Introduction to Online Learning Algorithms

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Outline

Example trace for Halving Algorithm

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.
outcome

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	
outcome	

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	1
outcome	

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	1
outcome	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	
outcome	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	
outcome	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	
outcome	1	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	

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	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	0

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	0
expert8	1	1	1	0	-
alg.	1	0	1	1	
outcome	1	1	1	0	

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	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
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- Number of mistakes is at most $\log_2 N$.

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- No stochastic assumptions.

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- Here, Potential = number of experts in pool.

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- Adversary chooses expert predictions and outcomes so as to maximize the number of mistakes.

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- If $N = 2^k$ then $k^* = k$, otherwise $k^* < \log N$
- k^* is the Min-Max number of mistakes: algorithm never makes more than k^* mistakes, adversary can force k^* mistakes.

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- Fits nicely in game theory

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- Unlike halving - no action is perfect.
- Basic idea - reduce probability of lossy actions,
  but not all the way to zero.
- **Modified Goal:**
  minimize REGRET =
  expected total loss
  minus
  minimal total loss of repeating one action.

The Hedge Algorithm

Consider action i at time t

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

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- $\eta > 0$ is the learning rate parameter. Halving: $\eta = \infty$

The Hedge Algorithm

Consider action i at time t

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

- Weight:

$$W_i^t = e^{-\eta L_i^t}$$

- $\eta > 0$ is the learning rate parameter. Halving: $\eta = \infty$
- Probability:

$$P_i^t = \frac{W_i^t}{\sum_{j=1}^{N} W_j^t}$$

Example trace for Hedge Algorithm

$\eta = 1$

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$$\eta = 1$$

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$
expert1	1
expert2	1
expert3	1
expert4	1
expert5	1
expert6	1
expert7	1
expert8	1
alg.	

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	L^1
expert1	1	.1
expert2	1	.8
expert3	1	.3
expert4	1	.1
expert5	1	.9
expert6	1	0
expert7	1	1
expert8	1	.8

alg.

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	L^1
expert1	1	.1
expert2	1	.8
expert3	1	.3
expert4	1	.1
expert5	1	.9
expert6	1	0
expert7	1	1
expert8	1	.8
alg.		.5

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	L^1	$\vec{W}^2$
expert1	1	.1	.90
expert2	1	.8	.45
expert3	1	.3	.74
expert4	1	.1	.90
expert5	1	.9	.41
expert6	1	0	1
expert7	1	1	.37
expert8	1	.8	.45
alg.		.5	

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	L^1	$\vec{W}^2$	L^2
expert1	1	.1	.90	.1
expert2	1	.8	.45	.5
expert3	1	.3	.74	.2
expert4	1	.1	.90	.7
expert5	1	.9	.41	1
expert6	1	0	1	.1
expert7	1	1	.37	.5
expert8	1	.8	.45	.2
alg.		.5		

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	L^1	$\vec{W}^2$	L^2
expert1	1	.1	.90	.1
expert2	1	.8	.45	.5
expert3	1	.3	.74	.2
expert4	1	.1	.90	.7
expert5	1	.9	.41	1
expert6	1	0	1	.1
expert7	1	1	.37	.5
expert8	1	.8	.45	.2
alg.		.5		.36

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	L^1	$\vec{W}^2$	L^2	$\vec{W}^3$
expert1	1	.1	.90	.1	0.82
expert2	1	.8	.45	.5	0.27
expert3	1	.3	.74	.2	0.61
expert4	1	.1	.90	.7	0.45
expert5	1	.9	.41	1	0.15
expert6	1	0	1	.1	0.91
expert7	1	1	.37	.5	0.22
expert8	1	.8	.45	.2	0.37
alg.		.5		.36	

