Stars, From Newton to Einstein

PHYS414 Computational Physics: Final Project

Abdullah Sihe

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"So far as hypotheses are concerned, let no one expect anything certain from astronomy, which cannot furnish it, lest he accept as the truth ideas conceived for another purpose, and depart from this study a greater fool than when he entered it." - Nicolaus Copernicus

Abstract

In this project, I calculate the structures of two types of stars, namely White Dwarfs (WDs) and Neutron Stars (NSs). The former will be studied using Newtonian gravity and the latter with General Relativity (GR).

Newton

Part A

We begin with the two equations of stellar structure that follow from the hydrostatic equilibrium of stars in Newtonian gravity.

$$\begin{aligned} \frac{dm(r)}{dr} &= 4\pi r^2 \rho(r) \\ \frac{dp(r)}{dr} &= -\frac{Gm(r)\rho(r)}{r^2}, \end{aligned} \tag{1}$$

where r is the radius of the star, $p(r)$ is the pressure, $m(r)$ is the mass contained within r, and $\rho(r)$ is the density. We need an expression for the density, and we are told that an adequate model for WDs is that of a polytropic EOS.

$$p = K\rho^{1+\frac{1}{n}}, \tag{2}$$

where n is called the polytropic index. Of course, p and ρ are still functions of r, but we'll suppress the dependence in what follows for visual clarity. This set of equations can be transformed into a more convenient form. Following the discussion on the Lane-Emden Equation Wikipedia [1], we can perform a series of manipulations on Equation (1), namely:

$$\begin{aligned} \frac{1}{\rho}\frac{dp}{dr} &= -\frac{Gm}{r^2} \\ \Rightarrow \frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) &= \frac{d}{dr}\left(-\frac{Gm}{r^2}\right) \\ &= \frac{2Gm}{r^3} - \frac{G}{r^2}\frac{dm}{dr} \\ &= \left(\frac{-2}{\rho r}\right)\left(\frac{-Gm\rho}{r^2}\right) - \frac{G}{r^2}\left(\frac{dm}{dr}\right) \\ &= \frac{-2}{\rho r}\left(\frac{dp}{dr}\right) - 4\pi G\rho \end{aligned} \tag{3}$$

Multiply both sides by r^2 and collect the pressure derivative terms on one side.

$$\begin{aligned} r^2\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) + \frac{-2r}{\rho}\frac{dp}{dr} &= -4\pi r^2 G\rho \\ \Rightarrow \frac{d}{dr}\left((r^2)\left(\frac{1}{\rho}\frac{dp}{dr}\right)\right) &= -4\pi r^2 G\rho \\ \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left((r^2)\left(\frac{1}{\rho}\frac{dp}{dr}\right)\right) &= -4\pi G\rho \end{aligned} \tag{4}$$

Now suppose that we write $\rho = \rho_c\theta^n$, which agrees with the hint given that $\theta = 0 \iff \rho = 0$, then our polytropic EOS in Equation (2) becomes

$$\begin{aligned} p &= K\left(\rho_c\theta^n\right)^{1+\frac{1}{n}} \\ &= K\rho_c^{1+\frac{1}{n}}\theta^{n+1}. \end{aligned} \tag{5}$$

Substituting this pressure and new density into the final expression of Equation (4), we get

$$
\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left((r^2)\left(\frac{1}{\rho_c\theta^n}\frac{d}{dr}\left(K\rho_c^{1+\frac{1}{n}}\theta^{n+1}\right)\right)\right) &= -4\pi G\rho_c\theta^n \\
\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left((r^2)\left(\frac{1}{\rho_c\theta^n}\left(K(n+1)\rho_c^{1+\frac{1}{n}}\theta^n\frac{d\theta}{dr}\right)\right)\right) &= -4\pi G\rho_c\theta^n \\
\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2K(n+1)\rho_c^{\frac{1}{n}}\frac{d\theta}{dr}\right) &= -4\pi G\rho_c\theta^n.
\end{aligned}
\tag{6}
$$

We can clean this expression up a bit by introducing a scaled radius ξ such that $r = \alpha\xi$. This allows us to write

$$
\begin{aligned}
\Rightarrow \frac{1}{\alpha^2\xi^2}\frac{1}{\alpha}\frac{d}{d\xi}\left(\alpha^2\xi^2K(n+1)\rho_c^{\frac{1}{n}}\frac{1}{\alpha}\frac{d\theta}{d\xi}\right) &= -4\pi G\rho_c\theta^n \\
\Rightarrow \frac{1}{\alpha^2\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) &= -\frac{4\pi G\rho_c\theta^n}{\left(K(n+1)\rho_c^{\frac{1}{n}}\right)}.
\end{aligned}
\tag{7}
$$

Therefore by defining

$$
\alpha^2 = \frac{K(n+1)\rho_c^{\frac{1}{n}-1}}{4\pi G}, \tag{8}
$$

we can write

$$
\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0, \tag{9}
$$

which is the Lane-Emden Equation. This equation can be solved numerically. Given the initial conditions that $\theta(0) = 1$ and $\theta'(0) = 0$, we can expand the solutions at the center in a series in Mathematica[1] for any n value. Thus

```
(* Substitute the solved coefficients back into the series to get the solution *)
(series/.solution)[[1]]
```

```
1 - ξ^2/6 + n ξ^4/120 + (5 n - 8 n^2) ξ^6/15120 + O[ξ]^7
```

$$
\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 + \frac{5n-8n^2}{1520} + \dots \tag{10}
$$

If we choose to look at the case $n = 1$, however, then we can actually solve the Lane-Emden equation analytically using Mathematica's `DSolve[]`. We get

```
(* Analytic Solution of the Equation setting n = 1 and imposing the initial conditions. *)
(* Quiet[] is used to supress the warning messages of dealing with the singluarity at ξ = 0. *)
Quiet[LaneEmden1 = DSolve[{1/ξ^2 ∂ξ (ξ^2∂ξ θ[ξ])+θ[ξ]==0, θ[0]== 1, θ'[0]== 0}, θ[ξ],ξ]]
Simplify[ExpToTrig/@LaneEmden1]
```

```
{{θ[ξ] → -(i e^(-i ξ) (-1 + e^(2 i ξ)))/(2 ξ)}}

{{θ[ξ] → Sin[ξ]/ξ}}
```

$$
\theta(\xi)_{n=1} = \frac{\sin\xi}{\xi}. \tag{11}
$$

We can immediately see that for this $n = 1$ case that ξ_1, the point such that $\theta(\xi_1)_{n=1} = 0$, is equal to π.

