Introduction to Online Learning Algorithms

Yoav Freund

December 31, 2019

Outline

$$t = 1$$

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$t = 3$

$t = 4$

$t = 5$

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$$\vec{P}^t$$

$$0 \leq \ell_i^t \leq 1$$

$i = 1, \ldots, N$

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$$\vec{P}^t$$

$$0 \leq \ell_i^t \leq 1$$

$i = 1, \ldots, N$

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$$i = 1, \ldots, N$$

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$$0 \leq \ell_i^t \leq 1$$

$i = 1, \ldots, N$

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$$\vec{P}^t$$

$$0 \leq \ell_i^t \leq 1$$

$i = 1, \ldots, N$

i

t

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

$$W_i^t = e^{-\eta L_i^t}$$

$\eta > 0$

$$\eta = \infty$$

$$P_i^t = \frac{W_i^t}{\sum_{j=1}^{N} W_j^t}$$

i

t

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$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

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$\eta > 0$

$$\eta = \infty$$

$$P_i^t = \frac{W_i^t}{\sum_{j=1}^{N} W_j^t}$$

$\eta = 1$

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L^{t}_{Hedge}

$1, 2, \ldots, t$

$$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L_i^t}{1 - e^{-\eta}}$$

$$\forall t, \quad L_{\text{Hedge}} \leq \min_i \left(\frac{\ln N + \eta L_i^t}{1 - e^{-\eta}} \right)$$

η

$$L^t_{\mathrm{Hedge}}$$

$1, 2, \ldots, t$

$$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L_i^t}{1 - e^{-\eta}}$$

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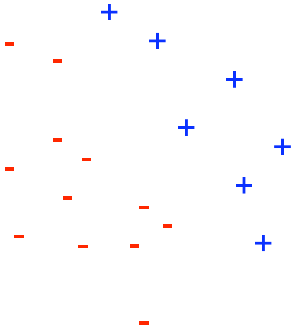
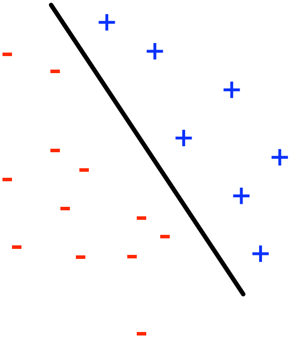
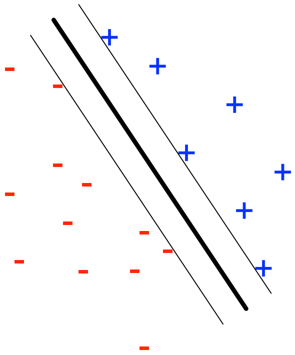
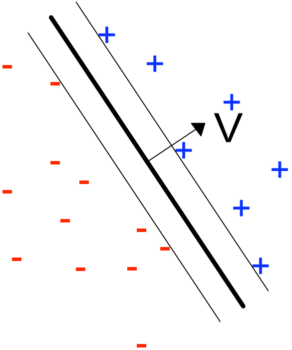
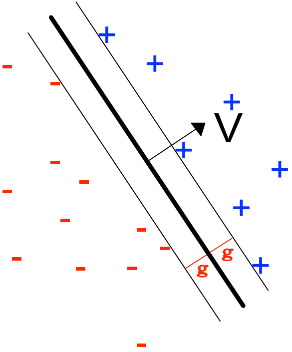







V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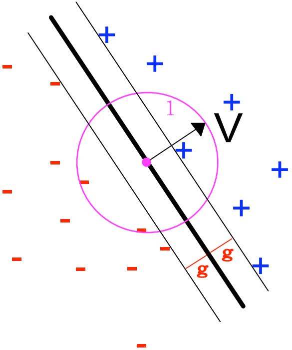
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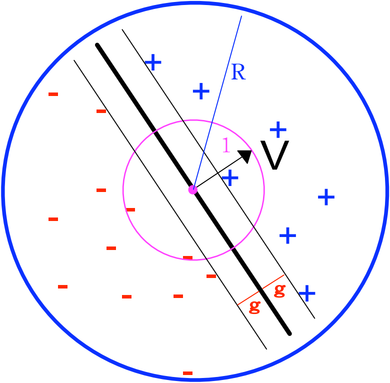

R
1
V
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g

$$\|\vec{V}\| = 1$$

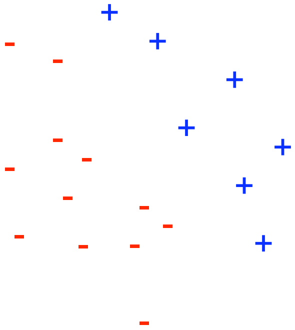
$(\vec{X}, y)$

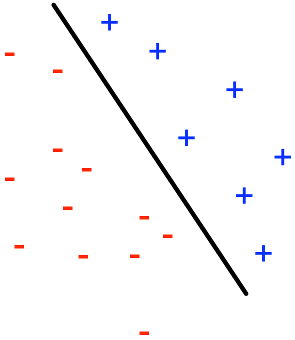
$$y \in \{-1, +1\}$$

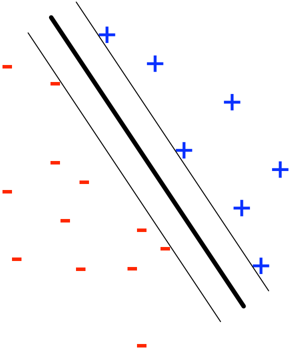
$$\forall \vec{X}, \ \ \|\vec{X}\| \leq R$$

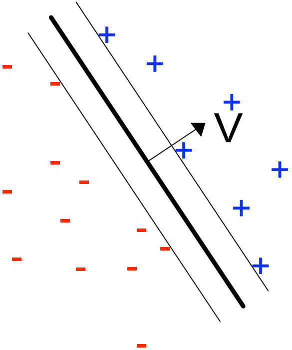
$\forall(\vec{X}, y),$

$$y(\vec{X} \cdot \vec{V}) \geq g$$





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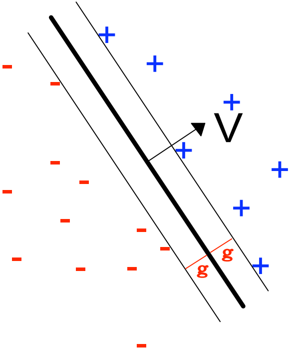
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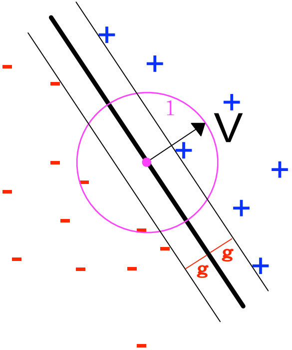

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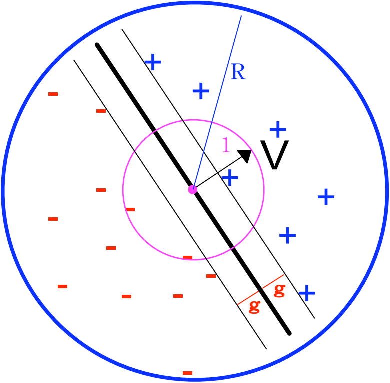

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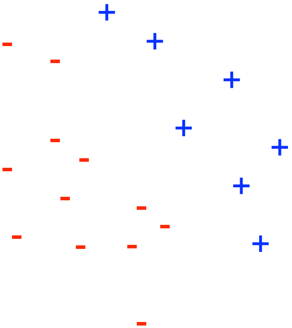

