# "Maths III"

*Lecture 6*

## Theorem: "Second shifting property"

→ If $L\{f(t)\} = F(S)$ and $g(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

, then ⟹ $L\{g(t)\} = e^{-as} F(S)$

**Example:** if 1) $f(t) = \begin{cases} \cos(t-2) & t > 2 \\ 0 & t < 2 \end{cases}$

2) $f(t) = \begin{cases} (t-3)^2 & t > 3 \\ 0 & t < 3 \end{cases}$ $\quad t>3 \quad t<3$

- Find $L\{f(t)\}$

→ 1) $L(\cos t) = \dfrac{S}{S^2+1}$ , $\therefore L\{f(t)\} = e^{-2S}\dfrac{S}{S^2+1}$

2) $L(t^2) = \dfrac{2!}{S^3}$ , $\therefore L\{f(t)\} = \dfrac{2!}{S^3} e^{-3S}$

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## Theorem: If $L\{f(t)\} = F(S)$, then $L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{dS^n} F(S)$

**Example:**

1) $L\{t\cos t\}$

→ $L(\cos t) = \dfrac{S}{S^2+1}$ , $\therefore L\{t\cos t\} = \dfrac{-d}{dS}\dfrac{S}{S^2+1}$

2) $L\{t^3 e^{2t}\}$

→ $L(t^3) = \dfrac{3!}{S^4}$ , $\therefore L\{t^3 e^{2t}\} = \dfrac{3!}{(S-2)^3}$ → "first shift"

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* Theorem: If $L\{f(t)\} = F(S)$
then: $L\{f^{(n)}(t)\} = S^n F(S) - S^{n-1} f(0) - S^{n-2} f'(0) \dots - S f^{(n-2)}(0) - f^{(n-1)}(0)$

1) $L\{f'(t)\} = SF(S) - F(0)$
2) $L\{f''(t)\} = S^2F(S) - Sf(0) - f'(0)$

- Example: $f(t) = \cos 3t$ , Find $L\{f'(t)\}$

$\rightarrow L(\cos 3t) = \dfrac{S}{S^2+9}$ , $\therefore L\{f'(t)\} = S \cdot \dfrac{S}{S^2+9} - 1$

$= \dfrac{S^2}{S^2+9} - 1$

$= \dfrac{-9}{S^2+9}$

- Another Solution:

$\rightarrow f'(t) = -3\sin 3t$

$\therefore L\{f'(t)\} = -3L\{\sin 3t\} = -3 \cdot \dfrac{3}{S^2+9} = \dfrac{-9}{S^2+9}$

## * Inverse of Laplace *

$\rightarrow L^{-1}\left\{\dfrac{1}{S}\right\} = 1$

$\rightarrow L^{-1}\left\{\dfrac{S}{S^2-a^2}\right\} = \cosh at$

$\rightarrow L^{-1}\left\{\dfrac{1}{S^2+a^2}\right\} = \dfrac{\sin at}{a}$

$\rightarrow L^{-1}\left\{\dfrac{1}{S^2-a^2}\right\} = \dfrac{\sinh at}{a}$

$\rightarrow L^{-1}\left\{\dfrac{S}{S^2+a^2}\right\} = \cos at$

$\rightarrow L^{-1}\left\{\dfrac{1}{S-a}\right\} = e^{at}$

- Example:

1) $L^{-1}\left\{\dfrac{5}{S^2-4}\right\} = 5L^{-1}\left\{\dfrac{1}{S^2-4}\right\} = 5\,\dfrac{\sinh 2t}{2}$

2) $L^{-1}\left\{\dfrac{-2S+6}{S^2+4}\right\} = -2L^{-1}\left\{\dfrac{S}{S^2+4}\right\} + 6L^{-1}\left\{\dfrac{1}{S^2+4}\right\}$

