Introduction to Online Learning Algorithms

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Outline

Example trace for Halving Algorithm

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.
outcome

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
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alg.	
outcome	

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
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alg.	1
outcome	

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	1
outcome	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	
outcome	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	
outcome	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	
outcome	1	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
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expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	0

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	0
expert8	1	1	1	0	-
alg.	1	0	1	1	
outcome	1	1	1	0	

Example trace for Halving Algorithm

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expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	0
expert8	1	1	1	0	-
alg.	1	0	1	1	0
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expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	0
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Mistake bound for Halving algorithm

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- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- We assume that at least one expert is perfect.
- Number of mistakes is at most $\log_2 N$.
- No stochastic assumptions whatsoever.
- Proof is based on combining a lower and upper bounds on the number of perfect experts.

The hedging problem

- N possible actions

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  - Algorithm suffers **expected** loss.
- **Goal:** minimize total expected loss
- Here we have stochasticity - but only in **algorithm**, not in **outcome**
- Fits nicely in game theory

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  but not all the way to zero.

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- Unlike halving - no action is perfect.
- Basic idea - reduce probability of lossy actions, but not all the way to zero.
- **Modified Goal:** minimize REGRET = expected total loss and minimal total loss of repeating one action.

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Consider action i at time t

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

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- $\eta > 0$ is the learning rate parameter. Halving: $\eta = \infty$

The Hedge Algorithm

Consider action i at time t

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

- Weight:

$$W_i^t = e^{-\eta L_i^t}$$

- $\eta > 0$ is the learning rate parameter. Halving: $\eta = \infty$
- Probability:

$$P_i^t = \frac{W_i^t}{\sum_{j=1}^N W_j^t}$$

Example trace for Hedge Algorithm

$\eta = 1$

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$$\eta = 1$$

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$
expert1	1
expert2	1
expert3	1
expert4	1
expert5	1
expert6	1
expert7	1
expert8	1

alg.

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$
expert1	1	.1
expert2	1	.8
expert3	1	.3
expert4	1	.1
expert5	1	.9
expert6	1	0
expert7	1	1
expert8	1	.8

alg.

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$
expert1	1	.1
expert2	1	.8
expert3	1	.3
expert4	1	.1
expert5	1	.9
expert6	1	0
expert7	1	1
expert8	1	.8
alg.		.5

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$
expert1	1	.1	.90
expert2	1	.8	.45
expert3	1	.3	.74
expert4	1	.1	.90
expert5	1	.9	.41
expert6	1	0	1
expert7	1	1	.37
expert8	1	.8	.45
alg.		.5	

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$
expert1	1	.1	.90	.1
expert2	1	.8	.45	.5
expert3	1	.3	.74	.2
expert4	1	.1	.90	.7
expert5	1	.9	.41	1
expert6	1	0	1	.1
expert7	1	1	.37	.5
expert8	1	.8	.45	.2
alg.		.5		

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$
expert1	1	.1	.90	.1
expert2	1	.8	.45	.5
expert3	1	.3	.74	.2
expert4	1	.1	.90	.7
expert5	1	.9	.41	1
expert6	1	0	1	.1
expert7	1	1	.37	.5
expert8	1	.8	.45	.2
alg.		.5		.36

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$	$\vec{W}^3$
expert1	1	.1	.90	.1	0.82
expert2	1	.8	.45	.5	0.27
expert3	1	.3	.74	.2	0.61
expert4	1	.1	.90	.7	0.45
expert5	1	.9	.41	1	0.15
expert6	1	0	1	.1	0.91
expert7	1	1	.37	.5	0.22
expert8	1	.8	.45	.2	0.37
alg.		.5		.36	

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$	$\vec{W}^3$	$\vec{\ell}^3$
expert1	1	.1	.90	.1	0.82	0
expert2	1	.8	.45	.5	0.27	.2
expert3	1	.3	.74	.2	0.61	.2
expert4	1	.1	.90	.7	0.45	.8
expert5	1	.9	.41	1	0.15	.8
expert6	1	0	1	.1	0.91	.2
expert7	1	1	.37	.5	0.22	.4
expert8	1	.8	.45	.2	0.37	.6
alg.		.5		.36		