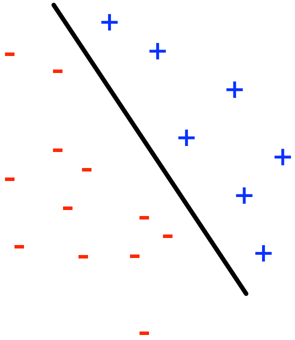

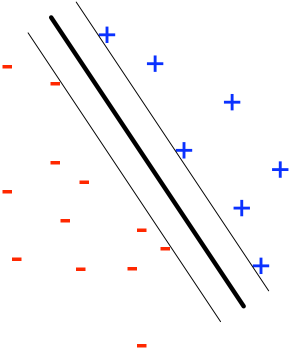

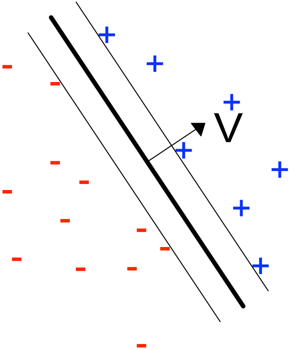
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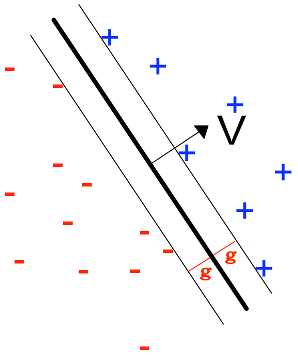
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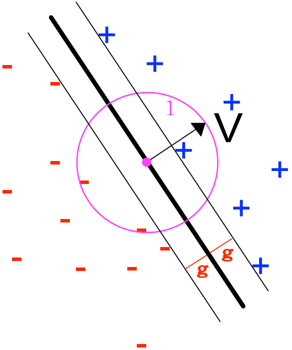

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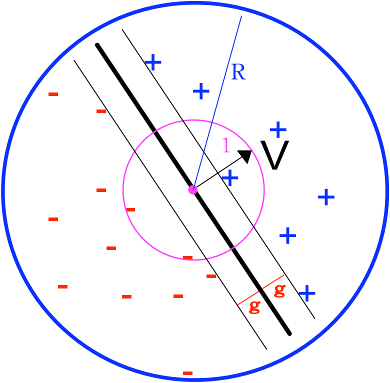

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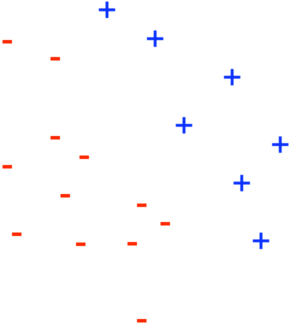

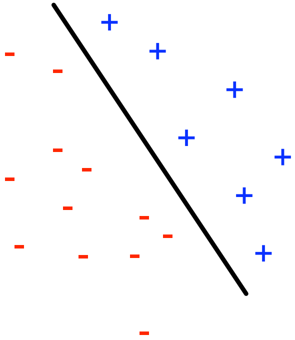

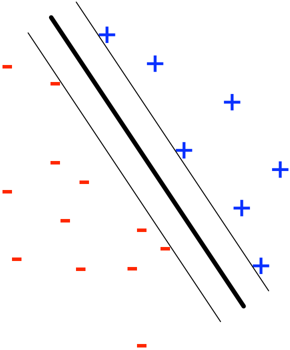

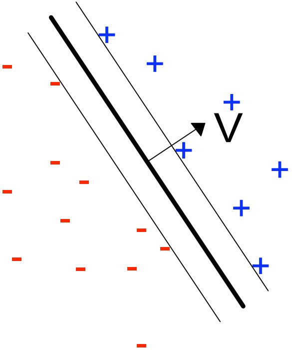
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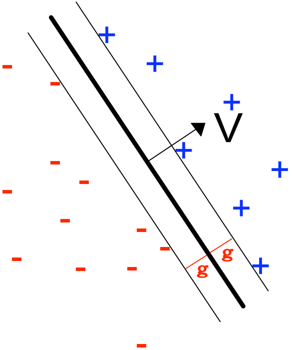

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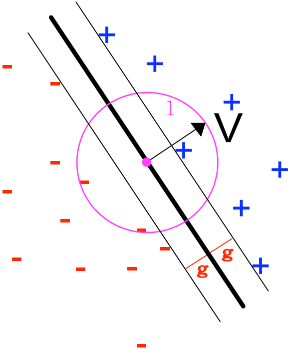

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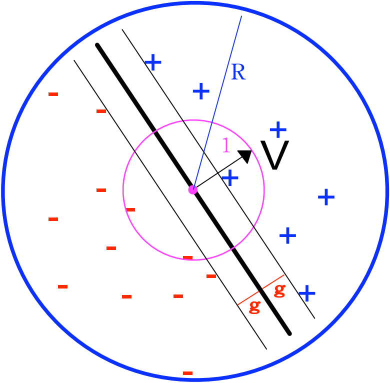

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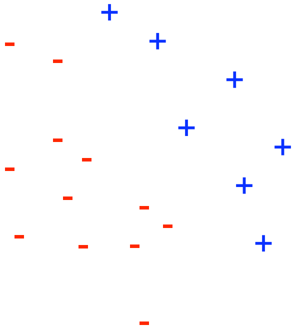

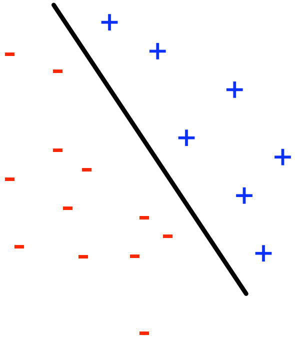

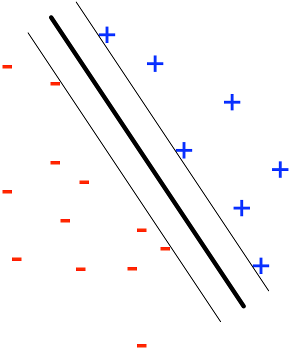

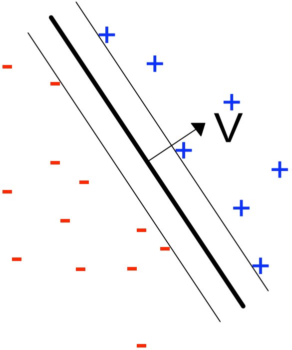
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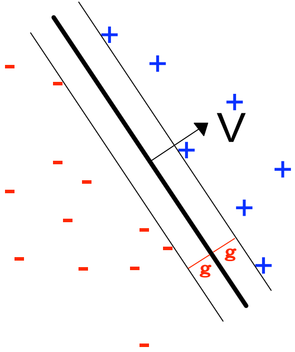
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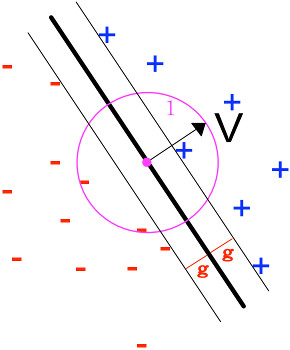

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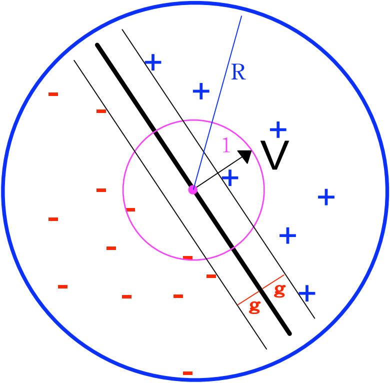

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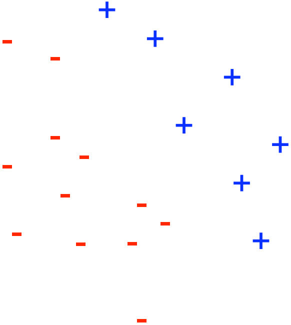