Now let's try to find an expression for the total mass M of the WD. We have a relation for the mass in Equation (1). If we can integrate out to R, the full radius of the WD, then we get M as a result. Now we

[1]The codes for the Mathematica parts of this project are fairly simple and short, therefore I will just be presenting their results without giving a detailed explanation of how they work.

can observe that

$$
\begin{aligned}
dm = 4\pi r^2 \rho(r) dr \Longrightarrow m &= \int_0^r 4\pi r'^2 \rho(r') dr' \\
\Rightarrow M &= \int_0^R 4\pi r'^2 \rho(r') dr'.
\end{aligned} \tag{12}
$$

Moving to the Lane Emden variables $(r' \to \alpha\xi')$, $(dr' \to \alpha d\xi')$, $(\rho(r') \to \rho_c\theta^n)$, and $(R \to \alpha\xi_n)$

$$
\begin{aligned}
\Rightarrow M &= 4\pi\rho_c\alpha^3 \int_0^{\alpha\xi_n} \xi'^2 \theta^n(\xi') d\xi' \\
&= 4\pi\rho_c\alpha^3 \int_0^{\xi_n} \xi'^2 \theta^n(\xi') d\xi' + \cancelto{0}{4\pi\rho_c\alpha^3 \int_{\xi_n}^{\alpha\xi_n} \xi'^2 \theta(\xi') d\xi'}.
\end{aligned} \tag{13}
$$

The second integral is 0 because $\theta(\xi) = 0$ when $\xi > \xi_n$. We can now recall Equation (9) and substitute $\xi^2\theta^n$ with the other term to get

$$
\begin{aligned}
M &= 4\pi\rho_c\alpha^3 \int_0^{\xi_n} \frac{d}{d\xi'} \left(-\xi'^2 \frac{d\theta}{d\xi'} \right) d\xi' \\
&= 4\pi\rho_c\alpha^3 \left(-\xi'^2 \theta'(\xi') \right) \Big|_{\xi_n} \\
&= -4\pi\rho_c\alpha^3 \xi_n^2 \theta'(\xi_n) \\
M &= 4\pi\rho_c R^3 \left(\frac{-\theta'(\xi_n)}{\xi_n} \right).
\end{aligned} \tag{14}
$$

Finally, let's find a relation between the masses and radii of a group of stars that all share the same polytropic EOS. This actually just amounts to rewriting ρ_c in Equation (14). For convenience, let us define $\alpha = \beta\rho_c^{\frac{1-n}{2n}}$. Then the relation between R and ξ_n can be rewritten as

$$
\begin{aligned}
R &= \beta\rho_c^{\frac{1-n}{2n}} \xi_n \\
\Rightarrow \rho_c &= \left(\frac{R}{\beta\xi_n} \right)^{\frac{2n}{1-n}}.
\end{aligned} \tag{15}
$$

Substituting this into Equation (14)

$$
\begin{aligned}
M &= 4\pi \left(\frac{R}{\beta\xi_n} \right)^{\frac{2n}{1-n}} R^3 \left(\frac{-\theta'(\xi_n)}{\xi_n} \right) \\
\Rightarrow M &= R^{\frac{3-n}{1-n}} \left(-4\pi \left(\frac{4\pi G}{(n+1)K} \right)^{\frac{n}{1-n}} \xi_n^{\frac{n+1}{n-1}} \theta'(\xi_n) \right),
\end{aligned} \tag{16}
$$

where the bracketed term is the constant of proportionality we were asked for.

Part B

We now wish to read the data in the `white_dwarf_data.csv` file to extract the masses and radii of several WDs. The masses are already provided to us in units of solar masses ($M_\odot$), however we are only additionally given the base-10 logarithm of the surface gravity ($\log(g)$) in CGS units. We can turn these values into radii, specifically to multiples of the Earth's radius ($R_\oplus$), using Newtonian gravity. Perhaps the most coherent way to do this is to convert all values to SI units, acquire the radii, then scale the radii to multiples of $R_\oplus$. For a given value $\log(g)$, we perform this procedure

$$\text{Take the } 10^{th} \text{ power, remove the logarithm} \qquad \log(g) \rightarrow 10^{\log(g)} \rightarrow g$$

$$\text{Convert to SI units} \qquad g \quad \left(\frac{cm}{s^s}\right) \quad [CGS] \longrightarrow \frac{1}{100} g \quad \left(\frac{m}{s^s}\right) \quad [SI]$$

$$\text{Solve for } r \text{ using Newtonian Gravity} \qquad g = \frac{GM}{r^2} \longrightarrow r = \sqrt{\frac{G(M \times M_\odot)}{g}}$$

$$\text{Write in units of } R_\oplus \qquad r \rightarrow \frac{r}{R_\oplus}$$

In this way, we have M in units of $M_\odot$ and r or R in units of $R_\oplus$. We will refer to this unit system as $MSRE$, and this will be relevant throughout the remainder of the Newton section. In terms of code, we write a function called `read_WD_data(filename)` that reads the data and stores it appropriately. Then, we perform the procedure outlined above and plot the $M(R)$ relation.

