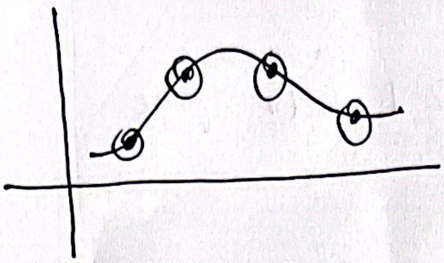
Polynomial Interpolation:



$$P_n(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

$\uparrow$
degree

$(a_0, a_1, \cdots a_n) \rightarrow$ constants / coefficients

→ $P_n(x)$ has $(n+1)$ coefficients

→ Polynomials can be thought as vectors in a vector space

→ Vector space is a set of vectors, that we can

add ⟶ $1 + x + 2x^2$
$+3x^3$
$(- - - -) \rightarrow$ new polynomial

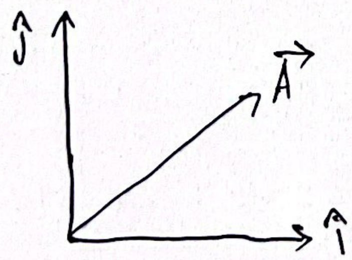
multiply by scalers, c ⟶ $5(1+x+2x^2)$ = new polynomial

satisfy 10 axioms

Basis of a vector space:

→ Basis is a set of vectors that spans the vector space

Example 1

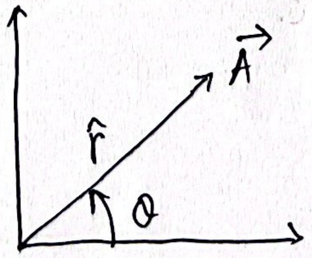


$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$\hat{i}$, $\hat{j}$ → basis

A_x, A_y ↑ constants / coefficients

Example-2



$$\vec{A} = \hat{r} \angle \theta$$

Any vector in a 2-d vectorspace can be written as a linear combination of:

① $\{\hat{i}, \hat{j}\}$
② $\{\hat{r}, \hat{\theta}\}$

both are basis of a 2-d vector space

$P_2(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$

$= a_0 \cdot 1 + a_1 x + a_2 x^2$

basis $\leftrightarrow \{1, x, x^2\} \rightarrow$ 3 dimensional space

→ By choosing different values of a, we can reproduce any polynomial we want of degree 2.

$P_n(x) = a_0 \cdot 1 + a_1 x^1 + a_2 x^2 + \cdots a_n x^n$

basis $\rightarrow \underbrace{\{1, x, x^2 \cdots x^n\}}_{\text{natural basis}} \rightarrow$ $(n+1)$ dimensional space

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Function Space → is a vector space

Function space can be considered as a vector space bcz

- add
- multiply by scalers, c
- ⋮
- 10 axioms

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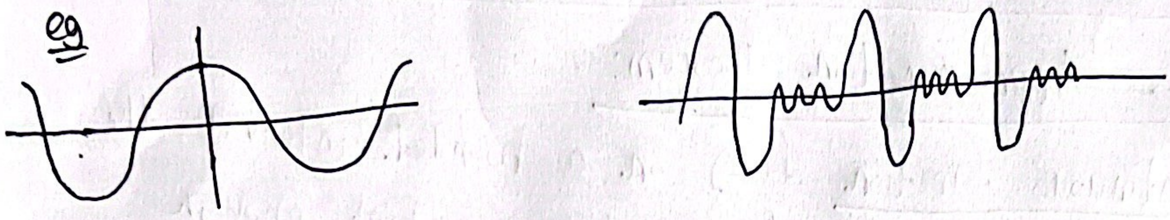
Intuition of fourier Series:

Any function can be constructed using sin and cos

$$f(x) = \Sigma \left(f' \sin(\ldots) + f' (\cos(\ldots)) \right)$$

eg



This implies there are ~~infinity~~ infinite amount of basis elements.

$$\underbrace{\{ \sin(\ldots), \cos(\ldots) \ldots\ldots \}}_{\infty \text{ dimension basis.}} \rightarrow \text{fourier basis}$$

Writing function using natural basis instead of fourier basis:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$$

example

$$\sin(x) = \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots}_{\text{still } \infty \text{ dim vector space, bcz it is a function.}}$$

But for polynomial, it should stop.

example

$$P_3(x) = x - \frac{x^3}{3!}$$

$P_n(x) \in V^{(n+1)}$ ← dimension

↑ vector space

$f(x) \in V^{\infty}$

Weierstrass Approximation Theorem:

For a continuous function $f(x)$, on a bounded interval, the following is always possible if you take a high enough degree polynomial:

For any $f \in C([0,1])$ and any $\epsilon > 0$, there exist a polynomial such that

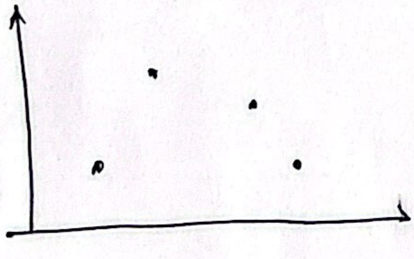
$$\max |f(x) - p(x)| \leq \epsilon$$

Remember, if $P_n(x) \approx P_\infty(x)$, $P(x)$ will act as $f(x)$

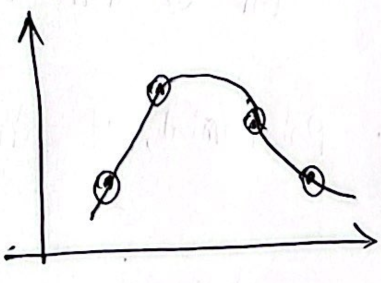
Polynomial Interpolation

→ There exists a $f(x)$ but we do not know how it looks like

→ But we know ~~some~~ the values of some points (x_1, y_1) (x_2, y_2) ...

