« Maths II »

* Lecture 3 *

- Follow Lecture 2 :-

* Example: Prove that the vectors $u_1=(1,2,3)$, $u_2=(0,1,2)$, $u_3=(0,0,1)$ generate the space R^3.

→ Let $u=(x,y,z) \in R^3$

$u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$

$(x,y,z) = \alpha_1(1,2,3) + \alpha_2(0,1,2) + a_3(0,0,1)$

$\therefore x = \alpha_1 \rightarrow (1)$

, $y = 2\alpha_1 + a_2 \rightarrow (2)$

, $z = 3\alpha_1 + 2\alpha_2 + \alpha_3 \rightarrow (3)$

from (2) : $\alpha_2 = y - 2\alpha_1 = y - 2x$

, from (3) : $\alpha_3 = z + x - 2y$.

$\therefore (x,y,z) = x u_1 + (y - 2x) u_2 + (z + x - 2y) u_3$

- For example: $(1,4,-1) \in R^3$

→ $(1,4,-1) = u_1 + 2u_2 - 8u_3$

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* Definition: If $V(F)$ is a vectorspace, then the vectors $u_1, u_2, \dots u_n \in V$ are:

a) Linearly independent if for the linear Combination $\alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_n u_n = 0$, the values for all the Constants $\alpha_1, \alpha_2, \dots \alpha_n$ equal to zero.

b) Linearly dependent if at Least α_i for some i is not equal to Zero.

- Example1: Prove that $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$ in the space R^3 are L.I « linearly independent »

→ $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$

$\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) = (0,0,0)$

$\therefore \alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$

$\therefore$ the three vectors are linearly independent.

- Example 2: Prove that the following vectors $u_1=(1,2,1)$, $u_2=(3,1,5)$, $u_3=(3,-4,7)$ are L.D «linearly dependent.»

→ $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \underline{0}$

$\alpha_1(1,2,1) + \alpha_2(3,1,5) + \alpha_3(3,-4,7) = (0,0,0)$

$\alpha_1 + 3\alpha_2 + 3\alpha_3 = 0 \rightarrow (1)$

$, 2\alpha_1 + \alpha_2 - 4\alpha_3 = 0 \rightarrow (2)$

$, \alpha_1 + 5\alpha_2 + 7\alpha_3 = 0 \rightarrow (3)$

from 1, 2, 3: the vectors are linearly dependent.

* Definition: The vectors $S = \{u_1, u_2, \ldots, u_n\}$ are called the Base for the vectorspace V if:

a) S are L.I

b) S generates the space V

- Examples for Bases of Some vector spaces:

1) for R^2: $e_1=(1,0)$, $e_2=(0,1)$

2) for R^3: $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$

3) for $M_{2\times2}(R)$: $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$

* Polynomials: $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$

for example: $P_2(x) = a_2 x^2 + a_1 x + a_0$, where $a_n \neq 0$

4) for $P_n(x)$: left for 5 marks.

- Example: Prove that the vectors $u_1=(1,-3,2)$, $u_2=(2,4,1)$, $u_3=(1,1,1)$ form a base for R^3.

→ 1) $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \underline{0}$

$\alpha_1(1,-3,2) + \alpha_2(2,4,1) + \alpha_3(1,1,1) = (0,0,0)$

$\alpha_1 + 2\alpha_2 + \alpha_3 = 0 \rightarrow (1)$

$-3\alpha_1 + 4\alpha_2 + \alpha_3 = 0 \rightarrow (2)$

$2\alpha_1 + \alpha_2 + \alpha_3 = 0 \rightarrow (3)$

from 1, 2, 3: $\alpha_1 = \alpha_2 = \alpha_3 = 0$ «L.I»

2) Let $u = (x, y, z) \in R^3$

$u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$

$(x, y, z) = \alpha_1(1,-3,2) + \alpha_2(2,4,1) + \alpha_3(1,1,1)$

, $\therefore x = \alpha_1 + 2\alpha_2 + \alpha_3$

, $y = -3\alpha_1 + 4\alpha_2 + \alpha_3$

, $z = 2\alpha_1 + \alpha_2 + \alpha_3$

Let $(x, y, z) = (0, 0, 1)$

, $\therefore$ the vectors generate the space R^3

$\therefore$ the three vectors form a base for R^3.

