Calcoolator: Linking Representations of Functions to Deepen Mathematical Understanding in Middle-School Students

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May 7, 2012

To use the Calcoolator, go to http://cs.brown.edu/people/jmkagan/calc/calc.html.

Introduction

Functions play a key role in mathematics learning, as children transition from the concreteness of arithmetic to the abstraction of geometry and algebra. However, the leap from the concrete to the abstract is often difficult for children, especially when teachers are under pressure to raise standardized test scores in favor of instilling a mathematical understanding. While textbooks, educational technology games, and calculators can all cement a student's comfort with functions, some teachers find their resources lacking (Hartter, 2009). Michele Pistocco, a sixth grade math teacher at the DelSesto Middle School in Providence, Rhode Island, explains that older students struggle to connect the symbolic representation, or equation ($f(x) = 3x + 6$), with a graph or even a table of values. What prevents children from unifying these concepts? There are several culprits at hand: first, a lack of thorough teaching at many schools in the United States (due to pressures of testing, curricular standards, or limited resources), and second, curricula that do not emphasize the relationships between multiple representations of functions but instead treat them as separate entities. Specifically, an emphasis is placed on the equation—-suddenly, there are letters where there used to be numbers, and no single solution (Booth, 1988). Our "Calcoolator," a web-based graphing calculator, seeks to address this problem by implementing several representations side-by-side and allowing them to be manipulated in kind. We argue that our design will endow students with an understanding of functions on several levels: the web application allows users to view the graph of a function, its symbolic notation, and a table of values associated with it. Manipulating one representation will automatically update the other representations: we believe that by viewing and exploring these representations together, students will understand how the representations relate to one another.

Motivation for the Calcoolator

The transition between the concrete and the abstract is one that can be difficult for students, and the concept of functions may be even more difficult. During pre-algebra courses, students may become comfortable with solving for x, but confusion sets in when they cannot solve for $f(x)$ (Carpenter, Franke, and Levi, 2003). Functions are often introduced as "input-output" machines (put in 7, get out 14, put in 8, get out 16: what's the rule?), but students often have trouble understanding the relationship between the formal rule $f(x) = 2x$ and the table of values. Michele Pistocco explained that she and her colleagues see this problem frequently among the middle school students they teach, a situation that becomes increasingly difficult as students approach first-year algebra. What does it mean to represent a function algebraically, rather than as a group of numbers? How can a function be represented as a graph? According to both Ms. Pistocco and extant literature, mathematics teaching tends to emphasize the symbolic notation, which is the most abstract of the three representations, over the graph and input/output chart. While this approach works for some students, it is problematic for others who either do not understand the notation or do not grasp functions themselves (Romberg, Fennema, and Carpenter, 1993).

Graphing calculators and similar programs certainly show multiple views of a function (functional notation, table of values, and graph), but each view is typically isolated on its own screen, preventing users from exploring the three representations simultaneously. We looked at a number of web-based graphing calculators and found the same issue: the three representations were not given equal weight. A lecture given by Brown University computer science professor Shriram Kristhnamurthi and discussions with two high school math teachers at the Winsor School in Boston, Massachusetts, suggested to us that a design that balances the three representations might improve students' understanding of functions and abstraction. Winsor teacher Laura Cohen, who has taught algebra, precalculus (trigonometry), and calculus courses, believes that a complete understanding of functions draws on all three representations. Even once students grasp the functional notation, they should still examine graphs to determine the function's long-term behavior. Tables can also yield this kind of information, especially as students learn about intercepts, intersections, or, later on, asymptotes. Moreover, different learning styles may lead students to prefer one representation or set of representations to another. Thus, we began to develop the Calcoolator, a web application that presents functional representations, tables, and graphs on the same screen. We expect that in its current instantiation the Calcoolator will be useful primarily to late-middle/early-high-school-age students who are exploring functions and their graphs. Specifically, we cater to pre-algebra and algebra classes, though the tool might also benefit pre-calculus courses interested in exploring the long-term behaviors of functions.

Design

Graphing calculators are used extensively in middle and high school math classes, with tools ranging from Texas Instruments graphing calculators to internet-based calculators (such as the HTML5 calculator), which offer more advanced graphing techniques. We sought not to replicate existing designs but rather develop our own prototype, which emphasizes the relationship between different views of the function. When manipulating one representation of a function, Calcoolator users will see the other representations change immediately. For example, changing the coefficient in the symbolic notation will cause not only the graph but also the table to update immediately.

We designed the Calcoolator such that the graphs and tables can be easily manipulated and users can explore their behaviors. Graphs of first and second-degree polynomials are altered with "control points," small circles stationed on the y-axis and another location (for lines) or the vertex and on one of the branches (for parabolas). Users can use these points to change the slope of lines, the width of a parabola, and y-intercept in both cases. As the graphs are adjusted, the function's equation and table of values are updated dynamically. (Third-degree and higher polynomials cannot be manipulated in the current instantiation.) Moreover, we aim for easy readability: the Calcoolator has a "snap to grid" feature that can be toggled on and off. When on, all points shown in the table lie on grid lines (for "nice, round numbers"); when off, the points can lie anywhere and scroll smoothly. This feature allows for either easy calculation or a deeper exploration of functional behavior. Scrubbing, additionally, allows students to see how the coefficients of each term affect the graph. We sought to solidify the relationship between each of the representations of the function by color-coding the graph, symbolic notation, and table (i.e. all representations associated with $f(x) = 2x + 3$ are purple). We believe this visual cue will better link the representations together.

