{1/2} \qquad where \qquad b = \pi r^2_{string}\sigma B^2(z=h)\sin^2\varphi$$

Note that the magnetic damping time constant, τ_{Eddy} varies quadratically with the strength of the magnetic field at the location of the strings vibrating over the magnetic poles of the pickup (z=h). Note also that τ_{Eddy} has only a weak dependence on the frequency of vibration, f. Numerically, the above formula gives a characteristic damping time of τ_{Eddy} ~ 3-4 seconds for the strings of a typical electric guitar.

The transverse displacement, $y_n(x,t)$ of a given mode, n = 1, 2, 3, 4, ... of vibration of the guitar string, for fixed end-fixed end supports at x = 0 and x = L, with magnetic damping present is given by:

$$y_n(x,t) = y_o \exp\left(-t/\tau_{Eddy}\right)\sin(k_n x)\sin(\omega_n t)$$

Thus, magnetic damping of the strings of an electric guitar, like viscous air damping, affects the sustain of an electric guitar. The height of the pickups on an electric guitar is usually adjustable. For strings that are typically located ~ a few mm above the magnetic poles of the guitar pickups, magnetic damping of the string vibrations is not very significant - it is comparable to, or less than the damping associated with other physical processes we have discussed above. However, if the pickup height is adjusted so as to obtain the maximum possible output signal from the pickup(s), by significantly reducing the distance between the guitar strings and the magnetic poles of the pickup(s), since the decay time constant associated with magnetic damping, τ_{Eddy} depends on the square of the magnetic field strength at the strings of the guitar, $B^2(z=h)$, then magnetic damping of the string vibrations in this situation can in fact become the dominant string damping mechanism, so much so that the sustain of the electric guitar is drastically reduced, especially since the magnetic field strength, B(z) varies as ~ $1/z^3$ from the poles of the permanent magnets of the pickups.

A magnetic force, $\mathbf{F}_{Eddy}(t)$ also exists on the vibrating string, which arises due to the motionally-induced Eddy current flowing in the vibrating string of an electric guitar in the magnetic field of a pickup on the guitar.

The direction of this force is given by the so-called cross-product of the direction of the current, $I_{Eddy}(t)$ with the direction of the magnetic field intensity, B at the string. By use of the so-called "right-hand rule" associated with taking the vector cross product, $d\mathbf{F}_{Eddy}(t) = I_{Eddy}(t)\, \mathbf{dx} \times \mathbf{B}(x, z=h)$, where **dx** is along the axis of the string, in the direction of the flow of current. Thus, the direction of the instantaneous magnetic force acting on the string is always opposite to the direction of the instantaneous transverse velocity of the string, $u_y(x, t)$ - i.e. the magnetic force, $dF_{Eddy}(t)$ *opposes* the transverse motion of the vibrating string (this is simply a manifestation of Lenz's law).

Note that if the entire length of the vibrating string were immersed in a strong, uniform magnetic field, oriented perpendicular to the plane of the vibrating string, the string would experience a viscous damping force, analogous to a guitar string vibrating in a viscous fluid, such as honey! Thus, when only a small section of the guitar string in the immediate vicinity of the guitar pickup(s) is affected by this magnetic damping effect, just this small segment of the guitar string experiences this viscous magnetic damping force.

The magnitude of the incremental force contribution, $dF_{Eddy}(t)$ acting on an infinitesimal length, dx of the string due to the interaction of the induced Eddy current with the magnetic field, B(z=h,x) of the magnetic pole of the pickup at the location of the string is given by:
The total force, $F_{Eddy}(t)$ is obtained by summing up/integrating the incremental force

$$dF_{Eddy}(t) = I_{Eddy}(t)B(z=h,x)\sin\varphi \; dx = \frac{\varepsilon(t)}{R_{string}} B(z=h,x)\sin\varphi \; dx$$

contributions over the entire length of the vibrating string:

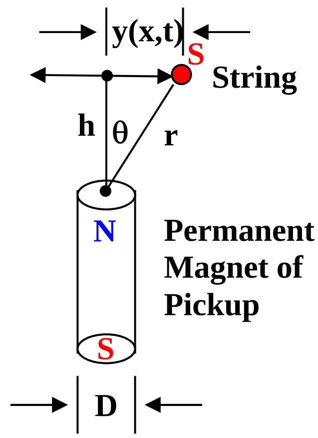
$$F_{magnetic}(t) = \int_{x=0}^{x=L} I_{Eddy}(t)B(z=h,x)\sin\varphi \; dx = \int_{x=0}^{x=L} \frac{\varepsilon(t)}{R_{string}} B(z=h,x)\sin\varphi \; dx$$

Again, because the magnetic field intensity decreases so rapidly in moving away in any direction from the immediate vicinity of the pole of the permanent magnet of the pickup, located directly underneath the guitar string, the bulk of the contribution of the integrand, $(\varepsilon(t)/R_{string})B(x)\sin\varphi\, dx$ of the above integral occurs in the x-region of the string in the vicinity of the pole of the permanent magnet. Since $\varepsilon(t) \cong u_y(x, t)B(x)\sin\varphi\, D$, then $F_{magnetic}(t) \cong u_y(x_{pu}, t)B^2(z=h,x_{pu})\sin\varphi\, D/R_{string}$. Thus, since $u_y(x, t) = \omega y_o \sin(kx)\cos(\omega t)$, this time-dependent magnetic force acting on the vibrating string depends linearly on the frequency, f of the vibrating string and the amplitude, y_o of the vibrating string, but depends quadratically on the magnetic field intensity at the location of the strings over the guitar pickup(s).

Even if the strings of an electric guitar are initially plucked horizontally, i.e. in the plane parallel to the strings of the guitar, the plane of the transverse displacement, y(x,t) of each of the strings will naturally begin to *precess* in both the horizontal and vertical direction with respect to the plane of the strings, with an angular precession frequency, Ω_p that is much slower than the frequency, f of the fundamental mode of vibration of the string. This precession of the plane of polarization of the string arises due to a small non-linear effect associated with the transverse vibrations of the string affecting the tension, T in the string.

This magnetic force, $F_{Eddy}(t)$ is maximum (zero) when the plane of the transverse displacement, y(x,t) is horizontal (vertical) - i.e. parallel (perpendicular) to the plane of the strings, where $\varphi = 90^o$ (0^o) and $\sin\varphi = 1$ (0), respectively.

There also exists ***another*** magnetic force due to the pickup(s) acting on the string(s) of an electric guitar. Since the strings of an electric guitar are magnetically permeable, the portion of the string(s) in proximity to the magnetic poles of the pickup(s) becomes magnetized, thus the string behaves as a linear magnetic quadrupole, as discussed above. Thus, when a (magnetized) electric guitar string undergoes transverse vibrations, with transverse amplitude, y(x, t) at a height, h above the permanent magnetic pole of a pickup, the separation distance, r between the (nearest) permanent magnetic pole of the electric guitar pickup (located directly under the equilibrium position of this string) and the string itself varies, as shown in the figure below:



Thus, the separation distance, r between vibrating string and permanent magnetic pole of a guitar pickup on an electric guitar varies with time, as

$$r(t) = y(x,t) / \sin\theta$$

However, the typical magnitude of transverse displacement, y(x,t) from the equilibrium position of the string is $|y_o| \sim$ 1-2 mm, whereas the typical diameter of the permanent magnets used in electric guitar pickups is D ~ 4-5 mm. Thus, the string vibrates mostly over the pole of the pickup, a distance z = h above it, where the magnetic field, B(z=h) from the pole of the permanent magnet is fairly constant with transverse amplitude, as long as $|y(x,t)| < D$.