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	L^1	$\vec{W}^2$	L^2	$\vec{W}^3$	L^3
expert1	1	.1	.90	.1	0.82	0
expert2	1	.8	.45	.5	0.27	.2
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expert4	1	.1	.90	.7	0.45	.8
expert5	1	.9	.41	1	0.15	.8
expert6	1	0	1	.1	0.91	.2
expert7	1	1	.37	.5	0.22	.4
expert8	1	.8	.45	.2	0.37	.6
alg.		.5		.36		

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	L^1	$\vec{W}^2$	L^2	$\vec{W}^3$	L^3
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expert2	1	.8	.45	.5	0.27	.2
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alg.		.5		.36		.30

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	L^1	$\vec{W}^2$	L^2	$\vec{W}^3$	L^3	total
expert1	1	.1	.90	.1	0.82	0	.2
expert2	1	.8	.45	.5	0.27	.2	1.5
expert3	1	.3	.74	.2	0.61	.2	.7
expert4	1	.1	.90	.7	0.45	.8	1.6
expert5	1	.9	.41	1	0.15	.8	2.7
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expert6	1	0	1	.1	0.91	.2	.3
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expert8	1	.8	.45	.2	0.37	.6	1.6
alg.		.5		.36		.30	1.16

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- $$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L^t_i}{1 - e^{-\eta}}$$
- Which implies

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- Which implies
$$\forall t, \quad L_{\text{Hedge}} \leq \min_i \left(\frac{\ln N + \eta L^t_i}{1 - e^{-\eta}} \right)$$
- Proof and choice of η: next class.

The Perceptron Problem

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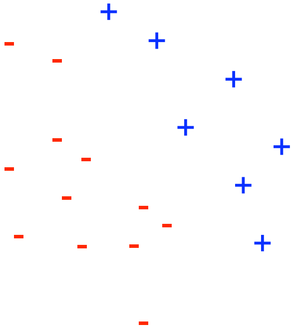

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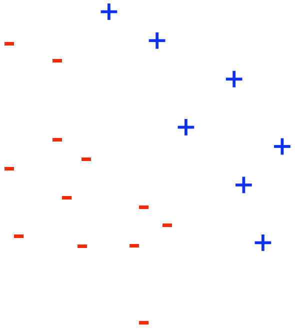

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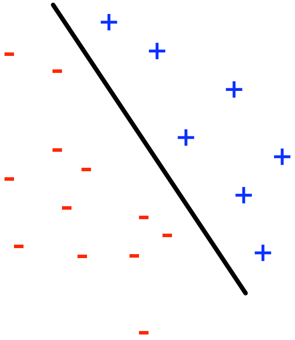

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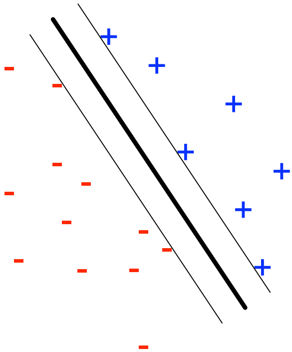

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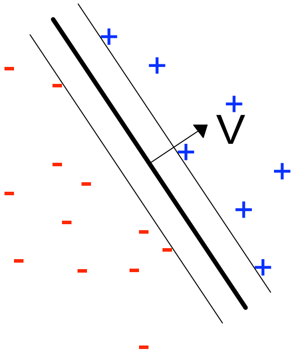

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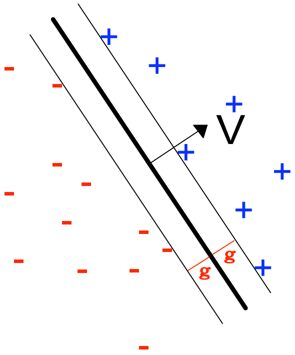

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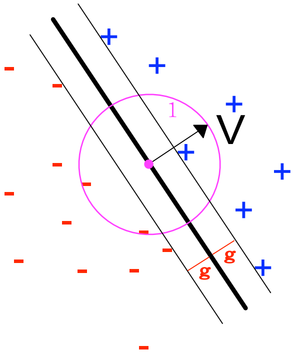