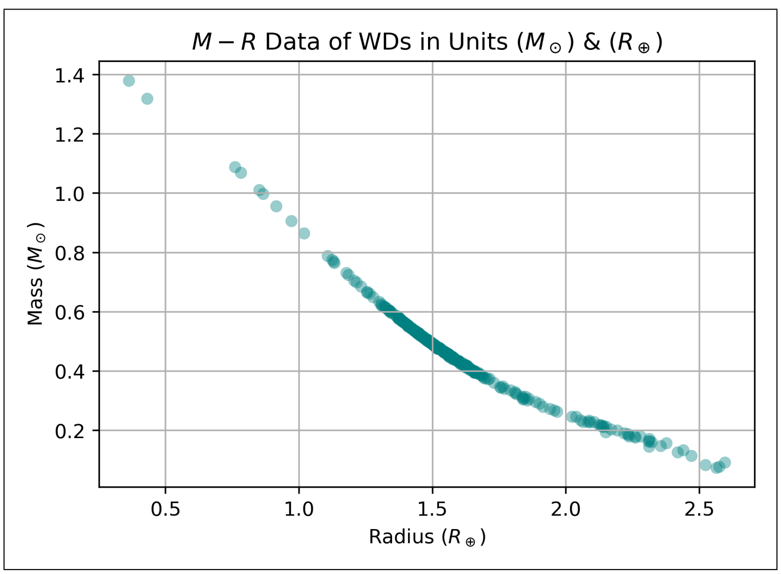


Figure 1: $M(R)$ Datapoints

The datapoints in Figure 1 have been made to be slightly transparent. Thus we can see that there are a lot of WDs occupying the center region of the plot. Consequently, we can be convinced that our unit conversions were correct since we know that WDs' masses and radii are somewhat close to unity in $MSRE$ units.

Part C

We are told that cold WDs have a more general EOS due to the pressure being dominated by electron degeneracy.

$$p = C\left[x(2x^2-3)(x^2+1)^{1/2} + 3\text{arcsinh}(x)\right], \quad x = \left(\frac{\rho}{D}\right)^{\frac{1}{q}} \tag{17}$$

However, if we limit ourselves to small masses, then $x \ll 1$. In this regime, we can actually expand Equation (17) and see that the leading term will be the polytropic EOS. Making the expansion in Mathematica

```
(* Expansion of P(x) to 7th order in x. *)
SeriesInX = Series[P[x], {x, 0, degree}]
```

$$\frac{8\,C\,x^5}{5} - \frac{4\,C\,x^7}{7} + O[x]^8$$

```
(* Replacing x with (ρ/D)^(1/q) *)
SeriesInRho = Normal[SeriesInX] /. x → D^(-1/q) ρ^(1/q)
```

$$\frac{8}{5}\,C\,D^{-5/q}\rho^{5/q} - \frac{4}{7}\,C\,D^{-7/q}\rho^{7/q}$$

$$p = K_*\rho^{1+\frac{1}{n_*}}, \quad K_* = \frac{8C}{5D^{\frac{5}{q}}}, \quad n_* = \frac{q}{5-q}. \tag{18}$$

Now before we move on to making any fits, we need to determine what mass is low enough. The polytropic EOS allows us to fit on Equation (16). So M and R are related by a power law for low masses. If we were to then plot these quantities on a *loglog* graph, any power law relations will correspond to a straight line.

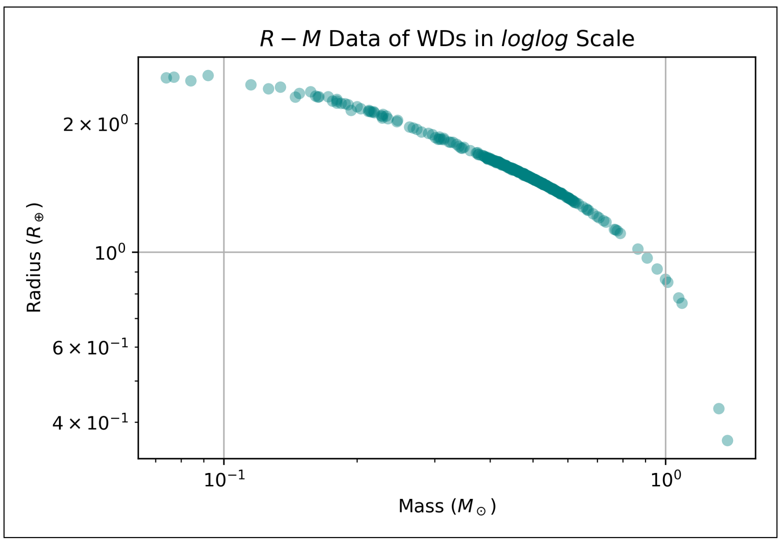


Figure 2: $R(M)$ Datapoints in *loglog* scale

As seen in Figure 2 above, linear behavior is visible starting from the left side of the graph (low-mass) up till $0.3 - 0.4M_\odot$. Beyond that, the points begin to curve downward, diverging from the power law dependence. It would thus seem that $0.35M_\odot$ would be a good cutoff value, and it is what we select. We could perform a nonlinear fitting on Equation (16) to find both n_* and K_* directly, or we could more efficiently find one first then the other. Notice that we can take the logarithm of both sides of Equation (16) and up with a simple linear fit in q.

$$\begin{aligned} \ln M &= \ln A + \left(1 + \frac{1}{n_*}\right)\ln R \\ &= \ln A + \left(\frac{15-4q}{5-2q}\right)\ln R \end{aligned} \tag{19}$$

We don't very much care what A turns out to be, only that we have an expression linear in q. We are told that q must be an integer from the theory. So we'll expect a value close to an integer value, and round it up (or down) to the nearest one. Another thing to note is that we see that M loses its R dependence at $n = 1$ and $n = 3$. This suggests that $n_* \in (1, \quad 3) \implies q \in (2.5, \quad 3.75)$ Performing the fit using SciPy's `curve_fit()` and imposing the boundaries on q results in

$$q = 2.969506741176875 \longrightarrow \quad q = 3 \quad \Rightarrow \quad n_* = 1.5, \tag{20}$$

which is indeed what the hint informed us would be the answer.

Having n_* now allows us to solve the Lane-Emden Equation! We do this numerically by transforming the Lane-Emden equation from a single second order ODE into two first order ODEs. Note that

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \implies \frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} + \theta^n = 0. \tag{21}$$

Define V_1 and V_2 such that

$$\begin{aligned} V_1 = \theta(\xi), \quad V_2 = \frac{d\theta}{d\xi} \\ \implies V_1' = V_2, \quad V_2' = -(V_1^n + \frac{2}{\xi}V_2). \end{aligned} \tag{22}$$

Consequently, we can impose the initial conditions as $(\theta(0) = 1, \quad \theta'(0) = 0) \iff (V_1(0) = 1, \quad V_2(0) = 0)$ and solve this system of ODEs using SciPy's `solve_ivp()`. We are interested in the point ξ_n where $\theta(\xi_n) = 0$, so we record that point and its index to also get $\theta'(\xi_n)$. Having done that, we plot the solution.

