Chapter II
Analysis of the main equations of mathematical physics

§1 Some elements of functional analysis

1.1 Vector spaces

Let us recall several notions of analysis. A set E is called vector space ($\mathbb{R}$ – vector space or $\mathbb{C}$ – vector space) if E is supplied with two operations. The first operation, called addition $+$: $E \times E \to E$, takes any two vectors (elements of the set E) $\mathbf{v}$ and $\mathbf{w}$ and assigns to them the third vector $\mathbf{v} + \mathbf{w}$ called the sum of these two vectors. The second operation called scalar multiplication $\cdot$: $\mathbb{R} \times E \to E$ or $\mathbb{C} \times E \to E$, takes any real number or complex number α and any vector $\mathbf{v}$ of E and assigns to them vector $\alpha\mathbf{v}$. These two operations adhere eight axioms:

A1. Associativity of addition:

$$\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in E \ , \ \ \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \ .$$

A2. Comutativity of addition:

$$\forall \mathbf{u}, \mathbf{v} \in E \ , \ \ \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \, .$$

A3. Identity element of addition (zero vector):

$$\exists \mathbf{0}_E \in E : \forall v \in E \ , \ \ \mathbf{v} + \mathbf{0}_E = \mathbf{0}_E + \mathbf{v} = \mathbf{v} \ .$$

A4. Invers elements of addition (additive inverse):

$$\forall \mathbf{v} \in E, \exists - \mathbf{v} \in E : \mathbf{v} + (-\mathbf{v}) = \mathbf{0}_E \ .$$

A5. Compatibility of scalar multiplication with the field multiplication in $\mathbb{R}$ or $\mathbb{C}$:

$$\forall \alpha, \beta \in \mathbb{K}, \forall \mathbf{v} \in E, \alpha(\beta \mathbf{v}) = (\alpha\beta)\mathbf{v} \ ,$$

here and below $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$.
A6. Identity element of scalar multiplication

$$\forall \mathbf{v} \in E \ , \ \ 1 \cdot \mathbf{v} = \mathbf{v}$$

(here 1 stands for the number 1).
A7. Distributivity of scalar multiplication with respect to the vector addition

$$\forall \alpha \in \mathbb{K}, \forall \mathbf{u}, \mathbf{v} \in E, \alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v} \ .$$

A8. Distributivity of scalar multiplication with respect to field addition in $\mathbb{R}$ or $\mathbb{C}$:

$$\forall \alpha, \beta \in \mathbb{K}, \forall \mathbf{v} \in E, (\alpha + \beta)\mathbf{v} = \alpha \mathbf{v} + \beta \mathbf{v} \ .$$

For instance, $\mathbb{R}^d$ is an $\mathbb{R}$ vector space, $\mathbb{C}^d$ is a $\mathbb{C}$ vector space.

1.2 Normed spaces

A vector space E is called normed vector space if it is supplied with a norm, that is a real-valued function defined on the vector space E and has the following properties:
N1. $\forall x \in E$, $\| x \| \geq 0$; $\| x \| = 0 \Leftrightarrow x = 0_E$;
N2. $\forall \alpha \in \mathbb{K}$, $\forall x \in E$, $\| \alpha x \| = | \alpha | \, \| x \|$;
N3. $\forall x, y \in E, \| x + y \| \leq \| x \| + \| y \|$ (triangle inequality).
The last property yields $\Big| ||x|| - ||y|| \Big| \leq ||x - y||$. As an example one can consider the so-called euclidean norm in $\mathbb{R}^d$:

$$||(x_1, ..., x_d)||_2 = \sqrt{|x_1|^2 + ... + |x_d|^2} \ .$$

Let $(x_n)_{n \in N}$ be a sequence, such that $x_n \in E$, where E is a normed space.

Definition 1.1. $(x_n)_{n \in N}$ **converges** to an element $\bar{x} \in E$ (denoted by $x_n \to \bar{x}$) if $||x_n - \bar{x}|| \to 0$.
Let us formulate some direct consequences of the corresponding assertions for the convergence of numerical sequences in $\mathbb{R}$.

Theorem 1.1. If $(x_n)_{n\in\mathbb{N}}$ converges to $\bar{x}$ then all subsequences $(x_{k_n})_{n\in N}$ converge to the same limit $\bar{x}$.
Theorem 1.2. If $(x_n)_{n\in\mathbb{N}}$ converges to some limit then its limit is unique.
Theorem 1.3. If $(x_n)_{n\in\mathbb{N}}$ converges then it is bounded, i.e. the real-valued sequence $(||x_n||)_{n\in\mathbb{N}}$ is bounded.
Theorem 1.4. If $x_n \to \bar{x}$ then $||x_n|| \to ||\bar{x}||$.
Proof. $\Big|||x_n|| - ||\bar{x}||\Big| \leq ||x_n - \bar{x}|| \to 0.$
Let E be a normed space, M, N two sets in E. M is called dense in N iff $\bar{M} \supset N$. Recall that $\bar{M}$ is the closure of M defined as in section 2 of Chater I replacing $\mathbb{R}^d$ by E. M is called nowhere dense in E iff it is not dense in any ball.

Definition 1.2. A sequence $(x_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence if

$$\forall\varepsilon > 0\ ,\ \ \exists n_0 \in \mathbb{N} \text{ such that } \forall n, m \geq n_0, ||x_n - x_m|| < \varepsilon\ \ .$$

Clearly any convergent sequence is a Cauchy sequence because if $x_n \to \bar{x}$ then

$$\forall\varepsilon > 0\ ,\ \ \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, ||x_n - \bar{x}|| < \varepsilon/2\ \ ,$$

and so, $\forall n, m \geq n_0$

$$||x_n - x_m|| \leq ||x_n - \bar{x}|| + ||x_m - \bar{x}|| < \varepsilon/2 + \varepsilon/2 = \varepsilon\ .$$

Definition 1.3. A normed space E is called complete space if every Cauchy sequence converges to some limit belonging to E.

Examples: $\mathbb{R}$ is complete. $\mathbb{Q}$ is not complete ($\mathbb{Q}$ is the field of rational numbers, $\{\frac{m}{l}|m, l \in \mathbb{Z}, l \neq 0\}$) because a sequence $(1 + \frac{1}{n})^n \to e$, where the terms $(1 + \frac{1}{n})^n$ are rational numbers while e is irrational. So this sequence is a Cauchy sequence, but its limit e is out of the space $\mathbb{Q}$, and so it is not a convergent sequence in $\mathbb{Q}$. This example shows that if some of Cauchy sequences have no limit we can “add” these missing limits and obtain a complete space: adding all irrational limits to $\mathbb{Q}$ we pass to $\mathbb{R}$, which is complete. The only problem is: how could we define these missing limits “living in $\mathbb{Q}$”, so that we a priori don’t know, what is an irrational number.