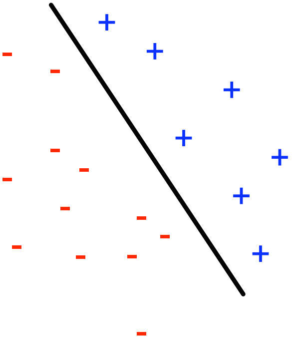

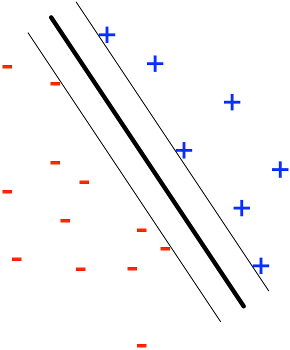

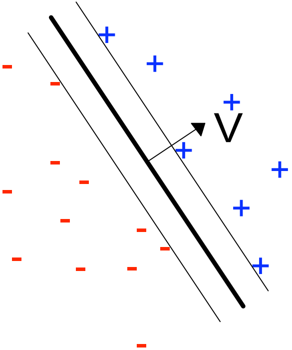
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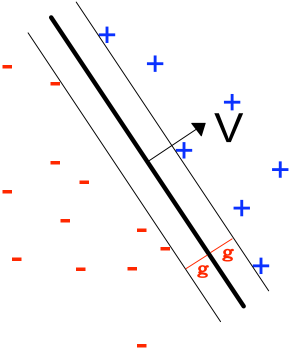

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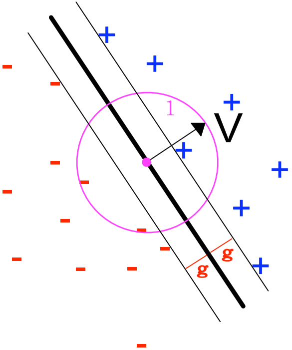

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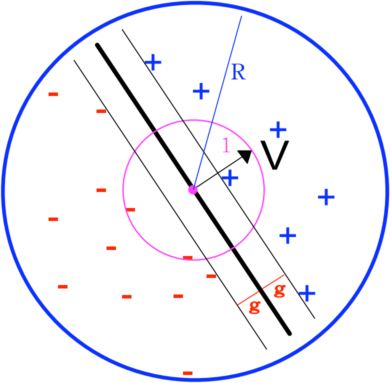
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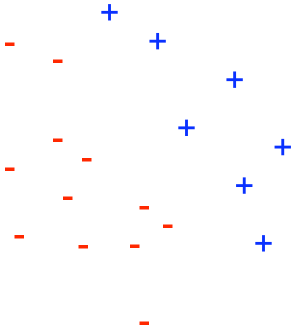

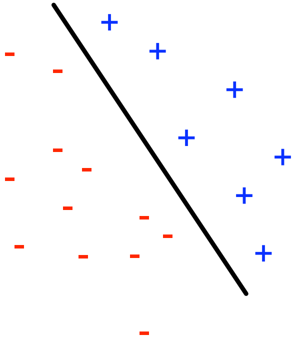

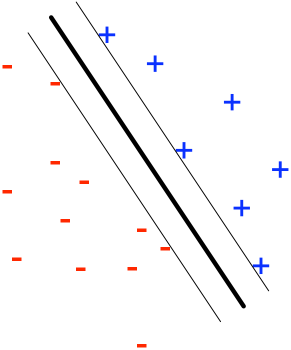

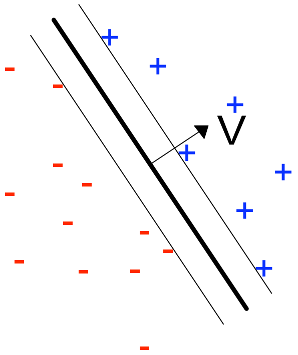
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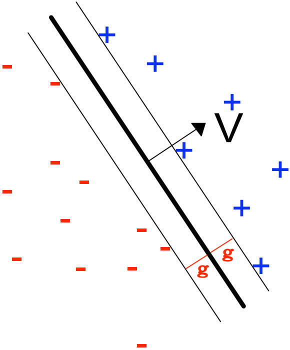
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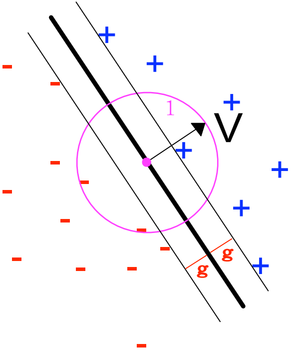

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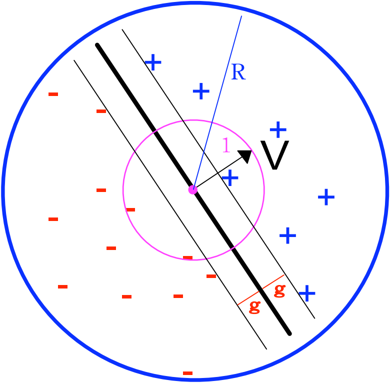
R
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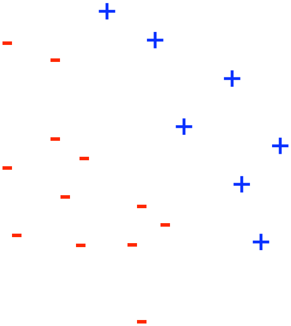

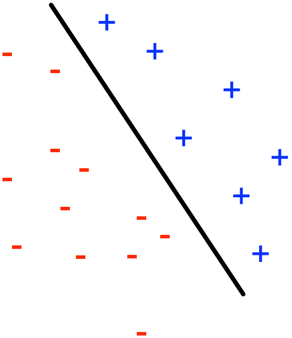

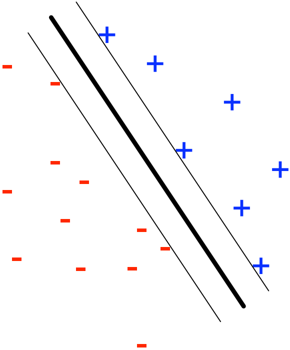

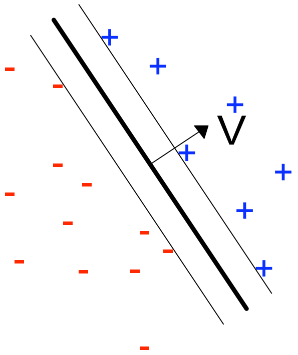
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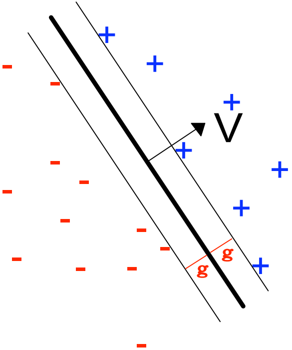

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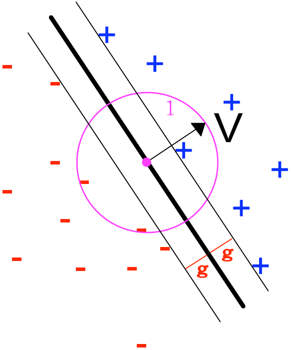

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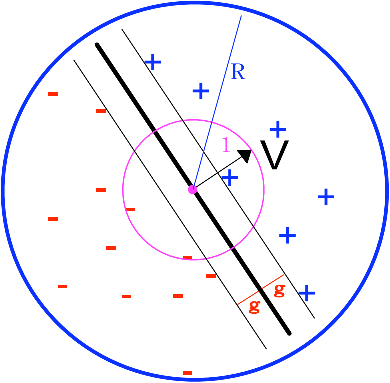

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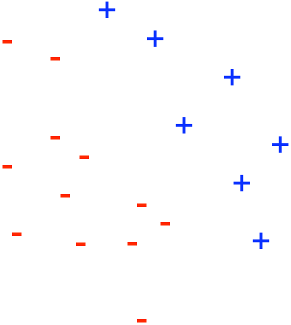