The attractive force between the magnetically permeable string and the pole of the permanent magnet is given (approximately) by:

$$F_{mag}(r) \cong -\frac{(2r_{string}D)}{2\mu_o} B^2(r) = -\frac{r_{string}D}{\mu_o} B^2(r)$$

where $2r_{string}D \cong$ area of magnetized string over the magnetic pole of the electric guitar pickup, μ_o = magnetic permeability of free space = $4\pi\times10^{-7}$ Henry/meter. This attractive magnetic force, for "horizontal" transverse vibrations (i.e. parallel to the plane of the strings of the electric guitar), will be approximately constant with separation distance, r if the equilibrium position of each of the strings is well-aligned with the corresponding magnetic pole of the pickup - sometimes this is not the case! If there is a misalignment of a string and its corresponding magnetic pole on the pickup, then B(r) will not be ~ constant, and hence $F_{mag}(r)$ will vary with the horizontal transverse vibrations of the string. Since the "fringe" portion of the magnetic field of a dipole magnet varies as $B(r) \sim 1/r^3$, then the magnetic force in this situation will vary as $F_{mag}(r) \sim 1/r^6$!

When the plane of polarization of the transverse vibration of a string on an electric guitar is vertical - i.e. perpendicular to the plane of the strings, then B(r) also varies significantly (since the string vibrations are now parallel to the axis of the permanent magnet), and hence so does $F_{mag}(r)$.

Thus, there are time-dependent magnetic forces acting locally on the strings of an electric guitar due to the interaction of the magnetic field(s) of the pickups of the electric guitar with the magnetized strings of the electric guitar.

If the pickup height is adjusted such that the strings are very close to the magnetic poles of the pickup, the magnetic force(s) acting on the string(s) can become large enough such that they noticably interfere with the natural vibration(s) of the strings - causing a noticable shift in the pitch (i.e. frequency) of the vibrating string - lowering it, and also altering the harmonic content of the string in a time-dependent manner. If the string height is adjusted such that they are too close to the pickups of the electric guitar, this can create an unpleasant, time-dependent warbling-type tone output from the guitar.
These warbling-type tones are also known as wolf-tones. Stratocaster guitars, having 3 pickups, are particularly susceptible to this problem, if the pickups are adjusted such that they are too close to the strings of the guitar, and especially so for notes plated on the G and B strings.

Overall Damping Time Constant

We have discussed a number of physical processes which cause damping of the vibrations of strings on a guitar - viscous air damping of the string, internal damping of the string vibrations, damping effects due to build-up of foreign material ("grunge") on the strings of the guitar, magnetic damping effects due to the Eddy currents induced in the metal strings of the electric guitar vibrating near the magnetic poles of the guitar pickups, and magnetic damping due to B-H hysteresis loss minor cycle-type effects due to the magnetically permeable strings of the electric guitar vibrating in the magnetic field(s) of the guitar pickups.

Because the damping of standing waves on a guitar occurs in the argument of exponential terms, e.g.