In order to define these missing limits staying inside an incomplete normed space E we will use a special algorithm called completion of E. Namely, we say that two Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ belong to the same equivalence class $\tilde{x}$ if $\lim\limits_{n\to\infty} ||x_n - y_n|| = 0$. Then the set of all Cauchy sequences is presented as a partition of the equivalence classes: every Cauchy sequence belongs to some equivalence class, and it cannot belong to two different classes. Indeed, if $\tilde{x}$, $\tilde{y}$ are some equivalence classes and if a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \in \tilde{x}$ and $(x_n)_{n\in\mathbb{N}} \in \tilde{y}$ then for **all** sequences $(x'_n)_{n\in\mathbb{N}} \in \tilde{x}$ and **all** sequences $(y'_n) \in \tilde{y}$ we have:

$$||x'_n - y'_n|| \leq ||x'_n - x_n|| + ||x_n - y'_n|| \to 0 \; .$$

So, $\tilde{x} = \tilde{y}$.
Every equivalence class can be associated to some sequence of this class, called a representant. In particular, we consider the equivalence classes $\tilde{c}$ containing constant sequences $(c)_{n\in\mathbb{N}} = (c, c, c, ...)$. These constant equivalence classes are in one-to-one correspondence with the elements of the normed space E.

Denote $\tilde{E}$ the set of equivalence classes, and $\tilde{E}_c$ the set of constant equivalence classes. We see that $\tilde{E}_c \sim E$ (equivalent to E).
Let us define now the operations of addition and scalar multiplication in $\tilde{E}$:

- $\tilde{x} + \tilde{y}$ is the equivalence class containing $(x_n + y_n)_{n\in\mathbb{N}}$, where $(x_n)_{n\in\mathbb{N}} \in \tilde{x}$ and $(y_n)_{n\in\mathbb{N}} \in \tilde{y}$.
- $\alpha\tilde{x}$ is the class containing $(\alpha x_n)_{n\in\mathbb{N}}$, where $\alpha \in \mathbb{R}$ or $\mathbb{C}$ and $(x_n)_{n\in\mathbb{N}} \in \tilde{x}$.

One can check that these operations satisfy axioms A1 – A8 with $\tilde{0}$ containing the zero sequence.
Then we introduce a norm in $\tilde{E}$: if $(x_n)_{n\in\mathbb{N}} \in \tilde{x}$ then $||\tilde{x}||_{\tilde{E}} = \lim\limits_{n\to\infty} ||x_n||$.[2] One can easily check that it satisfies the axioms N1 – N3 and is stable with respect to the choice of the representant $(x_n)_{n\in\mathbb{N}}$.

[2] Clearly, if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then $(||x_n||)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $\mathbb{R}$, and so it converges.

Indeed, if $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$ are two representants then

$$\lim_{n\to+\infty}||x_n|| = \lim_{n\to+\infty}||x'_n + (x_n - x'_n)|| = \lim_{n\to+\infty}||x'_n|| \, .$$

It can be proved that $\tilde{E}$ is complete, that $\tilde{E}_c$ (a "copy" of E) is dense in $\tilde{E}$ and is isometric to E (so that $||\tilde{c}||_{\tilde{E}} = ||c||_E$). So, the elements of $\tilde{E}$ are considered as the "missing" limits.

Definition 1.4. A normed space E is called separable space if it contains a countable dense subset.
Recall that a countable set is in one to one correspondance to the set $\mathbb{N}$. The example of a countable set: $\mathbb{Q}$. The example of a separable space: $\mathbb{R}$.
Note that a complete normed space is called Banach space.

1.3 Inner product spaces

Definition 1.5. A vector space E is called inner product space, or **pre-Hilbert space**, or Hausdorff pre-Hilbert space, if it is supplied with an inner product $(\cdot,\cdot)$ which is a map $E \times E \to \mathbb{K}$ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) satisfying the following three axioms:

I1. Linearity in the first argument:

$$\forall \alpha, \beta \in \mathbb{K}, \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in E \ ,$$

$$(\alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w}) = \alpha(\mathbf{u}, \mathbf{w}) + \beta(\mathbf{v}, \mathbf{w}) \ \ .$$

I2. Symmetry or conjugate symmetry:

$$\forall \mathbf{x}, \mathbf{y} \in E \ , \ \ (\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x}) \ \ (\text{case } \mathbb{K} = \mathbb{R}) \ \ ,$$

or

$$\forall \mathbf{x}, \mathbf{y} \in E \ , \ \ (\mathbf{x}, \mathbf{y}) = (\overline{\mathbf{y}, \mathbf{x}}) \ \ (\text{case } \mathbb{K} = \mathbb{C}) \ \ .$$

I3. $\forall \mathbf{x} \in E$, $(\mathbf{x}, \mathbf{x})$ is real and $(\mathbf{x}, \mathbf{x}) \geq \mathbf{0}$.
$(\mathbf{x}, \mathbf{x}) = \mathbf{0}$ iff $\mathbf{x} = \mathbf{0}_E$.

An inner product space is supplied with the norm

$$||\mathbf{x}|| = \sqrt{(\mathbf{x}, \mathbf{x})} \quad .$$

Example. $\mathbb{R}^d$ is an inner product space with $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{d} x_i y_i$. For $\mathbb{C}^d$ the inner product is defined by the formula

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{d} x_i \bar{y}_i \quad .$$

Let us recall the main **properties** of an inner product.

P1. Cauchy–Bunyakovsky–Schwarz (CBS) inequality.

$$\forall \mathbf{x}, \mathbf{y} \in E \ , \quad |(\mathbf{x}, \mathbf{y})| \leq ||\mathbf{x}||||\mathbf{y}|| \quad .$$

Proof.