$= -2\cos 2t + 6\,\dfrac{\sin 2t}{2}$

3) $L^{-1}\left\{\frac{3(S-1)^2}{2S^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{S^2-2S+1}{S^5}\right\}$

$$= \frac{3}{2}\left[L^{-1}\left\{\frac{1}{S^3}\right\} - 2L^{-1}\left\{\frac{1}{S^4}\right\} + L^{-1}\left\{\frac{1}{S^5}\right\}\right]$$

$$= \frac{3}{2}\left(\frac{t^2}{2!} - 2\frac{t^3}{3!} + \frac{t^4}{4!}\right)$$

4) $L^{-1}\left\{\frac{4S+15}{16S^2-25}\right\} = L^{-1}\left\{\frac{4S}{16S^2-25}\right\} + L^{-1}\left\{\frac{15}{16S^2-25}\right\}$

$$= \frac{1}{4}L^{-1}\left\{\frac{S}{S^2-\frac{25}{16}}\right\} + \frac{15}{16}L^{-1}\left\{\frac{1}{S^2-\frac{25}{16}}\right\}$$

$$= \frac{1}{4}\cosh\frac{5}{4}t + \frac{15}{16}\frac{\sinh\frac{5}{4}t}{\frac{5}{4}}$$

5) $L^{-1}\left\{\frac{S^2+1}{S^3+3S^2+2S}\right\}$

$\rightarrow \frac{S^2+1}{S^3+3S^2+2S} = \frac{S^2+1}{S(S^2+3S+2)} = \frac{S^2+1}{S(S+1)(S+2)} \rightarrow (1)$

, $\frac{A}{S} + \frac{B}{S+1} + \frac{C}{S+2} = \frac{A(S+1)(S+2)+B(S+2)S+C(S+1)S}{S(S+1)(S+2)} \rightarrow (2)$

, from (1), (2) : $S^2+1 = A(S+1)(S+2) + B(S+2)S + C(S+1)S$

$\rightarrow$ Let $S = 0, -1, -2$

, At $S=-1 \rightarrow 2 = -B$ , $\therefore B = -2$

, At $S=0 \rightarrow 1 = 2A$ , $\therefore A = \frac{1}{2}$

, At $S=-2 \rightarrow 5 = 2C$ , $\therefore C = \frac{5}{2}$

$\therefore L^{-1}\left\{\frac{1/2}{S}\right\} + L^{-1}\left\{\frac{-2}{S+1}\right\} + L^{-1}\left\{\frac{5/2}{S+2}\right\} = \frac{1}{2} - 2e^{-t} + \frac{5}{2}e^{-2t}$

*Theorem: If $L^{-1}\{F(S)\} = f(t)$, then $L^{-1}\{F(S-a)\} = e^{at} f(t)$

"inverse of first shift"

- Example:

1) Find $L^{-1}\left\{\frac{1}{S^2-2S+5}\right\}$

$\rightarrow = L^{-1}\left\{\frac{1}{(S-1)^2+4}\right\}, \quad L^{-1}\left\{\frac{1}{S^2+4}\right\} = \frac{\sin 2t}{2}$

$\therefore L^{-1}\left\{\frac{1}{(S-1)^2+4}\right\} = \frac{\sin 2t}{2} \cdot e^{t}$

* $S^2 - 2S + 5$
$= (S-1)^2 - 1 + 5$
$= (S-1)^2 + 4$

* $S^2 - 6S + 25$
$= (S-3)^2 - 9 + 25$
$= (S-3)^2 + 16$

2) $L^{-1}\left\{\frac{S}{S^2-6S+25}\right\}$

$\rightarrow = L^{-1}\left\{\frac{S}{(S-3)^2+16}\right\} = L^{-1}\left\{\frac{(S-3)+3}{(S-3)^2+16}\right\}$

$= L^{-1}\left\{\frac{(S-3)}{(S-3)^2+16}\right\} + L^{-1}\left\{\frac{3}{(S-3)^2+16}\right\}$

$= \cos 4t \cdot e^{3t} + 3\,\frac{\sin 4t}{4} \cdot e^{3t}$