Chapter 2
The Vibrating String

The subject of this text is music and color. Music is produced by musical instruments, some occurring naturally – such as the songs of birds – and others produced by man-made instruments – such as stringed instruments, wind instruments, and the percussive instruments of drum sets. Color is produced by sources of light such as natural sunlight and by man-made sources such as the floodlights for a stage.

Essentially, music and color are *subjective* manifestations of the corresponding *objective* physical phenomena – sound and light, respectively. Both sound and light are examples of **wave phenomena**. If we can understand the nature of waves along with the multitude of phenomena associated with waves, we will become more aware of much of the richness of our human experiences with sound and light and hence music and color.

There are many types of waves. We can observe the wave nature of some types of waves with our own eyes – such as waves along a vibrating string or waves on the surface of the ocean. On the other hand, the wave nature of many important waves are invisible; examples are sound waves and light waves. It is therefore reasonable for us to begin our study with waves along a string – the fundamental component of all stringed musical instruments.

2.1 Waves Along a Stretched String

Suppose that we have a long string and stretch it. The string is depicted as the uppermost solid line in Fig. 2.1. The **tension** in the string keeps the string straight. Next, we disturb the string by pulling the string upward a bit at a particular point along the string. The shape of the disturbance is a small triangle. What will happen next? The disturbance will move along the string as shown in the figure at one milli-second (1 ms) intervals: We set the time t equal to 1, 2, 3, 4, and 5 ms. Each of the vertical dotted lines marks a position along the string at a sequence of one-meter (1 m) intervals. We note that after each 1-s interval, the disturbance progresses a distance

L. Gunther, *The Physics of Music and Color*, DOI 10.1007/978-1-4614-0557-3_2,

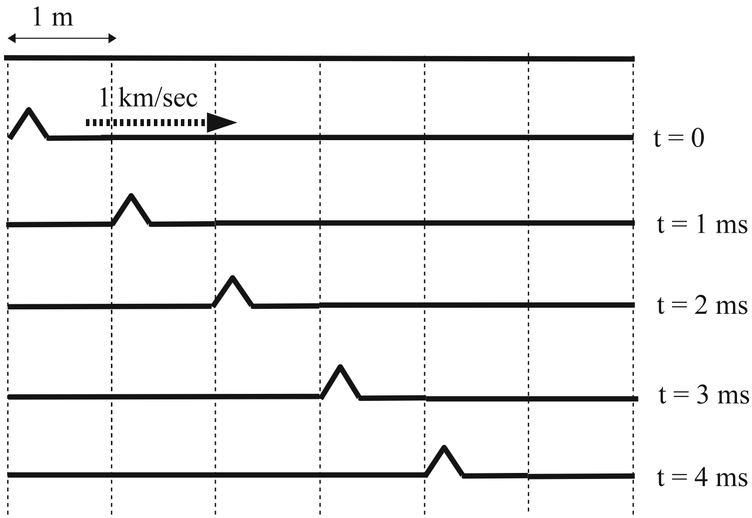

Fig. 2.1 A pulse traveling down the length of the stretched string

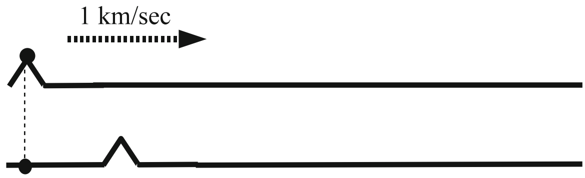

Fig. 2.2 The motion of a point – marked by a dot along the string

1 m to the right. Thus, the disturbance moves at a speed of 1 m/ms. This value is equivalent to 1,000 m/s. Note that this speed is quite large; in common units it is one kilometer per second (1 km/s), which is equivalent to 0.6 miles/s. Nevertheless, this value is close to the speed of a disturbance moving along a typical violin string.

A localized disturbance of this sort is called a **pulse** and is a simple example of **wave propagation**. The speed of the pulse is called the **wave velocity**. Later on in the chapter, we will investigate what determines the wave velocity for a stretched string.

We can easily show that the string itself does not move at a speed of 1 km/s, or 1,000 m/s, nor does the string itself move to the right. In order to see this, suppose we focus our attention on a single point along the string, say the point marked with a dot, shown in Fig. 2.2. We note that while the pulse is moving to the right, this point along the string has moved *downward*! We say that the wave is **transverse**, here meaning *perpendicular*. Suppose next that the height of the pulse is one millimeter (1 mm) (not drawn to scale above). Then the average speed of this point is 1 mm/s, a value much less than the wave velocity.

How can we account for the motion of the pulse? Think of the old familiar "telephone game," wherein we have a string of people. The first person whispers

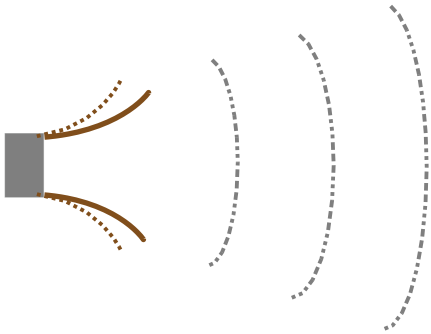

Fig. 2.3 Schematic of a loudspeaker

a message to the second person. The second person whispers the perceived message on to the third person, and so on. The last person announces the message received and the first person reveals the original message. One hopes that the message will not be garbled!

In the case of the string, the initial material of the pulse along the string pulls upward on the neighboring string material. The neighboring material pulls upward on its neighboring material, and so on, leading to the propagation of the pulse.

How does this description relate to other types of waves? The most important wave in the context of music is of course a sound wave – the focus of Chapter 3, THE VIBRATING AIR COLUMN. Sound waves can propagate through a variety of media – such as air or water or a solid. Let us try to produce such a wave: Imagine what would happen if you were to move your hand forward suddenly. You would compress the air immediately in front of your hand. That compressed region of air would compress the air immediately in front of it. This process will continue as in the case of a pulse propagating along a stretched string. You will have produced a **sound pulse**. The wave is said to be **longitudinal**, meaning that the motion of the air is along the same direction as the direction of propagation of the disturbance. Unfortunately, you cannot move your hands fast enough to hear this pulse.

If you were to be able to move your hand forward and backward at a rate that exceeds 20 times per second, you would in fact produce an audible sound. Your hand would be acting essentially like a loudspeaker, as shown in Fig. 2.3. At the left, we see the gray cone of the loudspeaker moving forward and backward. There are two positions shown – one as a pair of solid brown curves, the other as a pair of dotted brown curves. The sequence of three dotted pseudo-vertical curves represent the sound wave traveling through the air.

2.2 A Finite String Can Generate Music!

Consider now a guitar string strung on a guitar. The string considered in the previous section was assumed to be infinite; this string is finite with ends that are held fixed. See the uppermost line segment in Fig. 2.4, where we represent a string of length

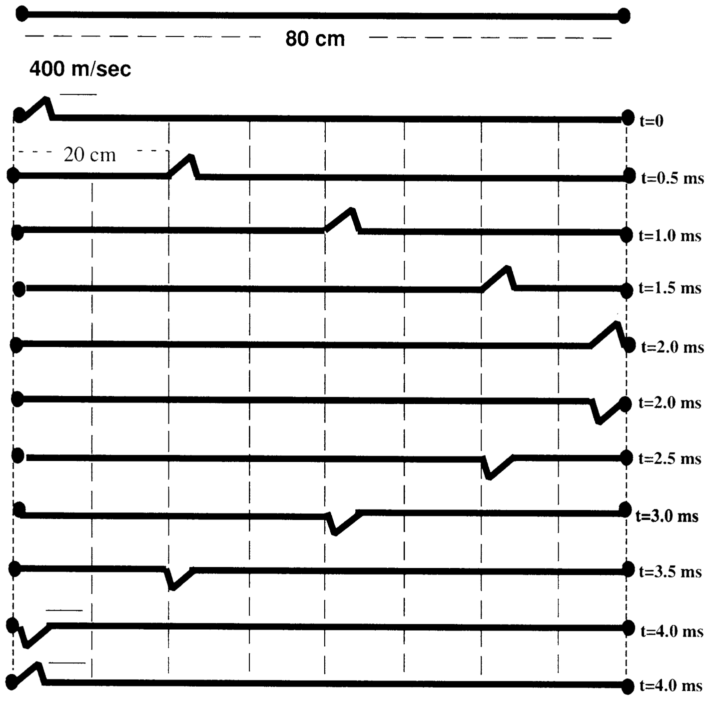

Fig. 2.4 A pulse traveling back and forth along a string with fixed ends

$l = 80\,\text{cm}$. We will assume that the wave velocity is $v = 400\,\text{m/s}$. Imagine what would happen to a pulse that is sent down the string, starting at one end, as in Fig. 2.4. The width of the pulse is exaggerated – the width is understood to be much less than a centimeter, so that it can be ignored in the calculations below.

Let us determine how long it will take for the pulse to reach the opposite end. We will use the relation

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} \quad \text{OR} \quad \text{Time} = \frac{\text{Distance}}{\text{Speed}}. \tag{2.1}$$

We will carry out the calculation using symbols – t for time, l for distance, and v for speed. We must be careful when we are given quantities that use different units for a given quantity. This issue is exemplified by the current situation, where we have a distance of 80 cm and a speed of 400 m/s. Thus both the centimeter and the meter are used for the dimension of length. In order to use (2.1), we must use the same unit of length for both quantities. We will choose to use the meter for both, recognizing that we could also use the centimeter for both without any error.

Since 1 m = 100 cm, the distance is 0.80 m. We then obtain

$$t = \frac{l}{v} = \frac{0.80\text{ m}}{400\,\text{m/s}} = 0.0020\,\text{s} = 2.0\,\text{ms}. \tag{2.2}$$

We note in the figure that the pulse reaches the opposite end in 2.0 ms. The pulse is then reflected back to the left along the string.

Look closely at the shape of the reflected pulse. Notice that the shape of the pulse is "reversed" in two ways: First, the original pulse approached pointing upward; the reflected pulse is pointing downward. Second, notice that the original pulse is steeper on the right side compared to the left side; on the other hand, the reflected pulse is steeper on the left side.

What will happen next? The pulse will reach the left end and be reflected back to the right. The same reversals as above will take place once again. The pulse is reversed from pointing downward to pointing upward; the steeper edge is reversed from being steeper on the left side to being steeper edge on the right side. The end result is a pulse that is exactly the same as the original pulse! The time for the round trip will be $2 \times 2.0\,\text{ms} = 4.0\,\text{ms}$.

Such a round trip is generally referred to as a **cycle**. Ultimately, the pulse will move back and forth, with one round trip every 4.0 ms. This time interval is called the **period**, with the symbol T. Thus,

$$T = \frac{2l}{v} = \frac{2(0.80)}{400} = 4 \times 10^{-3}\,\text{s} = 4\,\text{ms}. \tag{2.3}$$

The number of cycles per unit time is called the **frequency**, with the symbol f. In the current case, we have

$$f = \text{one cycle per 4 ms} = \frac{1\ \text{cycle}}{4 \times 10^{-3}\,\text{s}} = 250\ \text{cycles per second} \equiv 250\,\text{cps}. \tag{2.4}$$

An alternative term for the cycle per second as a unit of frequency is the **Hertz**,[1] which is abbreviated as Hz. Thus, **one cycle per second** = 1 cps = 1 **Hertz** = 1 Hz.

Note that the frequency and the period are inverses of each other:

$$f = \frac{1}{T}. \tag{2.5}$$

In the above case, $250\,\text{Hz} = 1/(4\,\text{ms})$.

One should note that there are many ways that the string could be excited. The most important example for a guitar is the **pluck**, which is shown in Fig. 2.5. The pluck is produced by pulling the string aside at one point and then releasing it from the rest. The figure shows the subsequent motion of the string.

We note that the time for a full cycle, the period T, is again 4 ms. The corresponding frequency is 250 Hz.

[1]Named after Heinrich Hertz (1857–1894). Hertz was a great physicist who first demonstrated the existence of electromagnetic waves, which will be discussed later in this book.

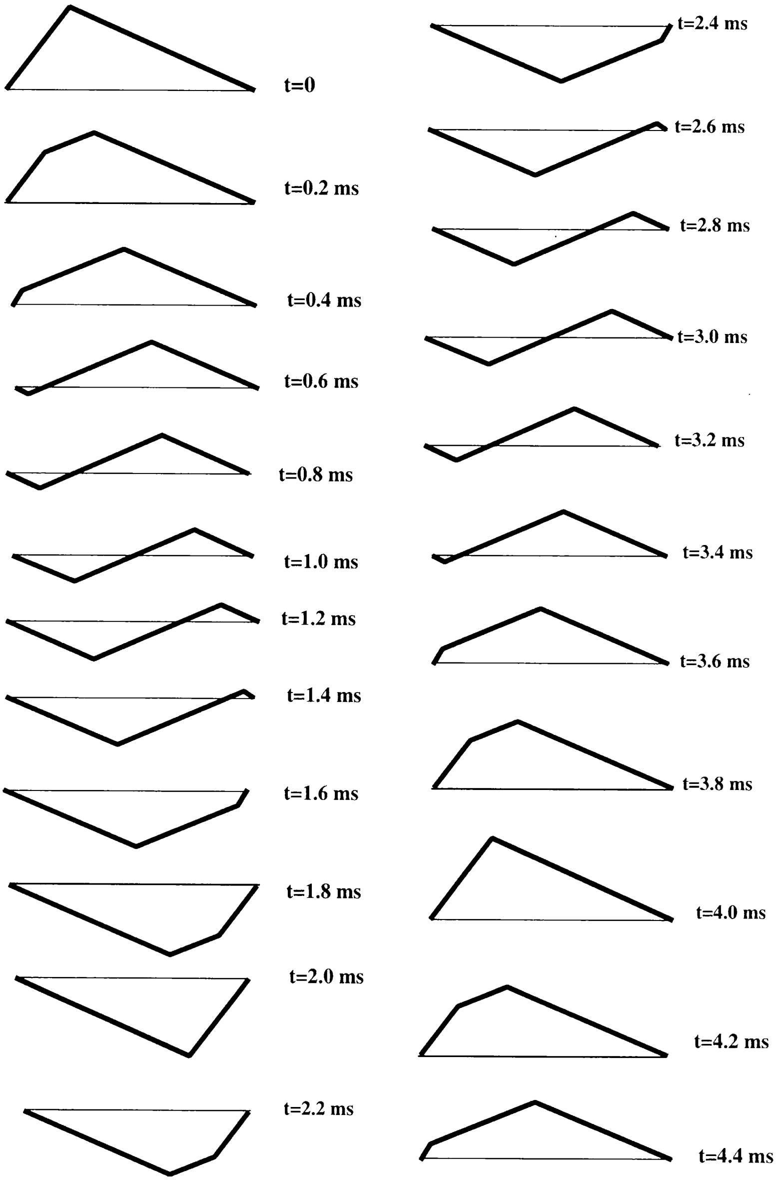

Fig. 2.5 The progressive wave along a plucked string

2.3 Pitch, Loudness, and Timbre

If you pluck a string, a sound is produced. You can identify several attributes of that sound. There is a definite **pitch**. Pitch designates the musical note to which the string is tuned. For example, the so-called G string of the violin (which is tuned to the G below middle C on the piano) produces the pitch G. If you loosen the string, by turning the tuning peg, you will immediately notice that the pitch will change – it will become lower. If you tighten the string, the pitch will become higher.

A second attribute of the sound is its **loudness** . By giving the string a bigger pull when you pluck it, you can produce a louder sound. Furthermore, the loudness decreases after the initial pluck, until the sound is inaudible.