For any $\lambda \in \mathbb{K}$, $\mathbf{x}, \mathbf{y} \in E$, $(\mathbf{x} - \lambda\mathbf{y}, \mathbf{x} - \lambda\mathbf{y}) \geq 0$, i.e.

$$(\mathbf{x}, \mathbf{x}) - \lambda(\mathbf{y}, \mathbf{x}) - \bar{\lambda}(\mathbf{x}, \mathbf{y}) + (\lambda\mathbf{y}, \lambda\mathbf{y}) \geq 0 \quad .$$

Setting $\lambda = \frac{(\mathbf{x},\mathbf{y})}{(\mathbf{y},\mathbf{y})}$ (for $\mathbf{y} \neq \mathbf{0}_E$), we get

$$\begin{aligned} (\mathbf{x}, \mathbf{x}) \quad &- \quad \frac{(\mathbf{x},\mathbf{y})(\overline{\mathbf{x},\mathbf{y}})}{(\mathbf{y},\mathbf{y})} - \frac{(\overline{\mathbf{x},\mathbf{y}})(\mathbf{x},\mathbf{y})}{(\mathbf{y},\mathbf{y})} \\ &+ \quad \frac{\left|(\mathbf{x},\mathbf{y})\right|^2}{(\mathbf{y},\mathbf{y})} \geq 0 \ , \text{ i.e.} \\ (\mathbf{x}, \mathbf{x}) \quad &- \quad \frac{\left|(\mathbf{x},\mathbf{y})\right|^2}{(\mathbf{y},\mathbf{y})} \geq 0 \ , \text{ and so } , \end{aligned}$$

$$\left|(\mathbf{x}, \mathbf{y})\right|^2 \leq ||\mathbf{x}||^2||\mathbf{y}||^2 \quad .$$

The case $\mathbf{y} = \mathbf{0}_E$ is evident.
The assertion is proved.

P2. Continuity of the inner product.
If $x_n \to \bar{x}$, $y_n \to \bar{y}$, then

$$(x_n, y_n) \to (\bar{x}, \bar{y}) \ .$$

Proof.

$$\begin{array}{lcl} \big|(x_n, y_n) - (\bar{x}, \bar{y})\big| & = & \big|(x_n - \bar{x}, y_n) + (\bar{x}, y_n - \bar{y})\big| \\ \leq \big|(x_n - \bar{x}, y_n)\big| & + & \big|(\bar{x}, y_n - \bar{y})\big| \\ \leq ||x_n - \bar{x}||\ ||y_n|| & + & ||\bar{x}||\ ||y_n - \bar{y}|| \ . \end{array}$$

The right-hand side tends to zero because

$$||x_n - \bar{x}|| \to 0 \ , \ \ ||y_n|| \to ||\bar{y}|| \ , \ \ ||y_n - \bar{y}|| \to 0 \ .$$

P3. Parallelogram identity.

$$\forall \mathbf{x}, \mathbf{y} \in E \ , \quad ||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 = 2\left(||\mathbf{x}||^2 + ||\mathbf{y}||^2\right) \quad .$$

Proof.

$$\begin{aligned} ||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 &= (\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}) \\ &= (\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{x}) + (\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{y}) \\ &+ (\mathbf{x}, \mathbf{x}) - (\mathbf{y}, \mathbf{x}) - (\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{y}) \\ &= 2||\mathbf{x}||^2 + 2||\mathbf{y}||^2 \quad . \end{aligned}$$

The completion of an inner product space is compatible with the inner product:

$$(\tilde{x}, \tilde{y})_{\tilde{E}} = \lim_{n \to +\infty} (x_n, y_n) \quad ,$$

where $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are representants of $\tilde{x}$ and $\tilde{y}$ respectively.

Definition 1.6. A complete inner product space E is called Hilbert space. We will consider separable Hilbert spaces.

1.4 Linear operators

Let us introduce linear operators. Let E, F be normed spaces. A mapping $A : E \to F$ is called linear operator
if $\forall \alpha, \beta \in \mathbb{K}, \mathbf{u}, \mathbf{v} \in E$,

$$A(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha A\mathbf{u} + \beta A\mathbf{v} \quad .$$

Linear operator A is called bounded operator if
$\exists M \geq 0$, $\forall \mathbf{u} \in E$, $||A\mathbf{u}||_F \leq M||\mathbf{u}||_E$.
Linear operator A is called continuous operator if for any convergent sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ of the space E, the sequence of the images $(A\mathbf{x}_n)_{n\in\mathbb{N}}$ converges to $A\mathbf{x}_0$, where $\mathbf{x}_0$ is the limit $(\mathbf{x}_n)_{n\in\mathbb{N}}$.

Theorem 1.5. A is continuous iff it is bounded.

Proof.

1. Assume that A is **bounded**. Let $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence converging to $\mathbf{x}_0$. So $\exists M \geq 0$ such that $||A(\mathbf{x}_n - \mathbf{x}_0)||_F \leq M||\mathbf{x}_n - \mathbf{x}_0||_E$. Taking into account that $||\mathbf{x}_n - \mathbf{x}_0||_E \to 0$ we derive that $A\mathbf{x}_n \to A\mathbf{x}_0$. So A is **continuous**.

2. Assume now that A is continuous. If A is not bounded then there exists a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in E such that $||\mathbf{x}_n||_E = 1$ and $||A\mathbf{x}_n|| > n$.
Consider the sequence $(\frac{1}{n}\mathbf{x}_n)_{n\in\mathbb{N}} \to \mathbf{0}_E$. However $||A\left(\frac{1}{n}\mathbf{x}_n\right)||_F > \frac{n}{n} = 1$. So $A(\frac{1}{n}\mathbf{x}) \nrightarrow A\mathbf{0}_E = \mathbf{0}_E$. It is incompatible with the continuity of A. So A cannot be unbounded. Theorem is proved.
If A is a bounded operator, we can define its norm

$$||A||_{\mathcal{L}(E,F)} = \sup_{\mathbf{x}\neq\mathbf{0}_E} \frac{||A\mathbf{x}||_F}{||\mathbf{x}||_E} = \sup_{\mathbf{x}:||\mathbf{x}||_E=1} ||A\mathbf{x}||_F \quad .$$

One can easily check that it satisfies axioms N1 – N3.
If $F = \mathbb{R}$ or $\mathbb{C}$ then A is called linear form or linear functional.

Theorem 1.6. (Riesz–Fréchet representation theorem.)
Let H be a Hilbert space. For every continuous linear functional $\varphi : H \to \mathbb{K}$ there exists a unique element $\mathbf{y} \in H$ such that for all $\mathbf{x} \in H$

$$\varphi(\mathbf{x}) = (\mathbf{x}, \mathbf{y})$$

and moreover

$$||\mathbf{y}||_H = ||\varphi||_{\mathcal{L}(H,\mathbb{K})} \, .$$

The space of continuous linear functionals is called **dual to** H and is denoted H^*.

Definition 1.7. A sequence $(\mathbf{x}_n) \in \mathbb{N}$ is called weakly convergent in H to the element $\mathbf{x}_0$ if for all continuous linear functionals $\varphi \in H^*$,

$$\varphi(\mathbf{x}_n) \to \varphi(\mathbf{x}_0) \; .$$

It means that for all elements $\mathbf{y} \in H$,

$$(\mathbf{x}_n, \mathbf{y}) \to (\mathbf{x}_0, \mathbf{y}) \; .$$

Clearly, every convergent sequence weakly converges to its limit. The inverse assertion is not true.

Theorem 1.7. (**Banach–Alaoglu theorem for a separable Hilbert space**.)

