Review of Linear Algebra

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Vectors

$$\begin{bmatrix} 3.6 \\ 5 \\ -4.1 \end{bmatrix} \in \mathbb{R}^3 \qquad \text{or} \qquad (3.6, 5, -4.1) \in \mathbb{R}^3$$

Linear Combination (LC) of vectors

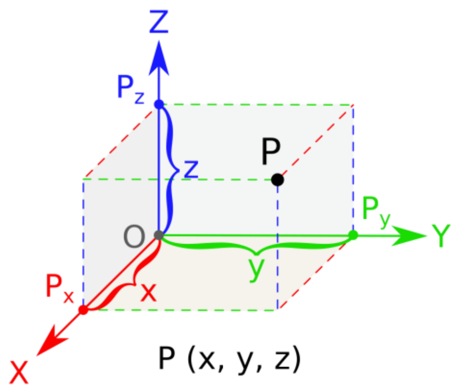
scalars (in $\mathbb{R}$)

v_1 $\quad$ v_2 $\quad$ v_3

vectors in $\mathbb{R}^d$

w is a LC of v_1, v_2, v_3

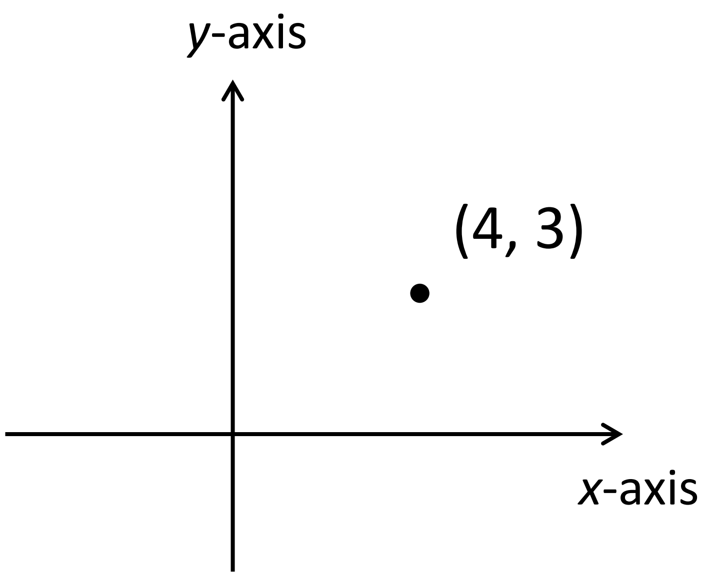
Graphical representation of vectors



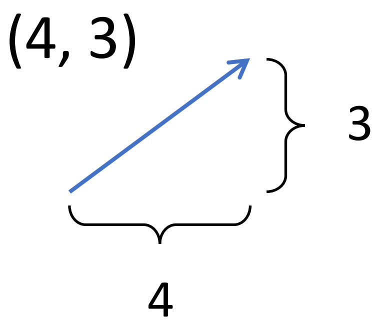
$\mathbb{R}^2$: the plane

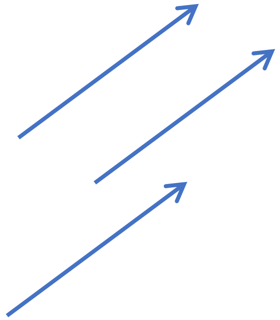
$\mathbb{R}^3$: three-dimensional space

point representation



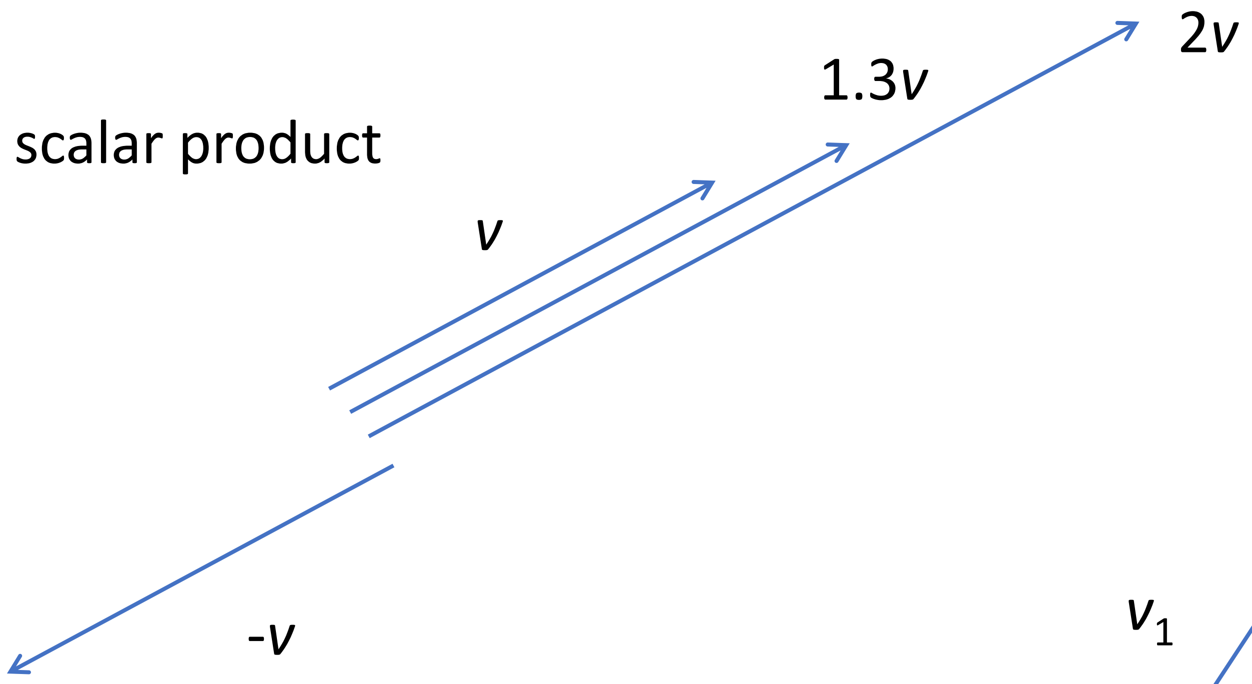
arrow representation

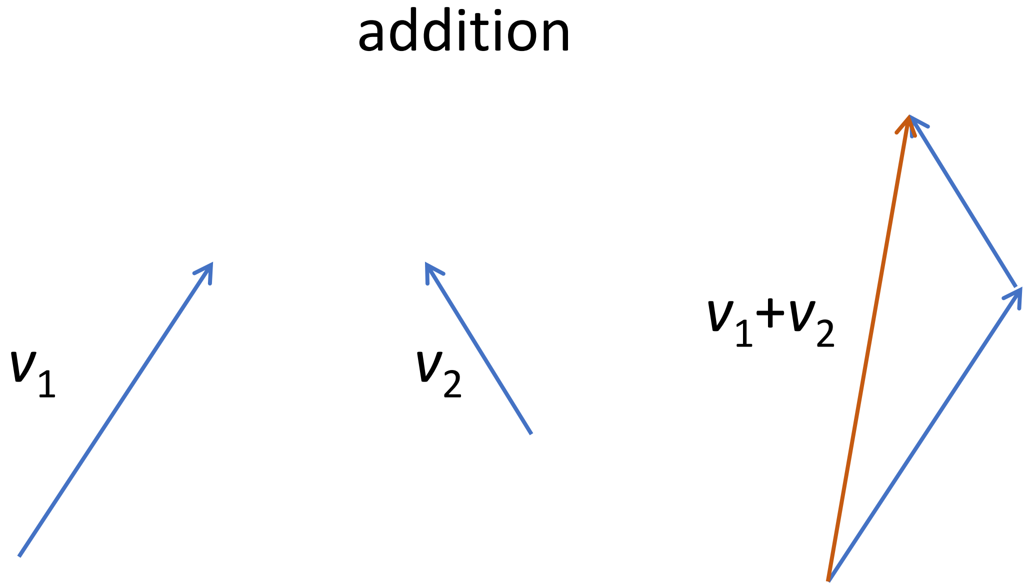




