18.330 Problem set 6 (spring 2020)

Submission deadline: 11:59pm on Tuesday, April 7

Exercise 1: Runge–Kutta methods

1. Consider a Runge–Kutta method starting at (t_n, x_n) that takes an Euler step of length $h/2$ to $(t_{n+1/2}, x_{n+1/2})$ and then uses the new evaluation at that point to take a complete Euler step from (t_n, x_n) of length h.

   Find the order of this method and write down its Butcher tableau. We will refer to it as the "midpoint method".

2. Define a type `RKMethod` to represent a general explicit Runge–Kutta method defined by a Butcher tableau as follows:

   ```
   struct RKMethod{T}
       c::Vector{T}
       b::Vector{T}
       a::Matrix{T}
       s::Int  # number of stages
   end
   ```

   Make it into a function by completing the function

   ```
   function (method::RKMethod)(f, x, t, h)
       ...
   end
   ```

   to execute one step of the corresponding Runge–Kutta method with initial condition x at time t and step size h. Your code should work for both scalar and vector x and a possibly vector-valued function $f = f(t, x)$ (Assume that a is a lower-triangular matrix, corresponding to an explicit method.)

3. Define RK methods `euler`, `midpoint` and `RK4` using their respective tableaus.

4. Write a routine `integrate` with the signature

   ```
   function integrate(method, f, x0, t0, t_final, h)
   ```

   where `method` is a RK method as defined above and h is a fixed step size.

   Make sure that the final step lands exactly at the final time by taking that final step as a special case.

5. Use each method on the ODE $\dot{x} = 1.5x$ with $x_0 = 2$ and integrate from $t = 0$ for a time $t_{\text{final}} = 3$.

   Find the rate of convergence of the numerical solution to the exact solution as $h \to 0$ for each method. Do they correspond to our analytical expectations?

6. Even Runge–Kutta methods may not be good enough without adaptivity: consider the ODE

$$\dot{u}(t) = \exp\left[t - u \sin(u)\right].$$

Integrate it using RK4 from $t = 0$ to $t = 5$ with a step size $h = 10^{-2}$. Now integrate it with a step size $h = 10^{-3}$.

Plot both solutions $x(t)$ as a function of t. What do you observe? What do you think is happening?

Exercise 2: Adaptivity in the Euler method

In this exercise we will invetigate adaptivity in ODE solvers by taking the simplest case: an adaptive Euler method.

1. Consider one step of the Euler method. Write down the local (single-step) error in terms of the step size h and the unknown constant C. Call the approximation obtained at the end of the step $x^{(1)}$.
2. Now consider taking two consecutive Euler steps of size $h/2$. Would you expect this to give a better or a worse approximation to the true solution? Write down the total error after taking the two steps, assuming that the constant C is the same for both. Call the approximation at the end of this combined step $x^{(2)}$.
3. Define Δx as the difference between the two approximations. Use this to find the step size h' that will give an error per unit time of a given size ϵ.
4. Use this derivation to write an adaptive Euler integrator `adaptive_euler(f, t0, t_final, epsilon)` that tries to keep the global error less than ϵ, using an update rule similar to the one we discussed in class. Add a multiplicative factor 0.9 to that rule to make the method behave better.
5. Use it to integrate the same ODE as in exercise 1. Plot the step size taken as a function of time.
6. Now integrate the equation for the van der Pol oscillator:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0,$$

with $\mu = 5$. Use initial conditions $x_0 = 0$ and $\dot{x}_0 = 1$.

7. Plot trajectories in $(x, \dot{x})$ phase space and (separately) the solution $x(t)$ as a function of t.

8. Plot the step size as a function of time. What do you observe? How do you interpret this?

Exercise 3: SIR model In this exercise we will study the SIR model of the dynamics of an infectious disease outbreak ("epidemic") in a population.

1. Use e.g. RK4 to solve the SIR equations:

$$\dot{S} = -\beta S\, I \tag{1}$$

$$\dot{I} = \beta S\, I - \gamma I \tag{2}$$

$$\dot{R} = \gamma I \tag{3}$$

   Here I is the proportion of the population which is infectious. β is the rate of contact between susceptible and infectious individuals, and γ the recovery rate.

   Use $S_0 = 0.99$ and $I_0 = 0.01.$

2. Make an interactive visualization, varying β and γ in, say, the range 0 to $1.$
3. What do you observe? Can you interpret this?
4. By summing the equations we see that $S + I + R$ should be constant (equal to the total population, assuming no births or deaths). For representative parameter values $\beta = 0.1$ and $\gamma = 0.05$, how well does the numerics conserve this quantity?