Restricted sequent calculus

Constructive Logic (15-317)
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This is the sequent calculus we have worked with so far:

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A \wedge B, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \wedge L_1 \quad \frac{\Gamma, A \wedge B, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \wedge L_2$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \vee R_1 \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \vee B, A \Rightarrow C \quad \Gamma, A \vee B, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \vee L$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, A \supset B, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \supset L$$

$$\frac{}{\Gamma \Rightarrow \top} \top R \qquad \frac{}{\Gamma, \bot \Rightarrow C} \bot L \qquad \frac{}{\Gamma, P \Rightarrow P} \mathit{init}_{\text{(for atomic } P)}$$

It looks a bit bulky and, after proving cut-elimination, you might have noticed that the repetition of the main formula in the premises can become quite a burden. Indeed, if we are automating proof search, keeping these formulas around is not a very good thing to do. The computer can choose to work on the same formula over and over again, like a dog chasing after its tail.

Of course we can come up with smart heuristics in an implementation, such as marking how many times a formula was used and keeping track of the sequents that have appeared in our proof, but as we discussed before, it is hard to guarantee soundness and completeness in an implementation level. The smarter the heuristic, the harder to check if it is not missing anything. We will thus follow the strategy from before: change the calculus and embed the fact that we should not use a formula again in the syntax.

This duplication of main formula is sometimes called auto- or implicit contraction: it is equivalent to having a system with the contraction rule (on left-side formulas) and conservatively duplicate a formula before decomposing it, just in case. Our goal is to have a contraction-free calculus: no contraction rule is allowed and no auto-contractions occur in the other rules.

To distinguish the sequents in our new simplified calculus, we will use the leaner arrow $\rightarrow$.

Since the right rules have no auto-contraction, they remain the same. We restrict our analysis thus to the left rules.

Conjunction Removing the auto-contracted formulas from the conjunction rule gives us the following:

$$\frac{\Gamma, A \rightarrow C}{\Gamma, A \wedge B \rightarrow C} \wedge L_1 \quad \frac{\Gamma, B \rightarrow C}{\Gamma, A \wedge B \rightarrow C} \wedge L_2$$

Are these rules ok? Actually no. They both result on information loss. You start from the fact that you know two things, but continue with one (and only one!) of them. If the conjunction formula is no longer there, the other conjunct is lost forever. It might as well just be the case that we need both of them in a proof. This happens, for instance, in the proof of $A \wedge (A \supset B) \Rightarrow B$. Using the proposed rules above, the sequent $A \wedge (A \supset B) \rightarrow B$ is not derivable.

Luckily, this can be easily fixed by simply keeping both conjuncts in the premise, and getting rid of the main formula at the same time:

$$\frac{\Gamma, A, B \rightarrow C}{\Gamma, A \wedge B \rightarrow C} \wedge L$$

Disjunction The simplified disjunction rule will be:

$$\frac{\Gamma, A \rightarrow C \quad \Gamma, B \rightarrow C}{\Gamma, A \vee B \rightarrow C} \vee L$$

In this case, there is no loss of information. Keeping the disjunction around and using it again will give us nothing new. So we keep this rule as is.

Implication Removing the duplication of the main formula on the $\supset L$ rule results in:

$$\frac{\Gamma \to A \quad \Gamma, B \to C}{\Gamma, A \supset B \to C} \supset L$$

On the right premise, having B certainly supersedes the fact $A \supset B$. The latter requires an A to use B whereas the former is simply B itself.

On the left premise, can a proof of A make use of $A \supset B$? Definitely. Maybe it is the case that A involves a choice, and we can choose one option at the first application of $\supset L$ and another option at the second application. If we were to throw this implication away, we would not be able to have both. This is what happens, for instance, in a proof of $\to \neg\neg(a \vee \neg a)$. Thus we have the following rule:

$$\frac{\Gamma, A \supset B \to A \quad \Gamma, B \to C}{\Gamma, A \supset B \to C} \supset L$$

Unfortunately we cannot get rid of the auto-contraction in this rule just yet.