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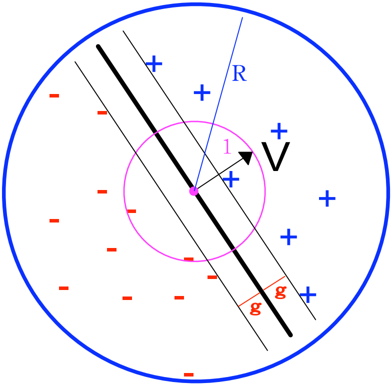

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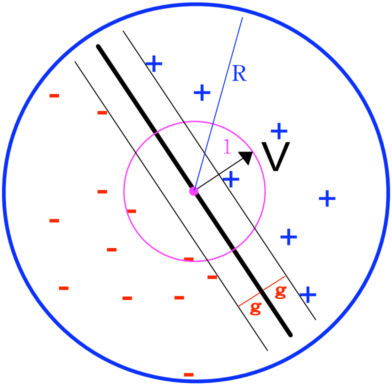
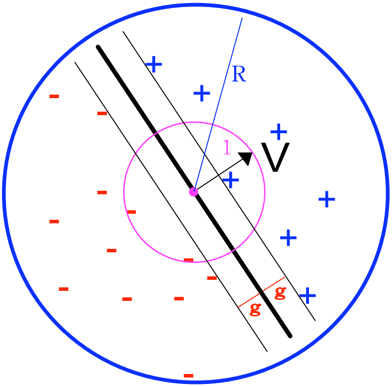


- $\|\vec{V}\| = 1$

The Perceptron Problem



- $\|\vec{V}\| = 1$
- Example = $(\vec{X}, y)$, $y \in \{-1, +1\}$.

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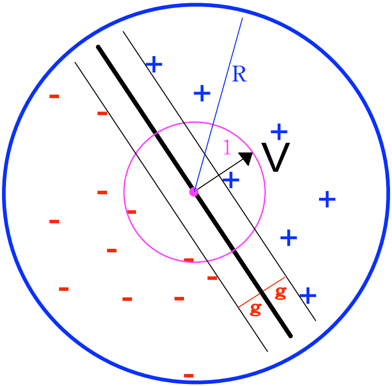

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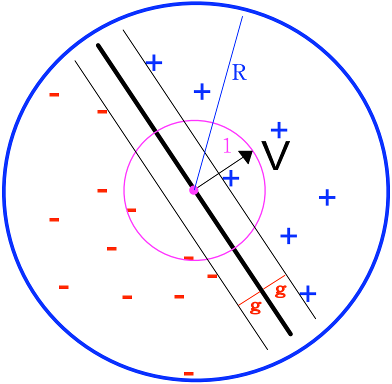

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- $\forall \vec{X}, \ \ \|\vec{X}\| \leq R$.
- $\forall (\vec{X}, y)$, $y(\vec{X} \cdot \vec{V}) \geq g$

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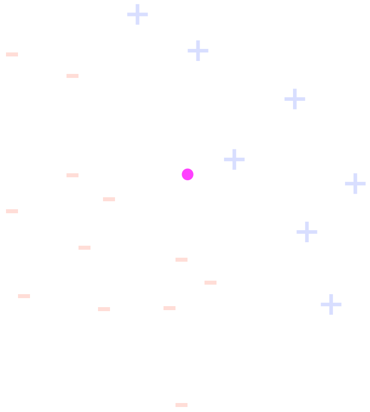
  - Update $\vec{W}_{i+1} = \vec{W}_i + y_i X_i$.

