Convexity

Gabriel Nivasch

March 22, 2020

Recall affine combination

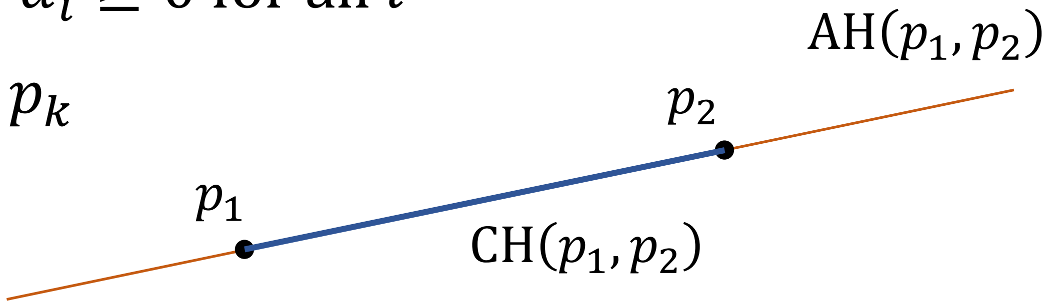
$$w = \alpha_1 p_1 + \cdots + \alpha_k p_k \qquad \text{where } \alpha_1 + \cdots + \alpha_k = 1$$

w is an AC of $p_1, \dots, p_k$

Convex Combination

$$w = \alpha_1 p_1 + \cdots + \alpha_k p_k \qquad \text{where: } \begin{array}{c} \alpha_1 + \cdots + \alpha_k = 1 \\ \alpha_i \geq 0 \text{ for all } i \end{array}$$

w is a ***convex combination*** of $p_1, \dots, p_k$



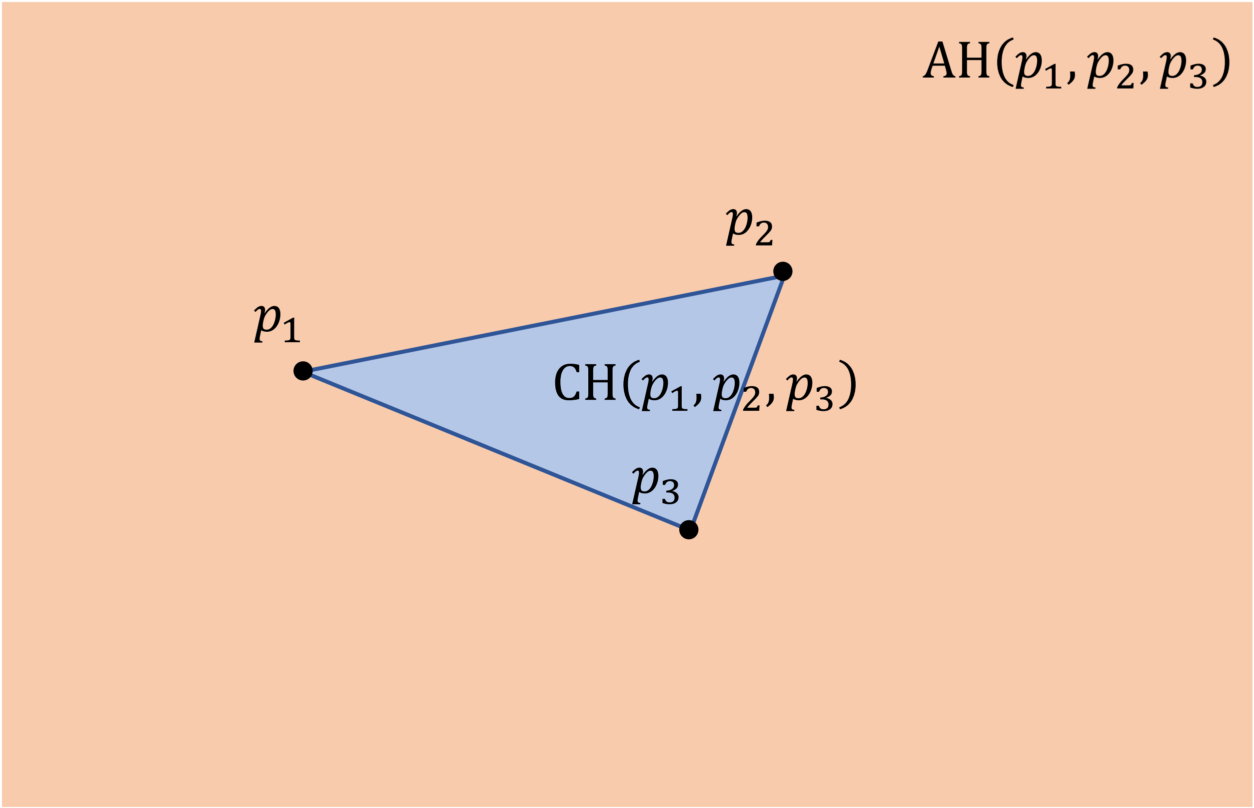
Convex Hull

Set of all convex combinations

(More precisely: $\mathrm{CH}(P)$ = set of all convex combinations of finitely many points of P.)