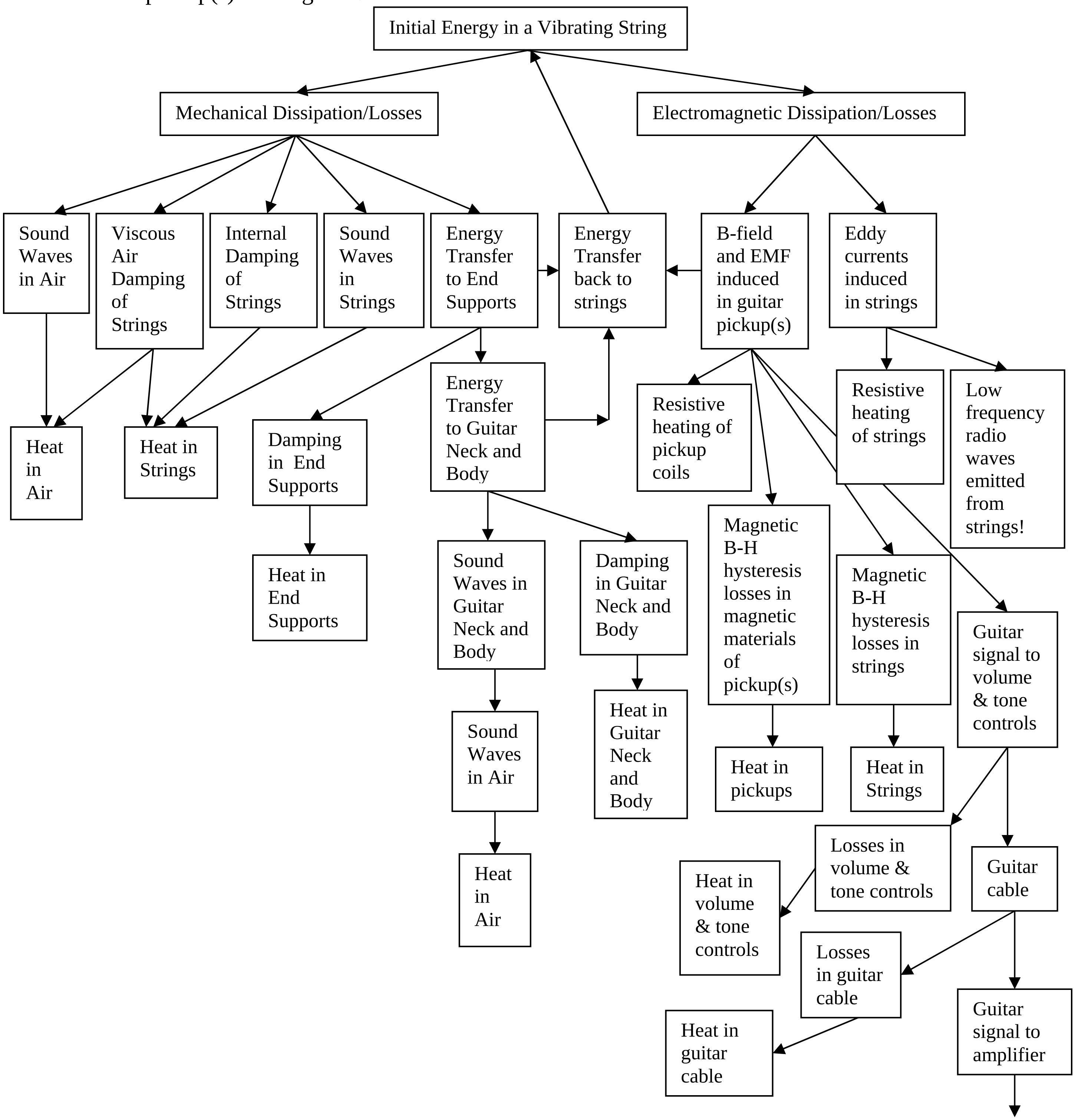
$$
\begin{aligned}
y(x,t) &= y_o e^{i(\omega t 0 kx)} e^{-t/\tau_1} e^{-t/\tau_2} e^{-t/\tau_3} e^{-t/\tau_4} \ldots\ldots e^{-t/\tau_n} \\
&= y_o e^{i(\omega t 0 kx)} e^{-t[1/\tau_1 + 1/\tau_2 + 1/\tau_3 + \ldots\ldots + 1/\tau_n]}
\end{aligned}
$$

For n damping time constants, then from the above expression we see that an overall damping time constant can be defined as

$$
\frac{1}{\tau} = \sum_{i=1}^{i=n} \frac{1}{\tau_i} = \frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_3} + \ldots. + \frac{1}{\tau_n}
$$

Energy Flow In a Vibrating String

The following "flow chart" diagram shows what happens to the initial energy associated with a vibrating string on an electric guitar. The initial energy in the string is dissipated by two basic categories of processes - mechanical losses and electromagnetic losses, the latter associated with the pickup(s) of the guitar.