One of the key themes underlying the Calcoolator is intuitiveness. The teachers we consulted believe that our design offers something lacking in many graphic calculators and tools meant to teach functions. The Calcoolator does not bias one representation over another, instead allowing students to compare, explore, and, ultimately, learn functions in a way they individually find intuitive. Not only did we aim to make our design easy to use, but we also wanted to develop a tool that allows students to learn and build connections on their own. According to Ms. Pistocco and the other teachers whom we consulted, the constraints of existing curricula and current resources limit students' ability to reason independently and investigate functions in such a manner. Moreover, we hope that the flexibility of our design will accommodate different learning styles (i.e. visual thinking, for which the graph might be most appropriate, as opposed to symbolic, for which the symbolic notation might be more suitable.) To this extent, the Calcoolator should help to resolve some the greater issues in middle school students' understanding of functions.

Technical Specifications

The Calcoolator is programmed in JavaScript, all running on the client side (i.e. in the web browser, not on a remote server). We used the Processing.js library for drawing the graph, and the Mathquill library for displaying equations and numbers.

The program is designed modularly. There is one main data model that synchronizes the state of each function across the three representation modules (equation, table, and graph). Each function is represented by a list of coefficients, which limits our current implementation to polynomials. When a representation receives input from the user that changes a function, the main data model is notified, and it tells all of the other representations about the change. This design allows for the easy addition of new representations, such as simulations.

User Testing and Improvements

We tested three classes of eleventh graders at the Winsor School in Boston, Massachusetts. All students were female and between 15 and 17 years old. They were all enrolled in a pre-calculus course, with one section designated honors. Students were given a laptop, connected to the Calcoolator, and experimented with the application. We tested on two separate occasions: once when only the graph was implemented, in order to get preliminary feedback on the control points and manipulation functionality, and later when most of the functionality was in place.

During both sets of tests, students were instructed to perform a set of tasks: graphing simple single and multiple degree polynomials, manipulating a graph in a particular way, and adjusting the numbers in the equations. We asked them to describe their processes as they clicked on and manipulated the graphs, tables, and functions, and to indicate what was intuitive and what was more difficult. We watched them work, as well, to determine how they used the application. After completing these tasks, the students were then free to explore the Calcoolator and generate feedback on their experience.

In general, students were very positive about the Calcoolator, indicating that it helped them to understand the relationships between the functional representations in a way they had not before. However, many advocated for more "calculator-like" features, such as tools to find the intersection between two curves or the x-intercepts of a function, suggesting that they would not use the Calcoolator simply to help them visualize functions. This suggestion was a source of conflict for us: we understand that intersections and intercepts are an important part of algebra, but we are not sure how to represent them in our design. If we seek to balance the three representations, it will be easier to depict intersections on graphs and tables, but less so with symbolic notation. While we will continue to solve this problem, we believe that with a feature such as an intercept finder, it will be more important to make the tool useful and usable for students than to balance it across representations in a counterintuitive or obscure way.

During the second test we asked them how best to orient the tables of values. We currently have two different orientations due to space constraints. In one version, all the tables are positioned along the bottom, with two vertical columns, one for x values, and one for $f(x)$ values. In the other version, all the tables are stacked on top of one another, but they consist of two rows (again, one for x values, and one for the $f(x)$). Students remarked that they preferred a mix of the two: vertical positioning of the tables, with a column for the x values, and another for the $f(x)$. They found this position more intuitive given that $x/f(x)$ pairings are typically presented in columns in their textbooks. While this presentation will not work due to the spacing of our present layout, we will work to refactor our design to accommodate these stacked, vertical tables.

We also presented the design to their teachers, Byron Parrish and Laura Cohen. As mentioned above, Ms. Cohen has taught primarily high school math courses, though Mr. Parrish has had significant experience teaching middle school as well. While they both agreed that the Calcoolator is a good resource for middle school students approaching functions for the first time, they shared some of the students' concerns. The Calcoolator will be more useful as a classroom tool if it helps the students "calculate" with functions (such as finding intersection points). Moreover, both Mr. Parrish and Ms. Cohen recommended expanding the functionality to include trigonometric functions so that the Calcoolator can be used in higher-level math classes. Unlike algebraic functions, trigonometric functions exhibit behaviors that are harder to model with the traditional "input-output" machine, due to the complex mathematics behind trigonometric relationships. Thus, being able to view multiple representations of the trigonometric functions at once, as well as examine individual values (per our table of values) would be helpful. Nonetheless, they believed that the Calcoolator would improve students' understanding of functions. In particular, they appreciated the table of values, believing that it could facilitate a connection between the symbolic notation and the graph, something that is often challenging for students.

