Computer Graphics

2D Transformations

By

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Based on Computer Graphics By Hearn & Baker

2D-Transformations

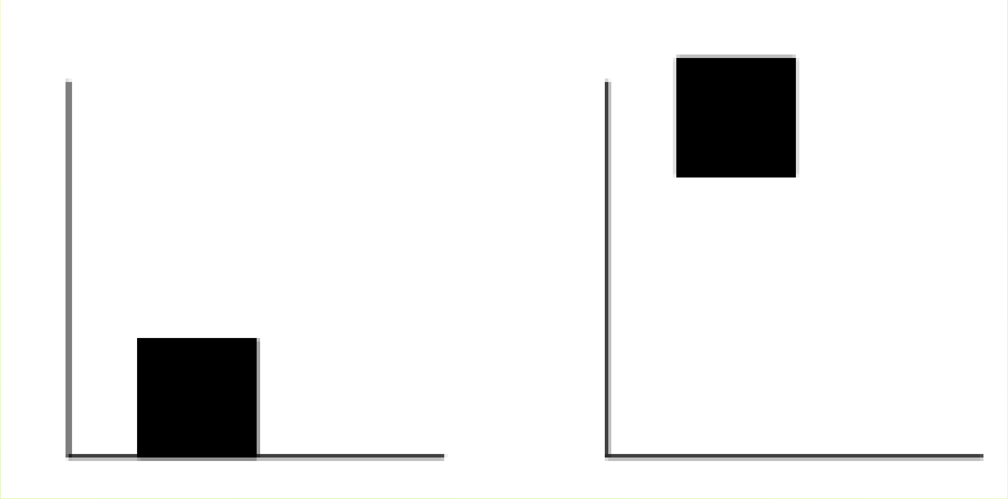
- Transformation means changes in orientation, size, and shape.

- The basic geometric transformations are translation, rotation, and scaling.

- Other transformations that are often applied to objects include reflection and shear.

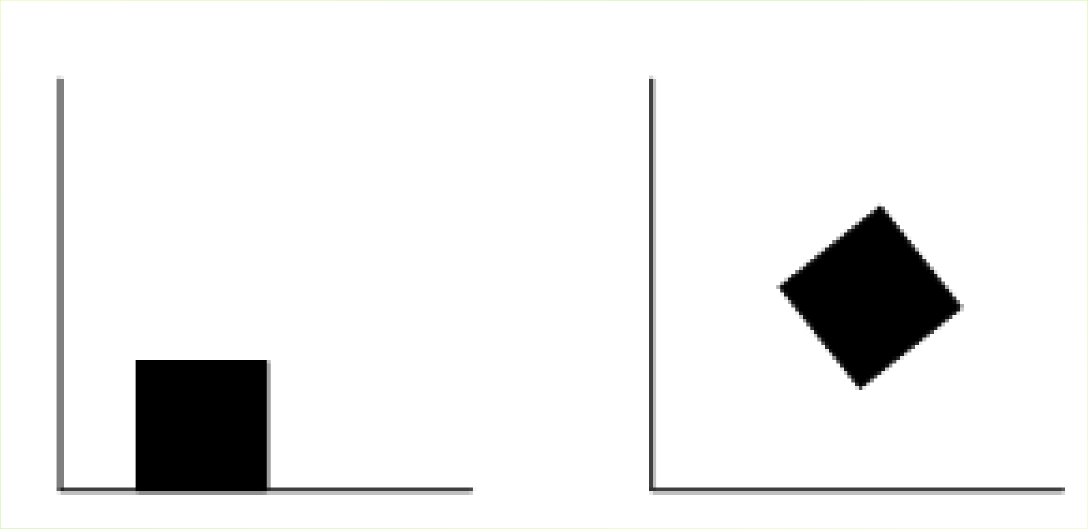
2D-Transformations

- Translation
  - moving things



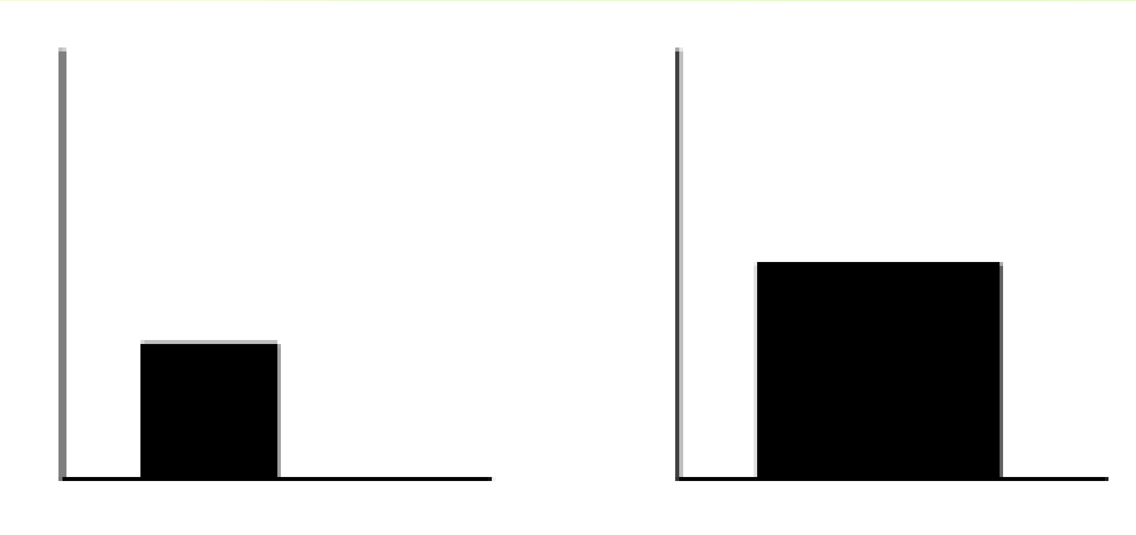
2D-Transformations

- Rotation
  - moving about a point by a given angle
  - rotation point may be origin or some other chosen point



2D-Transformations

- Scaling
  - changing size
  - relative to origin or some other chosen point



3D-Transformations

- Translation
  - Similar in 3D as 2D
  - We have t_x , t_y and now t_z to add to each point

3D-Transformations

- Scaling
  - similar in 3D as 2D
  - we have S_x, S_y and S_z scale factors
  - can be relative to origin or point in 3D space

2D-Transformations

- Homogenous coordinates

2D, 3 × 1 row matrices

$$P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

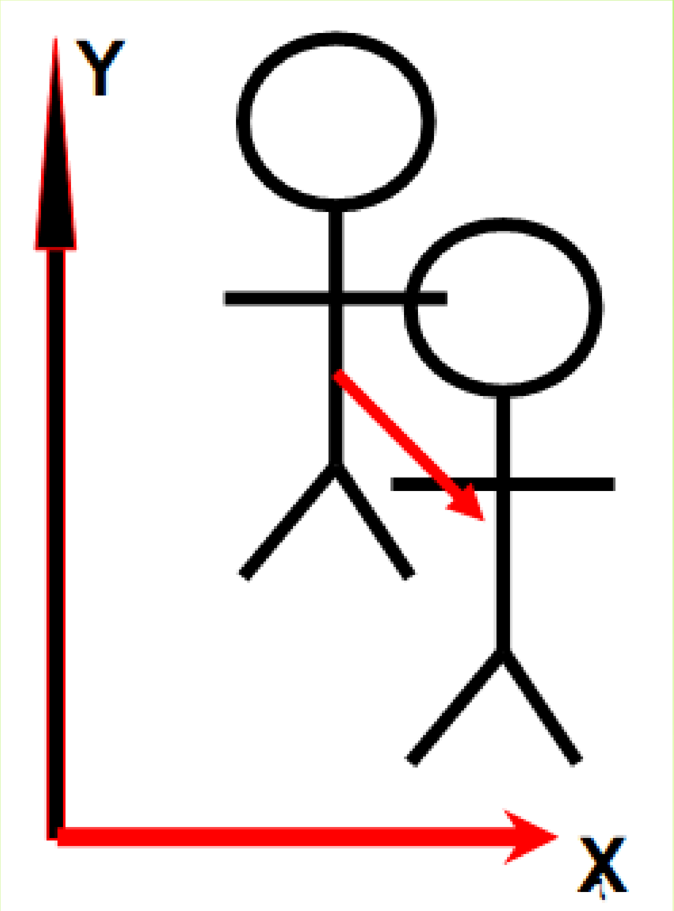
3D-Transformations

- Homogenous coordinates

3D, 4×1 row matrices $\quad P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

2D-Translation

- Moving all points of an object by the same x and y factors t_x and t_y.
- For example, $t_x = 10$ and $t_y = -10$.



