« Maths II »

Lecture 4

* Linear Transformations *

- Definition: if u, v are two vectorspaces on the same field, then the mapping $T: V \to U$ is called a Linear transformation if it satisfies the following conditions:

i) $T(x+y) = T(x) + T(y)$, $\forall\, x, y \in V$
ii) $T(ax) = a\,T(x)$, $\forall\, a \in F$

*** Theorem:** if u, v are two vectorspaces on the same field, then the mapping $T: V \to U$ is called a Linear transformation if: $T(ax+by) = aT(x) + bT(y)$ الشرط الجامع

*** Note:**
$\to T(0) = 0$
$\to T(-x) = -T(x)$

- Example: Determine wheather the following mappings are L.T or not:
i) $T(x, y, z) = (x, y, 0)$
ii) $T(x, y) = xy$

$\to$ i) $T: R^3 \to R^3$
Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$
$au + bv = (ax_1 + bx_2,\ ay_1 + by_2,\ az_1 + bz_2)$
$\therefore T(au+bv) = (ax_1 + bx_2,\ ay_1 + by_2,\ 0) \to$ L.H.S
$T(u) = T(x_1, y_1, z_1) = (x_1, y_1, 0)$
$T(v) = T(x_2, y_2, z_2) = (x_2, y_2, 0)$
$aT(u) + bT(v) = (ax_1 + bx_2,\ ay_1 + by_2,\ 0) \to$ R.H.S
$\because$ L.H.S = R.H.S , $\therefore$ L.T.

ii) $T: R^2 \to R$
Let $u = (x_1, y_1)$, $v = (x_2, y_2)$
$au + bv = (ax_1 + bx_2,\ ay_1 + by_2)$
$\therefore T(au+bv) = (ax_1 + bx_2)(ay_1 + by_2) = a^2x_1y_1 + abx_1y_2 + abx_2y_1 + b^2x_2y_2 \to$ L.H.S
$T(u) = x_1y_1$, $T(v) = x_2y_2$

, $\overline{aT(u) + bT(v)} = ax_1y_1 + bx_2y_2 \Rightarrow R.H.S$.
$\because L.H.S \neq R.H.S$, $\therefore$ Not L.T.

Ex1 Let $T: \mathbb{R}^2 \to \mathbb{R}$ be a linear transformation such that:
$\rightarrow T(1,1) = 3$, $T(0,1) = -2$. Assume that $\{(1,1),(0,1)\}$ is a base for $\mathbb{R}^2$.
Find $T(a,b)$ and then find $T(3,4)$.

$\rightarrow (a,b) = \alpha_1(1,1) + \alpha_2(0,1)$

$a = \alpha_1$, $b = \alpha_1 + \alpha_2 \Rightarrow \alpha_2 = b - a$.

, $T(a,b) = \alpha_1 T(1,1) + \alpha_2 T(0,1)$

$= 3\alpha_1 - 2\alpha_2$

$= 3a - 2(b - a)$

$= 3a - 2b + 2a$

$\therefore T(a,b) = 5a - 2b$

, $T(3,4) = 5\times3 - 2\times4 = 7$

Ex2: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ a L.T such that:
$\rightarrow T(1,1) = (0,2)$, $T(3,1) = (2,-4)$.
Then find $T(a,b)$, $T(7,4)$

$\rightarrow (a,b) = \alpha_1(1,1) + \alpha_2(3,1)$

$a = \alpha_1 + 3\alpha_2$, $b = \alpha_1 + \alpha_2 \Rightarrow \alpha_2 = b - \alpha_1$

$\therefore \alpha_1 = a - 3\alpha_2$

$\alpha_1 = a - 3(b - \alpha_1)$

$\alpha_1 = a - 3b + 3\alpha_1$

$2\alpha_1 = 3b - a$

$\therefore \alpha_1 = \frac{3b}{2} - \frac{a}{2}$, $\alpha_2 = b - \frac{3b}{2} + \frac{a}{2}$

$\therefore T(a,b) = \alpha_1 T(1,1) + \alpha_2(3,1)$

$= \frac{3b}{2} - \frac{a}{2}(0,2) + b - \frac{3b}{2} + \frac{a}{2}(2,-4)$

$= (0, 3b - a) + (2b - 3b + a, -4b + 6b - 2a)$

$\therefore T(a,b) = (a - b, 5b - 3a)$

, $\therefore T(7,4) = (7-4, 5\times4 - 3\times7) = (3,-1)$

* **Definition:** The Kernal of a L.T. $T: U \to V$ is the Collection of vectors whose image under T is $\underline{0}$, and denoted by « Ker(T) ».
* The domain of T is the space U and denoted by « dom(T) = U »
* The Range of T is the Collection of all images of all vectors of U under T and denoted by « Rang(T) = V ».

* **Note:-** $f(x) = x^2$, $f: R \to R$
  ↳ $dom(f) = R$, $Co\text{-}dom(f) = R$, range $= [0, \infty[$

Ex: Find $Ker(T)$, $T(U)$ for the following L.T:

i) $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3)$

ii) $T(x_1, x_2, x_3) = (x_1 + x_2, x_2, x_3)$

→ 1) $T: R^3 \to R^2$

Let $u = (x_1, x_2, x_3) \in R^3 \ni T(u) = \underline{0}$

$, T(x_1, x_2, x_3) = (0,0)$

$\therefore (x_1 - x_2, x_2 - x_3) = (0,0)$

$\therefore x_1 - x_2 = 0 \quad , x_2 - x_3 = 0$

$\therefore x_1 = x_2 = x_3$

$\therefore Ker(T) = \{(x_1, x_2, x_3) \in R^3 : x_1 = x_2 = x_3\}$

$; Ker(T) = \{(1,1,1), (2,2,2), (3,3,3), \ldots\}$

ii) $T: R^3 \to R^3$

Let $u = (x_1, x_2, x_3) \in R^3 \ni T(u) = \underline{0}$

$, T(x_1, x_2, x_3) = (0,0,0)$

$\therefore (x_1 + x_2, x_2, x_3) = (0,0,0)$

$\therefore x_1 + x_2 = 0 \quad , x_2 = 0 \quad , x_3 = 0$

$\therefore x_1 = x_2 = x_3 = 0$

$\therefore Ker(T) = \{(x_1, x_2, x_3) \in R^3 : x_1 = x_2 = x_3 = 0\}$

$Ker(T) = (0,0,0)$

* **Theorem:** Let $T_1, T_2: U \to V$ be two L.T, then $(T_1 + T_2), (\lambda T_1)$ are L.T. too.