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$	$\vec{W}^3$	$\vec{\ell}^3$
expert1	1	.1	.90	.1	0.82	0
expert2	1	.8	.45	.5	0.27	.2
expert3	1	.3	.74	.2	0.61	.2
expert4	1	.1	.90	.7	0.45	.8
expert5	1	.9	.41	1	0.15	.8
expert6	1	0	1	.1	0.91	.2
expert7	1	1	.37	.5	0.22	.4
expert8	1	.8	.45	.2	0.37	.6
alg.		.5		.36		.30

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$	$\vec{W}^3$	$\vec{\ell}^3$	total
expert1	1	.1	.90	.1	0.82	0	.2
expert2	1	.8	.45	.5	0.27	.2	1.5
expert3	1	.3	.74	.2	0.61	.2	.7
expert4	1	.1	.90	.7	0.45	.8	1.6
expert5	1	.9	.41	1	0.15	.8	2.7
expert6	1	0	1	.1	0.91	.2	.3
expert7	1	1	.37	.5	0.22	.4	1.9
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alg.		.5		.36		.30	

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$	$\vec{W}^3$	$\vec{\ell}^3$	total
expert1	1	.1	.90	.1	0.82	0	.2
expert2	1	.8	.45	.5	0.27	.2	1.5
expert3	1	.3	.74	.2	0.61	.2	.7
expert4	1	.1	.90	.7	0.45	.8	1.6
expert5	1	.9	.41	1	0.15	.8	2.7
expert6	1	0	1	.1	0.91	.2	.3
expert7	1	1	.37	.5	0.22	.4	1.9
expert8	1	.8	.45	.2	0.37	.6	1.6
alg.		.5		.36		.30	1.16

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- L^t_{Hedge}: Expected total loss of Hedge algorithm for time $1, 2, \ldots, t$

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- $$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L^t_i}{1 - e^{-\eta}}$$

Bound for Hedge Algorithm

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- $$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L^t_i}{1 - e^{-\eta}}$$
- Which implies

$$\forall t, \quad L_{\text{Hedge}} \leq \min_i \left(\frac{\ln N + \eta L^t_i}{1 - e^{-\eta}} \right)$$

Bound for Hedge Algorithm

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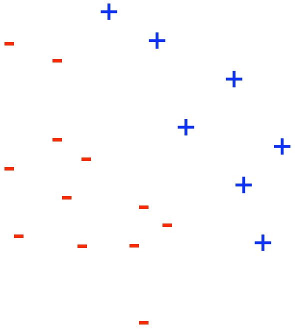
-
$$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L^t_i}{1 - e^{-\eta}}$$
- Which implies
$$\forall t, \quad L_{\text{Hedge}} \leq \min_i \left(\frac{\ln N + \eta L^t_i}{1 - e^{-\eta}} \right)$$
- Proof and choice of η: next class.

The Perceptron Problem

The Perceptron Problem



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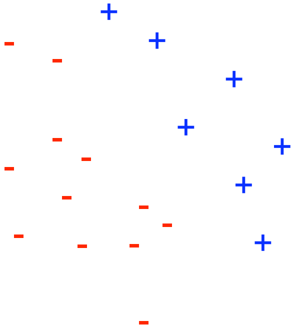

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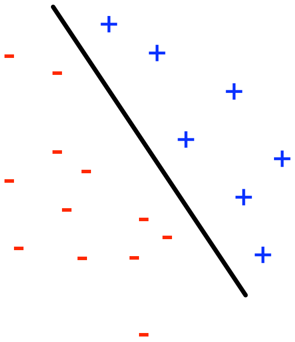

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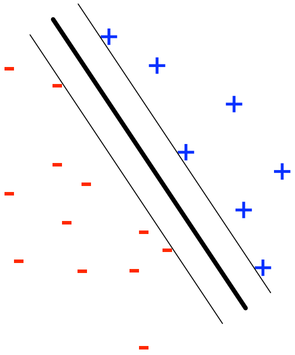

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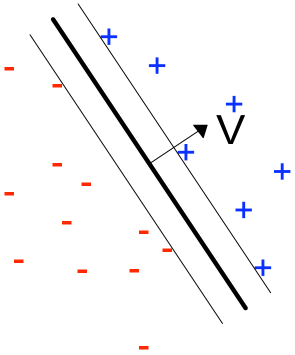