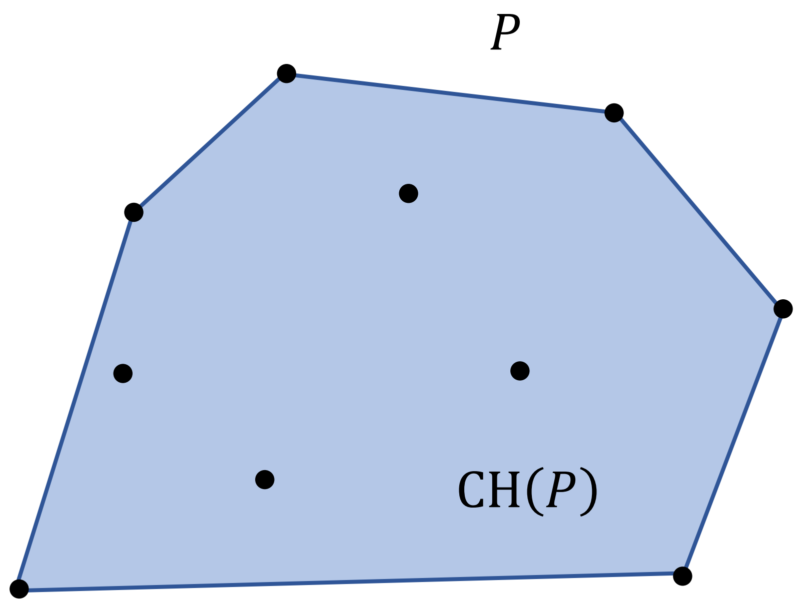
Convex hull

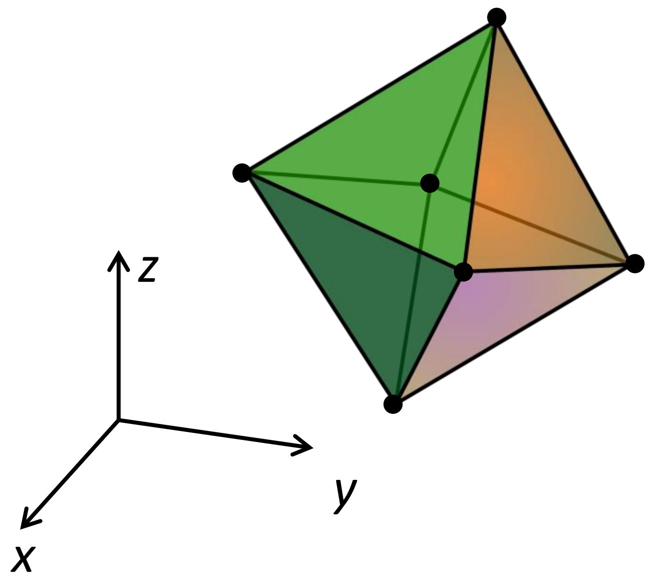
Examples:



Convex hull

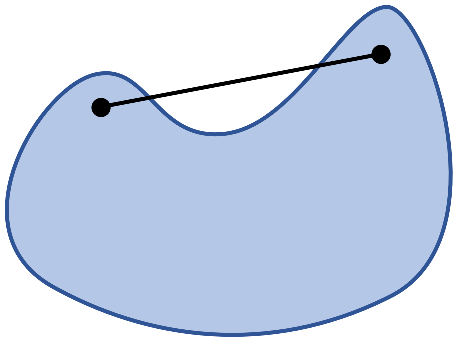
Examples:





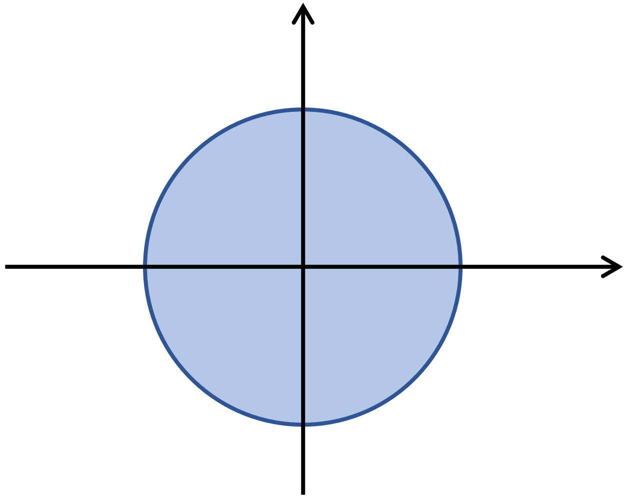
Convex set

A subset $S \subseteq \mathbb{R}^d$ is ***convex*** if for every two points $a, b \in X$, the whole segment ab is contained in S



