Stars, From
Newton to Einstein

January 13^{th}, 2025

Abstract

In this project, I calculate the structures of two types of stars, namely White Dwarfs (WDs) and Neutron Stars (NSs). The former will be studied using Newtonian gravity and the latter with General Relativity (GR).

Newton

Part A

We begin with the two equations of stellar structure that follow from the hydrostatic equilibrium of stars in Newtonian gravity

$$\begin{aligned} \frac{dm(r)}{dr} &= 4\pi r^2 \rho(r), \\ \frac{dp(r)}{dr} &= -\frac{Gm(r)\rho(r)}{r^2}. \end{aligned} \tag{1}$$

where r is the radius of the star, $p(r)$ is the pressure, $m(r)$ is the mass contained within r and $\rho(r)$ is the density. We need an expression for the density, and we are told that an adequate model for WDs is that of a polytropic EOS

$$p = K\rho^{1+\frac{1}{n}}, \tag{2}$$

where n is called the polytropic index. Of course, p and ρ are still functions of r, but we'll suppress the dependence in what follows for visual clarity. This set of equations can be transformed into a more convenient form. Following the discussion on the Lane-Emden Equation Wikipedia [1], we can perform a series of manipulations on Eq.(1), namely:

$$\begin{aligned} \frac{1}{\rho}\frac{dp}{dr} &= -\frac{Gm}{r^2} \\ \Rightarrow \frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) &= \frac{d}{dr}\left(-\frac{Gm}{r^2}\right) \\ &= \frac{2Gm}{r^3} - \frac{G}{r^2}\frac{dm}{dr} \\ &= \left(\frac{-2}{\rho r}\right)\left(\frac{-Gm\rho}{r^2}\right) - \frac{G}{r^2}\left(\frac{dm}{dr}\right) \\ &= \frac{-2}{\rho r}\left(\frac{dp}{dr}\right) - 4\pi G\rho \end{aligned} \tag{3}$$

Multiply both sides by r^2 and collect the pressure derivative terms on one side

$$\begin{aligned} r^2\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) + \frac{-2r}{\rho}\frac{dp}{dr} &= -4\pi r^2 G\rho \\ \Rightarrow \frac{d}{dr}\left((r^2)\left(\frac{1}{\rho}\frac{dp}{dr}\right)\right) &= -4\pi r^2 G\rho \\ \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left((r^2)\left(\frac{1}{\rho}\frac{dp}{dr}\right)\right) &= -4\pi G\rho \end{aligned} \tag{4}$$

Now suppose that we write $\rho = \rho_c\theta^n$, which agrees with the hint given that $\theta = 0 \iff \rho = 0$, then our polytropic EOS in Eq.(2) becomes

$$\begin{aligned} p &= K\left(\rho_c\theta^n\right)^{1+\frac{1}{n}} \\ &= K\rho_c^{1+\frac{1}{n}}\theta^{n+1} \end{aligned} \tag{5}$$

Substituting this pressure and new density into the final expression of Eq.(4), we get

$$
\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left((r^2)\left(\frac{1}{\rho_c\theta^n}\frac{d}{dr}\left(K\rho_c^{1+\frac{1}{n}}\theta^{n+1}\right)\right)\right) &= -4\pi G\rho_c\theta^n \\
\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left((r^2)\left(\frac{1}{\rho_c\theta^n}\left(K(n+1)\rho_c^{1+\frac{1}{n}}\theta^n\frac{d\theta}{dr}\right)\right)\right) &= -4\pi G\rho_c\theta^n \\
\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2K(n+1)\rho_c^{\frac{1}{n}}\frac{d\theta}{dr}\right) &= -4\pi G\rho_c\theta^n
\end{aligned}
\tag{6}
$$

We can clean this expression up a bit by introducing a scaled radius ξ such that $r = \alpha\xi$. This allows us to write

$$
\begin{aligned}
\Rightarrow \frac{1}{\alpha^2\xi^2}\frac{1}{\alpha}\frac{d}{d\xi}\left(\alpha^2\xi^2K(n+1)\rho_c^{\frac{1}{n}}\frac{1}{\alpha}\frac{d\theta}{d\xi}\right) &= -4\pi G\rho_c\theta^n \\
\Rightarrow \frac{1}{\alpha^2\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) &= -\frac{4\pi G\rho_c\theta^n}{\left(K(n+1)\rho_c^{\frac{1}{n}}\right)}
\end{aligned}
\tag{7}
$$

