Learning from Binary Labels with Instance-Dependent Corruption

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Abstract

Suppose we have a sample of instances paired with binary labels corrupted by arbitrary instance- and label-dependent noise. With sufficiently many such samples, can we optimally classify and rank instances with respect to the noise-free distribution? We provide a theoretical analysis of this question, with three main contributions. First, we prove that for instance-dependent noise, any algorithm that is consistent for classification on the noisy distribution is also consistent on the clean distribution. Second, we prove that for a broad class of instance- and label-dependent noise, a similar consistency result holds for the area under the ROC curve. Third, for the latter noise model, when the noise-free class-probability function belongs to the generalised linear model family, we show that the Isotron can efficiently and provably learn from the corrupted sample.

1 Learning with label noise: from constant to instance-dependent

Given an instance space \mathcal{X} , and training samples from some distribution \mathcal{D} over $\mathcal{X} \times \{\pm 1\}$, the goal in binary supervised learning is to learn a scorer $s \colon \mathcal{X} \to \mathbb{R}$ with low *risk* on future test samples drawn from \mathcal{D} . Depending on the choice of risk, one arrives at the practically pervasive problems of binary classification [Devroye et al., 1996], class-probability estimation [Buja et al., 2005], and bipartite ranking [Agarwal and Niyogi, 2005]. While the standard setup assumes that the train and test distributions are identical, often the training labels are *corrupted* in some way, so that the training samples are effectively from some $\bar{\mathcal{D}} \neq \mathcal{D}$. The case where the labels are flipped with constant or class-dependent probabilities have been well-studied of late [Natarajan et al., 2013, Scott et al., 2013, Menon et al., 2015, van Rooyen et al., 2015, Patrini et al., 2016].

Our interest is the case where training labels are flipped with unknown, *instance- and label-dependent* probabilities. This challenging setting was recently studied in [Manwani and Sastry, 2013, Ghosh et al., 2015], who established that certain non-convex losses are robust to such noise, provided the true distribution \mathcal{D} is separable. However, compared to the symmetric- and class-conditional noise case, three important questions remain unanswered. First, is suitable risk minimisation on the corrupted sample *consistent* for minimisation on the clean sample? Second, does consistency hold for more general risks, such as that for bipartite ranking? Third, if we have more knowledge as to the structure of \mathcal{D} , can we design efficient algorithms to provably learn from the corrupted samples?

In this paper, we provide positive answers to all questions under mild assumptions on the noise process, and an additional assumption on \mathcal{D} for the third question. Specifically:

• on the theoretical side, we prove that:

- under instance-dependent noise, the Bayes-optimal scorers for certain losses are unchanged (Corollary 3), and that any algorithm consistent for classification on the noisy distribution is also consistent on the clean distribution (Proposition 4);
- under a broad range of instance- and label-dependent noise, the corrupted class-probability function preserves the order of the clean one (Proposition 7), and we have consistency for the area under the ROC curve maximisation on the noisy distribution (Proposition 8);
- on the algorithmic side, we show that if \mathcal{D} has class-probability function belonging to the generalised linear model family, then under the aforementioned class of instance- and label-dependent noise, so does the corrupted class-probability function (Proposition 9); and thus, consistent classification and ranking is afforded by the Isotron [Kalai and Sastry, 2009] (Proposition 10).

Our analysis relies on the structure of the class-probability function under instance- and label-dependent noise (Lemma 1). Our results broadly generalise those for class-conditional label noise in Natarajan et al. [2013], Menon et al. [2015], where this viewpoint has similarly proven useful.

2 Background and notation

We fix some notation and introduce some relevant background material.

2.1 Learning from binary labels

Fix an instance space X. We denote by \mathcal{D} some distribution over $X \times \{\pm 1\}$, with $(X,Y) \sim D$ a pair of random variables. Any \mathcal{D} may be expressed via the *class-conditional distributions* $(P,Q) = (\mathbb{P}(X \mid Y = 1), \mathbb{P}(X \mid Y = -1))$ and *base* rate $\pi = \mathbb{P}(Y = 1)$, or equivalently via *marginal distribution* $M = \mathbb{P}(X)$ and *class-probability function* $\eta \colon x \mapsto \mathbb{P}(Y = 1 \mid X = x)$. We assume $\pi \in (0,1)$.

A scorer is any $s: \mathcal{X} \to \mathbb{R}$. A loss is any $\ell: \{\pm 1\} \times \mathbb{R} \to \mathbb{R}_+$. The ℓ -risk of a scorer s wrt \mathcal{D} is

$$R(s; \mathcal{D}, \ell) \doteq \mathbb{E}_{(\mathsf{X}, \mathsf{Y}) \sim \mathcal{D}} \left[\ell(\mathsf{Y}, s(\mathsf{X})) \right] = \mathbb{E}_{\mathsf{X} \sim M} \left[L(\eta(\mathsf{X}), s(\mathsf{X})) \right], \tag{1}$$

where $L(\eta, v) \doteq \eta \cdot \ell_1(v) + (1 - \eta) \cdot \ell_{-1}(v)$ is the *conditional risk* of ℓ . The *Bayes-optimal* scorers for a loss ℓ are those that minimise the ℓ risk. The ℓ -regret of a scorer $s \colon \mathcal{X} \to \mathbb{R}$ is the excess risk over that of any Bayes-optimal scorer $s^* \in \operatorname{argmin}_s R(s; \mathfrak{D}, \ell)$:

$$\operatorname{reg}(s; \mathcal{D}, \ell) \stackrel{.}{=} R(s; \mathcal{D}, \ell) - R(s^*; \mathcal{D}, \ell) = \mathbb{E}_{\mathsf{X} \sim M} \left[\operatorname{reg}(\eta(\mathsf{X}), s(\mathsf{X}), s^*(\mathsf{X})) \right]$$

where, in an overload of notation, $reg(\eta, s, s^*) \doteq L(\eta, s) - L(\eta, s^*)$.

2.2 Learning from corrupted binary labels

In the standard problem of learning from binary labels, we have access to a sample $S = \{(x_n, y_n)\}_{n=1}^N \sim \mathcal{D}^N$. Our goal is to learn a scorer s from this sample with low ℓ -risk with respect to \mathcal{D} . Fix some notional "clean" (not necessarily separable) distribution \mathcal{D} . In the problem of learning from *corrupted* binary labels, we have access to a sample $\bar{S} = \{(x_n, \bar{y}_n)\}_{n=1}^N \sim \bar{\mathcal{D}}^N$, for some $\bar{\mathcal{D}} \neq \mathcal{D}$ where $\mathbb{P}(X)$ is unchanged, but $\mathbb{P}(\bar{Y} \mid X = x) \neq \mathbb{P}(Y \mid X = x)$. Our goal remains to learn a scorer s from \bar{S} with low ℓ -risk with respect to \mathcal{D} . Examples include learning from symmetric label noise [Angluin and Laird, 1988], and class-conditional noise [Blum and Mitchell, 1998].

Note that we allow \mathcal{D} to be non-separable, i.e. $\eta(x) \cdot (1 - \eta(x)) > 0$ for some $x \in \mathcal{X}$; thus, even under \mathcal{D} , there is not necessarily certainty as to every instance's label. Our use of "noise" and "corruption" thus refers to an additional, exogenous uncertainty in the labelling process.

2.3 Existing work on learning from noisy labels

There is too large a body of work on label noise to fully summarise here (see e.g. Frénay and Kabán [2014] for a recent survey). Broadly, there have been three strands of theoretical analysis.

- (1) *PAC guarantees*. The first strand has focussed on PAC-style guarantees for learning under symmetric and class-conditional noise (e.g. [Bylander, 1994, Blum et al., 1996, Blum and Mitchell, 1998]), noise consistent with the distance to the margin (e.g. Angluin and Laird [1988], Bylander [1997, 1998], Servedio [1999]), noise with bounded error rate¹ (e.g. Kalai et al. [2005], Awasthi et al. [2014]) and arbitrary bounded instance dependent or Massart noise (e.g. Awasthi et al. [2015]). These works often assume the true distribution \mathcal{D} is linearly separable with some margin, the marginal over instances has some structure (e.g. uniform over the unit sphere, or log-concave isotropic), and that one employs linear scorers for learning.
- (2) Surrogate losses. The second strand has focussed on the design of surrogate losses robust to label noise. Long and Servedio [2008] showed that even under symmetric label noise, convex potential minimisation with such scorers will produce classifiers that are akin to random guessing. For class-conditional noise, Natarajan et al. [2013] provided a simple "noise-corrected" version of any loss. Ghosh et al. [2015] showed that losses whose components sum to a constant are robust to symmetric label noise. van Rooyen et al. [2015] showed that the linear or unhinged loss is robust to symmetric label noise. Patrini et al. [2016] showed that a range of "linear-odd" losses (LOLs) are approximately robust to asymmetric label noise, provided that the mean operator is not affected too much by corruption.
- (3) Consistency. The third strand, which is closest to our work, has focussed on showing consistency of appropriate risk minimisation in the regime where one has a suitably powerful function class [Scott et al., 2013, Natarajan et al., 2013, Menon et al., 2015]. For example, Natarajan et al. [2013] showed that minimisation of appropriately weighted convex surrogates on the corrupted distribution $\bar{\mathcal{D}}$ is in fact consistent for the purposes of classification on \mathcal{D} . This work has been restricted to the case of symmetricand class-conditional noise.

In the present paper, we do not make assumptions on \mathcal{D} for our theoretical analysis (unlike (1)), assume one is working with a suitably rich function class (unlike (1) and (2)), and work with general instance- and label-dependent noise models (unlike (2) and (3)).

2.4 The SIM family of class-probability functions

Recall the standard generalised linear model (GLM) family of class-probability functions.

Definition 1 (GLM). For any $u: \mathbb{R} \to [0,1], w^* \in \mathbb{R}^d$, the GLM class-probability function is

$$\operatorname{GLM}(u, w^*) \stackrel{\cdot}{=} x \mapsto u(\langle w^*, x \rangle).$$

For any $u : \mathbb{R} \to [0,1], w^* \in \mathbb{R}^d$, the generalised linear model (GLM) class-probability function is:

$$\operatorname{GLM}(u, w^*) \doteq x \mapsto u(\langle w^*, x \rangle).$$

Logistic regression e.g. assumes $\eta = \operatorname{GLM}(u, w^*)$ for $u(z) = 1/(1 + e^{-z})$. We often refer to $u(\cdot)$ as a link function. In this paper, we are interested in cases where $u(\cdot)$ is $\mathit{unknown}$, but is known to satisfy some mild properties. Specifically, we will study the "single-index model" (SIM) family of class-probability functions [Kalai and Sastry, 2009]. (See Appendix C for some examples.)

Definition 2 (SIM). For any $L, W \in \mathbb{R}_+$, the SIM family of class-probability functions is

$$SIM(L, W) \doteq \{GLM(u, w^*) \colon u \in \mathcal{U}(L), ||w^*|| \le W\}$$

¹For separable 𝔻, this is effectively the agnostic learning problem [Kearns et al., 1994].

3 The ILN model of label noise

We now outline the noise models forming the broad basis of this paper, starting with the most general.

3.1 An instance- and label-dependent noise model

In the general instance- and label-dependent noise model (*ILN model*), a conceptual sample from the true distribution \mathcal{D} has each of its labels flipped with an instance- and label-dependent probability.

Definition 3 (ILN model). Let $\rho_1, \rho_{-1} : \mathfrak{X} \to [0,1]$. Given any distribution \mathfrak{D} , under the ILN model we observe a distribution ILN $(\mathfrak{D}, \rho_{-1}, \rho_1)$ whose samples (X, \bar{Y}) are generated as follows: one draws a pair $(X, Y) \sim \mathfrak{D}$ as usual, but then flips the label with the *instance- and label-dependent* probability $\rho_Y(X)$.

In the sequel, we will always assume the following condition on the flip probability functions.

Assumption 1 (Bounded total noise). The label flip functions satisfy

$$(\forall x \in \mathcal{X}) \,\rho_1(x) + \rho_{-1}(x) < 1. \tag{2}$$

Assumption 1 simply encodes that there is always *some* signal to learn from for each instance. When the flip functions are constant, the requirement is that $\rho_+ + \rho_- < 1$, a standard condition in analysis of the class-conditional setting (e.g. Blum and Mitchell [1998], Scott et al. [2013]). To reiterate that the assumption is employed, we will refer to $\rho_{\pm 1}$ satisfying Assumption 1 as being "admissible".

3.2 Special cases: the IDN and BCN^+ model

There are a few special cases of the general ILN model that will be of interest to us; see Appendix D for more examples and discussion. The first one is the well-studied class-conditional noise (CCN) setting, where $\rho_{\pm 1}$ are constants independent of x. The second is where the noise is instance dependent only, which we call the IDN model.

Definition 4 (IDN model). Consider an ILN model $\operatorname{ILN}(\mathfrak{D}, \rho_{-1}, \rho_1)$ where $\rho_{-1} \equiv \rho_1 \equiv f$ for some function $f \colon \mathfrak{X} \to [0, 1/2)$. We term this problem learning with instance-dependent noise (IDN learning). We will write the corresponding corrupted distribution as $\operatorname{IDN}(\mathfrak{D}, f)$.

The third is where, roughly, the higher the inherent uncertainty (i.e. $\eta \approx 1/2$), the higher the noise.

Definition 5 (BCN⁺ model). Consider an ILN model ILN($\mathcal{D}, \rho_{-1}, \rho_1$) where $\rho_y = f_y \circ s$ for some functions $f_{\pm 1} \colon \mathbb{R} \to [0, 1]$, and a function $s \colon \mathcal{X} \to \mathbb{R}$ such that:

(a) s is order preserving for η i.e.

$$(\forall x, x' \in \mathfrak{X}) \eta(x) < \eta(x') \implies s(x) < s(x').$$

- (b) $f_{\pm 1}$ are non-decreasing when $\eta \leq 1/2$, and non-increasing when $\eta \geq 1/2$.
- (c) The flip function difference $\Delta(z) = f_1(z) f_{-1}(z)$ is non-increasing.

We term this problem learning with generalised boundary consistent noise (BCN⁺ learning). We will write the corresponding corrupted distribution as BCN⁺($\mathcal{D}, f_{-1}, f_1, s$); further, we will say that (f_{-1}, f_1, s, η) are BCN⁺-admissible if they satisfy the conditions detailed above.

Condition (a) above implies that $\eta=u\circ s$ for some non-decreasing u. Condition (b) encodes that $f_{\pm 1}$ are highest when $\eta\approx 1/2$, and lowest when $\eta\cdot(1-\eta)\approx 0$. Condition (c) is more opaque, but is trivially satisfied when the flip functions are identical or constant, and is needed to ensure a monotonicity property of $\bar{\eta}$; this will be discussed in §5.1.

A simple example of the BCN⁺ model (studied in e.g. Du and Cai [2015]) is when $s(x) = \langle w^*, x \rangle$ and $\eta(x) = [s(x) > 0]$ i.e. \mathcal{D} is linearly separable, and further $f_{\pm 1}(z) = g(|z|)$ for some monotone decreasing g. By Condition (b), one has higher noise for instances that are closer to the separator w^* . This is a reasonable model of noise in problems involving human annotation: the more intriniscally "hard" an instance, the higher noise we expect for it. A similar model was studied in Bootkrajang [2016] from a probabilistic perspective.

3.3 The corrupted class-probability function for the ILN model

The nature of the corrupted class-probability function $\bar{\eta}$ for the ILN model (Definition 3) will serve as the basis for learning from such corrupted samples.

Lemma 1. Pick any distribution \mathfrak{D} . Suppose $\bar{\mathfrak{D}} = \mathrm{ILN}(\mathfrak{D}, \rho_{-1}, \rho_1)$ for some admissible $\rho_{\pm 1} \colon \mathfrak{X} \to [0, 1]$. Then, $\bar{\mathfrak{D}}$ has corrupted class-probability function

$$(\forall x \in \mathcal{X}) \,\bar{\eta}(x) = (1 - \rho_1(x)) \cdot \eta(x) + \rho_{-1}(x) \cdot (1 - \eta(x)). \tag{3}$$

or equivalently,

$$(\forall x \in \mathcal{X}) \, \eta(x) = \frac{\bar{\eta}(x) - \rho_{-1}(x)}{1 - \rho_{1}(x) - \rho_{-1}(x)}.$$

Remark. Were it true that $\rho_1(x) + \rho_{-1}(x) = 1$ for some x, then we would have $\bar{\eta}(x) = \rho_{-1}(x)$ i.e. it is independent of the actual $\eta(x)$ value. Thus, Assumption 1 specifies that it is possible to infer something about $\eta(x)$ from $\bar{\eta}(x)$.