True There is no left rule for $\top$, therefore we do not need to analyse it.

False The left rule for $\bot$ does not copy formulas to premises because it does not have premises, so it remains unchanged.

After analysing all the rules, our new proposed calculus looks like this:

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \to C}{\Gamma, A \wedge B \to C} \wedge L$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee B} \vee R_1 \quad \frac{\Gamma \to B}{\Gamma \to A \vee B} \vee R_2 \qquad \frac{\Gamma, A \to C \quad \Gamma, B \to C}{\Gamma, A \vee B \to C} \vee L$$

$$\frac{\Gamma, A \to B}{\Gamma \to A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \to A \quad \Gamma, B \to C}{\Gamma, A \supset B \to C} \supset L$$

$$\frac{}{\Gamma \to \top} \top R \qquad \frac{}{\Gamma, \bot \to C} \bot L \qquad \frac{}{\Gamma, P \to P} \mathit{init}\text{(for atomic } P\text{)}$$

We formally show that it is correct by showing soundness and completeness w.r.t. the first calculus. The soundness proof is quite straightforward, using heavily the fact that weakening is admissible.

Theorem 1. *(Soundness) If* $\Gamma \to C$ *then* $\Gamma \Rightarrow C$.

Proof. As usual, we prove by structural induction on the proof tree.

BASE CASE:

$$\frac{}{\Gamma \to C}\ \mathit{init} \quad \rightsquigarrow \quad \frac{}{\Gamma \Rightarrow C}\ \mathit{init}$$

INDUCTION HYPOTHESIS: If there are derivations $\mathcal{D}$ and $\mathcal{E}$ of sequents $\Gamma_1 \to C$ and $\Gamma_2 \to D$, respectively, then there are derivations $\mathcal{D}'$ and $\mathcal{E}'$ of sequents $\Gamma_1 \Rightarrow C$ and $\Gamma_2 \Rightarrow D$, respectively.

INDUCTIVE CASES: We will not show the cases for the right-side rules or $\bot L$, as these are exactly the same for both calculi.

1. Case $\wedge L$

$$\frac{\begin{array}{c}\mathcal{D}\\ \Gamma, A, B \to C\end{array}}{\Gamma, A \wedge B \to C}\ \wedge L$$

By the inductive hypothesis, we have a derivation $\mathcal{D}'$ of $\Gamma, A, B \Rightarrow C$. By admissibility of weakening, this can be transformed into a derivation $\mathcal{D}''$ of $\Gamma, A \wedge B, A, B \Rightarrow C$ which, in turn, can be used to get a derivation of $\Gamma, A \wedge B \Rightarrow C$:

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}''\\ \Gamma, A \wedge B, A, B \Rightarrow C\end{array}}{\Gamma, A \wedge B, A \Rightarrow C}\ \wedge L_2}{\Gamma, A \wedge B \Rightarrow C}\ \wedge L_1$$

2. Case $\vee L$

$$\frac{\Gamma, A \to C \quad \Gamma, B \to C}{\Gamma, A \vee B \to C}\ \vee L$$

By the inductive hypothesis, we have derivations $\mathcal{D}'$ and $\mathcal{E}'$ of $\Gamma, A \Rightarrow C$ and $\Gamma, B \Rightarrow C$, respectively. By admissibility of weakening, these can be transformed into derivations $\mathcal{D}''$ and $\mathcal{E}''$ of $\Gamma, A \vee B, A \Rightarrow C$ and $\Gamma, A \vee B, B \Rightarrow C$. Using those, we get:

$$\dfrac{\begin{array}{c}\mathcal{D}'' \\ \Gamma, A \vee B, A \Rightarrow C\end{array} \quad \begin{array}{c}\mathcal{E}'' \\ \Gamma, A \vee B, B \Rightarrow C\end{array}}{\Gamma, A \vee B \Rightarrow C} \vee L$$