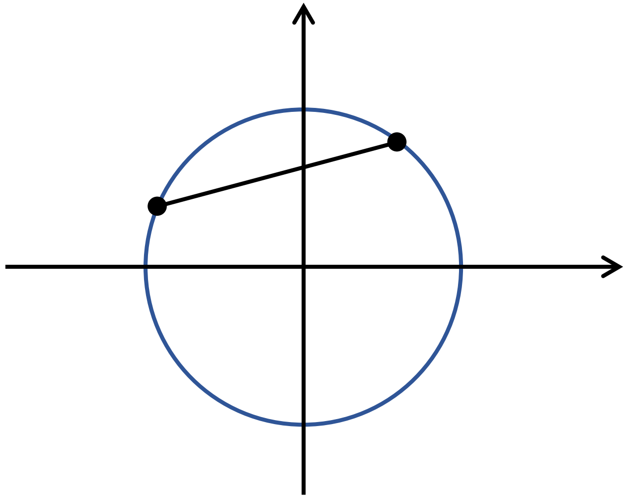
Not convex

$S = \{(x, y) : x^2 + y^2 \leq 1\}$



Convex

$S = \{(x, y) : x^2 + y^2 = 1\}$



Not convex

Segment ab: $\{\gamma a + (1 - \gamma)b : 0 \leq \gamma \leq 1\} \;\; = \mathrm{CH}(a, b)$

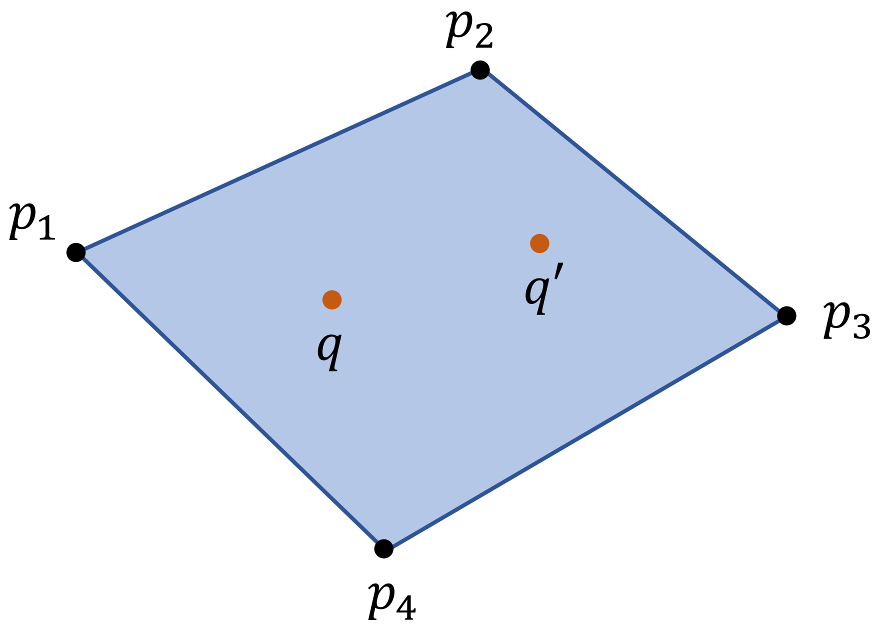
- Convex hull: Set of all convex combinations of finitely many points
- Convex set: A set that contains all segments between its points

Exercise 1: Prove that every convex hull is a convex set

Carathéodory's Theorem

Theorem: Let $P \subseteq \mathbb{R}^d$, and let $q \in \mathrm{CH}(P)$. Then q is a convex combination of at most $d + 1$ points of P.

Example in the plane:



$q \in \mathrm{CH}(p_1, p_2, p_4)$

$q' \in \mathrm{CH}(p_2, p_3, p_4)$

Carathéodory's Theorem

Theorem: Let $P \subseteq R^d$, and let $q \in \mathrm{CH}(P)$. Then q is a convex combination of at most $d+1$ points of P.