Lemma 1 generalises Natarajan et al. [2013, Lemma 7], Menon et al. [2015, Appendix C], who derived $\bar{\eta}$ for the case of CCN learning. See Appendix E for more special cases.

4 Classification consistency under instance-dependent noise

Suppose one minimises the ℓ^{01} -risk on the corrupted distribution. Does this imply minimisation of the ℓ^{01} -risk on the *clean* distribution i.e. is the former *consistent* for clean ℓ^{01} -risk minimisation? We will show that this is indeed true for instance-dependent noise, and for a range of losses ℓ beyond ℓ^{01} .

4.1 Relating clean and corrupted risks

Our first step is to relate the ℓ -risk on the clean and corrupted distributions. Following Ghosh et al. [2015], we will consider instance-dependent noise $IDN(\mathcal{D}, f)$, and losses ℓ that satisfy

$$(\forall v \in \mathbb{R}) \,\ell_{-1}(v) + \ell_1(v) = C \tag{4}$$

for some constant $C \in \mathbb{R}$. This condition was considered previously in Ghosh et al. [2015] to study noise-robustness, and is satisfied by the zero-one, ramp, and unhinged losses. Under these two assumptions, we can show the clean risk is an *instance-weighted* version of the corrupted risk. To simplify notation, for any $w \colon \mathcal{X} \to \mathbb{R}_+$, let the corresponding *weighted* ℓ -risk be

$$R^{\operatorname{wt}(w)}(s; \mathcal{D}, \ell) = \mathbb{E}_{\mathsf{X} \sim M} \left[w(\mathsf{X}) \cdot L(\eta(\mathsf{X}), s(\mathsf{X})) \right].$$

Then, we have the following, which is implicit in the proof of Ghosh et al. [2015, Theorem 1].

Proposition 2. Pick any distribution \mathbb{D} , and loss ℓ satisfying Equation 4. Suppose that $\bar{\mathbb{D}} = \mathrm{IDN}(\mathbb{D}, f)$ for admissible $f: \mathcal{X} \to [0, 1/2)$. Then, for any scorer $s: \mathcal{X} \to \mathbb{R}$,

$$R(s; \mathcal{D}, \ell) = R^{\operatorname{wt}(w)}(s; \bar{\mathcal{D}}, \ell) + A(\mathcal{D}, f)$$

where $w(x) = (1 - 2 \cdot f(x))^{-1}$, and $A(\mathcal{D}, f)$ is some term independent of s.

4.2 Relating clean and corrupted regrets

Proposition 2 has an important, non-obvious implication: under instance-specific noise, for losses satisfying Equation 4, the Bayes-optimal scorers on the clean and corrupted distributions coincide. This is a simple consequence of the fact that weighting a risk does *not* affect Bayes-optimal scorers.

Corollary 3. Pick any distribution \mathbb{D} , and loss ℓ satisfying Equation 4. Suppose that $\bar{\mathbb{D}} = \mathrm{IDN}(\mathbb{D}, f)$ for admissible $f: \mathcal{X} \to [0, 1/2)$. Then,

$$\underset{s \in \mathbb{R}^{\mathcal{X}}}{\operatorname{argmin}} R(s; \mathcal{D}, \ell) = \underset{s \in \mathbb{R}^{\mathcal{X}}}{\operatorname{argmin}} R(s; \bar{\mathcal{D}}, \ell).$$

For the case of $\ell=\ell^{01}$, Corollary 3 implies that the optimal classifiers on the two distribution coincide: $\operatorname{sign}(2\eta(x)-1)=\operatorname{sign}(2\bar{\eta}(x)-1)$. In fact, we can go further, and establish a relation between the clean and corrupted *regrets* of an arbitrary scorer.

Proposition 4. Pick any distribution \mathcal{D} , and loss ℓ satisfying Equation 4. Suppose that $\bar{\mathcal{D}} = \mathrm{IDN}(\mathcal{D}, f)$ for admissible $f: \mathcal{X} \to [0, 1/2)$ with $\rho_{\max} = \max_{x \in \mathcal{X}} f(x)$. Then, for any $s: \mathcal{X} \to \mathbb{R}$,

$$\operatorname{reg}(s; \mathfrak{D}, \ell) \le \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \operatorname{reg}(s; \bar{\mathfrak{D}}, \ell).$$

Further, if $\sup_{x \in \mathcal{X}} \operatorname{reg}(\bar{\eta}(x), s(x), s^*(x)) \leq R < +\infty$, then for any $\alpha \in [0, 1]$,

$$\operatorname{reg}(s; \mathcal{D}, \ell) \leq \frac{1}{(1 - 2 \cdot \rho_{\max})^{1 - \alpha}} \cdot R^{\alpha} \cdot \left(\mathbb{E}_{\mathsf{X} \sim M} \left[w(\mathsf{X}) \right] \right)^{\alpha} \cdot \left(\operatorname{reg}(s; \bar{\mathcal{D}}, \ell) \right)^{1 - \alpha}.$$

Since $(\mathbb{E}_{X \sim M}[w(X)])^{\alpha} \leq M^{\alpha}$ trivially, the dependence on ρ_{\max} is less stringent in the second bound above, at the expense of a possibly worse dependence on the corrupted regret. It is intuitive that one downweight the contribution of large weights that occur on instances with low marginal probability. Note that we trivially have R=1 for the case of $\ell=\ell^{01}$, since the loss is bounded.

By Proposition 4, for $\theta = \ell^{01}$, if we can find a sequence $\{s_n\}$ of scorers satisfying $\operatorname{reg}(s_n; \bar{\mathcal{D}}, \ell^{01}) \to 0$, then we also guarantee $\operatorname{reg}(s; \mathcal{D}, \ell^{01}) \to 0$, i.e. we have consistency of classification on the *clean* distribution. One can guarantee $\operatorname{reg}(s; \bar{\mathcal{D}}, \ell^{01}) \to 0$ by minimising an appropriate convex surrogate to ℓ^{01} on $\bar{\mathcal{D}}$, owing to standard classification calibration results [Zhang, 2004, Bartlett et al., 2006]. This surrogate does *not* have to satisfy Equation 4.

In the course of proving Proposition 2, we actually establish a more general relation between clean and corrupted risks (Proposition 15 in Appendix) that holds for ILN noise, and losses not satisfying Equation 4. This general result cannot however be used to prove a meaninguful regret bound beyond the IDN case with losses satisfying Equation 4; see Appendix A.3 for a discussion.

4.3 Beyond misclassification error?

Can the above consistency result be extended to generalised classification performance measures such as balanced error and F-score? Disappointingly, the answer is no. The reason is simple: for measures beyond the 0-1 loss, the following shows that Bayes-optimal classifier itself will not coincide on the clean and corrupted distributions, so that no analogue of Corollary 3 can possibly hold.

Proposition 5. Pick any distribution \mathcal{D} . Suppose that $\bar{\mathcal{D}} = \mathrm{IDN}(\mathcal{D}, f)$ for admissible $f : \mathcal{X} \to [0, 1/2)$. Then, for any $t \in [0, 1]$,

$$(\forall x \in \mathfrak{X}) \, \eta(x) > t \iff \bar{\eta}(x) > t + f(x) \cdot (1 - 2 \cdot t).$$

For any $t \neq 1/2$, Proposition 5 implies that classification on the corrupted distribution requires knowledge of the flipping function f(x) i.e. only the 0-1 threshold is preserved under corruption.

²This assumes minimisation over $\mathbb{R}^{\mathcal{X}}$, and may not hold with a restricted function class; see Appendix J.

³For a simpler proof of Proposition 4 that is specific to ℓ^{01} , see Appendix K.

4.4 Relation to existing work

The results of this section generalise those for the SLN model in Natarajan et al. [2013]. In particular, for the SLN model Proposition 2, reduces to Natarajan et al. [2013, Theorem 9], Corollary 3 to Natarajan et al. [2013, Corollary 10], and Proposition 4 to Natarajan et al. [2013, Theorem 11].

In the instance-dependent noise case, for linearly separable \mathcal{D} , Awasthi et al. [2015] observed that the Bayes-optimal classifier is unchanged, and Ghosh et al. [2015, Theorem 1] established that if $\min_s R(s; \mathcal{D}, \ell^{01}) = 0$, i.e. if \mathcal{D} is separable, then the 0-1 risk minimiser using *any* function class is unaffected. Corollary 3 above shows that if we work with a suitably rich function class, then we have equivalence of risk minimisers even for non-separable \mathcal{D} . Ghosh et al. [2015, Theorem 1] showed that one can make guarantees about the degradation of risk minimisation wrt losses ℓ satisfying Equation 4. This result only holds for the risk wrt the clean distribution \mathcal{D} , and does *not* give a regret bound. More precisely, Ghosh et al. [2015, Theorem 1] (generalised to the case of general instance- and label-dependent noise in ?), showed the following.

Theorem 6 ([Ghosh et al., 2015, Theorem 1]). *Pick any distribution* \mathcal{D} *and loss* ℓ *satisfying Equation* 4. *Let* $\bar{\mathcal{D}} = \mathrm{IDN}(\mathcal{D}, f)$ *for some admissible* $f: \mathcal{X} \to [0, 1/2)$. *Then, for any function class* $\mathcal{S} \subseteq \mathbb{R}^{\mathcal{X}}$,

$$R(\bar{s}^*; \mathcal{D}, \ell) \le \frac{R(s^*; \mathcal{D}, \ell)}{1 - 2 \cdot \max_x f(x)}$$

where

$$\begin{split} s^* &= \operatorname*{argmin}_{s \in \mathbb{S}} R(s; \mathcal{D}, \ell) \\ \bar{s}^* &= \operatorname*{argmin}_{s \in \mathbb{S}} R(s; \mathrm{IDN}(\mathcal{D}, f), \ell). \end{split}$$

Theorem 6 implies that for instance-dependent noise, the ℓ -risk minimiser (for suitable ℓ) will not differ considerably on the clean and corrupted samples. But a limitation of the result is that one cannot guarantee *consistency* wrt, e.g. 0-1 loss, of using the result of ℓ -risk minimisation on the corrupted samples. This is because the above only holds for the risk wrt the clean distribution \mathfrak{D} . It does *not* let us bound the clean regret $\operatorname{reg}(s; \mathfrak{D}, \ell)$ in terms of the corrupted regret $\operatorname{reg}(s; \operatorname{ILN}(\mathfrak{D}, \rho), \ell)$.

Proposition 5 generalises Natarajan et al. [2013, Theorem 9], Menon et al. [2015, Section F.1] which were for the CCN and SLN models. For the CCN model, Menon et al. [2015] established that the Bayes-optimal classifier for the *balanced error* is also unaffected by corruption. One might expect this to carry over to the instance-dependent case, but perhaps surprisingly, this is not the case.

5 AUC consistency under the BCN⁺ model

Bipartite ranking is concerned with the ranking risk

$$R_{\text{rank}}(s; \mathcal{D}) = \mathbb{E}_{\mathsf{X} \sim P, \mathsf{X}' \sim Q} \left[\ell_1^{01} (s(\mathsf{X}) - s(\mathsf{X}')) \right]$$

viz. one minus the area under the ROC curve (AUC) [Agarwal and Niyogi, 2005]. Can an analogous regret bound to Proposition 4 be established for this risk? Unfortunately, without further assumptions, this is not possible. The reason is as before: to establish a regret bound, the Bayes-optimal scorers must coincide. As the AUC is optimised by any scorer that is order preserving for η [Clémençon et al., 2008], the corrupted AUC will be optimised by any scorer that is order preserving for $\bar{\eta}$. For the two to coincide, we will have to ensure that $\bar{\eta}$ is order preserving for η . Intuitively, this will not be true in general, since there is no necessary relationship between $\rho_{\pm 1}$ and η ; see Appendix I.

5.1 Relating clean and corrupted AUC regret under the BCN⁺ model

It is of interest to determine conditions under which we *can* guarantee order preservation of η . Intuitively, this will require there being *some* dependence between the flip functions and η . Fortunately, the previously introduced BCN⁺ model is a feasible candidate.

Proposition 7. Pick any distribution \mathfrak{D} . Suppose $\bar{\mathfrak{D}} = \mathrm{BCN}^+(\mathfrak{D}, f_{-1}, f_1, s)$ where (f_{-1}, f_1, s, η) are BCN^+ -admissible. Then,

$$(\forall x, x' \in \mathfrak{X}) \eta(x) < \eta(x') \implies \bar{\eta}(x) < \bar{\eta}(x')$$

so that $\eta = \phi \circ \bar{\eta}$ for some non-decreasing ϕ .

Proving Proposition 7 relies on establishing a relation between $\bar{\eta}(x) - \bar{\eta}(x')$ and its counterpart on the clean distribution⁴. We emphasise that Condition (c) in the BCN⁺ model is vital to the result; see Appendix I for an example where removing this condition leads to a forfeit of order preservation.

Using Proposition 7, we can deduce that the BCN⁺ model affords a suitable AUC regret bound.

Proposition 8. Pick any distribution \mathbb{D} . Suppose that $\bar{\mathbb{D}} = \mathrm{BCN}^+(\mathbb{D}, f_{-1}, f_1, s)$ where (f_{-1}, f_1, s, η) are BCN^+ -admissible, and the total noise bound (Assumption 1) is

$$\rho_{\max} = \frac{1}{2} \cdot \max_{x \in \mathcal{X}} (\rho_1(x) + \rho_{-1}(x)) < \frac{1}{2}.$$

Then, for any scorer $s: \mathfrak{X} \to \mathbb{R}$,

$$\mathrm{reg}_{\mathrm{rank}}(s;\mathcal{D}) \leq \frac{\bar{\pi} \cdot (1 - \bar{\pi})}{\pi \cdot (1 - \pi)} \cdot \frac{1}{1 - 2 \cdot \rho_{\mathrm{max}}} \cdot \mathrm{reg}_{\mathrm{rank}}(s;\bar{\mathcal{D}})$$

where $\operatorname{reg}_{\operatorname{rank}}$ denotes the excess ranking risk of a scorer s.

Thus, under the BCN⁺ model, maximising AUC on the corrupted sample is consistent for maximisation on the clean sample. As before, one can appeal to surrogate regret bounds for the AUC [Agarwal, 2014] to deduce that appropriate surrogate minimisation on $\bar{\mathbb{D}}$ will ensure $\operatorname{reg}_{\operatorname{rank}}(s; \bar{\mathbb{D}}) \to 0$.

Remark. Proposition 8 is slightly surprising in the sense that the AUC can be expressed as an average of the balanced error across a range of thresholds [Flach et al., 2011]. Proposition 5 suggested that in general we do cannot have a regret bound for the clean and corrupted balanced errors. Note that such a bound would imply one for the AUC, but not vice-versa.

5.2 Relation to existing work

Proposition 7 is, to our knowledge, novel. Proposition 8 generalises Menon et al. [2015, Corollary 3], which established a risk *equivalence* between the clean and corrupted AUC for class-conditional noise. The reason an equivalence is possible in the CCN setting is that here, $\bar{\eta}(x) - \bar{\eta}(x)$ is just a scaling of $\eta(x) - \eta(x')$, so that both bounds in the proof above are tight (see Example 1 in Appendix).

6 Learning noisy SIMs with the Isotron

While the preceding sections establish consistency of corrupted risk minimisation, a practical concern is how precisely one ensures vanishing regret on the corrupted distribution. Certainly this is possible if one chooses *s* from the set of all measurable scorers (e.g. by employing a universal kernel with appropriately tuned parameters), but this may be infeasible in practical scenarios demanding the use of some restricted function class, e.g. linear scorers in a low-dimensional feature space.

We turn our attention to providing a simple algorithm guaranteeing $\operatorname{reg}(s; \bar{\mathcal{D}}, \ell^{01}) \to 0$ and $\operatorname{reg}_{\operatorname{rank}}(s; \bar{\mathcal{D}}) \to 0$ when the class of linear scorers is suitable for \mathcal{D} , and the noise possesses some structure. Specifically, we focus on \mathcal{D} such that $\eta \in \operatorname{SIM}$, so that $\eta = \operatorname{GLM}(u, w^*)$ for some (unknown) u, w^* . We then consider a BCN⁺ model of the noise, with $s^*(x) = \langle w^*, x \rangle$ determining the flip probability; for convenience, we shall call this the single index noise or "SIN" model.

Definition 6. Let $f_1, f_{-1} : \mathbb{R} \to [0, 1]$. Given any distribution \mathfrak{D} with $\eta = GLM(u, w^*)$ for some

⁴For some special cases, the proofs simplify considerably; see Appendix L.

$$(u, w^*)$$
, define SIN(\mathcal{D}, f_{-1}, f_1) \doteq BCN⁺($\mathcal{D}, f_{-1}, f_1, s^*$) where $s^*: x \mapsto \langle w^*, x \rangle$.

