Affine spaces

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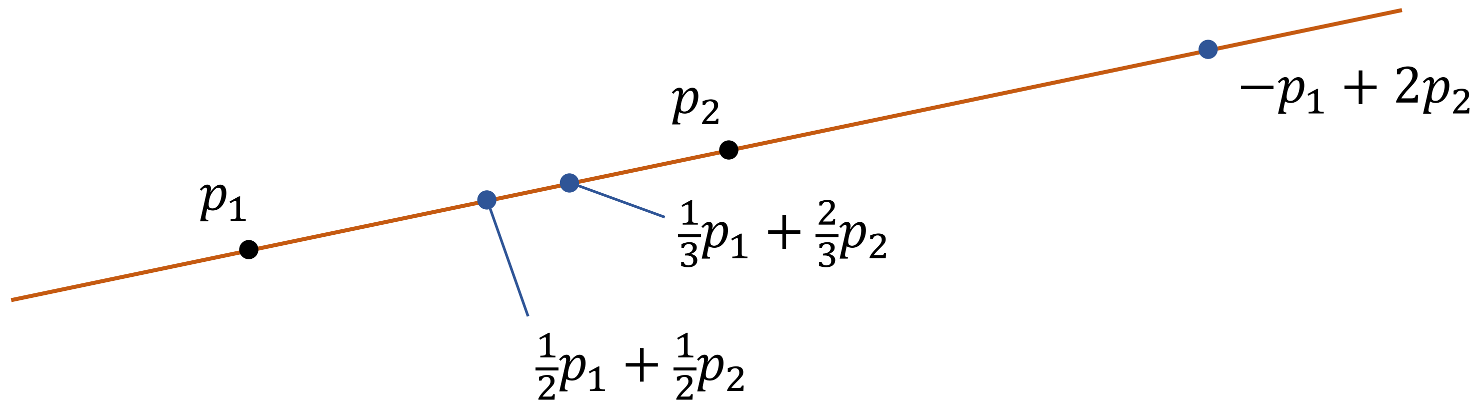
Affine combination of points

$p_1, \dots, p_k \in \mathbb{R}^d$

$w = \alpha_1 p_1 + \cdots + \alpha_k p_k$ where $\alpha_1 + \cdots + \alpha_k = 1$

w is an ***Affine Combination*** (AC) of $p_1, \dots, p_k$

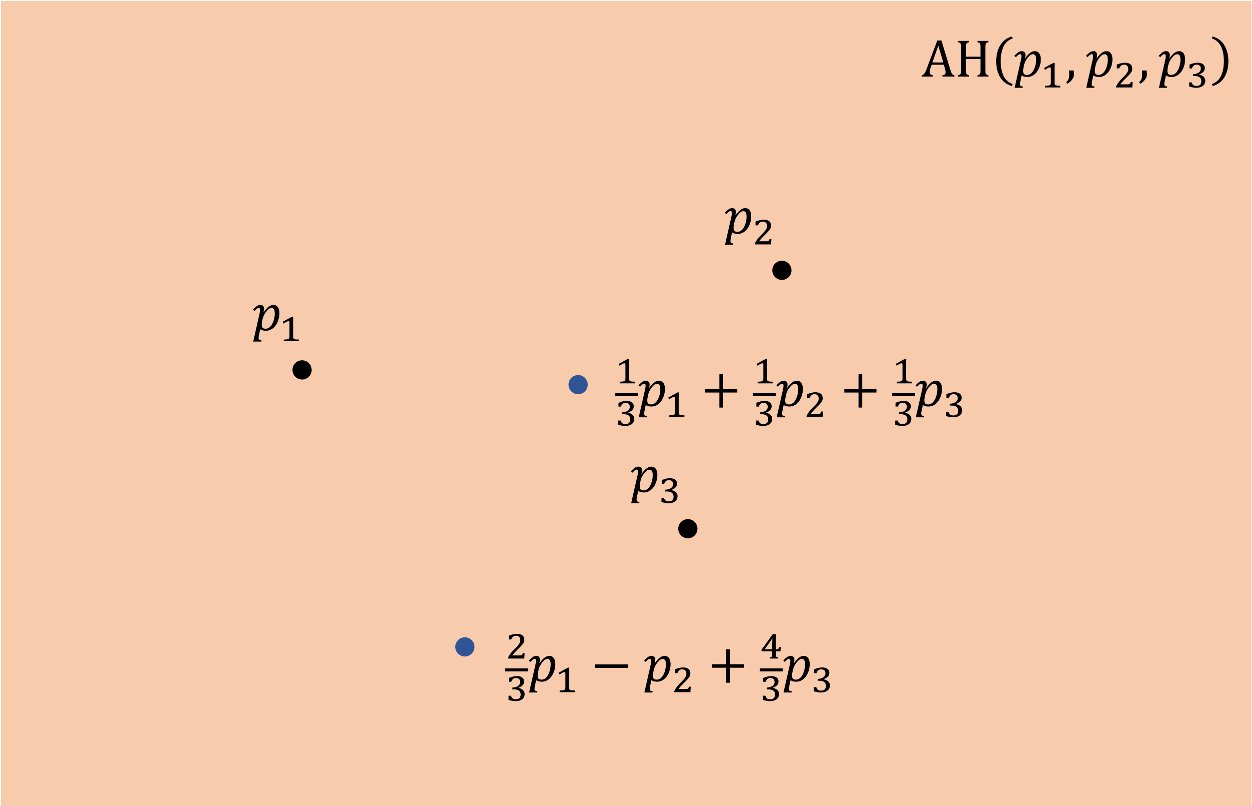
Examples:



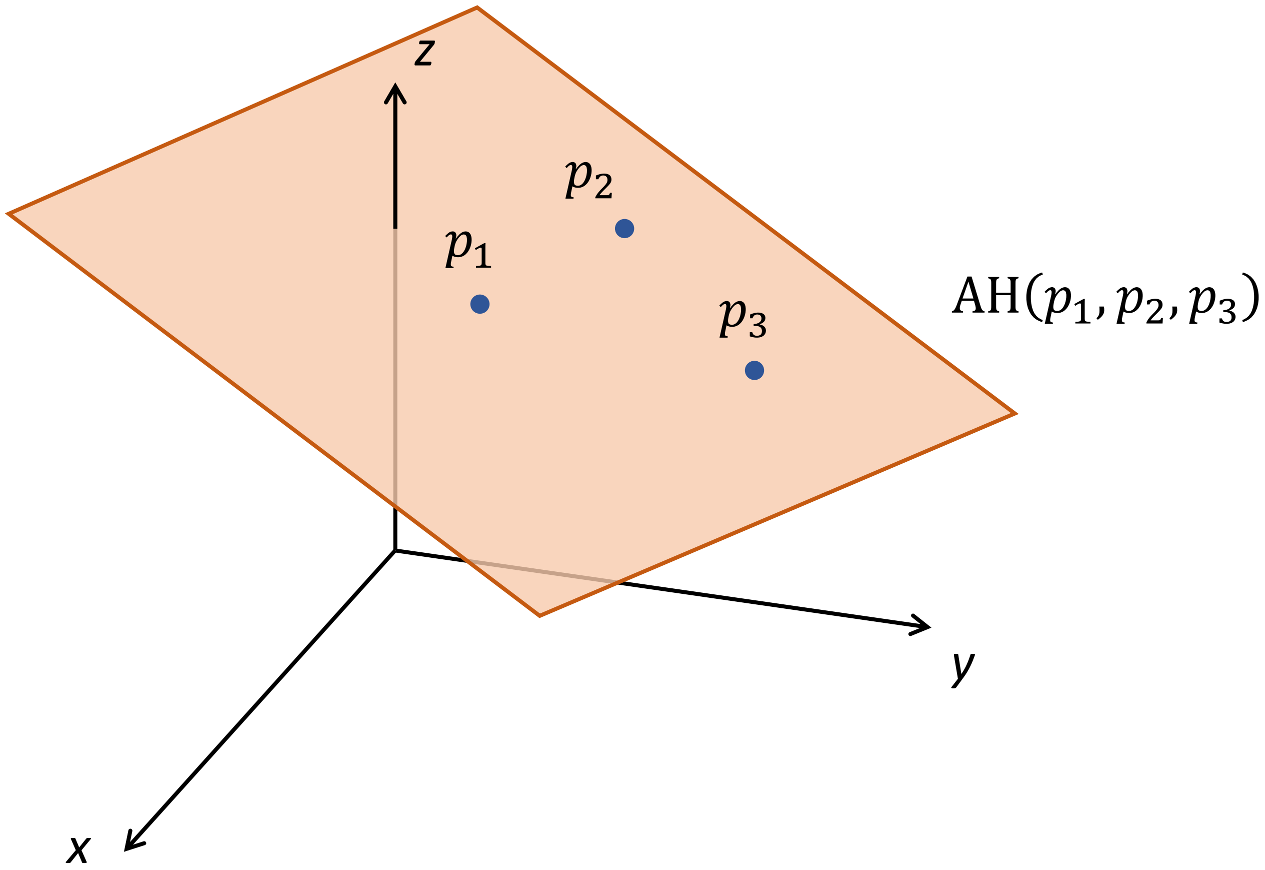
Affine Hull (AH) (or ***affine span***): Set of all affine combinations

