« Maths II »

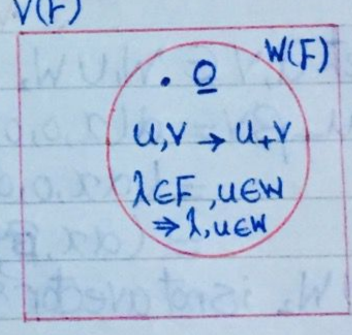
*** Lecture 2 ***

*** Theory:** for any vectorspace $V(F)$, we have:

i) $\underline{0}\,a = \underline{0}$, $a \in F$, $\underline{0} \in V$
ii) $0.u = \underline{0}$, $0 \in F$, $u \in V$
iii) $(-a)u = a(-u) = -au$, $a \in F$, $u \in V$
iv) $a(u-v) = au - av$, $a \in F$, $u, v \in V$
v) $au = 0 \longrightarrow a = 0$ or $u = 0$

*** Definition:** Let $V(F)$ is a vectorspace and $W \subseteq V$, then we say this W is avector Subspace of V if W is a vector space itself over F with the same two binary operations $(+, .)$ of V.

*** Definition:** Let $V(F)$ be avectorspace, $W \subseteq V$, then $W(F)$ is avector Subspace of $V(F)$ if it satisfies the following Conditions:-



i) $\forall\, u, v \in W \longrightarrow u + v \in W$
ii) $\underline{0} \in W$
iii) $\forall\, \lambda \in F,\ u \in W \longrightarrow \lambda u \in W$

$\Longrightarrow \forall\, u, v \in W,\ \alpha, \beta \in F \longrightarrow \boxed{\alpha u + \beta v \in W}$

Example 1: $W = \{(x, y, 0) : x, y \in R\}$, Is W avectorsubspace of R^3?

→ Let $u = (x, y, 0)$, $v = (x_1, y_1, 0)$
$u + v = (x + x_1, y + y_1, 0) \in W$
, $\underline{0} = (0, 0, 0) \in W$
, $\lambda u = \lambda(x, y, 0) = (\lambda x, \lambda y, 0) \in W$
$\therefore$ W is a vector Subspace of V

- Example 2: $W = \{(x,y) : x > y ,\ x,y \in R\}$, Is W a vector subspace of R^2?

→ Let $u = (3,2)$, $\lambda = -1$

$\lambda u = -1(3,2) = (-3,-2)$ but $-3 \not> -2$

$\therefore$ W is not a vector subspace.

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* Theory: The intersection «∩» of two vector subspaces is also a vector subspace, but their union «∪» doesn't give a vector subspace.

- Proof: Let $W_1(F)$, $W_2(F)$ be two vector subspaces, then:

→ $0 \in W_1$, $0 \in W_2 \Rightarrow 0 \in W_1 \cap W_2$, $W_1 \cap W_2 \neq \phi$

, Let $\alpha, \beta \in F$, $u, v \in W_1 \cap W_2 \Rightarrow \alpha u + \beta v \in W_1 \cap W_2$

, $u, v \in W_1$ and $u, v \in W_2$

$\therefore$ $\alpha u + \beta v \in W_1$ and $\alpha u + \beta v \in W_2$

$\therefore$ $\alpha u + \beta v \in W_1 \cap W_2$, $\therefore$ $W_1 \cap W_2$ is a vector subspace.

- Example: $W_1 = \{(x,0,0) : x \in R\}$, $W_2 = \{(0,y,0) : y \in R\}$

, $W_1 \cup W_2 = \{(x,0,0), (0,y,0) : x, y \in R\}$

→ Let $u, v \in W_1 \cup W_2$, $u = (x,0,0)$, $v = (0,y,0)$

$\alpha u + \beta v = \alpha(x,0,0) + \beta(0,y,0)$

$= (\alpha x,0,0) + (0,\beta y,0)$

$= (\alpha x, \beta y, 0) \notin W_1 \cup W_2$

$\therefore$ $W_1 \cup W_2$ is not a vector subspace.

