"Maths III"

Lecture 7

- follow Inverse of Laplace:-

$$L^{-1}\{F(S)\} = f(t)$$

$$\rightarrow L^{-1}\{e^{-as} F(S)\} = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$$

* Examples :-

1) Find $L^{-1}\left\{\frac{e^{-2S}}{S^2-25}\right\}$

$\rightarrow L^{-1}\left\{\frac{1}{S^2-25}\right\} = \frac{\sinh 5t}{5}$, $\therefore L^{-1}\left\{\frac{e^{-2S}}{S^2-25}\right\} = \begin{cases} \frac{\sinh 5(t-2)}{5} & t>2 \\ 0 & t<2 \end{cases}$

2) Find $L^{-1}\left\{\frac{e^{-3S}}{S^2-2S+5}\right\}$

$\rightarrow L^{-1}\left\{\frac{1}{S^2-2S+5}\right\} = L^{-1}\left\{\frac{1}{(S-1)^2+4}\right\}$, $L^{-1}\left\{\frac{1}{S^2+4}\right\} = \frac{\sin 2t}{2}$

$$\therefore L^{-1}\left\{\frac{1}{(S-1)^2+4}\right\} = e^t \frac{\sin 2t}{2}$$

$$\therefore L^{-1}\left\{\frac{e^{-3S}}{S^2-2S+5}\right\} = \begin{cases} e^{t-3} \frac{\sin 2(t-3)}{2} & t>3 \\ 0 & t<3 \end{cases}$$

$$\Longrightarrow L^{-1}\left\{\frac{d^n}{dS^n} F(S)\right\} = (-1)^n t^n f(t)$$

* Examples:-

1) $L^{-1}\left\{\frac{-2S}{(S^2+1)^2}\right\} = L^{-1}\left\{\frac{d}{dS} \frac{1}{(S^2+1)}\right\}$

, $L^{-1}\left\{\frac{1}{S^2+1}\right\} = \sin t$, $\therefore L^{-1}\left\{\frac{d}{dS}\frac{1}{S^2+1}\right\} = -t\sin t$

2) Find $L^{-1}\{\ln(\frac{S-3}{S+1})\}$

$\rightarrow F(S) = \ln(\frac{S-3}{S+1}) = \ln(S-3) - \ln(S+1)$

, $\frac{d}{ds} F(S) = \frac{1}{S-3} - \frac{1}{S+1}$

, $L^{-1}\{\frac{d}{ds} F(S)\} = L^{-1}\{\frac{1}{S-3}\} - L^{-1}\{\frac{1}{S+1}\}$

$-t f(t) = e^{3t} - e^{-t}$

$\therefore f(t) = \frac{e^{3t} - e^{-t}}{-t}$, $\therefore L^{-1}\{F(S)\} = \frac{e^{3t} - e^{-t}}{-t}$ || $f(t) = L^{-1}\{F(S)\}$

* Ordinary differential equations with Constant Coefficients:-

$\rightarrow L\{y'(t)\} = SY(S) - y(0)$

$\rightarrow L\{y''(t)\} = S^2Y(S) - Sy(0) - y'(0)$

* Examples:-

1) $y' + 2y = 0$, $y(0) = 1$

$\rightarrow L\{y'\} = SY(S) - y(0)$

$L\{y\} = Y(S)$

, $L\{y'\} + 2L\{y\} = 0$

$\therefore SY(S) - y(0) + 2Y(S) = 0$

$(S+2)Y(S) = 1$, $Y(S) = \frac{1}{S+2}$

$\therefore y(t) = L^{-1}\{Y(S)\} = L^{-1}\{\frac{1}{S+2}\} = e^{-2t}$

2) $y' - 4y = 1 \quad , y(0) = 1$

$\rightarrow \mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = \mathcal{L}\{1\}$

$SY(S) - y(0) - 4Y(S) = \frac{1}{S}$

$(S-4)\,Y(S) - 1 = \frac{1}{S}$

$(S-4)\,Y(S) = \frac{1}{S} + 1 = \frac{S+1}{S}$

$\therefore Y(S) = \frac{S+1}{S(S-4)} \rightarrow (1)$

$, \frac{A}{S} + \frac{B}{S-4} = \frac{A(S-4)+BS}{S(S-4)} \rightarrow (2)$

, from (1), (2): $A(S-4) + BS = S + 1$

$\rightarrow$ Let $S = 0, 4$

, At $S = 0 \rightarrow -4A = 1 \quad , \therefore A = \frac{-1}{4}$

, At $S = 4 \rightarrow 4B = 5 \quad , \therefore B = \frac{5}{4}$

$, \therefore Y(S) = \frac{-1}{4S} + \frac{5}{4(S-4)}$

$, y(t) = \mathcal{L}^{-1}\{Y(S)\} = \mathcal{L}^{-1}\{\frac{-1}{4S}\} + \mathcal{L}^{-1}\{\frac{5}{4(S-4)}\}$

$= \frac{-1}{4}\mathcal{L}^{-1}\{\frac{1}{S}\} + \frac{5}{4}\mathcal{L}^{-1}\{\frac{1}{S-4}\}$

$= \frac{-1}{4} + \frac{5}{4}e^{4t}$