Therefore by defining

$$
\alpha^2 = \frac{K(n+1)\rho_c^{\frac{1}{n}-1}}{4\pi G} \tag{8}
$$

we can write

$$
\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{9}
$$

which is the Lane-Emden Equation. This equation can be solved numerically. Given the initial conditions that $\theta(0) = 1$ and $\theta'(0) = 0$, we can expand the solutions at the center in a series in Mathematica[1] for any n value. Thus

```
(* Substitute the solved coefficients back into the series to get the solution *)
(series/.solution)[[1]]
```

$1 - \frac{\xi^2}{6} + \frac{n\,\xi^4}{120} + \frac{(5\,n - 8\,n^2)\,\xi^6}{15120} + O[\xi]^7$

$$
\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 + \frac{5n - 8n^2}{1520} + \ldots \tag{10}
$$

If we choose to look at the case $n = 1$, however, then we can solve actually the Lane-Emden equation analytically using Mathematica's DSolve[]. We get

```
(* Analytic Solution of the Equation setting n = 1 and imposing the initial conditions. *)
(* Quiet[] is used to supress the warning messages of dealing with the singluarity at ξ = 0. *)
Quiet[LaneEmden1 = DSolve[{1/ξ² ∂ξ (ξ²∂ξ θ[ξ])+θ[ξ]==0, θ[0]== 1, θ'[0]== 0}, θ[ξ],ξ]]
Simplify[ExpToTrig/@LaneEmden1]
```

$\left\{\left\{\theta[\xi] \to -\frac{i\,e^{-i\,\xi}\left(-1 + e^{2\,i\,\xi}\right)}{2\,\xi}\right\}\right\}$

$\left\{\left\{\theta[\xi] \to \frac{\text{Sin}[\xi]}{\xi}\right\}\right\}$

$$
\theta(\xi)_{n=1} = \frac{\sin\xi}{\xi} \tag{11}
$$

We can immediately see that for this $n = 1$ case that ξ_1, the point such that $\theta(\xi_1)_{n=1} = 0$ is equal to π.

Now let's try to find an expression for the total mass M of the WD. We have a relation for the mass in Equation (1). If we can integrate out to R, the full radius of the WD, then we get M as a result. Observe

[1]The codes for the Mathematica parts of this project are fairly simple and short, therefore I will just be presenting their results without giving a detailed explanation of how they work.

that

$$
\begin{aligned}
dm = 4\pi r^2 \rho(r) dr \Longrightarrow m &= \int_0^r 4\pi r'^2 \rho(r') dr' \\
\Rightarrow M &= \int_0^R 4\pi r'^2 \rho(r') dr'
\end{aligned}
\tag{12}
$$

move to Lane Emden variables $(r' \to \alpha\xi')$, $(dr' \to \alpha d\xi')$, $(\rho(r') \to \rho_c \theta^n)$, and $(R \to \alpha\xi_n)$

$$
\begin{aligned}
\Rightarrow M &= 4\pi \rho_c \alpha^3 \int_0^{\alpha\xi_n} \xi'^2 \theta^n(\xi') d\xi' \\
&= 4\pi \rho_c \alpha^3 \int_0^{\xi_n} \xi'^2 \theta^n(\xi') d\xi' + \cancelto{0}{4\pi \rho_c \alpha^3 \int_{\xi_n}^{\alpha\xi_n} \xi'^2 \theta(\xi') d\xi'}
\end{aligned}
\tag{13}
$$

The second integral is 0 because $\theta(\xi) = 0$ when $\xi > \xi_n$. We can now recall Equation (9) and substitute $\xi^2\theta^n$ with the other term to get

$$
\begin{aligned}
M &= 4\pi \rho_c \alpha^3 \int_0^{\xi_n} \frac{d}{d\xi'} \left(-\xi'^2 \frac{d\theta}{d\xi'} \right) d\xi' \\
&= 4\pi \rho_c \alpha^3 \left(-\xi'^2 \theta'(\xi') \right) \Big|_{\xi_n} \\
&= -4\pi \rho_c \alpha^3 \xi_n^2 \theta'(\xi_n) \\
M &= 4\pi \rho_c R^3 \left(\frac{-\theta'(\xi_n)}{\xi_n} \right)
\end{aligned}
\tag{14}
$$

