# Lecture 14
# Hessenberg/Tridiagonal Reduction

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Introduction to Numerical Methods

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# Introducing Zeros by Similarity Transformations

- Try computing the Schur factorization $A = QTQ^*$ by applying Householder reflectors from left and right that introduce zeros:

$$\underset{A}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}} \xrightarrow{Q_1^*} \underset{Q_1^* A}{\begin{bmatrix} \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_1^* A Q_1}{\begin{bmatrix} \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \end{bmatrix}}$$

- The right multiplication destroys the zeros previously introduced
- We already knew this would not work, because of Abel's theorem
- However, the subdiagonal entries typically decrease in magnitude

# The Hessenberg Form

- Instead, try computing an upper Hessenberg matrix $H$ similar to $A$:

$$\underset{A}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}} \xrightarrow{Q_1^*} \underset{Q_1^*A}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_1^*AQ_1}{\begin{bmatrix} \times & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \times & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \end{bmatrix}}$$

- This time the zeros we introduce are not destroyed

- Continue in a similar way with column 2:

$$\underset{Q_1^*AQ_1}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \end{bmatrix}} \xrightarrow{Q_1^*} \underset{Q_2^*Q_1^*AQ_1}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_2^*Q_1^*AQ_1Q_2}{\begin{bmatrix} \times & \times & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \times & \times & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & \times & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \end{bmatrix}}$$

# The Hessenberg Form

- After $m-2$ steps, we obtain the Hessenberg form:

$$\underbrace{Q_{m-2}^* \cdots Q_2^* Q_1^*}_{Q^*} A \underbrace{Q_1 Q_2 \cdots Q_{m-2}}_{Q} = H = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}$$

- For hermitian $A$, zeros are also introduced above diagonals

$$\underset{A}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}} \xrightarrow{Q_1^*} \underset{Q_1^* A}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ \mathbf{0} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_1^* A Q_1}{\begin{bmatrix} \times & \boldsymbol{\times} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \times & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \\ & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} & \boldsymbol{\times} \end{bmatrix}}$$

producing a tridiagonal matrix $T$ after $m-2$ steps

# Householder Reduction to Hessenberg

**Algorithm: Householder Hessenberg**

**for** $k = 1$ **to** $m - 2$

$$x = A_{k+1:m,k}$$
$$v_k = \text{sign}(x_1)\|x\|_2 e_1 + x$$
$$v_k = v_k / \|v_k\|_2$$
$$A_{k+1:m,k:m} = A_{k+1:m,k:m} - 2v_k(v_k^* A_{k+1:m,k:m})$$
$$A_{1:m,k+1:m} = A_{1:m,k+1:m} - 2(A_{1:m,k+1:m} v_k) v_k^*$$

- Operation count (*not* twice Householder QR):

$$\sum_{k=1}^{m} 4(m-k)^2 + 4m(m-k) = \underbrace{4m^3/3}_{QR} + 4m^3 - 4m^3/2 = 10m^3/3$$

- For hermitian $A$, operation count is twice QR divided by two $= 4m^3/3$

# Stability of Householder Hessenberg

- The Householder Hessenberg reduction algorithm is backward stable:

$$\tilde{Q}\tilde{H}\tilde{Q}^* = A + \delta A, \qquad \frac{\|\delta A\|}{\|A\|} = O(\epsilon_{\mathrm{machine}})$$

where $\tilde{Q}$ is an exactly unitary matrix based on $\tilde{v}_k$