2D-Translation

- Translation: Repositioning of an object from one coordinate location to another using translation vector (t_x, t_y) as follows:

$$x' = x + t_x$$
$$y' = y + t_y$$

- Where
  - t_x, t_y are the translation distances,
  - x, y are the coordinate of the old point
  - x', y' are the coordinate of the new point.

2D-Translation

Example:

Let $P = (4, 7), T = \left(t_x, t_y\right) = (7, 10)$

Then $\grave{P} = (11, 17)$

Note:

- To translate a line translate its endpoints.
- To translate polygon translate its vertices.
- Circle or Ellipse: Translate boundary points.

2D-Translation

We think of a point as a column vector, written as a column of numbers between parentheses.

$$(1,5) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

We add vectors by adding corresponding coordinates.

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 5+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

2D-Translation

We can multiply a matrix by a vector.

- 2×2 matrix multiplied by 2-D vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Translation matrix and Homogeneous Coordinates:

$$x^{`} = T(t_x, t_y) \cdot x$$

$$T(t_x, t_y) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Translation matrix and Homogeneous Coordinates:

- 3×3 homogenous matrix representation of translation

$$P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad T(t_x, t_y) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Translation matrix and Homogeneous Coordinates:

- 3×3 homogenous matrix representation of translation

$$P^{`} = T \cdot P$$

$$P^{`} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

2D - Scaling

2D Scaling

- A scaling transformation alters the size of an object.
- This operation can be carried out for polygons by multiplying the coordinate values (x, y) of each vertex by scaling factors S_x and S_y to produce the transformed coordinates $(x`, y`)$.

2D Scaling

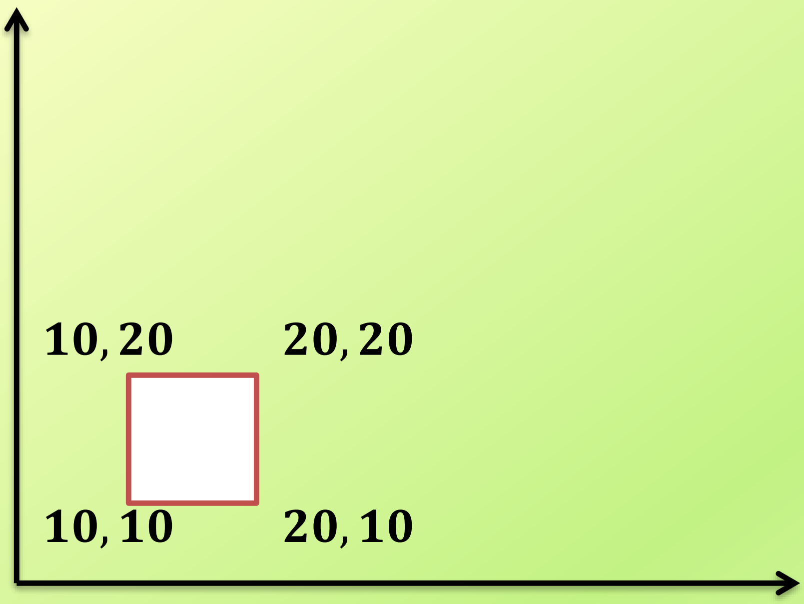
$$x` = S_x \cdot x$$

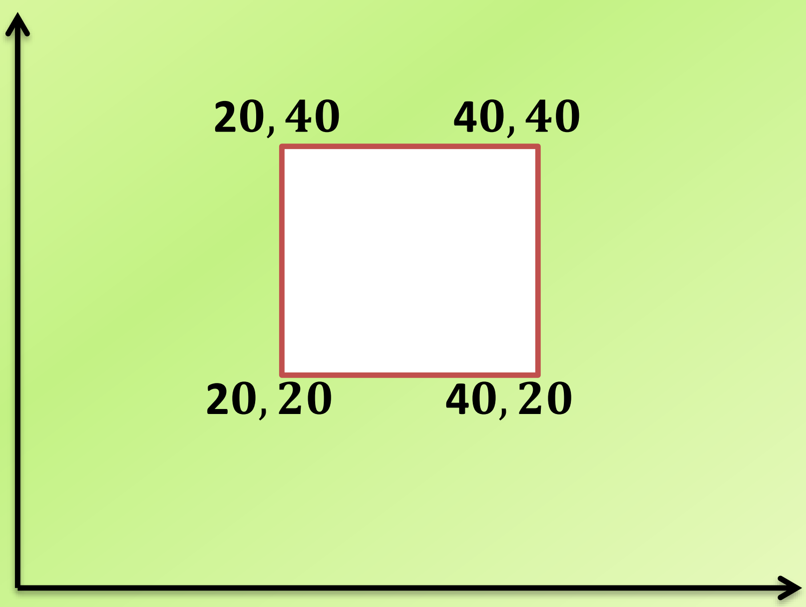
$$y` = S_y \cdot y$$

- The scaling is called **uniform** if $S_x = S_y$, so that the proportions are not affected.
- The scaling is called differential if $S_x \neq S_y$.

Scaling relative to the origin

- x and y values are multiplied by scaling factors S_x and S_y





2D-Scaling and Homogeneous Coordinates

- 3×3 homogenous matrix representation of scaling

$$P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad S(S_x, S_y)$$

$$= \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling matrix and Homogeneous Coordinates:

- 3×3 homogenous matrix representation of Scaling

$$P^{`} = S \cdot P$$

$$P^{`} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} S_x x \\ S_y y \\ 1 \end{bmatrix}$$

Scaling relative to a fixed point (x_f, y_f)

$$x' = x_f + (x - x_f)S_x$$
$$= x_f + xS_x - x_fS_x$$
$$= xS_x + x_f(1 - S_x)$$

$$y' = y_f + (y - y_f)S_y$$
$$= y_f + yS_y - y_fS_y$$
$$= yS_y + y_f(1 - S_y)$$

where $x_f(1 - S_x)$ and $y_f(1 - S_y)$ are fixed constants for all points in the object.

Scaling relative to a fixed point (x_f, y_f) 3 x 3 Homogeneous Matrix

- 3×3 homogenous matrix representation of scaling relative to a fixed point (x_f, y_f)

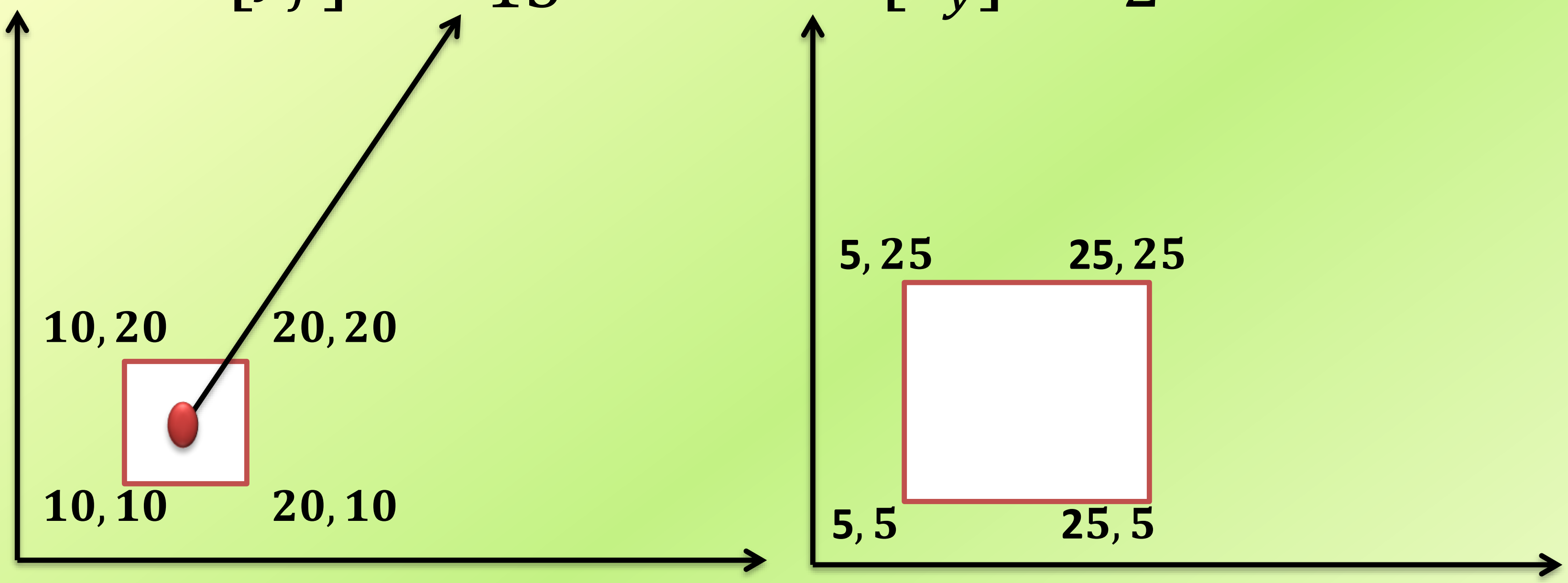
$$S = \begin{bmatrix} S_x & 0 & x_f(1 - S_x) \\ 0 & S_y & y_f(1 - S_y) \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling relative to a fixed point (x_f, y_f)
3 x 3 Homogeneous Matrix