3. Case $\supset L$

$$\dfrac{\Gamma, A \supset B \rightarrow A \quad \Gamma, B \rightarrow C}{\Gamma, A \supset B \rightarrow C} \supset L$$

By IH we have derivations $\mathcal{D}'$ and $\mathcal{E}'$ of $\Gamma, A \supset B \Rightarrow A$ and $\Gamma, B \Rightarrow C$, respectively. By admissibility of weakening, we can transform $\mathcal{E}'$ into a derivation $\mathcal{E}''$ of $\Gamma, A \supset B, B \Rightarrow C$. We can thus apply the $\supset L$ rule from the first calculus and obtain:

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma, A \supset B \Rightarrow A\end{array} \quad \begin{array}{c}\mathcal{E}'' \\ \Gamma, A \supset B, B \Rightarrow C\end{array}}{\Gamma, A \supset B \Rightarrow C} \supset L$$

□

The completeness proof is a little more involved.

Theorem 2. *(Completeness) If* $\Gamma \Rightarrow C$ *then* $\Gamma \rightarrow C$.

Proof. Again we proceed by structural induction on the proof tree (i.e. on the derivation).

BASE CASE:

$$\dfrac{}{\Gamma, C \Rightarrow C}\, init \quad \rightsquigarrow \quad \dfrac{}{\Gamma, C \rightarrow C}\, init$$

INDUCTION HYPOTHESIS: If there are derivations $\mathcal{D}$ and $\mathcal{E}$ of sequents $\Gamma_1 \Rightarrow C$ and $\Gamma_2 \Rightarrow D$, respectively, then there are derivations $\mathcal{D}'$ and $\mathcal{E}'$ of sequents $\Gamma_1 \rightarrow C$ and $\Gamma_2 \rightarrow D$, respectively.

INDUCTIVE CASES: We will not show the cases for the right-side rules or $\perp L$, as these are exactly the same for both calculi.

1. Case $\wedge L_1$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, A \wedge B, A \Rightarrow C\end{matrix}}{\Gamma, A \wedge B \Rightarrow C}\ \wedge L_1$$

   By IH we have a derivation $\mathcal{D}'$ of $\Gamma, A \wedge B, A \to C$. This can be used to derive $\Gamma, A \wedge B \to C$ in the following way:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, A, B \to A}\ id}{\Gamma, A \wedge B \to A}\ \wedge L \quad \begin{matrix}\mathcal{D}'\\ \Gamma, A \wedge B, A \to C\end{matrix}}{\Gamma, A \wedge B \to C}\ cut$$

   The $\wedge L_2$ case is analogous.

2. Case $\vee L$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, A \vee B, A \Rightarrow C\end{matrix} \quad \begin{matrix}\mathcal{E}\\ \Gamma, A \vee B, B \Rightarrow C\end{matrix}}{\Gamma, A \vee B \Rightarrow C}\ \vee L$$

   By IH we have derivations $\mathcal{D}'$ and $\mathcal{E}'$ of $\Gamma, A \vee B, A \to C$ and $\Gamma, A \vee B, B \to C$, respectively. We can use those and cut to construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A \to A}\ id}{\Gamma, A \to A \vee B}\ \vee R_1 \quad \begin{matrix}\mathcal{D}'\\ \Gamma, A \vee B, A \to C\end{matrix}}{\Gamma, A \to C}\ cut \quad \dfrac{\dfrac{\dfrac{}{\Gamma, B \to B}\ id}{\Gamma, B \to A \vee B}\ \vee R_2 \quad \begin{matrix}\mathcal{E}'\\ \Gamma, A \vee B, B \to C\end{matrix}}{\Gamma, B \to C}\ cut}{\Gamma, A \vee B \to C}\ \vee L$$

3. Case $\supset L$ (see homework)

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, A \supset B \Rightarrow A\end{matrix} \quad \begin{matrix}\mathcal{E}\\ \Gamma, A \supset B, B \Rightarrow C\end{matrix}}{\Gamma, A \supset B \Rightarrow C}\ \supset L$$

□

In order to make the transformations local and clearer for the completeness proofs, we have used the cut and id rules. To be precise, we need to show that they are admissible for this calculus. Admissibility of id is straightforward. The proof of cut-elimination follows the same lines as the one before: induction on the cut formula and structure of the trees above it; except that, in the case of this calculus we would need to use some invertibility lemmas. We will see what those are next class.

Using contraction

There is also another strategy to prove this result without using cut (directly at least). We can in fact get the transformations by using structural rules of contraction and weakening. The case for $\wedge L$ would thus be:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \Gamma, A \wedge B, A \rightarrow C\end{array}}{\Gamma, A \wedge B, A, B \rightarrow C}\,weak}{\Gamma, A \wedge B, A \wedge B \rightarrow C}\,\wedge L}{\Gamma, A \wedge B \rightarrow C}\,cont$$

Afterwards, of course, we need to show the admissibility of those rules in the simplified calculus. Weakening is simple, following the same lines as before. But contraction is more tricky. We only outline the proof here.

Theorem 3. *If $\mathcal{D} :: \Gamma, A, A \rightarrow C$ then $\mathcal{D}' :: \Gamma, A \rightarrow C$.*

Proof sketch. The proof is a nested induction on the structure of A first, and then $\mathcal{D}$.

- BASE CASE 1: A is an atom.

  We need to show that $\mathcal{D} :: \Gamma, A, A \rightarrow C$ implies $\mathcal{D}' :: \Gamma, A \rightarrow C$ for atomic A. The proof proceeds by induction on $\mathcal{D}$.

  - BASE CASE 1.1: $\mathcal{D}$ is $init$.

    There are two subcases, one where A is the main formula of $init$ and another where it is not. Both cases can be proved easily.

  - INDUCTIVE CASES 1.1: each case corresponds to the lowermost rule in $\mathcal{D}$.

    IH: The property holds for proof structurally smaller than $\mathcal{D}$.

    These cases follow the usual transformation where we apply the inductive hypothesis to the premises of a rule, and then use the rule itself to get the conclusion we want.

- INDUCTIVE CASES 1: We case on the structure of the contracted formula.

  IH: The property holds for formulas A and B structurally smaller than $A \star B$, for $\star \in \{\wedge, \vee, \supset\}$.

  At each case, we need an inner induction on $\mathcal{D}$. We show here only the case for $\supset$.

  - **Case** $A \supset B$**:**
    We proceed by structural induction on $\mathcal{D}$. If the lowermost rule of $\mathcal{D}$ cannot operate on $A \supset B$, then this case is analogous to the atomic one. The only really interesting cases are the ones where the lowermost rule of $\mathcal{D}$ can operate on the contracted formula. For the case of $A \supset B$, this would be:

$$\dfrac{\overset{\mathcal{E}}{\Gamma, A \supset B, A \supset B \to A} \quad \overset{\mathcal{F}}{\Gamma, A \supset B, B \to C}}{\Gamma, A \supset B, A \supset B \to C} \supset L$$

    Can we transform this into a proof of $\Gamma, A \supset B \to C$? Using cut we can, and then we are back at the problem of having to show cut-admissibility for this calculus.

□

Using the wrong rules

Interestingly, the wrong rules for $\supset L$ (without copying the $\supset$ formula to the left premise) and $\wedge L$ (keeping only one of the two conjuncts) seem to make it through the proof of completeness of the $\to$-calculus w.r.t. $\Rightarrow$ (at least the version using contraction). That may give us the impression that the rules are just fine. But we cannot forget that we used weakening, contraction, cut, etc. for these transformations, and we still need to show that these rules are admissible in the calculus. Eventually, one of these long proofs will fail (in maybe one or two out of dozens of cases), so we must always be careful about these things!