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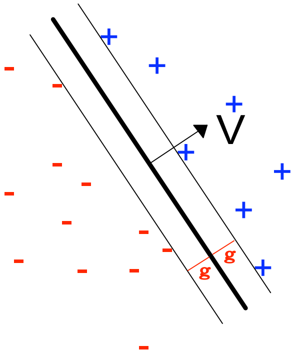

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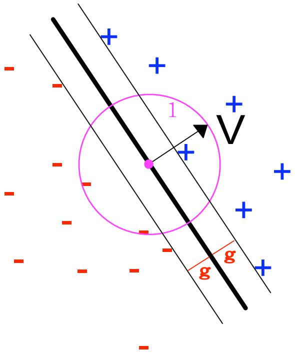

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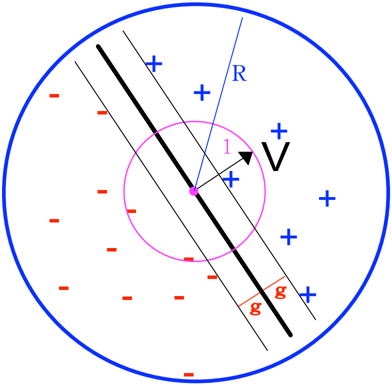

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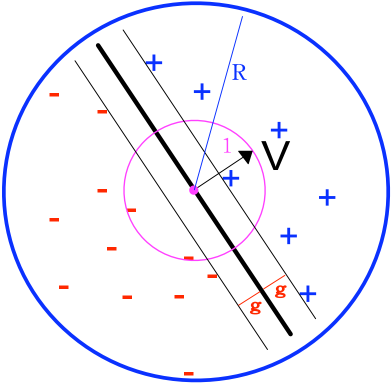
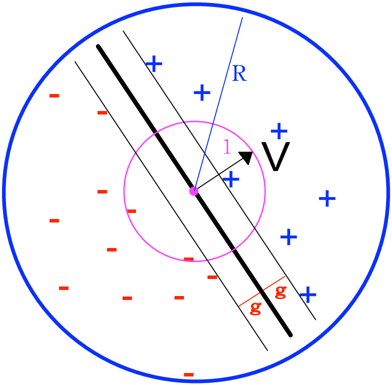


- $\|\vec{V}\| = 1$

The Perceptron Problem



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- Example = $(\vec{X}, y)$, $y \in \{-1, +1\}$.

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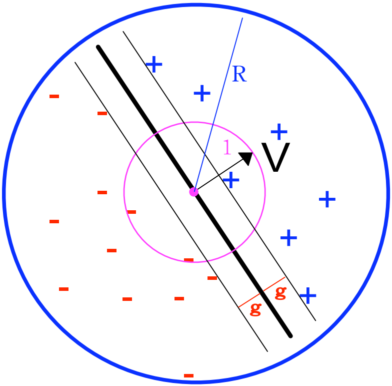

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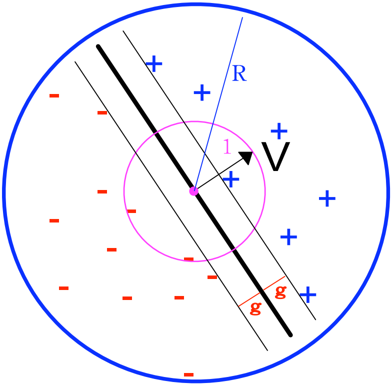

- $\|\vec{V}\| = 1$
- Example = $(\vec{X}, y)$, $y \in \{-1, +1\}$.
- $\forall \vec{X}, \quad \|\vec{X}\| \leq R$.
- $\forall(\vec{X}, y)$, $y(\vec{X} \cdot \vec{V}) \geq g$

The Perceptron learning algorithm

- An online algorithm. Examples presented one by one.