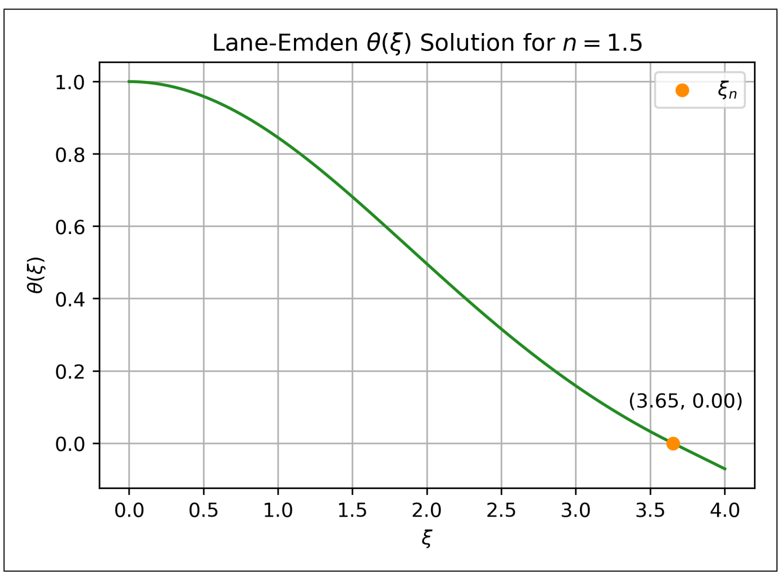


Figure 3: Numerical solution to the Lane-Emden equation for $n = 1.5$

The values of interest we get from this integration are $\xi_n = 3.65$ and $\theta'(\xi_n) = -0.204$. If we look back at Equation (16), we can see that the only unknown constant now is K_*. We can thus perform a fit in K_* (which will now be linear in K_*) to find its value. This is again done using `curve_fit()`, taking caution that G should be expressed in $MSRE$ units rather than SI. Doing this yields

$$K_* = 0.272 \quad MSRE \longleftrightarrow K_* = 2.846418 \times 10^7 \quad \text{SI}. \tag{23}$$

Before moving on, it may be first worth checking if our values are any good. There are two ways to check this. First, having all the constants, let's observe what the continuous $M(R)$ curve of Equation (16) looks like. This isn't interpolation. We are evaluating Equation (16) for many points r and plotting.

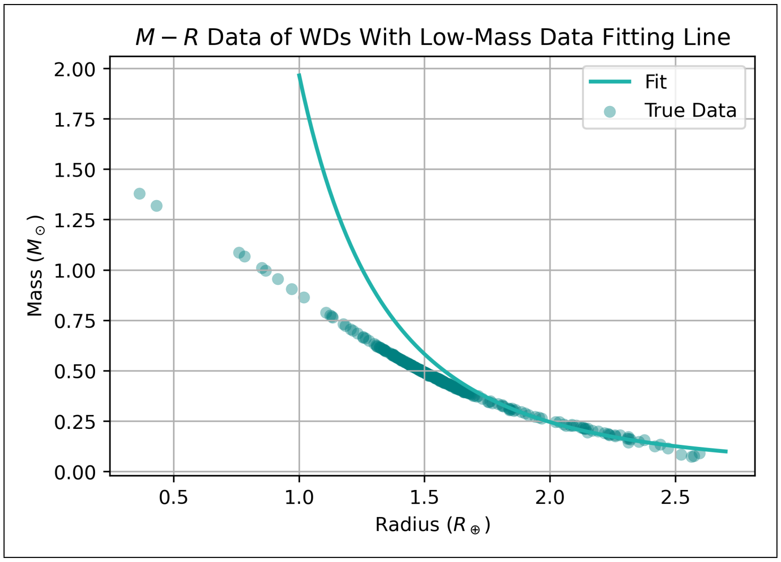


Figure 4: $M(R)$ Fit Curve for Low Masses

So the fit curve hugs the datapoints fairly well for small masses. That's reassuring. But let's also compare our values with those of much more capable people. Primarily, we consult O.R. Pols' book *Stellar Structure and Evolution (2011)*[2] (Pols) [2]. Our values can be summarized in a plot.

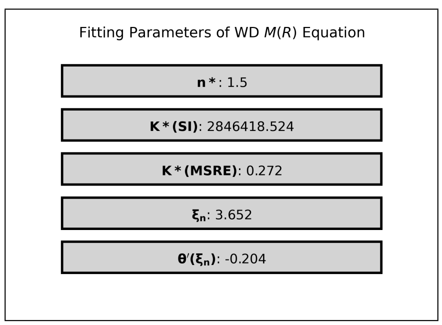


Figure 5: Computed Constants

Consulting Equation (3.35) and Table (4.1)[3] of (Pols) and comparing values:

Constant	**Measured Value**	**Accepted Value**
n_*	1.5	1.5
ξ_n	3.652	3.654
$\theta'(\xi_n)$	-0.204	-0.203
K_* SI	2846418	3161142
K_* MSRE	0.272	0.302

Table 1: Comparison of Measured and Accepted Values of WD Structure Constants

So we more or less see pretty good agreement between our values and the accepted ones.

[2]This book was the first result when I searched "Stellar Structure and Evolution" on Google. I had assumed the book must have been (KWW), but it was only until fairly late in the project that I realized it was written by Pols. It seems astrophysicists have some obsession with reusing the same title for their books.

[3]It is worth mentioning that (Pols) [2] does not provide $\theta'(\xi_n)$ but instead $\Theta_n = -\xi_n^2\theta'(\xi_n)$. Furthermore, his calculation for K_* contains $\mu_e^{5/3}$ directly where $\mu_e = 2$ is the number of nucleons per electron. So we multiply his given value of K_* by $2^{-5/3}$.

We finally end this part by plotting ρ_c vs M. Computing ρ_c can be easily done by solving for it in Equation (14)

$$\rho_c = \frac{M}{4\pi R^3}\left(\frac{-\xi_n}{\theta'(\xi_n)}\right) \tag{24}$$

The plot will be presented (as usual) in $MSRE$ units.