Affine combination of points

Examples:



Affine combination of points



Affinely Dependent points

Definition 1: Points $p_1, \dots, p_k$ are Affinely Dependent if one of them is an AC of the others.

Example: p_1, p_2, p_3

$$p_2 = 4p_1 - 3p_3$$

p_2 is an AC of p_1, p_3

p_1, p_2, p_3 are AD

Definition 2: Points $p_1, \dots, p_k$ are Affinely Dependent if there exist scalars $\alpha_1, \dots, \alpha_k$, not all zero, such that

$$\alpha_1 p_1 + \cdots + \alpha_k p_k = \vec{0}$$

$$\alpha_1 + \cdots + \alpha_k = 0$$

$$4p_1 - p_2 - 3p_3 = \vec{0}$$

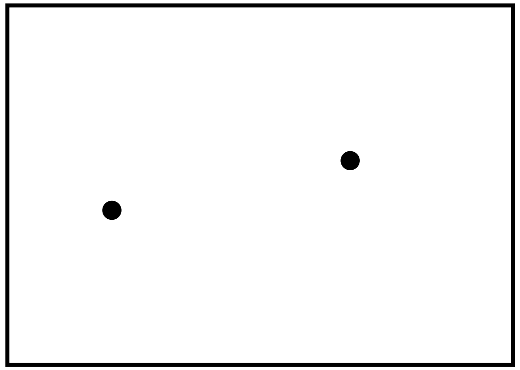
Affinely Dependent points

In other words, points $p_1, \dots, p_k$ are Affinely Independent if the only solution to the homogeneous system

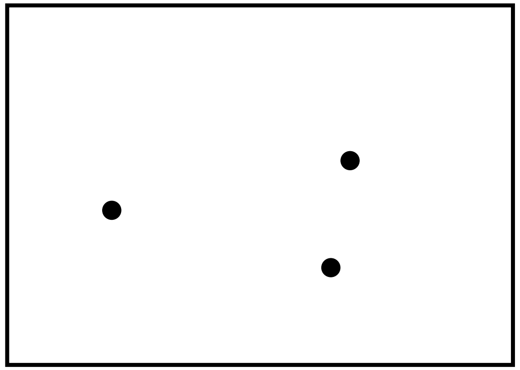
$$\begin{aligned} \alpha_1 p_1 + \cdots + \alpha_k p_k &= \vec{0} \\ \alpha_1 + \cdots + \alpha_k &= 0 \end{aligned}$$

is the trivial one $\alpha_1 = \cdots = \alpha_k = 0$

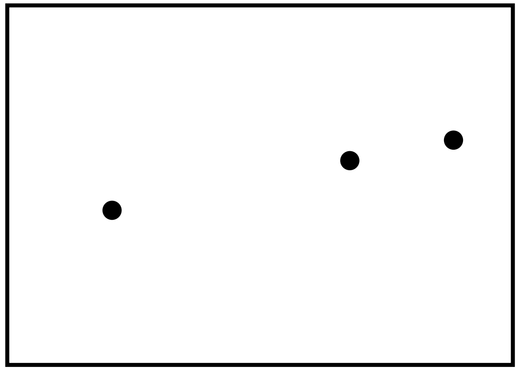
Examples in the plane:



AI



AI



AD

Exercise 1: Prove that the two definitions of AD are equivalent

Affine Combination: $\sum \alpha_i = 1$

Affinely Dependent: $\sum \alpha_i = 0$

Recall from Linear Algebra: $d+1$ vectors in $\mathbb{R}^d$ are always LD.

Exercise 2: Use this to prove that $d+2$ points in $\mathbb{R}^d$ are always AD.