A special case of the above is where \mathcal{D} is separable with some margin (see Appendix C), and one observes corrupted samples with instances closer to the separator having a higher chance of being corrupted. This is a seemingly reasonable model when labels are provided by human annotators. A similar model was considered in Du and Cai [2015], where it is assumed that the link $u(\cdot)$ is known.

6.1 Corruption runs in the SIN family

Proposition 7 established that for the BCN⁺ model, $\bar{\eta}$ is order preserving for η . When $\eta \in SIM$, this implies that for the SIN model with suitably Lipschitz label flipping functions, $\bar{\eta} \in SIM$ as well.

Proposition 9. Pick any distribution \mathfrak{D} with $\eta \in \mathrm{SIM}(L,W)$. Suppose that $\bar{\mathfrak{D}} = \mathrm{SIN}(\mathfrak{D}, f_{-1}, f_1)$ where (f_{-1}, f_1, η) are BCN^+ -admissible, and (f_{-1}, f_1) are (L_{-1}, L_1) -Lipschitz respectively. Then, $\bar{\eta} \in \mathrm{SIM}(L + L_{-1} + L_1, W)$. In particular, $\bar{\eta}(x) = \bar{u}(\langle w^*, x \rangle)$ where

$$\bar{u}(z) = (1 - f_1(z)) \cdot u(z) + f_{-1}(z) \cdot (1 - u(z)). \tag{5}$$

Examples of the form of $\bar{\eta}$ for specific $u(\cdot)$ are presented in Appendix M.

6.2 The Isotron: an efficient algorithm to learn noisy SIMs

Proposition 9 suggests that for a large class of noisy label problems, if one can learn a generic SIM, then the corrupted class-probability function may be estimated. Fortunately, SIMs can be provably learned with the Isotron [Kalai and Sastry, 2009], and its Lipschitz variant, the SLIsotron [Kakade et al., 2011]. The elegant algorithm consists of alternately updating the hyperplane w, and the link function u. The latter is estimated using the PAV algorithm [Ayer et al., 1955], which finds a solution to the isotonic regression problem:

$$(\hat{u}_1, \dots, \hat{u}_m) = \underset{u_1 \le u_2 \le \dots \le u_m}{\operatorname{argmin}} \sum_{i=1}^m (y_i - u_i)^2,$$

where we assume that the s_i 's are ordered such that $s_1 \leq s_2 \leq \ldots \leq s_m$, i.e. we wish for the u's to respect the ordering of the s's. The PAV algorithm provides a nonparametric estimate of $u(\cdot)$ at the specified points. At other points, one may use linear interpolation. The SLIsotron algorithm is identical to the Isotron, except that one calls LPAV, a variant of PAV that obeys a Lipschitz constraint.

One can provide precise theoretical guarantees on the output of the (SL)Isotron. Combined with Proposition 4, this lets one make guarantees about classification from corrupted labels.

Proposition 10. Pick any distribution \mathcal{D} over $\mathbb{B}^d \times \{\pm 1\}$ with $\eta \in \mathrm{SIM}(L,W)$. Suppose that $\bar{\mathcal{D}} = \mathrm{SIN}(\mathcal{D}, f_{-1}, f_1)$ where (f_{-1}, f_1) are Lipschitz. Then, for ℓ^{sq} being the square loss, we can construct $\hat{\eta}_S \colon \mathcal{X} \to [0, 1]$ from a corrupted sample $\bar{\mathsf{S}} \sim \bar{\mathcal{D}}^m$ using the SLIsotron, such that

$$\operatorname{reg}_{\operatorname{rank}}(\hat{\bar{\eta}}_{\bar{\mathsf{S}}}; \mathfrak{D}) \stackrel{\mathbb{P}}{\to} 0.$$

Further, if $f_{-1} = f_1$, we can construct a classifier $c_{\bar{S}} : x \mapsto \text{sign}(2\hat{\bar{\eta}}_{\bar{S}} - 1)$ such that

$$\operatorname{reg}(c_{\bar{\mathsf{S}}}; \mathfrak{D}, \ell^{01}) \stackrel{\mathbb{P}}{\to} 0.$$

A salient feature of the Isotron is that one need not know the precise form of either η , nor the label flipping functions. Even if one just knows that there exists some u such that $\eta = \operatorname{GLM}(u, w^*)$, and that the labels are subject to (Lipschitz) monotonic noise, one can estimate $\bar{\eta}$. Even when $u(\cdot)$ is known, \bar{u} will involve the typically unknown flipping functions. Thus, the Isotron solves a non-trivial estimation problem; see Appendix N for an illustrative example in the CCN setting.

Remark. Suppose one knows the precise form of u, but does not know w^* . For example, one may know that \mathcal{D} is separable with a certain margin. Then, under the *symmetric* BCN⁺ model, we can in fact infer

the label flipping function as

$$f(z) = \frac{\bar{u}(z) - u(z)}{1 - 2 \cdot u(z)}.$$

The estimation error in this term depends wholly on the error in estimating \bar{u} .

7 Experiments with Isotron and boundary-consistent noise

We now empirically verify that the Isotron can learn SIMs subject to noise from the BCN+ model.

7.1 Synthetic data

We first consider a \mathcal{D} such that M is a mixture of 2D Gaussians with identity covariance, and means (1,1) and (-1,-1). We picked $\eta\colon x\mapsto \llbracket s^*(x)>0 \rrbracket$ where $s^*(x)=x_1+x_2$. For flip functions $f_{\pm 1}\colon z\mapsto 1/(1+e^{|z|})$, we drew a sample $\bar{\mathsf{S}}$ of 5000 elements from $\bar{\mathcal{D}}=\mathrm{BCN}^+(\mathcal{D},f_{-1},f_1,s^*)$, the boundary-consistent corruption of \mathcal{D} . We then estimated $\bar{\eta}$ from $\bar{\mathsf{S}}$ using 1000 iterations of Isotron. Figure 1 shows this estimate closely matches the actual $\bar{\eta}$ (computed explicitly via Equation 3). Further, on a test sample of 5000 instances from \mathcal{D} , thresholding our estimate around 1/2 gives essentially perfect (99.46%) accuracy.

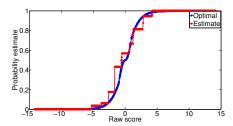


Figure 1: Isotron results for estimating $\bar{\eta}$ on synthetic data (§7.1).

7.2 Real-world data

We next ran experiments involving boundary consistent noise on the USPS and MNIST handwritten digit datasets. We converted both datasets into binary classification tasks by seeking to distinguish digits 0 and 9 for the former, and 6 and 7 for the latter. In each case, the binary classification task is nearly linearly separable: to make it fully separable, we took the optimal hyperplane w^* found by ordinary least squares, and discarded all instances with margin violations for a margin $\gamma=0.1$.

On the resulting separable dataset, we created a training sample S comprising 80% of instances, with the remaining 20% of instances used for testing. For each $(x,y) \in S$, we inject boundary-consistent noise by flipping the label with probability $f(x) = \left(1 + e^{\alpha \cdot |\langle w^*, x \rangle|}\right)^{-1}$ for some parameter $\alpha \in [0, \infty)$. The resulting corrupted sample \bar{S} mimics a scenario where the labels are provided by a human annotator more liable to make errors for the easily confusable digits. We then trained a regularised least squares model (using regularisation strength $\lambda = 10^{-8}$), and the Isotron (using 100 iterations) on \bar{S} . We measured the models' classification accuracy on the test set with *clean* labels.

Table 1 reports the mean and standard error of the accuracies of ridge regression and the Isotron over T=25 independent label corruptions for both datasets. We report results for $\alpha \in \{2^{-3},\ldots,2^3\}$, and for each α note the % of labels that end up being flipped. We find that for lower α , both methods perform comparably. This is unsurprising, as under low noise both should eventually find the optimal separator. For higher α , the Isotron offers a significant improvement over ridge regression (upto 17% on MNIST), in keeping with our analysis that it can effectively learn from instance-dependent noise.

α	Flip %	Ridge ACC	Isotron ACC	α	Flip %	Ridge ACC	Isotron ACC	
1/8	0.03 ± 0.01	0.9940 ± 0.0003	0.9974 ± 0.0002	1/8	0.04 ± 0.00	0.9958 ± 0.0001	0.9984 ± 0.0001	
1/4	0.17 ± 0.01	0.9947 ± 0.0004	0.9974 ± 0.0003	1/4	0.44 ± 0.01	0.9958 ± 0.0001	0.9979 ± 0.0001	
1/2	2.15 ± 0.09	0.9944 ± 0.0004	0.9937 ± 0.0006	$^{1/2}$	4.25 ± 0.04	0.9953 ± 0.0002	0.9966 ± 0.0003	
1	11.84 ± 0.17	0.9853 ± 0.0012	0.9700 ± 0.0021	1	15.97 ± 0.05	0.9871 ± 0.0005	0.9864 ± 0.0007	
2	26.57 ± 0.22	0.8988 ± 0.0053	0.9239 ± 0.0050	2	29.97 ± 0.09	0.9446 ± 0.0012	0.9565 ± 0.0013	
4	37.65 ± 0.24	0.7410 ± 0.0072	0.7863 ± 0.0138	4	39.49 ± 0.08	0.8262 ± 0.0022	0.8768 ± 0.0041	
8	43.76 ± 0.25	0.6185 ± 0.0078	0.6467 ± 0.0405	8	44.63 ± 0.08	0.6872 ± 0.0024	0.8088 ± 0.0291	
(a) USPS 0 vs 9					(b) MNIST 6 vs 7			

Table 1: Mean and standard error for 0-1 accuracies of ridge regression ("Ridge") and Isotron over T=25 independent corruption trials. See text in §7.2 for details of parameter α .

8 Conclusion

We have analysed the problem of learning with instance- and label-dependent noise, concluding that for instance-dependent noise, minimising the classification risk on the noisy distribution is consistent for classification on the clean distribution; for a broad class of instance- and label-dependent noise, a similar consistency result holds for the area under the ROC curve; and one can learn generalised linear models subject to the same noise model using the Isotron.

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Supplementary Material for "Learning from Binary Labels with Instance-Dependent Corruption"

Appendix A contains some helper results useful for the proofs of results in the main body. These proofs are provided in Appendix B. The subsequent sections contain further discussion and examples of material from the main body. (There are no new theoretical results in these later sections, but rather, simply expositions of special cases of results in the main body.)

A Additional helper results

A.1 Order preservation

We will make use of the following simple fact about order preservation, stated without proof.

Lemma 11. Suppose $f, g: \mathbb{R} \to \mathbb{R}$ are such that

$$(\forall x, y \in \mathbb{R}) f(x) < f(y) \implies g(x) < g(y).$$

Then, $f = u \circ g$ for some non-decreasing u.

Taking the contrapositive gives us an alternate useful statement.

Corollary 12. Suppose $f, g: \mathbb{R} \to \mathbb{R}$ are such that

$$(\forall x, y \in \mathbb{R}) g(x) \le g(y) \implies f(x) \le f(y).$$

Then, $f = u \circ g$ for some non-decreasing u.

Finally, we can make a more precise statement about behaviour when g(x) = g(y) under the above conditions.

Lemma 13. Suppose $f, g: \mathbb{R} \to \mathbb{R}$ are such that

$$(\forall x, y \in \mathbb{R}) f(x) < f(y) \implies g(x) < g(y).$$

Then,

$$(\forall x, y \in \mathbb{R}) g(x) = g(y) \implies f(x) = f(y).$$

$$(\forall x, y \in \mathbb{R}) g(x) < g(y) \implies f(x) \le f(y).$$

Proof. By the contrapositive,

$$(\forall x, y \in \mathbb{R}) g(x) \le g(y) \implies f(x) \le f(y).$$

If g(x) < g(y) then trivially $g(x) \le g(y)$ and the result follows. Suppose that g(x) = g(y). Then $g(x) \le g(y)$ and $g(y) \le g(x)$. Thus $f(x) \le f(y)$ and $f(y) \le f(x)$, i.e. f(x) = f(y).

Note that if we only know that $g(x) < g(y) \implies f(x) \le f(y)$, we cannot conclude that $f = u \circ g$, nor that $g = u \circ f$; we must be able to conclude something about the behaviour of f when g(x) = g(y).

A.2 Additional properties of \bar{D}

From Proposition 1, we can derive expressions for the corrupted base rate and class-conditional distributions.

Corollary 14. Pick any distribution \mathfrak{D} . Then, for any $\rho_1, \rho_{-1}: \mathfrak{X} \to [0,1]$, $\mathrm{ILN}(\mathfrak{D}, \rho_{-1}, \rho_1)$ has

$$\bar{\pi} = \pi - \mathbb{E}_{\mathsf{X} \sim M} \left[(\rho_1(\mathsf{X}) + \rho_{-1}(\mathsf{X})) \cdot \eta(\mathsf{X}) \right] + \mathbb{E}_{\mathsf{X} \sim M} \left[\rho_{-1}(\mathsf{X}) \right]$$

$$\bar{P}(x) = \bar{\pi}^{-1} \cdot \left((1 - \rho_1(x)) \cdot \pi \cdot P(x) + \rho_{-1}(x) \cdot (1 - \pi) \cdot Q(x) \right)$$

$$\bar{Q}(x) = (1 - \bar{\pi})^{-1} \cdot \left(\rho_1(x) \cdot \pi \cdot P(x) + (1 - \rho_{-1}(x)) \cdot (1 - \pi) \cdot Q(x) \right) .$$

Proof. Proposition 1 implies that the corrupted base rate is

$$\bar{\pi} = \mathbb{E}_{\mathsf{X} \sim M} \left[(1 - \rho_1(\mathsf{X})) \cdot \eta(\mathsf{X}) + \rho_{-1}(\mathsf{X}) \cdot (1 - \eta(\mathsf{X})) \right]$$
$$= \pi - \mathbb{E}_{\mathsf{X} \sim M} \left[(\rho_1(\mathsf{X}) + \rho_{-1}(\mathsf{X})) \cdot \eta(\mathsf{X}) \right] + \mathbb{E}_{\mathsf{X} \sim M} \left[\rho_{-1}(\mathsf{X}) \right) \right],$$

which is a complex translation of the clean base rate. Further, the corrupted class-conditional distributions are

$$\bar{P}(x) = \frac{\bar{\eta}(x) \cdot M(x)}{\bar{\pi}}$$

$$\begin{split} &= \frac{\left((1 - \rho_1(x) - \rho_{-1}(x)) \cdot \eta(x) + \rho_{-1}(x) \right) \cdot M(x)}{\bar{\pi}} \\ &= (1 - \rho_1(x) - \rho_{-1}(x)) \cdot P(x) \cdot \frac{\pi}{\bar{\pi}} + \frac{\rho_{-1}(x) \cdot M(x)}{\bar{\pi}} \\ &= \bar{\pi}^{-1} \cdot \left((1 - \rho_1(x)) \cdot \pi \cdot P(x) + \rho_{-1}(x) \cdot (1 - \pi) \cdot Q(x) \right), \end{split}$$

and similarly

$$\bar{Q}(x) = (1 - \bar{\pi})^{-1} \cdot (\rho_1(x) \cdot \pi \cdot P(x) + (1 - \rho_{-1}(x)) \cdot (1 - \pi) \cdot Q(x)).$$

For the class-conditionals, we can equally write

$$P(x) = (1 - \rho_1(x) - \rho_{-1}(x))^{-1} \cdot \pi^{-1} \cdot \left((1 - \rho_{-1}(x)) \cdot \bar{\pi} \cdot \bar{P}(x) - \rho_{-1}(x) \cdot (1 - \bar{\pi}) \cdot \bar{Q}(x) \right)$$

$$Q(x) = (1 - \rho_1(x) - \rho_{-1}(x))^{-1} \cdot (1 - \pi)^{-1} \cdot \left(-\rho_1(x) \cdot \bar{\pi} \cdot \bar{P}(x) + (1 - \rho_1(x)) \cdot (1 - \bar{\pi}) \cdot \bar{Q}(x) \right).$$
(6)

A.3 Relating clean and corrupt risks

We have the following general relationship between the risk on the clean and corrupted distributions.

