Stars, From Newton to Einstein

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Abstract

In this project, I calculate the structures of two types of stars, namely White Dwarfs (WDs) and Neutron Stars (NSs). The former will be studied using Newtonian gravity and the latter with General Relativity (GR).

Newton

Part A

We begin with the two equations of stellar structure that follow from the hydrostatic equilibrium of stars in Newtonian gravity

$$\begin{aligned} \frac{dm(r)}{dr} &= 4\pi r^2 \rho(r), \\ \frac{dp(r)}{dr} &= -\frac{Gm(r)\rho(r)}{r^2}. \end{aligned} \tag{1}$$

where r is the radius of the star, $p(r)$ is the pressure, $m(r)$ is the mass contained within r and $\rho(r)$ is the density. We need an expression for the density, and we are told that an adequate model for WDs is that of a polytropic EOS

$$p = K\rho^{1+\frac{1}{n}}, \tag{2}$$

where n is called the polytropic index. Of course, p and ρ are still functions of r, but we'll suppress the dependence in what follows for visual clarity. This set of equations can be transformed into a more convenient form. Following the discussion on the Lane-Emden Equation Wikipedia [?], we can perform a series of manipulations on Eq.(1), namely:

$$\begin{aligned} \frac{1}{\rho}\frac{dp}{dr} &= -\frac{Gm}{r^2} \\ \Rightarrow \frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) &= \frac{d}{dr}\left(-\frac{Gm}{r^2}\right) \\ &= \frac{2Gm}{r^3} - \frac{G}{r^2}\frac{dm}{dr} \\ &= \left(\frac{-2}{\rho r}\right)\left(\frac{-Gm\rho}{r^2}\right) - \frac{G}{r^2}\left(\frac{dm}{dr}\right) \\ &= \frac{-2}{\rho r}\left(\frac{dp}{dr}\right) - 4\pi G\rho \end{aligned} \tag{3}$$

Multiply both sides by r^2 and collect the pressure derivative terms on one side

$$\begin{aligned} r^2\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) + \frac{-2r}{\rho}\frac{dp}{dr} &= -4\pi r^2 G\rho \\ \Rightarrow \frac{d}{dr}\left((r^2)\left(\frac{1}{\rho}\frac{dp}{dr}\right)\right) &= -4\pi r^2 G\rho \\ \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left((r^2)\left(\frac{1}{\rho}\frac{dp}{dr}\right)\right) &= -4\pi G\rho \end{aligned} \tag{4}$$

Now suppose that we write $\rho = \rho_c\theta^n$, which agrees with the hint given that $\theta = 0 \iff \rho = 0$, then our polytropic EOS in Eq.(2) becomes

$$\begin{aligned} p &= K\left(\rho_c\theta^n\right)^{1+\frac{1}{n}} \\ &= K\rho_c^{1+\frac{1}{n}}\theta^{n+1} \end{aligned} \tag{5}$$

Substituting this pressure and new density into the final expression of Eq.(4), we get

$$
\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left((r^2)\left(\frac{1}{\rho_c\theta^n}\frac{d}{dr}\left(K\rho_c^{1+\frac{1}{n}}\theta^{n+1}\right)\right)\right) &= -4\pi G\rho_c\theta^n \\
\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left((r^2)\left(\frac{1}{\rho_c\theta^n}\left(K(n+1)\rho_c^{1+\frac{1}{n}}\theta^n\frac{d\theta}{dr}\right)\right)\right) &= -4\pi G\rho_c\theta^n \\
\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2K(n+1)\rho_c^{\frac{1}{n}}\frac{d\theta}{dr}\right) &= -4\pi G\rho_c\theta^n
\end{aligned}
\tag{6}
$$

We can clean this expression up a bit by introducing a scaled radius ξ such that $r = \alpha\xi$. This allows us to write

$$
\begin{aligned}
\Rightarrow \frac{1}{\alpha^2\xi^2}\frac{1}{\alpha}\frac{d}{d\xi}\left(\alpha^2\xi^2K(n+1)\rho_c^{\frac{1}{n}}\frac{1}{\alpha}\frac{d\theta}{d\xi}\right) &= -4\pi G\rho_c\theta^n \\
\Rightarrow \frac{1}{\alpha^2\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) &= -\frac{4\pi G\rho_c\theta^n}{\left(K(n+1)\rho_c^{\frac{1}{n}}\right)}
\end{aligned}
\tag{7}
$$

Therefore by defining

$$
\alpha^2 = \frac{K(n+1)\rho_c^{\frac{1}{n}-1}}{4\pi G} \tag{8}
$$

we can write

$$
\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{9}
$$

which is the Lane-Emden Equation. This equation can be solved numerically. Given the initial conditions that $\theta(0) = 1$ and $\theta'(0) = 0$, we can expand the solutions at the center in a series in Mathematica[1] for any n value. Thus

```
(* Substitute the solved coefficients back into the series to get the solution *)
(series/.solution)[[1]]
```

$1 - \frac{\xi^2}{6} + \frac{n\,\xi^4}{120} + \frac{(5n - 8n^2)\,\xi^6}{15120} + O[\xi]^7$

$$
\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 + \frac{5n - 8n^2}{1520} + \ldots \tag{10}
$$