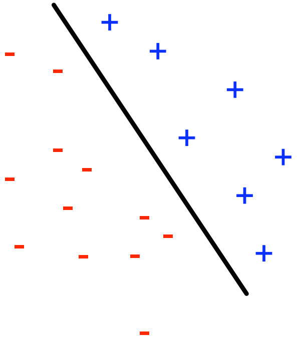

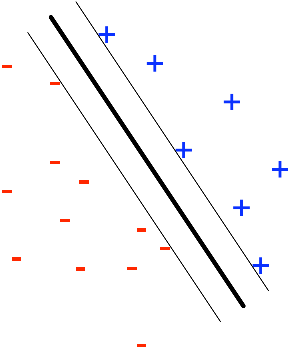

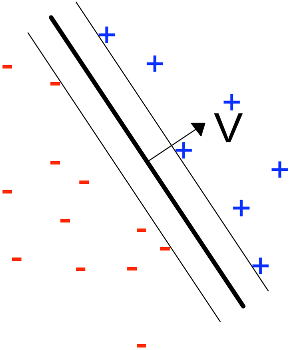
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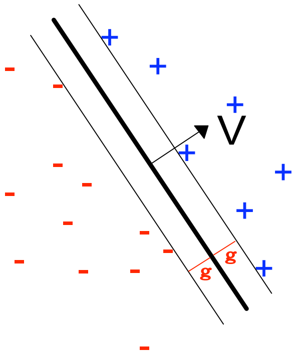

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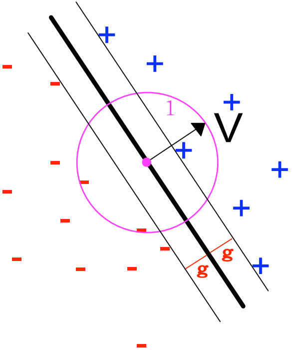

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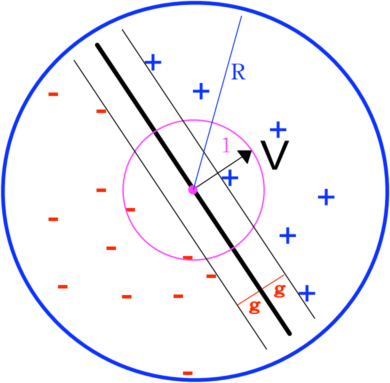

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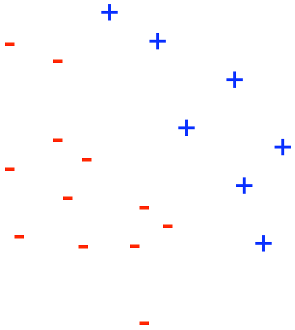

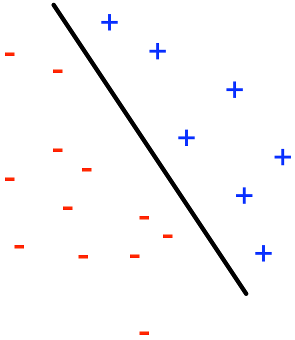

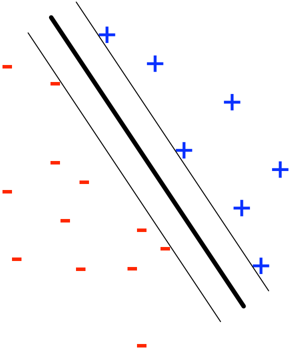

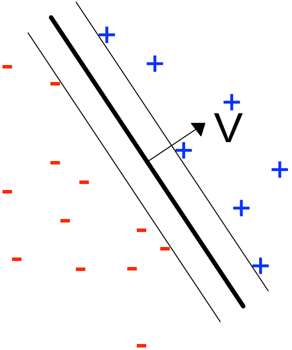
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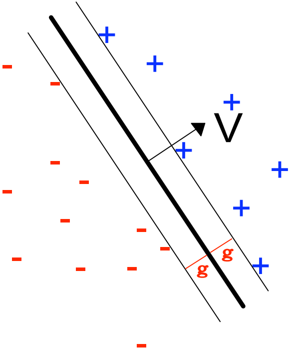

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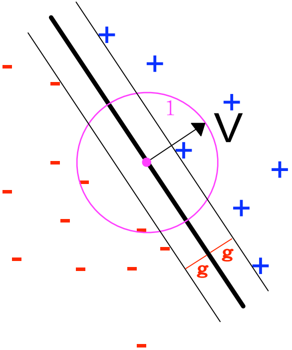

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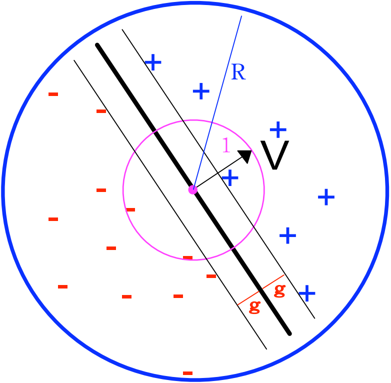

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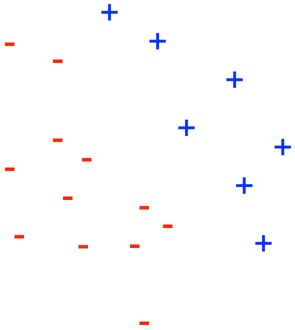

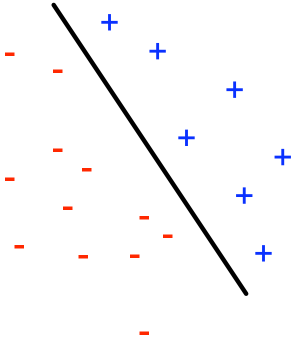

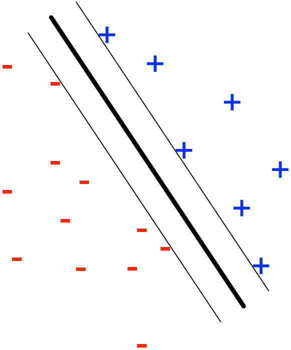

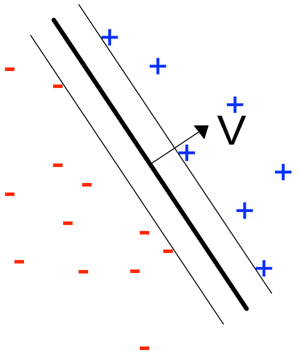
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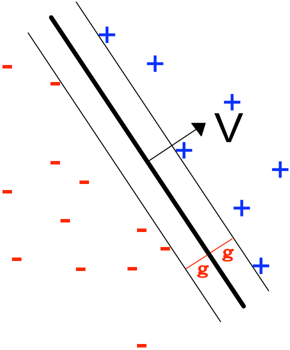

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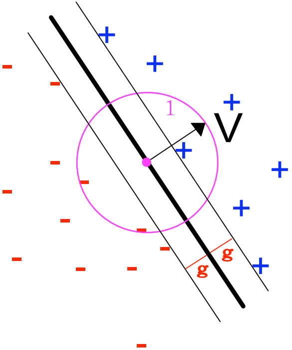

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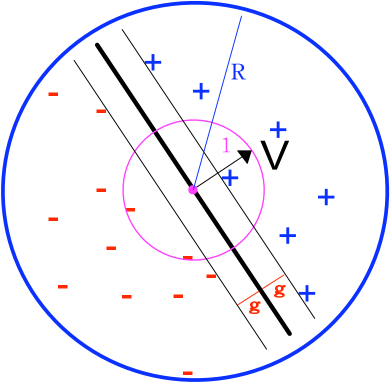

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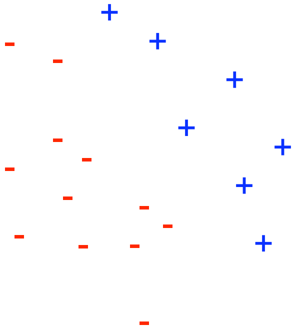

