« Maths II »

Lecture 1

* Linear Algebra *

* Chapter 1: Vector Space :-

- **Definition 1:**

→ Let S be a nonempty set « $S \neq \phi$ » and « $*$ » be a binary relation defined on it, then the pair $(S, *)$ is called a group if it satisfies the following conditions :-

1) $\forall a,b,c \in S$, $a*(b*c)=(a*b)*c$ → Associativity الدمج
2) $\exists e \in S \ni \forall a \in S$, $a*e=e*a=a$ → identity element.
3) $\forall a \in S \; \exists a' \in S \ni, a*a'=a'*a=e$ → inverse for each element.

→ if $(S,*)$ satisfies the condition: $a*b=b*a$, $\forall a,b \in S$, so this group is called abelian group. مجموعة إبالية

* $(Z,+)$, $\forall a,b \in Z$, $a+b \in Z$ → binary relation.
, $(N,-)$, $\forall a,b \in Z$, $b-a \notin Z$ → Not binary relation.

- **Definition 2:**

→ Let S be a nonempty set « $S \neq \phi$ » and « $*, o$ » be two binary relations defined on it, then the triple $(S, *, o)$ is called a Field if it satisfies the following conditions:-

1) $(S,*)$ is an abelian group.
2) $(S-\{o\}, o)$ abelian group, where o is identity element of the relation $(*)$
3) $\forall a,b,c \in S$, $a o (b*c) = (a o b) * (a o c)$

- Examples: 1) $(R,+,\cdot)$ is a field.
2) $(\mathbb{C},+,\cdot)$ is a field.

- explaination for example 1: $(R-\{0\}, .)$ is an abelian group

1) $\forall a,b,c \in R-\{0\}$, $[a.(b.c)=(a.b).c]$
2) $\exists e \in R-\{0\} \ni \forall a \in R-\{0\}$, $[a.e=e.a=a \Rightarrow e=1]$
3) $\forall a \in R-\{0\}\ \exists a^{-1} \in R-\{0\} \ni [a.a^{-1}=a^{-1}.a=1]$ « $a^{-1}=\frac{1}{a}$ »

* Note: We $-\{0\}$ because it has no inverse.

- Notice that:- $(2,3) \neq (3,2)$, $(2,3,4) \neq (2,4,3)$

- Definition 3:

→ Let V be a nonempty set of vectors, F be a field and $(*, o)$ be two binary relations defined by:

→ if $v,u \in V$; $* : V+V \to V$ « 5 Conditions. »
→ if $v \in V$; $a \in F$, $o : F.V \to V$ « 5 Conditions. », then

$(V,+,.)$ is a vector space defined on F if it satisfies these 10 Conditions.

- 10 Conditions:

1) $\forall u,v \in V \Longrightarrow u+v \in V$ « Closed relation. »

example: $V=\{(a,1) : a \in R\}$

1) $v=(b,1)$, 2) $u=(a,1)$
→ $u+v=(a+b,2) \notin V$ « Not closed relation. »

2) $\forall u,v,w \in V \Longrightarrow (u+v)+w=u+(v+w)$ Associative دامج
3) $\forall u,v, \in V \Longrightarrow u+v=v+u$ Abelian « Commutative » إبدالي
4) $\exists 0 \in V \ni \Longrightarrow 0+u=u+0=u$ Identity element.
5) $\exists -u \in V \Longrightarrow u+(-u)=0$ Inverse.
6) $\forall u \in V, a \in F \Longrightarrow a.u \in V$
7) $\forall u,v \in V, a \in F \Longrightarrow a.(u+v)=a.u+a.v$
8) $\forall u \in V, a,b \in F \Longrightarrow (a+b).u=a.u+b.u$
9) $\forall a,b \in F, u \in V \Longrightarrow a(b.u)=(a.b).u$
10) $1.u=u$, $\forall u \in V$.

* All these 10 Conditions must be satisfied to prove that a group is a vector space.