all the same vector

Operations in arrow representation

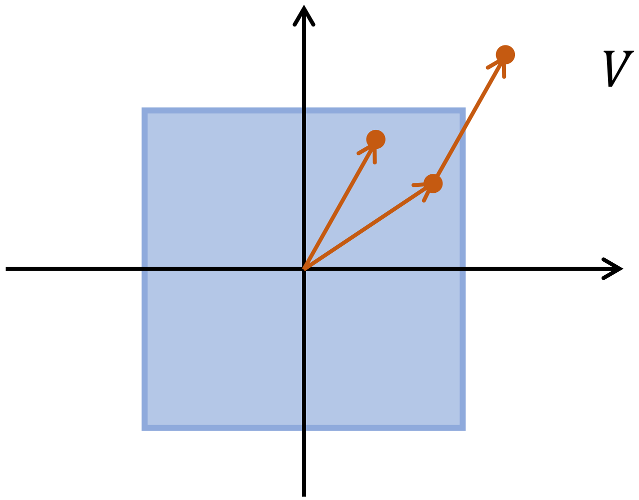




Linear subspaces

$V \subseteq \mathbb{R}^d$ is a ***linear subspace*** of $\mathbb{R}^d$ if:

- $\vec{0} \in V$
- V is closed under addition
- V is closed under scalar multiplication



$$V = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$$

not closed under addition

(also not under scalar multiplication)

Linear subspaces

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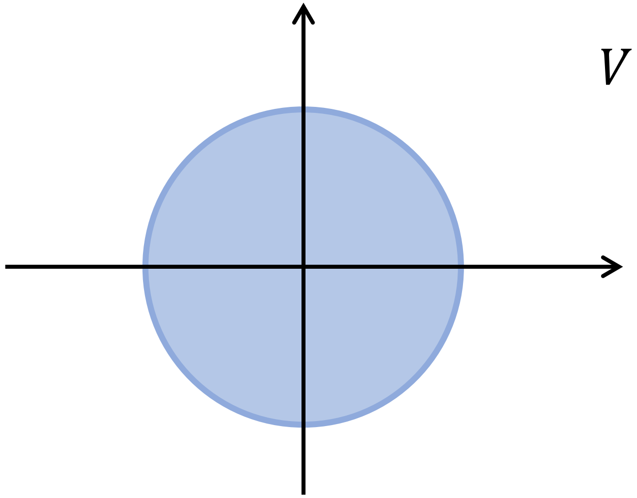

$V = \{(x, y) : x^2 + y^2 \leq 1\}$

- not closed under addition
- not closed under scalar multiplication

Linear subspaces

$V \subseteq \mathbb{R}^d$ is a ***linear subspace*** of $\mathbb{R}^d$ if:

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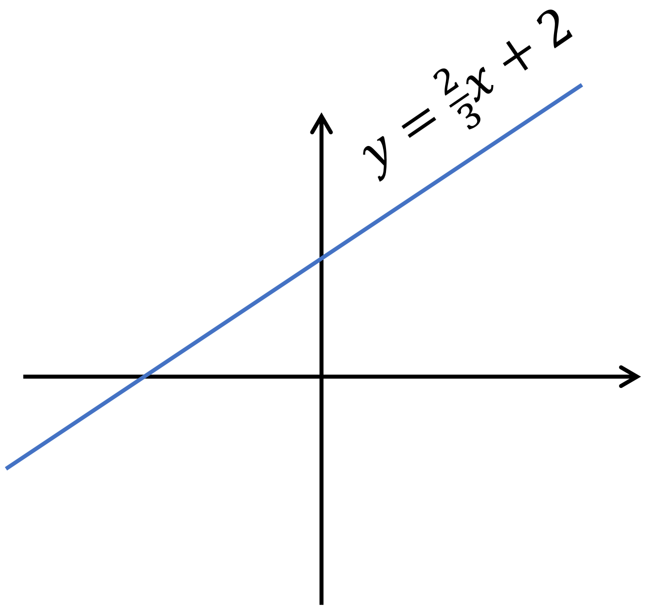
- V is closed under scalar multiplication



Line not through the origin

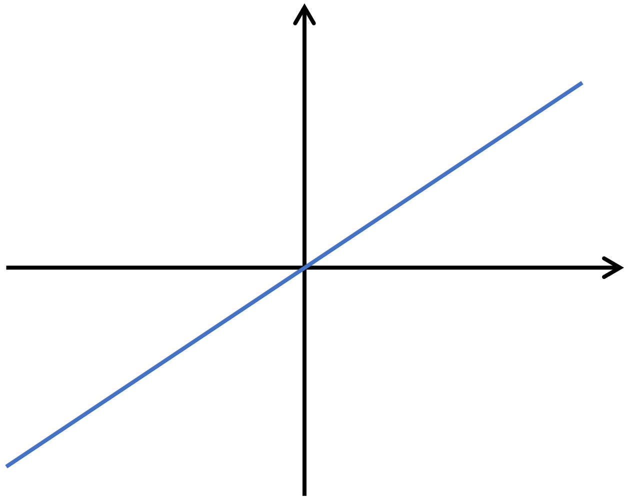
$$V = \left\{(x, y) : y = \tfrac{2}{3}x + 2\right\}$$

- does not contain origin
- not closed under addition
- not closed under scalar multiplication

Linear subspaces

$V \subseteq \mathbb{R}^d$ is a ***linear subspace*** of $\mathbb{R}^d$ if:

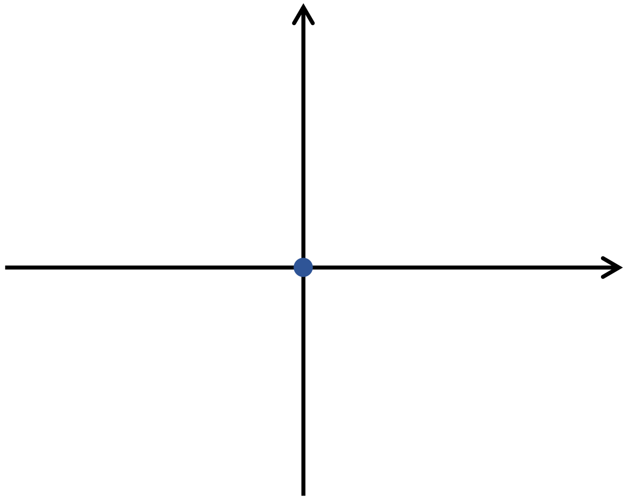
- $\vec{0} \in V$
- V is closed under addition
- V is closed under scalar multiplication



