# Exponential Weights Algorithms for Online Learning

Yoav Freund

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slides in Hedge/talk2.handout.pdf on:
https://github.com/yoavfreund/2020-Online-Learning

# Outline

**Hedge**$(\eta)$Algorithm

Hedging vs. Halving

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Bound on total loss

- Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$
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- Lower Bounds
- Repeated Matrix Games

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- Fits nicely in game theory

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- Basic idea - reduce probability of lossy actions, but not all the way to zero.
- **Modified Goal:** minimize difference between expected total loss and minimal total loss of repeating one action.

$$\sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t - \min_i \left( \sum_{t=1}^{T} \ell_i^t \right)$$

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  - outcome $o_i^t$ is revealed. $\ell_i^t = 0$ if $e_i^t = o_i^t$, $\ell_i^t = 1$ otherwise.

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- Plays a similar role to prior distribution in Bayesian algorithms.

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- Theorem (main theorem)

  *For any sequence of loss vectors* $\ell^1, \ldots, \ell^T$*, and for any* $i \in \{1, \ldots, N\}$*, we have*

$$L_{\textbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

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$$L_{\textbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

  - Proof: by combining upper and lower bounds on $\sum_{i=1}^{N} w_i^{T+1}$

# Upper bound on $\sum_{i=1}^N w_i^{T+1}$

## Lemma (upper bound)

*For any sequence of loss vectors* $\ell^1, \ldots, \ell^T$ *we have*

$$\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right) \leq -(1-e^{-\eta})L_{\mathbf{Hedge}(\eta)}.$$

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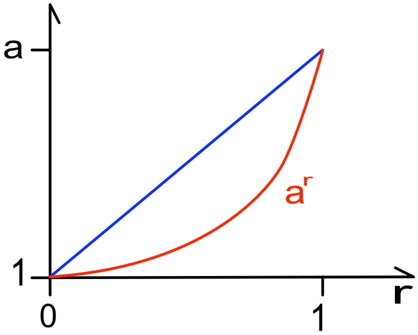


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# Proof of upper bound (slide 2)

Applying $a^r \leq 1 - (1-a)^r$ where $a = e^{-\eta}, r = \ell_i^t$

$$\sum_{i=1}^{N} w_i^{t+1} = \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t}$$

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$$\begin{aligned}\sum_{i=1}^{N} w_i^{t+1} &= \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t} \\ &\leq \sum_{i=1}^{N} w_i^t \left(1 - (1 - e^{-\eta})\ell_i^t\right)\end{aligned}$$

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&= \left(\sum_{i=1}^{N} w_i^t\right) \left(1 - (1 - e^{-\eta}) \frac{\mathbf{w}^t}{\sum_{i=1}^{N} w_i^t} \cdot \ell^t\right)
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&= \left(\sum_{i=1}^N w_i^t\right)\left(1 - (1 - e^{-\eta})\frac{\mathbf{w}^t}{\sum_{i=1}^N w_i^t} \cdot \ell^t\right) \\
&= \left(\sum_{i=1}^N w_i^t\right)(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell^t)
\end{aligned}
$$

# Proof of upper bound (slide 3)

- Combining

$$\sum_{i=1}^{N} w_i^{t+1} \leq \left(\sum_{i=1}^{N} w_i^t\right) \left(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell^t\right)$$

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- for $t = 1, \ldots, T$

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- for $t = 1, \ldots, T$
- yields

$$\sum_{i=1}^{N} w_i^{T+1} \quad \leq \quad \prod_{t=1}^{T} \left(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell^t\right)$$

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- for $t = 1, \ldots, T$
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$$\begin{aligned} \sum_{i=1}^{N} w_i^{T+1} &\leq \prod_{t=1}^{T}\left(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell^t\right) \\ &\leq \exp\left(-(1 - e^{-\eta})\sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t\right) \end{aligned}$$

since $1 + x \leq e^x$ for $x = -(1 - e^{-\eta})$.

# Lower bound on $\sum_{i=1}^N w_i^{T+1}$

For any $j = 1, \ldots, N$:

$$\sum_{i=1}^{N} w_i^{T+1} \geq w_j^{T+1} = w_j^1 e^{-\eta L_j}$$

# Combining Upper and Lower bounds

- Combining bounds on $\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right)$

$$\ln w_j^1 - \eta L_j \leq \ln \sum_{i=1}^{N} w_i^{T+1} \leq -(1 - e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t$$

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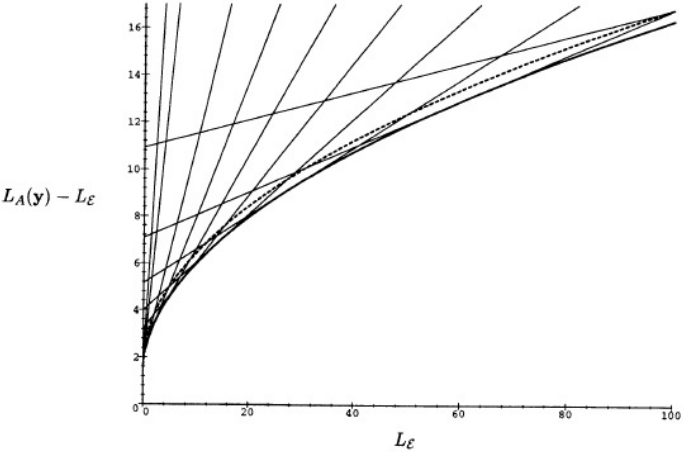
- Reversing signs, using $L_{\mathbf{Hedge}(\eta)} = \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t$ and reorganizing we get

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}$$

# Tuning $\eta$




$L_A(\mathbf{y}) - L_\mathcal{E}$
$L_\mathcal{E}$
0
2
4
6
8
10
12
14
16
0
20
40
60
80
100

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- Then

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}} \leq \min_i L_i + \sqrt{2\tilde{L}\ln N} + \ln N$$

## Tuning $\eta$ as a function of $T$

- trivially $\min_i L_i \leq T$, yielding

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- per iteration we get:

$$\frac{L_{\mathbf{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

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- The adversarial strategy is random, extremely simple, and does not depend on the hedging strategy!

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- The difference $L_{\mathbf{S}} - \min_i L_i$ is due to unlearnable random fluctuations!
- Detailed proof quite involved. See games paper.

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- A trivial random data, in which there is nothing to be learned forces any algorithm to suffer this total loss

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- Observing only the loss of chosen action - the multi-armed bandit problem. Will get to that later in the course.

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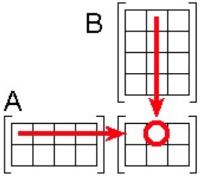


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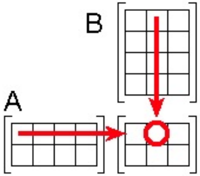


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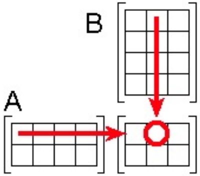


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$$\mathbf{M}(\mathbf{P}, \mathbf{Q}) = \mathbf{P}^T \mathbf{M} \mathbf{Q} = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}(i)\mathbf{M}(i,j)\mathbf{Q}(j)$$

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John von Neumann, 1928.

$$\min_{\mathbf{P}} \max_{\mathbf{Q}} \mathbf{M}(\mathbf{P}, \mathbf{Q}) = \max_{\mathbf{Q}} \min_{\mathbf{P}} \mathbf{M}(\mathbf{P}, \mathbf{Q})$$

In words: for mixed strategies, choosing second gives no advantage.

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- Our goal is to minimize regret.