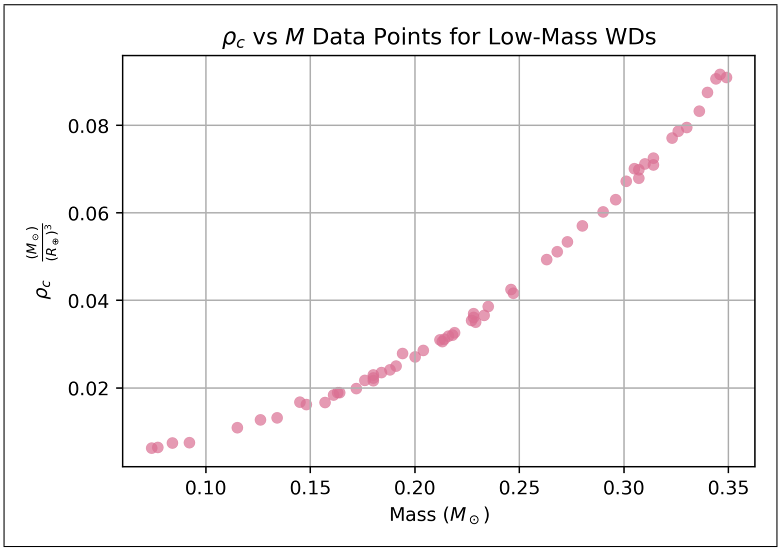


Figure 6: $\rho_c(M)$ Curve

Part D

We would now like to examine the general WD structure, not just that of low-mass WDs. This brings us back to the general EOS of Equation (17). We can no longer use the Lane-Emden equation. That being said, knowing that $C = 5K_* D^{5/q}/8$ simplifies our problem further into just needing to find a value for D. Let us begin with the fundamentals. We roughly know what ρ_c values our WDs have from the previous part. This also gives us some idea about what D is since D gets closer to ρ_c for heavier WDs. Then what we can do is start with a guess for D, solve the IVP equation governing the structure of WDs, and see if it matches the $M(R)$ curve we have from the data. We repeat this until we find a "good" D that corresponds to the data nicely. But first, what is the IVP we're trying to solve? Well it is the stellar structure equations along with the general EOS. We rewrite them all collectively below.

$$
\begin{aligned}
\frac{dm(r)}{dr} &= 4\pi r^2 \rho(r) \\
\frac{dp(r)}{dr} &= -\frac{Gm(r)\rho(r)}{r^2} \\
p = C\Big[x(2x^2-3)(x^2+1)^{1/2} &+ 3\text{arcsinh}(x)\Big], \quad x = \left(\frac{\rho(r)}{D}\right)^{\frac{1}{q}}
\end{aligned} \tag{25}
$$

The general EOS seems like a very nasty expression. I have been unable to figure out (or find) an analytic solution for x in terms of p, much less a solution for ρ. So do we just deal with an implicit equation? Well, luckily, while the general EOS is not very digestible in its current form, we can take its derivative to make it far more friendly. First note by the chain rule that

$$
\begin{aligned}
\frac{dp}{dr} &= \frac{dp}{dx}\frac{dx}{d\rho}\frac{d\rho}{dr} \\
\Rightarrow \frac{d\rho}{dr} &= \frac{dp}{dr}\left(\frac{dp}{dx}\right)^{-1}\left(\frac{dx}{d\rho}\right)^{-1}.
\end{aligned} \tag{26}
$$

So we have an explicit expression for $d\rho/dr$! We simply need to compute the intermediate derivatives. Of course, dp/dr is already given Equation (25). For the other two

$$
\begin{aligned}
\frac{dx}{d\rho} &= \frac{1}{q} D^{\frac{-1}{q}} \rho^{\frac{-q+1}{q}} \\
\\
\frac{dp}{dx} &= C\left(4x^2\sqrt{x^2+1} + \frac{3}{\sqrt{x^2+1}} + \sqrt{x^2+1}\left(2x^2-3\right) + \frac{x^2\left(2x^2-3\right)}{\sqrt{x^2+1}}\right) \\
&= \frac{8Cx^4}{\sqrt{x^2+1}} \quad \text{(Substitute x with its expression for ρ.)} \\
&= 5K_* D^{5/q} \frac{\left(\frac{\rho}{D}\right)^{4/q}}{\sqrt{\left(\frac{\rho}{D}\right)^{2/q}+1}}.
\end{aligned} \tag{27}
$$

Going through the slightly tedious algebra (which includes substituting $q = 3$) inevitably yields

$$
\frac{d\rho}{dr} = -\frac{3Gm(r)\rho(r)^{1/3}\sqrt{\left(\frac{\rho(r)}{D}\right)^{2/3}+1}}{5K_* r^2}. \tag{28}
$$

So we have all the IVP equations we need. Here is the procedure we perform to find D. First, begin with a guess value for D. Now choose a set of ρ_c values that span the range of the WD masses as your set of initial conditions. Solve the IVP, that is integrate out numerically in r until the pressure (or density) reaches 0. That point represents the total mass M and full radius R of the WD for that density ρ_c and that D value. When all of these $M(R)$ points are found, interpolate through them with a spline, and compare that with a spline of the raw data. Repeat for different values D until the one that minimizes the error is found. This is, admittedly, the costly method we were instructed not to use. However, while I conceptually understood the efficient interpolation procedure, I didn't know how to write the code in a way where the interpolation function also depended on D, not just on r. In any case, this method isn't that bad either. We have three main functions that accomplish this task. `general_EOS()` gives the pressure value for a set of given values ρ_c and D. This is used the to find the initial conditions $[m = 0, p_c, \rho_c]$ for the IVP. `solve_ivp_for_D()` is the function containing the ODEs. It contains certain conditions to ensure the solutions are well behaved near the end points. As one may expect, it will be handed to SciPy's `solve_ivp()`. Finally, we have `p_reaches_0()`, which is used in `solve_ivp()'s` events argument to stop the integration when the pressure goes to zero. This same logic will be used in Einstein as well.[4]

The ρ_c values we used were [0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1, 2, 4, 10, 30] in units $MSRE$. While the spacing may seem arbitrary, a look at the plot will show that they nicely cover the whole range. Finally, the value for D was determined through trial and error and searching for the smallest absolute difference with the raw data. We compare this 'optimal' value with two other nearby ones for illustration.