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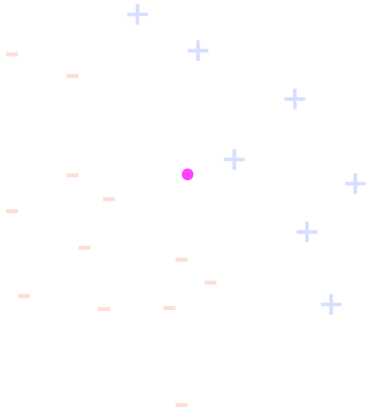
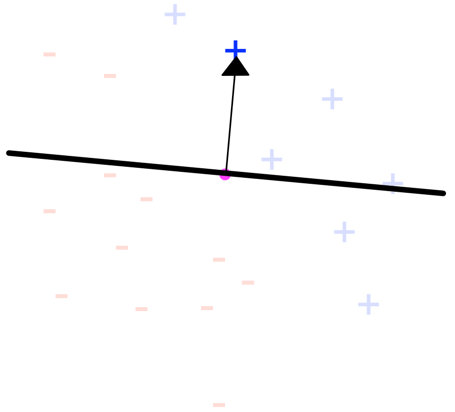
- An online algorithm. Examples presented one by one.
- start with $\vec{W}_0 = \vec{0}$.
- If mistake: $(\vec{W}_i \cdot \vec{X}_i) y_i \leq 0$
  - Update $\vec{W}_{i+1} = \vec{W}_i + y_i X_i$.

Example trace for the perceptron algorithm



Example trace for the perceptron algorithm



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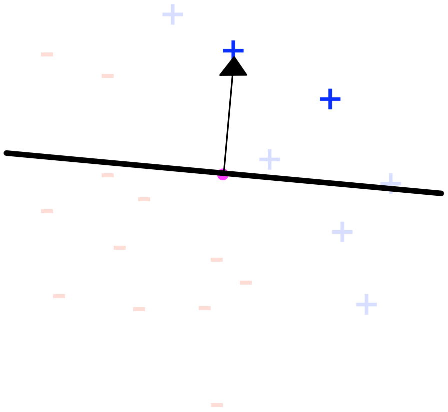

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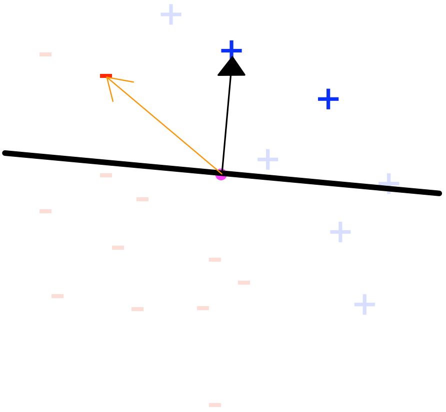

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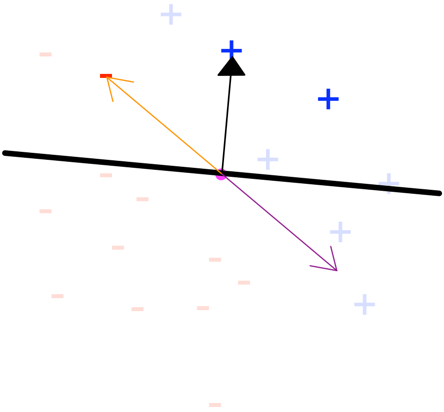

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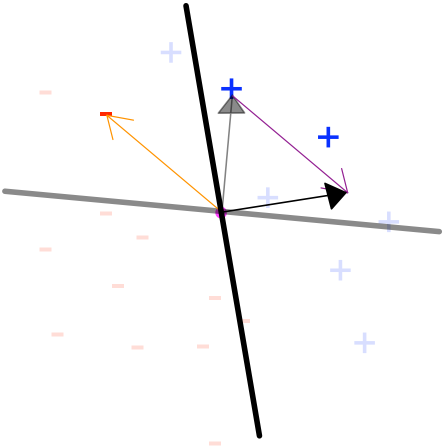

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- The number of mistakes that the perceptron algorithm can make is at most $\left(\frac{R}{g}\right)^2$.

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- Proof by combining upper and lower bounds on $\|\vec{W}\|$.

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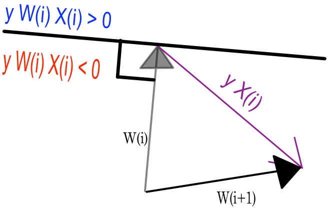

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Proof by induction

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  $\leq \|\vec{W}_i\|^2 + R^2 \leq (i + 1)R^2$

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  $\geq ig + g = (i + 1)g$

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Thus:

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- We observe n coin flips:
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- Natural Answer:

$$\frac{\#\mathbf{H}}{n} = \frac{4}{9}$$

What if the estimation has to be done online?

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