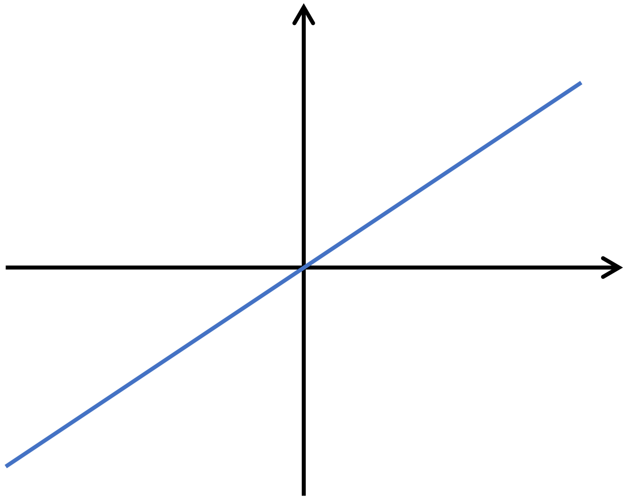
Line through the origin

✓ Linear subspace

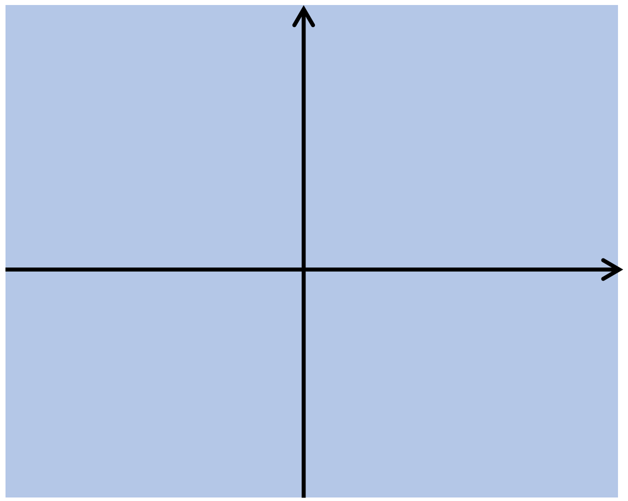
Linear subspaces of $\mathbb{R}^2$



Just the origin



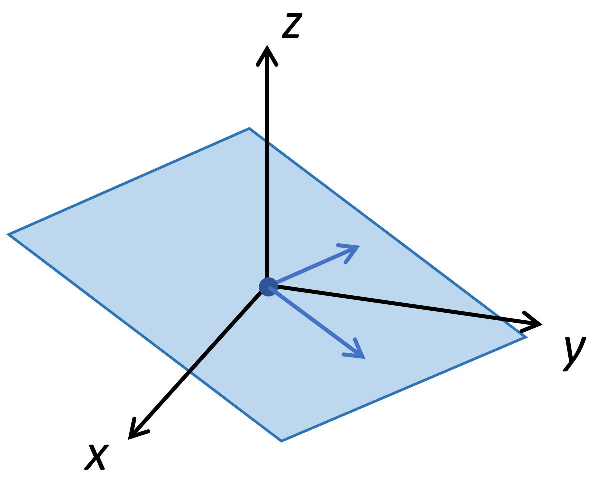
Line through the origin



The whole plane

Linear subspaces of $\mathbb{R}^3$

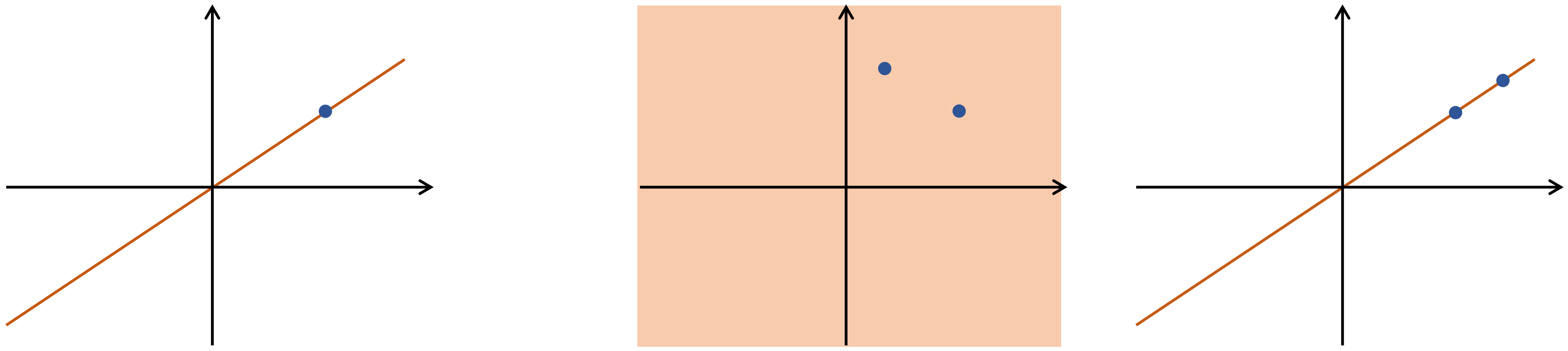
- Just the origin
- Line through the origin
- Plane through the origin
- The whole space



plane through the origin

Linear Span (LS) of vectors

Set of all Linear Combinations



Exercise 1: Prove that $\mathrm{LS}(v_1, \dots, v_k)$ is always a linear subspace.

Linearly Dependent (LD) and Linearly Independent (LI) vectors

Definition 1: Vectors $v_1, \dots, v_k$ are ***LD*** if one of them is a Linear Combination of the others

$$v_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad v_2 = \begin{bmatrix} 5 \\ -7 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$v_2 = 2v_1 - \tfrac{1}{2}v_3$$

v_2 is a LC of v_1, v_3

v_1, v_2, v_3 are LD

Definition 2: Vectors $v_1, \dots, v_k$ are ***LD*** if there exist scalars $\alpha_1, \dots, \alpha_k$, not all zero, such that

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = \vec{0}$$

$$2v_1 - v_2 - \tfrac{1}{2}v_k = \vec{0}$$

Def 2 is more convenient than Def 1. Def 2 is the real definition.

Example: Are these vectors LD?

$$v_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad v_2 = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

Do there exist scalars α_1, α_2, not both zero, such that $\alpha_1 v_1 + \alpha_2 v_2 = \vec{0}$?

No. The only solution is $\alpha_1 = \alpha_2 = 0$

They are LI

Exercise 2: Is a single vector v_1 LI or LD? (According to Definition 2)

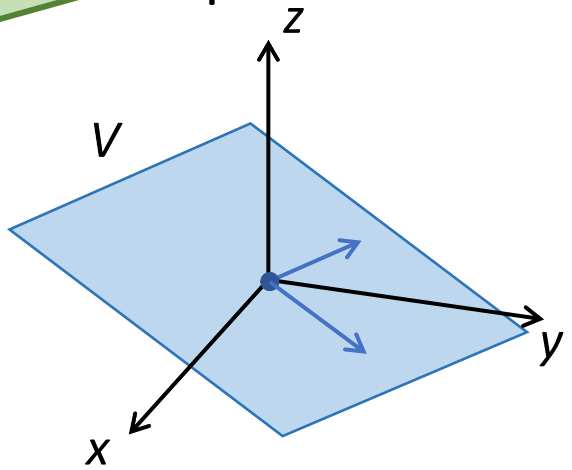
Basis

“There are enough vectors”

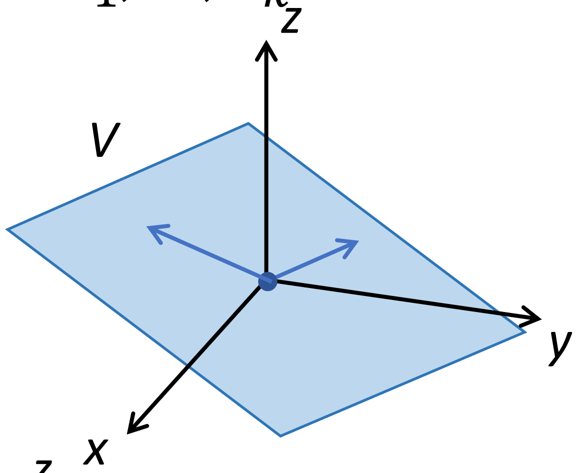
“There are not too many vectors”