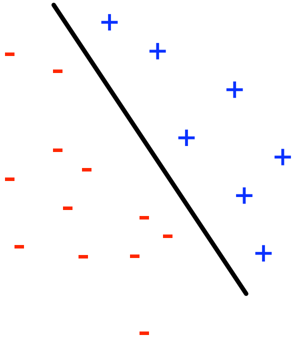

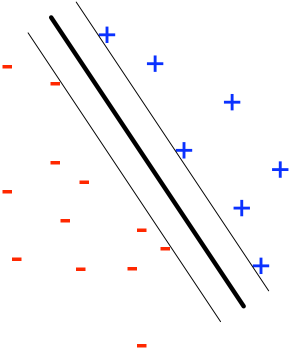

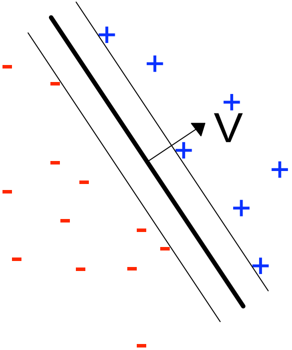
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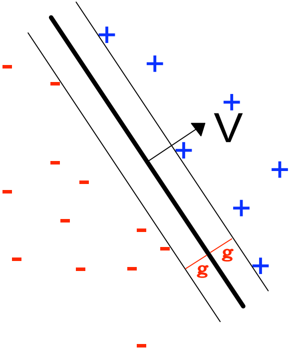
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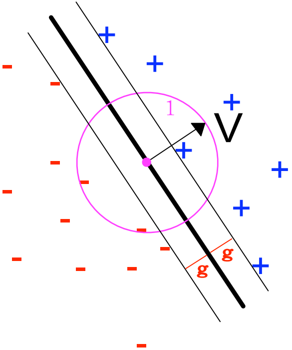

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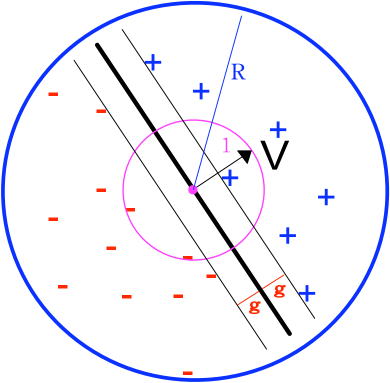
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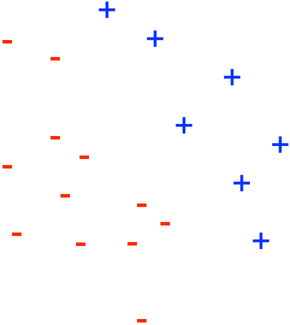

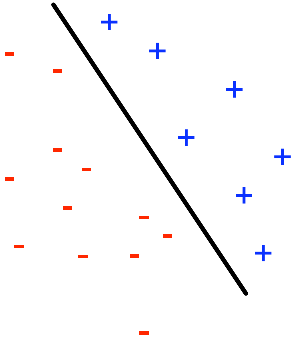

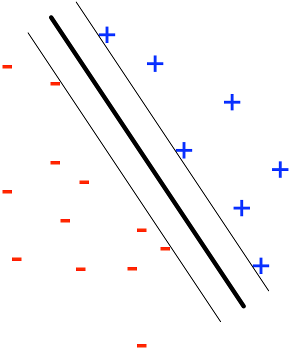

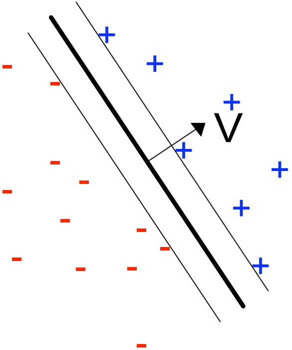
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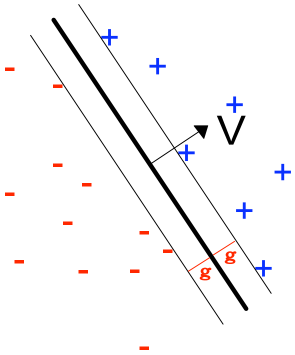

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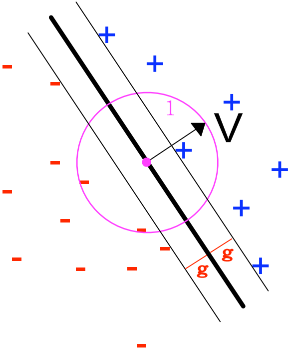

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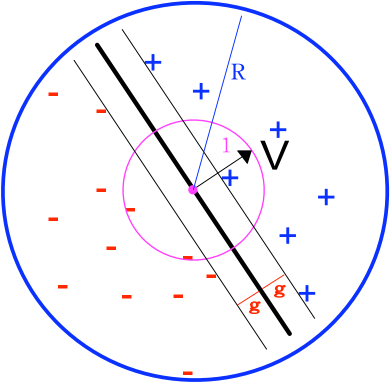

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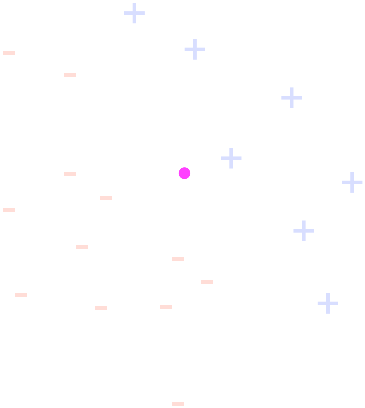
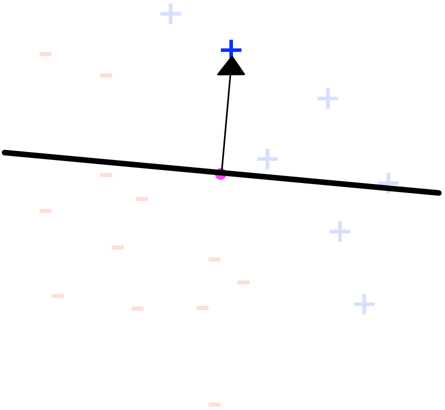
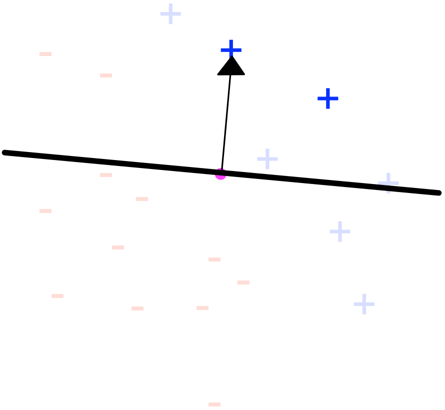
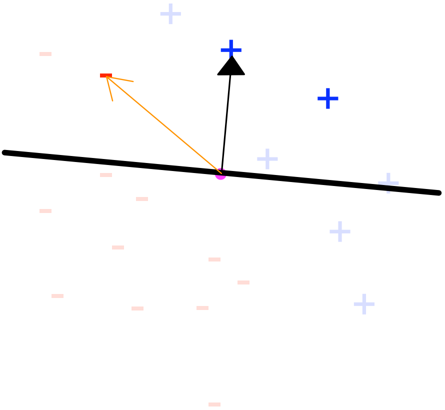
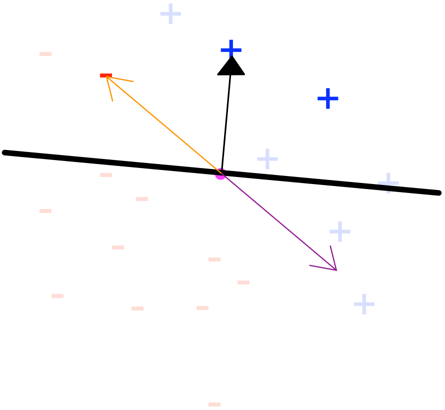
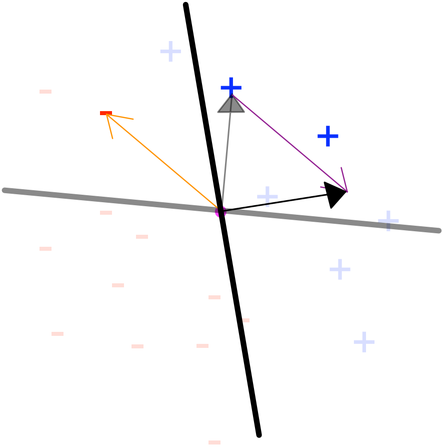