Further Improvements

We initially aimed to implement illustrative simulations as a fourth functional "representation." For example, a linear simulation might model the trajectories of two cars moving at different speeds to separate places, and ask the user how long it would take for them to reach their respective destinations. The user would be able to set the relevant information (speeds, directions, distances, etc.) and then watch two animated cars move accordingly across the screen. Similarly, a second-degree simulation would simulate the movement of projectile fired from the ground, allowing the user to set the starting velocity, height, etc. (If we were to implement trigonometric functionality, users could see how different launch angles will change the eventual trajectory, and therefore, the behavior of the function.) We chose to use "textbook" examples of linear and parabolic functions because they would be familiar to students. However, they fit the philosophy of the Calcoolator in that they offer another view of a function and its behavior.

Unfortunately, we were not able to add this feature due to time constraints, but we believe applications are important to building a mathematical understanding of a concept. Simulations, specifically, can make functions more accessible for students struggling with abstraction in a way that the other varieties may not (Roschelle et al., 2010). We were also reluctant to implement simulations in that they are not particularly extensible beyond two-degree polynomials: scenarios best modelled by higher-degree polynomials are likely to be obscure and unfamiliar to middle-high school aged students, whereas cars and projectiles are more familiar. Thus, while we hope to implement these simulations eventually, we will need to decide how best to address these challenges.

We would also like to implement "hover" functionality: when users mouse over a particular part of the graph, for example, the coordinates associated with that point will appear next to the mouse. Alternatively, when hovering over a location on the table of values, they will see that point indicated on the graph and substituted into the equation. For instance, for $f(x) = 2x$, when hovering over $x = 4$, $f(4) = 8$ would appear in the symbolic notation section of the Calcoolator.

Lastly, we did not have the opportunity to test younger students, and we hope to do so in the future, especially since middle-school students are our target audience. The eleventh graders were nevertheless helpful, especially since they used graphing calculators on a regular basis and could provide insight on the Calcoolator's design as well as its philosophy.

Future Use

Mr. Parrish and Ms. Cohen were both enthusiastic about the Calcoolator and are interested in bringing it into the classroom for its novel design and philosophy. We envision the application as a supplement—-but not a replacement—-to teaching and, potentially, to existing technology. For example, during a lecture on slope, teachers can bring up the Calcoolator to demonstrate how changing the slope of a line affects not only the graph but also the points. Ms. Cohen, who frequently uses graphing software in her classroom, was pleased because with the Calcoolator she would not have to flip between screens to see the points generated or copy the table of values from her calculator onto the whiteboard. Thus, the Calcoolator will free her up to explain concepts while students can view all relevant information simultaneously. Mr. Parrish and his students were excited about the Calcoolator's ability to regenerate an equation once the graph has been manipulated since it allows for more experimentation. "Sometimes I know what I want a graph to look like," one student remarked, "But I can't figure out what the equation should be. Or sometimes I have the equation but I can't visualize the graph." She then explained that she hopes the Calcoolator will build her intuition about the relationship between the graphs and the symbolic representations. Both teachers also thought it would be a helpful standalone online resource for quick graphing and comparison. To this extent, the Calcoolator can serve as a learning aid that will reinforce students' understanding. Moreover, it is online and free, which will make it accessible to a wide variety of students and classrooms.

On the other hand, students and teachers alike may be reluctant to use the Calcoolator because it does not supply all the functionality of a traditional graphing calculator. For example, when we showed the Calcoolator to students, some were surprised that there was no keypad. Others wanted an intercept-intersection finder, as mentioned above, even though we were unsure of whether those features were appropriate to the Calcoolator. We are concerned that potential users might pass up the Calcoolator in favor of a device that places more emphasis on one representation but has a greater range of functionality. It is also conceivable that users will miss the point of the Calcoolator (rather, fail to understand that it does not bias one representation over another). To address these issues, we will need to improve our documentation and choose our target audiences carefully.

Conclusion

Our Calcoolator is a tool that helps students build an appreciation for functions and the relationships between their various representations. It differs from other graphing calculators in that it presents all representations simultaneously while allowing users to explore and build their understanding. Although it is not meant to replace classroom teaching, it can enhance it, especially if students have trouble grasping the abstract nature of functions. We anticipate that with several improvements, it will be useful to middle and high school math teachers and their students; our user testing suggests that the application will be helpful in classrooms. As curricula develop and pressures on math classes increase, an application like the Calcoolator can encourage creative thinking in students, particularly for those frustrated by formulas or algorithms presented without much explanation.

Notes and Acknowledgements

Thank you to Michele Pistocco, Byron Parrish, and Lauren Cohen, and the students in Class VII at the Winsor School for allowing us to interview and test them.

Clara Kliman-Silver wrote the paper and user-tested the Calcoolator, Jonah Kagan and Jessica Liu designed and implemented the Calcoolator, and edited the manuscript.

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