Exercise 3: Prove that Affinely Dependent points are “unnecessary” for Affine Combinations:
If $q \in \mathbb{R}^d$ is an AC of $p_1, \dots, p_k \in \mathbb{R}^d$, and p_1 is an AC of $p_2, \dots, p_k$, then q is an AC of $p_2, \dots, p_k$

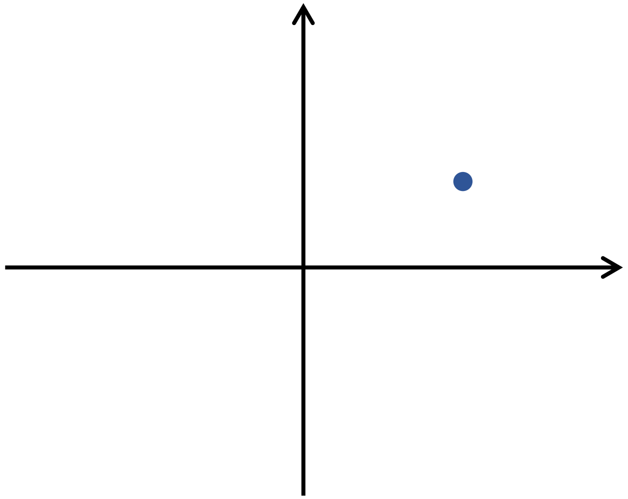
(Just like with LD and LC in Linear Algebra)

Affinely closed sets

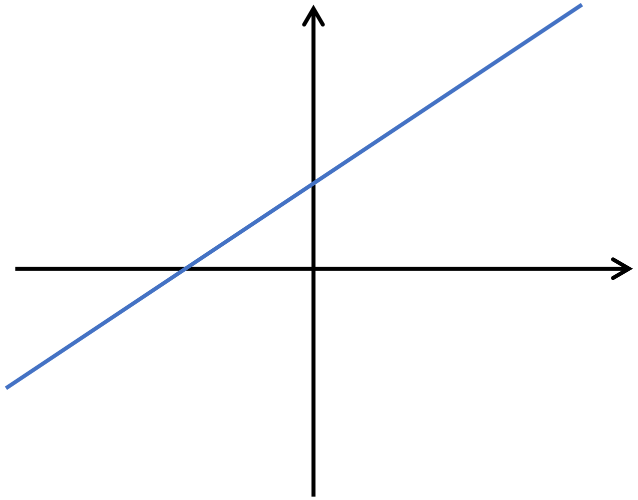
Definition: A subset $F \subseteq \mathbb{R}^d$ is ***affinely closed*** if every AC of finitely many points of F is also in F

up to $d + 1$ is enough

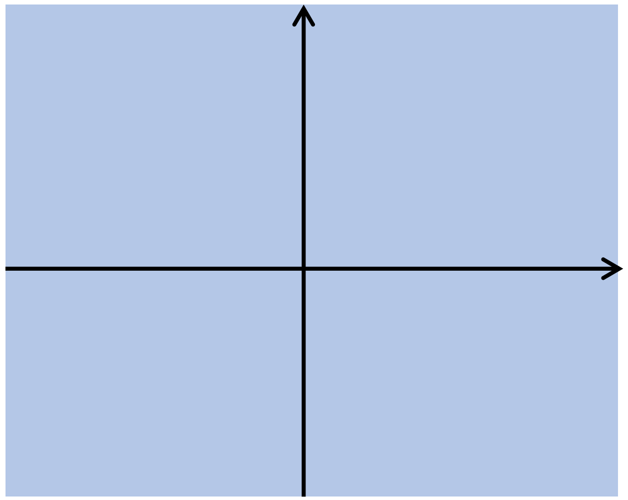
Affinely closed subsets of the plane:



points



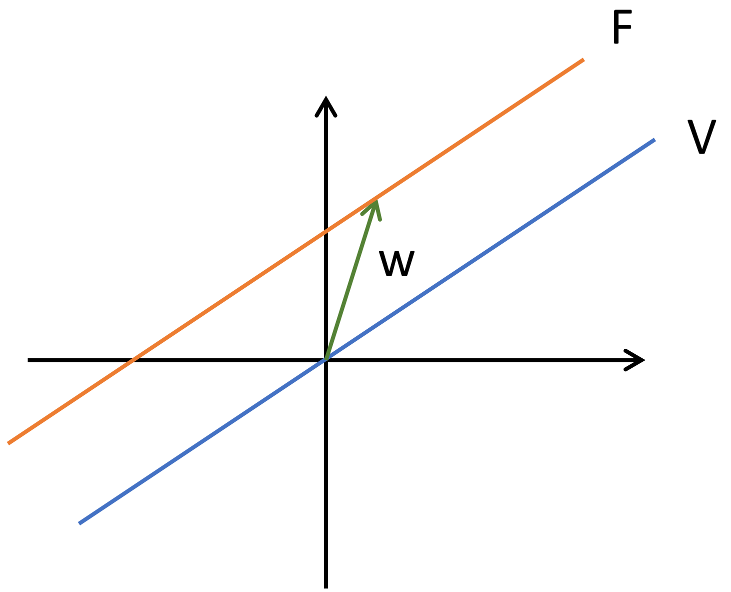
lines



the whole plane

Affinely closed sets

Affinely closed subsets of three-dimensional space:
● points ● lines ● planes ● the whole space



Affine subspaces

Definition: An ***affine subspace*** is a translated linear subspace:

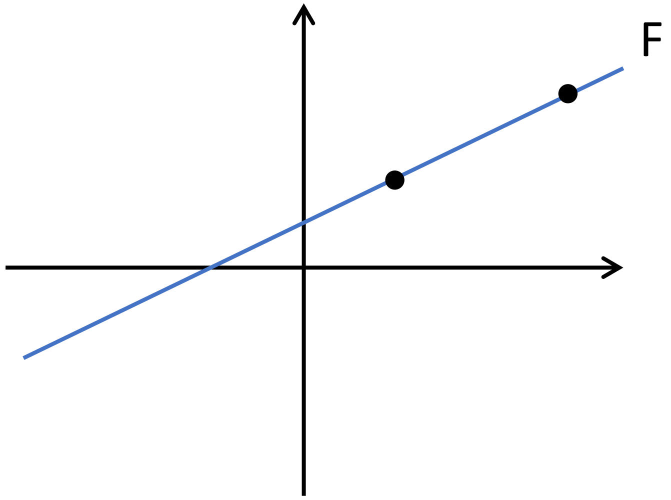
$$F = \{v + w | v \in V\} \qquad V \text{ linear subspace}$$

Exercise 4: Prove that the above two notions are equivalent:
A subset of $\mathbb{R}^d$ is affinely closed if and only if it is an affine subspace.

Representation of affine subspaces

There are two ways to represent affine subspaces:

- With points
- With equations



$F = \mathrm{AH}\big((1,1),(3,2)\big)$

two points

$F = \{(x,y) \in \mathbb{R}^2 \mid x - 2y = -1\}$

one equation

affine subspace = ***"flat"***

k-dimensional affine subspace = ***"k-flat"***

A k-flat in $\mathbb{R}^d$ can be represented by

- $k+1$ points
- $d-k$ equations

Hyperplanes

A $(d-1)$-dimensional flat in $\mathbb{R}^d$ is called a ***hyperplane***

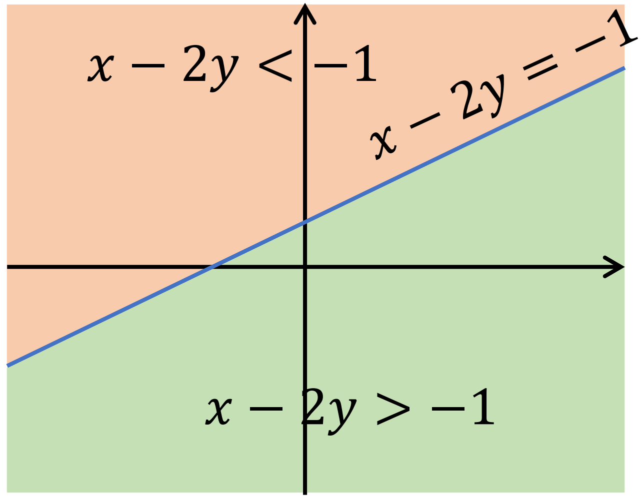
● Line in the plane ● Plane in space …

A hyperplane is defined by a single equation

It partitions $\mathbb{R}^d$ into two ***half-spaces***

$x - 2y < -1$ open half-space

$x - 2y \leq -1$ closed half-space



Affine transformation

An ***affine transformation*** is a linear transformation plus a translation

$$f: \mathbb{R}^d \to \mathbb{R}^k \quad \text{given by} \quad f(v) = Av + b$$

vector of size k

$k \times d$ matrix

Example in the plane: