Focused Linear Logic

Constructive Logic (15-317)
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Even though we have studied focusing in the context of constructive logic, it began with linear logic. It was first proposed by Jean-Marc Andreoli in 1992[1] with the goal of reducing redundancy in proofs. This is done, as you know, by ignoring irrelevant choices or permutations of rules, thus creating an equivalence class for (focused) proofs. The rules for first-order intuitionistic linear logic (ILL) are depicted in Figure 1.

Before getting to the focused sequent calculus, we will change to a *dyadic* system: the left context will be split into two, Γ and Δ. Differently from the focused calculus for constructive logic, Γ will hold the *unbounded resources* (those formulas we can use any number of times), while Δ will hold the *linear resources* (the formulas that must be used exactly once). In the dyadic system there will be no explicit rules of contraction and weakening. Instead, multiplicative rules will split only the linear context, but act additively on the unbounded one:

$$\frac{\Gamma; \Delta_1 \longrightarrow A \quad \Gamma; \Delta_2, B \longrightarrow C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \longrightarrow C} \multimap L \qquad \frac{\Gamma; \Delta_1 \longrightarrow A \quad \Gamma; \Delta_2 \longrightarrow B}{\Gamma; \Delta_1, \Delta_2 \longrightarrow A \otimes B} \otimes R$$

When removing the ! on the left, we move the formula to Γ, indicating that that is an unbounded resource. In this case, $!R$ and $init$ both require that there are no more linear resources left, and this will be indicated by an empty Δ, but Γ may contain formulas.

$$\frac{\Gamma, A; \Delta \longrightarrow C}{\Gamma; \Delta, !A \longrightarrow C} !L \qquad \frac{\Gamma; \cdot \longrightarrow A}{\Gamma; \cdot \longrightarrow !A} !R \qquad \frac{}{\Gamma; a \longrightarrow a} \textsf{init}$$

Finally, we need a rule that copies formulas from Γ to Δ when we want to actually use them:

$$\frac{\Gamma, A; \Delta, A \longrightarrow C}{\Gamma, A; \Delta \longrightarrow C} \textsf{copy}$$

[1] https://web2.qatar.cmu.edu/~greis/andreoli92jlc.pdf

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \& B}\ \&R \qquad \frac{\Gamma, A_i \longrightarrow C}{\Gamma, A_1 \& A_2 \longrightarrow C}\ \&L$$

$$\frac{}{\Gamma \longrightarrow \top}\ \top R \qquad \text{no } \top L \text{ rule}$$

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \multimap B}\ \multimap R \qquad \frac{\Gamma_1 \longrightarrow A \quad \Gamma_2, B \longrightarrow C}{\Gamma_1, \Gamma_2, A \multimap B \longrightarrow C}\ \multimap L$$

$$\frac{\Gamma \longrightarrow A\{x \mapsto \alpha\}}{\Gamma \longrightarrow \forall x.A}\ \forall R \qquad \frac{\Gamma, A\{x \mapsto t\} \longrightarrow C}{\Gamma, \forall x.A \longrightarrow C}\ \forall L$$

$$\frac{\Gamma_1 \longrightarrow A \quad \Gamma_2 \longrightarrow B}{\Gamma_1, \Gamma_2 \longrightarrow A \otimes B}\ \otimes R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \otimes B \longrightarrow C}\ \otimes L$$

$$\frac{}{\cdot \longrightarrow \mathtt{1}}\ \mathtt{1}R \qquad \frac{\Gamma \longrightarrow C}{\Gamma, \mathtt{1} \longrightarrow C}\ \mathtt{1}L$$

$$\frac{\Gamma \longrightarrow A_i}{\Gamma \longrightarrow A_1 \oplus A_2}\ \oplus R \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \oplus B \longrightarrow C}\ \oplus L$$

$$\text{no } \mathtt{0}R \text{ rule} \qquad \frac{}{\Gamma, \mathtt{0} \longrightarrow C}\ \mathtt{0}L$$

$$\frac{!\Gamma \longrightarrow A}{!\Gamma \longrightarrow !A}\ !R \qquad \frac{\Gamma, A \longrightarrow C}{\Gamma, !A \longrightarrow C}\ !L$$