- Definition :- The number of the base for any vector space is called " The dimension" of the space. « $\dim(V) = n$. »

* Notes :-

- Number of vectors is characteristic.
- if the number of given vectors isnot equal to the dimension of the space, so the vectors don't shape abase for space.
- if the number of given vectors is equal to the dimension of the space, then « Linearly independent » only enough to prove that it shapes a base for space.

- Important Example: Determine the base of R^3 from the following vectors: $u_1 = (1,-3,2)$, $u_2 = (2,4,1)$, $u_3 = (3,1,3)$, $u_4 = (1,1,1)$

$\rightarrow$ i) $u_1 \neq 0 \rightarrow$ L.I

ii) $\alpha_1 u_1 + \alpha_2 u_2 = 0$

$\alpha_1(1,-3,2) + \alpha_2(2,4,1) = (0,0,0)$

$\alpha_1 + 2\alpha_2 = 0 \rightarrow 1$

$-3\alpha_1 + 4\alpha_2 = 0 \rightarrow 2$

$2\alpha_1 + \alpha_2 = 0 \rightarrow 3$

, from 1, 2, 3 : $\alpha_1 = \alpha_2 = 0$, $\therefore u_1, u_2$ are L.I

iii) $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0$

$\alpha_1(1,-3,2) + \alpha_2(2,4,1) + \alpha_3(3,1,3) = (0,0,0)$

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0 \rightarrow 1$

$-3\alpha_1 + 4\alpha_2 + \alpha = 0 \rightarrow 2$

$2\alpha_1 + \alpha_2 + 3\alpha_3 = 0 \rightarrow 3$

, from 1,2,3 : u_1, u_2, u_3 are not the base.

iv) $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_4 u_4 = 0$

$\alpha_1(1,-3,2) + \alpha_2(2,4,1) + \alpha_4(1,1,1) = (0,0,0)$

$\alpha_1 + 2\alpha_2 + \alpha_4 = 0 \rightarrow 1$

$-3\alpha_1 + 4\alpha_2 + \alpha_4 = 0 \rightarrow 2$

$2\alpha_1 + \alpha_2 + \alpha_4 = 0 \rightarrow 3$

, from 1,2,3 : $\alpha_1 = \alpha_2 = \alpha_4 = 0$, $\therefore u_1, u_2, u_4$ are the base.

- Example 2 : Determine the base of R^4 if it Contains the vectors : $u_1 = (3,-2,0,0)$, $u_2 = (0,1,0,1)$

$\rightarrow$ $\because$ number of vectors is less than dimension (4)

$\therefore$ We use the Natural base for R^4 : $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$.

$\rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 e_1 = 0$.

$\alpha_1(3,-2,0,0) + \alpha_2(0,1,0,1) + \alpha_3(1,0,0,0) = (0,0,0,0)$

$3\alpha_1 + \alpha_3 = 0 \rightarrow 1$

$-2\alpha_1 + \alpha_2 = 0 \rightarrow 2$

$\alpha_2 = 0 \rightarrow 3$

, from 1,2,3 ; $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore u_1, u_2, e_1$ are base.

, $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 e_1 + \alpha_4 e_2 = 0$

$\alpha_1(3,-2,0,0) + \alpha_2(0,1,0,1) + \alpha_3(1,0,0,0) + \alpha_4(0,1,0,0) = (0,0,0,0)$

$3\alpha_1 + \alpha_3 = 0 \rightarrow 1$

$-2\alpha_1 + \alpha_2 + \alpha_4 = 0 \rightarrow 2$

$\alpha_2 = 0 \rightarrow 3$

, from 1,2,3 : $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_4$ $\therefore u_1, u_2, e_1, e_2$ are not the base.

, $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 e_1 + \alpha_4 e_3 = 0$

, $\alpha_1(3,-2,0,0) + \alpha_2(0,1,0,1) + \alpha_3(1,0,0,0) + \alpha_4(0,0,1,0) = (0,0,0,0)$

$3\alpha_1 + \alpha_3 = 0 \rightarrow 1$

$-2\alpha_1 + \alpha_2 = 0 \rightarrow 2$

$\alpha_4 = 0 \rightarrow 3$, $\alpha_2 = 0 \rightarrow 4$

, from 1,2,3,4 : $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

, $\therefore u_1, u_2, e_1, e_3$ are the base. $\rightarrow$ The base $S = \{u_1, u_2, e_1, e_3\}$