Proposition 15. Pick any distribution \mathbb{D} , and any loss ℓ . Suppose that $\bar{\mathbb{D}} = \mathrm{ILN}(\mathbb{D}, \rho_1, \rho_{-1})$ for admissible $\rho_{+1} \colon \mathcal{X} \to [0, 1]$. Then, for any scorer $s \colon \mathcal{X} \to \mathbb{R}$,

$$R(s;\mathcal{D},\ell) = \mathbb{E}_{(\mathsf{X},\bar{\mathsf{Y}})\sim\bar{\mathcal{D}}} \left[\tilde{\ell}(\bar{\mathsf{Y}},s,\mathsf{X}) \right]$$

where $\tilde{\ell}$: $\{\pm 1\} \times \mathbb{R}^{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ is a "generalised loss" given by

$$\tilde{\ell}_1(s,x) = w(x) \cdot ((1 - \rho_{-1}(x)) \cdot \ell_1(s(x)) - \rho_1(x) \cdot \ell_{-1}(s(x)))$$

$$\tilde{\ell}_{-1}(s,x) = w(x) \cdot (-\rho_{-1}(x) \cdot \ell_1(s(x)) + (1 - \rho_1(x)) \cdot \ell_{-1}(s(x)))$$

where $w(x) = (1 - \rho_1(x) - \rho_{-1}(x))^{-1}$.

Proof of Proposition 15. By Proposition 1, for ILN($\mathcal{D}, \rho_1, \rho_{-1}$),

$$(\forall x \in \mathfrak{X}) \, \eta(x) = \frac{\bar{\eta}(x) - \rho_{-1}(x)}{w(x)}$$

and

$$(\forall x \in \mathfrak{X}) \, 1 - \eta(x) = \frac{1 - \bar{\eta}(x) - \rho_1(x)}{w(x)},$$

where $w(x) = (1 - \rho_1(x) - \rho_{-1}(x))^{-1}$. Thus, the ℓ -risk of an arbitrary scorer is

$$R(s; \mathcal{D}, \ell) = \mathbb{E}_{X \sim M} \left[L(\eta(X), s(X)) \right]$$

$$= \mathbb{E}_{X \sim M} \left[\eta(X) \cdot \ell_1(s(X)) + (1 - \eta(X) \cdot \ell_{-1}(s(X))) \right]$$

$$= \mathbb{E}_{X \sim M} \left[w(X)^{-1} \cdot ((\bar{\eta}(X) - \rho_{-1}(X)) \cdot \ell_1(s(X)) + (1 - \bar{\eta}(X) - \rho_1(X)) \cdot \ell_{-1}(s(X))) \right]. \quad (7)$$

Observe that this may be re-expressed as

$$\begin{split} R(s;\mathcal{D},\ell) &= \mathbb{E}_{\mathsf{X} \sim M} \left[w(\mathsf{X})^{-1} \cdot ((\bar{\eta}(\mathsf{X}) - (\bar{\eta}(\mathsf{X}) + (1 - \bar{\eta}(\mathsf{X}))) \cdot \rho_{-1}(\mathsf{X})) \cdot \ell_{1}(s(\mathsf{X})) \right] + \\ & \mathbb{E}_{\mathsf{X} \sim M} \left[w(\mathsf{X})^{-1} \cdot ((1 - \bar{\eta}(\mathsf{X}) - (\bar{\eta}(\mathsf{X}) + (1 - \bar{\eta}(\mathsf{X}))) \cdot \rho_{1}(\mathsf{X})) \cdot \ell_{-1}(s(\mathsf{X}))) \right] \\ &= \mathbb{E}_{\mathsf{X} \sim M} \left[\bar{\eta}(\mathsf{X}) \cdot w(\mathsf{X})^{-1} \cdot ((1 - \rho_{-1}(\mathsf{X})) \cdot \ell_{1}(s(\mathsf{X})) - \rho_{1}(\mathsf{X}) \cdot \ell_{-1}(s(\mathsf{X}))) \right] + \\ & \mathbb{E}_{\mathsf{X} \sim M} \left[(1 - \bar{\eta}(\mathsf{X})) \cdot w(\mathsf{X})^{-1} \cdot (-\rho_{-1}(\mathsf{X}) \cdot \ell_{1}(s(\mathsf{X})) + (1 - \rho_{1}(\mathsf{X})) \cdot \ell_{-1}(s(\mathsf{X}))) \right] \\ &= \mathbb{E}_{(\mathsf{X}, \bar{\mathsf{Y}}) \sim \bar{\mathcal{D}}} \left[\tilde{\ell}(\bar{\mathsf{Y}}, s, \mathsf{X}) \right]. \end{split}$$

Proposition 15 is a generalisation of Natarajan et al. [2013, Lemma 1]. For the CCN setting, the "generalised loss" object of this Proposition simplifies to the "noise-corrected loss" studied in Natarajan et al. [2013], with Proposition 15 simply being the "method of unbiased estimators" described in that paper (see Appendix H).

Note that Proposition 15 cannot be used to establish a regret bound in general. This is because the "generalised loss" above only simplifies to a weighted version of ℓ under very specific cases (with an example being the IDN model and the partial losses summing to a constant).

A.4 Relating clean and corrupt thresholds

For a general ILN model, we have the following.

Proposition 16. Pick any distribution \mathcal{D} . Suppose that $\bar{\mathcal{D}} = \mathrm{ILN}(\mathcal{D}, \rho_{-1}, \rho_1)$ for admissible $\rho_{\pm 1} \colon \mathcal{X} \to [0, 1]$. Then, for any $t \in [0, 1]$,

$$(\forall x \in \mathcal{X}) \, \eta(x) > t \iff \bar{\eta}(x) > (1 - \rho_1(x) - \rho_{-1}(x)) \cdot t + \rho_{-1}(x).$$

Proof of Proposition 16. By Proposition 1,

$$\eta(x) = \frac{\bar{\eta}(x) - \rho_{-1}(x)}{1 - \rho_{1}(x) - \rho_{-1}(x)}.$$

Now if $\rho_1(x) + \rho_{-1}(x) < 1$ for every $x, 1 - \rho_1(x) - \rho_{-1}(x) > 0$. We thus have

$$\begin{split} \eta(x) > t &\iff \frac{\bar{\eta}(x) - \rho_{-1}(x)}{1 - \rho_{1}(x) - \rho_{-1}(x)} > t \\ &\iff \bar{\eta}(x) - \rho_{-1}(x) > (1 - \rho_{1}(x) - \rho_{-1}(x) \cdot t \text{ since } 1 - \rho_{1}(x) - \rho_{-1}(x) > 0 \\ &\iff \bar{\eta}(x) > (1 - \rho_{1}(x) - \rho_{-1}(x)) \cdot t + \rho_{-1}(x). \end{split}$$

A.5 Difference in $\bar{\eta}$ values

For the general ILN model, we have the following relation between the difference in $\bar{\eta}$ values and the corresponding η values.

Lemma 17. Pick any distribution \mathfrak{D} . Suppose $\bar{\mathfrak{D}} = \mathrm{ILN}(\mathfrak{D}, \rho_{-1}, \rho_1)$. Then,

$$(\forall x, x' \in \mathfrak{X}) \, \bar{\eta}(x) - \bar{\eta}(x') = (1 - \rho_{-1}(x') - \rho_{1}(x')) \cdot (\eta(x) - \eta(x')) + \Delta_{1}(x, x')$$
$$= (1 - \rho_{-1}(x) - \rho_{1}(x)) \cdot (\eta(x) - \eta(x')) + \Delta_{2}(x, x'),$$

where

$$\Delta_1(x, x') = (\rho_{-1}(x) - \rho_{-1}(x')) \cdot (1 - \eta(x)) + (\rho_1(x') - \rho_1(x)) \cdot \eta(x)$$

$$\Delta_2(x, x') = (\rho_{-1}(x) - \rho_{-1}(x')) \cdot (1 - \eta(x')) + (\rho_1(x') - \rho_1(x)) \cdot \eta(x').$$

Example 1. For the case of CCN learning $CCN(\mathcal{D}, \rho_{-1}, \rho_1)$, $\Delta_1 \equiv \Delta_2 \equiv 0$ and so we have the simpler expression

$$\bar{\eta}(x) - \bar{\eta}(x') = (1 - \alpha - \beta) \cdot (\eta(x) - \eta(x')),$$

from which order preservation is immediate.

Example 2. For the case of IDN learning IDN(\mathfrak{D}, f),

$$\Delta_1(x, x') = (f(x) - f(x')) \cdot (1 - 2 \cdot \eta(x))$$

$$\Delta_2(x, x') = (f(x) - f(x')) \cdot (1 - 2 \cdot \eta(x')).$$

Thus,

$$\bar{\eta}(x) - \bar{\eta}(x') = (1 - 2 \cdot f(x')) \cdot (\eta(x) - \eta(x')) + (f(x) - f(x')) \cdot (1 - 2 \cdot \eta(x))$$
$$= (1 - 2 \cdot f(x)) \cdot (\eta(x) - \eta(x')) + (f(x) - f(x')) \cdot (1 - 2 \cdot \eta(x')).$$

Order preservation here will depend on the structure of f.

Proof of Lemma 17. By Proposition 1,

$$\bar{\eta}(x) = (1 - \rho_1(x) - \rho_{-1}(x)) \cdot \eta(x) + \rho_{-1}(x).$$

We have

$$\bar{\eta}(x) - \bar{\eta}(x') = (1 - \rho_{-1}(x) - \rho_{1}(x)) \cdot \eta(x) - (1 - \rho_{-1}(x') - \rho_{1}(x')) \cdot \eta(x') + \rho_{-1}(x) - \rho_{-1}(x')$$

$$= (1 - \rho_{-1}(x') - \rho_{1}(x')) \cdot (\eta(x) - \eta(x')) + \Delta_{1}(x, x'), \tag{8}$$

where

$$\Delta_1(x,x') = (\rho_{-1}(x') + \rho_1(x') - \rho_{-1}(x) - \rho_1(x)) \cdot \eta(x) + (\rho_{-1}(x) - \rho_{-1}(x'))$$

= $(\rho_{-1}(x) - \rho_{-1}(x')) \cdot (1 - \eta(x)) + (\rho_1(x') - \rho_1(x)) \cdot \eta(x);$

alternately, we have

$$\bar{\eta}(x) - \bar{\eta}(x') = (1 - \rho_{-1}(x) - \rho_1(x)) \cdot (\eta(x) - \eta(x')) + \Delta_2(x, x') \tag{9}$$

where

$$\Delta_2(x,x') = (\rho_{-1}(x') - \rho_{-1}(x) + \rho_1(x') - \rho_1(x)) \cdot \eta(x') + (\rho_{-1}(x) - \rho_{-1}(x'))$$

= $(\rho_{-1}(x) - \rho_{-1}(x')) \cdot (1 - \eta(x')) + (\rho_1(x') - \rho_1(x)) \cdot \eta(x').$

For the BCN⁺ model, Lemma 17 can be converted to show that $\bar{\eta}$ is a monotone transform of s, the underlying score used in the noise model; furthermore, we have a simple bound on the differences in $\bar{\eta}$ values in terms of the corresponding difference in η values.

Lemma 18. Pick any distribution \mathbb{D} . Suppose $\bar{\mathbb{D}} = \mathrm{BCN}^+(\mathbb{D}, f_{-1}, f_1, s)$ where (f_{-1}, f_1, s, η) are BCN^+ -admissible. Then,

$$(\forall x, x' \in \mathfrak{X}) \, s(x) \leq s(x') \implies \bar{\eta}(x) - \bar{\eta}(x') \leq \max(1 - \rho_{-1}(x) - \rho_{1}(x), 1 - \rho_{-1}(x') - \rho_{1}(x')) \cdot (\eta(x) - \eta(x'))$$

where $\rho_{\pm 1}(x) = f_{\pm 1} \circ s$.

Proof of Lemma 18. For the BCN model, Lemma 17 is

$$(\forall x, x' \in \mathfrak{X}) \, \bar{\eta}(x) - \bar{\eta}(x') = (1 - f_{-1}(z') - f_1(z')) \cdot (u(z) - u(z')) + \Delta_1(z, z')$$
$$= (1 - f_{-1}(z) - f_1(z)) \cdot (u(z) - u(z')) + \Delta_2(z, z'),$$

where z = s(x), z' = s(x'), and

$$\Delta_1(z, z') = (f_{-1}(z) - f_{-1}(z')) \cdot (1 - u(z)) + (f_1(z') - f_1(z)) \cdot u(z)$$

$$\Delta_2(z, z') = (f_{-1}(z) - f_{-1}(z')) \cdot (1 - u(z')) + (f_1(z') - f_1(z)) \cdot u(z').$$

Suppose that s(x) = s(x'). Then clearly $\Delta_1 \equiv \Delta_2 \equiv 0$ and u(z) = u(z'), so $\bar{\eta}(x) = \bar{\eta}(x')$.

Suppose that s(x) < s(x') so that $\eta(x) \le \eta(x')$; or equivalently, z < z' so that $u(z) \le u(z')$. Our goal is to show that $\min(\Delta_1(z,z'),\Delta_2(z,z')) \le 0$; this will imply the desired bound, since we can just use the tighter of the implied bounds on Equation 8 and 9. By Condition (c) of BCN⁺-admissibility, for any z < z',

$$f_1(z) - f_{-1}(z) \ge f_1(z') - f_{-1}(z')$$

⁵By contrapositive of Condition (a) of BCN⁺-admissibility, if $s(x) \le s(x')$ then $\eta(x) \le \eta(x')$.

or equivalently

$$f_1(z') - f_1(z) < f_{-1}(z') - f_{-1}(z).$$

Thus, since $u(z) \geq 0$, we have

$$\Delta_1(z, z') \le (f_{-1}(z) - f_{-1}(z')) \cdot (1 - 2 \cdot u(z)),\tag{10}$$

and similarly,

$$\Delta_2(z, z') < (f_{-1}(z) - f_{-1}(z')) \cdot (1 - 2 \cdot u(z')). \tag{11}$$

We now argue why the minimum of these terms must be ≤ 0 . Consider the following three cases:

- (a) Suppose $f_{-1}(z) = f_{-1}(z')$. Then trivially both terms are ≤ 0 .
- (b) Suppose $f_{-1}(z) < f_{-1}(z')$. Then either $u(z) \le \frac{1}{2}$ or $u(z') \le \frac{1}{2}$; if both u values are larger than $\frac{1}{2}$, then by BCN-admissibility Condition (b) it must be true that $f_{-1}(z) \ge f_{-1}(z')$, a contradiction. Thus either $1 2 \cdot u(z) \ge 0$ or $1 2 \cdot u(z') \ge 0$, and so one of the terms must be ≤ 0 .
- (c) Suppose $f_{-1}(z) > f_{-1}(z')$. Then either $u(z) \geq \frac{1}{2}$ or $u(z') \geq \frac{1}{2}$; if both u values are smaller than $\frac{1}{2}$, then by BCN-admissibility Condition (b) it must be true that $f_{-1}(z) \leq f_{-1}(z')$, a contradiction. Thus either $1 2 \cdot u(z) \leq 0$ or $1 2 \cdot u(z') \leq 0$, and so one of the terms must be ≤ 0 .

Thus, we conclude $\min(\Delta_1(z,z'),\Delta_2(z,z')) \leq 0$, and so either

$$\bar{\eta}(x) - \bar{\eta}(x') \le (1 - \rho_{-1}(x) - \rho_{1}(x)) \cdot (\eta(x) - \eta(x'))$$

or

$$\bar{\eta}(x) - \bar{\eta}(x') \le (1 - \rho_{-1}(x) - \rho_{1}(x')) \cdot (\eta(x) - \eta(x'))$$

must be true; since $\eta(x) - \eta(x') \le 0$ and $\max(1 - \rho_{-1}(x) - \rho_{1}(x), 1 - \rho_{-1}(x) - \rho_{1}(x')) > 0$, this implies

$$\bar{\eta}(x) - \bar{\eta}(x') \le \max(1 - \rho_{-1}(x) - \rho_{1}(x), 1 - \rho_{-1}(x) - \rho_{1}(x')) \cdot (\eta(x) - \eta(x')).$$

Since $\eta(x) - \eta(x') \le 0$ and $\max(1 - \rho_{-1}(x) - \rho_{1}(x), 1 - \rho_{-1}(x) - \rho_{1}(x')) > 0$, we may bound the entire expression by 0, thus concluding that $\bar{\eta}(x) \le \bar{\eta}(x')$.

An immediate consequence of Lemma 18 is that $\bar{\eta}$ is order-preserving for the underlying scores.

Corollary 19. Suppose $\bar{\mathcal{D}} = \mathrm{BCN}^+(\mathcal{D}, f_{-1}, f_1, s)$ where (f_{-1}, f_1, s, η) are BCN^+ -admissible. Then,

$$(\forall x, x' \in \mathfrak{X}) s(x) < s(x') \implies \bar{n}(x) < \bar{n}(x')$$

and so $\bar{\eta} = \bar{u} \circ s$ for some non-decreasing \bar{u} .