The closed unit ball $\overline{B(\mathbf{0}_H, 1)}$ in a separable Hilbert space H is relatively weakly compact. It means that every sequence in $\overline{B(\mathbf{0}_H, 1)}$ has a weakly convergent subsequence.

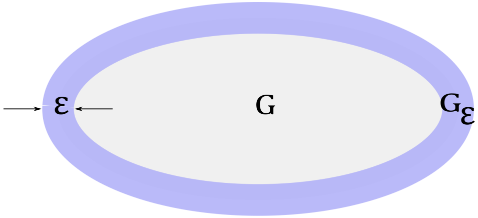
2.1 Auxiliary spaces

Let G be a bounded domain in $\mathbb{R}^d$. Let us introduce some vector spaces.
1) $C(\bar{G})$ is the set of continuous functions $\bar{G} \to \mathbb{R}$ (extendable to $C(\bar{B})$, where B is a ball in $\mathbb{R}^d$ containing $\bar{G}$).
2) $C^{(k)}(\bar{G})$ is the set of k times continuously differentiable functions $\bar{G} \to \mathbb{R}$, also extendable to $C^{(k)}(\bar{B})$.
Remark. For $G \subset B$, $\tilde{u}$: $B \to \mathbb{R}$ is an extension of a function u: $\bar{G} \to \mathbb{R}$ if $\forall x \in \bar{G}$, $\tilde{u}(x) = u(x)$. In this case u is called restriction of $\tilde{u}$.
3) $C_0(G)$, $C_0^{(k)}(G)$ are the subspaces of $C(\bar{G})$ and $C^{(k)}(\bar{G})$ such that their functions vanish in some neighborhood G_ε of the boundary ∂G:

$$G_\varepsilon = \{x \in G | \mathrm{dist}(x, \partial G) < \varepsilon\}$$

with some $\varepsilon > 0$ depending on the function from $C_0(G)$ or $C_0^{(k)}(G)$. Such function is called compact support function, i.e. $\overline{\mathrm{supp}\ f} \subset G$, where $\mathrm{supp}\ f = \{x \in G | f(x) \neq 0\}$. Here $\mathrm{dist}\ (x, A)$ is a distance between the point x and the set A. It is defined as follows:

$$\mathrm{dist}\ (x, A) = \inf_{y \in A} \|x - y\|_2 \ .$$



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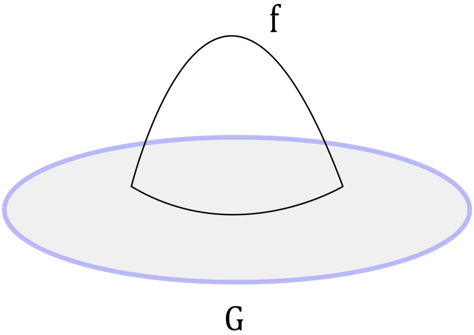

f
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Let us recall the definition of L^p space, $1 \le p < +\infty$ (Lebesgue spaces). For a bounded domain G, $L^p(G)$ is the vector space of functions $f : G \to \mathbb{R}$ having finite Lebesgue's integral

$$\int_G |f(x)|^p dx \quad .$$

$L^p(G)$ is supplied with the norm

$$\|f\|_{L^p(G)} = \left(\int_G |f(x)|^p dx \right)^{1/p} \quad . \tag{2.2.1}$$

$L^p(G)$ can be equivalently defined as a completion of $C(\bar{G})$ or $C_0^{(\infty)}(G)$ with respect to the norm (2.2.1).
So, $L^p(G)$ is a Banach space such that every function $f \in L^p(G)$ is a limit of a sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n \in C(\bar{G})$ or $C_0^{(\infty)}(G)$.

According to the Weierstrass theorem, every continuous function is the limit of a sequence of polynomials in the sense of $\|\cdot\|_{C(\bar{G})}$-norm, and so, also in the sense of L^p-norm, because

$$\begin{aligned}\|f\|_{L^p(G)} &= \left(\int_G |f(x)|^p dx\right)^{1/p} \leq \left(\int_G \max_{x\in\bar{G}}|f(x)|^p dy\right)^{1/p} \\ &= \max_{x\in\bar{G}}|f(x)|(\mathrm{mes}G)^{1/p} = \|f\|_{C(\bar{G})}(\mathrm{mes}G)^{1/p} \ .\end{aligned}$$

Also, every polynomial is the limit of a sequence of polynomials with rational coefficients.

So, the set $\mathbb{Q}[X]$ of polynomials with rational coefficients is dense in L^p. Thus, $L^p(G)$ is a separable space.

If $p = 2$, we can introduce an inner product in $L^2(G)$:

$$(f, g) = \int_G f(x)g(x)dx \ , \tag{2.2.2}$$

so $L^2(G)$ is a separable Hilbert space.

Finally let us define $L^\infty(G)$ as the space of functions $f : G \to \mathbb{R}$ such that $\exists M \geq 0$ such that $|f(x)| \leq M$ almost everywhere, i.e. for all $x \in G$ except for a subset E having measure zero. This space is supplied with the L^∞-norm:

$$\|f\|_{L^\infty(G)} = \mathrm{vrai}\ \max_{x \in G} |f(x)|$$

called also $\operatorname{ess\,sup}_{x\in G} |f(x)|$ which is

$$\inf_{E \subset G, \mathrm{mes}\ E = 0} \left\{ \sup_{x \in G \setminus E} |f(x)| \right\} .$$

$L^\infty(G)$ is a non-separable Banach space.

2.2 Sobolev space H^1

Consider a bounded domain G with Lipschitz boundary ∂G. It means that ∂G can be covered by a finite set of open bounded sets $O_1, ..., O_N$ such that in each O_i in some local coordinate system obtained from the original one by rotations, $\partial G \cap O_i$ is a graph of a Lipschitz function. (f is called L - Lipschitz on G if $\forall x, y \in G$, $|f(x) - f(y)| \leq L\|x - y\|_2$.)
Similarly we define $C^{(k)}$-smooth boundary.

Definition 2.1. The Sobolev space $H^1(G)$ is a completion of $C^{(\infty)}(\bar{G})$ with respect to the inner product

$$(f, g) = \int_G (f(x)g(x) + \nabla f(x) \cdot \nabla g(x))\, dx \ . \qquad (2.2.3)$$

Arguing as for L^p we can prove that $H^1(G)$ is separable Hilbert space. Note that in the definition $C^{(\infty)}(\bar{G})$ can be replaced by $C^{(1)}(\bar{G})$. The norm in $H^1(G)$ is defined as

$$\begin{aligned} \|f\|_{H^1(G)} &= \sqrt{(f,f)} = \sqrt{\int_G |f(x)|^2 + \|\nabla f\|_2^2 dx} \\ &= \sqrt{\|f\|^2_{L^2(G)} + \|\nabla f\|^2_{L^2(G)}}\,. \end{aligned}$$

So, $\|f\|_{H^1(G)} \geq \|f\|_{L^2(G)}$.