Example trace for the perceptron algorithm



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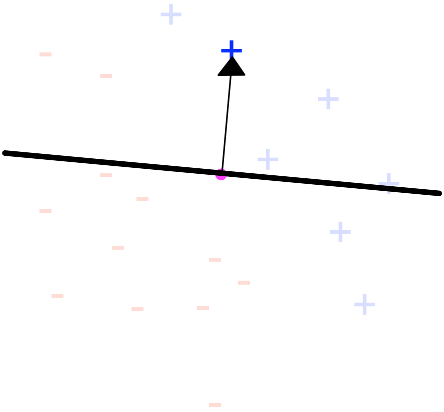

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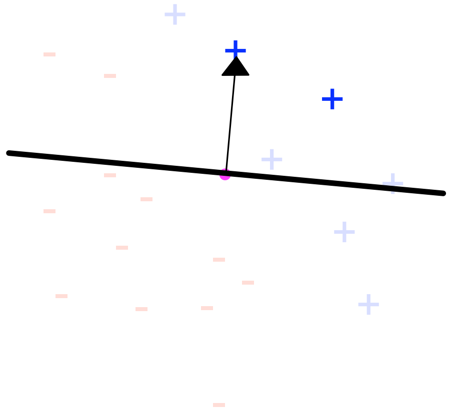

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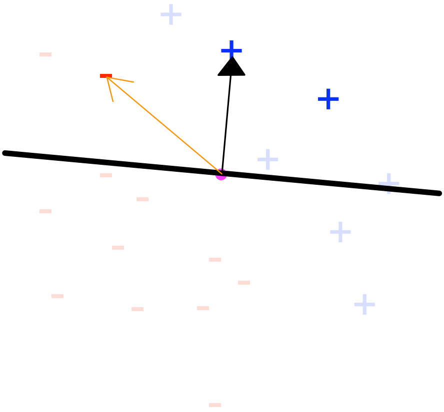

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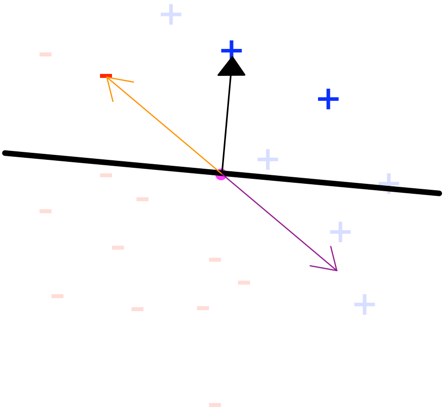

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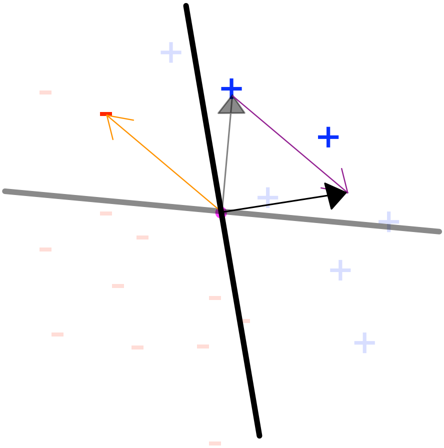

Bound on number of mistakes

- The number of mistakes that the perceptron algorithm can make is at most $\left(\frac{R}{g}\right)^2$.

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- Proof by combining upper and lower bounds on $\|\vec{W}\|$.

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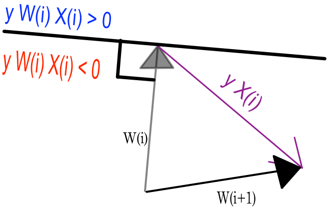

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  $\leq \|\vec{W}_i\|^2 + R^2 \leq (i + 1)R^2$

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  $\geq ig + g = (i+1)g$

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Thus:

$$i \leq \left(\frac{R}{g}\right)^2$$

Estimating the bias of a coin

- We observe n coin flips:
  H,T,T,H,H,T,H,T,T

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- Natural Answer:

$$\frac{\#\mathbf{H}}{n} = \frac{4}{9}$$

What if the estimation has to be done online?

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- What does “better” mean?

Next time

Analysis of the Hedge Algorithm.

- slides in talk1.handout.pdf on:
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- See you on Thursday!