Proof. By Lemma 18, if s(x) = s(x') then $\bar{\eta}(x) = \bar{\eta}(x')$. If s(x) < s(x') then $\eta(x) \le \eta(x')$ by BCN-admissibility Condition (a). Further, $1 - \rho_1(x) - \rho_{-1}(x) > 0$ by Assumption 1. Thus, $\bar{\eta}(x) - \bar{\eta}(x') \le 0$. The fact that $\bar{\eta} = \bar{u} \circ s$ follows from Corollary 12.

Remark. By definition of BCN admissibility, $\eta = u \circ s$ for some monotone u; and by Lemma 18, $\bar{\eta} = \bar{u} \circ s$, for some monotone \bar{u} . If we could establish that \bar{u} were *strictly* monotone, then we would immediately conclude $\eta = u \circ \bar{u}^{-1} \circ \bar{\eta}$, which would establish Proposition 7. But this is not true in general; fortunately, \bar{u} is only constant when u is (owing to the explicit bound in Lemma 18), and so we are still able to write $\eta = \phi \circ \bar{\eta}$ for some monotone ϕ .

A.6 Class-probability estimation guarantees with the Isotron

The basic SLIsotron guarantee is as follows.

Proposition 20 ([Kakade et al., 2011, Theorem 2]). Pick any \mathbb{D} over $\mathbb{B}^d \times \{\pm 1\}$ with $\eta \in SIM(1, W)$ for some $W \in \mathbb{R}_+$. Let $\{\hat{\eta}_{S,t}\}_{t=1}^{\infty}$ denote the estimates of η produced at each iteration of SLISotron, when applied to a training sample S. Then,

$$\mathbb{P}_{\mathsf{S} \sim D^m} \left(\min_t \operatorname{reg}(\hat{\eta}_{\mathsf{S},t}; \mathcal{D}, \ell^{\operatorname{sq}}) \le \left(\frac{dW^2}{m} \right)^{1/3} \cdot \left(\log \frac{Wm}{\delta} \right)^{1/3} \right) \ge 1 - \delta$$

where

$$\operatorname{reg}(\hat{\eta}; \mathcal{D}, \ell^{\operatorname{sq}}) = \mathbb{E}_{\mathsf{X} \sim M} \left[(\hat{\eta}(\mathsf{X}) - \eta(\mathsf{X}))^2 \right].$$

 $[\]overline{\ ^6 \text{If } \eta \in \operatorname{SIM}(L,W), \text{ then trivially } \eta \in \operatorname{SIM}(1,L \cdot W), \text{ because } \eta(x) = u(\langle w^*,x \rangle) = u((1/L) \cdot \langle (L \cdot w^*),x \rangle) = \tilde{u}(\langle \tilde{w}^*,x \rangle), \text{ where } \tilde{u} \text{ is a 1-Lipschitz function, and } ||\tilde{w}^*|| = L \cdot W.$

B Proofs of results in main body

Proof of Lemma 1. By definition of how corrupted labels \bar{Y} are generated,

$$\begin{split} \bar{\eta}(x) &= \mathbb{P}(\bar{\mathsf{Y}} = 1 \mid \mathsf{X} = x) \\ &= \sum_{y \in \{\pm 1\}} \mathbb{P}(\bar{\mathsf{Y}} = 1 \mid \mathsf{Y} = y, \mathsf{X} = x) \cdot \mathbb{P}(\mathsf{Y} = y \mid \mathsf{X} = x) \\ &= (1 - \rho_1(x)) \cdot \eta(x) + \rho_{-1}(x) \cdot (1 - \eta(x)). \end{split}$$

The second identity follows by rearranging.

Proof of Proposition 2. By Equation 7 from the proof of Proposition 15, for ILN($\mathcal{D}, \rho_1, \rho_{-1}$),

$$\begin{split} R(s; \mathcal{D}, \ell) &= \mathbb{E}_{\mathsf{X} \sim M} \left[w(\mathsf{X})^{-1} \cdot \left((\bar{\eta}(\mathsf{X}) - \rho_{-1}(\mathsf{X})) \cdot \ell_{1}(s(\mathsf{X})) + (1 - \bar{\eta}(\mathsf{X}) - \rho_{1}(\mathsf{X})) \cdot \ell_{-1}(s(\mathsf{X})) \right) \right] \\ &= \mathbb{E}_{\mathsf{X} \sim M} \left[w(\mathsf{X})^{-1} \cdot (\bar{\eta}(\mathsf{X}) \cdot \ell_{1}(s(\mathsf{X})) + (1 - \bar{\eta}(\mathsf{X})) \cdot \ell_{-1}(s(\mathsf{X}))) \right] - \\ &\mathbb{E}_{\mathsf{X} \sim M} \left[w(\mathsf{X})^{-1} \cdot (\rho_{-1}(\mathsf{X}) \cdot \ell_{1}(s(\mathsf{X})) + \rho_{1}(\mathsf{X}) \cdot \ell_{-1}(s(\mathsf{X}))) \right] \\ &= \mathbb{E}_{\mathsf{X} \sim M} \left[w(\mathsf{X})^{-1} \cdot (L(\bar{\eta}(\mathsf{X}), s(\mathsf{X})) - \rho_{-1}(\mathsf{X}) \cdot \ell_{1}(s(\mathsf{X})) + \rho_{1}(\mathsf{X}) \cdot \ell_{-1}(s(\mathsf{X}))) \right]. \end{split}$$

If $\rho_1 \equiv \rho_{-1} \equiv f$, $w(x) = 1 - 2 \cdot f(x)$ and

$$R(s; \mathcal{D}, \ell) = \mathbb{E}_{\mathsf{X} \sim M} \left[\frac{1}{1 - 2 \cdot f(\mathsf{X})} \cdot L(\bar{\eta}(\mathsf{X}), s(\mathsf{X})) \right] - \mathbb{E}_{\mathsf{X} \sim M} \left[\frac{f(\mathsf{X})}{1 - 2 \cdot f(\mathsf{X})} \cdot (\ell_1(s(\mathsf{X})) + \ell_{-1}(s(\mathsf{X}))) \right].$$

Thus, if the sum of the partial losses is a constant C,

$$R(s; \mathcal{D}, \ell) = R^{\operatorname{wt}(w)}(s; \bar{\mathcal{D}}, \ell) - C \cdot \mathbb{E}_{\mathsf{X} \sim M} \left[\frac{f(\mathsf{X})}{1 - 2 \cdot f(\mathsf{X})} \right].$$

Noting that the second term above does not depend on the scorer s, the result follows.

Proof of Corollary 3. By Proposition 2,

$$\underset{s}{\operatorname{argmin}} R(s; \mathcal{D}, \ell) = \underset{s}{\operatorname{argmin}} R^{\operatorname{wt}(w)}(s; \bar{\mathcal{D}}, \ell)$$
$$= \underset{s}{\operatorname{argmin}} R(s; \bar{\mathcal{D}}, \ell),$$

where the second line is because weighting does not affect the Bayes-optimal scorers for a risk. (Note that by definition, the weighting factor $w(x) = (1 - 2 \cdot f(x))^{-1} \ge 1$, and so no term is suppressed after weighting.)

Alternate proof of Corollary 3. If a loss ℓ satisfies Equation 4, its conditional risk is

$$L(\eta, v) = (2 \cdot \eta - 1) \cdot \ell_1(v) + C \cdot (1 - \eta).$$

Thus, the pointwise minimiser of the conditional risk is

$$\begin{split} \underset{v}{\operatorname{argmin}} \, L(\eta, v) &= \underset{v}{\operatorname{argmin}} \, (2 \cdot \eta - 1) \cdot \ell_1(v) \\ &= \underset{v}{\operatorname{argmin}} \, \begin{cases} \ell_1(v) & \text{if } \eta > 1/2 \\ -\ell_1(v) & \text{if } \eta < 1/2, \end{cases} \end{split}$$

implying a Bayes-optimal scorer of

$$(\forall x \in \mathcal{X}) \, s^*(x) = \underset{v}{\operatorname{argmin}} \, \begin{cases} \ell_1(v) & \text{if } \eta(x) > 1/2 \\ -\ell_1(v) & \text{if } \eta(x) < 1/2. \end{cases}$$

Now we recall for the IDN model, $\eta(x) > 1/2 \iff \bar{\eta}(x) > 1/2$. Thus, the two cases in the above scorer are the same for the clean and corrupted distributions. It follows that the Bayes-optimal scorer is retained.

Proof of Proposition 4. Let $s^* \in \underset{\circ}{\operatorname{argmin}} R(s; \mathcal{D}, \ell)$. By definition,

$$\begin{split} \operatorname{reg}(s; \mathfrak{D}, \ell) &= R(s; \mathfrak{D}, \ell) - R(s^*; \mathfrak{D}, \ell) \\ &= R^{\operatorname{wt}(w)}(s; \bar{\mathfrak{D}}, \ell) - R^{\operatorname{wt}(w)}(s^*; \bar{\mathfrak{D}}, \ell) \text{ by Proposition 2} \\ &= \mathbb{E}_{\mathsf{X} \sim M} \left[\frac{1}{1 - 2 \cdot \rho(\mathsf{X})} \cdot (L(\bar{\eta}(\mathsf{X}), s(\mathsf{X})) - L(\bar{\eta}(\mathsf{X}), s^*(\mathsf{X}))) \right] \\ &\leq \frac{1}{1 - 2 \cdot \rho_{\max}} \mathbb{E}_{\mathsf{X} \sim M} \left[L(\bar{\eta}(\mathsf{X}), s(\mathsf{X})) - L(\bar{\eta}(\mathsf{X}), s^*(\mathsf{X})) \right] \text{ by assumption on } \rho \\ &= \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot (R(s; \bar{\mathfrak{D}}, \ell) - R(s^*; \bar{\mathfrak{D}}, \ell)) \\ &= \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \operatorname{reg}(s; \bar{\mathfrak{D}}, \ell), \end{split}$$

where the last line is since by Corollary 3, we know that $s^* \in \operatorname{argmin} R(s; \bar{\mathcal{D}}, \ell)$ also. (Note that for the inequality step above, we can guarantee $L(\bar{\eta}(x), s(x)) \geq L(\bar{\eta}(x), s^*(x))$ for every $x \in \mathcal{X}$ because $s^* \in \operatorname{argmin} R(s; \bar{\mathcal{D}}, \ell)$, and so we do not have to worry about the direction of the inequality.)

To get the parameterised bound, suppose $w(x)=(1-2\cdot\rho(x))^{-1}$, and r(x) is the conditional regret $L(\bar{\eta}(x),s(x))-L(\bar{\eta}(x),s^*(x))$. Then for any $\alpha\in[0,1]$ the regret can be rewritten

$$\begin{split} \operatorname{reg}(s; \mathcal{D}, \ell) &= \mathbb{E}_{\mathsf{X} \sim M} \left[w(\mathsf{X}) \cdot r(\mathsf{X}) \right] \\ &= \int_{\mathcal{X}} m(x) \cdot w(x) \cdot r(x) \, dx \\ &= \int_{\mathcal{X}} m(x)^{\alpha} \cdot w(x) \cdot m(x)^{1-\alpha} \cdot r(x) \, dx \\ &= M \cdot R \cdot \int_{\mathcal{X}} m(x)^{\alpha} \cdot \frac{w(x)}{M} \cdot m(x)^{1-\alpha} \cdot \frac{r(x)}{R} \, dx \text{ for } M = \max_{x} w(x), R = \max_{x} r(x) \\ &\leq M^{1-\alpha} \cdot R^{\alpha} \cdot \int_{\mathcal{X}} \left(m(x) \cdot w(x) \right)^{\alpha} \cdot \left(m(x) \cdot r(x) \right)^{1-\alpha} \, dx \text{ since } x \leq x^{\alpha} \text{ for } \alpha \in [0, 1] \\ &< M^{1-\alpha} \cdot R^{\alpha} \cdot \left(\mathbb{E}_{\mathsf{X} \sim M} \left[w(\mathsf{X}) \right] \right)^{\alpha} \cdot \left(\operatorname{reg}(s; \bar{\mathcal{D}}, \ell) \right)^{1-\alpha}, \end{split}$$

where the last line is by Hölder's inequality⁷. The case $\alpha = 0$ gives the original bound of Proposition 4. \Box

Proof of Proposition 5. Plug in
$$\rho_{\pm 1} \equiv f$$
 into Proposition 16.

Proof of Proposition 7. If $\eta(x) < \eta(x')$, then certainly s(x) < s(x') since s is order preserving for η by BCN-admissibility Condition (a). Thus, by Lemma 18,

$$(\forall x, x' \in \mathfrak{X}) \, \bar{\eta}(x) - \bar{\eta}(x') \le \max(1 - \rho_{-1}(x) - \rho_{1}(x), 1 - \rho_{-1}(x') - \rho_{1}(x')) \cdot (\eta(x) - \eta(x')).$$

By the total noise assumption (Assumption 1), $1 - \rho_{-1}(x) - \rho_1(x) > 0$ for every x. Since $\eta(x) - \eta(x') < 0$ by assumption, we conclude that $\bar{\eta}(x) - \bar{\eta}(x') < 0$.

Proof of Proposition 8. From Clémençon et al. [2008], Agarwal [2014, Theorem 11],

$$\operatorname{reg}_{\mathrm{AUC}}(s; \mathcal{D}) = \frac{1}{2 \cdot \pi \cdot (1 - \pi)} \cdot \mathbb{E}_{\mathsf{X} \sim M, \mathsf{X}' \sim M} \left[|\eta(\mathsf{X}) - \eta(\mathsf{X}')| \cdot \mathbb{I}(\eta(\mathsf{X}) - \eta(\mathsf{X}'), s(\mathsf{X}) - s(\mathsf{X}')) \right]$$

where

$$\mathbb{I}(\Delta \eta, \Delta s) = [\![\Delta \eta \cdot \Delta s < 0]\!] + 1/2 \cdot [\![\Delta s = 0]\!].$$

⁷In its native form, this states that $\sum_i |x_i| \cdot |y_i| \leq (\sum_i |x_i|^{1/\alpha})^{\alpha} \cdot (\sum_i |y_i|^{1/(1-\alpha)})^{1-\alpha}$, so that $\sum_i |x_i|^{\alpha} \cdot |y_i|^{1-\alpha} \leq (\sum_i |x_i|)^{\alpha} \cdot (\sum_i |y_i|)^{1-\alpha}$.

By Proposition 7, for this noise model,

$$\eta(x) \neq \eta(x') \implies \operatorname{sign}(\eta(x) - \eta(x')) = \operatorname{sign}(\bar{\eta}(x) - \bar{\eta}(x')),$$

Thus, in this case, $\operatorname{sign}(\Delta \eta) = \operatorname{sign}(\Delta \bar{\eta})$, and so $\mathbb{I}(\Delta \eta, \Delta s) = \mathbb{I}(\Delta \bar{\eta}, \Delta s)$. When $\eta(x) = \eta(x')$, however, there is no guarantee on the relative values of $\bar{\eta}(x)$ and $\bar{\eta}(x')$. But if $\Delta \eta = 0$, then the first term in \mathbb{I} above is necessarily zero, while that for $\Delta \bar{\eta}$ can only be ≥ 0 . Thus, in general we have

$$\mathbb{I}(\Delta \eta, \Delta s) \le \mathbb{I}(\Delta \bar{\eta}, \Delta s),$$

and so

$$\operatorname{reg}_{\mathrm{AUC}}(s; \mathcal{D}) \leq \frac{1}{2 \cdot \pi \cdot (1 - \pi)} \cdot \mathbb{E}_{\mathsf{X} \sim M, \mathsf{X}' \sim M} \left[|\eta(\mathsf{X}) - \eta(\mathsf{X}')| \cdot \mathbb{I}(\bar{\eta}(\mathsf{X}) - \bar{\eta}(\mathsf{X}'), s(\mathsf{X}) - s(\mathsf{X}')) \right].$$

What remains then is the $|\eta(x) - \eta(x')|$ term. Now, by Lemma 18, when $\eta(x) \neq \eta(x')$,

$$\frac{\bar{\eta}(x) - \bar{\eta}(x')}{\eta(x) - \eta(x')} \ge \max(1 - \rho_{-1}(x) - \rho_{1}(x), 1 - \rho_{-1}(x') - \rho_{1}(x'))$$
$$\ge 1 - 2 \cdot \rho_{\max}.$$

If $\eta(x) = \eta(x')$, we trivially have $|\eta(x) - \eta(x')| \le |\bar{\eta}(x) - \bar{\eta}(x')| \cdot (1 - 2 \cdot \rho_{\text{max}})^{-1}$. We conclude that

$$\begin{split} \operatorname{reg}_{\operatorname{AUC}}(s; \mathfrak{D}) &\leq \frac{1}{2 \cdot \pi \cdot (1 - \pi)} \cdot \mathbb{E}_{\mathsf{X} \sim M, \mathsf{X}' \sim M} \left[|\eta(\mathsf{X}) - \eta(\mathsf{X}')| \cdot \mathbb{I}(\bar{\eta}(\mathsf{X}) - \bar{\eta}(\mathsf{X}'), s(\mathsf{X}) - s(\mathsf{X}')) \right] \\ &\leq \frac{1}{2 \cdot \pi \cdot (1 - \pi)} \cdot \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \mathbb{E}_{\mathsf{X} \sim M, \mathsf{X}' \sim M} \left[|\bar{\eta}(\mathsf{X}) - \bar{\eta}(\mathsf{X}')| \cdot \mathbb{I}(\bar{\eta}(\mathsf{X}) - \bar{\eta}(\mathsf{X}'), s(\mathsf{X}) - s(\mathsf{X}')) \right] \\ &= \frac{\bar{\pi} \cdot (1 - \bar{\pi})}{\pi \cdot (1 - \pi)} \cdot \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \operatorname{reg}_{\operatorname{AUC}}(s; \bar{\mathfrak{D}}). \end{split}$$

Proof of Proposition 9. The form of $\bar{\eta}$ follows from Equation 15 and Proposition 1.