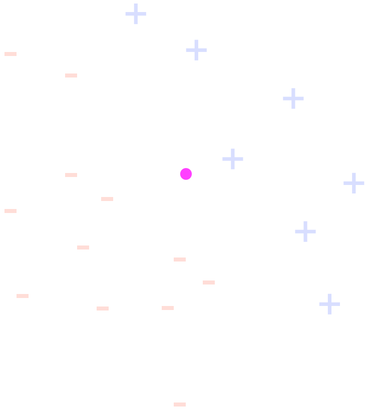


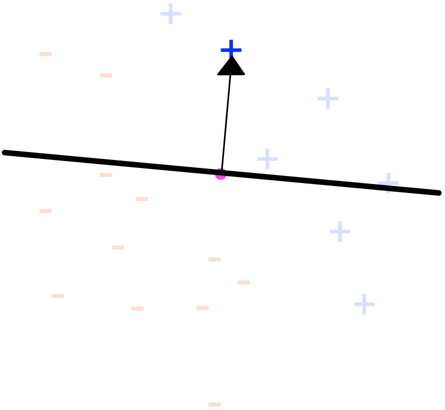


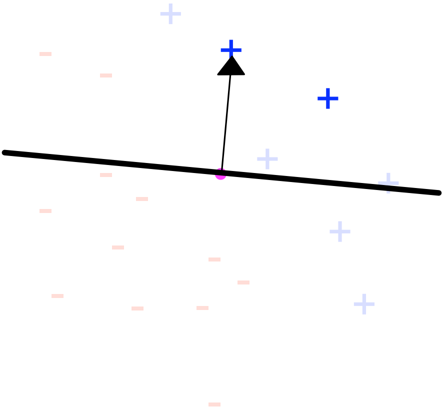


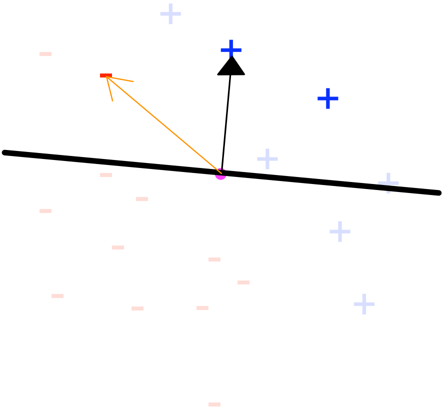


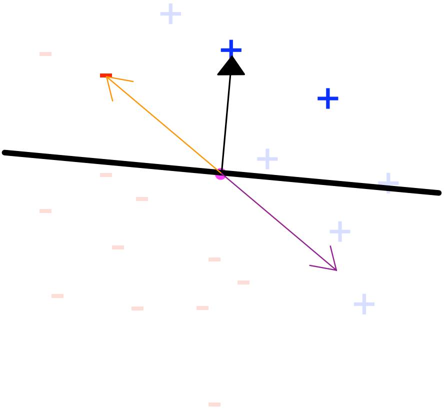
trace for the perceptron algorithm

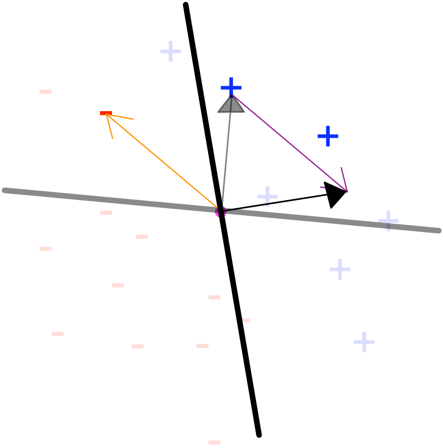












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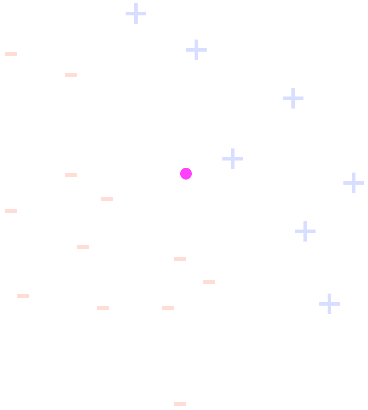