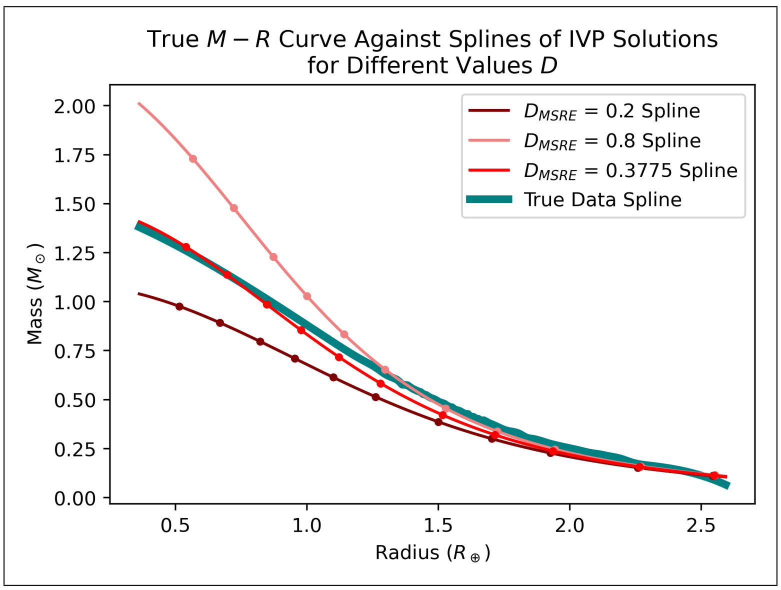


Figure 7: Fit Curves of IVP Solutions of Different D Values

We see in Figure 7 that the optimal value for D (in $MSRE$ units) is 0.3775. Also note that the dots represent the $M(R)$ points used in the interpolation, justifying our selection of ρ_c values. With D acquired, we can calculate the other quantities and present them.

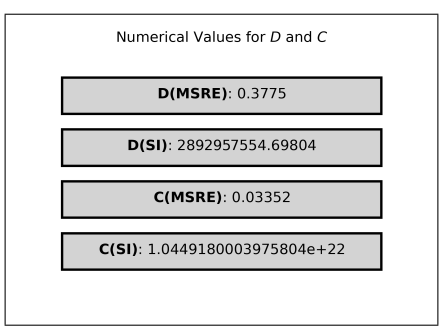


Figure 8: Numerical Values for D and C

[4] Ironically in practice it was the other way around. I used `p_reaches_0()` when solving the TOV equations in Einstein and copied it for Newton Part D.

Let us again tabulate these values against the theoretical/accepted ones.

Constant	**Measured Value**	**Accepted Value**
$D \quad SI$	2.892957554×10^9	1.947864339×10^9
$D \quad MSRE$	0.3775	0.2542
$C \quad SI$	$1.044918000 \times 10^{22}$	$6.002332193 \times 10^{21}$
$C \quad MSRE$	0.03352	0.01925

Table 2: Comparison of Measured and Accepted Values of D and C

We see that our values are not a perfect match with the accepted values, and this is most probably due to the discrepancy in the K_* value which forces D and C to have to 'compensate.' That being said, they are still not too far off.

Part E

With D found, we can now plot the $M(R)$ curve using many more values of ρ_c. We would like to also observe the Chandrasekhar maximal mass which occurs in the limit ($\rho \to \infty \quad \longleftrightarrow \quad r \to 0$). To this end, we solve IVPs for numerous values ρ_c in the interval [0.01, 5000]. We then interpolate these points and compute $M_{ch} = M(0)$.

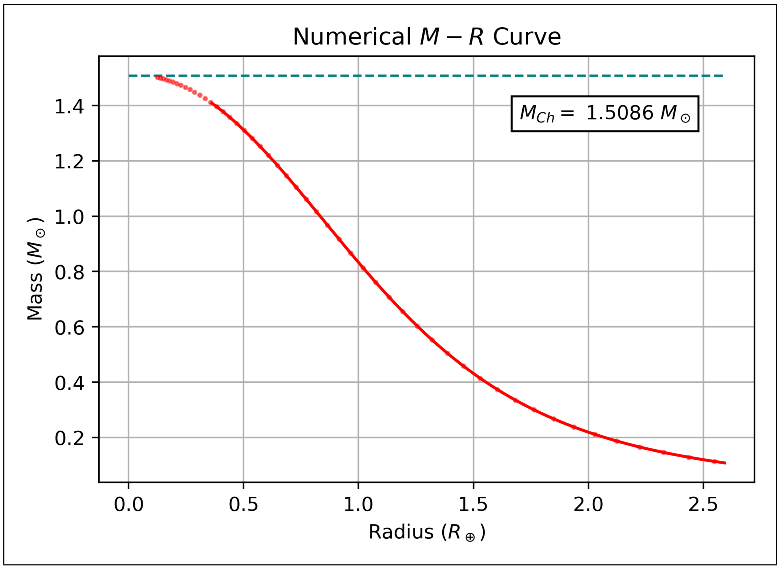


Figure 9: $M(R)$ Curve with IVP Solutions and Chandrasekhar Mass

We find that $M_{ch} = 1.5086 M_\odot$. Let's compare this with the theoretical value. Taking the $x \gg 1$ limit of Equation (17) in Mathematica gives

```
(* EOS for cold WDs in x where x = (ρ/D)^(1/q). *)
P[x] = C * (x (2 x^2 - 3) Sqrt[x^2 + 1] + 3 ArcSinh[x]);

(* Expansion of P(x) to 7th order in x. *)
SeriesInX = Series[P[x], {x, Infinity, degree}]
```

$$2 C x^4 - 2 C x^2 + \frac{1}{4} C \left(-7 + 6 \operatorname{Log}[4 x^2]\right) + O\left[\frac{1}{x}\right]^2$$

```
(* Replacing x with (ρ/D)^(1/q). At this point we also know that q = 3.*)
SeriesInRho = Normal[SeriesInX] /. x → D^(-1/3) ρ^(1/3)
```

$$-\frac{2 C \rho^{2/3}}{D^{2/3}} + \frac{2 C \rho^{4/3}}{D^{4/3}} + \frac{1}{4} C \left(-7 + 6 \operatorname{Log}\left[\frac{4 \rho^{2/3}}{D^{2/3}}\right]\right)$$

$$p = K\rho^{1+\frac{1}{n}}, \quad K = \frac{2C}{D^{4/3}}, \quad n = 3. \tag{29}$$