If we choose to look at the case $n = 1$, however, then we can solve actually the Lane-Emden equation analytically using Mathematica's DSolve[]. We get

```
(* Analytic Solution of the Equation setting n = 1 and imposing the initial conditions. *)
(* Quiet[] is used to supress the warning messages of dealing with the singluarity at ξ = 0. *)
Quiet[LaneEmden1 = DSolve[{1/ξ^2 ∂ξ (ξ^2∂ξ θ[ξ])+θ[ξ]==0, θ[0]== 1, θ'[0]== 0}, θ[ξ],ξ]]
Simplify[ExpToTrig/@LaneEmden1]
```

$\left\{\left\{\theta[\xi] \to -\frac{i\, e^{-i\xi}\left(-1 + e^{2i\xi}\right)}{2\xi}\right\}\right\}$

$\left\{\left\{\theta[\xi] \to \frac{\mathrm{Sin}[\xi]}{\xi}\right\}\right\}$

$$
\theta(\xi)_{n=1} = \frac{\sin\xi}{\xi} \tag{11}
$$

We can immediately see that for this $n = 1$ case that ξ_1, the point such that $\theta(\xi_1)_{n=1} = 0$ is equal to π.

Now let's try to find an expression for the total mass M of the WD. We have a relation for the mass in Equation (1). If we can integrate out to R, the full radius of the WD, then we get M as a result. Observe

[1]The codes for the Mathematica parts of this project are fairly simple and short, therefore I will just be presenting their results without giving a detailed explanation of how they work.

that

$$
\begin{aligned}
dm = 4\pi r^2 \rho(r) dr \Longrightarrow m &= \int_0^r 4\pi r'^2 \rho(r') dr' \\
\Rightarrow M &= \int_0^R 4\pi r'^2 \rho(r') dr'
\end{aligned}
\tag{12}
$$

move to Lane Emden variables $(r' \to \alpha\xi')$, $(dr' \to \alpha d\xi')$, $(\rho(r') \to \rho_c \theta^n)$, and $(R \to \alpha\xi_n)$

$$
\begin{aligned}
\Rightarrow M &= 4\pi \rho_c \alpha^3 \int_0^{\alpha\xi_n} \xi'^2 \theta^n(\xi') d\xi' \\
&= 4\pi \rho_c \alpha^3 \int_0^{\xi_n} \xi'^2 \theta^n(\xi') d\xi' + \cancelto{0}{4\pi \rho_c \alpha^3 \int_{\xi_n}^{\alpha\xi_n} \xi'^2 \theta(\xi') d\xi'}
\end{aligned}
\tag{13}
$$

The second integral is 0 because $\theta(\xi) = 0$ when $\xi > \xi_n$. We can now recall Equation (9) and substitute $\xi^2\theta^n$ with the other term to get

$$
\begin{aligned}
M &= 4\pi \rho_c \alpha^3 \int_0^{\xi_n} \frac{d}{d\xi'} \left(-\xi'^2 \frac{d\theta}{d\xi'} \right) d\xi' \\
&= 4\pi \rho_c \alpha^3 \left(-\xi'^2 \theta'(\xi') \right) \Big|_{\xi_n} \\
&= -4\pi \rho_c \alpha^3 \xi_n^2 \theta'(\xi_n) \\
M &= 4\pi \rho_c R^3 \left(\frac{-\theta'(\xi_n)}{\xi_n} \right)
\end{aligned}
\tag{14}
$$

Finally, let's find a relation between the masses and radii of a group od stars that all share the same polytropic EOS. This actually just amounts to rewriting ρ_c in Equation (14). For convenience, let us define $\alpha = \beta \rho_c^{\frac{1-n}{2n}}$. Then the relation between R and ξ_n can be rewritten as

$$
\begin{aligned}
R &= \beta \rho_c^{\frac{1-n}{2n}} \xi_n \\
\Rightarrow \rho_c &= \left(\frac{R}{\beta \xi_n} \right)^{\frac{2n}{1-n}}
\end{aligned}
\tag{15}
$$

Substituting this into Equation (14)

$$
\begin{aligned}
M &= 4\pi \left(\frac{R}{\beta \xi_n} \right)^{\frac{2n}{1-n}} R^3 \left(\frac{-\theta'(\xi_n)}{\xi_n} \right) \\
M &= R^{\frac{3-n}{1-n}} \left(-4\pi \left(\frac{4\pi G}{(n+1)K} \right)^{\frac{n}{1-n}} \xi_n^{\frac{n+1}{n-1}} \theta'(\xi_n) \right)
\end{aligned}
\tag{16}
$$

where the bracketed term is the constant of proportionality we were asked for.

Part B

We now wish to read the data in the WD .csv file to extract the masses and radii of several WDs. The masses are already provided to us in units of solar masses ($M_\odot$), however we are only additionally given the base-10 logarithm of the surface gravity ($\log(g)$) in CGS units. We can turn these values into radii, specifically to multiples of the earth's radius ($R_\oplus$), using Newtonian gravity. Perhaps the most coherent way to do this is to convert all values to SI units, acquire the radii, then scale the radii to multiples of $R_\oplus$. For a given value $\log(g)$, we perform this procedure

Take the 10th power to get rid of the logarithm

$$\log(g) \rightarrow 10^{\log(g)} \rightarrow g$$

Convert to SI units

$$g \quad \left(\frac{cm}{s^s}\right) \quad [CGS] \longrightarrow \frac{1}{100} g \quad \left(\frac{m}{s^s}\right) \quad [SI]$$

Solve for r using Newtonian Gravity

$$g = \frac{GM}{r^2} \longrightarrow r = \sqrt{\frac{G(M \times M_\odot)}{g}}$$

Write in units of $R_\oplus$

$$r \rightarrow \frac{r}{R_\oplus}$$