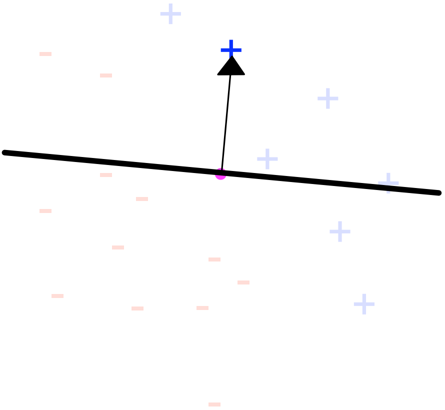

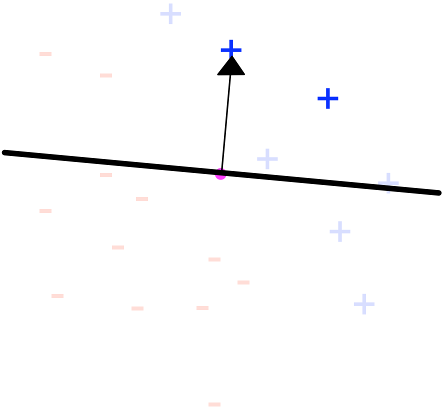

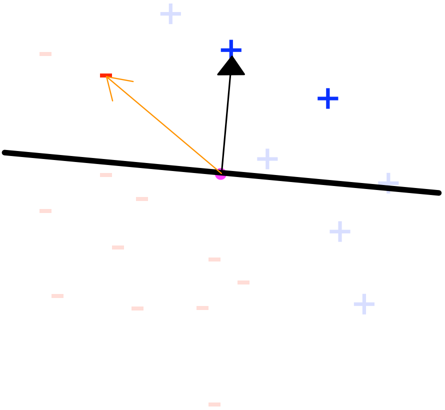

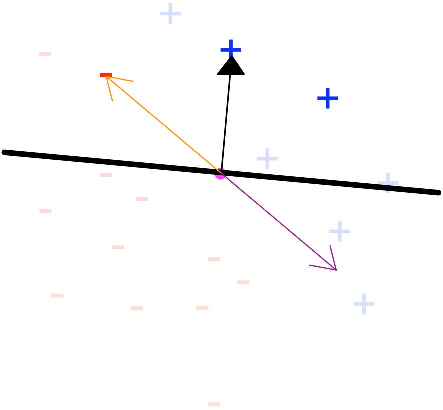

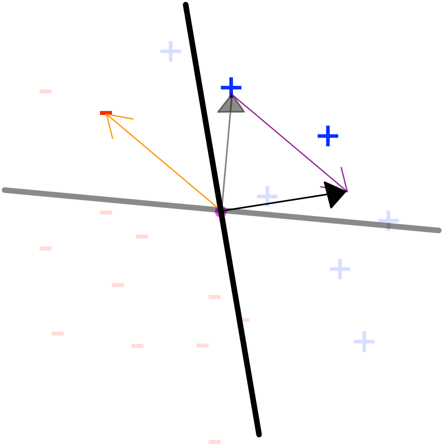

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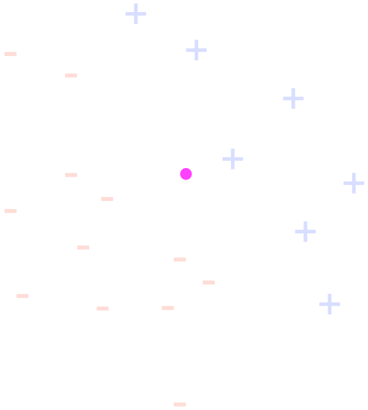
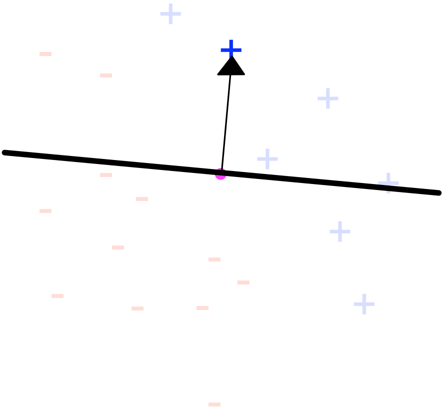
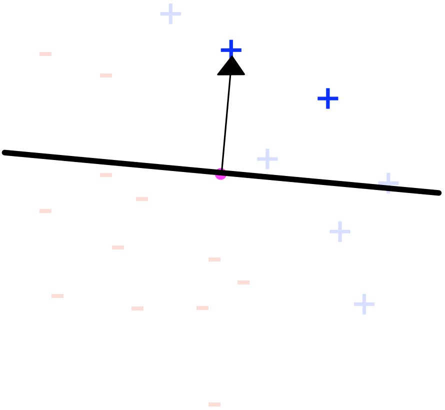
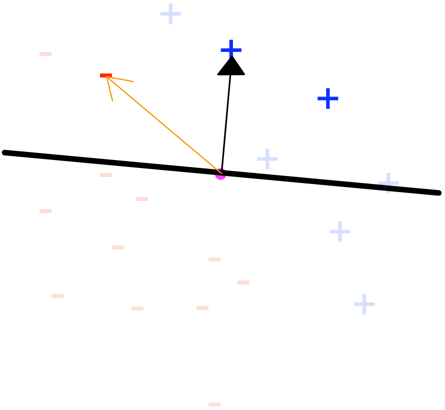
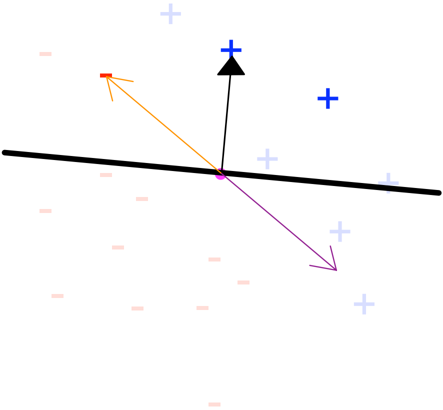
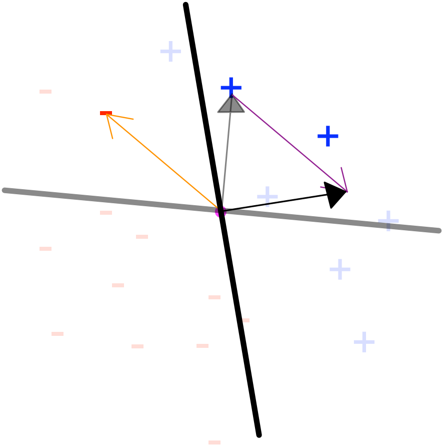