Finally, let's find a relation between the masses and radii of a group od stars that all share the same polytropic EOS. This actually just amounts to rewriting ρ_c in Equation (14). For convenience, let us define $\alpha = \beta \rho_c^{\frac{1-n}{2n}}$. Then the relation between R and ξ_n can be rewritten as

$$
\begin{aligned}
R &= \beta \rho_c^{\frac{1-n}{2n}} \xi_n \\
\Rightarrow \rho_c &= \left(\frac{R}{\beta \xi_n} \right)^{\frac{2n}{1-n}}
\end{aligned}
\tag{15}
$$

Substituting this into Equation (14)

$$
\begin{aligned}
M &= 4\pi \left(\frac{R}{\beta \xi_n} \right)^{\frac{2n}{1-n}} R^3 \left(\frac{-\theta'(\xi_n)}{\xi_n} \right) \\
M &= R^{\frac{3-n}{1-n}} \left(-4\pi \left(\frac{4\pi G}{(n+1)K} \right)^{\frac{n}{1-n}} \xi_n^{\frac{n+1}{n-1}} \theta'(\xi_n) \right)
\end{aligned}
\tag{16}
$$

where the bracketed term is the constant of proportionality we were asked for.

Part B

We now wish to read the data in the WD .csv file to extract the masses and radii of several WDs. The masses are already provided to us in units of solar masses ($M_\odot$), however we are only additionally given the base-10 logarithm of the surface gravity ($\log(g)$) in CGS units. We can turn these values into radii, specifically to multiples of the earth's radius ($R_\oplus$), using Newtonian gravity. Perhaps the most coherent way to do this is to convert all values to SI units, acquire the radii, then scale the radii to multiples of $R_\oplus$. For a given value $\log(g)$, we perform this procedure

Take the 10th power to get rid of the logarithm

$$\log(g) \rightarrow 10^{\log(g)} \rightarrow g$$

Convert to SI units

$$g \quad \left(\frac{cm}{s^s}\right) \quad [CGS] \longrightarrow \frac{1}{100}g \quad \left(\frac{m}{s^s}\right) \quad [SI]$$

Solve for r using Newtonian Gravity

$$g = \frac{GM}{r^2} \longrightarrow r = \sqrt{\frac{G(M \times M_\odot)}{g}}$$

Write in units of $R_\oplus$

$$r \rightarrow \frac{r}{R_\oplus}$$

In this way, we have M in units of $M_\odot$ and r or R in units of $R_\oplus$. We will refer to this unit system as $MSRE$, and this will be relevant throughout the remainder of the Newton section. In terms of code, we write a function called `read_WD_data(filename)` that reads the data and stores it appropriately. Then, we perform the procedure outlined above and plot the $M(R)$ relation.

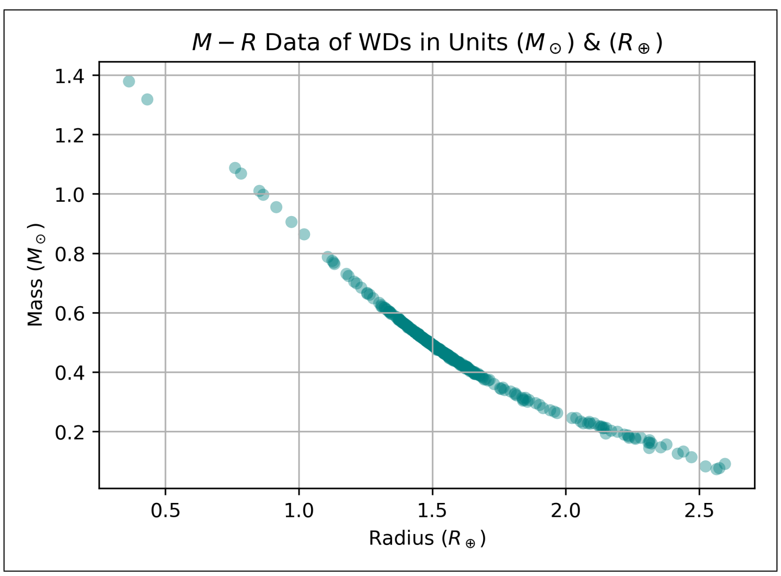

Figure 1: $M(R)$ Datapoints

The datapoints in Figure 1 have been made to be slightly transparent. Thus we can see that there are a lot of WDs occupying the center region of the plot. Consequently, we can be convinced that our unit conversions were correct since we know that WDs' masses and radii are somewhat close to unity in $MSRE$ units.