Let V be a linear subspace of $\mathbb{R}^d$. A basis of V is a sequence of vectors $v_1, \dots, v_k \in V$ that satisfies:

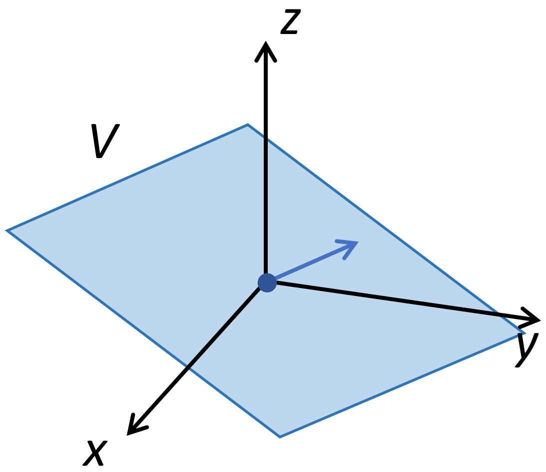
- $\mathrm{LS}(v_1, \dots, v_k) = V$
- $v_1, \dots, v_k$ are LI



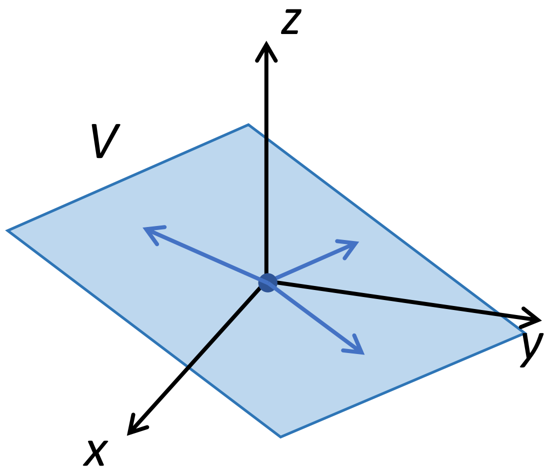
basis



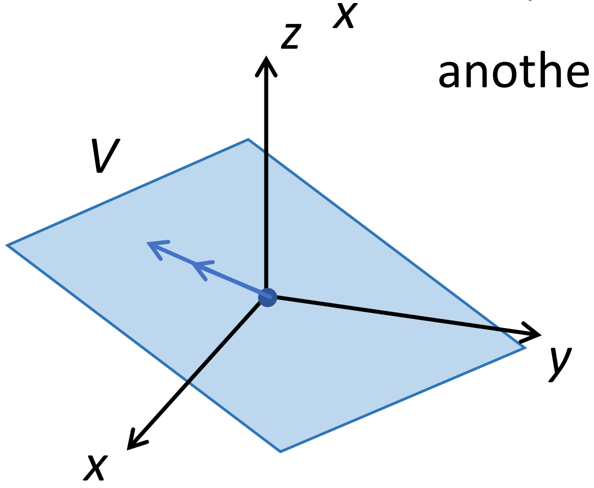
another basis



not basis



not basis



not basis

Dimension

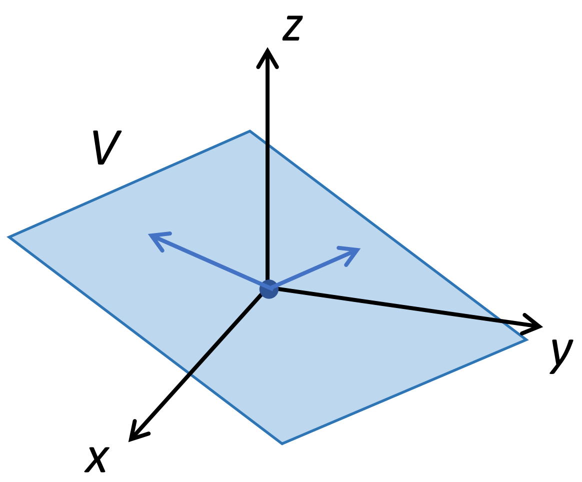
Theorem: All bases of V have the same number of vectors.

That number is called the ***dimension*** of V.

Representation of linear subspaces

There are two ways to represent linear subspaces:

- With vectors
- With homogeneous equations



$$V = \mathrm{LS}\left(\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}-1\\3\\2\end{bmatrix}\right)$$

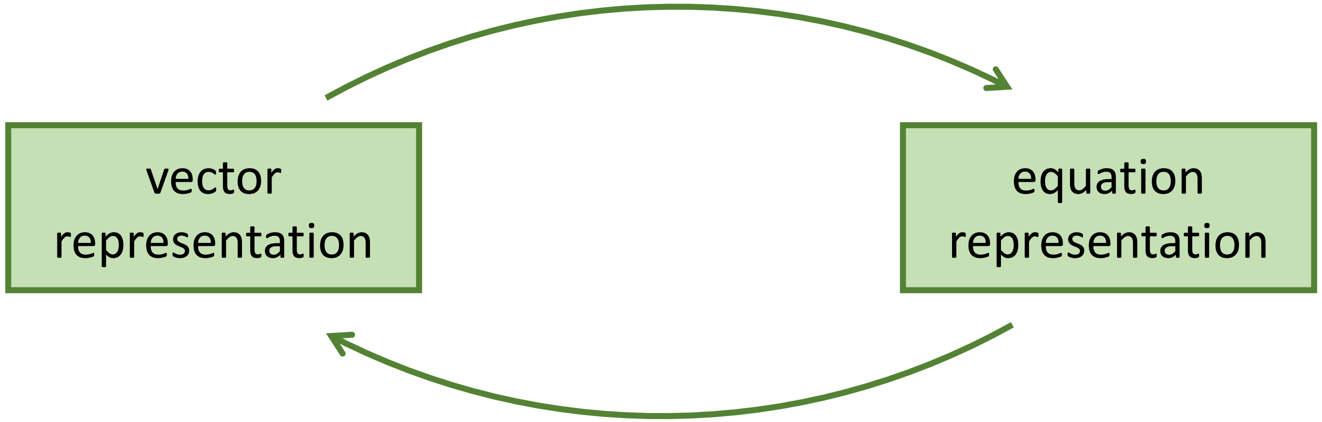
two vectors

$$V = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3 \,\middle|\, 7x + 5y - 4z = 0\right\}$$

one homogeneous equation

A k-dimensional linear subspace of $\mathbb{R}^d$ can be represented by:

