« Maths II »

Lecture 5

Ordinary differential Equations « ODE$_s$ »

Definitions:-

1) differential equation: it is an equation that contains differentials.
- Ex: $y' = \frac{dy}{dx} = x + 5$, $y'' + 2y'' + y' = \cos x$.

2) ordinary diff. equation: it is a diff. equation that contains only one independent variable. - ex: $y = y(x) \rightarrow y$: dependent تابع , x: independent مستقل.

3) partial diff. equation: it is a diff. equation that contains two or more independent variables.

4) order of diff. equation: it is the order of the highest derivative in the diff. equation.

5) Degree of diff. equation: it is the Algebric degree of the highest order derivative of the diff. equation.

* Ex: Determine the order and degree of the following ODE$_s$:-

1) $(y')^2 = \frac{3x}{4y} \rightarrow$ order 1 , degree 2

2) $[1 + (y')^2]^{3/2} = ky''$
$\rightarrow [1 + (y')^2]^3 = k^2 (y'')^2 \rightarrow$ order 2 , degree 2

3) $y''' = \sqrt{y'}$
$\rightarrow (y''')^2 = y' \rightarrow$ order 3 , degree 2

6) Solution of diff. equation: it is each function that satisfies the diff. equation
- ex: $y = e^{-\sin x}$ is a solution for: $y' + y\cos x = 0$

7) General Solution: it is the solution that contains constants.
- ex: $y'' = 0$, $y''' = A$, $y' = Ax + B$, $y = \frac{Ax^2}{2} + Bx + C$

8) particular Solution: it is the solution that contains specific values for the arbitrary constants.

EX: Determine the diff. equation for the following functions:

1) $y = A\cos nx + B\sin nx$

$\rightarrow y' = -nA\sin nx + nB\cos nx$

$y'' = -n^2A\cos nx - n^2B\sin nx$

$= -n^2(A\cos nx + B\sin nx) = -n^2y$

$\therefore y'' + n^2y = 0 \rightarrow$ diff. equation.

2) $y = Ae^{2x} + Be^{x} + C$

$\rightarrow y' = 2Ae^{2x} + Be^{x} \rightarrow (1)$

$y'' = 4Ae^{2x} + Be^{x} \rightarrow (2)$

$y''' = 8Ae^{2x} + Be^{x}$

from 1,2: $y'' - y' = 2Ae^{2x}$

$y''' = 4(2Ae^{2x}) + Be^{x}$

$= 4(y'' - y') + 2y' - y'' = 3y'' - 2y'$

* Methods of Solving ODEs *

1) Separation of variables: فصل المتغيرات

$\rightarrow y = f(x, y) \Rightarrow \phi(x)\,dx + \psi(y)\,dy = 0$

$\int \phi(x)\,dx + \int \psi(y)\,dy = C$

EX: Solve the following ODEs:

1- $x^3 dx + (y+1)^2 dy = 0$

$\rightarrow \int x^3 dx + \int (y+1)^2 dy = C$

$\therefore \dfrac{x^4}{4} + \dfrac{(y+1)^3}{3} = C$

2- $x^2(y+1)dx + y^2(x-1)dy = 0 \qquad \div (y+1)(x-1)$

$\rightarrow \dfrac{x^2}{x-1}dx + \dfrac{y^2}{y+1}dy = 0$

$\int \dfrac{x^2}{x-1}dx + \int \dfrac{y^2}{y+1}dy = c$

$\int \left(\dfrac{x^2-1+1}{x-1}\right)dx + \int \left(\dfrac{y^2-1+1}{y+1}\right)dy = c$

$\int \left[\dfrac{(x-1)(x+1)}{x-1} + \dfrac{1}{x-1}\right]dx + \int \left[\dfrac{(y-1)(y+1)}{y+1} + \dfrac{1}{y+1}\right]dy = c$

$\int \left[x+1 + \dfrac{1}{x-1}\right]dx + \int \left[y-1 + \dfrac{1}{y+1}\right]dy = c$

$\dfrac{x^2}{2} + x + \ln(x-1) + \dfrac{y^2}{2} - y + \ln(y+1) = c$

3- $y' = e^{2(x+y)}$

$\rightarrow y' = e^{2x} \cdot e^{2y}$

$\dfrac{dy}{dx} = e^{2x} \cdot e^{2y} \Rightarrow \int e^{-2y}dy = \int e^{2x}dx + c$

$\dfrac{-1}{2}\int -2e^{-2y}dy = \dfrac{1}{2}\int 2e^{2x}dx + c$

$\dfrac{-1}{2}e^{-2y} = \dfrac{1}{2}e^{2x} + c \rightarrow$ G.S.

2) Homogeneous ODEs: متجانس

- Definitions:

1) Homogeneous function: the function $f(x,y)$ is said to be homogeneous of degree n if: $f(\lambda x, \lambda y) = \lambda^n f(x,y)$.

- Ex: $f(x,y) = x^4 - x^3 y$

$\rightarrow f(\lambda x, \lambda y) = (\lambda x)^4 - (\lambda x)^3 (\lambda y)$

$= \lambda^4 x^4 - \lambda^3 x^3 \lambda y$

$= \lambda^4 (x^4 - x^3 y) = \lambda^4 f(x,y)$

$\therefore$ the equation is homogeneous of degree 4

2) Homogeneous diff equation: the diff. equation $y' = f(x,y)$ is said to be homo. if $f(x,y)$ is homo. of degree 0.

- Methods of Solving homo. diff. equations:-

$\rightarrow$ let $v = \dfrac{y}{x} \Rightarrow y = xv$

$\hookrightarrow dy = x\,dv + v\,dx \qquad (\div dx)$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

- Ex: Solve the following diff. equation:-

1) $y' = \dfrac{y^2 + 2xy}{x^2}$

$\rightarrow$ Let $v = \dfrac{y}{x} \Rightarrow y = vx \Rightarrow y' = x\dfrac{dv}{dx} + v$

$\therefore x\dfrac{dv}{dx} + v = \dfrac{(xv)^2 + 2x(xv)}{x^2} = v^2 + 2v$

$\therefore x\dfrac{dv}{dx} = v^2 + v \Rightarrow \int \dfrac{dv}{v(v+1)} = \int \dfrac{dx}{x} + c$

$\therefore \dfrac{1}{v(v+1)} = \dfrac{A}{v} + \dfrac{B}{v+1} = \dfrac{A(v+1) + Bv}{v(v+1)}$

$\therefore A(v+1) + Bv = 1$

at $v = 0$, $A = 1$

at $v = -1$, $-B = 1$, $\therefore B = -1$

$\therefore \dfrac{1}{v(v+1)} = \dfrac{1}{v} - \dfrac{1}{v+1} \Rightarrow$ General Solution.