Part C

We are told that cold WDs have a more general EOS due to the pressure being dominated by electron degeneracy.

$$p = C\left[x(2x^2-3)(x^2+1)^{1/2} + 3\mathrm{arcsinh}(x)\right], \quad x = \left(\frac{\rho}{D}\right)^{\frac{1}{q}} \tag{17}$$

However, if we limit ourselves to small masses, then $x \ll 1$. In this regime, we can actually expand Equation (17) and see that the leading term will be the polytropic EOS. Making the expansion in Mathematica

```
(* Expansion of P(x) to 7th order in x. *)
SeriesInX = Series[P[x], {x, 0, degree}]
```

$$\frac{8\,C\,x^5}{5} - \frac{4\,C\,x^7}{7} + O[x]^8$$

```
(* Replacing x with (ρ/D)^(1/q) *)
SeriesInRho = Normal[SeriesInX] /. x → D^(-1/q) ρ^(1/q)
```

$$\frac{8}{5}\,C\,D^{-5/q}\,\rho^{5/q} - \frac{4}{7}\,C\,D^{-7/q}\,\rho^{7/q}$$

$$p = K_*\rho^{1+\frac{1}{n_*}}, \quad K_* = \frac{8C}{5D^{\frac{5}{q}}}, \quad n_* = \frac{q}{5-q} \tag{18}$$

Now before we move on to making any fits, we need to determine what mass is low enough. The polytropic EOS allows us to fit on Equation (16). So M and R are related by a power law for low masses. If we were to then plot these quantities on a *loglog* graph, any power law regions will correspond to a straight line.

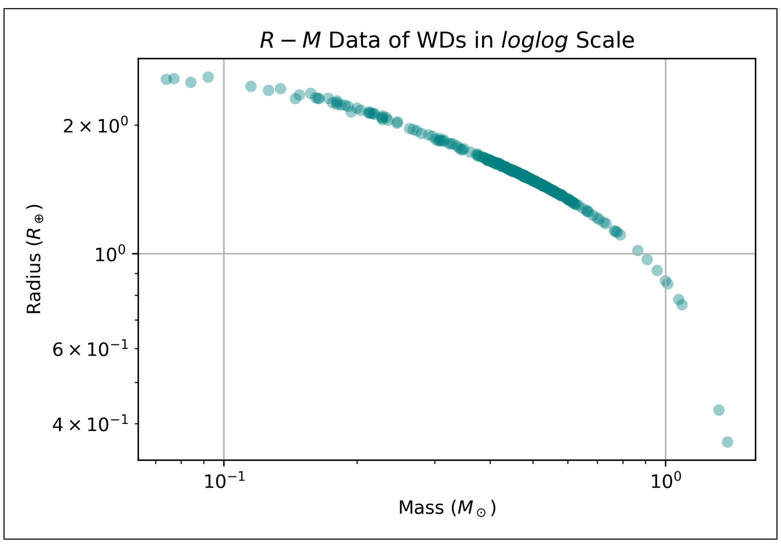

Figure 2: $R(M)$ Datapoints in *loglog* scale

As seen in Figure 2 above, we can see linear behavior start from the left side of the graph (low-mass) up till $0.3-0.4M_\odot$. Beyond that, the points begin to curve downward, diverging from the power law dependence. It would thus seem that $0.35M_\odot$ would be a good cutoff value, and it is the one we pick. We could perform a nonlinear fitting on Equation (16) to find both n_* and K_* directly, or we could more efficiently find one first then the other. Notice that we can take the logarithm of both sides of Equation (16) and up with a simple linear fit in q

$$\begin{aligned} \ln M &= \ln A + \left(1 + \frac{1}{n_*}\right)\ln R \\ &= \ln A + \left(\frac{15-4q}{5-2q}\right)\ln R \end{aligned} \tag{19}$$

We don't very much care what A turns out to be, only that we have an expression with q. We are told that q must be an integer from the theory. So we'll expect a value close to an integer value, and round it up (or down) to the nearest one. Another thing to note is that we see that M loses its R dependence at $n = 1$ and $n = 3$. This suggests that $n_* \in (1, 3) \Longrightarrow q \in (2.5, 3.75)$ Performing the fit using SciPy's `curve_fit()` and imposing the boundaries on q results in

$$q = 2.969506741176875 \longrightarrow \quad q = 3 \Rightarrow n_* = 1.5 \tag{20}$$

which is indeed what the hint informed us would be the answer.