$$P^{`} = \begin{bmatrix} S_x & 0 & x_f(1 - S_x) \\ 0 & S_y & y_f(1 - S_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

Scaling relative to fixed Point

- Scaling of a square with scale factors relative to fixed point that is center of square.
- Let $F = \begin{bmatrix} x_f \\ y_f \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$ and $S = \begin{bmatrix} S_x \\ S_y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$



2D - Rotation

2D Rotation

- To generate a rotation, we specify a rotation angle θ and the center of the rotation (x_r , y_r) (pivot point) about which the object is to be rotated.

2D Rotation

- If θ is positive then we have a counterclockwise rotation.
- if θ is negative then we have a clockwise rotation.

In Figure 1 the x and y axes have been rotated about the origin through an acute angle θ to produce the X and Y axes. Thus, a given point P has coordinates (x, y) in the first coordinate system and (X, Y) in the new coordinate system. To see how X and Y are related to x and y we observe from Figure 2 that

$$X = r\cos\phi \qquad Y = r\sin\phi$$

$$x = r\cos(\theta + \phi) \qquad y = r\sin(\theta + \phi)$$

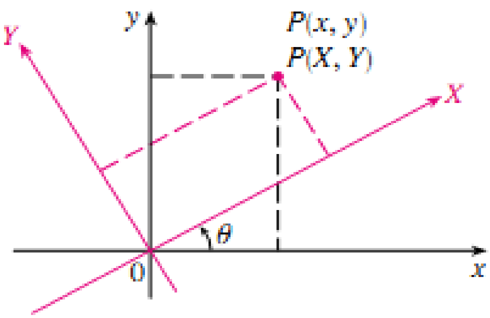


FIGURE 1

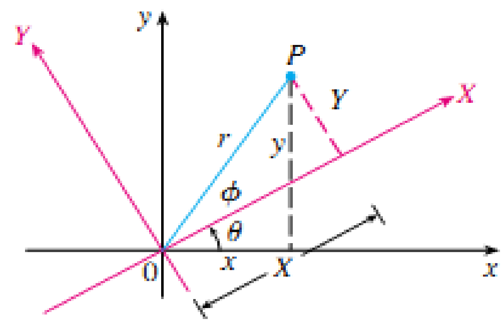


FIGURE 2

The addition formula for the cosine function then gives

$$x = r\cos(\theta + \phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

$$= (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta = X\cos\theta - Y\sin\theta$$

A similar computation gives y in terms of X and Y and so we have the following formulas:

2 $$\boxed{x = X\cos\theta - Y\sin\theta \qquad y = X\sin\theta + Y\cos\theta}$$

2D Rotation

$$\boldsymbol{x' = xcos(\theta) - ysin(\theta)}$$
$$\boldsymbol{y' = xsin(\theta) + ycos(\theta)}$$

- when the rotation angle equal to –θ

$$x' = xcos(\theta) + ysin(\theta)$$
$$y' = -xsin(\theta) + ycos(\theta)$$

Since

$$cos(-\theta) = cos(\theta)$$
$$sin(-\theta) = -\mathrm{sin}(\theta)$$

2D-Rotation

EXAMPLE 1 If the axes are rotated through 60°, find the XY-coordinates of the point whose xy-coordinates are (2, 6).

SOLUTION Using Equations 3 with $x = 2$, $y = 6$, and $\theta = 60°$, we have

$$X = 2\cos 60° + 6\sin 60° = 1 + 3\sqrt{3}$$

$$Y = -2\sin 60° + 6\cos 60° = -\sqrt{3} + 3$$

The XY-coordinates are $\left(1 + 3\sqrt{3}, 3 - \sqrt{3}\right)$.

2D Rotation of a point about a fixed pivot point (x_r, y_r)

$$x' = x\cos(\theta) - y\sin(\theta)$$
$$y' = x\sin(\theta) + y\cos(\theta)$$

$$x' = x_r + (x - x_r)\cos(\theta) - (y - y_r)\sin(\theta)$$
$$y' = y_r + (x - x_r)\sin(\theta) + (y - y_r)\cos(\theta)$$

2D Rotation of a point about a fixed pivot point (x_r, y_r)

$$x' = x_r + (x - x_r)cos(\theta) - (y - y_r)sin(\theta)$$
$$x' = x_r + xcos(\theta) - x_r cos(\theta) - ysin(\theta) + y_r sin(\theta)$$
$$x' = xcos(\theta) - ysin(\theta) + x_r - x_r cos(\theta) + y_r sin(\theta)$$

$$y' = y_r + (x - x_r)sin(\theta) + (y - y_r)cos(\theta)$$
$$y' = y_r + xsin(\theta) - x_r sin(\theta) + ycos(\theta) - y_r cos(\theta)$$
$$y' = xsin(\theta) + ycos(\theta) + y_r - x_r sin(\theta) - y_r cos(\theta)$$

2D Rotation of a point about a fixed pivot point (x_r, y_r)

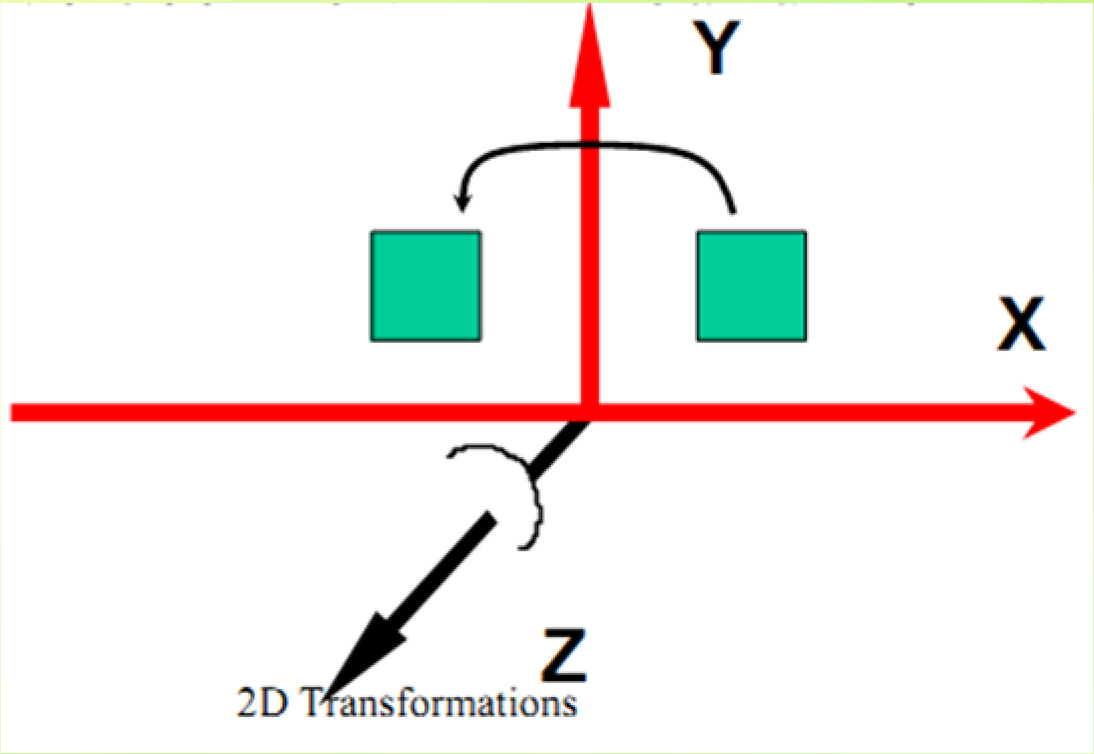
$$x' = x_r + (x - x_r)cos(\theta) - (y - y_r)sin(\theta)$$
$$y' = y_r + (x - x_r)sin(\theta) + (y - y_r)cos(\theta)$$

