«Maths II»

Lecture 6

3) Semi_Homogeneous D.E:-

→ it is an ODE of the form: $\frac{dy}{dx} = y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

* if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow$ parallel

, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow$ intersection.

Ex: Solve the following ODEs:-

1) $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$

→ $\frac{1}{1} \neq \frac{-1}{1} \Rightarrow$ intersection.

$$\begin{array}{l} y-x+1=0 \\ \underline{y+x+5=0} \\ 2y+6=0 \end{array} \rightarrow y=-3 \ , \ x=-2$$

∴ point of intersection: $(-2,-3)$

Let: $x = X-2$, $y = Y-3$

$dx = dX$, $dy = dY$

$$\frac{dY}{dX} = \frac{Y-3-(X-2)+1}{Y-3+X-2+5} = \frac{Y-X}{Y+X} \longrightarrow (I)$$

$Y-X$ → First degree

$Y+X$ → first degree

∴ Homogeneous D.E.

, Let $V = \frac{Y}{X} \Rightarrow Y = VX$

⟹

$\because Y = vX$

$\therefore \frac{dY}{dX} = X\frac{dv}{dX} + v$

<u>Substitute in (I)</u>: $X\frac{dv}{dX} + v = \frac{vX - X}{vX + X} = \frac{X(v-1)}{X(v+1)} = \frac{v-1}{v+1}$

$\therefore X\frac{dv}{dX} = \frac{v-1}{v+1} - v = \frac{v-1-v(v+1)}{v+1} = \frac{-(1+v^2)}{1+v}$

$\therefore \frac{1+v}{1+v^2}\,dv = -\frac{dX}{X} \rightarrow \int \frac{1+v}{1+v^2}\,dv = -\int \frac{dX}{X} + C$

$\therefore \int \left[\frac{1}{1+v^2} + \frac{v}{1+v^2}\right] dv = -\ln X + C$

$\therefore \tan^{-1} v + \frac{1}{2}\ln(1+v^2) = -\ln X + C$

Substitute v by $\frac{y+3}{x+2}$, X by $x+2$

$\therefore$ <u>General Solution</u>: $\tan^{-1}\left(\frac{y+3}{x+2}\right) + \frac{1}{2}\ln\left(1+\left(\frac{y+3}{x+2}\right)^2\right) = -\ln(x+2) + C$

2) $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$

$\rightarrow \frac{1}{1} = \frac{-1}{-1} \Rightarrow$ parallel

Let $z = y - x$

$\frac{dz}{dx} = \frac{dy}{dx} - 1$

$\therefore \frac{dz}{dx} + 1 = \frac{z+1}{z+5} \Rightarrow \frac{dz}{dx} = \frac{z+1}{z+5} - 1 = \frac{z+1-z-5}{z+5} = \frac{-4}{z+5}$

$\therefore (z+5)\,dz = -4\,dx$

$\int (z+5)\,dz = -4\int dx + C$

$\therefore \frac{z^2}{2} + 5z = -4x + C$

4) Exact Diff. Equations:- معادلات تامة

* it is an ODE of form: $M(x,y)\,dx + N(x,y)\,dy = 0$

, if $\frac{dM}{dy} = \frac{dN}{dx}$, So the equation is exact.

then we get that: $\int M(x,y)\,dx + \int N(x,y)\,dy = C$

- Ex: Solve the following ODEs:-

* if $y(x)$ So, its diff.: $\frac{dy}{dx}$

* if $z(x,y)$ So, its diff.: $\frac{\partial z}{\partial x}$

∂: partial

1) $(\sin y + y\sin x)\,dx + (x\cos y - \cos x)\,dy = 0$

$\rightarrow \underbrace{(\sin y + y\sin x)}_{M}\,dx + \underbrace{(x\cos y - \cos x)}_{N}\,dy = 0$

$\therefore M_y = \cos y + \sin x \quad , \quad N_x = \cos y + \sin x$

$\therefore$ The equation is exact.

$\int (\sin y + y\sin x)\,dx = x\sin y - y\cos x$ « every not x is Const. »

, $\int (x\cos y - \cos x)\,dy = x\sin y - y\cos x$ « every not y is Const. »

$\therefore$ General Solution: $x\sin y - y\cos x = C$

2) $(y^2 + 4x^3)\,dx + (2xy - 3y^2)\,dy = 0$

$\rightarrow \underbrace{(y^2 + 4x^3)}_{M}\,dx + \underbrace{(2xy - 3y^2)}_{N}\,dy = 0$

$\therefore M_y = 2y \quad , \quad N_x = 2y$, $\therefore$ The equation is exact.

$\int (y^2 + 4x^3)\,dx = xy^2 + x^4$

, $\int (2xy - 3y^2)\,dy = xy^2 - y^3$

$\therefore$ General Solution: $xy^2 + x^4 - y^3 = C$

5) Semi-Exact Diff Equations:-

$\rightarrow M(x,y)\,dx + N(x,y)\,dy = 0$

if $M_y \neq N_x$, $\therefore$ the equation is not exact.

then: Let $\mu M(x,y)\,dx + \mu N(x,y)\,dy = 0$

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \longrightarrow \text{Exact}$$

$$\therefore \mu M_y + M\frac{\partial \mu}{\partial y} = \mu N_x + N\frac{\partial \mu}{\partial x}$$

$$, \mu(M_y - N_x) = -M\frac{\partial \mu}{\partial y} + N\frac{\partial \mu}{\partial x}$$

a) if $\mu = \mu(x) \Rightarrow \frac{\partial \mu}{\partial y} = 0$, $\frac{\partial \mu}{\partial x} \Rightarrow \frac{d\mu}{dx}$

$$\mu(M_y - N_x) = N\frac{d\mu}{dx}$$

$$\therefore \int \frac{M_y - N_x}{N}\,dx = \int \frac{d\mu}{\mu}$$

$$\therefore \int \frac{M_y - N_x}{N}\,dx = \ln \mu$$

$$\therefore \mu = e^{\int \frac{M_y - N_x}{N}dx}$$

b) if $\mu = \mu(y) \Rightarrow \frac{\partial \mu}{\partial x} = 0$, $\frac{\partial \mu}{\partial y} \Rightarrow \frac{d\mu}{dy}$

$$\therefore \mu(M_y - N_x) = -M\frac{d\mu}{dy}$$

$$\therefore \int \frac{M_y - N_x}{-M}\,dy = \int \frac{d\mu}{\mu}$$

$$\therefore \int \frac{M_y - N_x}{-M}\,dy = \ln \mu$$

$$\therefore \mu = e^{\int \frac{M_y - N_x}{-M}dy}$$

- Example:- $(1-xy)dx+(xy-x^2)dy=0$

→ $M_y=-x \quad , \quad N_x=y-2x \Rightarrow$ Not exact

$M_y-N_x=-x-y+2x=x-y$

$, \quad \frac{M_y-N_x}{-M}=\frac{x-y}{-(1-xy)}=\frac{x-y}{xy-1} \rightarrow$ Not function in y

$, \quad \frac{M_y-N_x}{N}=\frac{x-y}{-x(x-y)}=\frac{-1}{x} \rightarrow$ function in x

$\therefore \mu=e^{\int \frac{M_y-N_x}{N}dx}=e^{\int -\frac{dx}{x}}=e^{-\ln x}=e^{\ln \frac{1}{x}}=\frac{1}{x}$

$\therefore \mu(1-xy)dx+\mu(xy-x^2)dy=0$

$\therefore \left(\frac{1}{x}-y\right)dx+(y-x)dy=0$

$\therefore M_y=-1 \quad , \quad N_x=-1 \Rightarrow$ Exact

$\int\left(\frac{1}{x}-y\right)dx=\ln x-xy \quad , \quad \int(y-x)dy=\frac{y^2}{2}-xy$

$\therefore$ G.S. : $\ln x-xy+\frac{y^2}{2}=c$

* Try By yourself: $(2x^3y+y)dx-(x^2y^2+3x)dy=0$

→ Hint: $\mu=\frac{1}{y^4}$