Having n_* now allows us to solve the Lane-Emden Equation! We do this numerically by transforming the Lane-Emden equation from a single second order ODE into two first order ODEs. Note that

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \Longrightarrow \frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} + \theta^n = 0 \tag{21}$$

Define V_1 and V_2 such that

$$\begin{gathered} V_1 = \theta(\xi), \quad V_2 = \frac{d\theta}{d\xi} \\ \Longrightarrow V_1' = V_2, \quad V_2' = -(V_1^n + \frac{2}{\xi}V_2) \end{gathered} \tag{22}$$

Consequently, we can impose the initial conditions as $(\theta(0) = 1, \theta'(0) = 0) \Longleftrightarrow (V_1(0) = 1, V_2(0) = 0)$ and solve this system of ODEs using SciPy's `solve_ivp()`. We are interested in the point ξ_n where $\theta(\xi_n) = 0$, so we record that point and its index to also get $\theta'(\xi_n)$. Having done that, we plot the solution.

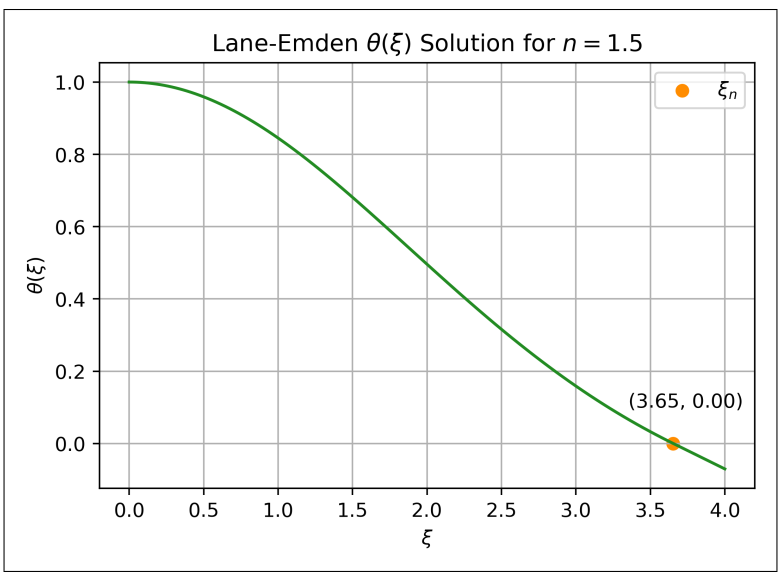

Figure 3: Numerical solution to the Lane-Emden equation for $n = 1.5$

The values of interest we get from this integration roughly are $\xi_n = 3.65$ and $\theta'(\xi_n) = -0.204$. If we look back at Equation (16), we can see that the only unknown constant now is K_*. We can thus perform a fit in K_* (which will now be linear) to find its value. This is again done using `curve_fit()`, taking caution that G should be expressed in $MSRE$ units rather than SI. Doing this yields

$$K_* = 0.272 \quad MSRE \longleftrightarrow K_* = 2.846418 \times 10^7 \quad \text{SI} \tag{23}$$

Before moving on, it may be first worth checking if our values are any good. There are two ways to check this. First, having all the constants, let's observe what the continuous $M(R)$ curve of Equation (16) looks like.