Definition 2.2. $H_0^1(G)$ is a completion of $C_0^{(\infty)}(G)$ with respect to the inner product (2.2.3).
So $H_0^1(G)$ is a subspace of $H^1(G)$ "vanishing at the bord".

Note that definition 2.2 is still valid for any bounded domain G.

Let f be a function of $H^1(G)$. Then there exists a sequence $f_n \in C^{(\infty)}(\bar{G})$ convergent to f (because $C^{(\infty)}(\bar{G})$ is dense in $H^1(G)$). This sequence is a Cauchy sequence in $H^1(G)$ and so, $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N}$ such that $\forall n$, $m \geq n_0$,

$$\sqrt{\|f_n - f_m\|^2_{L^2(G)} + \sum_{i=1}^{d} \|\frac{\partial f_n}{\partial x_i} - \frac{\partial f_m}{\partial x_i}\|^2_{L^2(G)}} < \varepsilon \quad .$$

So, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(G)$ and $\left(\frac{\partial f_n}{\partial x_i}\right)_{n\in\mathbb{N}}$ for all $i = 1, ..., d$ are as well Cauchy sequences in $L^2(G)$. As $L^2(G)$ is complete, these sequences have limits: $f_n \to f$, $\frac{\partial f_n}{\partial x_i} \to w_i$, f, $w_i \in L^2(G)$. Functions w_i are called weak partial derivatives of f, denoted $\frac{\partial f}{\partial x_i}$.

Theorem 2.1. $\forall f \in H^1(G)$, $v \in C_0^{(\infty)}(G)$, $\forall i = 1, ..., d$

$$\int\limits_G f(x) \frac{\partial v}{\partial x_i}(x) dx = -\int\limits_G \frac{\partial f}{\partial x_i}(x) v(x) dx \quad . \tag{2.2.4}$$

Proof.
Let $(f_n)_{n \in \mathbb{N}}$ be a sequence convergent to f in $H^1(G)$. For every n, integrating by parts and taking into consideration that v vanishes in the neighborhood of ∂G, we get:

$$\int\limits_G f_n(x) \frac{\partial v}{\partial x_i}(x) dx = -\int\limits_G \frac{\partial f_n}{\partial x_i}(x) v(x) dx \quad .$$

Passing to the limit in this equality, we get (2.2.4).

Remark 2.1. Let v be a function of $H_0^1(G)$. Then, considering a sequence $(v_n)_{n \in \mathbb{N}}$ in $C_0^{(\infty)}(G)$ convergent to v and passing to the limit we prove that (2.2.4) is still valid for $v \in H_0^1(G)$.

Theorem 2.2. If $f \in H^1(G)$ and $w_i \in L^2(G)$ such that $\forall v \in H_0^1(G)$,

$$\int\limits_G f(x) \frac{\partial v}{\partial x_i} dx = -\int\limits_G w_i(x) v(x) dx \ .$$

Then $w_i = \frac{\partial f}{\partial x_i}$ (so $\frac{\partial f}{\partial x_i}$ is uniquely defined).

Proof.
Theorem 2.1 yields

$$\int\limits_G f(x) \frac{\partial v}{\partial x_i}(x) dx = -\int\limits_G \frac{\partial f}{\partial x_i} v(x) dx \ .$$

So,

$$\int\limits_G \left(w_i - \frac{\partial f}{\partial x_i} \right) v(x) dx = 0$$

for all $v \in H_0^1(G)$. $H_0^1(G)$ is dense in $L^2(G)$ because $C_0^{(\infty)}(G)$ is dense in $L^2(G)$. So, $w_i - \frac{\partial f}{\partial x_i}$ is orthogonal to all functions of $L^2(G)$. So, $w_i - \frac{\partial f}{\partial x_i} = 0$.

Comment. Theorems 2.1, 2.2 show that the definition of the weak partial derivatives $\frac{\partial f}{\partial x_i}$ is equivalent to the following one: $w_i \in L^2(G)$ is a weak partial derivative $\frac{\partial f}{\partial x_i}$ of the function $u \in L^2(G)$ if and only if $\forall v \in C_0^{(\infty)}(G)$,

$$\int_G f(x) \frac{\partial v}{\partial x_i}(x) dx = -\int_G w_i v(x) dx \quad .$$

This definition allows us to introduce the Sobolev space $H^1(G)$ as the space of functions of $L^2(G)$ such that for all $i = 1, ..., d$ they have weak partial derivatives $\partial/\partial x_i$ belonging to $L^2(G)$.
For the functions of $H^1(G)$ with Lipschitz ∂G one can introduce the notion of the trace of a function on ∂G. Let us show it in the case of G-parallelepiped

$$\Pi = \{x = (x_1, ..., x_d) | 0 < x_i < l_i \ , \ \ i = 1, ..., d\} \quad .$$

Denote $x' = (x_2, ..., x_d)$,

$$\Gamma = \{x_1 = 0,\ 0 < x_i < l_i\ ,\ \ i = 2, ..., d\}\ \ ,$$

a part of ∂G. Let u be a function of $H^1(\Pi)$, define the trace of u on Γ.

Lemma 2.1. Let $u \in C^{(\infty)}(\bar{\Pi})$, then $\forall \delta > 0$, $\delta < l_1$,

$$\int\limits_{\Gamma} u^2(0, x')dx' \leq \frac{2}{\delta} \int\limits_{Q_\delta(\Gamma)} u^2(x)dx + \delta \int\limits_{Q_\delta(\Gamma)} (\partial u/\partial x_1)^2 dx\ ,$$

$$Q_\delta(\Gamma) = \{x \in \Pi | x_1 \in (0, \delta)\}\ \ .$$

Proof.

$$\int\limits_{\Gamma} u^2(0,x')dx' = \frac{1}{\delta} \int\limits_{Q_\delta(\Gamma)} \left[-u(x) + \int\limits_0^{x_1} \frac{\partial u(\tau,x')}{\partial \tau} d\tau \right]^2 dx$$

$$\leq \frac{2}{\delta} \int\limits_{Q_\delta(\Gamma)} u^2(x)dx + \frac{2}{\delta} \int\limits_{\Gamma} \left\{ \int\limits_0^{\delta} \left(\int\limits_0^{x_1} \frac{\partial u(\tau,x')}{\partial \tau} d\tau \right)^2 dx_1 \right\} dx'$$

$$\leq \frac{2}{\delta} \int\limits_{Q_\delta(\Gamma)} u^2(x)dx + \frac{2}{\delta} \int\limits_{\Gamma} \left\{ \int\limits_0^{\delta} x_1 \int\limits_0^{\delta} \left(\frac{\partial u(\tau,x')}{\partial \tau} \right)^2 d\tau dx_1 \right\} dx' \quad ,$$

where we used the CBS inequality

$$\left(\int\limits_0^{x_1} \varphi(\tau)d\tau \right)^2 = \left(\int\limits_0^{x_1} 1\varphi(\tau)d\tau \right)^2 \leq \int\limits_0^{x_1} 1^2 d\tau \cdot \int\limits_0^{x_1} (\varphi(\tau))^2 d\tau \leq x_1 \int\limits_0^{\delta} (\varphi(\tau))^2 d\tau$$

and Young's inequality $(A+B)^2 \leq 2A^2 + 2B^2$.

