Lecture 14
Hessenberg/Tridiagonal Reduction

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Introduction to Numerical Methods

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Introducing Zeros by Similarity Transformations

- Try computing the Schur factorization $A = QTQ^*$ by applying Householder reflectors from left and right that introduce zeros:

$$\underset{A}{\begin{bmatrix} \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \end{bmatrix}} \xrightarrow{Q_1^*} \underset{Q_1^*A}{\begin{bmatrix} \boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_1^*AQ_1}{\begin{bmatrix} \boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \end{bmatrix}}$$

- The right multiplication destroys the zeros previously introduced
- We already knew this would not work, because of Abel's theorem
- However, the subdiagonal entries typically decrease in magnitude

The Hessenberg Form

- Instead, try computing an upper Hessenberg matrix H similar to A:

$$\underset{A}{\begin{bmatrix} \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \end{bmatrix}} \xrightarrow{Q_1^*} \underset{Q_1^*A}{\begin{bmatrix} \times&\times&\times&\times&\times \\ \boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_1^*AQ_1}{\begin{bmatrix} \times&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \times&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \end{bmatrix}}$$

- This time the zeros we introduce are not destroyed
- Continue in a similar way with column 2:

$$\underset{Q_1^*AQ_1}{\begin{bmatrix} \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ &\times&\times&\times&\times \\ &\times&\times&\times&\times \\ &\times&\times&\times&\times \end{bmatrix}} \xrightarrow{Q_1^*} \underset{Q_2^*Q_1^*AQ_1}{\begin{bmatrix} \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ &\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &\mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &\mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_2^*Q_1^*AQ_1Q_2}{\begin{bmatrix} \times&\times&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \times&\times&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &\times&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \end{bmatrix}}$$

The Hessenberg Form

- After $m-2$ steps, we obtain the Hessenberg form:

$$\underbrace{Q_{m-2}^* \cdots Q_2^* Q_1^*}_{Q^*} A \underbrace{Q_1 Q_2 \cdots Q_{m-2}}_{Q} = H = \begin{bmatrix} \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ &\times&\times&\times&\times \\ &&\times&\times&\times \\ &&&\times&\times \end{bmatrix}$$

- For hermitian A, zeros are also introduced above diagonals

$$\underset{A}{\begin{bmatrix} \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times \end{bmatrix}} \xrightarrow{Q_1^*} \underset{Q_1^*A}{\begin{bmatrix} \times&\times&\times&\times&\times \\ \boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ \mathbf{0}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_1^*AQ_1}{\begin{bmatrix} \times&\boldsymbol{\times}&\mathbf{0}&\mathbf{0}&\mathbf{0} \\ \times&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \\ &\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times}&\boldsymbol{\times} \end{bmatrix}}$$

producing a tridiagonal matrix T after $m-2$ steps

Householder Reduction to Hessenberg

Algorithm: Householder Hessenberg

for $k = 1$ **to** $m-2$

$\quad x = A_{k+1:m,k}$

$\quad v_k = \text{sign}(x_1)\|x\|_2 e_1 + x$

$\quad v_k = v_k / \|v_k\|_2$

$\quad A_{k+1:m,k:m} = A_{k+1:m,k:m} - 2v_k(v_k^* A_{k+1:m,k:m})$

$\quad A_{1:m,k+1:m} = A_{1:m,k+1:m} - 2(A_{1:m,k+1:m} v_k) v_k^*$

- Operation count (*not* twice Householder QR):

$$\sum_{k=1}^{m} 4(m-k)^2 + 4m(m-k) = \underbrace{4m^3/3}_{QR} + 4m^3 - 4m^3/2 = 10m^3/3$$

- For hermitian A, operation count is twice QR divided by two $= 4m^3/3$

Stability of Householder Hessenberg

- The Householder Hessenberg reduction algorithm is backward stable:

$$\tilde{Q}\tilde{H}\tilde{Q}^* = A + \delta A, \qquad \frac{\|\delta A\|}{\|A\|} = O(\epsilon_{\text{machine}})$$

where $\tilde{Q}$ is an exactly unitary matrix based on $\tilde{v}_k$