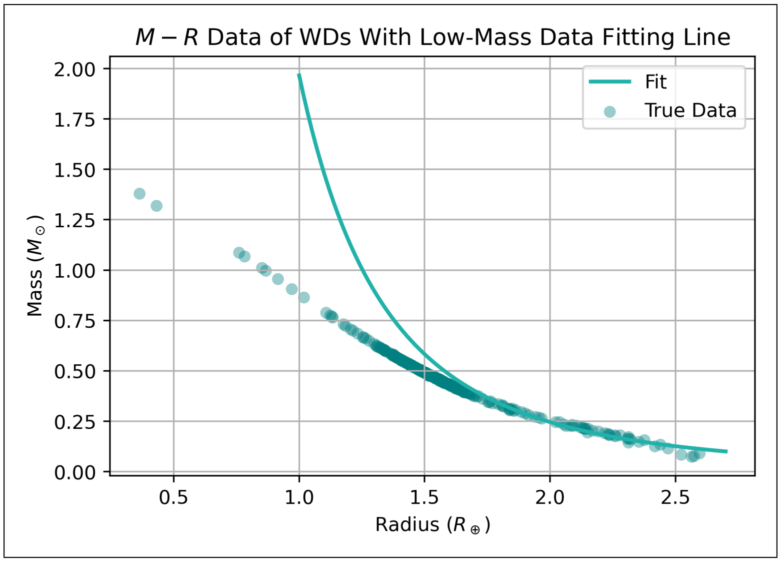

Figure 4: $M(R)$ Fit Curve for Low Masses

So the fit curve hugs the datapoints fairly well for small masses. That's reassuring. But let's also compare our values with those of much more capable people. Primarily, we consult O.R. Pols' book *Stellar Structure and Evolution (2011)*[2] (Pols). Our values can be summarized in a plot.

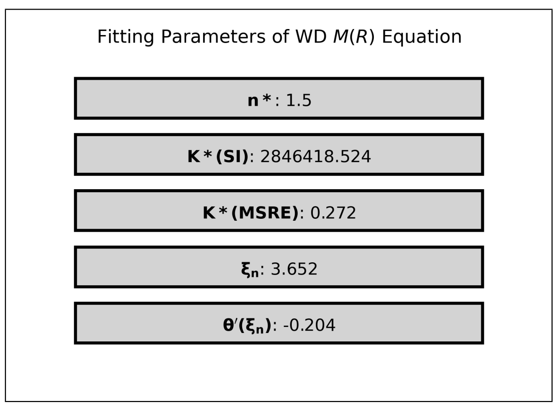

Figure 5: Computed Constants

Consulting Equation (3.35) and Table (4.1)[3] of Pols and comparing values:

Constant	Measured Value	Accepted Value
n_*	1.5	1.5
ξ_n	3.652	3.654
$\theta'(\xi_n)$	-0.204	-0.203
K_* SI	2846418	3161142

Table 1: Comparison of Measured and Accepted Values of WD Structure Constants

[2]This book was the first result when I searched "Stellar Structure and Evolution" on Google. I had assumed the book must have been (KWW), but it was only until fairly late in the project that I realized it was written by Pols. It seems astrophysicists have some obsession with reusing the same title for their books.

[3]It is worth mentioning that (Pols) does not provide $\theta'(\xi_n)$ but instead $\Theta_n = -\xi_n^2\theta'(\xi_n)$. Furthermore, his calculation for K_* contains $\mu_e^{5/3}$ directly where $\mu_e = 2$ is the number of nucleons per electron. So we multiply his given value of K_* by $2^{-5/3}$.

So we more or less see pretty good agreement between our values and the generally accepted ones.

We finally end this part by plotting ρ_c vs M. Computing ρ_c can be easily done by solving for it in Equation (14)

$$\rho_c = \frac{M}{4\pi R^3}\left(\frac{-\xi_n}{\theta'(\xi_n)}\right) \tag{24}$$

The plot will be presented (as usual) in $MSRE$ units.

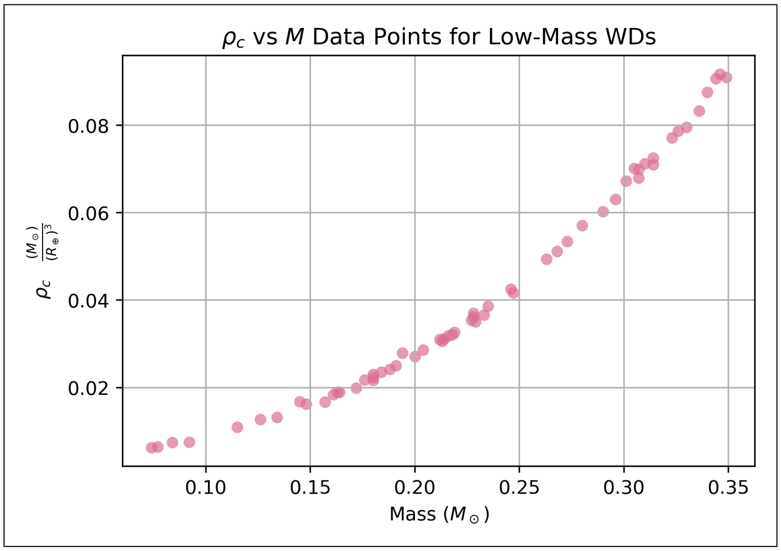

Figure 6: Computed Constants

Part D

References

[1] *BLAH BLAH BLAH*