So,

$$\int\limits_{\Gamma} u^2(0,x')dx' \le \frac{2}{\delta} \int\limits_{Q_\delta(\Gamma)} u^2(x)dx + \delta \int\limits_{Q_\delta(\Gamma)} \left(\frac{\partial u}{\partial x_1}\right)^2 dx\ .$$

This lemma shows that every Cauchy sequence $(u_n)_{n\in\mathbb{N}} \in C^{(\infty)}(\bar{\Pi})$ generates the sequence of traces $(u_n(0,x'))_{n\in\mathbb{N}}$ which is a Cauchy sequence in $L^2(\Gamma)$ and so, it converges to some function of $L^2(\Gamma)$.

Now, let $(u_n)_{n\in\mathbb{N}}$ be convergent to $u \in H^1(\Pi)$, denote $u|_\Gamma$ the function of $L^2(\Gamma)$ which is the limit of the Cauchy sequence of traces $(u_n(0,x'))_{n\in\mathbb{N}}$. This function $u|_\Gamma$ is called the trace of the function u.

§3 Poincaré's inequalities

3.1 Poincaré–Friedrichs inequality

Theorem 3.1 Poincaré–Friedrichs inequality. There exists $C_{\mathrm{PF}} \geq 0$ such that $\forall u \in H_0^1(G)$,

$$\|u\|_{L^2(G)} \leq C_{\mathrm{PF}} \|\nabla u\|_{L^2(G)} \ , \tag{2.3.1}$$

where $\|\nabla u\|_{L^2(G)} = \sqrt{\int\limits_G \sum\limits_{i=1}^{d} \left(\frac{\partial u}{\partial x_i}\right)^2 dx}$,

$$C_{\mathrm{PF}} = \frac{\mathrm{diam}\ G}{\sqrt{2}} \ ,$$

$(\mathrm{diam}\ G = \sup_{x,y \in G} |x - y|)$. This inequality is called Poincaré–Friedrichs (PF) inequality.

Proof.

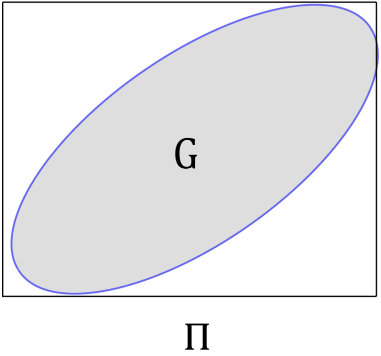
1) Let $\Pi \supset G$ be the parallelepiped

$$\Pi = (a_1, b_1) \times ... \times (a_d, b_d) \text{ and } \forall i = 1, ..., d,\ b_i - a_i \leq \text{diam } G \ .$$

Without loss of generality assume $a_1 = 0$.

Consider $u \in C_0^{(\infty)}(\bar{\Pi})$. Then

$$u(x_1, x_2, ..., x_d) = \int\limits_0^{x_1} \frac{\partial u}{\partial \tau}(\tau, x')d\tau \ , \quad x' = (x_2, ..., x_n) \ .$$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So, applying the CBS inequality, we obtain

$$\begin{array}{rcl}
\|u\|^2_{L^2(\Pi)} & = & \int\limits_{\Pi} \left(\int\limits_0^{x_1} \frac{\partial u}{\partial \tau}(\tau, x') d\tau \right)^2 dx \\
& \leq & \int\limits_{\Pi} \left\{ \int\limits_0^{x_1} 1^2 dx_1 \int\limits_0^{x_1} \left(\frac{\partial u}{\partial \tau}(\tau, x') \right)^2 d\tau \right\} dx_1 dx' \\
& \leq & \int\limits_{a_2}^{b_2} \ldots \int\limits_{a_d}^{b_d} \int\limits_{a_1}^{b_1} x_1 \int\limits_{a_1}^{b_1} \left(\frac{\partial u}{\partial \tau}(\tau, x') \right)^2 d\tau dx_1 dx' \\
& = & \frac{(b_1 - a_1)^2}{2} \int_{\Pi} \left(\frac{\partial u}{\partial x_1} \right)^2 dx \leq C^2_{\mathrm{PF}} \|\nabla u\|^2_{L^2(\Pi)} \ .
\end{array}$$

2) Let u be a function of $C_0^{(\infty)}(G)$. Extend it by zero to $\Pi \backslash G$. The extension $\tilde{u} \in C_0^{(\infty)}(\Pi)$ and we obtain for it (2.3.2). On the other hand, $\|\tilde{u}\|_{L^2(\Pi)} = \|u\|_{L^2(G)}$ and $\|\nabla \tilde{u}\|_{L^2(\Pi)} = \|\nabla u\|_{L^2(G)}$. So, we get (2.3.1) for $u \in C_0^{(\infty)}(G)$.

3) Let u be a function of $H_0^1(G)$. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of functions of $C_0^{(\infty)}(G)$ convergent to u; for all n,

$$\|u_n\|_{L^2(G)} \leq C_{\mathrm{PF}} \|\nabla u_n\|_{L^2(G)} \ .$$

Passing to the limit and using the continuity of a norm, we get (2.3.1).

The theorem is proved.

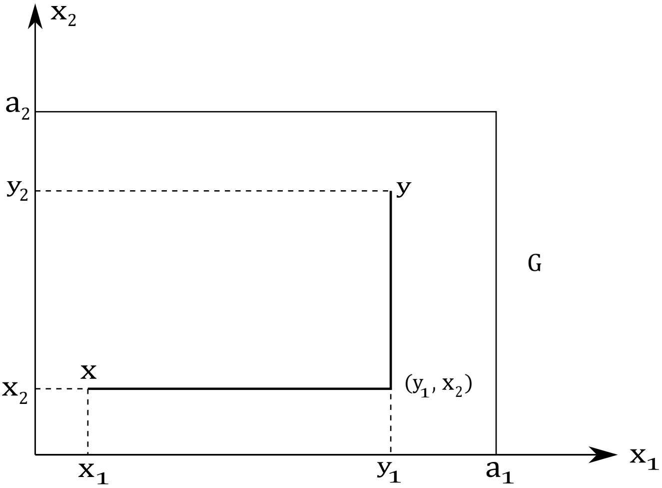
3.2 Poincaré's inequality in a parallelepiped

Let G be a parallelepiped $(0, a_1) \times ... \times (0, a_d) \subset \mathbb{R}^d$.