Under Assumption 2, the noise model $SIN(\mathcal{D}, f_{-1}, f_1) = BCN^+(\mathcal{D}, f_{-1}, f_1, s^*)$. By Corollary 19,

$$s^*(x) < s^*(x') \implies \bar{u}(s^*(x)) \le \bar{u}(s^*(x')),$$

so that \bar{u} is a monotone function, and thus a valid GLM link.

Next, applying the triangle inequality to Lemma 17, and using z = s(x), z' = s(x'),

$$\begin{aligned} |\bar{\eta}(x) - \bar{\eta}(x')| &= |\bar{u}(z) - \bar{u}(z')| \\ &\leq |1 - f_{-1}(z') - f_1(z')| \cdot |u(z) - u(z')| + |f_{-1}(z) - f_{-1}(z')| \cdot |1 - u(z)| + |f_1(z) - f_1(z')| \cdot |u(z)| \\ &\leq (L + L_{-1} + L_1) \cdot |z - z'|, \end{aligned}$$

using the fact that $|1-f_{-1}(z')-f_1(z')|<1$ by the total noise assumption (Assumption 1), $|1-u(z)|\leq 1$ and $|u(z)|\leq 1$ since $\mathrm{Im}(u)=[0,1]$, and the Lipschitz assumptions on $u,f_{\pm 1}$. It follows that \bar{u} is $(L+L_{-1}+L_1)$ -Lipschitz.

Proof of Proposition 10. By Proposition 9, $\bar{\eta} \in \text{SIM}(L + L_2 + L_3, W)$. Thus, as a member of the SIM family, it is suitable for estimation using SLIsotron.

Proposition 20 implies that one can always choose an iteration of SLIsotron with low regret. Let $\hat{\eta}_{S,t}$ denote the estimate produced by SLIsotron at iteration t. If in an abuse of notation we let $\hat{\eta}_{S}$ denote the estimate $\hat{\eta}_{S,t^*}$, where t^* is an appropriately determined iteration, then we have that $\operatorname{reg}(\hat{\eta}_{S}; \mathcal{D}, \ell^{\operatorname{sq}}) \stackrel{\mathbb{P}}{\to} 0$.

For AUC consistency, standard surrogate regret bounds [Agarwal, 2014] imply that

$$\mathrm{reg}_{\mathrm{AUC}}(\hat{\bar{\eta}}_{\bar{\mathsf{S}}};\bar{\mathcal{D}}) \leq \frac{1}{2 \cdot \bar{\pi} \cdot (1 - \bar{\pi})} \cdot \sqrt{\mathrm{reg}(\hat{\bar{\eta}}_{\bar{\mathsf{S}}};\bar{\mathcal{D}},\ell^{\mathrm{sq}})}.$$

By Proposition 8, we conclude that

$$\begin{split} \operatorname{reg}_{\operatorname{AUC}}(\hat{\bar{\eta}}_{\bar{\S}}; \mathcal{D}) & \leq \frac{\bar{\pi} \cdot (1 - \bar{\pi})}{\pi \cdot (1 - \pi)} \cdot \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \operatorname{reg}_{\operatorname{AUC}}(\hat{\bar{\eta}}_{\bar{\S}}; \bar{\mathcal{D}}) \\ & \leq \frac{1}{2 \cdot \pi \cdot (1 - \pi)} \cdot \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \sqrt{\operatorname{reg}(\hat{\bar{\eta}}_{\bar{\S}}; \bar{\mathcal{D}}, \ell^{\operatorname{sq}})}. \end{split}$$

The Isotron guarantee implies the RHS tends to 0 with sufficiently many samples. Thus, $\operatorname{reg}_{\mathrm{AUC}}(\hat{\eta}_{\bar{5}}; \mathcal{D}) \to 0$. For classification consistency, standard surrogate regret bounds [Zhang, 2004, Bartlett et al., 2006, Reid and Williamson, 2009] imply that we can bound the 0-1 regret in terms of the square loss regret:

$$reg(2\hat{\bar{\eta}}_{\bar{\mathsf{S}}} - 1; \bar{\mathcal{D}}, \ell^{01}) \leq \sqrt{reg(\hat{\bar{\eta}}_{\bar{\mathsf{S}}}; \bar{\mathcal{D}}, \ell^{sq})}.$$

By Proposition 4, for symmetric noise, thresholding our estimate of $\bar{\eta}$ around 1/2 will be consistent wrt the clean distribution:

$$\begin{split} \operatorname{reg}(c_{\bar{\mathsf{S}}}; \mathcal{D}, \ell^{01}) &= \operatorname{reg}(2\hat{\eta}_{\bar{\mathsf{S}}} - 1; \mathcal{D}, \ell^{01}) \\ &\leq \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \operatorname{reg}(2\hat{\eta}_{\bar{\mathsf{S}}} - 1; \bar{\mathcal{D}}, \ell^{01}) \\ &= \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \sqrt{\operatorname{reg}(\hat{\eta}_{\mathsf{S}}; \bar{\mathcal{D}}, \ell^{\operatorname{sq}})}. \end{split}$$

The Isotron guarantee implies the RHS tends to 0 with sufficiently many samples. Thus, in the case of symmetric BCN⁺ noise, thresholding $\bar{\eta}$ around $^{1}/_{2}$ will be consistent wrt the clean distribution.

C Examples of the SIM family

Two simple examples of the SIM family are presented below. The first was established in Kalai and Sastry [2009].

Example 3. Suppose that \mathcal{D} corresponds to a concept that is linearly separable with margin $\gamma > 0$ i.e.

$$\eta(x) = [\![\langle w^*, x \rangle > 0]\!],$$

and

$$\mathbb{P}(\{(x,y) \mid y \cdot \langle w^*, x \rangle < \gamma\}) = 0.$$

Then, $\eta \in \mathrm{SIM}((2\gamma)^{-1}, ||w^*||)$. The reason is that we can equally think of η as

$$\eta(x) = u_{\max(\gamma)}(\langle w^*, x \rangle)$$

where

$$u_{\max(\gamma)}(z) = \begin{cases} 1 & \text{if } z > \gamma \\ \frac{z+\gamma}{2\gamma} & \text{if } z \in [-\gamma, +\gamma] \\ 0 & \text{if } z < -\gamma. \end{cases}$$
 (12)

The function u is clearly $(2\gamma)^{-1}$ -Lipschitz.

Example 4. Suppose that \mathcal{D} corresponds to a concept that can be modelled using logistic regression i.e.

$$\eta(x) = \frac{1}{1 + e^{-\langle w^*, x \rangle}}.$$

Then, $\eta \in SIM(1, ||w^*||)$.

D Special cases of the ILN model

Several special cases of the ILN model are of interest. (Table 2 summarises.)

D.1 Instance-independent noise models

The following have been the focus of a vast literature.

Definition 7 (Noise-free learning). Suppose we have an ILN model $ILN(\mathcal{D}, \rho_{-1}, \rho_1)$ where $\rho_{\pm 1} \equiv 0$. Then, we have the standard problem of learning from (noise free) binary labels.

Definition 8 (SLN model). Suppose we have an ILN model $ILN(\mathcal{D}, \rho_{-1}, \rho_1)$ where $\rho_{\pm 1} \equiv \rho$ for some constant $\alpha < \frac{1}{2}$. Then, we have the problem of learning with symmetric label noise (SLN learning), also known as the problem of learning with random classification noise (RCN learning) [Long and Servedio, 2008, van Rooyen et al., 2015]. We will write the corresponding corrupted distribution as $SLN(\mathcal{D}, \alpha)$.

Definition 9 (CCN model). Suppose we have an ILN model ILN($\mathcal{D}, \rho_{-1}, \rho_1$) where $\rho_1 \equiv \alpha, \rho_{-1} \equiv \beta$ for some constants $\alpha, \beta < 1$. Then, we have the problem of learning with class-conditional label noise (CCN learning) [Angluin and Laird, 1988, Blum and Mitchell, 1998, Scott et al., 2013, Natarajan et al., 2013]. We will write the corresponding corrupted distribution as $CCN(\mathcal{D}, \beta, \alpha)$.

D.2 Boundary-consistent noise models

The following is a far-reaching generalisation of the above.

Definition 10 (BCN model). Suppose we have an ILN model $\mathrm{ILN}(\mathcal{D}, \rho_{-1}, \rho_1)$ where $\rho_y = f_y \circ s$ for some functions $f_{\pm 1} \colon \mathbb{R} \to [0, 1]$, and a function $s \colon \mathcal{X} \to \mathbb{R}$ such that:

(a) s is order preserving for η i.e.

$$(\forall x, x' \in \mathfrak{X}) \eta(x) < \eta(x') \implies s(x) < s(x'),$$

or equivalently,

$$(\exists u \colon \mathbb{R} \to [0,1] \text{ monotone}) \eta = u \circ s.$$

The u above is not required to be *strictly* monotone, so it may not be true that $s = v \circ \eta$ for some $v : [0,1] \to \mathbb{R}$; as a simple example, suppose that $\eta(x) = [s(x) > 0]$.

(b) $f_{\pm 1}$ are non-decreasing on $(-\infty, u^{\dagger}(1/2)]$ and non-increasing on $[u^{\dagger}(1/2), \infty)$, where

$$u^{\dagger}(1/2) = \sup_{z \in \mathbb{R}} \left\{ z \colon u(z) \le \frac{1}{2} \right\}$$

is the generalised inverse of u at $\frac{1}{2}$; or more compactly, if $f_{\pm 1}$ are differentiable,

$$(\forall z \in [0,1]) f'_{+1}(z) \cdot (z - u^{\dagger}(1/2)) \le 0.$$
(13)

In the case where \mathcal{D} is linearly separable, and s is such that $u^{\dagger}(1/2) = 0$, such a model was considered in Du and Cai [2015], where it was termed learning with boundary consistent noise (BCN learning). (A similar model was studied in Bootkrajang [2016] from a probabilistic perspective.) We borrow this terminology for the case of general \mathcal{D} . We will write the corresponding corrupted distribution as $BCN(\mathcal{D}, f_{-1}, f_1, s)$; further, we will say that (f_{-1}, f_1, s, η) are BCN-admissible if they satisfy the conditions detailed above.

The above model in turn has several special cases that are of interest.

^aDu and Cai [2015] considers $\mathbb{P}(\bar{Y} \neq Y \mid X = x)$, which is precisely our $\rho_Y(x)$.

D.2.1 The probabilistically transformed noise model

A simple noise model is where the noise is some monotone transformation of the underlying η .

Definition 11 (PTN model). Suppose we have an ILN model ILN $(\mathcal{D}, \rho_{-1}, \rho_1)$ where $\rho_y = f_y \circ \eta$ for some functions $f_{\pm 1} \colon [0,1] \to [0,1]$ such that (f_{-1},f_1,η,η) are BCN-admissible. In this model, labels are flipped with higher probability for those instances with high inherent uncertainty (i.e. with η values close to $\frac{1}{2}$). We term this problem learning with probabilistically transformed noise (PTN learning). We will write the corresponding corrupted distribution as $\operatorname{PTN}(\mathcal{D}, f_{-1}, f_1)$; further, we will say that (f_{-1}, f_1) are $\operatorname{PTN-admissible}$ if they satisfy the conditions detailed above.

When $f_{\pm 1} \equiv f$ above, we flip labels with probability proportional to the distance of η from 1/2. In the general case, it is intuitive that we will need some conditions on $f_{\pm 1}$ to ensure order preservation.

D.2.2 The Bylander model

Bylander [1997] describes the following model, termed monotonic or probabilistically-consistent noise. One has a distribution \mathcal{D} that is linearly separable with margin $\gamma > 0$, i.e. $\eta(x) = u_{\max(\gamma)}(\langle w^*, x \rangle)$ with $u_{\max(\gamma)}$ as per Equation 12. One observes samples from a distribution $\bar{\mathcal{D}}$ that satisfies the following conditions:

$$\langle w^*, x \rangle \ge \langle w^*, x' \rangle \implies \frac{\eta(x)}{1 - \eta(x)} \ge \frac{\eta(x')}{1 - \eta(x')}$$
$$\langle w^*, x \rangle \ge -\langle w^*, x' \rangle \implies \frac{\eta(x)}{1 - \eta(x)} \ge \frac{1 - \eta(x')}{\eta(x')},$$

or equivalently,

$$\langle w^*, x \rangle \ge \langle w^*, x' \rangle \implies \eta(x) \ge \eta(x')$$

 $\langle w^*, x \rangle \ge -\langle w^*, x' \rangle \implies \eta(x) \ge 1 - \eta(x').$

The contrapositive of these implications is

$$\eta(x) < \eta(x') \implies \langle w^*, x \rangle < \langle w^*, x' \rangle$$

$$\eta(x) < 1 - \eta(x') \implies \langle w^*, x \rangle < -\langle w^*, x' \rangle.$$

The first of these implications means that $\eta(x) = \phi(\langle w^*, x \rangle)$ for some non-decreasing ϕ . The second of these implications is satisfied if $\phi(-z) = 1 - \phi(z)$. We formalise this as follows.

Definition 12 (BYLN model). Suppose we have an ILN model $\text{ILN}(\mathcal{D}, \rho_{-1}, \rho_1)$ where $\rho_y = f_y \circ s$, where $f_1 \equiv f_{-1} \equiv f$ such that

- (a) (f, f, s, η) is BCN-admissible,
- (b) f is symmetric around $u^{\dagger}(1/2)$, i.e. $f(z) = g(|z u^{\dagger}(1/2)|)$ for some non-increasing function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$.

We term this model for general \mathcal{D} as learning with Bylander noise (BYLN learning). We will write the corresponding corrupted distribution as BYLN(\mathcal{D}, f, s); further, we will say that (f, s, η) are BYLN-admissible if they satisfy the conditions detailed above.

In the case where \mathcal{D} is linearly separable, and s is such that $u^{\dagger}(1/2) = 0$, the BYLN model is as considered in Bylander [1997, 1998], Servedio [1999].

D.2.3 The BCN^+ model

The BCN⁺ model introduced in Definition 5 is seen to be the BCN model augmented with an additional assumption.

Assumption 2. The difference $\Delta(z) = f_1(z) - f_{-1}(z)$ between the positive and negative flip functions is non-increasing.

This assumption proves crucial in guaranteeing that $\bar{\eta}$ is order-preserving for η (see Appendix I). It is trivially satisfied in special cases of the BCN⁺ model.

Example 5. For the case of CCN noise $CCN(\mathcal{D}, \rho_{-1}, \rho_1)$, the flip functions are constant, and so Assumption 2 is trivially satisfied.

Example 6. For the case of symmetric BCN noise $BCN(\mathcal{D}, f, f, s)$, the difference between the flip functions is a constant, and so Assumption 2 is trivially satisfied.

D.3 Instance-dependent model

The final special case of ILN is a generic instance- (but not label-) dependent noise model, previously considered in Ghosh et al. [2015].

Definition 13 (IDN model). Suppose we have an ILN model ILN $(\mathcal{D}, \rho_{-1}, \rho_1)$ where $\rho_{-1} \equiv \rho_1 \equiv f$ for some function $f \colon \mathcal{X} \to [0, 1/2)$. We term this problem learning with instance-dependent noise (IDN learning). We will write the corresponding corrupted distribution as IDN (\mathcal{D}, f) .