* $R^2 = R\times R = \{(a,b) : a,b \in R\}$
* $R^3 = R\times R\times R = \{(a,b,c) : a,b,c \in R\}$
* $M_{2\times2}(R) = \left\{\begin{bmatrix} a & c \\ b & d \end{bmatrix} : a,b,c,d \in R\right\}$
* $(a,b) + (c,d) = (a+c, b+d)$
* $\lambda(a,b) = (\lambda a, \lambda b)$
* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} a+a_1 & b+b_1 \\ c+c_1 & d+d_1 \end{bmatrix}$
* $\lambda\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$

$\forall \rightarrow$ for all $\rightarrow$ لكل
$\in \rightarrow$ belong to $\rightarrow$ ينتمي إلى
$\exists \rightarrow$ there is $\rightarrow$ يوجد
$\ni \rightarrow$ Such that $\rightarrow$ حيث أن
$R \rightarrow$ real numbers $\rightarrow$ الأعداد الحقيقية
$N \rightarrow$ natural numbers $\rightarrow$ الأعداد الطبيعية

* Solved Examples * الخلاصة

1) Prove that: $V = \{(x,y,z), x,y,z \in R\}$ Not a vector space.
where: 1- $(x_1,y_1,z_1) + (x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$
2- $\lambda(x,y,z) = (0,0,0)$

→ Let: $u = (x,y,z)$, $v = (A,b,C)$
$1u = u = (x,y,z) \neq (0,0,0)$
$\therefore$ this space is not a vector space.

2) Prove that: $V = \{(x,0), x \in R\}$ Not a vector space.
where: 1) $(x,0) + (y,0) = (x-y, 0)$
2) $\alpha(x,0) = (\alpha x, 0)$

→ Let: $u = (x,0)$, $v = (y,0)$
$u+v = v+u$
$L.H.S = u+v = (x,0)+(y,0) = (x-y, 0)$
, $R.H.S. = v+u = (y,0)+(x,0) = (y-x, 0) \neq L.H.S.$
$\therefore$ this space is not a vector space. لا يتحقق الشرط الإبدالي

3) Prove that $V = \left\{ \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}, a,b \in R \right\}$ Not a vector space.

→ Let $u = \begin{pmatrix} a_1 & 1 \\ 1 & b_1 \end{pmatrix}$, $v = \begin{pmatrix} a_2 & 1 \\ 1 & b_2 \end{pmatrix}$

$u + v = \begin{pmatrix} a_1 + a_2 & 2 \\ 2 & b_1 + b_2 \end{pmatrix} \notin V$

$\therefore$ V is not a vector space.

4) Discuss if $V = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a,b \in R \right\}$ A vector space or Not.

→ Let $u = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$, $v = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$

$u + v = v + u = \begin{pmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{pmatrix}$ ✓

, $\alpha u = \alpha \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & 0 \\ 0 & \alpha b_1 \end{pmatrix}$ ✓

$\therefore$ V is a vector space.

5) Discuss if $V = \left\{ \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}, a,b \in R \right\}$ A vector space or not.

→ Let $u = \begin{pmatrix} a_1 & 0 \\ 1 & b_1 \end{pmatrix}$, $v = \begin{pmatrix} a_2 & 0 \\ 1 & b_2 \end{pmatrix}$

$u + v = \begin{pmatrix} a_1 + a_2 & 0 \\ 2 & b_1 + b_2 \end{pmatrix} \notin V$, $\therefore$ V is not a vector space.

6) Discuss if $V = \{(x,y,z), x,y,z \in R\}$ A vector space or not.
$, \alpha(x,y,z) = (\alpha x, y, z)$

→ Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

$(a+b)u = au + bu$

L.H.S. $= (a+b)(x,y,z) = ((a+b)x, y, z)$

R.H.S. $= a(x,y,z) + b(x,y,z) = (ax,y,z) + (bx,y,z) = (ax+bx, 2y, 2z)$
$\neq$ L.H.S

$\therefore$ V is not a vector space

* Note :-

$$V = C_{[a,b]}$$

1) $(f+g)(x) = f(x) + g(x)$
2) $(\lambda f)(x) = \lambda(f(x))$

→ Let $u = f(x)$
$(f+g)(x) = f(x) + g(x)$
, $(\lambda f)(x) = \lambda(f(x))$

$\therefore V$ is a vector space.

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* Sheet: prove that $(R^3, +, \cdot)$ is a vector space. «10 Conditions.»
