Lossless compression and cumulative log loss

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- A natural way for describing a distribution.

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t	h	e	r	e		a
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t	h	e	r	e		a	r	e		n
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- To decode use the same prediction algorithm

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- Widely used in practice.

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- Code = discriminating binary expansion of a point in $[l_t, u_t)$.

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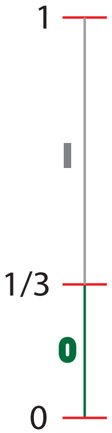

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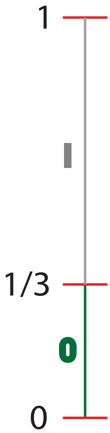

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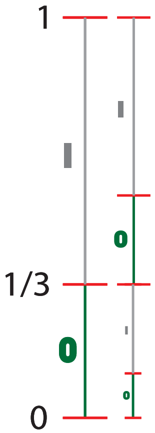

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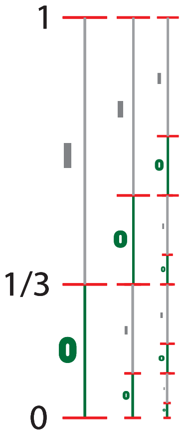

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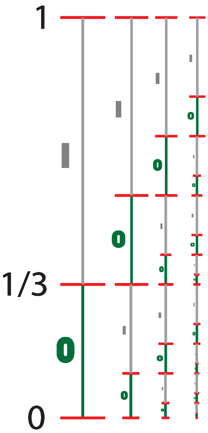

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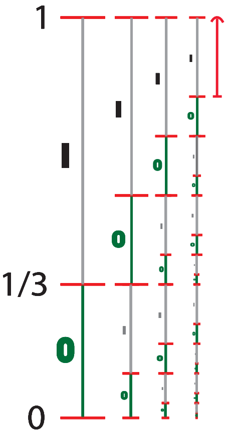

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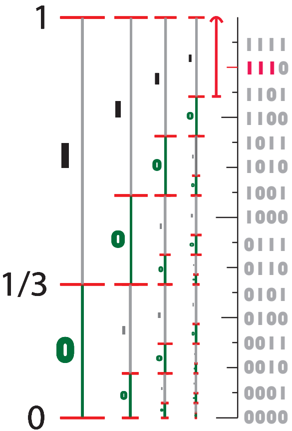

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- Holds for all sequences.

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$$\begin{aligned} & \sum_{c_1, \dots c_T} p(c_1, \dots c_T) \lceil -\log_2 p(c_1, \dots, c_T) \rceil \\ \leq \quad & 1 - \sum_{c_1, \dots c_T} p(c_1, \dots c_T) \log_2 p(c_1, \dots, c_T) \end{aligned}$$

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$$(1 - o(1))H(p_T) = (1 - o(1))\ T\ H(p)$$
- The proof of Shannon's lower bound is not trivial (Can be a student lecture).

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- There are other losses with this property, for example, square loss.

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- High risk prediction: Setting $p_t \in \{0, 1\}$ results in Bonus $b_T = 2^T$ if always correct, zero otherwise.
- If forecaster predicts with the true probabilities then

$$E(\log b_T) = T - H(p_T)$$

and that is the maximal expected value for $E(\log b_T)$

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- Taking logs, we get cumulative log loss.

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- Is the key idea of MDL (Minimal Description Length) modeling.
  - Good prediction model = model that minimizes the total code length
- Often inappropriate because based on lossless coding. **Lossy** coding often more appropriate.

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- **Goal:** Total loss of algorithm minus loss of best predictor should be at most $\log_2 N$

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- $\lceil L_A{}^T \rceil$ is the code length if A is combined with arithmetic coding.

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- **Goal:** minimize regret:

$$-\sum_{t=1}^{T} \log p_A^t(c^t) + \min_{i=1,\ldots,N} \left(-\sum_{t=1}^{T} \log p_i^t(c^t) \right)$$

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\end{aligned}
$$

- Dividing by T we get $\frac{L_A^T}{T} = \min_i \frac{L_i^T}{T} + \frac{\log N}{T}$

Bound better than for two part codes

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- More generally, the regret is smaller if many of the experts perform well.

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