Quantum bouncing ball

from $x = 0$ and (kinetic) energy E

$$x_{max} =$$
$$A = \frac{E}{mg}$$

$$\begin{aligned}
1^{\text{st}} \text{ bounce at time } t_1 \qquad &= \frac{1}{2}\tau \\
2^{\text{nd}} \text{ bounce at time } t_2 = t_1 + \tau &= \frac{3}{2}\tau \\
3^{\text{rd}} \text{ bounce at time } t_3 = t_2 + \tau &= \frac{5}{2}\tau
\end{aligned}$$

since $A = \frac{1}{2}g(\frac{1}{2}\tau)^2$

$$\text{bounce period } \tau = \sqrt{\frac{8a}{g}} = \sqrt{\frac{8E}{mg^2}}.$$

if we set $t = 0$ at the time of a bounce, the flight up until the next bounce follows

$$\begin{aligned}
x(t) &= \frac{1}{2}gt(\tau - t) \qquad : \qquad 0 < t < \tau \\
\dot{x}(t) &= \frac{1}{2}g\tau - gt \qquad : \qquad 0 < t < \tau
\end{aligned}$$

Action per bounce: we have

$$\begin{aligned}
x(t) &= \frac{1}{2}gt(\tau - t) \\
p(t) &= \frac{1}{2}mg(\tau - 2t)
\end{aligned}$$

Hence

$$p(x) = p_0\sqrt{1 - (\frac{x}{A})} \qquad \text{with} \qquad p_0 \equiv \frac{1}{2}mg\tau$$

$$
\begin{aligned}
\text{Action} = \oint p dx &= 2\int_0^A p_0\sqrt{1-(\frac{x}{A})dx} \\
&= \frac{4}{3}p_0 A = \frac{4}{3}\cdot\frac{1}{2}mg\tau\cdot\frac{1}{8}g\tau^2 \\
&= \frac{1}{12}mg^2\tau^2
\end{aligned}
$$

Quantization condition:

$$
\begin{aligned}
&\oint p dx = nh \quad : \quad n = 1, 2, 3, ... \\
\Rightarrow \qquad & \frac{1}{12}mg^2\tau^3 = nh \\
\Rightarrow \qquad & \tau_n = [\frac{12nh}{mg^2}]^{\frac{1}{3}} \quad : \quad n = 1, 2, 3, ...
\end{aligned}
$$

$$
\begin{aligned}
\text{by } \tau = \sqrt{\frac{8E}{mg^2}} \Longrightarrow E_n &= \frac{mg^2}{8}\tau_n^2 \\
&= \xi n^{\frac{2}{3}} \text{ with } \xi \equiv [\frac{9}{64}mg^2h^2]^{\frac{1}{3}}
\end{aligned}
$$

And since $E = mgA = mgx_{max}$

$$
\begin{aligned}
x_{max} = \frac{E}{mg} &= \frac{\xi n^{\frac{2}{3}}}{mg} \\
&= \frac{[\frac{9}{64}mg^2h^2]^{\frac{1}{3}}n^{\frac{2}{3}}}{mg} \\
&= (\frac{9}{64}\frac{h^2n^2}{m^2g})^{\frac{1}{3}}
\end{aligned}
$$