4. Root finding

Last time

- Approximating functions by polynomials
- Taylor series
- Taylor polynomials + Lagrange remainder

Goal for today

- Methods for finding **roots** of functions
- Convergence of those methods

Solving equations

- Suppose we need to solve the equation $g(x) = h(x)$
- Write $f(x) := g(x) - h(x)$
- Equivalent to solve $f(x) = 0$ instead
- Solutions x^* such that $f(x^*) = 0$: **roots** or **zeros** of f
- We may want to find a single root or all roots

Roots of polynomials

- Suppose $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is **polynomial**
- How many roots does p have?
- **Fundamental theorem of algebra**:
  a degree-n polynomial has exactly n roots in $\mathbb{C}$
- So can factor p as

$$p(x) = (x - x_1)^{m_1} (x - x_2)^{m_2} \cdots (x - x_k)^{m_k}$$

  where x_i are the roots and $\sum_{i=1}^{k} m_i = n$
- m_i is the **multiplicity** of root x_i

Roots of polynomials II

- Some of the roots may be **multiple roots**
- E.g. $p = (x-1)^2(x^2-2)$ has roots 1, 1, $\sqrt{2}$ and $-\sqrt{2}$
- Roots may be complex even if coefficients a_i are real
- E.g. x^2+1 has roots $x_{1,2} = \pm i$ and *no* real roots

Can we find roots of polynomial analytically?

- Suppose p is a degree-n polynomial with coefficients a_i
- $n = 2$ (quadratic): exact formula for roots in terms of a_i
- $n = 3, 4$: more complicated formulae exist
- Abel–Ruffini theorem:
  for $n \geq 5$, in general ***no such formula exists****!*

Numerical methods for root finding

- Need **numerical methods** to **approximate** roots
- Some methods for *all* roots of a polynomial simultaneously, e.g. Aberth method (PS 2)
- Here we focus on methods to find single root

Fixed-point iteration

- Numerical method for roots must be **iterative**:

$$x_{n+1} = g(x_n)$$

  - algorithm g and initial guess x_0
- Produces a sequence $x_0, x_1, x_2, \ldots$
- *If* iteration converges, $x_n \to x^*$ as $n \to \infty$, then

$$g(x^*) = x^*$$

  provided g is **continuous**
- x^* is a **fixed point** of g; solves $f(x) := g(x) - x = 0$

Existence of fixed point

- If g is continuous and maps $[a, b]$ into itself, $\exists$ a fixed point.
- If $|g'(x)| \leq k \; \forall x \in [a, b]$ with $k < 1$, then unique.
- Called a **contraction mapping**

Dynamics of iterations

- Such an iteration is a **discrete-time dynamical system**
- Assume there is a fixed point x^*
- What condition do we need for (x_n) to converge?
- Look at distance $\delta_n := x_n - x^*$

Dynamics II

- We have

$$\delta_{n+1} = x_{n+1} - x^*$$

$$= g(x_n) - x^* = g(x^* + \delta_n) - x^* \simeq \delta_n g'(x^*)$$

- So (asymptotically) $\delta_{n+1} = \alpha\,\delta_n$ with $\alpha = g'(x^*)$
- $\delta_{n+1} = \alpha\,\delta_n \Rightarrow \delta_n = \alpha^n$
- Decays (**stable fixed point**) if $|\alpha| < 1$
- Grows (**unstable fixed point**) if $\alpha > 1$

Rate of convergence

- For $\sqrt{}$, Babylonian converges **faster** than bisection
- How can we make this precise?
- If $x_n \to x^*$ then $x_n - x^* \to 0$
- Estimate **distance** or **error** $|x_n - x^*|$ as function of n
- In bisection, error reduces by constant factor 2 at each step

Rate of convergence II

- If there are constants λ and α such that

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^\alpha} = \lambda$$

  then the iteration is of **order** α
- $\alpha = 1$: **linearly convergent**
- $\alpha = 2$: **quadratic convergence**
- Note that “linear” actually means that the sequence converges exponentially fast!

Newton method

- Want to solve $f(x) = 0$; have guess x_0
- Let $x = x_0 + \delta$; solve $f(x_0 + \delta) = 0$
- Taylor theorem: $f(x_0) + \delta f'(x_0) + \frac{1}{2}\delta^2 f''(\xi) = 0$
- So $\delta \simeq \frac{-f(x_0)}{f'(x_0)}$
- **Newton method**: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- **Quadratically convergent**!
- Babylon is special case

Newton method viewed as a fixed-point iteration

- How **design** quadratically-convergent algorithm for $f(x) = 0$?
- Look for fixed-point algorithm with $g(x) := x - \phi(x) f(x)$
- Impose $g'(x^*) = 0$ at root $f(x^*) = 0$
- $g'(x) = 1 - \phi'(x) f(x) - \phi(x) f'(x)$
- So $g'(x^*) = 1 - \phi(x^*) f'(x^*) = 0$
- So need $\phi(x^*) = \frac{1}{f'(x^*)}$
- Take $\phi(x) := \frac{1}{f'(x)}$ – gives Newton method

Convergence of Newton method

- Newton is very powerful *when it works*
- Generalises
- When “sufficiently close” to a root, Newton converges fast
- Otherwise, *not guaranteed* to converge; may fail
- There are known conditions for Newton to converge:
  - Smale α theory
  - Radii polynomials
- Interval Newton method: Use interval arithmetic to guarantee convergence

Summary

- There are many methods for finding **roots** of functions
- Rewrite as **fixed-point iteration**
- Do not always converge
- Methods differ in **convergence rate**