« Maths II »

Lecture 7

Follow methods of Solving ODEs:-

6) Linear DEs:

- form: $y' + p(x)y = Q(x)$

→ at first, we find $M = e^{\int P(x)\,dx}$
then get General Solution: $My = \int M\,Q(x)\,dx + C$

Ex: Solve the following ODEs:

1) $y' - y\cot x = 2x\sin x$

$$\rightarrow y' + \underbrace{(-\cot x)}_{p(x)}y = \underbrace{2x\sin x}_{Q(x)}$$

$$M = e^{\int p(x)\,dx} = e^{-\int \frac{\cos x}{\sin x}dx} = e^{-\ln \sin x} = e^{\ln \frac{1}{\sin x}} = \frac{1}{\sin x}$$

$\therefore$ G.S. : $My = \int M\,Q(x)\,dx + C$

$$\frac{y}{\sin x} = \int 2x\,dx + C = x^2 + C$$

$\therefore y = \sin x(x^2 + C)$ → General Solution.

2) $\frac{dx}{dy} + \frac{1}{y\ln y}x = \frac{1}{y}$

$$\rightarrow \frac{dx}{dy} + p(y)x = Q(y)$$

$$M = e^{\int p(y)\,dy} = e^{\int \frac{1}{y\ln y}dy} = e^{\int \frac{1/y}{\ln y}dy} = e^{\ln(\ln y)} = \ln y$$

G.S. : $Mx = \int M\,Q(y)\,dy + C$

$$x\ln y = \int \frac{1}{y}\ln y\,dy + C = \frac{(\ln y)^2}{2} + C$$

7) Semi-linear ODEs «Bernoulli equation»:-

- form: $\frac{dy}{dx} + p(x)\,y = Q(x)\,y^n$
- We divide the equation by «y^n»

$$\therefore y^{-n}\frac{dy}{dx} + p(x)\,y^{1-n} = Q(x) \longrightarrow *$$

- then Let $z = y^{1-n}$

$$\therefore z' = (1-n)\,y^{-n}\cdot y' \Rightarrow y^{-n}y' = \frac{1}{(1-n)}z'$$

- then Sub. in $*$: $\frac{1}{(1-n)}z' + p(x)z = Q(x)$

$$\therefore z' + (1-n)\,p(x)z = (1-n)\,Q(x)$$

$$\therefore M = e^{\int (1-n)p(x)dx}$$

, General Solution: $Mz = (1-n)\int M\,Q(x)\,dx + c$

3) $y' - \frac{1}{3x}y = y^4 \ln x \qquad (\div y^4)$

$$\rightarrow y^{-4}y' - \frac{1}{3x}y^{-3} = \ln x \longrightarrow *$$

Let $z = y^{-3} \Rightarrow z' = -3y^{-4}y'$

Sub. in $*$: $\frac{-1}{3}z' - \frac{1}{3x}z = \ln x \qquad (\times(-3))$

$$\therefore z' + \underbrace{\frac{1}{x}}_{p(x)}z = \underbrace{-3\ln x}_{Q(x)}$$

$$\therefore M = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

, G.S.: $Mz = \int M\,Q(x)\,dx + c$

$$xz = -3\int x\ln x\,dx + c \longrightarrow **$$

Let $I = \int x \ln x \, dx$

$$u = \ln x \quad , \quad dv = x\,dx$$
$$du = \frac{dx}{x} \quad \quad v = \frac{x^2}{2}$$

$+$, $\times(-\int)$

$$\therefore I = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}$$

Sub. in $**$: $xz = -3\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right] + C$

$\therefore$ General Solution: $xy^{-3} = -3\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right] + C$

4) $y' - y = xy^5$ $\quad (\div y^5)$

$\rightarrow y^{-5}y' - y^{-4} = x \longrightarrow *$

Let $z = y^{-4} \Rightarrow z' = -4y^{-5}y'$

Sub. in $*$: $\frac{-1}{4}z' - z = x$ $\quad (\times(-4))$

$$\therefore \underset{}{z'} + \underset{P(x)}{4}z = \underset{Q(x)}{-4x}$$

$$\therefore \mu = e^{\int 4dx} = e^{4x}$$

$\therefore$ G.S. : $ze^{4x} = -\int 4x e^{4x}\,dx + C$

$$u = x \quad , \quad dv = 4e^{4x}dx$$
$$du = dx \quad \quad v = e^{4x}$$

$+$, $\times(-\int)$

$$\therefore ze^{4x} = -\left[xe^{4x} - \int e^{4x}dx\right] + C$$
$$\therefore y^{-4}e^{4x} = -\left[xe^{4x} - \frac{e^{4x}}{4}\right] + C$$

b) Generalized Bernoulli :-

- form: $\underbrace{f'(y)y'}_{z'} + P(x)\underbrace{f(y)}_{z} = Q(x)$

5) $\sin y\, y' = \cos y(1 - x\cos y)$

$\rightarrow \sin y\, y' = \cos y - x\cos^2 y$

$\sin y\, y' - \cos y = -x\cos^2 y \qquad (\div \cos^2 y)$

$\therefore \frac{\sin y}{\cos^2 y} y' - \frac{1}{\cos y} = -x \longrightarrow *$

Let $z = \frac{1}{\cos y} \Rightarrow z' = \sec y \tan y\, y' = \frac{\sin y}{\cos^2 y} y'$

Sub. in $*$: $z' - z = -x$

$\therefore \mu = e^{\int -dx} = e^{-x}$

$\therefore$ G.S.: $e^{-x} z = \int x e^{-x} dx + c$

$u = x \quad , \quad dv = e^{-x}$

$du = dx \quad \quad v = e^{-x}$

$\times(-\int)$

$\therefore e^{-x} z = x e^{-x} - \int e^{-x} dx + c$

$, e^{-x} z = x e^{-x} - e^{-x} + c$

c) Riccati Equation:-

- form: $y' = a(x) + b(x)y + c(x)y^2$

given that: $y_1 = y_1(x)$ is a Solution.

6) $e^{-x} y' = 1 - e^{2x} + 2e^{x} y - y^2$, given that: $y_1 = e^{x}$ is a Solution

$\rightarrow$ Let $y = e^{x} + \frac{1}{v}$ be a G.S.

$\therefore y' = e^{x} - \frac{1}{v^2} v'$

$, e^{-x}(e^{x} - \frac{v'}{v^2}) = 1 - e^{2x} + 2e^{x}(e^{x} + \frac{1}{v}) - (e^{x} + \frac{1}{v})^2$

$\therefore 1 - \frac{e^{-x}}{v^2} v' = 1 - e^{2x} + 2e^{2x} + \frac{2e^{x}}{v} - (e^{2x} + \frac{2e^{x}}{v} + \frac{1}{v^2})$

$$\therefore \frac{-e^{-x}}{\cancel{v^2}} v' = \frac{-1}{\cancel{v^2}}$$

$$\therefore e^{-x} \frac{dv}{dx} = 1$$

$$, \; dv = e^{x} \, dx$$

$$\int dv = \int e^{x} \, dx + C$$

$$\therefore v = e^{x} + C$$

$$\therefore y = e^{x} + \frac{1}{(e^{x} + C)} \Rightarrow \text{General Solution.}$$