The third attribute is what we identify with the quality of the sound produced by the particular instrument – the **timbre**. Timbre is one of the factors that enables you to distinguish the G played on the violin from an equally loud G played on a piano or a trumpet or any other instrument. You can vary the timbre of the plucked string itself by changing the point at which you pluck as follows: first pluck the string near its center and listen carefully to the quality of the sound. Then pluck the string very near one end, trying to produce the same loudness. The pitch will be the same but there will be a slightly different timbre to the sound. When plucked near the end, the resulting sound has a slight high-pitched ring or "twang," which is not present in the sound produced by plucking near its center. Similarly, if a narrow pulse is cycling back and forth along the string, a sound will be produced having the same pitch but different timbre. A bowed string produces a wave that moves back and forth the length of the string with a different characteristic shape; yet again, we will hear a sound with the same pitch.[2]

We have not been very precise, at this point in defining pitch, loudness, and timbre. To be more precise, you must first understand what physical phenomena give rise to the "perceptual" qualities we have discussed.

2.4 The Relation Between Frequency and Pitch

Recall that in discussing pitch, we said that if the string being plucked was loosened, the pitch would become lower. Imagine loosening the string of Fig. 2.5 and then plucking it, so that at the moment of release it has exactly the same shape as that in the first frame of the figure. However, it will take more time to complete one cycle. The period will increase, with a consequent decrease in the number of oscillations per second; that is, the frequency will decrease. This is in agreement with (2.5).

Let us suppose that the string is loosened just enough to increase the period to 5 ms. Then the new frequency is $f = 1/0.005$ second per cycle $= 200$ cps $=$ 200 Hz.

How much loosening does this change require? To answer this question, we need a quantitative measure of the "tautness" or tension of the string and how that tension is related to frequency. We will return to this question in Sect. 2.8. What we want you to consider at the moment is the qualitative result of this little experiment.

[2]The sound of the violin is strongly affected by the other physical components of the instruments, along with their respective vibrations.

Loosening the string decreases the frequency of the oscillations, as it lowers the pitch. Correspondingly, tightening the string increases the frequency and raises the pitch. So there is a relation between the physical quantity, frequency, and the psychological attribute, pitch. This relationship is the basis for the tuning of musical instruments. In Chap. 10, we will note that the loudness of a note also affects the sense of pitch.

The strings of a piano are tuned to a definite set of frequencies. First, one sets the A above the middle key on a piano – referred to as "middle-C" – at a frequency of 440 Hz. The "middle C" on a piano is set at a frequency of approximately 262 Hz. The lowest C is set correspondingly to a frequency of approximately 33 Hz, and so on.

We see that one way to produce different frequencies is to vary the tension. Are there other ways? If we combine (2.3) and (2.5), we obtain the relation

$$f = \frac{v}{2\ell}. \tag{2.6}$$

We will see later in the chapter that an increase in the tension produces an increase in the wave velocity. As a consequence, according to (2.6) the frequency will increase and so will the pitch. We will also see later that changing the nature of the string itself will change the wave velocity. Finally, we see that decreasing the length of the string will increase the frequency. All three factors are used to produce the huge range of frequencies of a piano – from 27.5 Hz to $\sim$4,186 Hz.[3]

Various stringed instruments are tuned accordingly. For example, in order that the A string on the violin be in tune with the corresponding A string on the piano, their frequencies should be equal.

Why the particular frequency of 440 Hz is chosen for the "A" is a matter of history. In fact, this frequency has been rising steadily over the past 200 or more years, so much so that in Bach's time it is believed to have been about 415 Hz. Why the notes of the Western scale have the frequencies to which we have just alluded will be the subject of Chapter 11, TUNING, INTONATION, AND TEMPERAMENT: CHOOSING FREQUENCIES FOR MUSICAL NOTES. The development of scales is a fascinating story of the interdependence of scientific understanding and esthetics.

2.5 The Wave Motion of a Stretched Rope

It is difficult to study the motion of the strings of musical instruments without special equipment because the wave velocities and the frequencies are very large. It is possible to check the relation (2.6) by performing a simple, but illustrative experiment. Get a long piece (2 or 3 m) of heavy rope or clothesline or a long tightly

[3]The lower frequency is precisely four octaves (a factor of $2^4 = 16$ below 440 Hz), while the latter frequency corresponds to tuning according to equal temperament. (See Chap. 11)

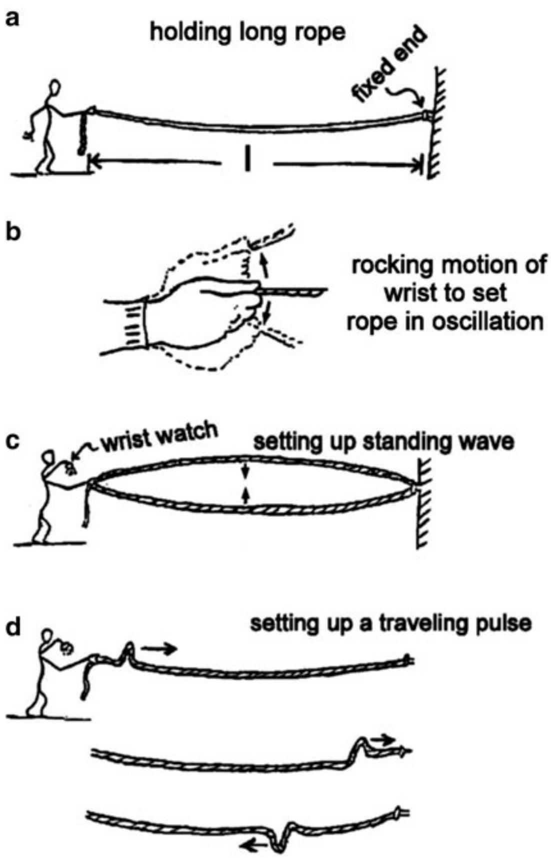

Fig. 2.6 Exciting a long rope. (drawing by Gary Goldstein)

wound spring (as used to close screen doors). Secure one end to a fixed point – say a doorknob on a closed door. Pull the free end so that the rope is stretched loosely to its full extent, as shown in Fig. 2.6a. Estimate the length, ℓ, of the stretched rope.

You are going to set up wave motion of the rope by shaking the held end up and down while using your wrist as a pivot, as shown in Fig. 2.6b. By shaking very slowly at first and gradually increasing the rate of shaking, you will soon reach a rate that sets up a wave of the form shown in Fig. 2.6c. The whole rope will be oscillating up and down at that rate. Notice that once you set up that motion, it is easy to maintain the motion. It is as if the system has "locked in" to that mode of oscillation.

While maintaining the motion of Fig. 2.6c, use the seconds hand on your wrist watch to determine the period. (You might have a friend to assist you.) This can be done easily by counting, say ten cycles and observing how many seconds have elapsed. Remember that a cycle is completed when the rope has returned to some initial configuration, so whenever it reaches the lowest point in its motion it has completed a cycle. If, for example, the rope completes ten cycles in 8 s, the period would be $8/10$ s. The frequency would be $10/8 = 5/4$ Hz.

Next, let the rope return to rest. It is important not to vary the tension, so do not change your position. Now you are going to set up a disturbance in the rope of the form shown in Fig. 2.6d. This is accomplished by very quickly jerking your hand up and down while quickly returning to the starting position. It is best to keep your hand as rigid as possible. Observe what happens. The short disturbance or **pulse** moves rapidly to the end of the rope, is reflected, and returns to your hand upside down. If your hand remains rigid the pulse will reflect at your hand, turn right side up and move to the far end again. The pulse might make many round trips before it disappears. You have set up a **traveling wave**. (A pulse travels across the string.)

Note that any particular segment of rope material moves up and down, while the wave pattern, the pulse in this case, moves down the length of the rope. These two directions are perpendicular to each other. The waves are **transverse**.

Now time the pulse by measuring the time required for the pulse to complete several round trips. For example, if the pulse makes five round trips in 4 s, then the time for a single circuit would be $4/5$ s or 0.8 s. If you are careful, you will find that the time required for one round trip is the same as the period of oscillatory wave motion that you determined before, for the standing wave.

Measuring the length of the stretched rope will then enable you to determine the velocity of propagation for the traveling wave. In our example, the round trip time and the period were 0.8 s. If the rope were 2 m long, a round trip would be 4 m and the velocity of propagation would be $4\,\text{m}/0.8\,\text{s} = 5\,\text{m/s}$. Determine the velocity for your rope, using (2.3).

2.6 Modes of Vibration and Harmonics

One might ask whether the string can be excited so as to produce a vibration that does not have a frequency of 250 Hz. The answer is yes. In the course of demonstrating this fact, we will describe what are referred to as the modes of vibration of the string.

Now that you have become familiar with working the rope you can learn how to excite its modes of vibration. Start by exciting the same standing wave that you did before (Fig. 2.6c). Count the cycles rhythmically while the rope is oscillating. That is, say the numbers out loud every time the rope reaches bottom – "one-two-three-four-one-two…". Now start shaking your hand at twice the original tempo. You will have doubled the rate of oscillations and hence the frequency. The rope will "lock into" a different mode of oscillation. It will now appear as in Fig. 2.7a. The period of this oscillation is one-half the period of the preceding oscillation, wherein a pulse is traveling back and forth along the rope, as in Fig. 2.6.

For the sake of identification, we call the mode of Fig. 2.6c the **fundamental mode** or the **first harmonic** of the string. The mode of oscillation you are now producing is called the **second harmonic** (Fig. 2.7a). While the rope is oscillating

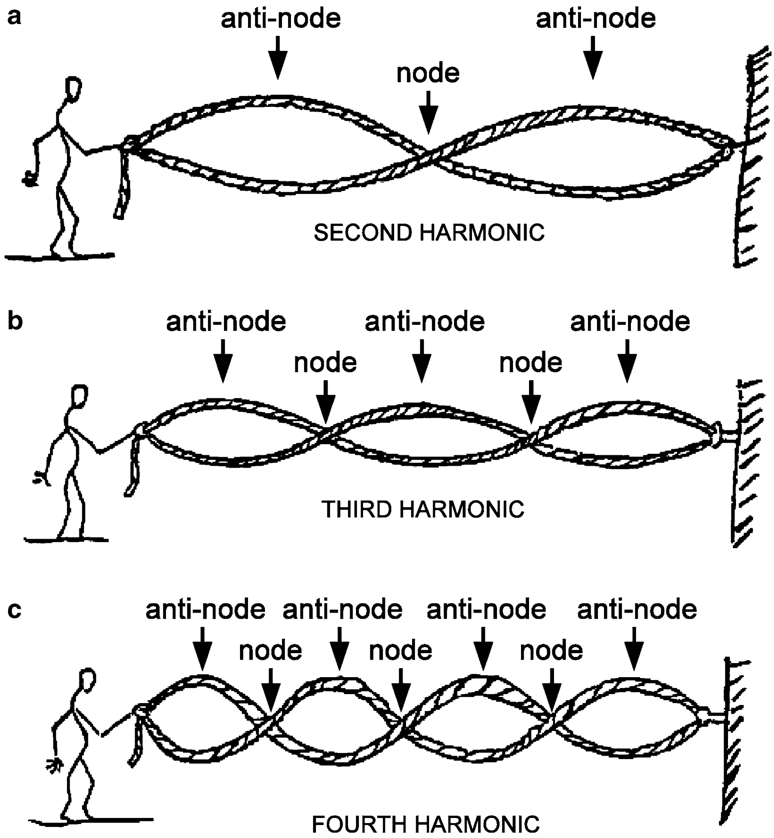

Fig. 2.7 Higher harmonics of the vibrating string (drawing by Gary Goldstein)

in this mode, notice that near the midpoint the rope is hardly moving at all. This point at which no motion occurs is called a **node**. For the second harmonic, there is one node between the end points, whereas the fundamental mode (Fig. 2.6c) had no nodes between the end points.

Observe also that there are two points along the rope which achieve the greatest displacement from equilibrium (either above or below), one at about $1/4$ the distance from your hand, the other at $3/4$ the distance. These points along the rope at which the maximum displacement occurs are called **antinodes**. The second harmonic has two antinodes, whereas the fundamental mode has one antinode at the midpoint of the rope (see Fig. 2.6 again).

Now, by shaking your hand at triple the rate for the fundamental mode you can excite the mode shown in Fig. 2.7b, the third harmonic. This is somewhat harder to excite than the preceding mode, but once you get near the right rate of shaking, the rope will respond very strongly and will "lock in" to that mode. The third harmonic has three times the frequency of the fundamental. You will observe that there are

Table 2.1 Harmonics

Mode	Frequency	No of nodes	No of antinodes
Fundamental = 1st harmonic	f_1	0	1
2nd harmonic	$f_2 = 2f_1$	1	2
3rd harmonic	$f_3 = 3f_1$	2	3
4th harmonic	$f_4 = 4f_1$	3	4
5th harmonic	$f_5 = 5f_1$	4	5
6th harmonic	$f_6 = 6f_1$	5	6
nth harmonic	$f_n = nf_1$	$n-1$	n

two nodes in this mode – one at $1/3$ the distance to the fixed end, the other at $2/3$ that distance. There are three antinodes.

You should now see the pattern. By exciting the fourth harmonic (Fig. 2.7c), which has a frequency four times the fundamental frequency, you will produce a mode having three nodes and four antinodes. (It is appreciably harder to excite this mode; the higher modes are progressively more difficult.) The fifth harmonic would have a frequency five times the fundamental frequency, and the wave pattern would have four nodes and five antinodes. We summarize this information in Table 2.1.

The frequencies of the harmonics are written as multiples of the fundamental frequency f_1. We have included a general mode, the nth harmonic, where n symbolizes any integer $(1, 2, 3, 4, \ldots)$. Letting $n = 7$, for example, tells you that the 7th harmonic has frequency $7f_1$, $(7-1) =$ six nodes and seven antinodes.

From all of the preceding you now see that the rope, or a stretched string, has many different modes of vibration. These modes of vibration have frequencies which are integral multiples of the fundamental frequency – the modes are harmonic. Then the periods for each of the modes will be different from one another. Recall, however, that the time required for a traveling pulse to make a round trip (Fig. 2.6d) was equal to the period of oscillation of the rope in the fundamental mode (Fig. 2.6c). Therefore, the relation between wave velocity, length, and frequency (2.6) should be rewritten to show explicitly that the fundamental frequency is involved.

We have

$$f_1 = \frac{v}{2\ell}. \tag{2.7}$$

For the other harmonics, the frequencies are multiples of the fundamental frequency, so

$$\begin{aligned} f_2 &= 2f_1 = 2 \times \frac{v}{2\ell} \\ f_3 &= 3f_1 = 3 \times \frac{v}{2\ell} \\ f_4 &= 4f_1 = 4 \times \frac{v}{2\ell} \end{aligned} \tag{2.8}$$

and so on. This series is summarized by writing the frequency of nth harmonic, f_n, as

$$f_n = n\ f_1 = n \times \frac{v}{2\ell}. \tag{2.9}$$

The fact that the rope or stretched string can be set into oscillation in many different modes will be of continual importance. It forms the basis for much of the subsequent discussion.

Notice that the wave patterns for these modes do not move either to the right or to the left. We refer to such a wave as a **standing wave**. In contrast, the wave described initially in this chapter that moves along an endless string is called a **traveling wave**. We will discuss such waves more fully in the next section.

