Why restarting Arnoldi/Lanczos is not trivial

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Overview

The Arnoldi (and Lanczos) algorithms iteratively construct an $m \times n$ orthonormal basis Q_n for the Krylov space $\mathcal{K}_n = \text{span}\left\{b, Ab, \ldots, A^{n-1}b\right\}$ for a starting vector b and an $m \times m$ matrix A. For very large m (e.g. huge sparse A), however, practical application of this algorithm eventually hits a maximum n where one runs out of memory for Q_n [and the computational cost also grows as $\Theta(mn^2)$]. Even Lanczos runs into this problem, because roundoff errors lead to "ghost eigenvalues" if one does not explicitly store Q_n and periodically re-orthogonalize. The solution is periodic **restarting**: shrink down to a k-dimensional subspace containing your "best guesses" for the solution vectors, and continue Arnoldi from there. It turns out that the algorithms to perform restarting properly—called **implicitly restarted** Arnoldi or Lanczos—are surprisingly complicated and subtle. The purpose of these notes is not to explain how implicit restarting works. Rather, it is to briefly explain why naive restarting methods *don't* work. That is, why is restarting so hard?

Restarting in general

In general, restarting means finding a smaller orthonormal basis

$$\underbrace{\hat{Q}_k}_{m\times k} = \underbrace{Q_n}_{m\times n}\underbrace{\hat{Q}}_{n\times k}$$

for a subspace $\hat{\mathcal{K}}_k \subset \mathcal{K}_n$ ($k < n$), where $\hat{Q}^*\hat{Q} = I \implies \hat{Q}_k^*\hat{Q}_k = I$, and then treating this as the k-th step of an Arnoldi process and continuing from there normally (until you restart again). However, for this to work, $\hat{Q}_k$ needs to preserve a **key property** of the Arnoldi process:

$$AQ_n = Q_n H_n + r_n e_n^*,$$

where $H_n = Q_n^* A Q_n$ is upper-Hessenberg $n \times n$, $r_n = h_{n+1,n} q_{n+1} \perp \mathcal{K}_n$, and $e_n^* = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$. This is the property that allows subsequent steps to continue the upper-Hessenberg property (which for Lanczos is tridiagonal and

crucial to its three-term recurrence structure). Hence, we would like to obtain the *same* structure for $A\hat{Q}_k$

To compute $A\hat{Q}_k$, it is convenient to define the $n \times n$ unitary matrix $Q = \begin{bmatrix} \hat{Q} & \hat{Q}_\perp \end{bmatrix}$ where the $n \times (n-k)$ matrix $\hat{Q}_\perp$ is any orthonormal basis for the orthogonal complement of $\hat{Q}$'s column space. Then we can write

$$\begin{aligned} A\hat{Q}_k &= AQ_n\hat{Q} = Q_nH_n\hat{Q} + r_ne_n^*\hat{Q} \\ &= Q_nQQ^*H_n\hat{Q} + r_ne_n^*\hat{Q} \\ &= \begin{bmatrix} \hat{Q}_k & Q_n\hat{Q}_\perp \end{bmatrix} \begin{bmatrix} \hat{Q}^*H_n\hat{Q} \\ \hat{Q}_\perp^*H_n\hat{Q} \end{bmatrix} + r_ne_n^*\hat{Q} \\ &= \boxed{\hat{Q}_k \underbrace{\left(\hat{Q}^*H_n\hat{Q}\right)}_{\hat{H}_k} + Q_n\hat{Q}_\perp\hat{Q}_\perp^*H_n\hat{Q} + r_ne_n^*\hat{Q}}. \end{aligned}$$

This looks messy, but we can simplfy it quite a bit if we make a good choice for $\hat{Q}$. With the *right* choice of $\hat{Q}$, in fact it *is* possible to have the $\hat{Q}_k\hat{H}_k + \hat{r}_ke_k^*$ structure, allowing us to restart Arnoldi and Lanczos, but finding such a $\hat{Q}$ is surprisingly subtle.

