19. Linear algebra

Last time

- Mid-term review
- Designing auxiliary problems
- Convergence

Goals for today

- Conceptual review of linear algebra
- Emphasising **geometry**

- Vectors
- Linear transformations
- Matrices
- Mostly think about $\mathbb{R}^2$

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- **Algebra**: Manipulating, solving, understanding “structure”
- Study of **vectors** and **linear transformations** between them

Why linear algebra?

- Have repeatedly seen systems of linear equations crop up:
    - First-order Taylor expansion
    - Newton’s method (linear equations to solve nonlinear equations)
    - Coefficients of interpolating polynomial (Vandermonde matrix)

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- Discretising many PDEs gives linear systems
- Numerical linear algebra: fundamental importance in science and engineering

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$$ax + by = c$$

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- (Relative) simplicity allows us to obtain much information

- **Geometry**: "flat" objects – lines, planes, hyperplanes
- Flatness makes much easier to understand than general curved surfaces

– "rigidity"

Vectors

- **Vectors** are the cornerstone of linear algebra
- What is a vector?

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- What is a vector?

- Directed / oriented piece of line
- Displacement through space
- Column of numbers

Operations on vectors

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2. **Addition**: $v_1 + v_2$
   - Geometry: Move along one vector and then the other

Linear combinations

- In general, + and scalar multiplication *only* operations
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- Define **span** $\langle v_1, \ldots, v_n \rangle$ as set of *all* linear combinations

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  except if v_2 is in $\langle v_1 \rangle$

  i.e. if $v_2 = \beta v_1$ – **linearly dependent**

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- All spans contain 0
- A span is a **vector subspace**

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- $\ell(t) = p + tv$ – parametric equation
- E.g. Line joining points a and b:

$$(1 - \alpha)a + \alpha b$$

- $\alpha = 0$ gives a and $\alpha = 1$ gives b

Vector equations

- Suppose $\mathbf{u} := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ span $\mathbb{R}^2$
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- x, y: "how much **u** and **v** need to reach **b**"

Vector equations II

- Suppose $x\mathbf{u} + y\mathbf{v} = \mathbf{b}$
- Rewrite in terms of components:

$$x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + y \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

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- So linear system can be interpreted as finding correct linear combination of vectors

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- One very important subclass are **linear** maps

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- A linear map **preserves** operations $+$ and $*$
- Doesn't matter whether we do them before or after applying the map

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- So linear map maps lines to lines

  and moves along them at the same speed

Linear maps III

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- Suppose x and y span $\mathbb{R}^2$
- Suppose f is linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$
- Then we know *exactly* what f does for *every* point on the plane!

Geometry of linear maps

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- Squishes unit square into parallelogram
- Or rotates it
- Or reflects it

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- Specifying action on any two (non-collinear) vectors

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- In $\mathbb{R}^2$ encode via action on **standard basis vectors**
  $e_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Matrices

- e_1 mapped to $a_1 := f(e_1)$
- e_2 mapped to $a_2 := f(e_2)$
- Need 4 parameters; encode as **matrix**

$$\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$

- First **column** is image of $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Second **column** is image of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $Ae_1 = a_{1,1}e_1 + a_{2,1}e_2$

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- Let matrix A represent linear map f
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- Let A have columns a_1 and a_2
- Let x be $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- Then $f(x) = f(x_1 e_1 + x_2 e_2) = x_1 f(e_1) + x_2 f(e_2) = x_1 a_1 + x_2 a_2$
- So $A\,x$ is **linear combination of columns of** A

Matrix–vector multiplication

- For matrix A and vector x define $A \cdot x$ as

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- Linear system becomes

$$A\, x = b$$

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- Since $(AB)(e_1) = A(B\,e_1) = A\,b_1$

Summary

- Vectors are basic objects
- Linearly independent spans form a basis
- Define matrices as recording action of a linear map