Mathematics

Limits and Continuity

Lecture 1

* $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$
* $\lim_{x \to a} \frac{x^n - a^n}{x^m - a^m} = \frac{n}{m} a^{n-m}$
* $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$
* $\ln x = 5 \Rightarrow x = e^5$
* $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \times \lim_{x \to 0} \frac{1}{\cos x}$
  $= 1 \times 1 = 1$
* $\log_e x = \ln x$
* $\ln e^x = x$
* $e^{\ln x} = x$
* $\sin^{-1}(\sin x) = x$
* $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$
* $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

Examples

1. $\lim_{x \to 2} \frac{x+3}{x+2} = \frac{5}{4}$

2. $\lim_{x \to 3} \frac{x^2+1}{\sqrt{2x+3}} = \frac{9+1}{\sqrt{9}} = \frac{10}{3}$

3. $\lim_{x \to \infty} \frac{x+7 \div x^2}{x^2 - 5x + 2 \div x^2} = \lim_{x \to \infty} \frac{1/x + 7/x^2}{1 - 5/x + 2/x^2} = \frac{0}{1} = 0$

4- $\lim_{x\to\infty} \frac{2x^3+5x-1 \div x^3}{x^3-5 \div x^3} = \lim_{x\to\infty} \frac{2+5/x^2-1/x^3}{1-5/x^3} = \frac{2}{1} = 2$

5- $\lim_{x\to\infty} \frac{3x^3+5}{x+2} = \lim_{x\to\infty} \frac{3+5/x^3}{1/x^2+2/x^3} = \frac{3}{0} = \infty$

6- $\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{2\sin^2(x/2)}{x} = \lim_{x\to 0} \frac{\sin x/2}{x/2} \times \lim_{x\to 0} \sin \frac{x}{2}$

$= 1 \times 0 = 0$

7- $\lim_{x\to 2} \frac{x^5-32}{x-2} = \lim_{x\to 2} \frac{x^5-2^5}{x-2} = 5 \times 2^{5-1} = 80$

8- $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x+5} = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x} \times \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{5} = e \times 1 = e$

9- $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{x}$, $x = 2y$, $\lim_{y\to\infty} \left(1+\frac{1}{y}\right)^{2y} = \lim_{y\to\infty} \left(\left(1+\frac{1}{y}\right)^{y}\right)^{2} = e^2$

* The function $f(x)$ is Continious at a if : $f(a) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

Discuss the Continuity

1- $f(x) = \frac{1}{x}$

$\lim_{x\to a^+} \frac{1}{x} = +\infty$, $\lim_{x\to a^-} \frac{1}{x} = -\infty$. ∴ Not Continious

2- $f(x) = \sqrt{x}$

$\lim_{x\to 0^+} \sqrt{x} = 0$, $\lim_{x\to 0^-} \sqrt{x} = \infty$, ∴ Not Continous.

3- $f(x) = \begin{cases} 5x+3 & x \geq 0 \\ 3 & x < 0 \end{cases}$

$f(0) = 5 \times 0 + 3 = 3$

$\lim_{x \to 0^+} f(x) = 5 \times 0 + 3 = 3$, $\lim_{x \to 0^-} f(x) = 3$

$\therefore f(x)$ is Continous.