Proof: q is a CC of some points of P.
We will show that, as long there are at least $d+2$ points in the CC, we can drop one point.

$$q = \sum \alpha_i p_i \qquad \sum \alpha_i = 1 \qquad \alpha_i \geq 0$$

$$\sum \beta_i = 0$$

p_i are AD $\quad \gamma \times \left(\vec{0} = \sum \beta_i p_i \right) \quad +$

$q = \cdots \longleftarrow$ Another AC

$\nwarrow$ Still a CC

Increase γ gradually from 0 until first time a coefficient becomes 0

QED

Exercise 2: Let

$$p_1 = (1,2,1,1)$$
$$p_2 = (2,0,1,0)$$
$$p_3 = (4,3,2,1)$$
$$p_4 = (3,2,3,1)$$
$$p_5 = (2,3,3,2)$$
$$p_6 = (3,1,1,3)$$

and let $q \in \mathrm{CH}\{p_1, \ldots, p_6\}$ be given by

$$q = \tfrac{1}{10}p_1 + \tfrac{3}{10}p_2 + \tfrac{1}{10}p_3 + \tfrac{2}{10}p_4 + \tfrac{2}{10}p_5 + \tfrac{1}{10}p_6$$

Using the argument in the proof of Carathéodory's theorem, express q as a convex combination of 5 of the 6 points.

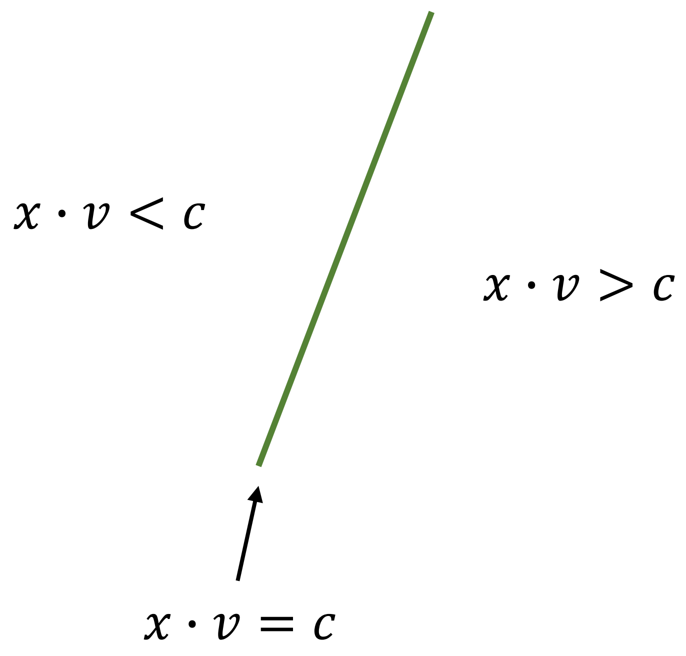
Separation Theorem

"Every two disjoint convex sets can be separated by a hyperplane"

Version 1: Let $P, Q \subseteq \mathbb{R}^d$ be disjoint convex sets. Then there exist $v \in \mathbb{R}^d$ and $c \in \mathbb{R}$ such that

$$x \cdot v \leq c \quad \text{for every} \quad x \in P$$
$$x \cdot v \geq c \quad \text{for every} \quad x \in Q$$

This is "weak" separation



Separation Theorem

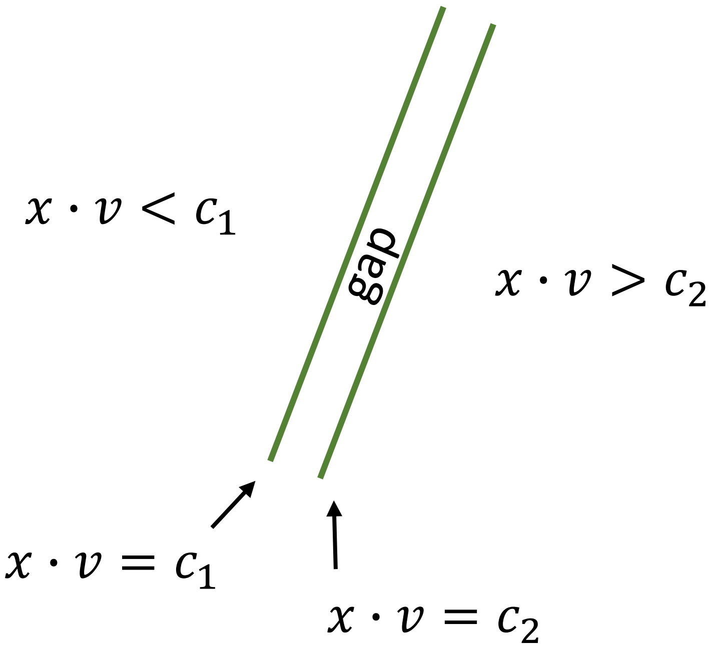
"Every two disjoint convex sets can be separated by a hyperplane"

