"Maths III"

Lecture 5

* Prove that: $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$

, given that $z(x,y)$ is homogeneous.

$\rightarrow$ $\because$ $z(x,y)$ is homogeneous.

, $\therefore$ $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ «Euler»

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}\rightarrow(1)$$ «diff with respect to x.»

$$x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}=n\frac{\partial z}{\partial y}\rightarrow(2)$$ «diff with respect to y.»

- multiply (1) by x: $x^2\frac{\partial^2 z}{\partial x^2}+x\frac{\partial z}{\partial x}+xy\frac{\partial^2 z}{\partial x\partial y}=nx\frac{\partial z}{\partial x}\rightarrow(3)$
- multiply (2) by y: $xy\frac{\partial^2 z}{\partial y\partial x}+y^2\frac{\partial^2 z}{\partial y^2}+y\frac{\partial z}{\partial y}=ny\frac{\partial z}{\partial x}\rightarrow(4)$

By adding (3),(4): $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial y\partial x}+y^2\frac{\partial^2 z}{\partial y^2}+nz=n^2z$

$$\therefore\ x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$

Chapter "3"

* Laplace Transforms *

- definition:

$$L(f(t))=\int_0^\infty e^{-st}f(t)\,dt=\lim_{T\to\infty}\int_0^T e^{-st}f(t)\,dt=F(S)$$

- **Linear:-**

Prove that: $L(C_1 f_1(t) + C_2 f_2(t)) = C_1 L(f_1(t)) + C_2 L(f_2(t))$

$$\rightarrow L(C_1 f_1(t) + C_2 f_2(t)) = \int_0^\infty e^{-st}(C_1 f_1(t) + C_2 f_2(t))\,dt$$

$$= C_1 \int_0^\infty e^{-st} f_1(t)\,dt + C_2 \int_0^\infty e^{-st} f_2(t)\,dt$$

$$= C_1 L(f_1(t)) + C_2 L(f_2(t))$$

* Important Laws:-

$$f(t) \longrightarrow F(S)$$

1. $L(1) = \dfrac{1}{S}$
2. $L(t^n) = \dfrac{n!}{S^{n+1}}$, $n = 0, 1, 2, \ldots$
3. $L(e^{at}) = \dfrac{1}{S-a}$
4. $L\{\sin at\} = \dfrac{a}{S^2+a^2}$
5. $L\{\cos at\} = \dfrac{S}{S^2+a^2}$
6. $L\{\sinh at\} = \dfrac{a}{S^2-a^2}$
7. $L\{\cosh at\} = \dfrac{S}{S^2-a^2}$

* Prove that: $L(1) = \dfrac{1}{S}$

$$\rightarrow L(1) = \int_0^\infty e^{-st}\, 1\, dt = \lim_{T\to\infty} \int_0^T e^{-st}\,dt$$

$$= \lim_{T\to\infty} \frac{-1}{S} e^{-st}\Big|_0^T$$

$$= \frac{-1}{S} \lim_{T\to\infty} [e^{-sT} - e^0]$$

$$= \frac{-1}{S}[0 - 1] = \frac{1}{S}$$

$e^{-\infty} = 0$

* Prove that: $L(e^{at}) = \frac{1}{S-a}$

$\rightarrow L(e^{at}) = \int_0^\infty e^{-ST} \cdot e^{at}\, dt = \int_0^\infty e^{-(S-a)t}\, dt$

$$= \lim_{T\to\infty} \int_0^T e^{-(S-a)t}\, dt$$

$$= \lim_{T\to\infty} \left[\frac{-1}{S-a} e^{-(S-a)t}\right]_0^T$$

$$= \frac{-1}{S-a} \lim_{T\to\infty} \left[e^{-(S-a)T} - 1\right]$$

$$= \frac{1}{S-a}$$

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* $e^{iat} = \cos at + i \sin at \rightarrow$ "Euler's formula"

- Proof of Euler's formula: -

$\rightarrow L(e^{iat}) = \frac{1}{S-ai} = \frac{S+ai}{(S-ai)(S+ai)}$

$$= \frac{S+ai}{S^2+a^2}$$

$$= \frac{S}{S^2+a^2} + i\frac{a}{S^2+a^2}$$

$$= L(\cos at + i \sin at)$$

$$= L(\cos at) + i L(\sin at)$$

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* Solved Examples *

1- $L(\cos 4t) = \dfrac{s}{s^2+16}$

3- $L(t^3) = \dfrac{3!}{s^4}$

2- $L(\sin 4t) = \dfrac{4}{s^2+16}$

4- $L(\sinh 3t) = \dfrac{3}{s^2-9}$

5- $L(\sin t \cos t) = L(\frac{1}{2}\sin 2t)$

$= \frac{1}{2} L(\sin 2t)$

$= \frac{1}{2} \times \dfrac{2}{s^2+4}$

6- $L(\cos^2 5t) = L(\frac{1}{2}(1+\cos 10t))$

$= \frac{1}{2}(L(1) + L(\cos 10t))$

$= \frac{1}{2}\left(\frac{1}{s} + \dfrac{s^2}{s^2+100}\right)$

7- $L(e^{-5t} + \cos 3t + 2\sinh 7t) = L(e^{-5t}) + L(\cos 3t) + 2L(\sinh 7t)$

$= \dfrac{1}{s+5} + \dfrac{s}{s^2+9} + 2 \times \dfrac{7}{s^2-49}$

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* Theorem "first shifting property":-

→ if $L(f(t)) = F(s)$, then $L(e^{at} f(t)) = F(s-a)$

* Examples:-

1- $L(e^{3t}\cos 2t)$

$\rightarrow \because L(\cos 2t) = \frac{S}{S^2+4}$, $\therefore L(e^{3t}\cos 2t) = \frac{S-3}{(S-3)^2+4}$

2- $L(t^2 e^{-3t})$

$\rightarrow \because L(t^2) = \frac{2!}{S^3}$, $\therefore L(t^2 e^{-3t}) = \frac{2!}{(S+3)^3}$

3- $L(e^{-6t}\sin 4t)$

$\rightarrow \because L(\sin 4t) = \frac{4}{S^2+16}$, $\therefore L = \frac{4}{(S+6)^2+16}$

- Sheet "Page 30."