$$\frac{\Gamma \longrightarrow A\{x \mapsto t\}}{\Gamma \longrightarrow \exists x.A}\ \exists R \qquad \frac{\Gamma, A\{x \mapsto \alpha\} \longrightarrow C}{\Gamma, \exists x.A \longrightarrow C}\ \exists L$$

$$\frac{\Gamma, !A, !A \longrightarrow C}{\Gamma, !A \longrightarrow C}\ !cL \qquad \frac{\Gamma \longrightarrow C}{\Gamma, !A \longrightarrow C}\ !wL \qquad \frac{}{a \longrightarrow a}\ \mathit{init}$$

Figure 1: Sequent calculus for intuitionistic linear logic (ILL)

Without explicit structural rules, we can now focus on focusing.

As you may remember, the focusing discipline requires that the connectives are split into those that are right invertible (negatives) and those that are left invertible (positives). Since we have two conjunctions, one will be positive and the other will be negative, so there are no neutral connectives. Notice how the need for a "neutral" class arose precisely because, for our constructive logic calculus, we took the right rule for one linear logic conjunction and the left rule for the other. Atoms, again, can have arbitrary polarities. The polarities of the linear logic connectives are:

$$\begin{aligned} \text{Negative: } & \&, \top, \multimap, \forall, a^- \\ \text{Positive: } & \otimes, \mathtt{1}, \oplus, \mathtt{0}, !, \exists, a^+ \end{aligned}$$

Naturally, the asynchronous phase consists of applying all invertible rules.

Asynchronous phase

$$\frac{\Gamma; \Delta \longrightarrow A \quad \Gamma; \Delta \longrightarrow B}{\Gamma; \Delta \longrightarrow A \& B}\ \&R \qquad \frac{}{\Gamma; \Delta \longrightarrow \top}\ \top R \qquad \frac{\Gamma; \Delta, A \longrightarrow B}{\Gamma; \Delta \longrightarrow A \multimap B}\ \multimap R$$

$$\frac{\Gamma; \Delta, A, B \longrightarrow C}{\Gamma; \Delta, A \otimes B \longrightarrow C}\ \otimes L \quad \frac{\Gamma; \Delta \longrightarrow C}{\Gamma; \Delta, \mathtt{1} \longrightarrow C}\ \mathtt{1}L \quad \frac{\Gamma; \Delta, A \longrightarrow C \quad \Gamma; \Delta, B \longrightarrow C}{\Gamma; \Delta, A \oplus B \longrightarrow C}\ \oplus L \quad \frac{}{\Gamma; \Delta, \mathtt{0} \longrightarrow C}\ \mathtt{0}L$$

$$\frac{\Gamma; \Delta \longrightarrow A\{x \mapsto \alpha\}}{\Gamma; \Delta \longrightarrow \forall x.A}\ \forall R \qquad \frac{\Gamma; \Delta, A\{x \mapsto \alpha\} \longrightarrow C}{\Gamma; \Delta, \exists x.A \longrightarrow C}\ \exists L \qquad \frac{\Gamma, A; \Delta \longrightarrow C}{\Gamma; \Delta, !A \longrightarrow C}\ !L$$

Note that we do not force the application of invertible rules on formulas from the context Γ. Since these are always available, we would never exhaust them. Instead, we force focus on those formulas if we choose them.

A focused phase will only begin if C and all other formulas in Δ need invertible rule applications. This means that everything in Δ is negative (indicated by Δ^-) and C is either an atom or positive formula (indicated by C_a^+). On the rules below, P is a positive atom or a positive formula, N is a negative atom or negative formula, and A is any formula.

$$\frac{\Gamma; \Delta^- \longrightarrow [P]}{\Gamma; \Delta^- \longrightarrow P}\ \mathsf{focus}_r \qquad \frac{\Gamma; \Delta^-, [N] \longrightarrow C_a^+}{\Gamma; \Delta^-, N \longrightarrow C_a^+}\ \mathsf{focus}_l \qquad \frac{\Gamma, A; \Delta^-, [A] \longrightarrow C_a^+}{\Gamma, A; \Delta^- \longrightarrow C_a^+}\ \mathsf{focus!}$$

The synchronous phase then consists of applications of non-invertible rules. The focus is kept persistent on the premises, as usual, except for $!R$. We will discuss why later on.