Version 2: Let $P, Q \subseteq \mathbb{R}^d$ be disjoint compact convex sets. Then there exist $v \in \mathbb{R}^d$ and $c_1, c_2 \in \mathbb{R}$ with $c_1 < c_2$ such that

$x \cdot v \leq c_1$ for every $x \in P$

$x \cdot v \geq c_2$ for every $x \in Q$

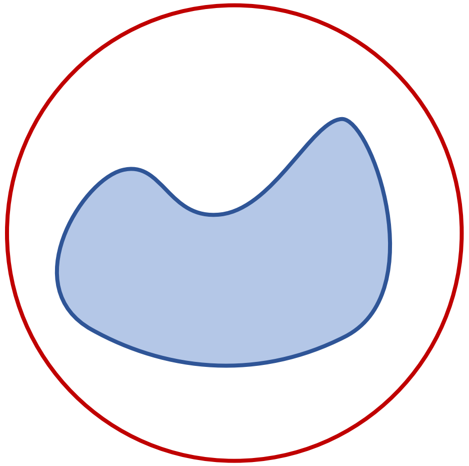
This is "strong" separation



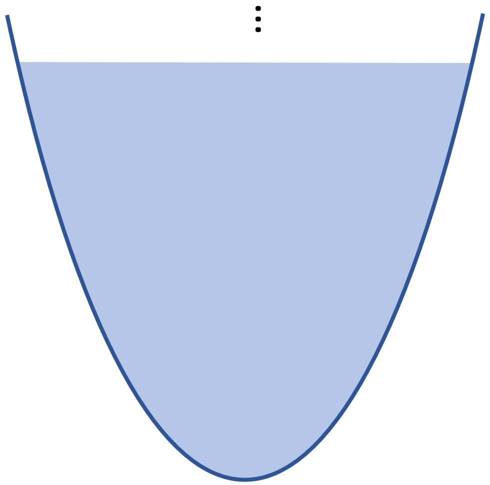
Background on topology

Compact set: Closed + bounded

Bounded set: Contained in some ball of finite radius



bounded



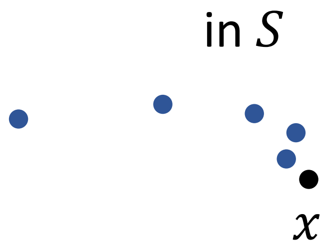
not bounded

Background on topology

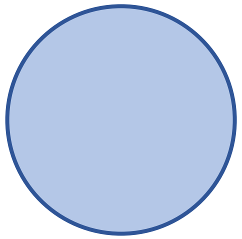
Compact set: Closed + bounded

Closed set: Contains all its limit points

Limit point: Let $S \subseteq \mathbb{R}^d$. A point $x \in \mathbb{R}^d$ (not necessarily in S) is a ***limit point*** of S if for every $\varepsilon > 0$ the set S contains a point $q \neq x$ with $\|q - x\| \leq \varepsilon$

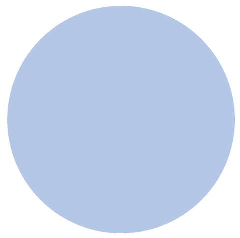


$\{(x,y) | x^2 + y^2 \leq 1\}$



closed

$\{(x,y) | x^2 + y^2 < 1\}$



not closed

Back to Separation Theorem

Version 1: Let $P, Q \subseteq \mathbb{R}^d$ be disjoint convex sets. Then there exist $v \in \mathbb{R}^d$ and $c \in \mathbb{R}$ such that

$$x \cdot v \leq c \quad \text{for every} \quad x \in P$$
$$x \cdot v \geq c \quad \text{for every} \quad x \in Q$$

Version 2: Let $P, Q \subseteq \mathbb{R}^d$ be disjoint <u>compact</u> convex sets. Then there exist $v \in \mathbb{R}^d$ and $c_1, c_2 \in \mathbb{R}$ with $c_1 < c_2$ such that

$$x \cdot v \leq c_1 \quad \text{for every} \quad x \in P$$
$$x \cdot v \geq c_2 \quad \text{for every} \quad x \in Q$$

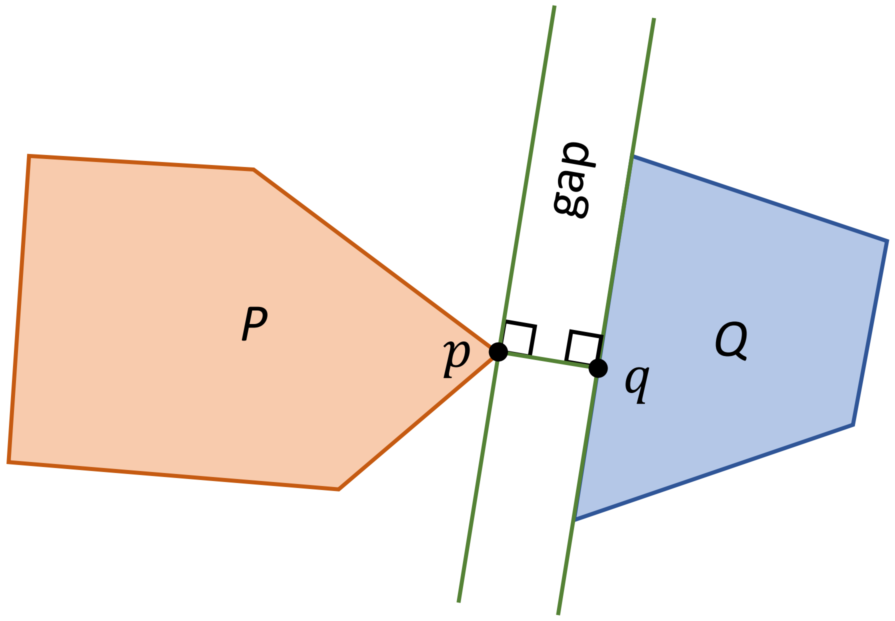
We will sketch the proof of Version 2

Back to Separation Theorem

Version 2: Let $P, Q \subseteq \mathbb{R}^d$ be disjoint <u>compact</u> convex sets. Then there exist $v \in \mathbb{R}^d$ and $c_1, c_2 \in \mathbb{R}$ with $c_1 < c_2$ such that

$x \cdot v \leq c_1$ for every $x \in P$

$x \cdot v \geq c_2$ for every $x \in Q$



Proof sketch:

There exist $p \in P, q \in Q$ with minimal distance

Take hyperplanes through p and q perpendicular to the segment pq

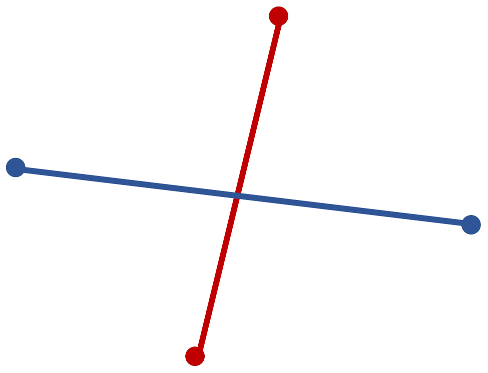
Exercise 3: Neither P nor Q can contain a point in the gap

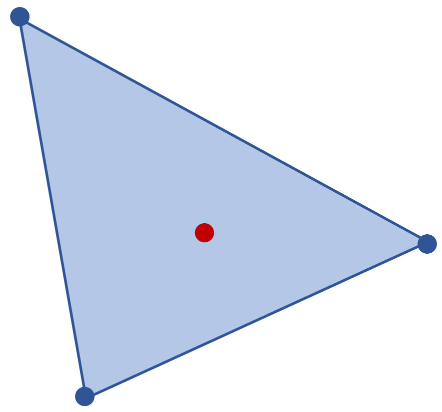
QED

Radon's Lemma

Lemma: Let P be a set of $d+2$ points in $\mathbb{R}^d$. Then P can be partitioned into two subsets whose convex hulls intersect.

Examples in the plane:





Exercise 4: Make all possible types of examples in $\mathbb{R}^3$

Radon’s Lemma

Lemma: Let P be a set of $d+2$ points in $\mathbb{R}^d$. Then P can be partitioned into two subsets whose convex hulls intersect.

Proof: P is AD.

$$\vec{0} = \sum \beta_i p_i \qquad\qquad \sum \beta_i = 0$$

Some coefficients are positive, some are negative

Move terms with negative coefficients to the LHS

Now on both sides all coefficients are positive, and on both sides the coefficients add up to the same value z