Theorem 3.2. Poincaré's inequality. There exists a positive constant C_P such that $\forall u \in H^1(G)$,

$$\|u\|^2_{L^2(G)} \leq \frac{1}{\operatorname{mes}(G)} \left(\int_G u(x)dx \right)^2 + C_P \|\nabla u\|^2_{L^2(G)} \quad .$$

Proof.



1. Consider $u \in C^{(\infty)}(\bar{G}), x, y \in G$. Then

$$\begin{aligned}(u(y)-u(x))^2 &= \left(\int\limits_{x_1}^{y_1} \frac{\partial u}{\partial \theta_1}(\theta_1, x_2, ..., x_d)d\theta_1 + ... + \int\limits_{x_d}^{y_d} \frac{\partial u}{\partial \theta_d}(y_1, ..., y_{d-1}, \theta_d)d\theta_d\right)^2 \\ &\leq d\left\{\left(\int\limits_{x_1}^{y_1} \frac{\partial u}{\partial \theta_1}d\theta_1\right)^2 + ... + \left(\int\limits_{x_d}^{y_d} \frac{\partial u}{\partial \theta_d}d\theta_d\right)^2\right\}\end{aligned}$$

(by Cauchy–Schwarz inequality $(a_1 + ... + a_d)^2 \leq d(a_1^2 + ... + a_d^2)$), and so it is smaller than (CSB):

$$d\left\{a_1 \int\limits_0^{a_1} \left(\frac{\partial u}{\partial \theta_1}\right)^2 d\theta_1 + ... + a_d \int\limits_0^{a_d} \left(\frac{\partial u}{\partial \theta_d}\right)^2 d\theta_d\right\} .$$

Integrating the inequality in $x, y \in G$, we get:

$$\int_G \int_G (u(y) - u(x))^2 dx dy$$
$$\leq d \left\{ a_1^2 \text{ mes } G \int_G \left(\frac{\partial u}{\partial x_1}(x) \right)^2 dx + ... + a_d^2 \text{ mes } G \int_G \left(\frac{\partial u}{\partial x_d}(x) \right)^2 dx \right\}$$
$$\leq d \text{ mes } G \max_{1 \leq j \leq d} a_j^2 \|\nabla u\|^2_{L^2(G)} \quad .$$

The left-hand side is:

$$\int_G \int_G (u(y))^2 dx dy - 2 \int_G \int_G u(y) u(x) dx dy + \int_G \int_G (u(x))^2 dx dy$$
$$= 2 \text{ mes } G \|u\|^2_{L^2(G)} - 2 \left(\int_G u(x) dx \right)^2 \quad .$$

Finally we get

$$\|u\|^2_{L^2(G)} \leq \frac{1}{\mathrm{mes}\ G}\left(\int_G u(x)dx\right)^2 + \frac{d}{2}\max_{1\leq j\leq d} a_j^2 \|\nabla u\|^2_{L^2(G)}\ .$$

So, $C_P = \frac{d}{2}\max\limits_{1\leq j\leq d} a_j^2$.

2. Consider $u \in H^1(G)$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that $u_n \in C^{(\infty)}(\bar{G})$ and $u_n \to u$ in H^1-norm. Then we get:

$$\|u_n\|_{L^2(G)} \to \|u\|_{L^2(G)}\ ,\ \ \|\nabla u_n\|_{L^2(G)} \to \|\nabla u\|_{L^2(G)}\ .$$

In the first part of the proof we have obtained:

$$\|u_n\|^2_{L^2(G)} \leq \frac{1}{\text{mes }(G)} \left(\int\limits_G u_n(x)dx \right)^2 + C_P \|\nabla u_n\|^2_{L^2(G)} \ .$$

The continuity of a norm yields:

$$\|u_n - u\|^2_{L^2(G)} + \|\nabla u_n - \nabla u\|^2_{L^2(G)} = \|u_n - u\|^2_{H^1(G)} \to 0 \ ,$$

and so,

$$\Big| \int\limits_G u_n dx - \int\limits_G u dx \Big| = \Big| \int\limits_G (u_n - u) 1 dx \Big| \leq \sqrt{\int\limits_G 1^2 dx} \|u_n - u\|_{L^2(G)} \to 0 \ .$$

So, finally, $\int\limits_G u_n dx$ also converges to $\int\limits_G u(x)dx$.

Theorem is proved.

§4 Stationary conductivity equation

Consider the following problem:

$$-\mathrm{div}(A(x)\nabla u) = f(x)\ , \quad x \in G\ , \tag{2.4.1}$$

$$u\big|_{\partial G} = 0\ \ , \tag{2.4.2}$$

where G is a bounded domain in $\mathbb{R}^d$, f is a function of $L^2(G)$ and

$$A(x) = (A_{ij}(x))_{1\le i,j\le d} \ \text{ is } \ d \times d$$

symmetric, positive definite matrix, satisfying
$(\mathbf{i})$ $\ \forall x \in G\ ,\ \ \forall i,j \in \{1,...,d\}\ ,\ \ A_{ij}(x) = A_{ji}(x)\ ,$
$(\mathbf{ii})$ $\ \exists \kappa > 0 : \forall x \in G\ ,\ \ \forall \boldsymbol{\xi} = (\xi_1,...,\xi_d)\ ,$

$$\sum_{i,j=1}^{d} A_{ij}(x)\xi_j\xi_i \ge \kappa \sum_{i=1}^{d} \xi_i^2\ .$$

Currently we assume that $A_{ij} \in C^{(1)}(\bar{G})$, however further we will define a weak solution, and it will be valid for any bounded measurable functions A_{ij}.
Namely, the notion of the classical solution, satisfying equation (2.4.1) in each point $x \in G$ and vanishing on ∂G, can be generalized for the case of non-smooth coefficients by means of a **weak formulation**.
Function $u \in H_0^1(G)$ is called a **weak solution** of problem (2.4.1), (2.4.2), if $\forall v \in C_0^{(\infty)}(G)$ the following integral identity holds:

$$\int\limits_G A(x)\nabla u \cdot \nabla v dx = \int\limits_G f(x)v(x)dx \quad . \qquad (2.4.3)$$

Evidently, if A_{ij} are smooth functions and u is a classical solution then multiplying equation (2.4.1) by a test function $v \in C_0^{(\infty)}(G)$ we integrate it over G, integrate the left-hand side by parts and obtain (2.4.3).

Note that (2.4.1) can be rewritten as

$$-\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}\right)=f(x)\ , \tag{2.4.1$'$}$$

and (2.4.3) as

$$\int\limits_G \sum_{i,j=1}^{d} A_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx=\int\limits_G f(x)v(x)dx\ . \tag{2.4.3$'$}$$

Note that due to the density of $C_0^{(\infty)}(G)$ in $H_0^1(G)$, (2.4.3$'$) is valid $\forall v\in H_0^1(G)$. So, weak solution is more general: it can be considered in the case of discontinuous coefficients.

Let us prove the following

Theorem 4.1. For any function $f \in L^2(G)$ a weak solution u exists, is unique and satisfies the inequality

$$\|u\|_{H^1(G)} \leq C_D \|f\|_{L^2(G)} \quad , \tag{2.4.4}$$

where the constant

$$C_D = \frac{\sqrt{2}}{\kappa} \max\{C_{PF}, C_{PF}^2\} \quad . \tag{2.4.5}$$

Proof.

1. Consider the space $H_0^1(G)$ supplied with a new inner product

$$[u, v] = \int_G \sum_{i,j=1}^{d} A_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx \; .$$

It is easy to check its linearity with respect to the first argument and symmetry. Moreover $[v, v] \geq 0$ for all $v \in H_0^1(G)$ because of the property **(ii)** of coefficients A_{ij}. If $[v, v] = 0$ then due to **(ii)** $\|\nabla v\|^2_{L^2(G)} = 0$, and then due to the PF inequality $\|v\|^2_{L^2(G)} = 0$. So, $v = 0$.

The norm $\big[|v|\big] = \sqrt{[v,v]}$ is equivalent to the norm $H^1(G)$, i.e. there exist constants $C_1, C_2 > 0$ such that for all $v \in H_0^1(G)$,

$$C_1 \|v\|_{H^1(G)} \leq \big[|v|\big] \leq C_2 \|v\|_{H^1(G)} \ .$$

Denote H the space $H_0^1(G)$ supplied with the inner product $[u,v]$.

2. Consider the linear functional Φ: $H \to \mathbb{R}$, such that

$$\Phi(v) = \int\limits_G f(x)v(x)dx \ \ .$$

Clearly, $\forall \alpha$, $\beta \in \mathbb{R}$, $\forall v_1, v_2 \in H$,

$$\Phi(\alpha v_1 + \beta v_2) = \alpha \Phi(v_1) + \beta \Phi(v_2) \ \ .$$

Φ is a bounded (i.e. continuous) functional. Indeed, $\forall v \in H$,

$$|\Phi(v)| \leq \|f\|_{L^2(G)} \|v\|_{L^2(G)}$$

due to CBS inequality, so

$$|\Phi(v)| \leq \|f\|_{L^2(G)} C_{PF} \|\nabla v\|_{L^2(G)} \ , \tag{2.4.6}$$

$$|\Phi(v)| \leq \|f\|_{L^2(G)} C_{PF} \frac{1}{\sqrt{\kappa}} \left[|v|\right] \tag{2.4.7}$$

(we applied the PF inequality and then **(ii)**).

3. So, due to the Riesz–Frechet theorem, there exists a unique $u \in H$ such that

$$\forall v \in H \ , \ \ \Phi(v) = [u, v] \ ,$$

i.e. $\forall v \in H_0^1(G)$ (2.4.3) holds, and the existence and uniqueness of a weak solution is proved.

4. Taking $v = u$, we get:

$$[u, u] = \Phi(u) \leq C_{PF} \frac{1}{\sqrt{\kappa}} \|f\|_{L^2(G)} \sqrt{[u, u]}$$

(see (2.4.7)). So,

$$\big[|u|\big] \leq \frac{C_{PF}}{\sqrt{\kappa}} \|f\|_{L^2(G)} \, .$$

On the other side,

$$\begin{array}{rcl} \big[|u|\big] = \sqrt{[u, u]} & \geq & \sqrt{\kappa} \sqrt{\|\nabla u\|^2_{L^2(G)}} = \sqrt{\kappa} \|\nabla u\|_{L^2(G)} \\ & \geq & \sqrt{\kappa} \frac{1}{C_{PF}} \|u\|_{L^2(G)} \, , \end{array}$$

and so,

$$\big[|u|\big] \geq \frac{\sqrt{\kappa}}{\sqrt{2}} \min\left\{1, \frac{1}{C_{PF}}\right\} \|u\|_{H^1(G)} \, .$$

So, the theorem is proved:

$$\|u\|_{H^1(G)} \leq \frac{\sqrt{2}}{\kappa} C_{PF} \max\{C_{PF}, 1\} \|f\|_{L^2(G)} \quad .$$

Remark 4.1. Consider formally equation (2.4.1) with the right-hand side

$$f(x) = f_0(x) - \sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i}(x) \quad , \tag{2.4.8}$$

where $f_0, f_i (i = 1, ..., d)$ belong to $L^2(G)$. Let us define a weak solution of the problem (2.4.1), (2.4.2) as a function $u \in H_0^1(G)$ such that $\forall v \in C_0^{(\infty)}$,

$$\int_G A(x) \nabla u \cdot \nabla v dx = \int_G f_0(x) v(x) + \sum_{i=1}^{d} f_i(x) \frac{\partial v}{\partial x_i}(x) dx \; . \tag{2.4.9}$$

Then Theorem 4.1 remains valid for this case and (2.4.4) is modified as

$$\|u\|_{H^1(G)} \leq C'_D \left\{ C_{PF}\|f_0\|_{L^2(G)} + \sqrt{\sum_{i=1}^{d} \|f_i\|^2_{L^2(G)}} \right\} ,$$

$$C'_D = \frac{\sqrt{2}}{\kappa} \max\{1, C_{PF}\} \, . \tag{2.4.10}$$

Remark 4.2. In the case of regular right-hand side f and piecewise smooth coefficients A_{ij} having discontinuities at some smooth surfaces Σ, the weak solution satisfies equation (2.4.1) everywhere out of Σ and the interface conditions

$$[u]_\Sigma = 0 \, , \quad \left[\sum_{i,j=1}^{d} A_{ij} \frac{\partial u}{\partial x_j} n_i \right]_\Sigma = 0$$

at Σ. If ∂G is smooth then the weak solution satisfies the boundary condition (2.4.2), and so the weak solution coincides with the classical one. The theorems providing this assertion are called Agmon–Duglis–Nirenberg (ADN) theorems.