We close this section by introducing other widely used terms – the **overtone** and the **partial**. By definition, the first overtone is the second harmonic; the second overtone is the third harmonic; and so on. The term **partial** is used to refer to one of the modes of a musical instrument **whether or not** the frequencies form a harmonic series. An example is the sound of a gong, whose mode frequencies do not form a harmonic series.

2.7 The Sine Wave

The shape of the pattern along a string that is vibrating in one of its modes is very specific – being the curve produced by plotting the trigonometric **sine function**. In addition, if we plot the displacement of any point along the string vs. time, we will obtain a graph of the sine function. In fact, of all periodic curves in nature, the sine curve is very unique in its physical ramifications, as we will see many times in the course of our study of sound and light and therefore of music and color. Thus, we now turn to an examination of the sine curve.

You probably have paid attention to how various radio stations are identified. For example, a popular radio station for classical music in the Boston area is WCRB 102.5FM. The number 102.5 stands for a frequency of 102,500,000 Hz. (The letters 'FM' stand for 'frequency modulation', which is a special means of transmitting information using waves; it will be discussed later in the text.) Or, as another example, you might have heard that most current symphonic orchestras are tuned to a frequency of 440 cycles per second. What these numbers fully represent is the subject of this section.

Let us begin by returning to the long stretched string. Suppose that you were to take hold of the string and move it up and down repeatedly at a constant rate in time. If the pattern of your motion is repeated again and again, we say that the pattern is periodic. As an example, let us display the pattern of motion for the most important such motion; it is called a **sine wave** pattern. See Fig. 2.8.

The graph displays the displacement of the hand as it varies in time. Note that there is a pattern that extends over a 1-s interval. It is repeated three times over the entire 3 s interval. This interval is called the **period** or the motion. The maximum

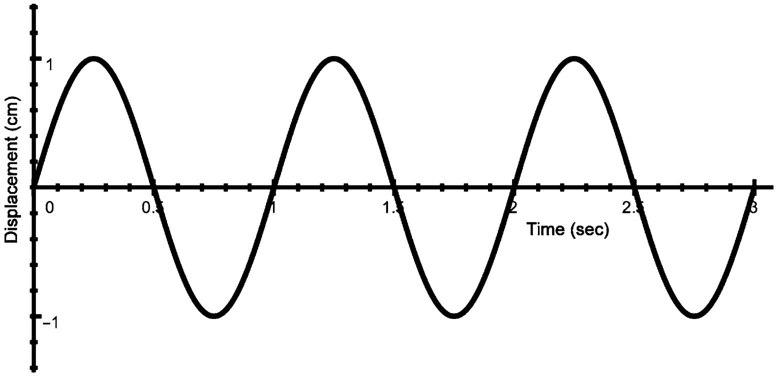

Fig. 2.8 Sine wave of displacement vs. time

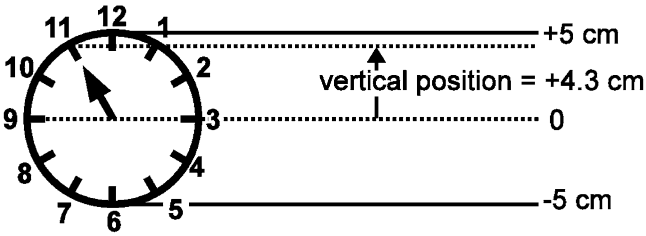

Fig. 2.9 Sweeping hand of a clock defining a sine wave

displacement is 1 cm and is called the **amplitude** of the motion. The pattern is **sinusoidal** and represents the **sine function** of trigonometry. Let us review the nature of the sine function.

You might recall that the sine of an angle is the ratio of the "side opposite" to the hypotenuse of a right triangle. Thus,

$$\sin\theta = \frac{b}{c}. \tag{2.10}$$

We can produce a graph of the sine function by a simple method involving the constant circular motion of the seconds hand of a clock. The seconds hand sweeps around, making a full circle every 60 s. Let us measure the vertical position of the tip of the hand as it sweeps around. We do this by first drawing a base line across the face passing through the center and the 3 and 9 o'clock marks, as shown in Fig. 2.9. Suppose the hand extends 5 cm from the center. Then the *vertical position* of the tip, relative to the base line, will vary from the lowest point (at the 6 o'clock mark) of -5 cm, to the highest point (at the 12 o'clock mark) of $+5$ cm. When the hand points at 11 o'clock, for example, as shown in Fig. 2.9, the vertical position will be $+4.3$ cm.

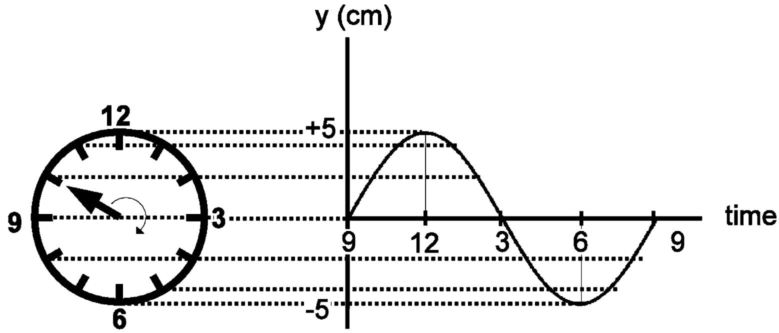

Fig. 2.10 Clock defining one cycle of a sine wave

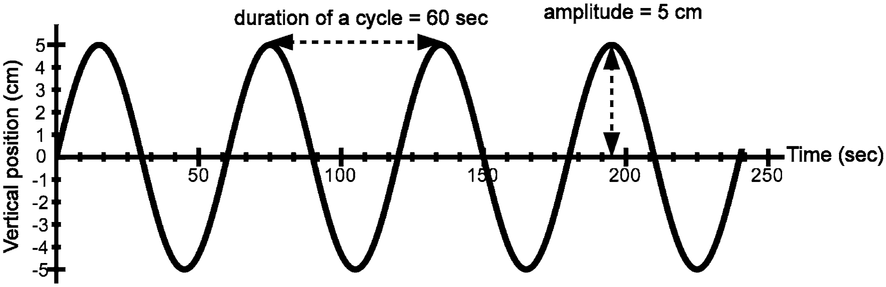

Fig. 2.11 Vertical position of clock hand vs. time elapsed

Now we will plot the vertical position as the hand sweeps around, starting at the 9 o'clock position when the vertical position is 5.0 cm. Five seconds later, the hand will be at 10 o'clock and the vertical position will be +2.5 cm. Then in 5 s more, the hand will be at 11 o'clock and the vertical position will be +4.3 cm, and so on. The procedure of plotting the vertical position as a function of time elapsed is illustrated in Fig. 2.10, for the first 60 s; we obtain one cycle of the sine wave. Continuing this plotting gives the curve in Fig. 2.11.

The form of the curve repeats exactly every 60 s, when the hand has returned to its initial position. If we continued plotting the vertical position of the hand indefinitely, the curve would continue on repeating itself indefinitely (or until the clock stopped). The full curve is the sine function. It is **periodic** in that it repeats itself indefinitely. Note that the angle changes steadily, going through a full circle of 360° in 60 s. Thus, the rate of change of the angle is 360°/min, or 6°/s. Then, the value of the sine function after 5 s will be $\sin 30° = 0.5$. The result is a vertical position equal to $0.5(5.0) = 2.5$ cm.

Although we have obtained this curve by a particular procedure, its significance is far more general. Being a graphical representation of a function, it represents a mathematical prescription: Given some numerical value of the variable, the sine of that variable has a definite numerical value. The variable may represent a time (as in the example we have used), or it may represent a position along a string, or it may represent an angle. What is important is the shape of the curve and its periodicity.

The particular sine function of Fig. 2.10 can be characterized by two numbers. The first of these is the maximum height of the curve, called the **amplitude**, which is 5 cm for this case. The second is the length of one cycle, which is a time interval of 60 s for this example. When the variable is time, the length of one cycle, its duration, is the **period** of the motion. We will soon consider examples of sine functions representing some vertical position as a function of distance rather than time. In that circumstance the length of one cycle will be a distance and is called the **wavelength**.

2.8 The Simple Harmonic Oscillator

While the sine curve is central to the behavior of the modes of a vibrating string, it shows itself in a simpler way in the behavior of a **simple harmonic oscillator** (**SHO** for short). The SHO is a system that is fundamental for understanding all vibrating systems and therefore deserves significant attention. It consists of a spring having negligible mass that is attached at one end to some fixed support and a rigid object that has the essential mass of the system – referred to as the **mass** of the SHO – at the other end. See Fig. 2.12.

When isolated, an SHO will come to rest at its **equilibrium state**, in which the spring is neither stretched nor compressed. To displace the mass from its equilibrium position, a force must be applied. That force F is proportional to the displacement y from the equilibrium position of the mass, as shown in Fig. 2.12b, in which a **downward** displacement corresponds to positive y, while an **upward** displacement corresponds to a **negative** y. Because the graph of y vs F is a straight line, we say that the relation between y and F is **linear**. We have

$$\text{Displacement} \quad \propto \quad \text{Force.} \tag{2.11}$$

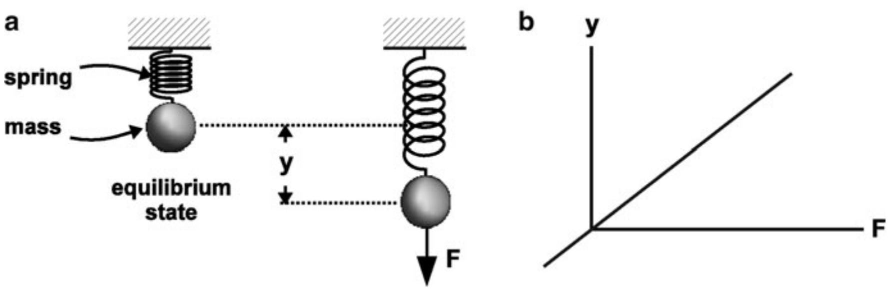

Fig. 2.12 The Simple Harmonic Oscillator

Mathematically we write

$$y = \frac{1}{k}F = \frac{F}{k}, \tag{2.12}$$

where the constant k is known as the **spring constant** or the force constant. The relation is known as **(Robert) Hooke's Law**.

If the force is measured in lbs and the displacement in inches, the spring constant is expressed in lbs-per-in, which will be written as "lbs/in." In this text, the force will often be expressed in Newtons (abbreviated as N) (one Newton is about 4.5 lbs) and the displacement will be expressed in meters. Then the spring constant will be expressed in Newtons per meter, or N/m.

Sample Problem 2-1

Suppose it takes a force of 5 N to stretch a given spring 2 m. Find the spring constant.

Solution

The spring constant is given by

$$k = \frac{F}{y} = \frac{5}{2} = 2.5\,\mathrm{N/m}. \tag{2.13}$$

Note that if that same spring is stretched by a force equal to 10 N, the displacement will be

$$y = \frac{F}{k} = \frac{10}{2.5} = 4\,\mathrm{m}. \tag{2.14}$$

Doubling the force leads to a doubling of the displacement.

If the mass is pulled from its equilibrium position as in Fig. 2.12 and released, it will oscillate in time at a certain frequency.

The *linear* relation between y and F is **unique** in leading to two characteristics:

1. A **sinusoidal** displacement in time, as shown in Fig. 2.13
2. A frequency that is **independent of the amplitude of oscillation**

Let us imagine suspending a mass to a spring and letting it oscillate. We see in Fig. 2.13 that the displacement of the mass exhibits a sine wave pattern. Its amplitude is 2 cm. The period is 2 s. The corresponding frequency is $f = 1/T = 1/2 = 0.5\,\mathrm{Hz}$.

In fact, the two characteristics of oscillatory motion automatically imply a linear relation. Real springs do not obey **Hooke**'s Law precisely, as shown in Fig. 2.14. However, they do so approximately for small enough displacements.

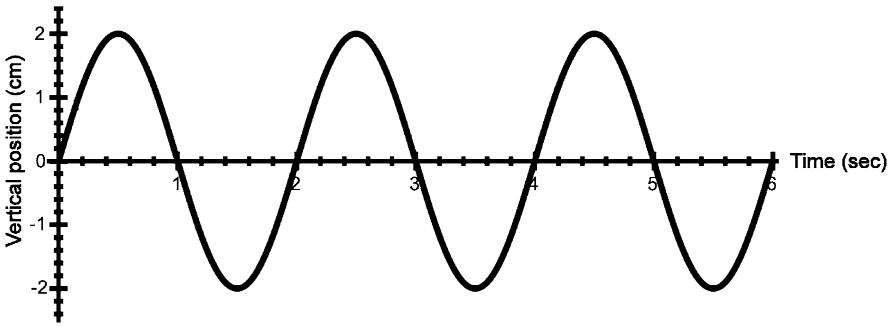

Fig. 2.13 Displacement of an SHO vs. time

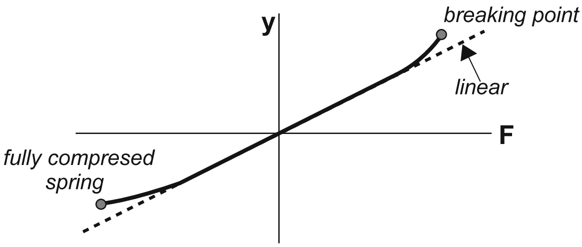

Fig. 2.14 Displacement vs. force for a real spring

2.8.1 The Vibration Frequency of a Simple Harmonic Oscillator

The spring constant and the mass of an SHO determine its vibration frequency. We will see later that the fundamental frequency of a vibrating string is proportional to the square root of the ratio of a restoring force to a mass. This qualitative relationship holds for an SHO too. It can be shown that the frequency of vibration of the SHO is given by[4]

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{Frequency of SHO.} \tag{2.15}$$

The spring constant reflects the restoring force.

In using this formula, we are not free to express the units of k and mass independently. The choice of units must be consistent. Thus, if we express the spring constant in N/m, the mass must be expressed in kilograms (abbreviated as "kg"). Then the frequency we obtain is expressed in Hz.

[4]This expression for the frequency as well as the fact that the motion is sinusoidal can be derived rigourously mathematically using a combination of Hooke's Law $F = kx$ and Newton's Second Law of Motion $F = ma$, where a is the acceleration of the mass. Eliminating the force leads to a direct relation between the displacement and the acceleration: $x = ma/k$. This subject is discussed in Appendix E. In Appendix F, you can see how the sinusoidal motion evolves by using a technique of Numerical Integration.

Sample Problem 2-2

Suppose that an SHO has a spring constant equal to 25 N/m and a mass of 500 g. Find the frequency and period of vibration of the SHO.

Solution

We must first express the mass in kg: 500 g = 0.500 kg. Then

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{25}{0.500}} = 1.1\,\text{Hz}. \qquad (2.16)$$

Correspondingly, the **period** of vibration is given by

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{Period of SHO.} \qquad (2.17)$$

so that $T = 1/1.1 = 0.9\,\text{s}$.

We note that generally, the frequency increases if the spring constant increases and/or the mass decreases. However, if the spring constant is doubled or the mass is halved, the frequency is not doubled. Instead, it is increased by a factor of $\sqrt{2}$: Taking off from the previous numerical example, suppose that the spring constant is 50 N/m and the mass is 0.500 kg. A simple calculation leads to a frequency of 1.6 Hz, which equals $\sqrt{2}$ multiplied by the frequency of the previous example.