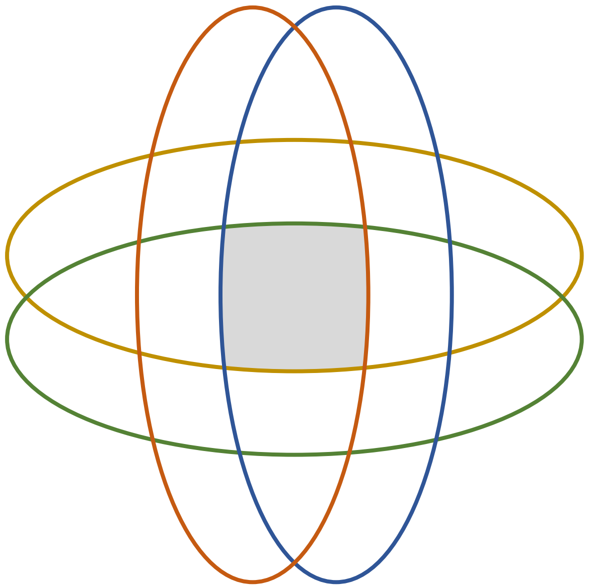
Divide both sides by z

QED

Helly's Theorem

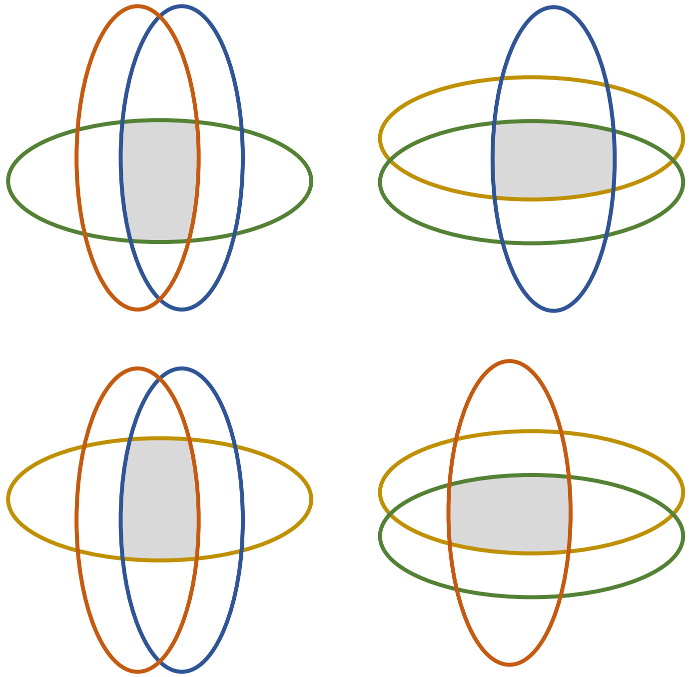
Theorem: Let $C_1, \ldots, C_n$, $n \geq d + 1$, be convex sets in $\mathbb{R}^d$.
Suppose every $d + 1$ have a common intersection.
Then they all have a common intersection.

Example:



They all intersect

Every 3 intersect:



Exercise 5: Show that the convexity requirement is necessary

Helly’s Theorem

Theorem: Let $C_1, \dots, C_n$, $n \geq d+1$, be convex sets in $\mathbb{R}^d$.
Suppose every $d+1$ have a common intersection.
Then they all have a common intersection.

Proof sketch:

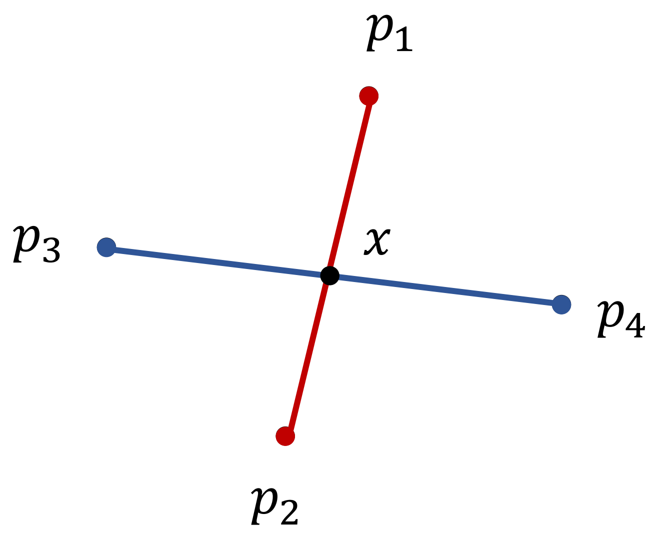
- Case $n = d+1$: Nothing to prove
- Case $n = d+2$: We’ll prove below
- Cases $n \geq d+3$: By induction on n (**Exercise:** See Wikipedia)

Helly's Theorem

Claim: Let $C_1, \dots, C_{d+2}$ be convex sets in $\mathbb{R}^d$.
Suppose every $d+1$ have a common intersection.
Then they all have a common intersection.

Proof: We're given that for each i, all sets except for C_i have a point in common.
Call that point p_i.

We have $d+2$ points $p_1, \dots, p_{d+2}$. Apply Radon's lemma to them.
Get point x.