$$x' = xcos(\theta) - ysin(\theta) + x_r - x_r cos(\theta) + y_r sin(\theta)$$
$$y' = xsin(\theta) + ycos(\theta) + y_r - x_r sin(\theta) - y_r cos(\theta)$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x_r - x_r cos(\theta) + y_r sin(\theta) \\ \sin\theta & \cos\theta & y_r - x_r sin(\theta) - y_r cos(\theta) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D-Rotations

- rotation to date has all been of the X-Y plane
- i.e. we have been rotating around the Z-axis



2D-Rotations - Homogeneous Coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

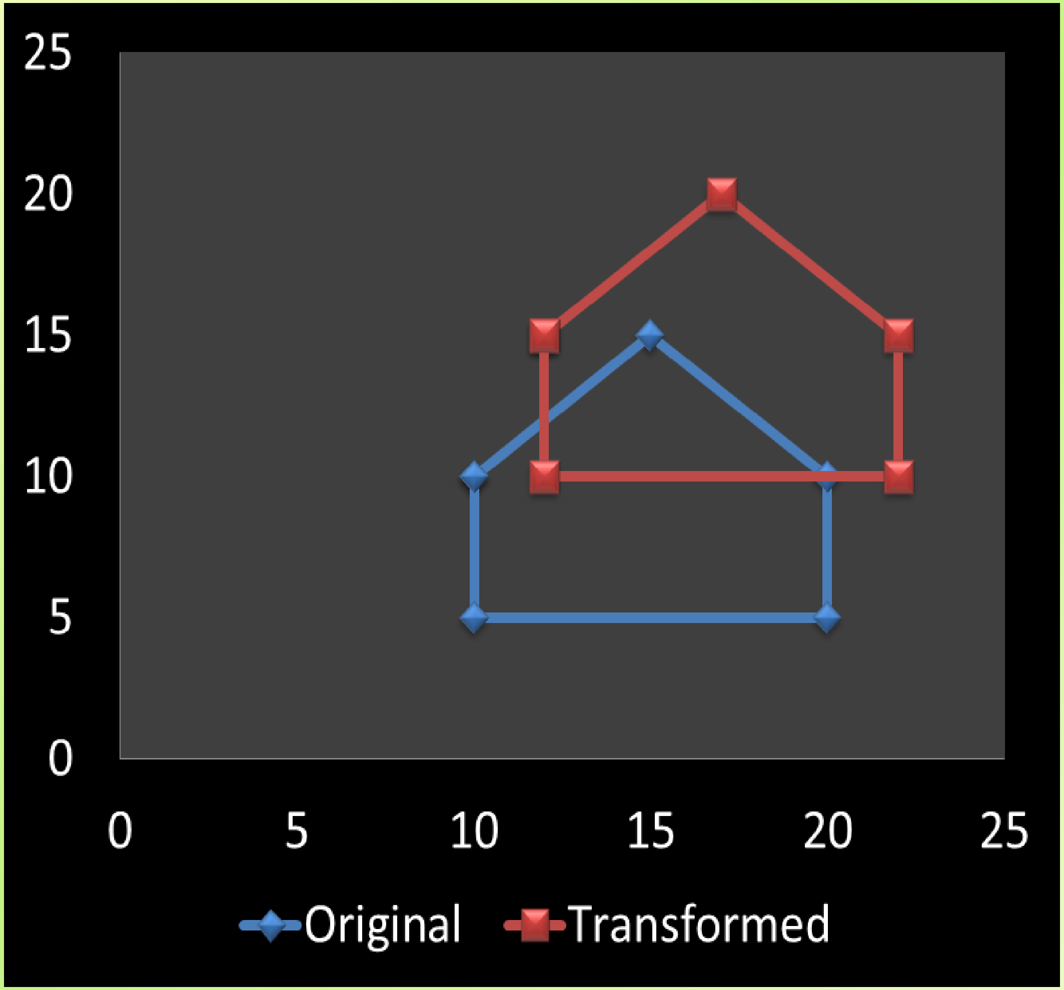
$$P' = R(\theta) \cdot P$$

A shape is defined by $\begin{pmatrix} 10 & 15 & 20 & 20 & 10 \\ 10 & 15 & 10 & 5 & 5 \end{pmatrix}$ find the transformed coordinates after translating it by 2 in the horizontal direction and 5 in the vertical direction.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

y =

12	17	22	22	12
15	20	15	10	10
1	1	1	1	1

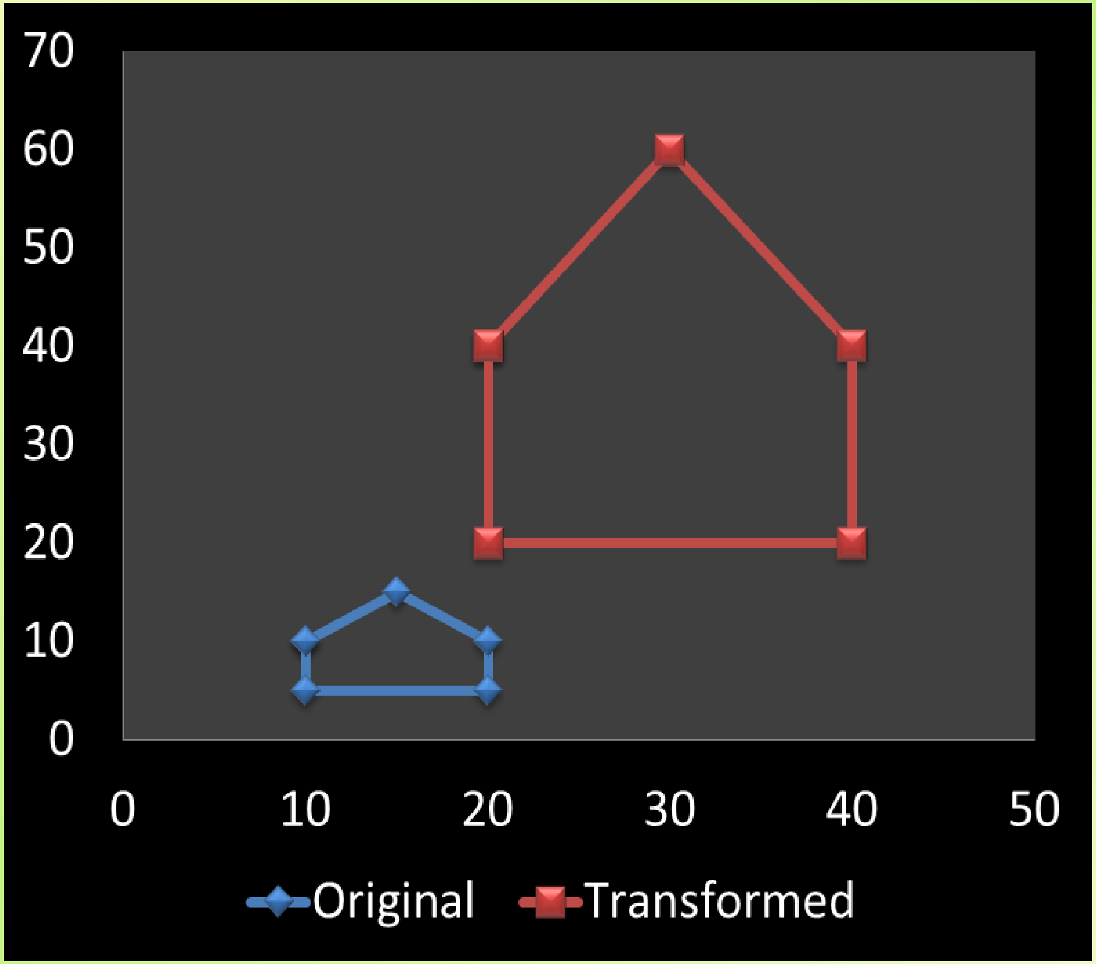


A shape is defined by $\begin{pmatrix} 10 & 15 & 20 & 20 & 10 \\ 10 & 15 & 10 & 5 & 5 \end{pmatrix}$ find the transformed coordinates after scaling it by two in x direction and by 4 in y direction.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

