«Maths III»

Lecture 8

* Fourier Series *

- Basics:

→ If $f(-x) = f(x)$, $\therefore f$ is even function.

→ If $f(-x) = -f(x)$, $\therefore f$ is odd function.

→ $\sin x$: odd function.

→ $\cos x$: even function.

- example: $f(x) = x^2 + 4$

→ $f(-x) = (-x)^2 + 4$

$= x^2 + 4 = f(x)$

$\therefore f(x)$ is even.

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* Note:

→ $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & f \text{ even} \\ 0 & f \text{ odd} \end{cases}$$

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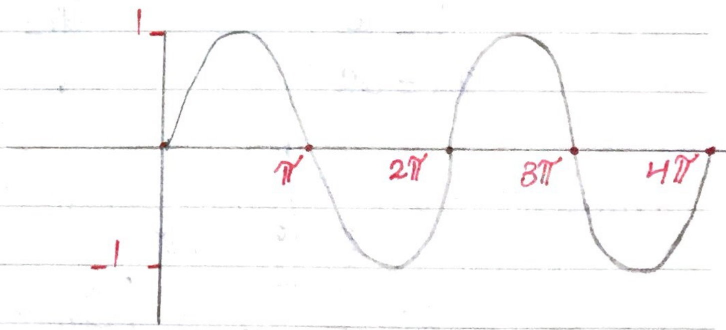
* **Definition:** A function is periodic with period T if $f(x+T) = f(x)$.

→ $\sin x$ } Periodic with 2π
→ $\cos x$ }



* **Definition**: A fourier Series of periodic function on [-L, L] (f with Period 2L):

→ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$

→ $a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$

→ $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$

→ $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$

* **Remarks**:-

1) if f is odd $\Rightarrow a_n, a_0 = 0$
2) if f is even $\Rightarrow b_n = 0$

- **Examples**:

1) find a fourier Series for $f_x = x$, $-2 < x < 2$, $f(x+4) = f(x)$

→ $\because f$ is odd , $\therefore a_0, a_n = 0$

, $b_n = \frac{1}{2} \int_{-2}^{2} x \sin \frac{n\pi x}{2}\, dx$

$= \int_0^2 x \sin \frac{n\pi x}{2}\, dx$

$u = x \qquad dv = \sin \frac{n\pi x}{2}$

$du = dx \qquad v = \frac{-2}{n\pi} \cos \frac{n\pi x}{2}$

$\therefore b_n = \frac{-2x}{n\pi} \cos \frac{n\pi x}{2} \Big|_0^2 + \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi x}{2}\, dx$

$= \frac{-2x}{n\pi} \cos \frac{n\pi x}{2} \Big|_0^2 + \left(\frac{2}{n\pi}\right)^2 \sin \frac{n\pi x}{2} \Big|_0^2$

* $\sin n\pi = 0$
* $\cos n\pi = (-1)^n$

$= \frac{-4}{n\pi} \cos n\pi$

$= \frac{-4}{n\pi} (-1)^n = \frac{4}{n\pi} (-1)^{n+1}$

$\therefore f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

2) $f(x) = \frac{1}{2}(\pi - x)$ $\qquad -\pi < x < \pi$

$\rightarrow L = \pi$, Neither even nor odd.

$$, a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(\pi - x)\, dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \pi\, dx - \cancel{\int_{-\pi}^{\pi} x\, dx}\right] \qquad \rightarrow x \text{ is odd}, \therefore \int_{-\pi}^{\pi} x\, dx = 0$$

$$= \frac{2\pi}{2\pi}\int_{0}^{\pi} dx$$

$$= \pi$$

$$, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(\pi - x)\cos nx\, dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \pi \cos nx\, dx - \cancel{\int_{-\pi}^{\pi} x \cos nx\, dx}\right] \qquad \rightarrow \text{odd}$$

$$= \int_{0}^{\pi} \cos nx\, dx$$

$$= \frac{\sin nx}{n}\Big|_{0}^{\pi} = 0$$

$$, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(\pi - x)\sin nx\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - x)\sin nx$$

$$u = \pi - x \qquad , dv = \sin nx$$

$$du = -dx \qquad , v = \frac{-\cos nx}{n}$$

$$\therefore b_n = \frac{1}{2\pi}\left[\frac{-(\pi - x)}{n}\cos nx\right]_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} \cos nx\, dx$$

$$= \frac{(-1)^n}{n}$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

* Example page 114 $\rightarrow$ ملغي

3) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi - x & 0 \le x < \pi \end{cases}$

$\rightarrow L = \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} f\,dx} + \int_0^{\pi} f\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{1}{\pi}\left(\pi x - \frac{1}{2}x^2\right)\Big|_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$, \quad a_n = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx$$

$u = \pi - x \qquad , \; dv = \cos nx$

$, \; du = -dx \qquad v = \frac{\sin nx}{n}$

$$\therefore a_n = \frac{1}{\pi}\left[\cancel{\frac{(\pi - x)}{n}\sin nx\Big|_0^{\pi}} + \frac{1}{n}\int_0^{\pi} \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left(\frac{-1}{n^2}\cos nx\right)\Big|_0^{\pi}$$

$$= \frac{-1}{n^2\pi}\left(\cos n\pi - 1\right)$$

$$= \frac{1}{n^2\pi}\left(1 - (-1)^n\right)$$

$$, \quad b_n = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin nx\,dx$$

$u = \pi - x \qquad dv = \sin nx$

$du = -dx \qquad v = -\frac{\cos nx}{n}$

$$\therefore b_n = \frac{1}{\pi}\left[\frac{-(\pi - x)}{n}\cos nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} \cos nx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \left(\frac{1}{n} \times \frac{\sin nx}{n}\right)\Big|_0^{\pi}\right]$$

$$= \frac{\pi}{\pi n} = \frac{1}{n}$$

$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right)$$

4) $f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 \le x < 1 \end{cases}$

→ at $f(x) = -1$ $\quad -1 < x < 0$

$f(-x) = 1$ $\quad 1 < -x < 0 \rightarrow 0 < x < 1$

$\therefore f(x) = -(-1) = -f(x)$

→ at $f(x) = 1$ $\quad 0 \leqslant x < 1$

$f(-x) = -1$ $\quad 0 \le -x < -1 \rightarrow -1 < x < 0$

$\therefore f(x) = -(1) = -f(x)$

∴ The function is odd for all x

$\therefore a_n, a_0 = 0$

$$, b_n = \frac{1}{1}\left[\int_{-1}^{0} -\sin n\pi x \, dx + \int_{0}^{1} \sin n\pi x \, dx\right]$$

$$= \frac{\cos n\pi x}{n\pi}\Big|_{-1}^{0} + \frac{-\cos n\pi x}{n\pi}\Big|_0^1$$

$$= \frac{1-(-1)^n}{n\pi} - \frac{(-1)^n - 1}{n\pi}$$

$$= \frac{2-2(-1)^n}{n\pi}$$

$$= \frac{2(1-(-1)^n)}{n\pi}$$

$$\therefore f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\sin n\pi x\right)$$