These values allow us to rewrite Equation (16) as

$$\begin{aligned} M(\rho \to \infty) &= \left(-4\pi \left(\frac{4\pi G}{(3+1)\left(\frac{2C}{D^{4/3}}\right)}\right)^{\frac{3}{1-3}} \xi_3^{\frac{3+1}{3-1}} \theta'(\xi_3)\right) \\ &= 4\pi \left(\frac{2C}{\pi G D^{4/3}}\right)^{3/2} \Theta_3, \end{aligned} \tag{30}$$

where Θ_n is as defined in (Pols) [2] to be $-\xi_n^2 \theta'(\xi_n)$. As before, we can get this value from (Pols) Table (4.1). Namely $\Theta_3 = 2.01824$. Plugging in the values and making the unit conversions yields $M = M_{ch} = 1.456 M_\odot$, which is in agreement with the one reported in (Pols) and fairly close to our found value.

Einstein

Part A & Part B

We now shift our focus from WDs to NSs. These are unbelievably dense stellar objects capable of containing a few solar masses inside a 10−20 km radius. That is like putting the entire mass of the sun inside a ball formed with Manhattan as its diameter. The powerful gravity of such objects breaks down Newton's Mechanics and forces us to use Einstein's GR. Though an explicit EOS for NS structure is unknown, we are told that a polytrope with $n = 1$ is a good approximation.

$$p = K_{NS}\rho^2, \tag{31}$$

where $K_{NS} \sim 100$ in geometric units. Now we need our stellar structure equations which were the hydrostatic equilibrium equations in the previous section. In GR, the equations are modified to account for relativistic effects and are titled the Tolman-Oppenheimer-Volkoff (TOV) equations.

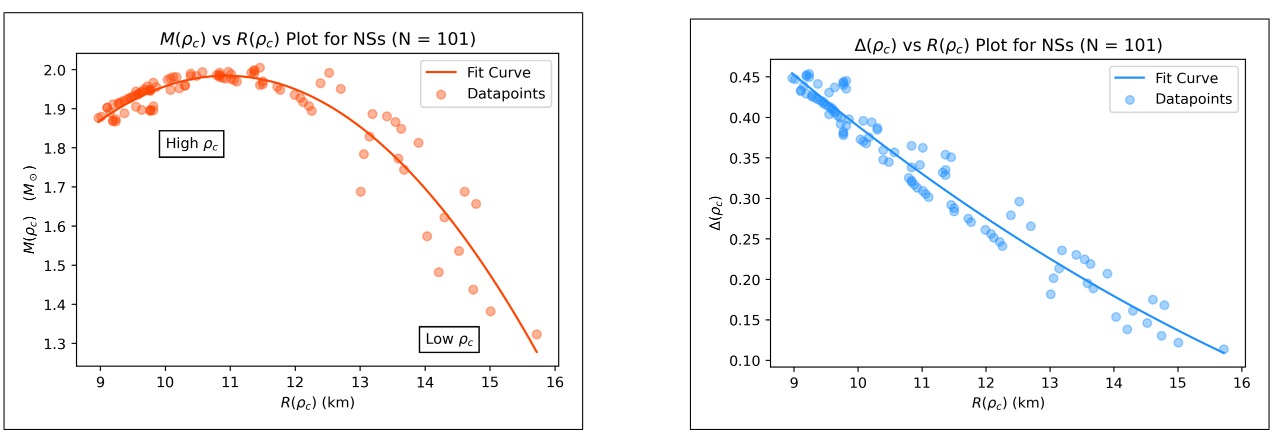
$$\begin{aligned} m' &= 4\pi r^2 \rho \\ \nu' &= 2\frac{m + 4\pi r^3 p}{r(r-2m)} \\ p' &= -\frac{m + 4\pi r^3 p}{r(r-2m)}(\rho + p) = -\frac{1}{2}(\rho + p)\nu' \end{aligned} \tag{32}$$

We also note down the equations for the rest mass m_P', or Baryonic mass, and the fractional binding energy Δ.

$$\begin{aligned} m_P' &= 4\pi \left(1 - \frac{2m}{r}\right)^{-1/2} r^2 \rho \\ \Delta &= \frac{M_P - M}{M}, \end{aligned} \tag{33}$$

where capital M and M_P, as before, represent the total mass quantities of the NS.

We are tasked with finding the $M(R)$ and $\Delta(R)$ curves. The process here is the same as in Newton, as we alluded. We have this system of ODEs with nonzero initial conditions for the density and pressure. We start with a density initial condition ρ_c, integrate out till the pressure and/or density reach 0 and record M, M_P, and R at that point. Repeat this for multiple values ρ_c, and we now have a curve. We are informed that, in geometric units, $\rho \sim 10^{-3}$. So we compute the points for $\rho \in [0.9, \quad 9] \times 10^{-3}$. After these points are found, we perform a cubic polynomial fit to the points.



(a) $M(R)$ Curve

(b) $\Delta(R)$ Curve

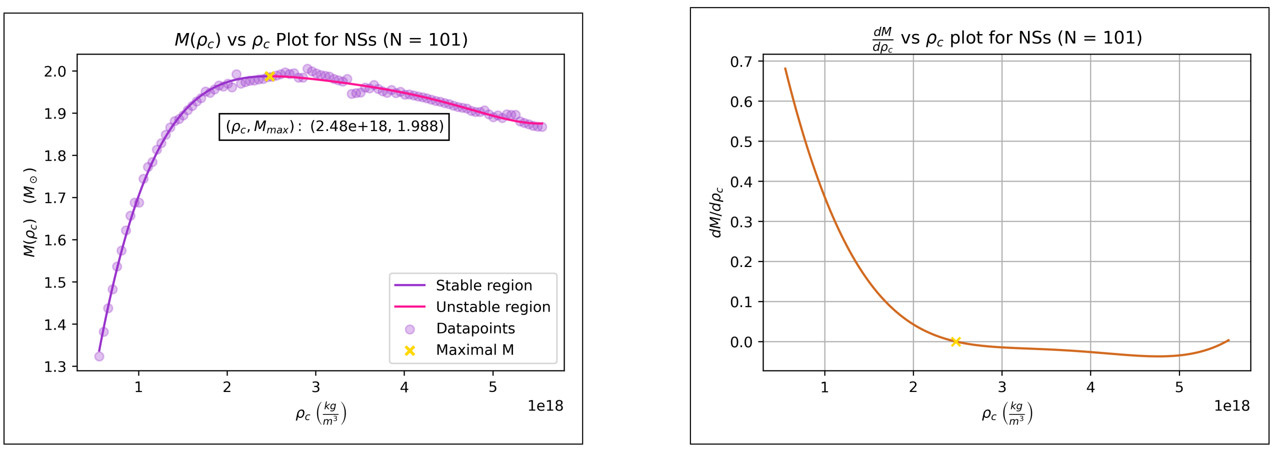
Figure 10: $M(R)$ and $\Delta(R)$ Curves for NSs