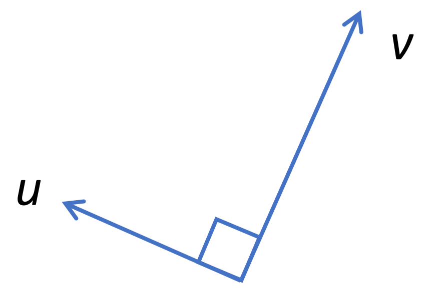
- k LI vectors
- $d - k$ LI homogeneous equations



Scalar product

$$(u_1, \ldots, u_k) \cdot (v_1, \ldots, v_k) = u_1 v_1 + \cdots + u_k v_k$$

$$(3, -1, 4) \cdot (2, 9, 0) = -3$$



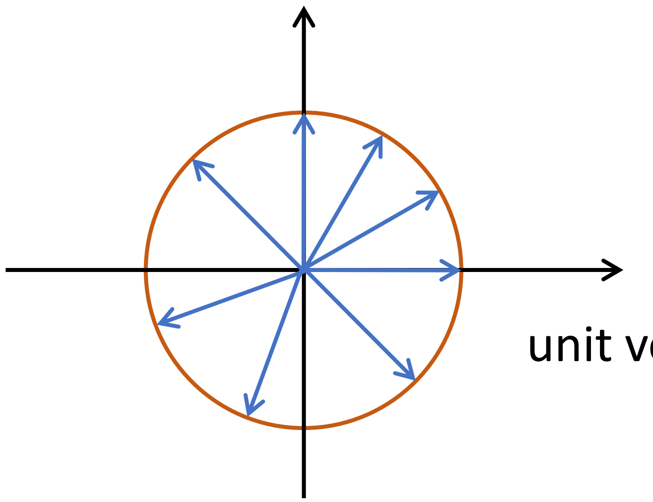
If $u \cdot v = 0$ then u, v are called ***orthogonal***, and we write $u \perp v$

Norm

$$\|v\| = \sqrt{v \cdot v}$$

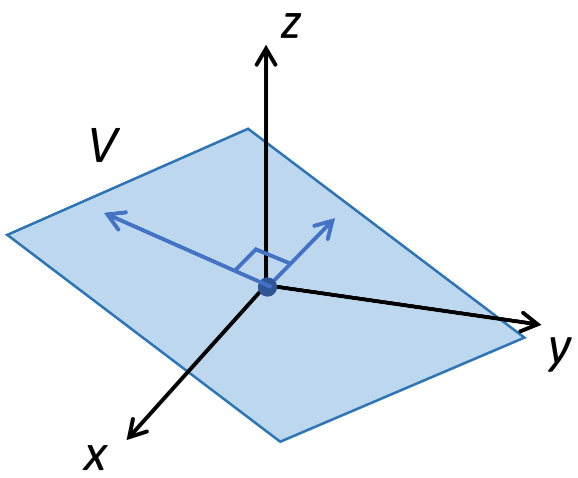
If $\|v\| = 1$ then v is a ***unit vector***