y =

20	30	40	40	20
40	60	40	20	20
1	1	1	1	1



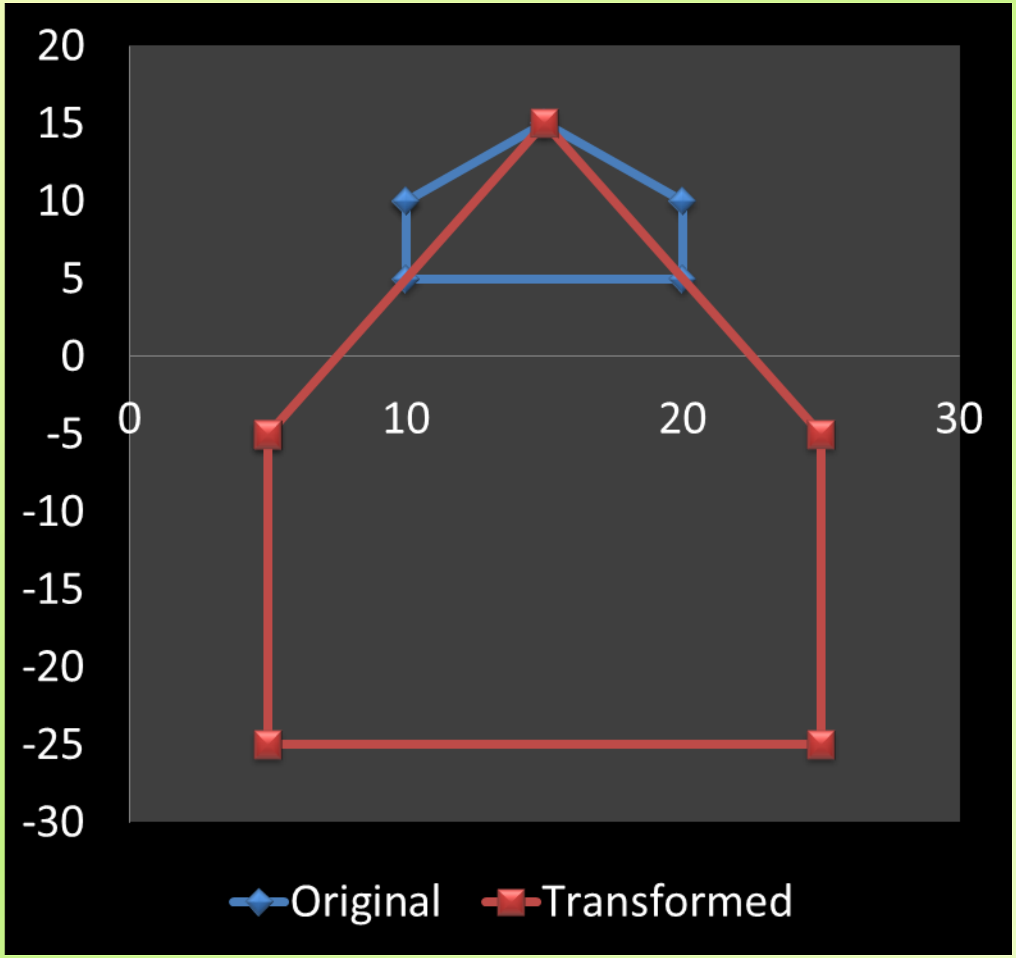
A shape is defined by $\begin{pmatrix} 10 & 15 & 20 & 20 & 10 \\ 10 & 15 & 10 & 5 & 5 \end{pmatrix}$ find the transformed coordinates after scaling it by two in x direction and by 4 in y direction relative to (15, 15).

$$S = \begin{bmatrix} S_x & 0 & x_f(1 - S_x) \\ 0 & S_y & y_f(1 - S_y) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -15 \\ 0 & 4 & -45 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

y =

5	15	25	25	5
-5	15	-5	-25	-25
1	1	1	1	1



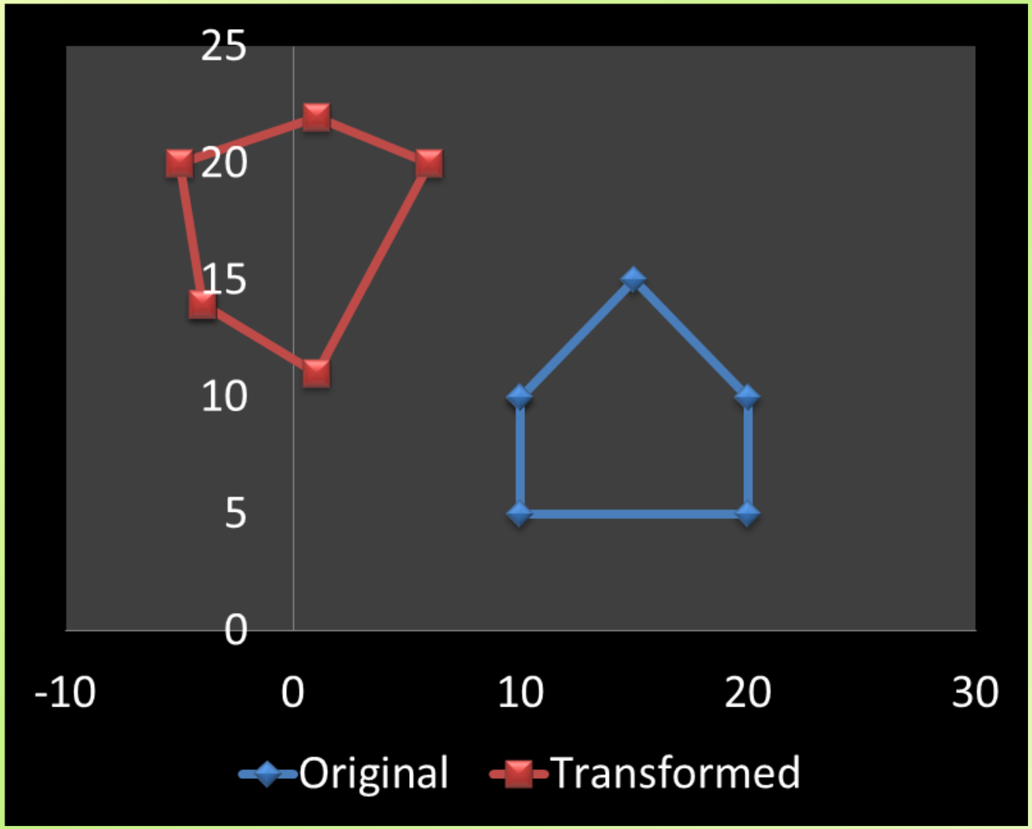
A shape is defined by $\begin{pmatrix} 10 & 15 & 20 & 20 & 10 \\ 10 & 15 & 10 & 5 & 5 \end{pmatrix}$ find the transformed coordinates after rotating it by 60 degree about the origin.

$$T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.8660 & 0 \\ 0.8660 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

y =

-4	-5	1	6	1
14	20	22	20	11
1	1	1	1	1



A shape is defined by $\begin{pmatrix} 10 & 15 & 20 & 20 & 10 \\ 10 & 15 & 10 & 5 & 5 \end{pmatrix}$ find the transformed coordinates after rotating it by 30 degree about (15,15).

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x_r - x_r cos(\theta) + y_r sin(\theta) \\ \sin\theta & \cos\theta & y_r - x_r sin(\theta) - y_r cos(\theta) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

T =

0.8660	-0.5000	9.5096
0.5000	0.8660	-5.4904
0	0	1.0000

y =

13	**15**	**22**	**24**	**16**
8	**15**	**13**	**9**	**4**
1	**1**	**1**	**1**	**1**

Reflection

Reflection

- A reflection is a transformation that produces a mirror image of an object.
- The mirror image is generated relative to an axis or point of reflection.