Naive restarting

The most obvious way to restart is to use Ritz vectors. Recall the Rayleigh–Ritz procedure: search for $x \in \mathcal{K}_n$ and $\nu \in \mathbb{C}$ such that $Ax - \nu x \perp \mathcal{K}_n$, or equivalently $x = Q_nz$ where $H_nz = \nu z$. This is how we estimate the eigenvectors and eigenvalues at the n-th step of Arnoldi. It seems natural that we should want our "restarted" basis $\hat{Q}_k$ to contain the Ritz vectors $x = Q_nz$ that are our best estimates so far for the desired eigenvectors. For example, suppose we are looking for the k biggest-$|\lambda|$ eigenvalues, then a natural choice of restarting basis would be the Ritz vectors $Q_n\hat{Z}$ corresponding to the biggest $|\nu|$. If we orthogonalize these via QR as $\hat{Z} = \hat{Q}\hat{R}$, we get

$$\hat{Q} = \hat{Z}\hat{R}^{-1}$$

and

$$H_n\hat{Q} = H_n\hat{Z}\hat{R}^{-1} = \hat{Z}\underbrace{\begin{bmatrix} \nu_1 & & & \\ & \nu_2 & & \\ & & \ddots & \\ & & & \nu_k \end{bmatrix}}_{\hat{\Lambda}}\hat{R}^{-1} = \hat{Z}\hat{\Lambda}\hat{R}^{-1} = \hat{Q}\hat{R}\hat{\Lambda}\hat{R}^{-1}.$$

Two nice things happen! In the boxed term $\hat{Q}_k\hat{H}_k$ above, we get

$$\hat{H}_k = \hat{Q}^*H_n\hat{Q} = \hat{Q}^*\hat{Q}\hat{R}\hat{\Lambda}\hat{R}^{-1} = \hat{R}\hat{\Lambda}\hat{R}^{-1},$$

which is a product of upper-triangular matrices, and hence is upper-triangular—this certainly satisfies the requirement that $\hat{H}_k$ should be upper-Hessenberg! Also, from the second boxed term:

$$\hat{Q}_\perp^* H_n \hat{Q} = \hat{Q}_\perp^* \hat{Q} \hat{R} \hat{\Lambda} \hat{R}^{-1} = 0,$$

since $\hat{Q}_\perp^* \hat{Q} = 0$ by construction. So, the second boxed term above disappears! Unfortunately, the third boxed term is

$$r_n e_n^* \hat{Q} = r_n \left(\text{last row of } \hat{Q} \right).$$

While $r_n \perp \mathcal{K}_n \implies r_n \perp \hat{\mathcal{K}}_n$ (that is, $\hat{Q}_k^* r_n = 0$) as desired, in general the last row of $\hat{Q}$ will **not** be a multiple of e_k^*. So, this doesn't work.

The same problem arises for naive restarting of the Lanczos case $A = A^*$. In this case, the upper-Hessenberg matrix H_n is Hermitian. Hence the upper-triangular matrix $\hat{H}_k = \hat{\Lambda}$ is diagonal ($\hat{R} = I$ since the eigenvectors $\hat{Z}$ are orthonormal). But there is still no reason why the last row of $\hat{Q}$ should be a multiple of $\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$, so it doesn't work.

Implicit restarting

In fact, it is possible to choose a $\hat{Q}_k$ such that it *mostly* contains the Ritz vectors that we want and *does* preserve the Arnoldi/Lanczos property. One hint of this is that our naive choice above was actually *too good* in two ways: $\hat{H}_k$ was upper-triangular instead of just upper-Hessenberg, and r_n was orthogonal to $\mathcal{K}_n$ and not just $\hat{\mathcal{K}}_k$. This gives us "wiggle room:" if we do a little "worse" in making $\hat{H}_k$ only upper-Hessenberg and $\hat{r}_k$ only $\perp \hat{\mathcal{K}}_k$, we then have enough freedom to make the last row of $\hat{Q}$ a multiple of e_k^*.

In particular, instead of taking eigenvectors $\hat{Z}$ of H_n, a better solution is to do exactly $n - k$ steps of shifted QR iteration on H_n and let $\hat{Q}$ be the resulting eigenvector/Schur-vector estimate. This is a good estimate for the Ritz eigenvectors that we want, and it turns out to be *just right* to preserve the Arnoldi property. Proving that this is true requires care and tedious calculation, but is relatively straightforward. I won't go through it in detail, but if you google "implicitly restarted Arnoldi" or "implicitly restarted Lanczos" you can find a number of reviews that go through the algebra. In practice, you are unlikely to ever need to know the details: most people use "canned" implementations of Arnoldi and Lanczos such as ARPACK. But please **resist the temptation to do naive restarting!**