D.4 Relation between the noise models

The above noise models are related to each other as follows:

$$\begin{split} \operatorname{SLN}(\mathcal{D},\alpha) &= \operatorname{CCN}(\mathcal{D},\alpha,\alpha) \\ \operatorname{CCN}(\mathcal{D},\beta,\alpha) &= \operatorname{PTN}(\mathcal{D},\beta \cdot \mathbb{1},\alpha \cdot \mathbb{1}) \\ \operatorname{PTN}(\mathcal{D},f_{-1},f_1) &= \operatorname{BCN}(\mathcal{D},f_{-1},f_1,\eta) \\ \operatorname{BYLN}(\mathcal{D},f,s) &= \operatorname{BCN}(\mathcal{D},f,f,s) \\ \operatorname{BCN}(\mathcal{D},f_{-1},f_1,s) &= \operatorname{ILN}(\mathcal{D},f_{-1}\circ s,f_1\circ s) \\ \operatorname{BCN}(\mathcal{D},f,f,s) &= \operatorname{IDN}(\mathcal{D},f\circ s) \\ \operatorname{IDN}(\mathcal{D},f) &= \operatorname{ILN}(\mathcal{D},f,f). \end{split}$$

Here, $\mathbb{1}$ refers to the function which is 1 everywhere.

Remark. As seen above, the BCN model reduces to the PTN model when u is invertible. Note however that when u is not invertible, the BCN model is more powerful than the PTN model. For example, if $\mathfrak D$ is separable with a margin, then under the PTN model, all deterministically positive instances are flipped with some probability, and similarly all deterministically negative instances. However, under the BCN model, instances closer to the optimal decision boundary, regardless of their label, will have a higher chance of being flipped.

Noise model	Notation	Description		
Instance- and label-dependent noise	$\mathrm{ILN}(\mathfrak{D}, \rho_{-1}, \rho_1)$	Flip probability function of instance and label		
Instance-dependent noise	$\mathrm{IDN}(\mathcal{D},f)$	Flip probability function of instance only		
Class-conditional noise	$CCN(\mathcal{D}, \beta, \alpha)$	Flip probability depends on label only		
Symmetric label noise	$SLN(\mathcal{D}, \alpha)$	Constant flip probability		
Boundary-conditional noise	$BCN(\mathcal{D}, f_{-1}, f_1, s)$	Flip probability function of <i>score</i> on instance and label, where score is consistent with un derlying class-probability function		
Bylander noise	$\mathrm{BYLN}(\mathcal{D},f,s)$	Flip probability function of <i>score</i> on instance only, where score is consistent with underlying class-probability function		
Probabilistically transformed noise	$\mathrm{PTN}(\mathcal{D}, f_{-1}, f_1)$	Flip probability function of underlying class- probability function and label		

Table 2: Summary of noise models.

E Special cases of \bar{D}

We list the components of $\bar{\mathcal{D}}$ in some special cases.

Example 7. For the class-conditional noise model $CCN(\mathcal{D}, \beta, \alpha)$, we have

$$\bar{\eta}(x) = (1 - \alpha - \beta) \cdot \eta(x) + \beta$$

$$\bar{\pi} = \pi \cdot (1 - \alpha - \beta) + \beta$$

$$\bar{P}(x) = \bar{\pi}^{-1} \cdot ((1 - \alpha) \cdot \pi \cdot P(x) + \beta \cdot (1 - \pi) \cdot Q(x))$$

$$\bar{Q}(x) = (1 - \bar{\pi})^{-1} \cdot (\alpha \cdot \pi \cdot P(x) + (1 - \beta) \cdot (1 - \pi) \cdot Q(x)).$$
(14)

This is in agreement with Natarajan et al. [2013, Lemma 7], Menon et al. [2015, Appendix C].

Example 8. For the boundary-conditional noise model $BCN(\mathcal{D}, f_{-1}, f_1, s)$, with $\eta = u \circ s$ for some non-decreasing u, we have

$$\bar{\eta}(x) = \bar{u}(s(x)) \text{ where}$$

$$\bar{u}(z) = (1 - f_1(z)) \cdot u(z) + f_{-1}(z) \cdot (1 - u(z))$$

$$= (1 - f_1(z) - f_{-1}(z)) \cdot u(z) + f_{-1}(z).$$
(15)

The monotonicity of \bar{u} is studied in Lemma 18.

Example 9. For the PTN model $PTN(\mathcal{D}, f_{-1}, f_1)$, we have By Proposition 1, and the fact that $\rho_y = g \circ \eta$ for the PTN model,

$$\bar{\eta}(x) = \varphi(\eta(x))$$

$$\varphi(z) = (1 - f_{-1}(z) - f_1(z)) \cdot z + f_{-1}(z).$$
(16)

Example 10. For the instance-dependent noise model $IDN(\mathcal{D}, f)$, we have

$$\bar{\eta}(x) = (1 - 2 \cdot f(x)) \cdot \eta(x) + f(x)$$
$$\bar{\pi} = \pi + \mathbb{E}_{\mathbf{X} \sim M} \left[f(\mathbf{X}) \cdot (1 - 2 \cdot \eta(\mathbf{X})) \right]$$

$$\bar{P}(x) = \bar{\pi}^{-1} \cdot ((1 - f(x)) \cdot \pi \cdot P(x) + f(x) \cdot (1 - \pi) \cdot Q(x))$$
$$\bar{Q}(x) = (1 - \bar{\pi})^{-1} \cdot (f(x) \cdot \pi \cdot P(x) + (1 - f(x)) \cdot (1 - \pi) \cdot Q(x)).$$

For the class-conditionals, we can equally write

$$\begin{split} P(x) &= (1 - 2 \cdot f(x))^{-1} \cdot \pi^{-1} \cdot \left((1 - f(x)) \cdot \bar{\pi} \cdot \bar{P}(x) - f(x) \cdot (1 - \bar{\pi}) \cdot \bar{Q}(x) \right) \\ Q(x) &= (1 - 2 \cdot f(x))^{-1} \cdot (1 - \pi)^{-1} \cdot \left(-f(x) \cdot \bar{\pi} \cdot \bar{P}(x) + (1 - f(x)) \cdot (1 - \bar{\pi}) \cdot \bar{Q}(x) \right). \end{split}$$

F Boundary consistent noise and flip probabilities

Given an instance $x \in \mathcal{X}$, let F(x) denote the probability that the instance has its label flipped. It is easy to check that

$$F(x) = \mathbb{P}(\mathsf{Y} \neq \bar{\mathsf{Y}} \mid \mathsf{X} = x)$$

$$= \mathbb{P}(\mathsf{Y} \neq \bar{\mathsf{Y}} \mid \mathsf{Y} = 1, \mathsf{X} = x) \cdot \mathbb{P}(\mathsf{Y} = 1 \mid \mathsf{X} = x) + \mathbb{P}(\mathsf{Y} \neq \bar{\mathsf{Y}} \mid \mathsf{Y} = -1, \mathsf{X} = x) \cdot \mathbb{P}(\mathsf{Y} = -1 \mid \mathsf{X} = x)$$

$$= \rho_1(x) \cdot \eta(x) + \rho_{-1}(x) \cdot (1 - \eta(x))$$

$$= (\rho_1(x) - \rho_{-1}(x)) \cdot \eta(x) + \rho_{-1}(x).$$

F.1 Guaranteeing maximisation at $\frac{1}{2}$ for BCN model

In the case of boundary consistent noise,

$$F(x) = \varphi(s(x))$$

where

$$\varphi(z) = (f_1(z) - f_{-1}(z)) \cdot u(z) + f_{-1}(z).$$

Suppose we want F to be increasing when $\eta < 1/2$, and decreasing otherwise. Observe that

$$\varphi'(z) = (f_1(z) - f_{-1}(z)) \cdot u'(z) + (f'_1(z) - f'_{-1}(z)) \cdot u(z) + f'_{-1}(z)$$

= $(f_1(z) - f_{-1}(z)) \cdot u'(z) + f'_1(z) \cdot u(z) + f'_{-1}(z) \cdot (1 - u(z)).$

When $u(z)<\frac{1}{2}$, the second and third terms are guaranteed to be positive (by Condition (b) of BCN-admissibility). Since u'(z)>0, for the first term to be positive we need $\Delta(z)\geq 0$. Similarly, when $u(z)>\frac{1}{2}$, the second and third terms are guaranteed to be negative; for the first term to be negative we need $\Delta(z)\leq 0$. Thus, a sufficient condition for F to be maximised when $\eta=\frac{1}{2}$ is for

$$\Delta(z) \cdot (2 \cdot u(z) - 1) \le 0. \tag{17}$$

F.2 Relation to Assumption 2

Note that above, we do *not* require Assumption 2 (i.e. $\Delta(z)$ is decreasing). Indeed, Assumption 2 by itself does not guarantee that F is maximised when $\eta = \frac{1}{2}$. As a simple example, for the CCN model,

$$F(x) = (\alpha - \beta) \cdot \eta(x) + \beta.$$

Evidently, this is maximised at either $\eta(x) = 0$ or $\eta(x) = 1$, depending on whether $\alpha > \beta$ or not.

On the other hand, $\Delta(z)$ satisfying Equation 17 by itself does not guarantee that $\bar{\eta}$ is order preserving for η . Consider for example a case where $f_1(z) = [z \le 0] \cdot \frac{1}{2} \cdot e^z$, $f_{-1}(z) \equiv 0$, and $\eta(x) = 1/(1 + \exp(-s(x)))$. Then, $\bar{\eta}$ will not be order preserving for η .

If $\Delta(z)$ satisfies both Equation 17 and Assumption 2, then we will have that F is maximised when $\eta = \frac{1}{2}$, and also that $\bar{\eta}$ is order preserving for η .

G Label swapping and Assumption 2

Assumption 2 implies an asymmetry in the treatment of positive and negative labels. This is at first glance surprising, since intuitively we would expect our results to hold even if we swap the labels. In particular, suppose we have some $\mathcal{D} = (M, \eta)$ with a $BCN^+(\mathcal{D}, f_{-1}, f_1, s)$ noise model. Then,

$$1 - \bar{\eta}(x) = f_1(s(x)) \cdot \eta(x) + (1 - f_{-1}(s(x))) \cdot (1 - \eta(x)).$$

Now consider $\mathcal{D}' = (M, 1 - \eta)$ so that the positive and negative labels are swapped. Then, it is not hard to see that for a BCN⁺(\mathcal{D}' , $f_1, f_{-1}, -s$) noise model,

$$\bar{\eta}'(x) = (1 - f_{-1}(-s(x))) \cdot (1 - \eta(x)) + f_1(-s(x)) \cdot \eta(x).$$

Therefore, if the flip functions $f_{\pm 1}$ are even (i.e. symmetric around the origin), we have

$$\bar{\eta}'(x) = 1 - \bar{\eta}(x).$$

So,

$$\eta'(x) < \eta'(x') \iff \eta(x) > \eta(x')$$

 $\implies \bar{\eta}(x) > \bar{\eta}(x')$
 $\implies \bar{\eta}'(x) < \bar{\eta}'(x').$

Thus, order preservation is retained. This may seem peculiar since for \mathcal{D}' , we have the opposite of Assumption 2 holding. But note that

$$\bar{\eta}'(x) = \varphi(-s(x))$$

where

$$\varphi(z) = (1 - f_{-1}(z)) \cdot (1 - u(z)) + f_1(z) \cdot u(z)$$

= 1 - (f_{-1}(z) \cdot (1 - u(z)) + (1 - f_1(z)) \cdot u(z))
= 1 - (f_{-1}(z) \cdot v(z) + (1 - f_1(z)) \cdot (1 - v(z)))

where v(z)=1-u(z). For $\varphi(z)$ to be non-decreasing, the second term above must be non-increasing. This term is precisely that arising from the standard BCN model, but with a link function v that is non-increasing, and with flip functions satisfying the opposite of Assumption 2. Thus, it is not hard to see that we can guarantee the opposite of the standard BCN model, so that the term is non-increasing.

H The generalised loss object under the CCN model

For the class-conditional noise model $CCN(\mathcal{D}, \rho_{-1}, \rho_1)$, the generalised loss of Proposition 15 is simply

$$\tilde{\ell}_1(s,x) = w^{-1} \cdot ((1 - \rho_{-1} \cdot \ell_1(s(x)) - \rho_1 \cdot \ell_{-1}(s(x)))$$

$$\tilde{\ell}_{-1}(s,x) = w^{-1} \cdot (-\rho_{-1} \cdot \ell_1(s(x)) + (1 - \rho_1) \cdot \ell_{-1}(s(x)))$$

where $w = 1 - \rho_{-1} - \rho_1$. The dependence on x is only via the corresponding s(x) value. Thus, we may equally consider the noise-corrected loss

$$\bar{\ell}_1(v) = w^{-1} \cdot ((1 - \rho_{-1} \cdot \ell_1(v) - \rho_1 \cdot \ell_{-1}(v))$$
$$\bar{\ell}_{-1}(v) = w^{-1} \cdot (-\rho_{-1} \cdot \ell_1(v) + (1 - \rho_1) \cdot \ell_{-1}(v)),$$

with Proposition 15 then reducing to

$$R(s; \mathcal{D}, \ell) = R(s; \bar{\mathcal{D}}, \bar{\ell}),$$

as shown in Natarajan et al. [2013, Lemma 1], who termed the approached of minimising $\bar{\ell}$ as the "method of unbiased estimators".

Remark. Natarajan et al. [2013, Lemma 1] was generalised in a different direction by van Rooyen and Williamson [2015], who considered problems with general label spaces. The noise model in van Rooyen and Williamson [2015] is still instance independent, unlike Proposition 15.

I Failure of order preservation under $\bar{\eta}$

We illustrate that for noise models other than BCN⁺, order preservation under $\bar{\eta}$ is not guaranteed.

I.1 Failure of order preservation for the BCN model

Order preservation is not guaranteed for the BCN model without Condition (c) of the BCN⁺ model.

Example 11. Suppose $f_1(z) \equiv 0$, $f_{-1}(z) = a \cdot [z \leq 0]$ for some a < 1, and s is such that $\eta(x) = \frac{1}{1+e^{-s(x)}}$. Certainly (f_{-1}, f_1, s) is BCN-admissible. It is easy to check that

$$\bar{\eta}(x) = \varphi(s(x))$$

where

$$\begin{split} \varphi(z) &= (1-a\cdot [\![z\leq 0]\!]) \cdot \frac{e^z}{1+e^z} + a\cdot [\![z\leq 0]\!] \\ &= \begin{cases} (1-a)\cdot \frac{e^z}{1+e^z} + a & \text{if } z\leq 0 \\ \frac{e^z}{1+e^z} & \text{if } z>0, \end{cases} \end{split}$$

which is easily checked to not be monotone in z.

The difference $\Delta(z) = f_1(z) - f_{-1}(z)$ above is non-decreasing. Swapping the flip functions thus makes the function non-increasing, satisfying Assumption 2. We can confirm that in this case, $\bar{\eta}$ will indeed be order-preserving for η .

Example 12. Suppose $f_{-1}(z) \equiv 0$, $f_1(z) = a \cdot [z \le 0]$ for some a < 1, and s is such that $\eta(x) = \frac{1}{1 + e^{-s(x)}}$. Certainly (f_{-1}, f_1, s) is BCN-admissible. It is easy to check that

$$\bar{\eta}(x) = \varphi(s(x))$$

where

$$\varphi(z) = (1 - a \cdot [z \le 0]) \cdot \frac{e^z}{1 + e^z}$$
$$= \begin{cases} a \cdot \frac{e^z}{1 + e^z} & \text{if } z \le 0\\ \frac{e^z}{1 + e^z} & \text{if } z > 0, \end{cases}$$

which is easily checked to be monotone in z.

I.2 Failure of order preservation for the IDN model

For the IDN model, order preservation will not be guaranteed in general. Consider the simple case where $f(x) = \frac{1}{2}\eta(x)$. This means that there is more noise for positive instances. Then, we have

$$\bar{\eta}(x) = (1 - \eta(x)) \cdot \eta(x) + \frac{1}{2} \cdot \eta(x)$$
$$= \eta(x) \cdot \left(\frac{3}{2} - \eta(x)\right).$$

This will not be order preserving for η , since $\varphi(z) = z \cdot (2-z)$ is not monotone on [0,1].

