* Lecture « 7 » *

Integration.

→ $\int \cos^n x \, dx$

→ $\int \sin^n x \, dx$

→ $\int \cos^n x \sin^m x \, dx$

n, m

odd: $\sin^2 x + \cos^2 x = 1$

Even: $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

or

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

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* Solved Examples *

1) $\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx$

$= \int (1 - \cos^2 x) \sin x \, dx$

$= \int \sin x - \cos^2 x \sin x \, dx$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

2) $\int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \cos^2 x \cos x \, dx$

$= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx$

$= \int (\sin^4 x \cos x - \sin^6 x \cos x) \, dx$

$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$

3) $\int \sin^4 x \, dx = \int \left[\frac{1}{2}(1 - \cos 2x)\right]^2 dx$

$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx$

$= \frac{1}{4}\int (1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)) \, dx$

$= \frac{1}{4}\int (\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x) \, dx$

$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{2}\frac{\sin 4x}{4}\right) + C$

4) $\int \tan^3 x \, dx = \int \tan^2 x \tan x \, dx$

$= \int (\sec^2 x - 1) \tan x \, dx$

$= \int (\sec^2 x \tan x - \tan x) \, dx$

$= \int \left(\sec^2 x \tan x - \frac{\sin x}{\cos x}\right) dx$

$= \frac{\tan^2 x}{2} + \ln \cos x + C$

* $\tan^2 x + 1 = \sec^2 x$

5) $\int \frac{1}{4x^2 + 25} \, dx = \frac{1}{4} \int \frac{dx}{x^2 + \left(\frac{5}{2}\right)^2}$

$= \frac{1}{4} \times \frac{2}{5} \tan^{-1} \frac{2x}{5} + C$

6) $\int \frac{dx}{25 - 9x^2} = \frac{1}{9} \int \frac{dx}{\left(\frac{5}{3}\right)^2 - x^2}$

$= \frac{1}{9} \times \frac{3}{5} \tanh^{-1} \frac{3x}{5} + C$

7) $\int \frac{dx}{\sqrt{3 - 5x^2}} = \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{\frac{3}{5} - x^2}}$

$= \frac{1}{\sqrt{5}} \sin^{-1} \frac{x\sqrt{5}}{\sqrt{3}} + C$

* $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$
* $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a} + C$
* $\int \frac{dx}{\sqrt{x^2 - a}} = \cosh^{-1} \frac{x}{a} + C$
* $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$
* $\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} + C$

8) $\int \frac{dx}{x^2 - 2x + 5} = \int \frac{dx}{(x-1)^2 - 1 + 5}$

$= \int \frac{dx}{(x-1)^2 + 4}$

$= \int \frac{dx}{(x-1)^2 + 2^2}$

$= \frac{1}{2} \tan^{-1} \frac{x-1}{2} + C$

* $x^2 - 2x + 5$
$= (x-1)^2 - 1 + 5$
$= (x-1)^2 + 4$
* $x^2 + 2x + 3$
$= (x+1)^2 - 1 + 3$
$= (x+1)^2 + 2$
* $x^2 - x + 1$
$= (x - 1/2)^2 - \frac{1}{4} + 1$
$= (x - 1/2)^2 + \frac{3}{4}$

9) $\int \frac{dx}{\sqrt{x^2-x+1}} = \int \frac{dx}{\sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}} = \sinh^{-1} \frac{x-1/2}{\sqrt{3}/2} + C = \sinh^{-1} \frac{2x-1}{\sqrt{3}} + C$

10) $\int \frac{x-1}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{2x-2}{x^2+2x+5}\,dx$

$= \frac{1}{2}\int \frac{2x-2+2-2}{x^2+2x+5}\,dx$

$= \frac{1}{2}\left[\int \frac{2x+2}{x^2+2x+5}\,dx - 4\int \frac{dx}{x^2+2x+5}\right]$