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* Linear Combinations *

* Definition: The vector $u \in V(F)$ is a linear combination of the vectors $u_1, u_2, \ldots, u_n \in V$ if it can be expressed by: $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n$, where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$

- Example: $u_1 = (1,1,1)$, $u_2 = (1,2,3)$, $u_3 = (2,-1,1)$

prove that: $u = (1,-2,5)$ is a linear combination of u_1, u_2, u_3

→ $u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$

$(1,-2,5) = (\alpha_1 + \alpha_2 + 2\alpha_3 , \alpha_1 + 2\alpha_2 - \alpha_3 , \alpha_1 + 3\alpha_2 + \alpha_3)$

, $\therefore \alpha_1 + \alpha_2 + 2\alpha_3 = 1 \rightarrow (1)$

, $\alpha_1 + 2\alpha_2 - \alpha_3 = -2 \rightarrow (2)$

, $\alpha_1 + 3\alpha_2 + \alpha_3 = 5 \rightarrow (3)$

, from 1, 2, 3 : $\alpha_1 = -6$, $\alpha_2 = 3$, $\alpha_3 = 2$

$\therefore u = -6u_1 + 3u_2 + 2u_3$, $\therefore u$ is a linear Combination.

- Example 2 : $u_1 = (1,2,-1)$, $u_2 = (6,4,2)$, prove that : $u = (4,-1,8)$ is not a linear Combination of u_1, u_2.

→ $u = \alpha_1 u_1 + \alpha_2 u_2$

$(4,-1,8) = (\alpha_1 + 6\alpha_2 , 2\alpha_1 + 4\alpha_2 , -\alpha_1 + 2\alpha_2)$

, $\therefore \alpha_1 + 6\alpha_2 = 4 \rightarrow (1)$

, $2\alpha_1 + 4\alpha_2 = -1 \rightarrow (2)$

, $-\alpha_1 + 2\alpha_2 = 8 \rightarrow (3)$

, from 1, 2, 3 : u isn't a linear Combination.

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* Definition : if $u_1, u_2, \ldots, u_n \in V$ and for all vectors of V, we Can express them as a linear Combination of $u_1, u_2, \ldots, u_n$, then we Say that $u_1, u_2, \ldots u_n$ generates the vector space V.

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* Section Examples *

1) $W = \{(x,y,z),\ x^2 + y^2 + z^2 \leq 1\}$, Is W a vector Subspace of R^3?

→ $0 = (0,0,0)$, $0^2 + 0^2 + 0^2 = 0 \leq 1 \in W$

, Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

$u = (1,0,0)$, $v = (0,1,0)$

, Let $\alpha, \beta \in R$, $\alpha u + \beta v = \alpha(1,0,0) + \beta(0,1,0) = (\alpha,0,0) + (0,\beta,0)$

$= (\alpha, \beta, 0)$

, $\alpha^2 + \beta^2 + 0^2 \not\leq 1$, $\notin W$

$\therefore W$ isn't a vector subspace.

2) $W = \{(x,y,z,\alpha) : x = y\}$, Is W a vector subspace for R^4?

→ 1) $0 = (0,0,0,0)$, $x = y = 0 \in W$

2) Let $u = (x,x,y,z)$, $v = (a,a,y_2,z_2)$

, $u + v = (x+a, x+a, y+y_2, z+z_2) \in W$

3) Let $\alpha \in F$, $u = (x,x,y,z)$

, $\alpha u = \alpha(x,x,y,z) = (\alpha x, \alpha x, \alpha y, \alpha z) \in W$

∴ W is a vector subspace.

3) $W = \{(a,0,0), a \in R\}$, Is W a vector subspace of R^3?

→ $0 = (0,0,0) \in W$

, Let $u = (x,0,0)$, $v = (y,0,0)$, $\alpha, \beta \in R$

, $\alpha u + \beta v = \alpha(x,0,0) + \beta(y,0,0)$

$= (\alpha x + \beta y, 0, 0) \in W$

∴ W is a vector subspace.

4) $W = \{(a,b,c) : b = a + c\}$, Is W a vector subspace of R^3?

→ $0 = (0,0,0) \in W$

, Let $u = (x_1,y_1,z_1)$, $v = (x_2,y_2,z_2)$, $\alpha, \beta \in R$

, $\alpha u + \beta v = \alpha(x_1,y_1,z_1) + \beta(x_2,y_2,z_2)$

$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \in W$

, $y_1 = x_1 + z_1$, $y_2 = x_2 + z_2$

$y_1 + y_2 = x_1 + x_2 + z_1 + z_2 \rightarrow (1)$

, $\alpha y_1 + \beta y_2 = \alpha x_1 + \beta x_2 + \alpha z_1 + \beta z_2 \rightarrow (2)$

, from 1, 2 , ∴ W is a vector subspace.

5) $W = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \alpha_2 + \alpha_3 = 4\}$, Is W a vector subspace of R^4?

→ $0 = (0,0,0,0)$, $0 + 0 \neq 4$ $\in W$

∴ W is not a vector subspace.