Ultimately, all of the initial energy associated with the vibrating strings of the guitar winds up as heat, somewhere in the universe. Most of this initial energy is dissipated within the guitar itself. Only a small fraction of the initial energy of the vibrating string is converted into an electrical signal, which is then sent to the guitar amplifier. The voltage amplitude of the signal output from an electric guitar is typically on the order of $|V_{signal}|$ ~ 100 mV = 0.100 Volts (after initial, fast transients have died out). The (peak) power associated with this signal is $P_{signal} = |V_{signal}|^2/R_{load}$, where R_{load} is the magnitude of the input impedance of a guitar amplifier, typically 1 Meg-Ohm = 10^6 Ohms (note that this is also the nominal/industry standard input impedance of oscilloscopes). Thus, the peak power associated with a guitar signal is typically $P_{signal} \sim 10^{-8} = 10 \times 10^{-9}$ = 10 nano-Watts!

Non-Linear Effects in Vibrating Strings

Up to now, our discussions of various effects on string vibrations have all been associated with so-called *linear* processes. We now wish to discuss some non-linear effects of string vibrations. The first non-linear effect is a simple dependence of the natural frequency, f_n of the mode of vibration, n of the string upon its amplitude of vibration, $y_n(x,t)$.

Non-Linear Effect of String Tension

For a string of cross sectional area $A_{string} = \pi r2_{string}$, length, L, Young's modulus, Y_{string}, and material density, ρ_{string} stretched with tension, T between rigid/fixed end supports, located at x = 0 and x = L, the equation of motion for transverse waves on this string is given by:

$$\mu \frac{\partial^2 y(x,t)}{\partial t^2} = \rho_{string} A_{string} \frac{\partial^2 y(x,t)}{\partial t^2} = T \frac{\partial^2 y(x,t)}{\partial x^2}$$

where the x-direction is along the axis of the string and the y-direction is transverse to the axis of the string, e.g. in the horizontal plane - thus we say the string vibration is *polarized* in the (x,y) plane. The normal modes of vibration of transverse standing waves on the string are given by:

$$y_n(x,t) = |\, y_{on} \,| \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

where $\omega_n = 2\pi f_n = k_n v_x = (n\pi/L)(T/\mu)^{1/2} = (n\pi/L)(T/\rho_{string}A_{string})^{1/2}$.

For a given mode of vibration of the string, n with transverse displacement, $y_n(x,t)$ the string is displaced from its equilibrium (i.e. zero-amplitude) configuration. The length of the string increases slightly, by an amount:

$$\delta L_n(t) = \int_{x=0}^{x=L} \left[1 + \left(\frac{\partial y_n(x,t)}{\partial x} \right)^2 \right]^{1/2} dx - L \cong \frac{n\pi^2 \,|\, y_{on} \,|^2}{8L^2} \left(1 - \cos(2\omega_n t)\right)$$

The string tension, T therefore increases by an amount, $\delta T_n(t) \cong Y_{string}A_{string}(\delta L_n(t)/L)$:

$$\delta T_n(t) = Y_{string} A_{string} \left[\frac{n\pi^2 \,|\, y_{on} \,|^2}{8L^3} \right] \left(1 - \cos(2\omega_n t)\right)$$

which has both a steady and an oscillating, time-dependent term. If the (overall) transverse displacement, y(x,t) involves many modes of vibration, their contributions to the (overall) string tension are simply summed, because all cross-terms vanish during the integration in the above string-length formula. Then the overall tension increase, δT(t) is given by:

$$\delta T(t) = \sum_{n=1}^{n=\infty} \delta T_n(t) = Y_{string} A_{string} \sum_{n=1}^{n=\infty} \left[\frac{n\pi^2 \mid y_{on} \mid^2}{8L^3} \right] \left(1 - \cos\left(2\omega_n t\right)\right)$$