SYNCHRONOUS PHASE

$$\frac{\Gamma; \Delta, [A_i] \longrightarrow C}{\Gamma; \Delta, [A_1 \& A_2] \longrightarrow C} \,\&L \qquad \frac{\Gamma; \Delta_1 \longrightarrow [A] \quad \Gamma; \Delta_2, [B] \longrightarrow C}{\Gamma; \Delta_1, \Delta_2, [A \multimap B] \longrightarrow C} \multimap L$$

$$\frac{\Gamma; \Delta_1 \longrightarrow [A] \quad \Gamma; \Delta_2 \longrightarrow [B]}{\Gamma; \Delta_1, \Delta_2 \longrightarrow [A \otimes B]} \otimes R \qquad \frac{}{\Gamma; \cdot \longrightarrow [\mathsf{1}]} \,\mathsf{1}R \qquad \frac{\Gamma; \Delta \longrightarrow [A_i]}{\Gamma; \Delta \longrightarrow [A_1 \oplus A_2]} \oplus R$$

$$\frac{\Gamma; \Delta, [A\{x \mapsto t\}] \longrightarrow C}{\Gamma; \Delta, [\forall x.A] \longrightarrow C} \forall L \qquad \frac{\Gamma; \Delta \longrightarrow [A\{x \mapsto t\}]}{\Gamma; \Delta \longrightarrow [\exists x.A]} \exists R \qquad \frac{\Gamma; \cdot \longrightarrow A}{\Gamma; \cdot \longrightarrow [!A]} \,!R$$

Naturally, if we encounter a formula that can be decomposed by an invertible rule during a synchronous phase, we loose focus.

$$\frac{\Gamma; \Delta^- \longrightarrow N}{\Gamma; \Delta^- \longrightarrow [N]} \,\mathsf{blur}_r \qquad \frac{\Gamma; \Delta^-, P \longrightarrow C_a^+}{\Gamma; \Delta^-, [P] \longrightarrow C_a^+} \,\mathsf{blur}_l$$

The only thing missing to complete the calculus is *init*. Since we have polarized atoms, we have two *init*s just as in the focused calculus for constructive logic.

$$\frac{}{\Gamma; a^+ \longrightarrow [a^+]} \,\mathsf{init}^+ \qquad \frac{}{\Gamma; [a^-] \longrightarrow a^-} \,\mathsf{init}^-$$

Soundness of focusing is obtained straightforwardly by mapping focused proofs to linear logic proofs, removing the extra syntactical burden and focus/blur steps. Completeness, of course, is more complicated. Andreoli's paper contains the full proof, starting from Appendix A.2. It proceeds by induction on the proof depth and uses some invertibility lemmas (Lemma 6). Note that this proof is for the one-sided version of classical linear logic. The up/down arrows indicate a synchronous ($\Downarrow$) or asynchronous ($\Uparrow$) phase, and Θ contains unbounded formulas and Γ contains linear formulas.

Now, why do we need to loose focus on $!R$? The more natural or at least uniform rule would be:

$$\frac{\Gamma; \cdot \longrightarrow [A]}{\Gamma; \cdot \longrightarrow [!A]} \,!R^*$$

But this rule makes the calculus incomplete. Using it we cannot prove, for example, the identity: $!a^+ \multimap !a^+$. In fact, we cannot prove any formula $!P \multimap !P$ for a positive formula P. Let's see what happens. We need to start with $\multimap R$ and $!L$:

$$\cfrac{\cfrac{P; \cdot \longrightarrow !P}{\cdot; !P \longrightarrow !P}\ !L}{\cdot; \cdot \longrightarrow !P \multimap !P}\ \multimap R$$

At this point, we need to choose a formula to focus on. If we focus on P, by moving it to the right of ; (in the empty Δ), then we can decompose it to its atomic subparts, but it would be impossible to remove the exponential from $!P$. If we focus on $!P$ and remove the exponential first, and it turns out that one of the atomic subparts of P is positive, we would need to close the proof. But we can't since P on the left is not yet decomposed.