Reflection

- Reflection about the diagonal line

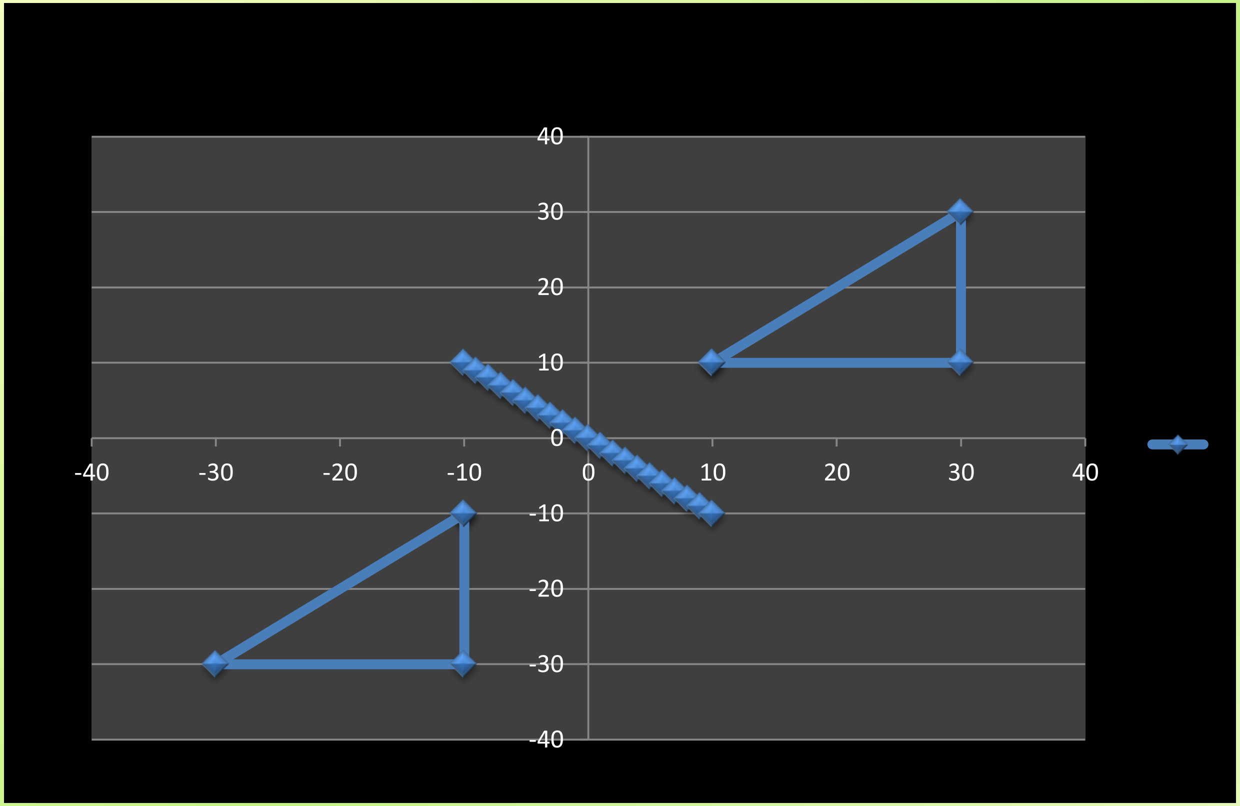
$$y = -x$$

- Transformation Matrix

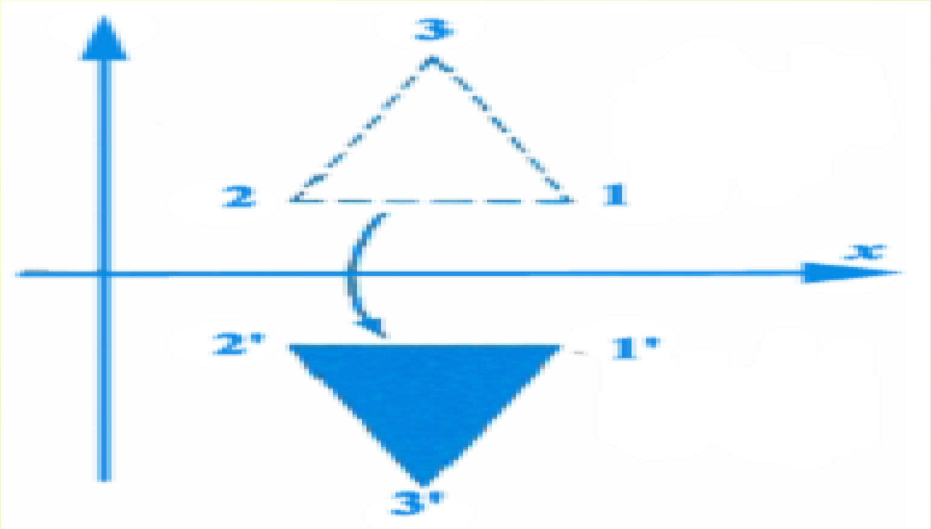
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection

10	10
30	10
30	30
10	10
-10	-10
-10	-30
-30	-30
-10	-10



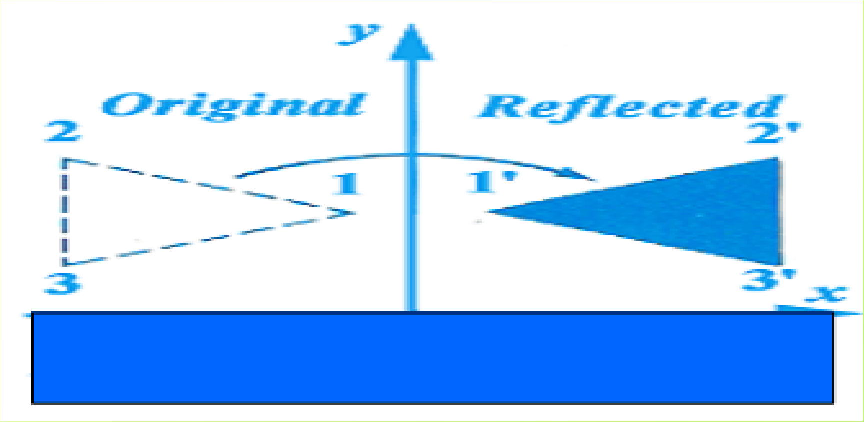
Reflection



- Reflection about the x-axis
- Transformation matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

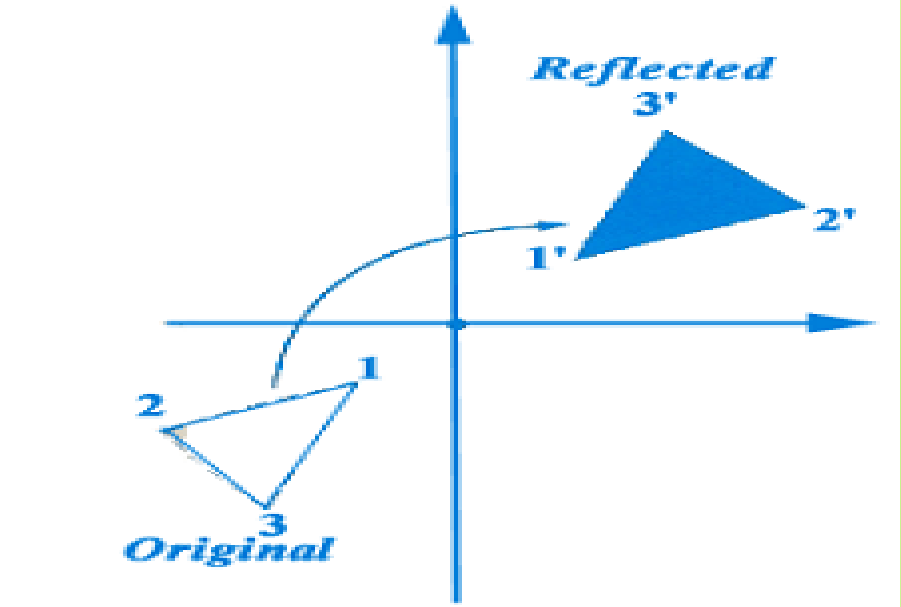
Reflection



- Reflection about the y-axis
- Transformation matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

