* Lecture «8» *

- Integration by Trigonometric Substitution :- التكامل بالتعويض المثلثي

* Important Uses :-

1. $\sqrt{a^2 - x^2} \longrightarrow x = a\sin\theta$
2. $\sqrt{a^2 + x^2} \longrightarrow x = a\tan\theta$
3. $\sqrt{x^2 - a^2} \longrightarrow x = a\sec\theta$

* You must Know that :-

$\longrightarrow \cos^2 x + \sin^2 x = 1$

$\longrightarrow 1 + \tan^2 x = \sec^2 x$

. Examples .

1) $\int \sqrt{a^2 - x^2}\, dx$

$= \int \sqrt{a^2 - (a^2\sin^2\theta)}\, dx$ $\quad\Big\|\quad x = a\sin\theta,\ dx = a\cos\theta\, d\theta$

$= \sqrt{a^2}\int \sqrt{1 - \sin^2\theta}\, dx$

$= a\int \cos\theta\, (a\cos\theta)\, d\theta$

$= a^2 \int \cos^2\theta\, d\theta$ $\quad\Big\|\quad \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$= \frac{a^2}{2}\int (1 + \cos 2\theta)\, d\theta$

$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

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2) $\int \sqrt{9-x^2}\,dx = \int \sqrt{9-(9\sin^2\theta)}\,dx$

$= 3\int \sqrt{1-\sin^2\theta}\,dx$

$x = 3\sin\theta$
$dx = 3\cos\theta\,d\theta$

$= 3\int \cos\theta\,(3\cos\theta)\,d\theta$

$= 9\int \cos^2\theta\,d\theta$

$= \frac{9}{2}\int (1+\cos 2\theta)\,d\theta$

$= \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

3) $\int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2+a^2\tan^2\theta}\,dx,$

$= a\int \sqrt{1+\tan^2\theta}\,dx$

$x = a\tan\theta$
$dx = a\sec^2\theta\,d\theta$

$= a\int \sec\theta\,(a\sec^2\theta)\,d\theta$

$= a^2\int \sec^3\theta\,d\theta$

*Note: $\int \sec^3\theta$ in example 6

4) $\int \frac{dx}{\sqrt{16+x^2}} = I$

$x = 4\tan\theta$
$dx = 4\sec^2\theta\,d\theta$

$\rightarrow \sqrt{16+x^2} = \sqrt{16+16\tan^2\theta} = 4\sqrt{1+\tan^2\theta} = 4\sec\theta$

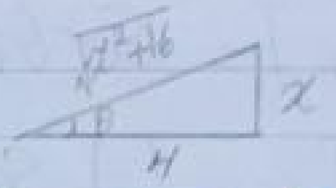
$\therefore I = \int \frac{4\sec^2\theta\,d\theta}{4\sec\theta} = \int \sec\theta\,d\theta \quad \times \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$

$= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta$

$= \ln(\sec\theta+\tan\theta) + C$

$$\therefore I = \ln\left(\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right) + C$$

$\tan\theta = \dfrac{x}{4}$



$\cos\theta = \dfrac{4}{\sqrt{x^2+16}}$, $\sec\theta = \dfrac{\sqrt{x^2+16}}{4}$

Another Solution:-

$$\int \frac{dx}{\sqrt{16+x^2}} = \sinh^{-1}\frac{x}{4} + C$$

5) $\int \sqrt{x^2-a^2}\,dx = \int \sqrt{a^2\sec^2\theta - a^2}\,dx$ | $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\,d\theta$

$$= a\int \sqrt{\sec^2\theta - 1}\,dx$$

$$= a\int \tan\theta\,(a\sec\theta\tan\theta)\,d\theta$$

$$= a^2\int \tan^2\theta\sec\theta\,d\theta$$

$$= a^2\int (\sec^2\theta - 1)\sec\theta\,d\theta$$

$$= a^2\left[\int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta\right]$$

Note: $\int \sec^3\theta\,d\theta$ in ex. 6

$\cdot \int \sec\theta\,d\theta$ in ex 5

6) $\int \sec^3\theta\,d\theta = \int \sec^2\theta\sec\theta\,d\theta = I$

$u = \sec\theta$ $\quad dv = \sec^2\theta$

$du = \sec\theta\tan\theta$ $\quad v = \tan\theta$

$\times$ $\quad \times(-\int)$

$$I = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\,d\theta$$

$$= \sec\theta\tan\theta - \int(\sec^3\theta - \sec\theta)\,d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$$

$\therefore I + I = \sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)$

$\therefore I = \frac{1}{2}\left[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\right] + C$

* Definite Integration *

$\rightarrow \int_1^2 (x^2 + x)\,dx = \frac{x^3}{3} + \frac{x^2}{2}\Big|_1^2 = \left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + \frac{1}{2}\right) = \frac{23}{6}$

$$\int_{-a}^{a} f(x)\,dx$$

- 0 if $f(x)$ is odd
- $2\int_0^a f(x)\,dx$ if $f(x)$ is even.

. Examples .

1) $\int_0^{\frac{\pi}{4}} e^{\tan x}\sec^2 x\,dx = e^{\tan x}\Big|_0^{\frac{\pi}{4}} = e - 1$

2) $\int_1^2 \frac{x}{\sqrt{x^2+4}}\,dx = \frac{1}{2}\int_1^2 \frac{2x}{\sqrt{x^2+4}}\,dx = \sqrt{x^2+4}\Big|_1^2 = \sqrt{8} - \sqrt{5}$

3) $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x\,dx = \frac{\sin^3 x}{3}\Big|_0^{\frac{\pi}{2}} = \frac{1}{3}$

4) $\int_1^4 (2e^{2x} + 3)\,dx = e^{2x} + 3x\Big|_1^4 = e^8 + 12 - e^2 - 3 = e^8 - e^2 + 9$

5) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^2 x \cos x\,dx = \frac{2\sin^3 x}{3}\Big|_0^{\frac{\pi}{2}} = \frac{2}{3}$

6) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin^2 x \cos^3 x\,dx = \text{zero}$ «f(x) is odd.»