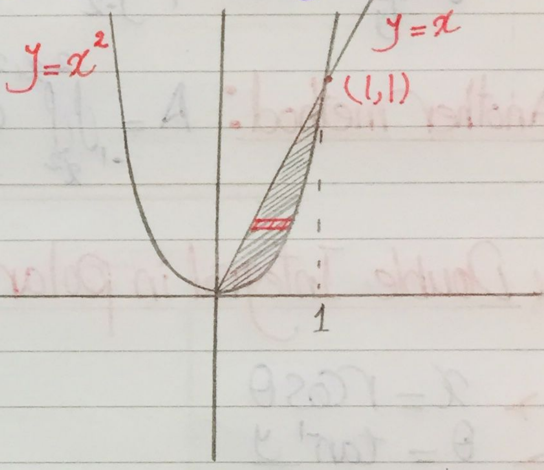
# "Maths III"

*Lecture 4*

## * The area by double integration :-

→ The area of a closed bounded plane region R is $A = \iint_R dA$

- **Example:** Find the area of region R bounded by $y = x$ and $y = x^2$




→ $x^2 = x$

$x^2 - x = 0 \quad , \quad x(x-1) = 0$

$\therefore x = 0 \quad or \quad x = 1$

$\therefore$ point of intersection : (1,1)

$$A = \int_0^1 \int_y^{\sqrt{y}} dx\, dy = \int_0^1 (\sqrt{y} - y)\, dy$$

$$= \frac{2}{3} y^{3/2} - \frac{1}{2} y^2 \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

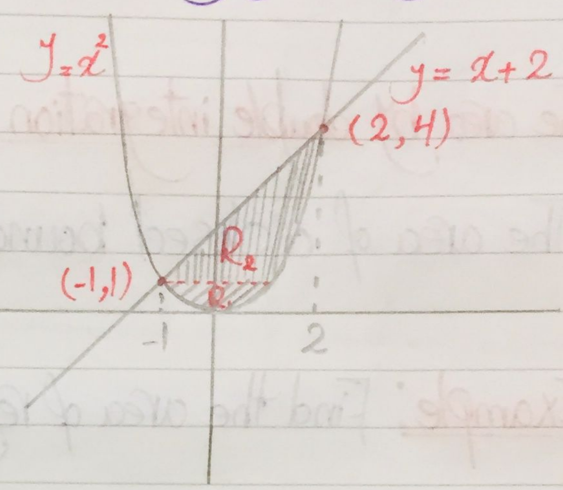
- **Another method:**

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 (x - x^2)\, dx$$

$$= \frac{1}{2} x^2 - \frac{1}{3} x^3 \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

- Example 2: Find the area of the region R bounded by $y = x^2$, $y = x + 2$




$\rightarrow x^2 = x + 2$ , $x^2 - x - 2 = 0$

$\therefore x = -1$ or $x = 2$

$\therefore$ points of intersection: $(-1,1)$, $(2,4)$

$$A = \iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$$

$$= \int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4\int_{y-2}^{\sqrt{y}} dx\,dy$$

- Another method: $A = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$

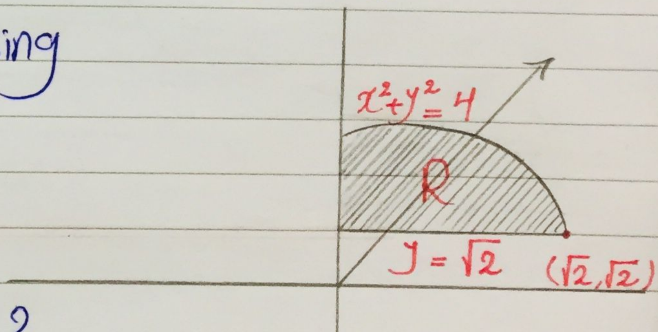
## * Double Integral in polar form:-

$\rightarrow x = r\cos\theta$ , $\rightarrow y = r\sin\theta$

$\rightarrow \theta = \tan^{-1}\frac{y}{x}$ , $\rightarrow r = \sqrt{x^2+y^2}$

$$\Longrightarrow \iint_R f(x,y)\,dA = \int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

- Example: we want to find limits of integration of that given figure.




$\rightarrow$ We imagine that ray from origin point passing by the region.

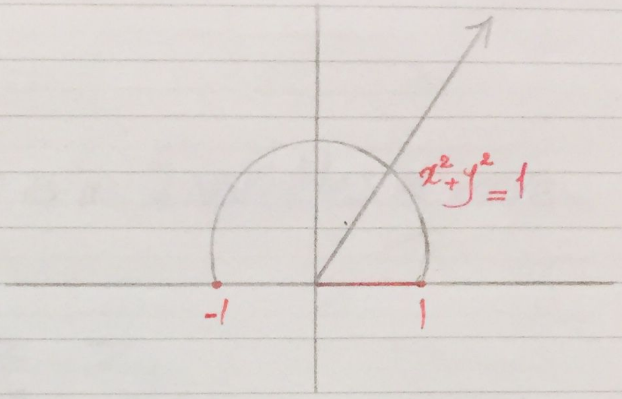
$\because y = \sqrt{2}$ , $\therefore r\sin\theta = \sqrt{2}$

$\therefore r = \frac{\sqrt{2}}{\sin\theta} = \sqrt{2}\csc\theta$ , $\because x^2+y^2 = 4 \Rightarrow r = 2$

, $\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\pi}{4}$

$$\therefore \iint_R f(r\theta)\,dA = \int_{\pi/4}^{\pi/2}\int_{\sqrt{2}\csc\theta}^{2} f(r,\theta)\,r\,dr\,d\theta$$

- Evaluate: $\iint\limits_R e^{x^2+y^2}\, dA$, R bounded by $x$-axis and $y=\sqrt{1-x^2}$




$\rightarrow y^2 = 1-x^2 \Rightarrow \underbrace{y^2+x^2=1}$

Circle of radius 1 passing by origin point.

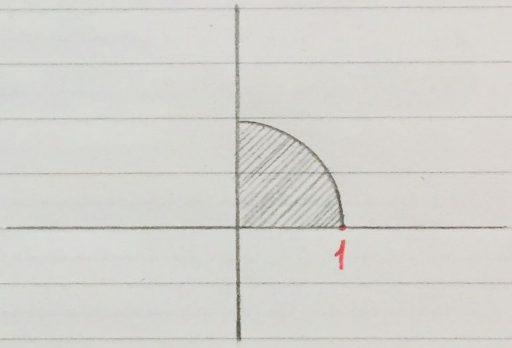
$$\therefore \iint\limits_R e^{x^2+y^2}\, dA = \int_0^{\pi}\int_0^1 e^{r^2}\, r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} e^{r^2}\Big|_0^1\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (e-1)\, d\theta$$

$$= \frac{1}{2}(e-1)\theta\Big|_0^{\pi} = \frac{\pi}{2}(e-1)$$

- Example: Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} dy\, dx$




$\rightarrow 0 \le y \le \sqrt{1-x^2}$ , $0 \le x \le 1$

$$\int_0^{\pi/2}\int_0^1 r^2\, r\, dr\, d\theta$$