$$\|(3, -1, 4)\| = \sqrt{26}$$



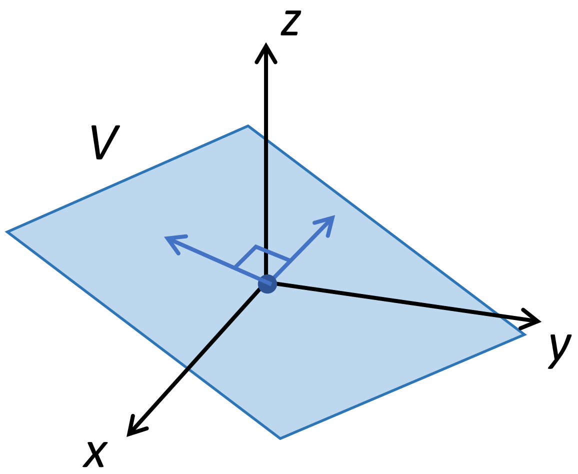
unit vectors in the plane

Orthogonal basis



Vectors are orthogonal to one another

Orthonormal basis

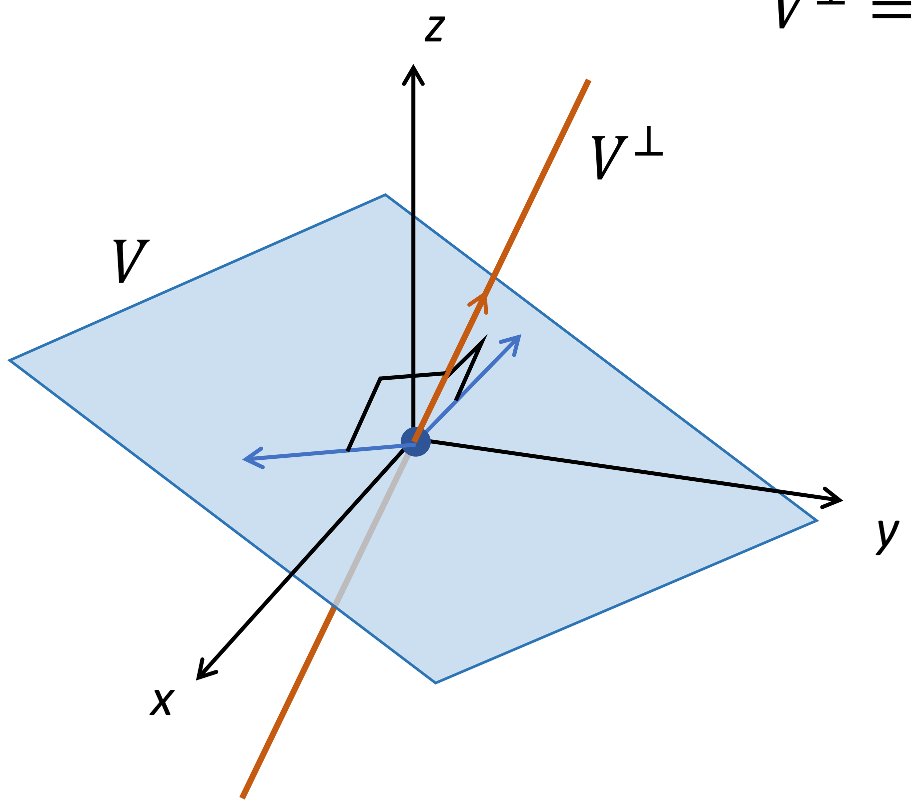


Vectors are orthogonal to one another
and are unit vectors

Orthogonal complement

$$V^{\perp} = \{u \in \mathbb{R}^d \,|\, u \perp w \text{ for every } w \in V\}$$

$V \subseteq \mathbb{R}^d$



Every vector of V is orthogonal to every vector of $V^{\perp}$

$$\dim V + \dim V^{\perp} = d$$

Matrices

$$\begin{bmatrix} -3 & -2 & -5 \\ 2 & 4 & 6 \\ 2 & 0 & 2 \\ 1 & -5 & 4 \end{bmatrix}$$

4x3 matrix

dimension = 2
(row rank)

Row space: $\mathrm{LS}\big((-3,-2,-5),(2,4,6),(2,0,2),(1,-5,4)\big)$

Linear subspace of $\mathbb{R}^3$

dimension = 2
(column rank)

Column space: $\mathrm{LS}\left(\begin{bmatrix} -3 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -5 \\ 6 \\ 2 \\ 4 \end{bmatrix}\right)$ Linear subspace of $\mathbb{R}^4$

Theorem: The row rank and column rank of a matrix are always equal

"Rank" of the matrix

Invertible matrices

Let A be a square matrix. A is called ***invertible*** if there exists B such that $AB = I$.

Theorem: If A, B are square matrices and $AB = I$, then also $BA = I$.

Determinants

$$\det\begin{bmatrix} 4 & 5 & -4 \\ 1 & 2 & 3 \\ -8 & 3 & 2 \end{bmatrix} = -226$$

Theorem: Let A be a square matrix. Then the following conditions are equivalent:

1. $\det A \neq 0$
2. A is invertible
3. The rows of A are LI
4. The columns of A are LI
5. The system $Ax = b$ has a unique solution

Orthogonal matrices

A $d \times d$ matrix is called ***orthogonal*** if its rows from an orthonormal basis of $\mathbb{R}^d$

If A is orthogonal, then its columns also from an orthonormal basis of $\mathbb{R}^d$

If A is orthogonal, then $\det A = \pm 1$

Linear Transformations

An $m \times n$ matrix represents a ***Linear Transformation*** (LT) from $\mathbb{R}^n$ to $\mathbb{R}^m$