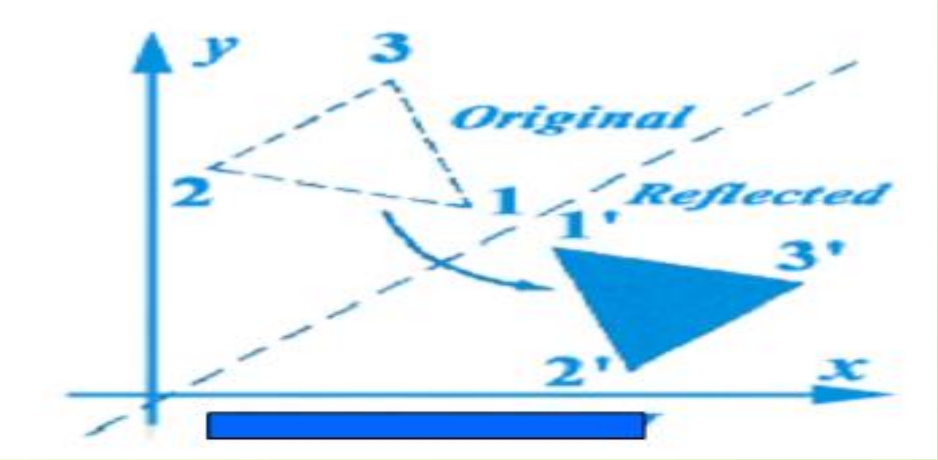
Reflection



- Reflection about the Origin
- Transformation matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection



- Reflection about the diagonal line y= x
- Transformation matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

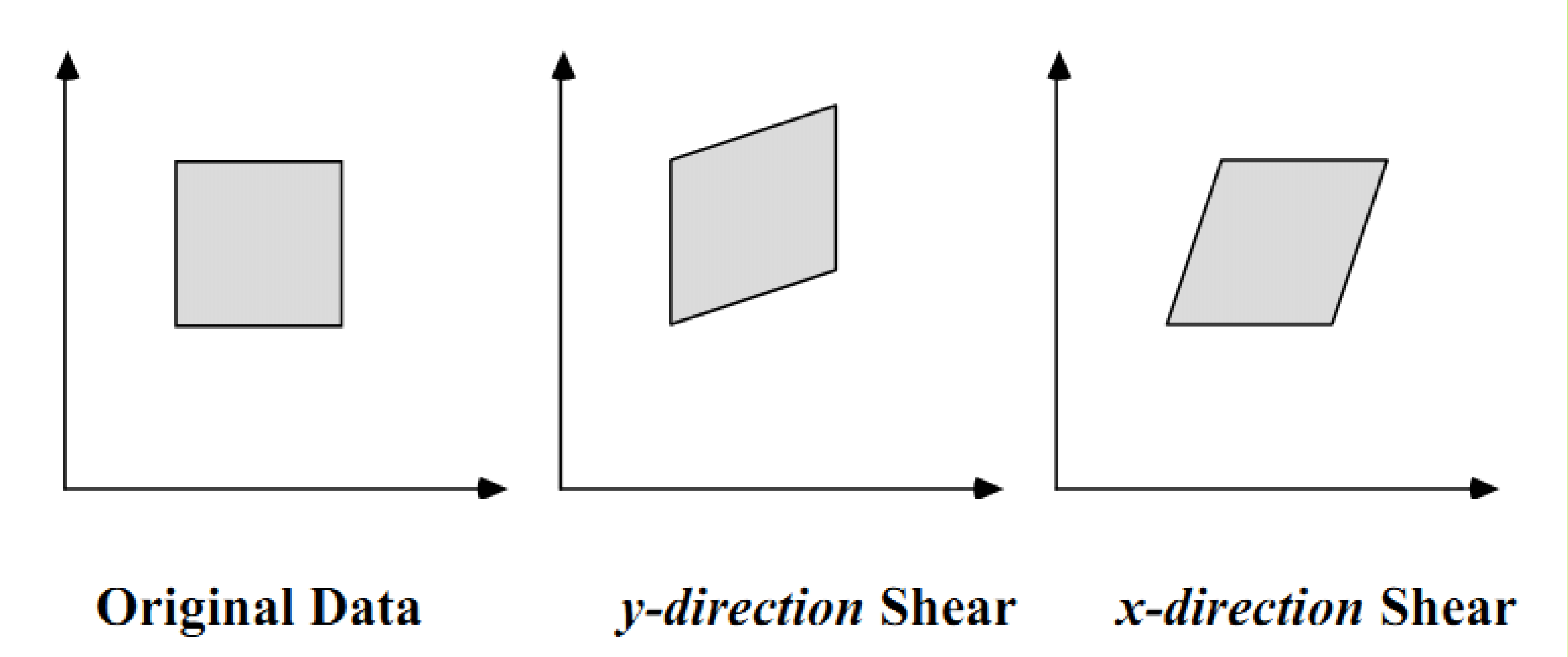
Shear

- These transformations produce shape distortions that represent a twisting, or shear effect, as if an object were composed of layers that are caused to slide over each other " in another word Italic representation of an object".

Shear

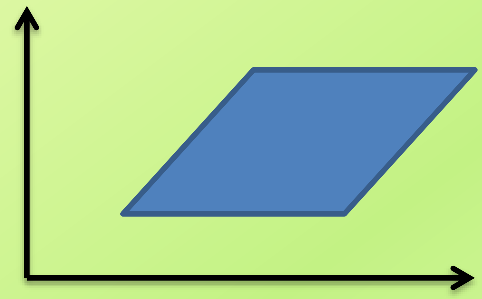
- A transformation that distorts the shape of an object such that the transformed shape appears as if the object were composed of internal layers that had been caused to slide over each other is called a shear.
- Two common shearing transformations are those that shift coordinate x values and those that shift y values.

Shear



X-direction Shear

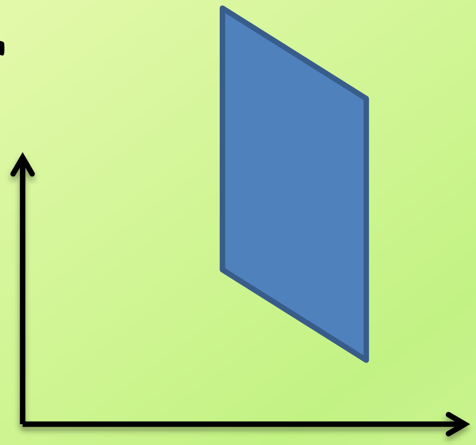
- Transformation matrix



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x + sh_x y$$
$$y' = y$$

y-direction Shear



- Transformation matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x$$
$$y' = y + sh_y x$$