$x \in C_1$? Yes because $p_3 \in C_1,\ p_4 \in C_1$

$x \in C_2$? Yes because $p_3 \in C_2,\ p_4 \in C_2$

$x \in C_3$? Yes because $p_1 \in C_3,\ p_2 \in C_3$

$x \in C_4$? Yes because $p_1 \in C_4,\ p_2 \in C_4$

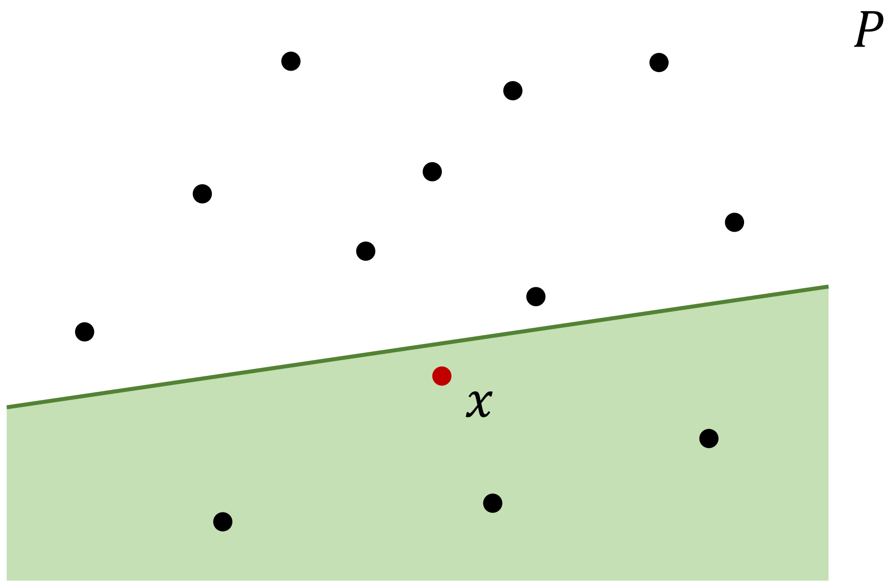
QED Claim

Helly's Theorem

Exercise: Show that Helly's Theorem does not hold if we have infinitely many convex sets.

Centerpoint Theorem

Definition: Let $P \subset \mathbb{R}^d$ be a finite point set. Let $x \in \mathbb{R}^d$ be a point.
The ***half-space depth*** of x in P is the minimum number of points of P contained in a half-space that contains x.

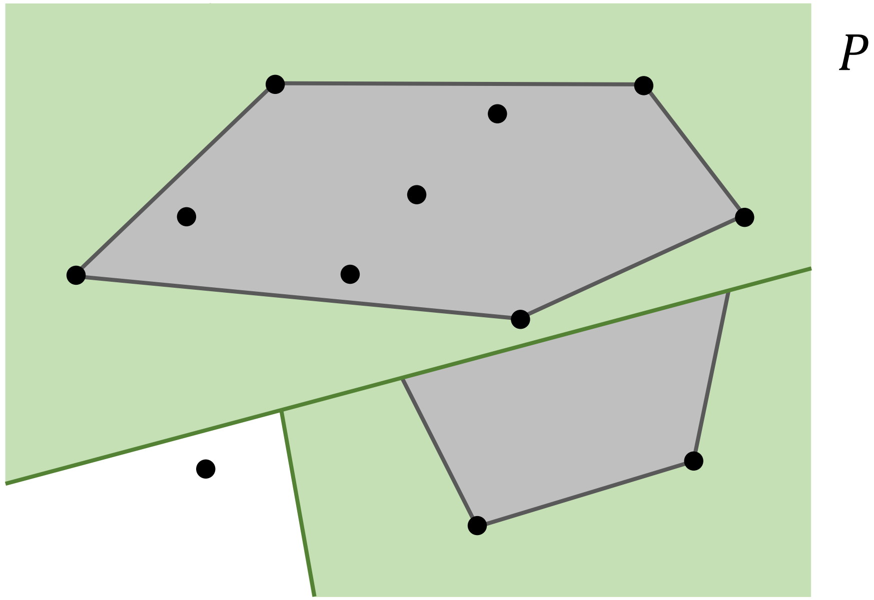


Half-space depth of x = 3

Theorem: Let $n = |P|$. Then there exists a point $x \in \mathbb{R}^d$ with half-space depth at least $n/(d+1)$.

Centerpoint Theorem

Proof: Take all subsets of P of size larger than $\frac{d}{d+1}n$ that can be separated by a hyperplane.



$n = 12$

$d = 2$

Size at least 9

Take their convex hulls. We obtain a finite family of convex sets.

Every $d + 1$ of them intersect (by the PHP).

By Helly's Lemma, they all intersect at a point x.

x is the desired point.

QED