$= \frac{1}{2}\ln(x^2+2x+5) - 2\int \frac{dx}{(x+1)^2+4}$

$= \frac{1}{2}\ln(x^2+2x+5) - \tan^{-1}\frac{x+1}{2} + C$

* Integration By Parts *

$\rightarrow \int u\,dv = uv - \int v\,du$

Examples

1) $\int \sin^{-1}x\,dx$

$u = \sin^{-1}x \quad\quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} \quad\quad v = x$

$\times(-\int)$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}x + \sqrt{1-x^2} + C$

2) $\int \tan^{-1}x\,dx$

$$\begin{array}{lcl} u=\tan^{-1}x & \searrow^{+} & dv = dx \\ du = \dfrac{1}{1+x^2} & \xleftarrow{\times(-\int)} & v = x \end{array}$$

$$= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$$

3) $\int \ln x\,dx$

$$\begin{array}{lcl} u=\ln x & \searrow^{+} & dv = dx \\ du = \dfrac{1}{x} & \xleftarrow{\times(-\int)} & v = x \end{array}$$

$$= x\ln x - \int dx = x\ln x - x + C$$

4) $\int x e^{ax}\,dx$

$$\begin{array}{lcl} u=x & \searrow^{+} & dv = e^{ax} \\ du = 1 & \xleftarrow{\times(-\int)} & v = \dfrac{e^{ax}}{a} \end{array}$$

$$= \frac{x}{a}e^{ax} - \frac{1}{a}\int e^{ax}\,dx = \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax} + C$$

5) $\int x^3 e^{ax}\,dx$

$$\begin{array}{lcl} u=x^3 & \searrow^{+} & dv = e^{ax} \\ du = 3x^2 & \xleftarrow{\times(-\int)} & v = \dfrac{e^{ax}}{a} \end{array}$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\int x^2 e^{ax}\,dx$$

$$\begin{array}{lcl} u=x^2 & \searrow^{+} & dv = e^{ax} \\ du = 2x & \xleftarrow{\times(-\int)} & v = \dfrac{e^{ax}}{a} \end{array}$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left[\frac{x^2 e^{ax}}{a} - \frac{2}{a}\int x e^{ax}\,dx\right]$$

$$\begin{array}{lcl} u=x & \searrow^{+} & dv = e^{ax} \\ du = 1 & \xleftarrow{\times(-\int)} & v = \dfrac{e^{ax}}{a} \end{array}$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left[\frac{x^2 e^{ax}}{a} - \frac{2x}{a^2} - \frac{1}{a}\int e^{ax}\,dx\right]$$

$$= \frac{x^3}{a} e^{ax} - \frac{3}{a}\left[\frac{x^2}{a} e^{ax} - \frac{2x}{a^2} - \frac{1}{a^2} e^{ax} + C\right]$$

6) $\int x^n \ln x \, dx$

$u = \ln x \qquad dv = x^n \, dx$

$du = \frac{1}{x} \qquad v = \frac{x^{n+1}}{n+1}$

$$= \frac{\ln x}{n+1} x^{n+1} - \frac{1}{n+1}\int x^n \, dx$$

$$= \frac{\ln x}{n+1} x^{n+1} - \frac{1}{(n+1)^2} x^{n+1} + C$$

, Then find $\int x^5 \ln x \, dx$.

$$\therefore \int x^5 \ln x = \frac{\ln x}{6} \cdot x^6 - \frac{1}{36} \cdot x^6 + C$$

7) $\int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$

$u = \sin^{n-1} x \qquad dv = \sin x \, dx$

$du = (n-1) \sin^{n-2} x \cos x \qquad v = -\cos x$

$\boxed{\int \sin^n x \, dx = I}$

$$I = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$$

$$I = -\cos x \sin^{n-1} x + (n-1)\left[\int \sin^{n-2} x (1 - \sin^2 x) \, dx\right]$$

$$I = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx$$

$$I + (n-1) I = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \, dx$$

$$\therefore n I = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \, dx$$

$$\therefore I = \frac{-1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x \, dx.$$