"Maths III"

Lecture 3

Revision: if $z=\sqrt{x^2+y^2}$ prove that: $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$

$\rightarrow z(\lambda x,\lambda y)=\sqrt{(\lambda x)^2+(\lambda y)^2}$

$\qquad = \lambda\sqrt{x^2+y^2}=\lambda z$

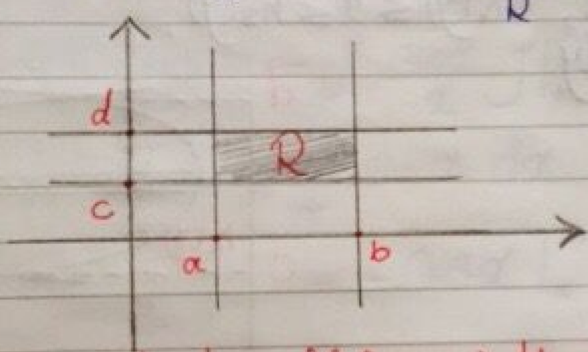
$\therefore$ z is homogeneous of first degree.

$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$ «Euler's Theorem»

Chapter 2 — Multiple Integrals.

Theorem: if $f(x,y)$ is continuous throughout the rectangular R, $a\le x\le b$, $c\le y\le d$, then: $\iint\limits_R f(x,y)\,dA=\int_c^d\int_a^b f(x,y)\,dx\,dy$

$$=\int_a^b\int_c^d f(x,y)\,dy\,dx$$



- **example:** Evaluate $\iint f(x,y)\,dA$, $f(x,y)=100-6x^2y^2$, $0\le x\le 2$, $-1\le y\le 1$

$\rightarrow \int_0^2\int_{-1}^1(100-6x^2y)\,dx\,dy=\int_{-1}^1(100x-2x^3y)\Big|_0^2\,dy$

$$=\int_{-1}^1(200-16y)\,dy$$

$$=200y-8y^2\Big|_{-1}^1$$

$$=(200-8)-(-200-8)=400$$

or $\int_0^2\int_{-1}^1(100-6x^2y)\,dy\,dx$

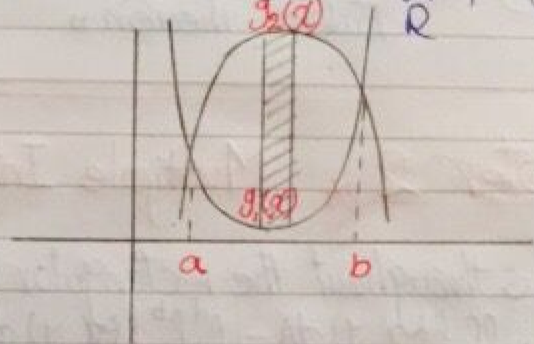
$=\int_0^2(100y-3x^2y^2)\Big|_{-1}^1\,dx=\int_0^2[(100-3x^2)-(-100-3x^2)]\,dx$

$= \int_0^2 200\, dx = 200x \Big|_0^2 = 400$

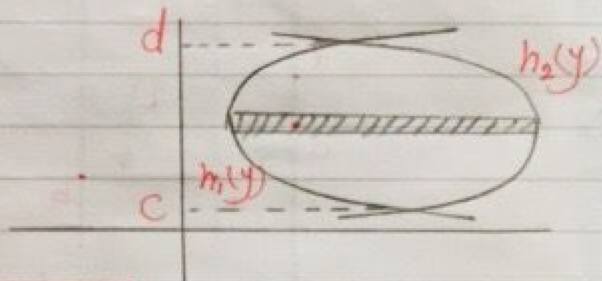
* double integrals over general regions:-

* Theorem: Let $f(x,y)$ be continuous function on R.

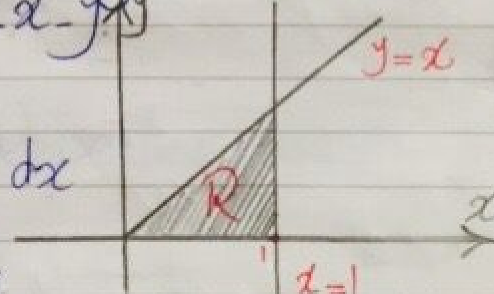
1) if R is defined by $a \le x \le b$, $g(x) \le y \le g_2(x)$ where $g(x)$ and $g_2(x)$ are continuous on $[a,b]$, then: $\iint f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$



2) if R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$,
then: $\iint_R = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$



* Example: find the volume of the prism whose base is the triangle in the x-y plane bounded by the x-axis and the lines $y=x$ and $x=1$ and whose top lines in the plane $z = f(x,y) = 3 - x - y$



$\rightarrow V = \iint_R f(x,y)\, dA = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx$

$= \int_0^1 (3y - xy - \frac{1}{2}y^2) \Big|_0^x dx$

$= \int_0^1 (3x - x^2 - \frac{1}{2}x^2)\, dx$

$= \int_0^1 (3x - \frac{3}{2}x^2)\, dx = \frac{3}{2}x^2 - \frac{1}{2}x^3 \Big|_0^1 = \frac{3}{2} - \frac{1}{2} = 1$

* Another method: $\int_0^1\int_y^1 (3-x-y)\,dx\,dy$

$$= \int_0^1 \left(3x - \frac{1}{2}x^2 - xy\right)\Big|_y^1 dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y\right) - \left(3y - \frac{1}{2}y^2 - y^2\right) dy$$

$$= \int_0^1 \frac{5}{2} - 4y + \frac{3}{2}y^2\, dy$$

$$= \frac{5}{2}y - 2y^2 + \frac{1}{2}y^2 \Big|_0^1$$

$$= \frac{5}{2} - 2 + \frac{1}{2} = 1$$

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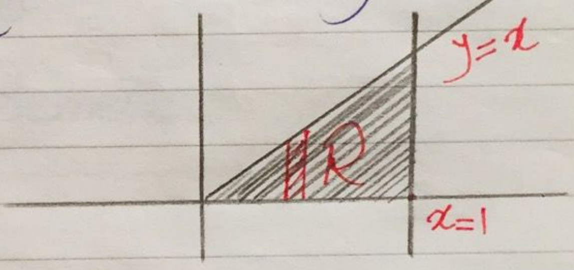
* Example 2: Calculate $\int\int_R \frac{\sin x}{x}\,dx\,dy$, where R is the triangle in the x-y plane bounded by the x-axis and lines $y = x$, $x = 1$

→ $\int_0^1\int_0^x \frac{\sin x}{x}\,dy\,dx = \int_0^1 \frac{\sin x}{x} \cdot y\Big|_0^x dx$



$$= \int_0^1 \sin x\, dx$$

$$= -\cos x\Big|_0^1$$

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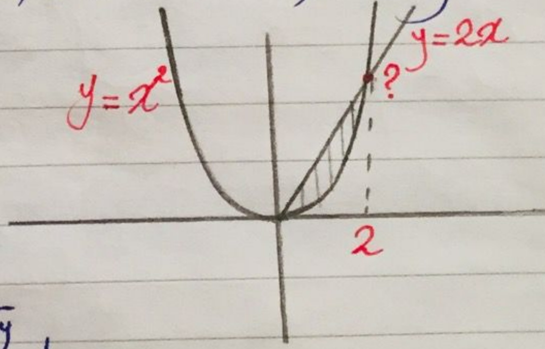
* Example 3: $\int_0^2\int_{x^2}^{2x} (4x+2)\,dy\,dx$, $0 \le x \le 2$, $x^2 \le y \le 2x$

→ point of intersection: $x^2 = 2x$

$\therefore x^2 - 2x = 0$

$x = 0$ or $x = 2$



∴ Point: $(2,4)$

$$\int_0^4\int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy = \int_0^4 2x^2 + 2x\Big|_{y/2}^{\sqrt{y}} dy$$

$$= \int_0^4 (2y + 2\sqrt{y}) - \left(\frac{1}{2}y^2 + y\right) dy$$

$$= \int_0^4 y - \frac{1}{2}y^2 + 2\sqrt{y}\, dy$$

$$= \frac{1}{2}y^2 - \frac{1}{6}y^3 + \frac{4}{3}y^{3/2} \Big|_0^4$$

$$= \frac{1}{2}\times 4^2 - \frac{1}{6}\times 4^3 + \frac{4}{3}\times 4^{3/2} = 8$$

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