Part C

Given the structures we found, we want to study the stability of NSs under the criterion given to us.

$$\begin{aligned} \frac{dM}{d\rho_c} &> 0 \rightarrow \text{stable} \\ \frac{dM}{d\rho_c} &< 0 \rightarrow \text{unstable} \end{aligned} \tag{34}$$

Of course, we don't actually have a 'true' $M(\rho_c)$ curve to analyze, but a collection of datapoints. These datapoints are interpolated through to give us a curve. But since we are interested in not only the curve but its derivative as well, we can make our life far easier by interpolating with a polynomial fit. That way, the $dM/d\rho_c(\rho_c)$ curve can be easily found by taking the derivative of the polynomial describing $M(\rho_c)$. A good polynomial degree for the data was found to be degree $= 5$. We present the plots below, along with the point representing the maximal mass for a stable NS.



(a) $M(\rho_c)$ Curve

(b) $dM/d\rho_c(\rho_c)$ Curve

Figure 11: $M(\rho_c)$ and $dM/d\rho_c(\rho_c)$ Curves for NSs

We can clearly see the stable and unstable regions for NS masses. Namely, for $K_{NS} = 100$, we have that $M_{max} = 1.988 M_\odot$.

Part D

Now we would like to find the allowed values of K_{NS} based on the observation that the most massive and stable observed NS had a mass of $2.14M_{\odot}$. There's nothing conceptually new to be done here. We will in effect just be repeating Parts A, B, and C for different K_{NS} values. For each K_{NS} value, we find the maximal mass and record it. Then we plot $M_{max}(K)$ vs K. This is computationally a bit heavy since we have to solve many IVPs but it isn't too bad. The plot is given below.

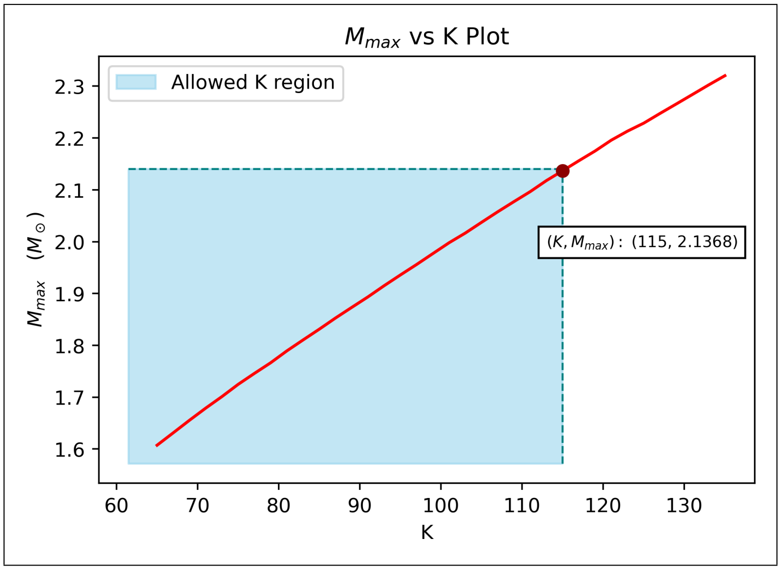


Figure 12: $M_{max}(K)$ Curve for NSs

Looking at the plot, we see that the highest integer K_{NS} value is $K_{NS} = 115$ with $M_{max}(115) = 2.137M_{\odot}$. The curve is a neat linear function with positive slope, suggesting that values of $K_{NS} < 115$ are also valid.

Part E

In the final part of this project, we are trying to compute the constant term present in the time dilation factor $\nu(r)$. Outside the NS ($r > R$), we can express $\nu(r)$ in a more simpler form than that in Equation (32).

$$\nu' = \frac{2M}{r(r-2M)} \quad (r > R). \tag{35}$$

We can solve this expression in Mathematica using `DSolve[]` with the initial condition that $\nu(R)$ is a known constant.

```
(* Solve the DE of time-dialation factor ν(r) when r > R. Initial condition is that ν(R) is equal to some constant. *)
solution = DSolve[{ν'[r] == (2 M)/(r (r - 2 M)), ν[R] == νR}, ν[r], r]

(* Combine logarithms to reach a form like the on in the project document. *)
logrule = {-Log[r]+Log[-2M + r] :> Log[1-2M/r], Log[R]- Log[-2M + R] :> -Log[1-2M/R]};
simplifiedSolution= solution//.logrule;

(* Print. *)
simplifiedSolution
```

```
{{ν[r] → νR - Log[r] + Log[-2 M + r] + Log[R] - Log[-2 M + R]}}
{{ν[r] → νR + Log[1 - 2 M/r] - Log[1 - 2 M/R]}}
```

$$\nu(r > R) = \ln\left(1 - \frac{2M}{r}\right) - \ln\left(1 - \frac{2M}{R}\right) + \nu(R). \tag{36}$$

So we recover the same expression we were asked to find.

With that, we humbly end this project.

References

[1] *Wikipedia contributors, "Lane–Emden equation," Wikipedia, Dec. 28, 2024. https://en.wikipedia.org/wiki/Lane%E2%80%93Emden_equation*

[2] *O. R. Pols, Stellar Structure and Evolution. Utrecht, Netherlands: Utrecht University, 2011.*