We can obtain an estimate of the amplitude of the velocity – the maximum speed – of an SHO over the course of one oscillation. It is on the order of the average speed. The latter is simply the total distance traveled by the mass in one cycle divided by the period. Thus, with A = amplitude

$$\text{Average speed} = \frac{4A}{T} = 4Af. \qquad (2.18)$$

The actual velocity amplitude is $2\pi Af$, which is a bit greater, as it should be.

2.9 Traveling Sine Waves

The modes of vibration of a string of fixed length are such that there is a pattern that remains stationary except for oscillations in the overall amplitude. The pattern does not move to the right or the left. To the contrary, a wave that progresses in one or the other direction is referred to as a **traveling wave**. Here is a simple way to produce a traveling sine wave.

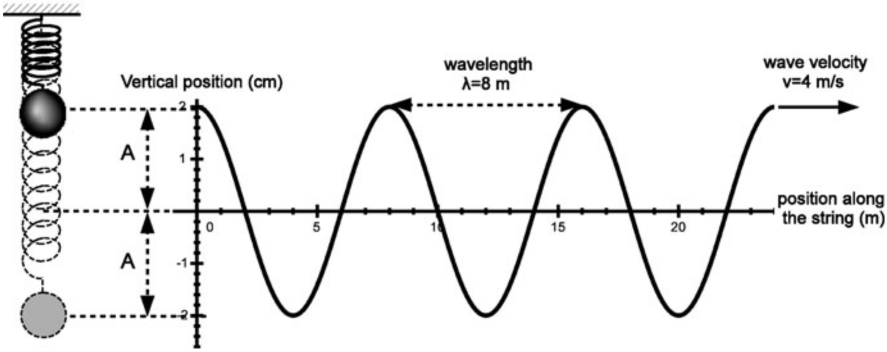

Fig. 2.15 Traveling wave produced by an SHO

Suppose that we attach the mass of an SHO to the end of a long stretched string. The mass is set into oscillation with an amplitude of 2 cm, as above. In Fig. 2.15, we see the mass at an instant when it has its maximum upward displacement of 2 cm. To its right, we see the sinusoidal pattern of the wave along the string. Note that this curve represents the actual material of the string at this instant in time. Furthermore, **whereas the displacement of the oscillator is sinusoidal in *time*, the pattern of the string is sinusoidal in *space*.**

We have assumed that the wave velocity is 4 m/s. As a consequence, during one cycle of oscillation lasting 2 s, the wave moves along the string a distance $x = vt = 4(2) = 8$ m. Each cycle that has its left end in contact with the mass is replaced by another such cycle. The three cycles along the string were produced by three cycles of oscillation of the mass. The period in space is called the **wavelength** and is here equal to 8 m.

We see that the sinusoidal wave in space is characterized by the following four parameters:

1. The **amplitude** A – here equal to 2 cm
2. The **wave velocity** v – here equal to 4 m/s
3. The **wavelength** λ – here equal to 8 m
4. The **period** T – here equal to 2 s

Clearly, we have a simple relation among the velocity, the wavelength, and the period:

$$v = \frac{\lambda}{T}. \tag{2.19}$$

The frequency and period are inverses of each other: $f = 1/T$. Therefore, we have the relation

$$\lambda f = v. \tag{2.20}$$

Notice that this equation can be rewritten as $\lambda = v/f$. As a consequence, for a given velocity, the wavelength decreases as the frequency increases. Alternatively, if the frequency is constant and the velocity decreases, the wavelength must decrease.

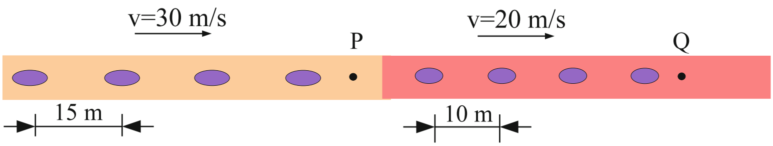

Fig. 2.16 Spacing vs. speed of cars in traffic

This latter result can be understood in terms of the following traffic situation. Suppose a line of cars is traveling along a one-lane road over a long time so that the traffic flow is stationary. The number of cars passing a given point must then be constant so that there is neither a pile up of cars someplace nor any buildup of empty space. Then, if the cars speed up, they must be further apart. Similarly, if the cars slow down, the space between the cars must decrease. A consequence of this decrease in space and the need for safety is that the cars usually slow down even more. Figure 2.16 illustrates how the spacing between neighboring cars decreases when the speed decreases. The rate at which cars pass point P is $30/15 = 2$ cars per second, which is equal to the rate at which cars pass point Q, that is, $20/10 = 2$ cars per second.

2.9.1 Applications

1. In the case of the standing wave, we recall that $\lambda = 2\ell$. Then, as we have already shown.
$$v = 2\ell f_1 \ \text{ or } \ f_1 = \frac{v}{2\ell}.$$
2. The audible range of frequencies is from 20 to 20,000 Hz. In the case of a traveling sound wave in air, with $v = 340$ m/s, the corresponding range of wavelengths is
$$\max\lambda = 340/20 = 17\,\text{m} \ \textbf{to} \ \min\lambda = 340/20{,}000 = 0.017\,\text{m} = 1.7\,\text{cm}.$$
3. As we will see in Chapter 5, Electricity, Magnetism, and Electromagnetic Waves, light is a visible electromagnetic wave having a range of frequencies from 4.0×10^{14} Hz to 7.0×10^{14} Hz. In the case of a light wave traveling in vacuum, the wave velocity is given the symbol c and is equal to 3.0×10^{8} m/s. The corresponding range of wavelengths is
$$\max\lambda = \frac{3.0 \times 10^8}{4.0 \times 10^{14}} = 7.5 \times 10^{-7}\,\text{m} = 750\,\text{nm}$$

to

$$\min\lambda = \frac{3.0 \times 10^8}{7.0 \times 10^{14}} = 430\,\text{nm}.$$

2.10 Modes of Vibration: Spatial Structure

The modes of vibration of a stretched string are related to traveling sine waves. Later on in this chapter we will see that when two sine waves of the same wavelength head toward each other, they superpose to produce a standing wave with the same wavelength. The standing waves of the modes of vibration are portions of sine waves. An examination of their shapes reveals the relationship between these shapes and the corresponding wavelengths. In Fig. 2.17 we show the first six harmonics of the vibrating string. The patterns display the extreme shapes at two times, one-half cycle apart.

The second harmonic is a full cycle of a sine wave. Thus, the length of the string is equal to the wavelength. The fundamental (first harmonic) is a half cycle, so that the length is equal to one-half of a wavelength.

The wavelength for the fundamental is

$$\lambda_1 = 2\ell. \tag{2.21}$$

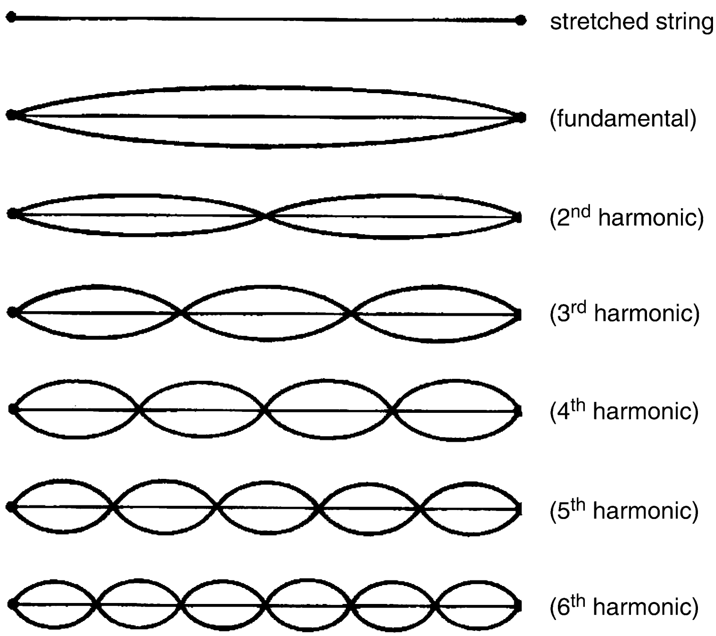

Fig. 2.17 Modes of vibration of the stretched string

For the second harmonic, the shape of the string encompasses a full cycle of a sine wave, so the second harmonic has a wavelength

$$\lambda_2 = \frac{\lambda_1}{2} = \ell. \tag{2.22}$$

In the third harmonic mode, the string has the form of one and one-half cycles of a sine wave. Thus

$$\lambda_3 = \frac{2}{3}\ell \;\text{ or }\; \frac{3}{2}\lambda_3 = \ell. \tag{2.23}$$

Lastly, the fourth harmonic has the wavelength

$$\lambda_4 = \frac{1}{2}\ell. \tag{2.24}$$

This sequence of wavelengths can be rewritten in a way that allows us to generalize these examples:

$$\begin{aligned}
\lambda_1 &= \frac{2\ell}{1} = 2\ell \\
\lambda_2 &= \frac{2\ell}{2} = \ell \\
\lambda_3 &= \frac{2\ell}{3} = \frac{2}{3}\ell \\
\lambda_4 &= \frac{2\ell}{4} = \frac{1}{2}\ell.
\end{aligned} \tag{2.25}$$

Written in this way it is obvious that the fifth harmonic will have wavelength $\lambda = 2\ell/5$, and so on. For the nth harmonic, then, we will have the relation

$$\lambda_n = \frac{2}{n}\ell. \tag{2.26}$$

Now recall that according to (2.9) the frequency for the nth harmonic is given by $nv/2\ell$. The formula for the frequency is then

$$f_n = \frac{v}{\lambda_n}. \tag{2.27}$$

We can write this equation also as

$$\lambda_n f_n = v. \tag{2.28}$$

The equation becomes identical to (2.10), which applies to the two sine waves that when added together produce the standing wave.

We can appreciate this process if we realize that when we excite a standing wave by shaking one end of the string, we send a sine wave down the string. This wave is reflected off the other fixed end. Upon reflection, we have a second sine wave with the identical wavelength traveling back toward the hand. This sine wave adds together with the sine wave that we are sending with our hand to produce the standing wave.

2.11 The Wave Velocity of a Vibrating String

It is well known to players of string instruments that the pitch and hence fundamental frequency of a string increases with increasing tension. This fact is connected with the increase in the wave velocity with increasing tension. Similarly, one notices that for given string material, the pitch of a string decreases with increasing string thickness. This fact is connected with the decrease in the wave velocity with increasing mass of string, for given length of string. This section is concerned with the parameters that determine the wave velocity and the precise relationship among them.

It turns out that the wave velocity depends upon two parameters that characterize the string. First, we have the **Tension** , with the symbol $\mathcal{T}$. The tension acts to restore the string to its equilibrium shape and favors a greater wave velocity. The second parameter is the **mass per unit length**, with the Greek letter μ as a symbol. This parameter is also called the **linear mass density**. The mass of an SHO reflects its resistance to having its velocity change – that is, being accelerated. Similarly, the linear mass density of a string reflects the string's resistance to having any point along the string undergo a change in velocity. This set of changes is what constitutes a wave. Just as an increase in the mass of an SHO decreases its vibration frequency, an increase in the linear mass density decreases the wave velocity.

We will now discuss these two parameters in greater detail.

Let us turn our attention to tension. This parameter is measured in units of force such as the "pound" (lb) or the **Newton**(named after Isaac Newton), abbreviated as *N*. (The two are related as follows: 1 lb = 4.5 N.) A common device for measuring tension is a "spring scale".[5] On average, a string of a stringed instrument is under a tension of about 50 lbs (thus about 200 N). We will later show that the total tension of the strings of a piano is on the order of 70,000 lbs!

The physical parameter **force** has direction as well as magnitude. Thus, for example, the gravitational force of the earth on a person is downward, toward the center of the earth. On the other hand, *tension has no directionality*. This fact is illustrated in the following Fig. 2.18, wherein a string is being pulled on by two spring scales, one to the right and one to the left.

[5]If you attach a spring to one end of a string that is under tension, the tension is proportional to the consequent displacement of the spring.

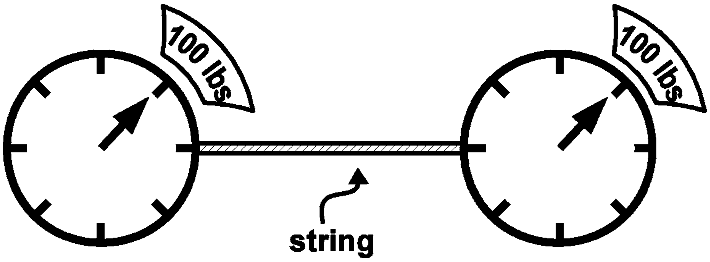

Fig. 2.18 String pulled from two directions

Both scales read a force of 100 lbs. This implies that the right scale pulls on the string with a force of 100 lbs to the right, while the left scale pulls on the string with a force of 100 lbs to the left. These two forces cancel each other out, so that the net force acting on the string is zero and the string can remain stationary in this situation.

The resulting tension, $\mathcal{T}$, in the string might not be obvious: It might seem that the two applied forces add to produce a tension of 200 lbs. In fact the tension is 100 lbs. We write

$$\mathcal{T} = 100\,\text{lbs}. \tag{2.29}$$

Note

The Nature of Tension

How can we understand the above enigma, that the tension is not 200 lbs? We will be able to answer this question by examining what the tension represents. Hence, let us imagine two people facing each other with their right arms outstretched and clasping each other's hand. Call them, Richard and Lisa. Richard's shoulder pulls his arm with a force of 10 lbs and so does Lisa's shoulder pull her arm with a force of 10 lbs. Richard's hand pulls Lisa's hand toward him with a force of 10 lbs and correspondingly, Lisa's hand pulls Richard's hand toward her with a force of 10 lbs. We say that there is a tension of 10 lbs at the point where the two hands are clasped.

Now let us turn to the string in Fig. 2.18. We consider two segments of this rope, labeled L and R, respectively, shown at the top of Fig. 2.19(a), whose boundary is marked by the letter "P". At the bottom of Fig. 2.19(a) we focus on segment L, noting the two forces acting on this segment that balance each other – 100 lbs by the scale to its left and 100 lbs by the R segment to its right. [In fact, the L segment pulls on the R segment towards L with a force of 100 lbs and correspondingly, the R segment pulls on the L segment towards R with a force of 100 lbs.] Since the point P is arbitrary, the tension is uniform all along the string and is said to be 100 lbs.

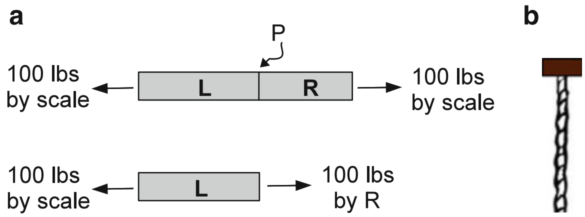

Fig. 2.19 Hanging rope

This is not so for a string with weight that is hanging from a support, as seen in Fig. 2.19(b). At any point along the string, the tension equals the weight of string below that point. Thus, the tension vanishes at the bottom end and equals the total weight of the string at the top end, where the string is supported. **Question**: If the string weighs 8 oz, what is the tension at the midpoint of the string?

We now turn to the **linear mass density**. Suppose we have a spool of string and cut off a meter length of string and find that it has a mass of 5 g. Then the linear mass density of the string in the spool is $5\,\text{g/m} = 0.005\,\text{kg/m}$. A 2-m length of such a string would have a mass of 10 g. As a result, the linear mass density will be $10\,\text{g}/2\,\text{m} = 0.005\,\text{kg/m}$. The result is that there is no change in the linear mass density. Generally, for a length of string ℓ with a mass m, we have the relation

$$\mu = \frac{m}{\ell}. \tag{2.30}$$

The linear mass density of the string of the spool is independent of the length of the string.