Shear Example: x-direction shear

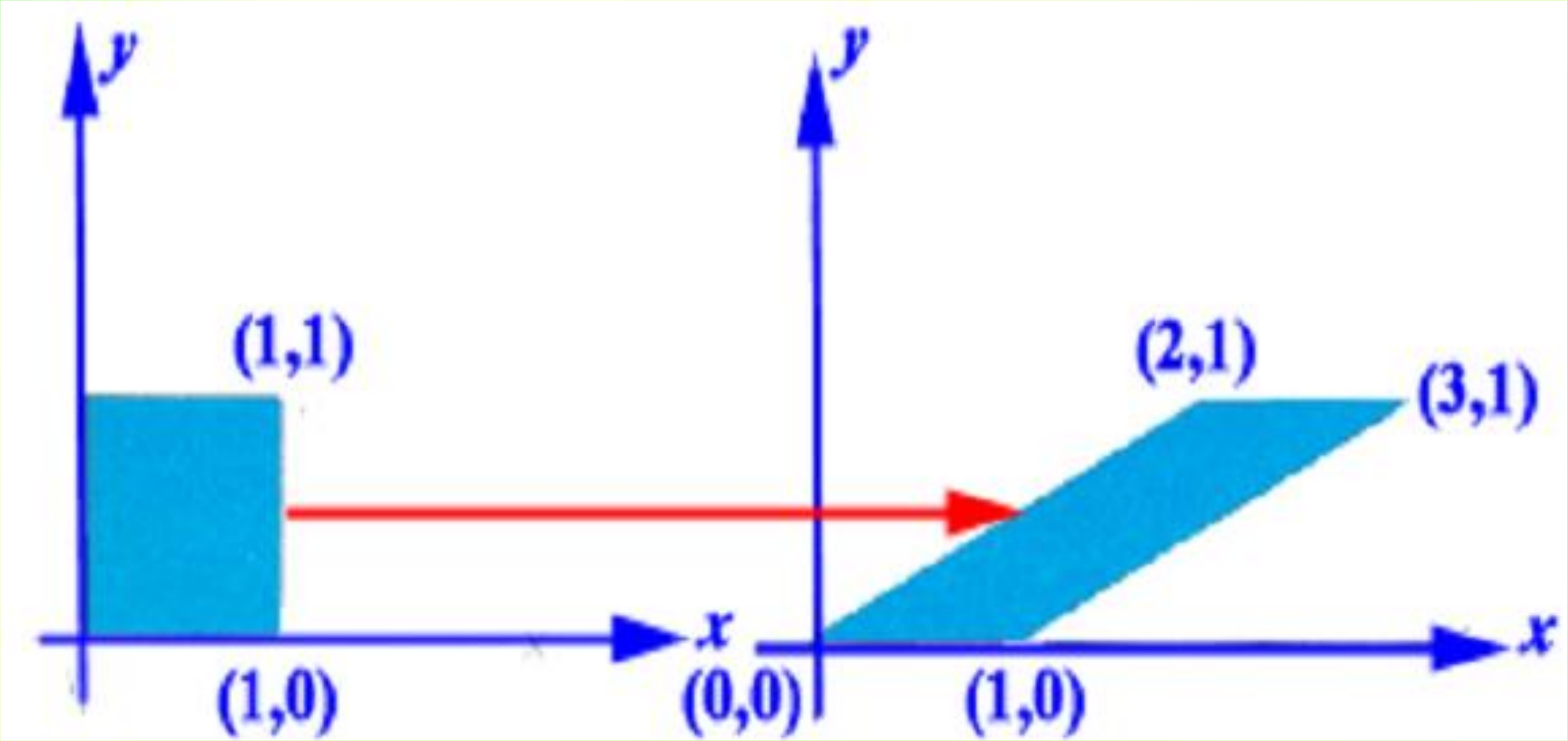
Given sh_x = 2, relative to the x-axis (y = 0), and a square with coordinates (0,0), (0,1), (1,1), (1,0). Write the x-direction shear matrix, then use it to compute the new vertices of the square.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x + 2y$$
$$y' = y$$

0	0		0	0
0	1		2	1
1	1		3	1
1	0		1	0

Shear Example: x-direction shear



Shear Example: y-direction shear

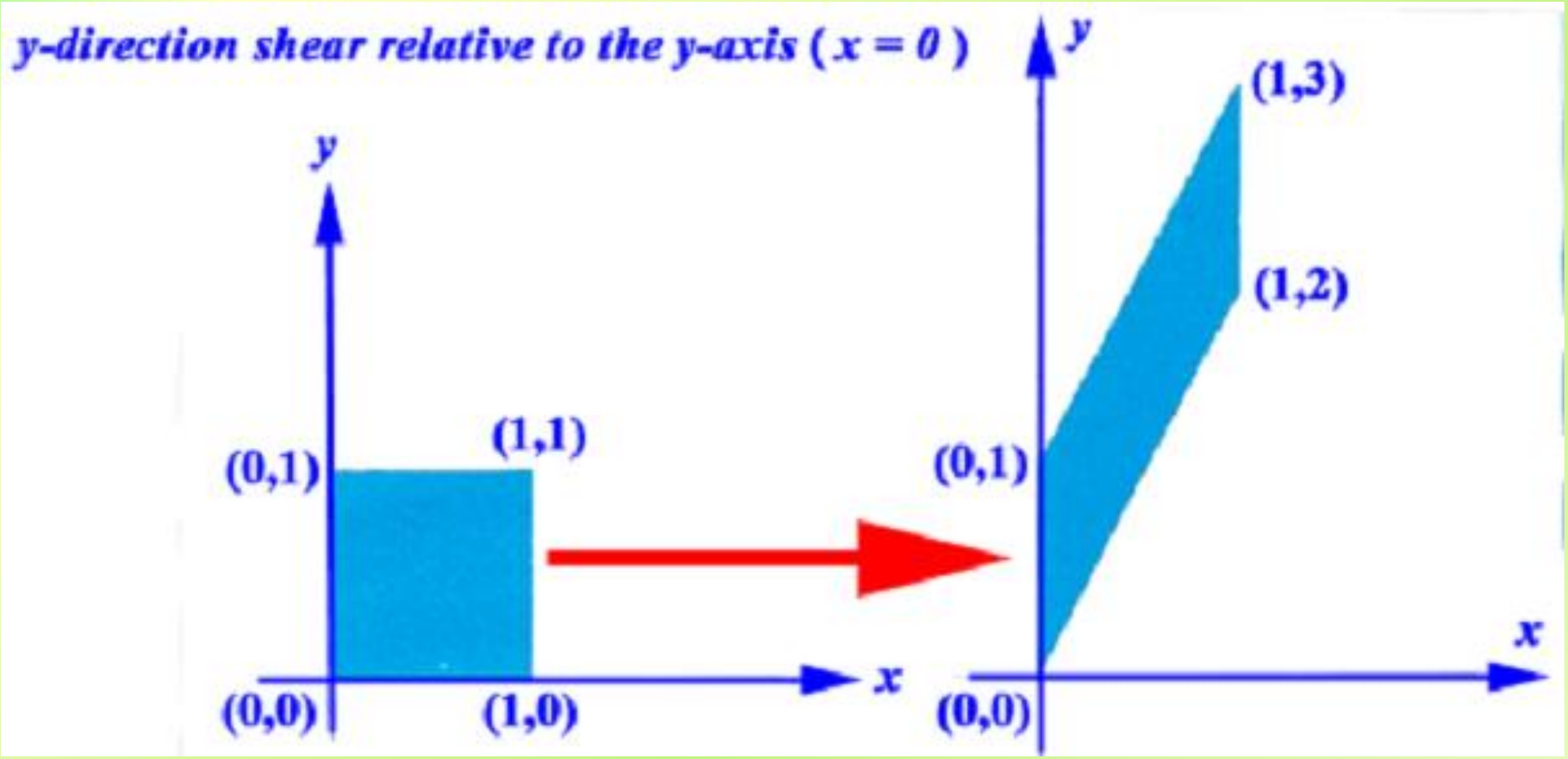
Given sh_y = 2, relative to the y-axis (x = 0), and a square with coordinates (0,0), (0,1), (1,1), (1,0). Write the y-direction shear matrix, then use it to compute the new vertices of the square.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x$$
$$y' = y + sh_y x$$

0	0		0	0
0	1		0	1
1	1		1	3
1	0		1	2

Shear Example: y-direction shear



x-direction relative Shear

- x -direction shearing relative to other reference line $y \neq 0$

$$x' = x + sh_x(y - y_{ref})$$

$$y' = y$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & -sh_x y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

x-direction relative Shear

- Example: x-direction shear relative to reference line $y \neq 0$. Given $sh_x = 0.5$, $y_{ref} = -1$, and the square with coordinates (0,0), (0,1), (1,1), (1,0).

Solution:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- So (0,0) will become (0.5,0);
- (0,1) will become (1,1);
- (1,1) will become (2,1); and
- (1,0) will become (1.5,0)

y-direction relative Shear

- y -direction shearing relative to other reference line $x \neq 0$

$$x' = x$$

$$y' = y + sh_y\left(x - x_{ref}\right)$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y x_{ref} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

y-direction relative Shear

- *Example:* y-direction shearing *relative to the line* $x = x_{ref}$. *Given* $sh_y = 3.5$, $x_{ref} = -12$, *and the square with coordinates (0,0), (0,1), (1,1), (1,0).*

Solution

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3.5 & 1 & 42 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- *(0,0) will become (0,42);*
- *(0,1) will become (0,43);*
- *(1,1) will become (1,46.5); and*
- *(1,0) will become (1,45.5).*

Transformations

- Last update on 17-November 2014