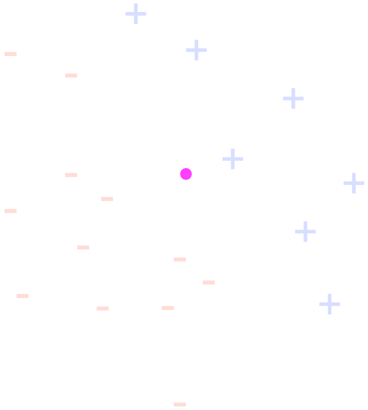


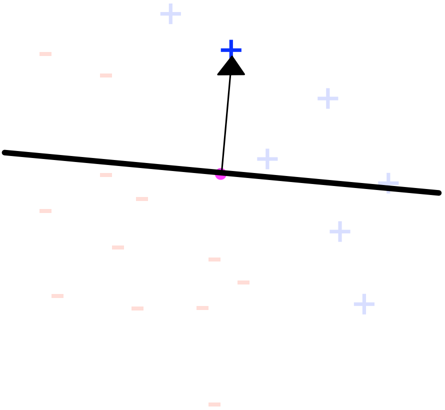


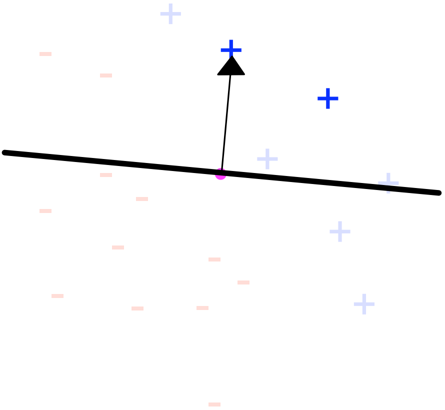


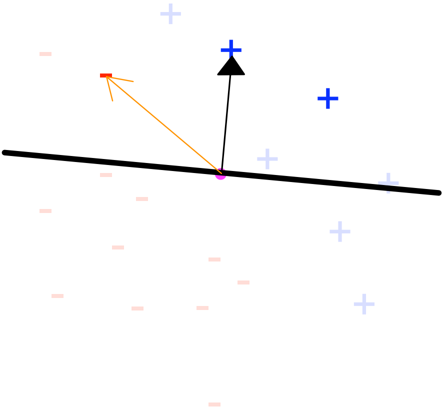


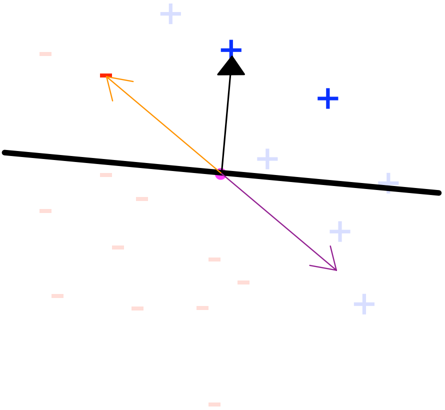
trace for the perceptron algorithm

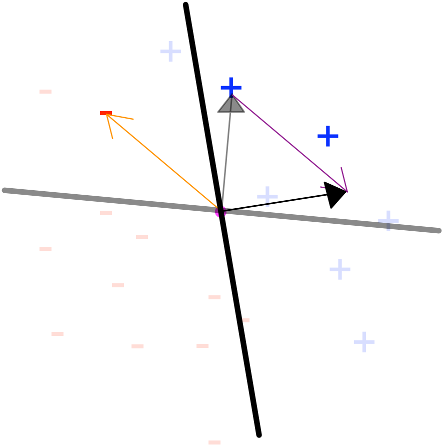




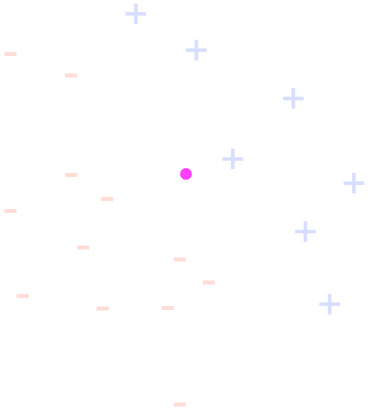


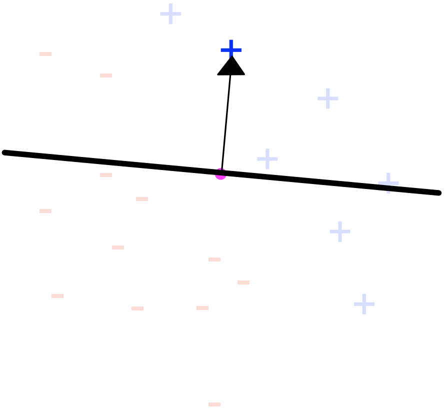


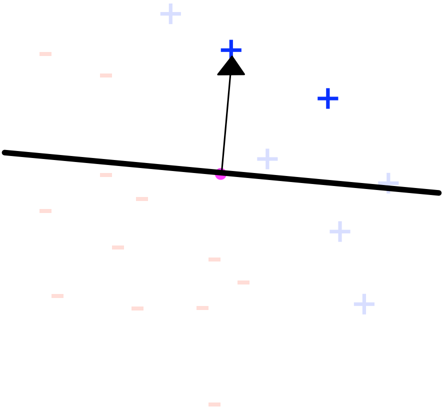


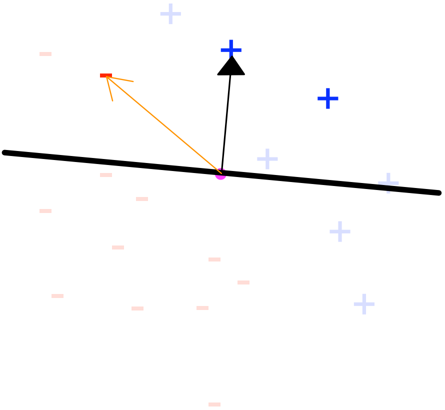


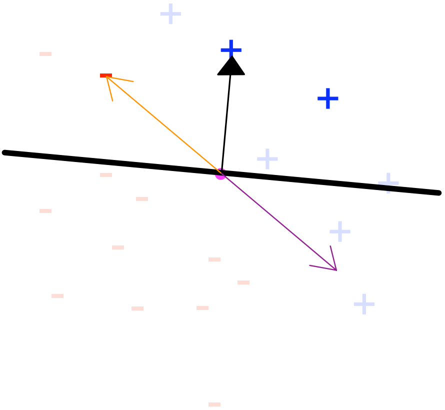
trace for the perceptron algorithm

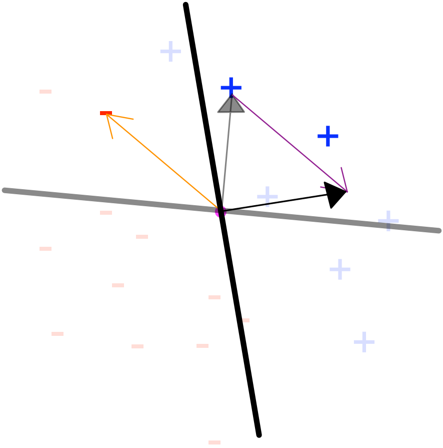












trace for the perceptron algorithm

$$\left(\frac{R}{g}\right)^2$$

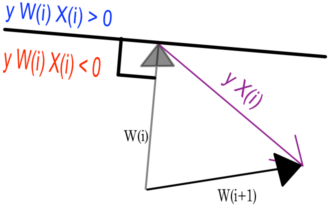
$$||\vec{W}||$$

$$\left(\frac{R}{g}\right)^2$$

$$\|\vec{W}\|$$

$$(\vec{W}_i \cdot X_i)y < 0$$

$$\|\vec{W}_{i+1}\|^2 = \|\vec{W}_i + y_i\vec{X}_i\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$$

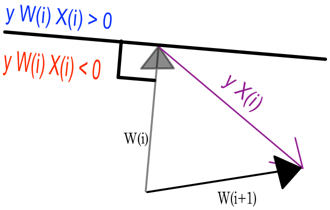


y W(i) X(i) > 0
y W(i) X(i) < 0
y X(i)
W(i)
W(i+1)

an Lemma

$$(\vec{W}_i \cdot X_i)y < 0$$

$$\|\vec{W}_{i+1}\|^2 = \|\vec{W}_i + y_i\vec{X}_i\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$$



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y X(i)
W(i)
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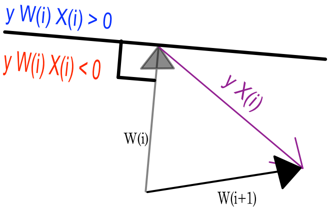
y W(i) X(i) > 0
y W(i) X(i) < 0
y X(i)
W(i)
W(i+1)

an Lemma

$$\|\vec{W}_i\|^2 \leq iR^2$$

$$i = 0$$

$$\|\vec{W}_0\|^2 = 0$$

i

$i + 1$

$$\|\vec{W}_{i+1}\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$$

$$\leq \|\vec{W}_i\|^2 + R^2 \leq (i+1)R^2$$

$$\|\vec{W}_i\|$$

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$$\|\vec{W}_i\|$$

$$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$$

$$\|\vec{V}\| = 1$$

$$\vec{W}_i \cdot \vec{V}$$

i

$$\vec{W}_i \cdot \vec{V} \geq ig$$

$$i = 0$$

$$\vec{W}_0 \cdot \vec{V} = 0$$

i

$i + 1$

$$\vec{W}_{i+1} \cdot \vec{V} = \left(\vec{W}_i + \vec{X}_i y_i\right) \vec{V}$$

$$= \vec{W}_i \cdot \vec{V} + y_i \vec{X}_i \cdot \vec{V}$$

$$\geq ig + g = (i + 1)g$$

$$\|\vec{W}_i\|$$

$$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$$

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$$\left\| \vec{W}_i \right\|$$

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$$\vec{W}_i \cdot \vec{V}$$

i

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i

$i + 1$

$$\vec{W}_{i+1} \cdot \vec{V} = \left(\vec{W}_i + \vec{X}_i y_i \right) \vec{V}$$

$$= \vec{W}_i \cdot \vec{V} + y_i \vec{X}_i \cdot \vec{V}$$

$$\geq ig + g = (i + 1)g$$

$$\|\vec{W}_i\|$$

$$(ig)^2 \leq \|\vec{W}_i\|^2 \leq iR^2$$

$$i \leq \left(\frac{R}{g}\right)^2$$

$$(ig)^2 \leq \|\vec{W}_i\|^2 \leq iR^2$$

$$i \leq \left(\frac{R}{g}\right)^2$$

$$(ig)^2 \leq \|\vec{W}_i\|^2 \leq iR^2$$

$$i \leq \left(\frac{R}{g}\right)^2$$

n

$$\frac{\#\mathbf{H}}{n} = \frac{4}{9}$$

n

$$\frac{\#\mathbf{H}}{n} = \frac{4}{9}$$

n

$$\frac{\#\mathbf{H}}{n} = \frac{4}{9}$$

p_0

p_1

p_2

• • •

p_0

p_1

$$\frac{\#\mathbf{H} + 1}{n + 2}$$

$$\frac{\#\mathbf{H} + 1/2}{n + 1}$$

observe a hf sequence of coin flips, and

p_0

p_1

p_2

• • •

p_0

p_1

$$\frac{\#\mathbf{H}+1}{n+2}$$

$$\frac{\#\mathbf{H} + 1/2}{n + 1}$$

observe a bf sequence of coin flips, and

p_0

p_1

p_2

• • •

p_0

p_1

$$\frac{\#\mathbf{H} + 1}{n + 2}$$

$$\frac{\#\mathbf{H} + 1/2}{n+1}$$

observe a hf sequence of coin flips, an

p_0

p_1

p_2

• • •

p_0

p_1

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p_0

p_1

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• • •

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p_0

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observe a bf sequence of coin flips, an

p_0

p_1

p_2

• • •

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p_1

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observe a hf sequence of coin flips, and

p_0

p_1

p_2

• • •

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p_1

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observe a bf sequence of coin flips, and

p_0

p_1

p_2

• • •

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p_1

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