We are now ready to reveal the relation between the wave velocity and the two parameters, $\mathcal{T}$ and μ. It is given by

$$v = \sqrt{\frac{\mathcal{T}}{\mu}} \tag{2.31}$$

which is the wave velocity along a string.

Note that the wave velocity involves a square root of a force-like parameter (here $\mathcal{T}$) divided by a quantity which reflects mass or inertia (here μ):

$$\text{Wave velocity} = \sqrt{\frac{\text{Force-like parameter}}{\text{Mass-like parameter}}}. \tag{2.32}$$

The force-like parameter is usually referred to as the **restoring force**.

This result is universal for all types of waves.

In this case, tension is the restoring force.

Now we can see fully how the fundamental frequency depends upon the three parameters of a vibrating string – the length, the tension, and the linear mass density. Using (2.6) and (2.31), we obtain

$$f = \frac{v}{2\ell} = \frac{\sqrt{\frac{\mathcal{T}}{\mu}}}{2\ell}. \tag{2.33}$$

Thus, we have found that **the fundamental frequency is proportional to the square root of the tension, inversely proportional to the square root of the linear mass density, and inversely proportional to the length**.

Sample Problem 2-3

Suppose that a violin string has a length of 33 cm and a linear mass density of 6 g/m, and has a fundamental frequency of 440 Hz. Find the wave velocity, the mass of the string, and the tension in the string.

Solution

$$v = 2f\ell = 2(440\,\mathrm{Hz})(0.33\,\mathrm{m}) = 290\,\mathrm{m/s}$$

$$\begin{aligned} m &= \mu\ell = 6 \times 10^{-3} \times 0.33 \\ &= 1.98 \times 10^{-3}\,\mathrm{kg} = 1.98\,\mathrm{g}. \end{aligned}$$

To obtain the tension is a bit more complicated because it appears within a square root:

$$v = \sqrt{\frac{\mathcal{T}}{\mu}}$$

so that

$$v^2 = \mathcal{T}/\mu$$

and

$$\mathcal{T} = \mu v^2 = (0.006\,\mathrm{kg/m})(290)^2 = 506\,\mathrm{N}.$$

2.11.1 *Application of the Above Relations to the Piano*

We will now review in a bit of detail the methods whereby the huge range of frequencies (and hence pitches) of piano strings, from 27.5 to 4,186 Hz, can be obtained:

To increase the pitch, one can

- Increase the tension
- Decrease the linear mass density, or
- Decrease the length

We can obtain an idea of the **total tension** on the block of metal that supports the strings as follows:

From the relation

$$v = \sqrt{\frac{\mathcal{T}}{\mu}},$$

we obtain $\mathcal{T} = \mu v^2$.

In order to estimate the average wave velocity, we will use the relation $v = 2Rf$. Since the average frequency is about 500 Hz and the average length of the strings is about 0.5 m, we obtain $v \sim 2(0.5)(500) = 500$ m/s. Now for the linear mass density, which in kg/m is the mass in kilogram of a 1 m length of string. That mass is the product of the mass density (8 g/cc = 8,000 kg/m^3 for the steel of most piano strings) and the volume of 1 m of string. From observation, the strings have an average radius of about 0.5 mm. The above volume V is thus 1 m times the area of a circle of radius 0.5 mm. Since 1 mm = 10^{-3} m, we obtain

$$V = (1\text{ m})\pi R^2 = (1\text{ m})(0.5 \times 10^{-3}\ m)^2 = 8 \times 10^{-7}\ \text{m}^3.$$

Hence, $\mu \sim 8{,}000(8 \times 10^{-7}) = 6.4 \times 10^{-3}\,\text{kg/m}^3 = 6.4\,\text{g/m}^3$. The average tension is then

$$\mathcal{T} = \mu v^2 \sim (6 \times 10^{-3})(500)2 = 1{,}500\,\text{N} \sim 300\,\text{lbs}.$$

This is the average tension in a single string. Most keys have a few strings (i.e., there is more than one string per note), so that the total number of keys is about 230. Our estimate for the total tension is then $230 \times 300 = 69{,}000$ lbs.

We have seen that the wave velocity of a vibrating string involves a characteristic force parameter (the tension) and a mass parameter (the linear mass density). We will next see how the behavior of an SHO can help us understand the expression for the wave velocity of a vibrating string. The SHO has a force parameter (the spring constant) and a mass that together determine its vibration frequency.

2.12 The Connection Between an SHO and a Vibrating String**

What has the SHO in common with a vibrating string? To simplify our analysis, we will examine the vibration of a string that is pulled aside at its midpoint and then released and allowed to vibrate.

We have seen that an SHO is characterized simply by a mass that is displaced and experiences a restoring force proportional to its displacement. The entire length of string is the corresponding mass. There is variable displacement all along the length of the string; so the system is more complicated. As an approximation we will let the displacement of the midpoint correspond to the displacement of the SHO. (See Fig. 2.20.)

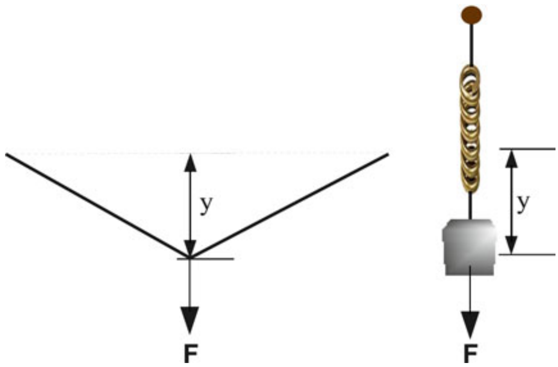

Fig. 2.20 The plucked string vs. the SHO

It can be shown that for small displacements, the restoring force on the string is proportional to the displacement. "Small" means displacement much less than the length of the string. A simpler description of this restriction is that the slope of the string during vibration must be very small. The result is

$$F = \frac{4y}{\ell}\mathcal{T} = \frac{4\mathcal{T}}{\ell}y. \tag{2.34}$$

That is, the restoring force is given by the tension $\mathcal{T}$ reduced by a factor $(4y/\ell)$, which is typically much less than unity. [For example, a guitar string of length 650 mm might vibrate with an amplitude of but a few mm.] Alternatively, we can express this relation as

$$F = \frac{4\mathcal{T}}{\ell}y. \tag{2.35}$$

We see that the restoring force is proportional to the displacement y, as in the case of an SHO. This is the essential reason that a string vibrates sinusoidally. The effective spring constant is defined by the relation $F = ky$. Thus, it is given by

$$k = \frac{4\mathcal{T}}{\ell}. \tag{2.36}$$

Sample Problem 2-4

Suppose that a string has a tension equal to 200 N and a length equal to 33 cm(=0.33 m). Find the effective spring constant.

Solution

$$k = \frac{4\mathcal{T}}{\ell} = \frac{4(200)}{0.33} = 2{,}400\,\mathrm{N/m}.$$

Sample Problem 2-5

Suppose that the previous string is displaced at its midpoint by a distance of 1 mm(=1/1,000 m). Find the restoring force.

Solution

$$F = kx = (2{,}400)(0.001) = 2.4\,\text{N}.$$

We can now combine the exact expression (2.17) for the period of an SHO with our expression (2.36) for the spring constant of the plucked string so as to obtain an expression for the **period of a plucked string**.

$$T_{\text{SHO}} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{along with} \qquad k \approx \frac{4\mathcal{T}}{\ell} \tag{2.37}$$

to obtain

$$T_{\text{SHO}} \sim 2\pi\sqrt{\frac{m}{4\mathcal{T}/\ell}} = \pi\sqrt{\frac{m\ell}{\mathcal{T}}}. \tag{2.38}$$

Finally we are ready to obtain an approximate expression for the wave velocity along a string:

$$v_{\text{approx}} = \frac{2\ell}{T_{\text{SHO}}} = \frac{2\ell}{\pi\sqrt{\frac{m\ell}{\mathcal{T}}}}. \tag{2.39}$$

Substituting μ for the ratio m/ℓ, we obtain:

$$v_{\text{approx}} = \frac{2}{\pi}\sqrt{\frac{\mathcal{T}}{\mu}} \qquad \text{wave velocity along a string.} \tag{2.40}$$

Why the difference between the two equations for the wave velocity, (2.31) and (2.40), amounting to a ratio of $2/\pi \sim 0.6$? The spring constant is the ratio of force to displacement. The displacement of an SHO is well defined with a specific value. In contrast, for a string, the displacement varies from zero at the ends of the string to its maximum value at the string's center. As a consequence, we have overestimated the average displacement and underestimated the effective spring constant. This leads to an overestimate of the fundamental period.[6]

[6]The exact expression for the fundamental period of a plucked string is

$$T = 2\sqrt{\frac{m\ell}{\mathcal{T}}}. \tag{2.41}$$

2.13 Stiffness of a String

So far, we have assumed that the vibrating string is completely flexible. No force is necessary to bend the string. The term **stiffness** is used to characterize the force necessary to bend a string.

The physical parameter that determines stiffness is the same as that which determines the force necessary to stretch a string. It is called **Young's modulus**. Why is this so? Because when a string is bent, one side of the string is stretched while the other side is compressed. This can be seen in Fig. 2.21, where a string of length $L = \pi R$ is bent into a semi-circle. The thickness of the string is $R_2 - R_1$. The outer perimeter has a length πR_2, while the inner perimeter has a length πR_1. While the outer perimeter is stretched by an amount $\pi(R_2 - R)$, the inner perimeter is compressed by an amount $\pi(R - R_1)$. Thus, the difference in the perimeters is $\pi(R_2 - R_1)$, or π times the thickness.

Note: The shape of the wave for the vibrating stiff string in the nth partial is sinusoidal even in the presence of stiffness. It is a portion of a sine wave having a wavelength

$$\lambda = 2\ell/n. \tag{2.42}$$

While we will not discuss Young's modulus because the subject is beyond the scope of this text, we can see qualitatively what effect stiffness might have on the wave velocity along a string and more importantly on the frequency spectrum of the modes.

First, we expect that stiffness contributes to bringing the string back from a curved shape toward a straight shape. It is a restoring force . Therefore, we expect that the wave velocity will increase. Next, as the wavelength decreases, the degree of bending increases. Therefore, we expect the wave velocity to increase with decreasing wavelength – or alternatively, to increase with increasing frequency.[7]

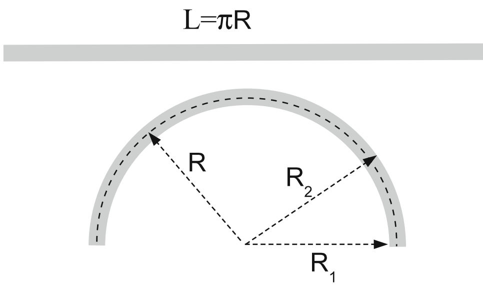

Fig. 2.21 A thick string bent into a semi-circle

[7]Mathematically, the wave velocity can be expressed as

$$v = \sqrt{\frac{\mathcal{T}}{\mu} + \frac{\overline{\mathcal{B}}}{\rho\lambda^2}}, \tag{2.43}$$

Finally, since the wavelength is inversely proportional to n, the effect of stiffness increases with increasing n. In fact, it can be shown that the frequency of the nth partial is given by

$$f_n = n\frac{\sqrt{\frac{\mathcal{T}}{\mu}}}{2\ell}\sqrt{1+\mathcal{B}n^2}, \tag{2.45}$$

where $\mathcal{B}$ is a number that is inversely proportional to the square of the length of the string.[8] Thus, the longer the string, the smaller the effect of stiffness. Relatively, the longer a string, the easier it is to bend it. Therefore, for a given mode, longer strings have less of an effect due to stiffness. Also, the constant is proportional to the fourth power of the radius of the string. As a result, thicker strings are stiffer, as we would expect. Typically, the constant $\mathcal{B}$ is much less than unity, so that the effect is small for $n = 1$. On the other hand, for large n the effect will be much more significant. Alternatively, we can write

$$f_n = nf_0\sqrt{1+\mathcal{B}n^2}. \tag{2.46}$$

Here, the parameter $f_0 = v/2\ell = \sqrt{\mathcal{T}/\mu}/(2\ell)$ is the fundamental frequency f_1 in the absence of stiffness. Note that the fundamental frequency is different in the presence of stiffness: Instead, $f_1 = f_0\sqrt{1+\mathcal{B}}$. The equation for f_n shows us plainly that the frequency spectrum is no longer a harmonic series.

To gain a sense of the order of magnitude of the constant $\mathcal{B}$, we find that for a steel wire of radius 1 mm and a length of 1 m, under a tension of 100 Newtons, $\mathcal{B} = 0.008$. For the fundamental mode, the correction to the frequency is less than 1%. However, as the mode number increases, the correction increases too, and not proportionately. We can see the effect dramatically in the graph of Fig. 2.22. The dark curve represents the spectrum with stiffness, while the straight line in magenta represents the spectrum without stiffness included.

Note that the effect on the first two modes is not great. For the third mode, the corresponding frequencies are 1,500 and 1,600 Hz, respectively – a significant difference. For the fifth mode, the difference is dramatic: 2,500 vs. 3,000 Hz.

---

where $\overline{\mathcal{B}}$ is a constant. If the tension is absent, as is the case for a suspended rod of metal – e.g., one prong of a tuning fork – the speed of transverse vibrations is given by

$$v = \sqrt{\frac{\overline{\mathcal{B}}}{\rho\lambda^2}}. \tag{2.44}$$

We see from this equation that the ratio $\overline{\mathcal{B}}/\lambda^2$ is the restoring force. Sometimes this force is referred to as the **bending force**. This is the equation for the speed of **transverse waves** along a solid rod. The rod also exhibits **longitudinal sound waves**, which will be discussed in the next chapter.

[8]Explicitly, $\mathcal{B} = (\pi Y/\mathcal{T})(\pi a^2/2\ell)^2$, where a is the radius of the string and Y is **Young's modulus**. Young's modulus determines how much of an elongation $\Delta\ell$ results from tension. Thus, if a string is under a tension $\mathcal{T}$, the relative change in its length is given by $\Delta\ell/\ell = \mathcal{T}/(\pi a^2 Y)$.

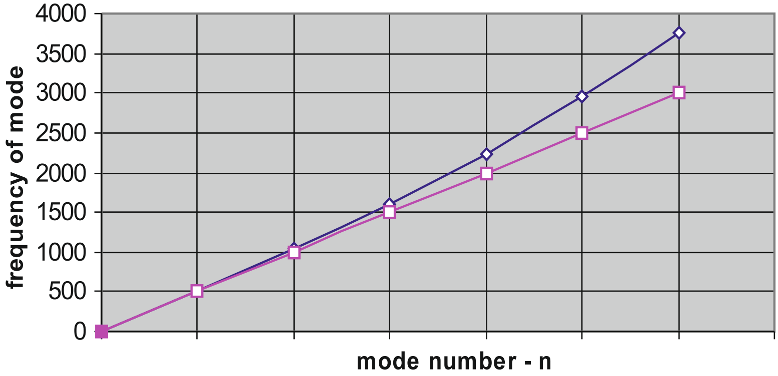

Fig. 2.22 Frequency of modes for string with (*dark curve*) and without stiffness (*magentacurve*)

2.14 Resonance

Consider the process of exciting a mode of vibration of a string. It requires that you move the end of the string up and down at a frequency equal to the frequency of that mode. If your hand moves at a frequency that is different from a mode frequency, the degree of excitation of the mode will not be great. However, the closer the frequency match is, the greater the ultimate amplitude of the excited mode.

We say that there is a **resonance** between the two systems – your hand and the string – when there is a frequency match, or practically speaking, close to a frequency match, so that there is a high degree of excitation.

For a simple example of resonance, consider two SHOs having an identical frequency f. We connect them with a fine string as shown in Fig. 2.23. Clearly, either SHO can excite vibrations in the other through the coupling between them.

Digression on the Modes of Two Coupled SHOs

Suppose that both SHOs are released from rest with the same initial displacement. Clearly, they will oscillate up and down at their common mode frequency f, because the coupling between them will be **inactive**. We say that the SHOs oscillate **in phase**. Now, suppose that the two SHOs are released from rest with the same initial amplitude but NOW with **displacements in opposite directions**. The two SHOs will oscillate up and down, always in opposite directions. We say that they oscillate **out of phase**. In this case, there will be strong coupling between the two SHOs.

What we have described above are the two modes of a pair of coupled SHOs. They are depicted in Fig. 2.24. The IN-PHASE mode has a frequency $f_{\text{in}} = f$,

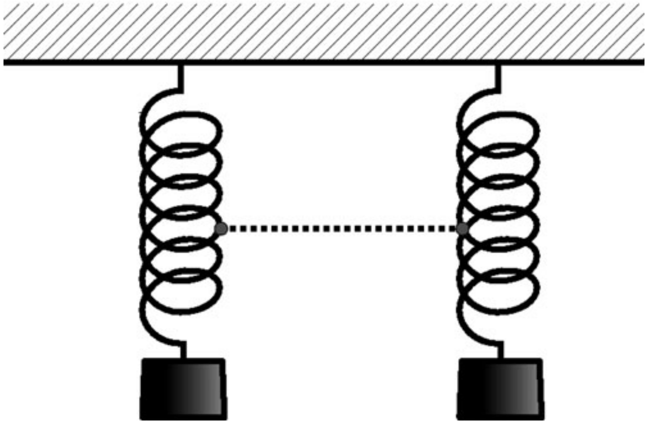

Fig. 2.23 Resonance between two coupled SHOs

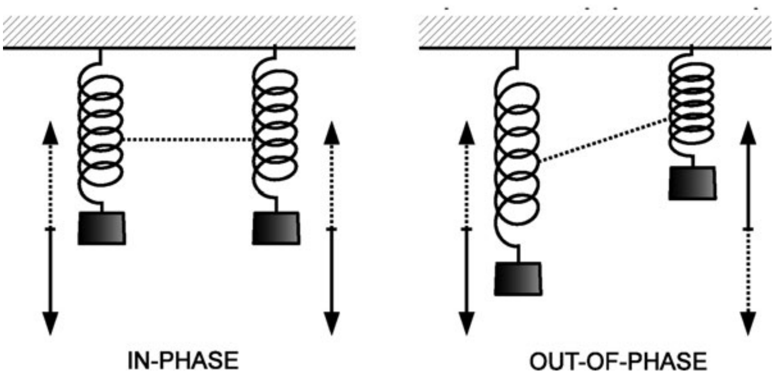

Fig. 2.24 Two modes of two coupled SHOs

while the OUT-OF-PHASE mode has a frequency f_{out} that is slightly larger. The *difference* Δf between these two frequencies increases with increasing coupling between the strings and vanishes in its absence.

It can be shown that any particular motion of the two SHOs can be expressed as a sum of the two modes. A very interesting example is the following:

Suppose that both SHOs are released from rest, with the **left** SHO displaced downward by an amount A from its equilibrium position, while the **right** SHO is kept in its equilibrium position. We see that the initial condition is a sum of the initial conditions described above for the two modes. As a consequence, the subsequent motion is a sum of the in-phase and the out-of-phase modes, with equal amplitudes $A/2$ of each in the sum. The resulting subsequent motion is quite interesting. (We neglect attenuation, for simplicity.)

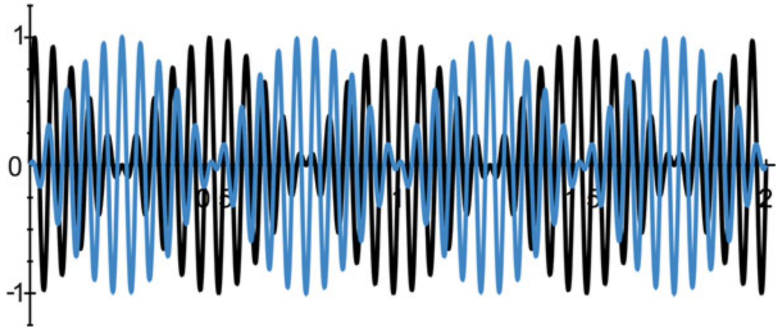

Fig. 2.25 Exchange of oscillation between two coupled SHOs

The left SHO will begin oscillating with an amplitude A. In time, that amplitude will decrease, while the right SHO will begin to oscillate. The amplitude of the left SHO will eventually momentarily vanish, while at the same time, the amplitude of the right SHO will equal A: The left SHO will have passed its energy (initially potential energy) entirely onto the right SHO!

Subsequently, the roles will be reversed, with the right SHO passing its energy back to the left SHO. Ultimately, the two SHOs will exchange energy sinusoidally at an **exchange frequency** f_{ex} that is exactly equal to the frequency Δf! The time dependence of the displacements of the two oscillators is shown in Fig. 2.25. The black curve represents the oscillation of the left SHO, while the blue curve represents the oscillation of the right SHO. Two cycles of exchange are shown. Note how the left SHO comes to rest at the time 0.5 units, where the right SHO has its maximum oscillation.[9]

2.15 General Vibrations of a String: Fourier's Theorem

Suppose that you do not move a string up and down at exactly any of the mode frequencies. The string will vibrate; however, the pattern of vibration will not resemble any one of the modes unless there is a near frequency match. Furthermore, a plucked string does not vibrate with the pattern of any of the modes, even though the pattern vibrates periodically at the fundamental frequency! **How is the general vibration of a string related to the various modes of vibration?** The answer

[9]Note that a musical instrument can have a few vibrating components, such as does the violin – the two components being a vibrating string and a vibrating wooden plate. It can be desirable to have mode frequencies of the components match: the original source of vibration – as from a bowed string – might not transfer the vibration into the air efficiently. The second component – here the wooden plate – might be able to do so efficiently. With efficient transfer of vibration to the air, the second component will not fully return the vibration back to the original source and the above annoying phenomenon will be reduced.

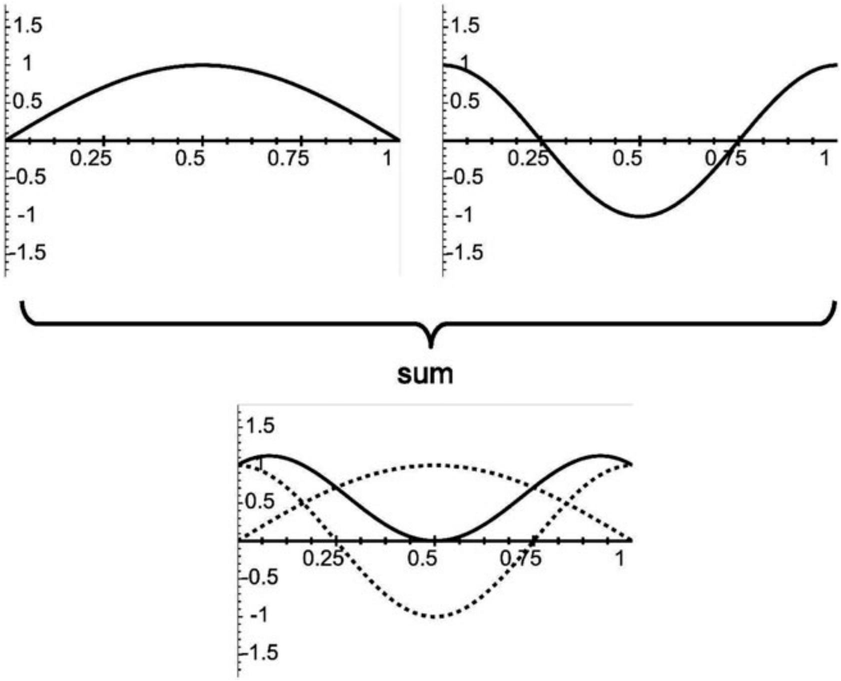

Fig. 2.26 Summing two sine waves

lies in a mathematical theorem due to **Jean Baptiste Joseph Fourier**, a French mathematician who lived from 1768 to 1830. Here is **Fourier's Theorem** in words:

> Any pattern can be expressed mathematically as a sum of sine waves, the amplitudes of each sine wave in the sum being unique.

This theorem is analogous to one of the most important theorems in number theory that any number can be expressed as a product of prime numbers. (For example, $60 = 2 \times 2 \times 3 \times 5$. The representation of the number by such a product is unique. In particular, the number of times that any prime number appears in the product is unique.) And so it is with the amplitudes of the various sine waves in the sum which represents the pattern.

Given a particular pattern, there are mathematical as well as electronic means for obtaining the unique mixture of sine waves associated with the pattern, a process known as **Fourier analysis**. Each individual sine wave is referred to as a **Fourier component**. To specify a Fourier component, we need to know three factors: its **frequency**, its **amplitude**, and its **relative phase**.

The set of frequencies in the mixture of sine waves is called the **Fourier spectrum** or simply the **frequency spectrum**. The amplitudes of the sine waves in the sum are called **Fourier amplitudes**. The "sum" is obtained by a straightforward graphical sum of the curves representing the waves. For example, in Fig. 2.26 we exhibit the sum of two Fourier components A and B, with the resultant SUM.

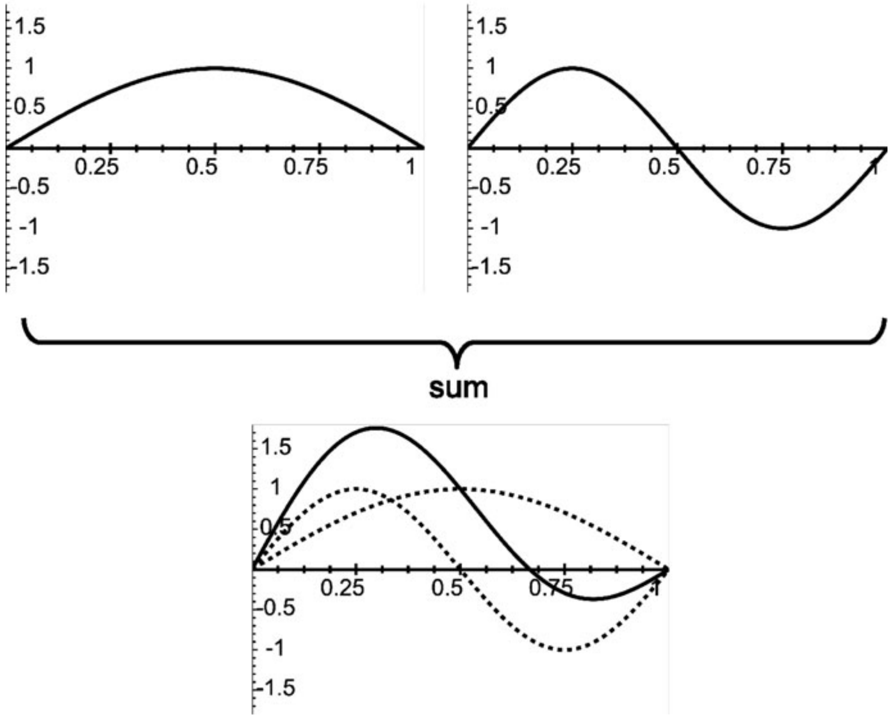

Fig. 2.27 Summing two sine waves with different phase relation from above

The **relative phase** refers to the relative positions of the waves. To appreciate the significance and importance of the relative phase, in Fig. 2.27 we exhibit the sum of the same two Fourier components as Fig. 2.26, except that component B has been shifted by a quarter of a wavelength so as to produce component C.

The reverse process of adding the given mixture of sine waves corresponding to a specific pattern so as to produce that pattern is called **Fourier synthesis**.

Generally, the frequency spectrum will include all frequencies, from zero to infinity. This is not the case for a finite vibrating string. Here, Fourier's theorem leads to the result that any pattern of vibration is a sum of the modes of vibration of the string, with a unique set of amplitudes for each mode in the sum. Hence, in this case the Fourier spectrum is a harmonic series.

There is a **corollary** to Fourier's theorem that is central to understanding the basis for obtaining a sense of pitch from musical instruments:

The Fourier spectrum of any periodic wave – and hence any sound wave that has a well-defined single pitch – must be a harmonic series with a fundamental frequency equal to the frequency of the periodic wave.

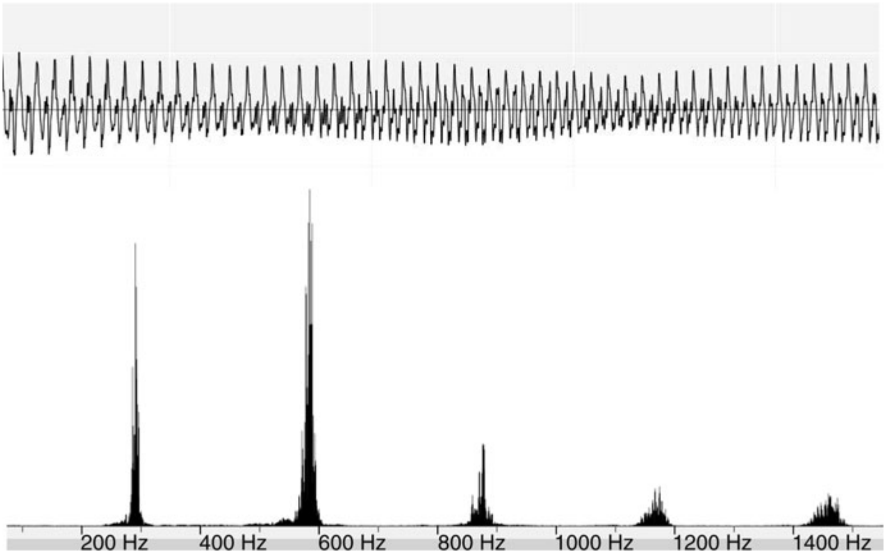

Fig. 2.28 VIOLIN wave and frequency spectrum

As a consequence, the frequency spectrum for the modes of vibration of a musical instrument that produces a well-defined pitch must be a harmonic series. In contrast, the frequency spectrum of a gong, a tuning fork, or a drum is not a harmonic series. Unless one excites one mode alone of these instruments, the sound produced will be perceived to have more than one sense of pitch. This fact is exhibited in the spectra of some musical instruments. In Fig. 2.28, we see the wave and spectrum of a short segment of sound from a violin. We can see the peaks at the harmonics with a fundamental frequency of about 280 Hz. Note the variation in the envelope of the wave, corresponding to varying loudness. More interesting are the numerous spikes surrounding the main peak. These partially reflect the **vibrato**, which is briefly discussed below. In contrast, we see in Fig. 2.29 the wave and spectrum of a segment of sound from a flute. The absence of much contribution from harmonics above the second is evident.[10]

NOTE: We must remember that the spectrum for any given instrument varies, depending upon how a note is played by the musician.

[10]The waves and spectra were produced using mp3s of instrumental sounds downloaded into the program **AmadeusPro**.

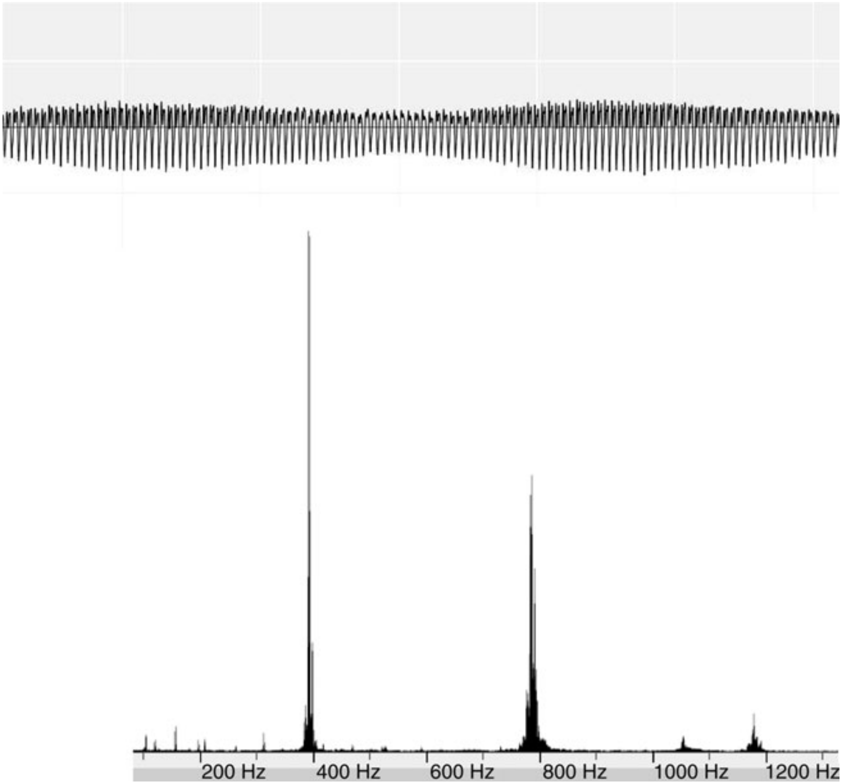

Fig. 2.29 FLUTE wave and frequency spectrum

Note

The sound of a violin exhibits characteristics that are analogous to the two most important modes of communicating audio signals with radio – AM (**amplitude modulation**) and FM (**frequency modulation**).[11]

Consider the central fundamental frequency of the above violin wave – 380 Hz. Let us suppose that the envelope of the wave oscillates at a frequency of 2 Hz. This is the **amplitude modulation frequency**. An AM radio wave of WEEI in Boston, MA, has a frequency of the **carrier wave** of 550 kHz. This frequency corresponds to the 380 Hz of the violin. If the station wants to transmit an audio signal of 400 Hz, the amplitude modulation frequency is 400 Hz.

[11] See the extremely informative applet on this website (2-15-2011): http://engweb.info/courses/wdt/lecture07/wdt07-am-fm.html#FM_Applet You can vary the modulation frequency as well as the amplitude of modulation and observe the changing wave form as well the resulting frequency spectrum.

Now let us turn to the frequency modulation, which is strongly produced by **vibrato**[12]: The fundamental frequency of the violin is determined by the position of a finger on the violin string that restricts the length that is free to vibrate. If the violinist rocks the finger back and forth on the string at a frequency of 5 Hz, the length that is free to vibrate will oscillate at this frequency and the sound will be frequency modulated at this frequency.[13] In the case of FM radio, an **FM radio wave** from WCRB-FM in Boston would have a **carrier frequency** of 89.7 MHz; the audio signal of 400 Hz would be the **FM modulation frequency**.

NOTE: The general term for the frequency of a mode of vibration of a system is the **partial**. The first partial is always equivalent to the fundamental frequency. For a vibrating string without stiffness, the set of partials forms a harmonic series.

In order to illustrate the results of a Fourier analysis of a vibrating system, consider the vibration of a string that is plucked at its midpoint, as shown previously in Fig. 2.14. It can be shown using mathematical analysis that the pattern of vibration is a sum of all the odd modes of vibration of the string. Suppose that at some instant the amplitude is $\pi^2/8$ at its midpoint. The shape of the string is triangular. The amplitudes A_1, A_2, A_3, ... of the sine waves that reproduce this pattern are given by:

$$\text{Fundamental: } A_1 = 1$$

$$\text{3rd harmonic: } A_3 = -\frac{1}{9}$$

$$\text{5th harmonic: } A_5 = \frac{1}{25}$$

$$\text{General odd harmonic: } A_n = \frac{(-1)^{(n-1)/2}}{n^2}, \qquad \text{where } n = 1, 3, 5, \ldots$$

We illustrate this result in Fig. 2.30, where we show how adding sine waves produces a triangular wave pattern. The black curve is the desired triangular wave. The blue curve is the first harmonic – $n = 1$ or the function $\sin(\pi t)$. The red curve

[12] We realize that vibrato is what gives the violin its sweet tone. However, vibrato is probably a very important factor in a number of other specific ways: For example, see Sect. 10.7 for a discussion of the fusion of harmonics and Sect. 11.7, wherein we discuss the important role that vibrato certainly plays in allowing us not to be affected by the impossibility of performing combinations of musical pure tones that are consistently consonant.

[13] It can be shown that the resulting frequency spectrum consists of a central peak at 380 Hz along with side peaks at frequencies, $380 \pm 5 = 375$ and 385, $380 \pm 10 = 370$ and 390, and $380 \pm 15 = 365$ and 395, ... The weight of these side frequencies falls off as we move to greater distances from the fundamental and depends upon the amplitude of the rocking motion.

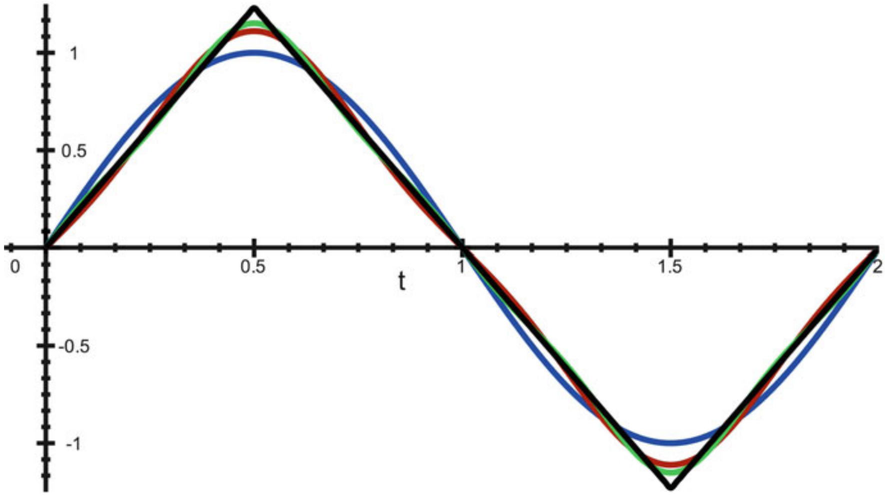

Fig. 2.30 Sum of Fourier components to produce a triangular wave

is the sum of the first and third harmonics – $n = 1$ and $n = 3$ – that is, the function $\sin(\pi t) - (1/9)\sin(3\pi t)$. We see that these two terms alone are within about 10% of reproducing the triangular wave. The green curve is the sum of the first, third, and fifth harmonic.

2.15.1 Frequency of a Wave with Missing Fundamental

It is probably obvious that a wave that includes the fundamental in its spectrum has the frequency of the fundamental. For example, the frequency of a mixture 100, 200, and 500 Hz is 100 Hz. However, consider the mixture 200 and 500 Hz. What is the frequency in this case?

Suppose we focus our attention on the displacement of a specific point along a vibrating string. According to Fourier's theorem, its wave pattern in time is a sum of sine waves, all of which are members of the harmonic series of the string's mode spectrum. It can be shown that the pattern is always **periodic, with a frequency equal to the largest common denominator (LCD) of all the frequencies in the Fourier spectrum**. For this last example, the LCD is 100 Hz, which is the frequency of wave, even though the fundamental 100 Hz is missing. We can appreciate this result as follows: During one cycle of oscillation over the period of $1/100 = 0.01$ s there will be exactly two cycles of the 200 Hz component and five cycles of the 500 Hz component.

2.16 Periodic Waves and Timbre

We can now appreciate a major factor that distinguishes the timbre of one musical instrument from another: Two instruments that are producing the same *steady* musical note are producing periodic patterns having the same frequency. It is this frequency that determines the *pitch* of the note. However, the two sets of relative amplitudes of the Fourier components are different. This difference is one of the important factors that distinguishes the timbres of musical instruments.

There are two other factors that contribute to our ability to distinguish one instrument from another when the notes are not steady but have a beginning and end; they are the **attack** and the **decay** parts of the note, which are depicted in Fig. 2.31. The variation of the amplitude – defining with the growth and final decay – is called the **envelope**. It is given by the pair of dashed curves in the figure.

Experiments have shown that in the absence of differing envelopes, it is often difficult to distinguish the sounds of different instruments.

2.17 An Application of Fourier's Theorem to Resonance Between Strings

When a string is disturbed, generally a mixture of modes is excited. The Fourier amplitudes depend upon the manner in which the string is excited. This fact has important ramifications with regards to resonance between strings: Consider two strings, one tuned to 440 Hz and the second to 660 Hz. (These two frequencies correspond to the **fundamental** frequencies of the respective strings.) The frequency spectra are:

440 Hz string: 440, 880, 1,320, 1,760, ...
660 Hz string: 660, 1,320, 1,980, 2,640, ...

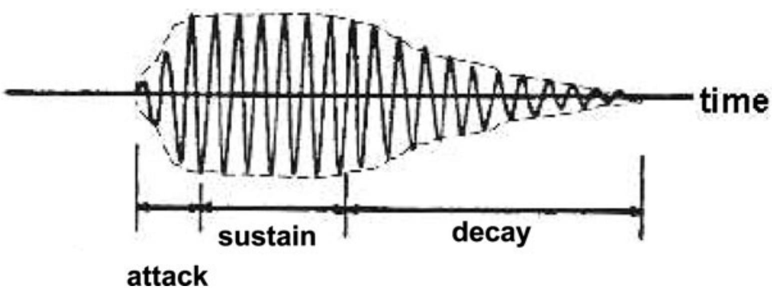

Fig. 2.31 The attack, decay, and envelope of a wave

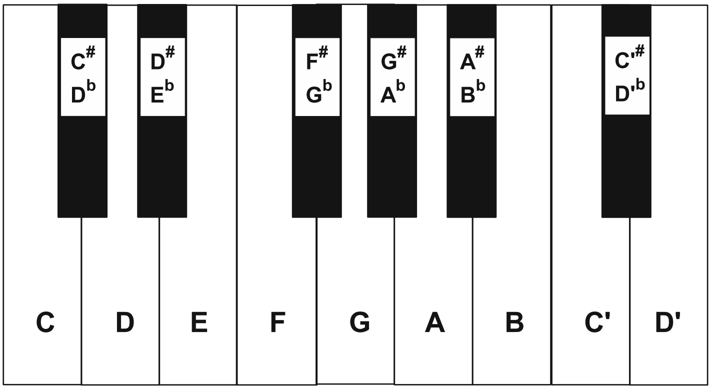

Fig. 2.32 Piano keyboard

We see that the third harmonic of the 440 Hz string and the second harmonic of the 660 Hz string have the same frequency. Thus, a general excitation of the 440 Hz string can strongly excite the second harmonic of the 660 Hz string or, a general excitation of the 660 Hz string can excite the third harmonic of the 440 Hz string.

Can you find a **second** matching pair of frequencies for the above strings? Discuss resonance between a 440 Hz string and a second string tuned to its octave at 880 Hz.

Resonance among the strings of a stringed instrument enriches tone quality. It therefore provides us with a partial explanation as to why good intonation of a string player – that is, playing notes "in tune" – improves tone quality. Conversely, by becoming more aware of the resonant response among strings, a string player can improve intonation.

Home Exercise with a Piano

If you have a piano available, you can observe the resonances discussed above as follows. Let us refer to the **piano keyboard** depicted in Fig. 2.32:

You will note that there is a pattern of the keys that repeats itself. Each cycle of keys is called an octave, with the white keys labeled from A through G. Focus on the key labeled C. This C is called *"middle-C"*. The A above middle-C is the key that is first tuned by a piano tuner, presently usually at a frequency of 440 Hz. We will call this key $A - 440$. The A above it, being one octave above, is tuned at double this frequency. The E above $A - 440$ is tuned at a frequency that is close to 660 Hz. (See Chapter 11, TUNING, INTONATION, AND TEMPERAMENT: CHOOSING FREQUENCIES FOR MUSICAL NOTES for more details on the choice of frequencies.)

Now, hold the $A - 880$ key down so as to free the string from a damper which prevents it from vibrating. Next, give the $A - 440$ key a sharp, "staccato" blow, so

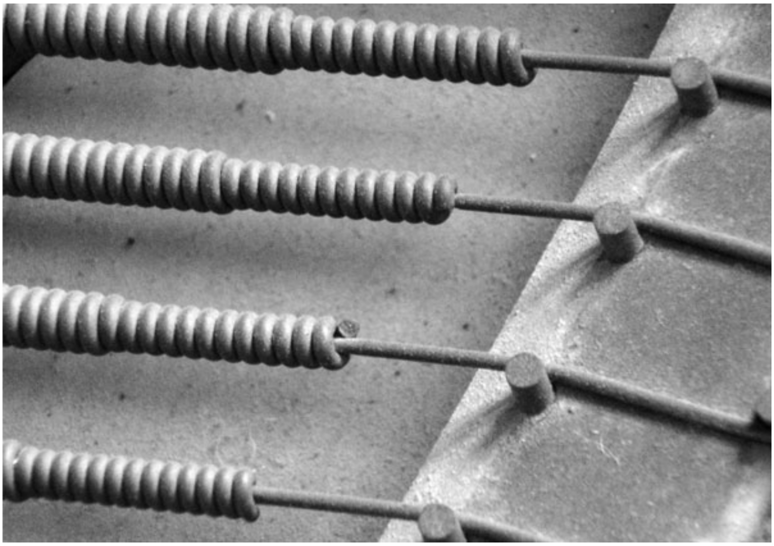

Fig. 2.33 Wound piano strings

that the $A - 440$ tone will sound long enough as to excite the $A - 880$ string, but short enough so that you can eventually hear the sound of the $A - 880$ string. To confirm that you are hearing a sound produced by the $A - 880$ string, release the $A - 880$ key so as to dampen that string's vibration.

Repeat the above by exchanging the roles of the two strings. Next, repeat all of the above with a second pair of strings – say, the $A - 440$ and $E - 660$ strings.

Note

Let us now recall that stiffness causes the frequency spectrum of a string not to be a harmonic series. The effect is strongest for the thick strings of a piano at the low end, where increased thickness is necessary to produce the low frequencies. As a result, stiffness reduces the degree of resonance among these strings. In order to reduce the effect of stiffness, these strings are constructed out of a central core of steel that is surrounded by a coil, as seen in Fig. 2.33.

In an effort to reduce the mismatch of common harmonics, pianos are **stretch tuned** – a feature discussed further in a problem of Chap. 11. Furthermore, it is interesting to note that while we tend to regard resonance as a desirable characteristic, the reduced resonance in a piano is often regarded as an attractive feature of the sound of a piano.

2.18 A Standing Wave as a Sum of Traveling Waves

A standing wave is not a traveling sine wave since it is moving neither to the right nor to the left. According to Fourier's theorem, a standing wave must be a sum of sine waves. In fact, it is a sum of two sine waves having the same wavelength and amplitude but traveling in opposite directions. This fact is depicted in Fig. 2.34. Diagrams (a)–(e) one-eighth of a cycle apart. Each diagram depicts the position of the two component sine waves and their sum. The figure reflects a special property of sine waves: **The sum of two sine waves having the same wavelength is a sine wave having the same common wavelength. The amplitude of the resultant sine wave depends upon the amplitudes of the components and their relative phase.** In our case, the components have exactly the same amplitude. In (a), the components are in phase and the resultant wave has an amplitude that is double that of the components. In (c), the components are out of phase so that the components cancel each other. Note that while the displacement vanishes everywhere in (c), the string does have an instantaneous velocity. This situation can be compared to the SHO whose mass is passing through the equilibrium position.

We can now understand how we are able to set up a standard wave along a string of finite extent: With our hand, we propagate a sine wave down the string. The reflected wave is a sine wave traveling in the opposite direction, which when added to the original wave forms a standing wave! (Of course, the observed standing wave is only a portion of an infinite standing wave.)

2.19 Terms

- Amplification
- Amplitude analyzer
- Antinode
- Attenuation
- Bending force
- Centi- 10^{-2}
- Chladni plate
- Cycle
- Damping
- Direction of propagation
- Dispersion
- Dispersive
- Displacement
- Dissipation
- Equilibrium state
- Excitation
- Force constant (or 'spring constant') **k**
- Fourier Analysis
- Fourier component
- Fourier spectrum
- Fourier Synthesis
- Fourier theorem
- Frequency **f**
- Fundamental mode
- Fusion of harmonics
- Giga- 10^9
- Gong sound
- Harmonic
- Harmonic series
- Hertz (Hz) (a unit of frequency)
- Integral multiples

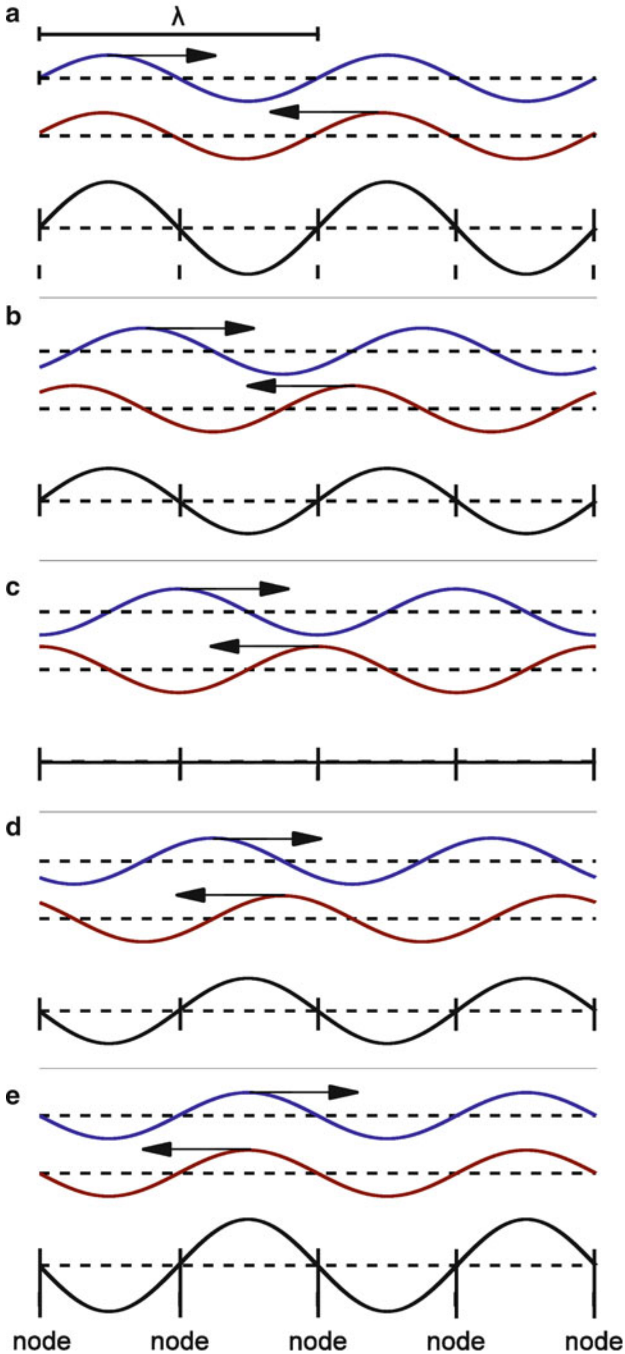

Fig. 2.34 Standing wave from two traveling waves

- Kilo- 10^3
- Largest common denominator
- Linear mass density μ
- Longitudinal wave
- Medium for wave propagation
- Mega- 10^6
- Micro- 10^{-6}
- Milli- 10^{-3}
- Nano- 10^{-9}
- Newton (N) (a unit of force)
- Nodal line
- Node
- Octave of notes oscillation
- Overtone
- Period **T**
- Periodic wave (in time or space)
- Phase relation
- Pitch
- Pluck
- Pulse
- Resonance
- Restoring force
- Simple harmonic oscillator (SHO)
- Sinusoidal
- Sonometer
- Spectrum
- Standing wave
- Stiffness
- Stretch tuning
- Stroboscope
- Tension $\mathcal{T}$
- Timbre or tone quality
- Transverse wave
- Travelling wave
- Tuning fork
- Wave propagation
- Wave velocity **v**

2.20 Important Equations

$$x = vt. \tag{2.47}$$

$$f = \frac{1}{T}. \tag{2.48}$$

$$f_1 = \frac{v}{2\ell}, f_2 = 2\frac{v}{2\ell} = \frac{v}{\ell}, f_3 = 3\frac{v}{2\ell}, \ldots. \tag{2.49}$$

$$\lambda f = v. \tag{2.50}$$

Linear mass density:

$$\mu = \frac{m}{\ell}. \tag{2.51}$$

$$v = \sqrt{\frac{\mathcal{T}}{\mu}}. \tag{2.52}$$

Hooke's Law:

$$F = ky. \tag{2.53}$$

Frequency of a Simple Harmonic Oscillator:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{2.54}$$

General form of the wave velocity:

$$v = \sqrt{\frac{\text{Restoring force}}{\text{Mass density}}}. \tag{2.55}$$

2.21 Problems for Chap. 2

1. Suppose you see a flash of lightning and then hear the sound of its thunder 5 s later. Assuming a speed of light equals 3×10^8 m/s and a speed of sound equal to 340 m/s:
   (a) Estimate the distance between you and the lightning flash.
   (b) How long did it take for the light of the lightning flash to travel from the lightning to you?
2. Consider a violin string of length 31.6 cm. Waves travel on this string with a velocity of 277 m/s.
   (a) What is the largest period a wave can have if it is to be accommodated by the string as a standing wave?
   (b) Waves of other periods can also exist as standing waves on the string. What are some of these periods
3. Suppose that there are two strings, with one string excited at the fundamental and the other excited at the second harmonic. Both are vibrating at 800 Hz with the same internodal distance of 0.45 m.
   (a) Draw a diagram depicting the two strings at the same scale.
   (b) Show that the wave velocity v is the same for the two strings and calculate its value.
4. Let us assume that a musical instrument has strings of length 76 cm. The player can press the string against the "fingerboard" so as to reduce the length of string that is free to vibrate. Suppose that when the string is fingered a distance of 8 cm from one end, leaving a length of 68 cm free to vibrate, the string vibrates with a frequency of 300 Hz.
   Where would the finger have to be placed to obtain a frequency of 311 Hz?
5. How many antinodes does a string vibrating in its fifth *overtone* have?
6. What is the fundamental frequency of a string that has five antinodes when vibrating at 450 Hz?
7. **Ernst Chladni**, who lived from 1756 to 1827, studied the vibrations of a metal plate by sprinkling sand on its surface and exciting one of its modes of vibration. A mode is distinguished by having **nodal lines**, along which the displacement of the metal plate vanishes. The frequency spectrum is *not* a harmonic series. Generally, vibrations of the plate are composed of

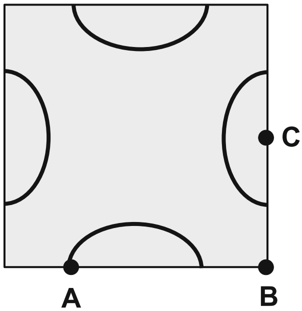

Fig. 2.35 Chladni plate

superpositions of many modes. Excitation of a single mode is facilitated by bowing the plate with a violin bow at some position along the edge and holding the plate at another position along the edge. If the plate is held horizontal, the sand particles dance around as the surface vibrates, being tossed into the air wherever there are vibrations and ultimately settling close to nodal lines. (Where you bow cannot be a node. Why so?) Such plates – in the context of its modes – are called **Chladni plates**.

A pattern of sand on a square Chladni plate is shown in the Fig. 2.35. This pattern could result from

(a) Bowing the plate at A and holding it at B
(b) Bowing the plate at C and holding it at B
(c) Bowing the plate at A and holding it at C
(d) Bowing the plate at C and holding it at A
(e) Bowing the plate at B and holding it at C

8. A pendulum swings back and forth at 20 Hz. Find its period and frequency.
9. (a) What characteristic of the relation between the displacement of an SHO and an applied force is central to its behavior and distinguishes it from other oscillators?
   (b) Specify **at least two** characteristics of the oscillation of a SHO that make it **unique**.
10. (a) Suppose an SHO has a spring constant of 32 N/m and mass of 500 g. Find its vibration frequency.
    (b) How must the mass be changed so as to double the frequency; to halve the frequency?
    (c) How must the **spring constant** be changed so as to double the frequency; to **halve** the frequency?
11. A simple harmonic oscillator (SHO) has a period of 0.002273 s. What is the frequency of the oscillator?
12. A telephone wire electrician needs to determine the tension in a 16 m segment of wire that is suspended between two poles. The wire is known to have a linear

mass density of 0.2 kg/m. He plucks the wire at one end of the segment and finds that the pulse returns in 8 s.

(a) Find the wave velocity.
(b) Find the tension.

13. A piano wire of length 2 m has a mass of 8 g and is kept under a tension of 160 N.

    (a) Find the wave velocity along the wire.
    (b) To double the wave velocity:

        i. The tension can be changed to _______.
        ii. The mass can be changed to _______.

14. (a) A tightrope walker tends to avoid walking at a pace equal to a multiple of the fundamental frequency of the tightrope. **Explain why.**
    Describe what would happen if he were to do so.
    Now suppose that the tightrope is 25 m long, has a mass per unit length of 0.2 kg/m, and has a tension of 2,000 N.
    (b) Calculate the speed of wave propagation along the rope.
    (c) Calculate the rope's fundamental frequency and its two **lowest** overtone frequencies.
15. (a) What is stiffness in a string?
    (b) Does it exist in the absence of tension?
    (c) How does stiffness affect the frequency spectrum of a vibrating string?
    (d) Because of their longer strings, grand pianos need less stretch tuning than upright pianos. Give **two** reasons why this is so.
16. Express the wavelength of a standing wave in terms of the distance between nodes. Now do so in terms of the distance between antinodes.
17. (a) Find the wavelength of a sound wave in water (with a wave velocity of 1,400 m/s) that has a frequency of 10 kHz.
    (b) Find the frequency of a light wave in vacuum that has a wavelength of 0.5 μm ($=5 \times 10^{-7}$ m).
18. (a) Suppose that a guitar string is plucked at its midpoint. Which Fourier components **CANNOT** be excited?
    (b) Repeat the previous question when the string is plucked at a point 1/3 from one end.
19. Explain how a vibrating 440 Hz string can cause a 550 Hz string to vibrate.
20. Find the frequency and period of a periodic wave whose **only** Fourier components are equal to the following:

    (a) 500 Hz, 1,000 Hz.
    (b) 500 Hz, 800 Hz, and 1,000 Hz.

21. What is special about the Fourier frequency spectrum of a periodic wave?

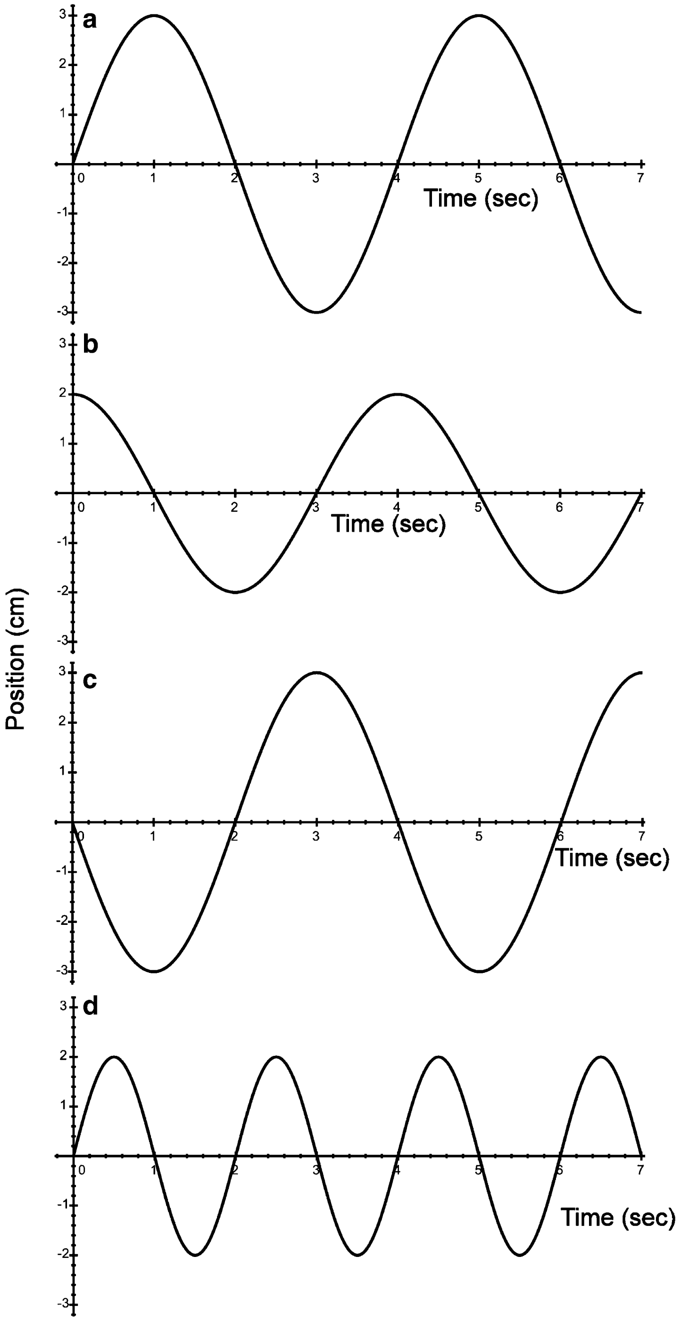

Fig. 2.36 Four different sinusoidal waves

22. Using (2.26), (2.27), and (2.43), derive equation (2.45).
23. In Fig. 2.36 are depicted the displacement vs. time of four wave patterns, labeled A–D:

    (a) What are the respective amplitudes of patterns A and B?
    (b) Which two waves differ **only** in phase?
    (c) What is the frequency of pattern A?

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The Physics of Music and Color
Gunther, L.
2012, XXII, 540 p., Hardcover
ISBN: 978-1-4614-0556-6