$\boxed{A}$

matrix

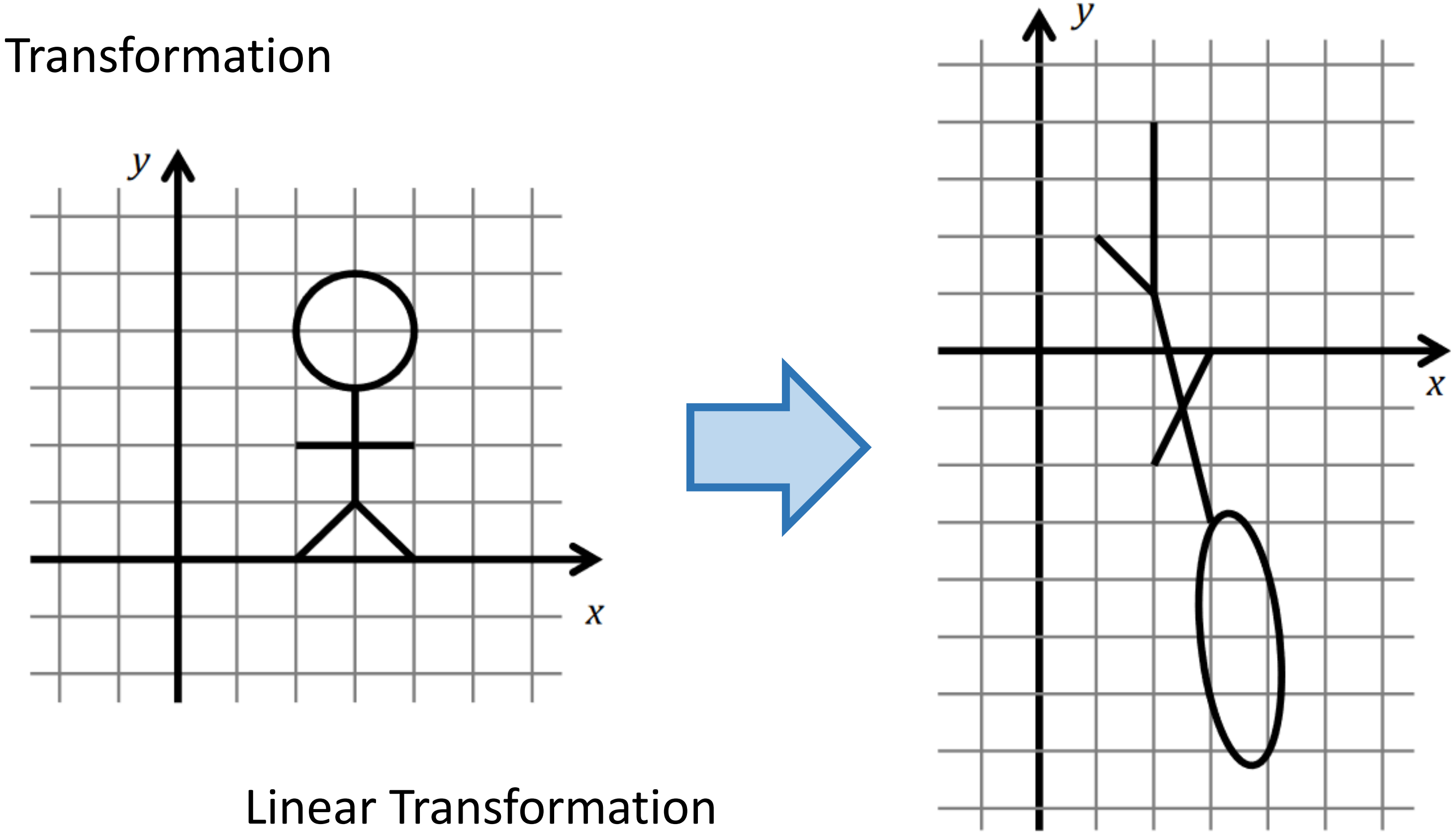
$\boxed{v \mapsto Av}$

Linear Transformation

Example:

$$\begin{bmatrix} 1/2 & 1/2 \\ 1 & -2 \end{bmatrix}$$

matrix



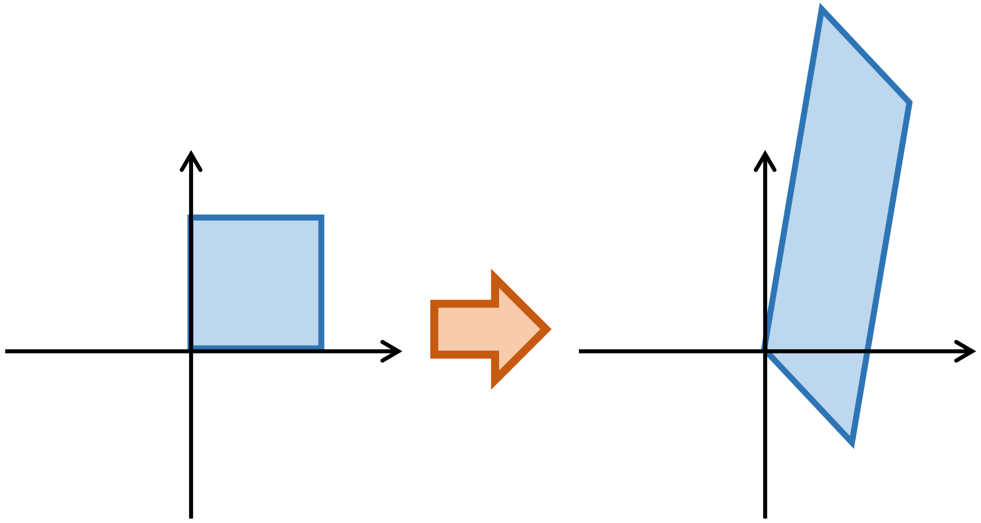
Linear Transformation

Linear Transformations

Let A be a square matrix. Let f be the corresponding Linear Transformation

f multiplies volumes by a factor of $|\det A|$

Example: $\det A = -2$

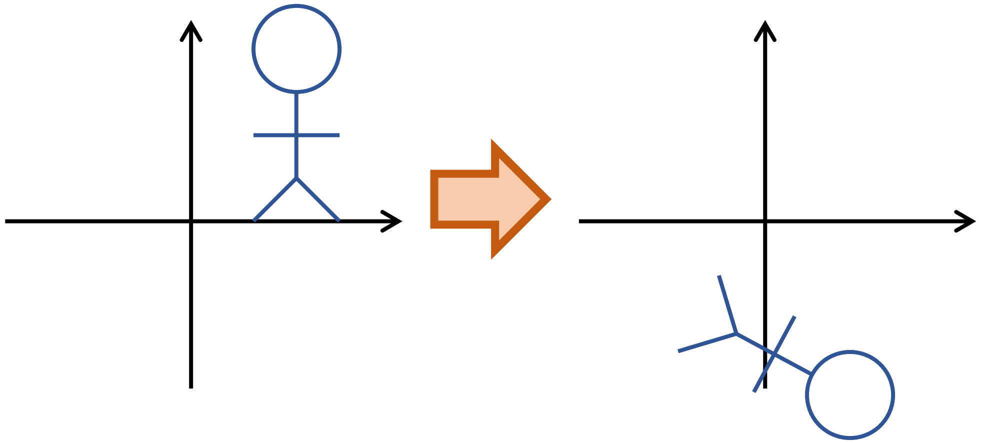


f multiplies areas by a factor of 2

Linear Transformations

Let A be a square matrix. Let f be the corresponding Linear Transformation

If A is an orthogonal matrix, then f is rigid (preserves distances and angles)



- If $\det A = +1$ then f is a rotation
- If $\det A = -1$ then f is a rotation + mirror reflection