J On Bayes-optimal scorers coinciding on clean and corrupted distributions

Corollary 3 is a statement about the minimisers when using all measurable scorers $\mathbb{R}^{\mathfrak{X}}$. When using e.g. linear scorers, one does not have the same equivalence in general, unless the Bayes-optimal scorer happens to lie in our chosen class. For example, with the unhinged loss and kernelised linear scorers $\langle w, \Phi(x) \rangle_{\mathfrak{H}}$ for some RKHS \mathfrak{H} , under the IDN model we have optimal weight

$$\begin{split} w^* &= \mathbb{E}_{(\mathsf{X},\bar{\mathsf{Y}}) \sim \bar{\mathcal{D}}} \left[\mathsf{Y} \cdot \Phi(\mathsf{X}) \right] \\ &= \mathbb{E}_{\mathsf{X} \sim M} \left[\Phi(\mathsf{X}) \cdot (2 \cdot \bar{\eta}(\mathsf{X}) - 1) \right] \\ &= \mathbb{E}_{\mathsf{X} \sim M} \left[(1 - 2 \cdot f(\mathsf{X})) \cdot \Phi(\mathsf{X}) \cdot (2 \cdot \eta(\mathsf{X}) - 1) \right], \end{split}$$

which possesses an additional weighting term compared to the optimal weight on \mathcal{D} . Nonetheless, we can expect the scores resulting from this solution to have the correct sign for classification. The score on an instance $x' \in \mathcal{X}$ is

$$s^*(x') = \mathbb{E}_{\mathsf{X} \sim M} \left[(1 - 2 \cdot f(\mathsf{X})) \cdot k(\mathsf{X}, x') \cdot (2 \cdot \eta(\mathsf{X}) - 1) \right]$$

for kernel function $k(x,x') = \langle \Phi(x), \Phi(x') \rangle_{\mathcal{H}}$. If $k(x,x') = \delta_x(x')$, this would reduce to $(1-2 \cdot f(x')) \cdot (2 \cdot \eta(x') - 1) \cdot m(x')$, which has the same sign as $2 \cdot \eta(x') - 1$.

⁸This is not actually a valid kernel for an RKHS, since delta functions are not square integrable.

Proof of Proposition 4 specialised to 0-1 loss

The regret for 0-1 loss is [Devroye et al., 1996, Theorem 2.2], [Reid and Williamson, 2009, Lemma 8]

$$\operatorname{reg}(s; \mathcal{D}, \ell^{01}) = \mathbb{E}_{\mathsf{X} \sim M} \left[\left| \eta(\mathsf{X}) - \frac{1}{2} \right| \cdot \left[(2\eta(\mathsf{X}) - 1) \cdot s(\mathsf{X}) < 0 \right] \right]. \tag{18}$$

The following shows that for 0-1 loss and label-independent noise, there is a simple relationship between the regrets on the clean and corrupted distributions. A key ingredient is the following.

Proposition 21. Pick any distribution \mathfrak{D} . Suppose that $\bar{\mathfrak{D}} = \mathrm{IDN}(\mathfrak{D}, f)$ for admissible $f: \mathfrak{X} \to [0, 1/2)$. Then,

$$(\forall x \in \mathcal{X}) \, \eta(x) - \frac{1}{2} = \frac{1}{1 - 2 \cdot f(x)} \cdot \left(\bar{\eta}(x) - \frac{1}{2} \right).$$

Proof. By Proposition 1,

$$\bar{\eta}(x) = (1 - 2 \cdot f(x)) \cdot \eta(x) + f(x).$$

Thus,

$$\begin{split} \bar{\eta}(x) - \frac{1}{2} &= \eta(x) - \frac{1}{2} + f(x) \cdot (1 - 2 \cdot \eta(x)) \\ &= \eta(x) - \frac{1}{2} + 2 \cdot f(x) \cdot \left(\frac{1}{2} - \eta(x)\right) \\ &= \left(\eta(x) - \frac{1}{2}\right) \cdot (1 - 2 \cdot f(x)). \end{split}$$

Thus, since $f(x) \neq \frac{1}{2}$,

$$\eta(x) - \frac{1}{2} = \frac{1}{1 - 2 \cdot f(x)} \cdot \left(\bar{\eta}(x) - \frac{1}{2}\right).$$

The above required that no instance has label flipped with probability $\frac{1}{2}$, which is a mild and intuitive condition. If we further assume that the flip probability for every instance is less than $\frac{1}{2}$, this simple relationship implies the Bayes-optimal classifier is unaffected.

Corollary 22. Pick any distribution \mathfrak{D} . Suppose that $\overline{\mathfrak{D}} = \mathrm{IDN}(\mathfrak{D}, f)$ for admissible $f \colon \mathfrak{X} \to [0, 1/2)$. Then,

$$(\forall x \in \mathcal{X}) \, \eta(x) > \frac{1}{2} \iff \bar{\eta}(x) > \frac{1}{2}$$

and so

$$\underset{s}{\operatorname{argmin}} R(s; \mathfrak{D}, \ell^{01}) = \underset{s}{\operatorname{argmin}} R(s; \bar{\mathfrak{D}}, \ell^{01}).$$

Proof of Corollary 22. By Proposition 21, if $f(x) < \frac{1}{2}$ for every x, so that $1 - 2 \cdot f(x) > 0$, the two class-probability functions have the same sign around $\frac{1}{2}$. Thus, $\eta(x) > \frac{1}{2} \iff \bar{\eta}(x) > \frac{1}{2}$.

Alternately, simply plug in $t = \frac{1}{2}$ to Proposition 16.

We are now in a position to provide the regret bound.

Proposition 23. Pick any distribution \mathfrak{D} . Suppose that $\overline{\mathfrak{D}} = \mathrm{IDN}(\mathfrak{D}, f)$ for admissible $f : \mathfrak{X} \to [0, 1]$ such that

$$(\forall x \in \mathfrak{X}) f(x) \le \rho_{\max} < \frac{1}{2}.$$

Then, for any scorer $s: \mathfrak{X} \to \mathbb{R}$ *,*

$$\operatorname{reg}(s; \mathcal{D}, \ell^{01}) \le \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \operatorname{reg}(s; \bar{\mathcal{D}}, \ell^{01}).$$

Proof. If $f(x) \le \rho_{\max}$, then $1 - 2 \cdot f(x) \ge 1 - 2 \cdot \rho_{\max}$, and so by Proposition 21,

$$\eta(x) - \frac{1}{2} \le \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \left(\bar{\eta}(x) - \frac{1}{2}\right).$$

Now, by Equation 18, for any scorer s,

$$\begin{split} \operatorname{reg}(s; \mathcal{D}, \ell^{01}) &= \mathbb{E}_{\mathsf{X} \sim M} \left[\left| \eta(\mathsf{X}) - \frac{1}{2} \right| \cdot \left[\left(2\eta(\mathsf{X}) - 1 \right) \cdot s(\mathsf{X}) < 0 \right] \right] \\ &\leq \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \mathbb{E}_{\mathsf{X} \sim M} \left[\left| \bar{\eta}(\mathsf{X}) - \frac{1}{2} \right| \cdot \left[\left(2\eta(\mathsf{X}) - 1 \right) \cdot s(\mathsf{X}) < 0 \right] \right] \\ &= \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \mathbb{E}_{\mathsf{X} \sim M} \left[\left| \bar{\eta}(\mathsf{X}) - \frac{1}{2} \right| \cdot \left[\left(2\bar{\eta}(\mathsf{X}) - 1 \right) \cdot s(\mathsf{X}) < 0 \right] \right] \\ &= \frac{1}{1 - 2 \cdot \rho_{\max}} \cdot \operatorname{reg}(s; \bar{\mathcal{D}}, \ell^{01}), \end{split}$$

where the penultimate line is because $\eta(x)>\frac{1}{2}\iff \bar{\eta}(x)>\frac{1}{2}$ by Corollary 22.

L Simplified proofs of Proposition 7

We present some simplified proofs of Proposition 7 in some special cases. Appendix L.1 considers the case of the symmetric PTN model. Appendix L.2 considers the case when $f_{\pm 1}$ are differentiable.

L.1 Proof for PTN model

We now show that the PTN model of Example 11 will guarantee $\bar{\eta}$ is order preserving for η . Suppose $\bar{\mathbb{D}} = \text{PTN}(\mathbb{D}, g, g)$ for some $g \colon [0, 1] \to [0, 1/2)$. Recall from Equation 16 that

$$\bar{\eta}(x) = \varphi(\eta(x))$$

where $\varphi(z) = (1 - 2 \cdot g(z)) \cdot z + g(z)$. Therefore, we just need to establish strict monotonicity of φ . For differentiable g, strict monotonicity is easy to establish: this is because

$$\varphi'(z) = 1 - 2 \cdot g(z) - 2 \cdot g'(z) \cdot z + g'(z)$$

= 1 - 2 \cdot g(z) + \quad g'(z) \cdot (1 - 2 \cdot z).

Since $1 - 2 \cdot g(z) \ge 1 - 2 \cdot \rho_{\max}$, and $g'(z) \ge 0 \iff z \le \frac{1}{2}$ by definition of the PTN model (see Example 11), we have $\varphi'(z) \ge 1 - 2 \cdot \rho_{\max} > 0$.

For non-differentiable g, we must explicitly check that $x < y \implies \varphi(x) < \varphi(y)$. We have

$$\varphi(x) - \varphi(y) = x - y + g(x) \cdot (1 - 2 \cdot x) - g(y) \cdot (1 - 2 \cdot y)$$

= $x - y - g(x) \cdot (2 \cdot x - 1) + g(y) \cdot (2 \cdot y - 1)$.

Consider the three possible cases.

• Suppose $x \leq \frac{1}{2} \leq y$. Then, $1 - 2 \cdot x \geq 0$ and $2 \cdot y - 1 \geq 0$, and so

$$\begin{split} \varphi(x) - \varphi(y) &= x - y + g(x) \cdot (1 - 2 \cdot x) + g(y) \cdot (2 \cdot y - 1) \\ &\leq x - y + 2 \cdot \max(g(x), g(y)) \cdot (y - x) \\ &= (x - y) \cdot (1 - 2 \cdot \max(g(x), g(y))) \\ &< 0. \end{split}$$

since x - y < 0 and $1 - 2 \cdot \max(g(x), g(y)) > 0$.

• Suppose $\frac{1}{2} \le x < y$. Then, since g is decreasing on $[\frac{1}{2}, 1]$, g(x) > g(y), and so

$$\varphi(x) - \varphi(y) < x - y - 2 \cdot g(x) \cdot (x - y)$$

$$= (x - y) \cdot (1 - 2 \cdot g(x))$$

$$< 0,$$

since x - y < 0 and $1 - 2 \cdot g(x) > 0$.

• Suppose $x < y \le \frac{1}{2}$. Then, g(x) < g(y) and so

$$\varphi(x) - \varphi(y) < x - y + 2 \cdot g(y) \cdot (y - x)$$

$$= (x - y) \cdot (1 - 2 \cdot g(y))$$

$$< 0.$$

since x - y < 0 and $1 - 2 \cdot g(y) > 0$.

Thus, we conclude that $\eta(x) < \eta(x') \implies \varphi(\eta(x)) < \varphi(\eta(x')) \implies \bar{\eta}(x) < \bar{\eta}(x')$.

L.2 Proof for differentiable $f_{\pm 1}$

For the case of differentiable $f_{\pm 1}$, the following is one consequence of Assumption 2.

Lemma 24. Pick any \mathbb{D} . Suppose $f_{\pm 1}$ are differentiable, and (f_{-1}, f_1, s, η) are BCN-admissible (Equation 13), and additionally satisfy Assumption 2. Then,

$$(f'_{-1}(z) + f'_{1}(z)) \cdot u(z) \le f'_{-1}(z)$$

where $\eta = u \circ s$.

Proof. Observe that

$$\begin{split} (f'_{-1}(z) + f'_1(z)) \cdot u(z) - f'_{-1}(z) &= f'_{-1}(z) \cdot (u(z) - 1) + f'_1(z)) \cdot u(z) \\ &= f'_{-1}(z) \cdot (u(z) - {}^{1\!/}\!{}_2) + f'_1(z)) \cdot (u(z) - {}^{1\!/}\!{}_2) + \frac{1}{2} \cdot (f'_1(z) + f'_{-1}(z)) \\ &= (f'_{-1}(z) + f'_1(z)) \cdot (u(z) - {}^{1\!/}\!{}_2) + \frac{1}{2} \cdot (f'_1(z) - f'_1(z)). \end{split}$$

The first term is ≤ 0 by Condition (b) of BCN-admissibility. The second term is ≤ 0 by Assumption 2. Thus, the result is shown.

We use this to show the desired order preserving property of $\bar{\eta}$. Recall from Equation 15 that for a BCN model,

$$\bar{\eta}(x) = \varphi(s(x))$$

where

$$\varphi(z) = (1 - f_1(z) - f_{-1}(z)) \cdot u(z) + f_{-1}(z).$$

Assuming all terms are differentiable, we have

$$\varphi'(z) = (1 - f_1(z) - f_{-1}(z)) \cdot u'(z) + f'_{-1}(z) - (f'_1(z) + f'_{-1}(z)) \cdot u(z).$$

Since $1 - f_1(z) - f_{-1}(z) > 0$ by Assumption 1, $u'(z) \ge 0$ by monotonicity of u, and the last term is ≥ 0 by Lemma 24, we have $\varphi'(z) \ge 0$. Further, $\varphi'(z) = 0$ only if u'(z) = 0, meaning φ is strictly monotone whenever u is, i.e.

$$(\forall x, y \in \mathbb{R}) u(x) < u(y) \implies \varphi(x) < \varphi(y),$$

which in turn means that

$$(\forall x, x' \in \mathfrak{X}) \eta(x) < \eta(x') \implies \bar{\eta}(x) < \bar{\eta}(x').$$

M Examples of corrupted SIM members

We present two examples of SIM members corrupted by noise following the SIN model.

Example 13. Suppose we are in the CCN regime, so that $f_1 \equiv \alpha, f_{-1} \equiv \beta$ for admissible $\alpha, \beta < 1$. Then, as per Equation 14,

$$\bar{u}(z) = (1 - \alpha - \beta) \cdot u(z) + \beta.$$

That is, the corrupted class-probability function is a scaled and translated version of the original class-probability function. If further u(z) = [z > 0], so that \mathcal{D} is separable, we have

$$\bar{u}(z) = \begin{cases} 1 - \alpha & \text{if } z > 0\\ \beta & \text{if } z < 0. \end{cases}$$

That is, the corrupted class-probability function takes on two unique values, depending on which side of the optimal hyperplane one is on.

Example 14. Suppose we are in the Bylander regime, so that $f_1 \equiv f_{-1} \equiv f$ and f(z) = g(|z|) for some arbitrary monotone decreasing function g. Then,

$$\bar{u}(z) = (1 - 2 \cdot f(z)) \cdot u(z) + f(z).$$

If further assume u(z) = [z > 0], so that \mathcal{D} is separable, we have

$$\bar{u}(z) = \begin{cases} 1 - f(z) & \text{if } z > 0 \\ f(z) & \text{if } z < 0 \end{cases}$$
$$= \begin{cases} 1 - g(z) & \text{if } z > 0 \\ g(-z) & \text{if } z < 0. \end{cases}$$

Observe that if g satisfies g(-z) = 1 - g(z), then this is

$$\bar{u}(z) = g(-z).$$

That is, a structured form of monotonic noise on a linearly separable distribution yields a distribution scorable by some generalised linear model. In the case where $g(z) = 1/(1 + e^z)$ for example, we end up with a logistic regression model. This observation has been made previously, e.g. Du and Cai [2015].

N Application of Isotron to CCN setting

To see the challenge in estimating $\bar{\eta}$, recall the following example.

Example 15. Suppose that $\eta(x) = u(\langle w^*, x \rangle)$ for some known u. Suppose that $f_1 \equiv \rho_1, f_{-1} \equiv \rho_{-1}$ for admissible $\rho_1, \rho_{-1} < 1$. Then,

$$\bar{\eta}(x) = (1 - \rho_1 - \rho_{-1}) \cdot u(\langle w^*, x \rangle) + \rho_{-1},$$

or for simplicity

$$\bar{\eta}(x) = \alpha \cdot u(\langle w^*, x \rangle) + \beta.$$

If we had access to clean samples, then we could minimise the canonical loss corresponding to the link function u over the class of linear scorers (a simple convex objective) in order to recover w^* asymptotically. For example, if we know u is a sigmoid, we would minimise the logistic loss on clean samples.

Can we similarly learn $\bar{\eta}$ from corrupted samples? If we knew the parameters α , β , then the same procedure could be applied. However, we unfortunately do not know these in general, and must be estimated as well. Estimating α , β means that, effectively, we are also estimating the link function. We thus are apparently faced with the challenging problem of having to *learn the link function as well as the weight*. (Note that the resulting problem is entirely equivalent to learning a neural network with a single hidden unit.)

To solve this problem, one might resort to an alternating procedure wherein one alternately takes a gradient step in the direction of w, and then in α , β . This approach is simple, but it is unclear whether the procedure is consistent. Thus, the Isotron solves a non-trivial estimation problem.