"Maths III"

Lecture 2

* Examples on Chain Rule:-

1) if $w = xy$, $x = \cos t$, $y = \sin t$, find $\frac{dw}{dt}$

$$\rightarrow \frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$, \frac{dw}{\partial x} = y, \quad \frac{\partial w}{\partial y} = x, \quad \frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t$$

$$\therefore \frac{dw}{dt} = y(-\sin t) + x\cos t$$
$$= \sin t(-\sin t) + \cos t . \cos t$$
$$= -\sin^2 t + \cos^2 t$$
$$= \cos 2t$$

2) find $\frac{dw}{dt}$, $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$

$$\rightarrow \frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$
$$= y(-\sin t) + x\cos t + 1$$
$$= \sin t(-\sin t) + \cos^2 t + 1$$
$$= -\sin^2 t + \cos^2 t + 1$$

3) Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ interms of r and s if: $w = x^2 + y^2$, $x = r - s$, $y = r + s$

$$\rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$$
$$= 2x \times 1 + 2y \times 1 = 2x + 2(r + s) = 4r$$

$$, \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$
$$= 2x(-1) + 2y \times 1 = -2(r - s) + 2(r + s) = 4s$$

4) if $u=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$, show that: $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

$\rightarrow$ $$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$=\cos\theta\cdot\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}$$

$$,\left(\frac{\partial u}{\partial r}\right)^2=\cos^2\theta\cdot\left(\frac{\partial u}{\partial x}\right)^2+2\cos\theta\sin\theta\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+\sin^2\theta\cdot\left(\frac{\partial u}{\partial y}\right)^2$$

$$,\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$=-r\sin\theta\cdot\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y}$$

$$,\left(\frac{\partial u}{\partial \theta}\right)^2=r^2\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2-2r^2\sin\theta\cos\theta\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+r^2\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2$$

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial t}\right)^2=(\sin^2\theta+\cos^2\theta)\left(\frac{\partial u}{\partial x}\right)^2+(\sin^2\theta+\cos^2\theta)\left(\frac{\partial u}{\partial y}\right)^2$$

* Implicit differentiation *

* **theorem:** if $F(x,y)=0$ and y is function of x, then $\frac{dy}{dx}=-\frac{F_x}{F_y}$

- example:

Find $\frac{dy}{dx}$ if $x^2+y^2=6xy$

$\rightarrow$ $F(x,y)=x^2+y^2-6xy=0$

$F_x=2x-6y$, $F_y=2y-6x$

$$\therefore\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{2x-6y}{2y-6x}$$

* theorem: if $F(x,y,z)=0$ and $z=f(x,y)$, then: $\frac{\partial z}{\partial x}=\frac{-F_x}{F_z}$, $\frac{\partial z}{\partial y}=\frac{-F_y}{F_z}$

- example:

if $x^3+y^3+z^3+6xyz=1$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

$\longrightarrow F(x,y,z)=x^3+y^3+z^3+6xyz-1=0$

$F_x=3x^2+6yz$
$F_y=3y^2+6xz$
$F_z=3z^2+6xy$

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* Homogeneous Functions *

* The function $F(x_1,x_2,\dots,x_n)$ is homogeneous of degree P if $F(\lambda x_1,\dots,\lambda x_n)=\lambda F(x_1,\dots,x_n)$

- example:

$f(x,y)=x^2+6xy+y^2$

$\longrightarrow F(\lambda x,\lambda y)=(\lambda x)^2+6(\lambda x)(\lambda y)+(\lambda y)^2$
$=\lambda^2(x^2+6xy+y^2)$
$=\lambda^2 f(x,y)$

$\therefore$ The function is homogeneous of second degree.

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* Euler's Theorem: if $F(x_1,\dots,x_n)$ is homogeneous of degree P, then: $x_1\frac{\partial F}{\partial x_1}+x_2\frac{\partial F}{\partial x_2}+\dots+x_n\frac{\partial F}{\partial x_n}=PF$

- example:

if $F(x,y)=x^4y^2\sin^{-1}(y/x)$, Prove that: $x.\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}=6F$

$\longrightarrow F(\lambda x,\lambda y)=(\lambda x)^4(\lambda y)^2\sin^{-1}(\lambda y/\lambda x)$
$=\lambda^6x^4y^2\sin^{-1}(\frac{y}{x})=\lambda^6F(x,y)$

$\therefore$ The function is homogeneous of degree 6
then from Euler's theorem:

$$x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}=6F$$

- example 2:

if $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, prove that: $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

$\longrightarrow \tan u=\frac{x^3+y^3}{x-y}\Rightarrow (V)$

$, V(\lambda x,\lambda y)=\frac{\lambda^3x^3+\lambda^3y^3}{\lambda x-\lambda y}=\lambda^2\frac{x^3+y^3}{x-y}$

$\therefore V$ is homogeneous of degree 2

$$\therefore x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=V$$

$, V=\tan u$

$$\frac{\partial V}{\partial x}=\sec^2 u\frac{\partial u}{\partial x},\quad \frac{\partial V}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}$$

$$\therefore x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{\tan u}{\sec^2 u}$$

$$=2\frac{\sin u}{\cos u}\cdot\cos^2 u$$

$$=2\sin u\cos u$$

$$=\sin 2u$$