The predominant effect of this quasi-steady increase in string tension, δT(t) is to raise the vibrational frequencies of *all* modes of vibration by the same factor - the fractional shift in frequency is given by:

$$\frac{\delta\omega(t)}{\omega} = \frac{\delta f(t)}{f} = \left(\frac{Y_{string} A_{string}}{L} \right) \left(\frac{\delta T(t)}{T} \right)$$

However, an additional shift in frequency for each mode arises because of the time-dependence of the tension. This causes a further (small) frequency shift for each mode of vibration,

$$\frac{\delta\omega_n(t)}{\omega_n} = \frac{\delta f_n(t)}{f_n} = 2\left(\frac{Y_{string} A_{string}}{L} \right) \left(\frac{\delta T_n(t)}{T} \right)$$

which destroys the harmoniticity of the modes of vibration - i.e. we no longer have the precise relation between the harmonics and the fundamental, $f_n = nf_1$, n = 2, 3, 4, ... !

Thus, when a string of a guitar is plucked with a large initial transverse amplitude, as the string excitation decays with time, there is a slow glide in the pitch/frequency of vibration back toward its small-amplitude value. The degree of this pitch-glide, or “twang” depends on the square of the amplitude, the length, L tension, T and Young’s modulus of the string. Shorter-scale guitars with light gauge, low-tension strings, or guitars with de-tuned strings are more susceptable to such pitch-glide/twang effects.

Non-Linear Effect of String Tension on a Non-Rigid End Support

Because of the fact that the string tension, T is not constant with time, if one (or both) of the end supports of the strings on a guitar - i.e. the bridge or the nut are not perfectly rigid, then since the strings pass over these not-completely rigid end supports at an angle, then the oscillating tension T(t) causes a transverse motion of these end supports, vibrating at a frequency, $2f_n$. This can excite the vibrational mode 2n, or in fact interfere (constructively or destructively, depending on the exact nature of the motion of the not-completely rigid end support), if the 2n vibrational mode is already present.

Since the angle at which the string passes over the end support varies as f_n, then there is also the possibility of exciting (and or interfering with) the vibrational mode 3n, with frequency $3f_n$.

Non-Linear Effect of String Tension on a Polarization of the String Vibration

If a vibrational mode, n of a string has initial polarization in the y-direction, vibrating in the (x,y) plane at frequency, f_n then this creates an oscillating/time-dependent tension, T(t) which has frequency, $2f_n$. The oscillating tension creates a time-dependent force on the vibrating string, but in the (x,z) plane, that is proportional to

$$|y_{on}|^2 \left(\cos\omega_n t + \cos 3\omega_n t\right)$$

This process is known as *parametric amplification*, because one of the parameters of the system (in this case the tension, T) provides the driving force. As the vibrational mode with z-polarization grows, it either takes energy from, or feeds back energy to the vibrational mode with y-polarization, depending on their relative phases.

The (initial) rate of rotation/precession of the polarization of the string Ω_n (units = radians/sec), for a single mode of vibration, n on a string, is given by:

$$\Omega_n = \alpha\left(\frac{Y_{string} A_{string}}{T}\right)\left(\frac{|y_{on}|^2 |z_{on}|^2}{L^2}\right)\omega_n$$

where α is a numerical factor of order of unity. Typically, the precession frequency of a guitar string, $f_n^{prec} = \Omega_n / 2\pi \sim 1$ Hz, though as can be seen from the above expression, it clearly depends on the string material, the string tension, T, the length of the string, L, the initial amplitude(s) in the y- and z- directions, and on the mode of vibration, n. Note that as the excitation of the string vibration decays with time, the rate of precession of the string also decreases with time.

Exercises:

References for Waves and Further Reading:

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