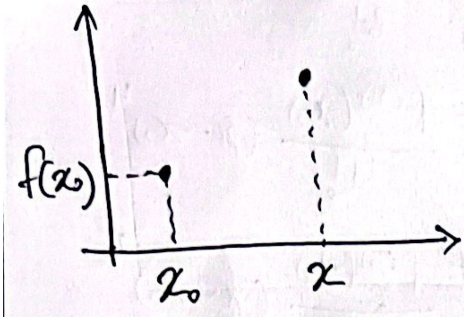


After interpolation



After interpolation, we will have a graph which would exactly do through these points.

Taylor Series:



If I know $f^{(1)}(x_0)$

$f^{(2)}(x_0)$

$f^{(3)}(x_0)$

$\vdots$

$f^{(\infty)}(x_0)$

Can I predict the value of the function at other point, x? $\rightarrow$ Yes.

$$\underbrace{f(x)}_{\text{new value at point } x} = \underbrace{f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots}_{\text{We have all info at point } x_0}$$

Example:

Expand the function $\sin(x)$ using taylor series, centered at $x_0 = 0$

Solution:

$f(x) = \sin(x) \longrightarrow f(0) = \sin(0) = 0$

$f^{(1)}(x) = \cos(x) \longrightarrow f^{(1)}(0) = \cos(0) = 1$

$f^{(2)}(x) = -\sin(x) \longrightarrow f^{(2)}(0) = -\sin(0) = 0$

$f^{(3)}(x) = -\cos(x) \longrightarrow f^{(3)}(0) = -\cos(0) = -1$

$f^{(4)}(x) = \sin(x) \longrightarrow f^{(4)}(0) = \sin(0) = 0$

$f^{(5)}(x) = \cos(x) \longrightarrow f^{(5)}(0) = \cos(0) = 1$

$$f(x) = f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4 + \frac{f^{(5)}(x_0)}{5!}(x-x_0)^5 + \cdots$$

$\boxed{x_0 = 0}$

$$f(x) = f(0) + f^{(1)}(0)(x-0) + \frac{f^{(2)}(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 + \cdots$$

$$= 0 + 1(x-0) + \frac{0}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4 + \frac{1}{5!}(x-0)^5 + \cdots$$

$$\therefore f(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

→ we knew all derivatives at $x_0 = 0$

→ Now we can find the value at, lets say for example, $x = 0.1$

If we take the 1st term, $f(\overset{0.1}{x}) = x$

2nd term, $f(x) = x - \frac{1}{3!}x^3$

3rd term, $f(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

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$\therefore$ if we want to find the value of the function at $x=0.1$

Taking 1st term, $f(0.1) = 0.1 = 0.1$

2nd term, $f(0.1) = 0.1 - \frac{1}{3!}(0.1)^3 =$

3rd term, $f(0.1) = 0.1 - \frac{1}{3!}(0.1)^3 + \frac{1}{5!}(0.1)^5 =$

~~Taylors~~ Taylor's Series:

$$f(x) = \underbrace{f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2}_{\text{If we take upto the third term, it becomes a polynomial of degree 2}} + \underbrace{\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots}_{\text{This terms are part of the error.}}$$

If we take upto the third term, it becomes a polynomial of degree 2

This terms are part of the error.

$\Rightarrow$ We can formalize the above thing using Taylor's Theorem

Taylor's theorem:

Let f be $(n+1)$ times differentiable on $(a+b)$ and let $f^{(n)}$ be continuous on $[a,b]$. If $x, x_0 \in [a,b]$ then there exists $\xi \in (a,b)$, such that:

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{Taylor's polynomial of degree, } n} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Lagrange form of the remainder}}$$

Taylor's polynomial of degree, n

Lagrange form of the ~~remainder~~ remainder

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Important point:

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

→ We do not know the actual value of ξ, if we did, we could find the exact value of the error, and hence, the exact value of the function. We only know that ξ is a value between (a,b), but we dont know the exact value.

→ What we can find, however, is the maximum bound of the error.

→ In other words, if ξ is a value between (a,b), what will be the maximum value of the error?

Example:

using the first 3 terms only.

$$f(x) = \sin(x) = \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!}} - \frac{x^7}{7!} + \cdots$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} + 0 \cdot x^6$$

$$P_6(x)$$

Using ~~From~~ Taylor's Theorem:

$$f(x) = \underbrace{P_6(x)}_{\text{Taylor's polynomial of degree 6}} + \underbrace{\frac{f^{(7)}(\xi)}{7!}(x-x_0)^7}_{\text{Lagrange form of the remainder}}$$

$$\Rightarrow \underbrace{\left| f(x) - P_6(x) \right|}_{\text{difference}} = \underbrace{\left| \frac{f^{(7)}(\xi)}{7!} (x-x_0)^7 \right|}_{\text{error}}$$

The taylor series was found while being centered at $x_0 = 0$

Now, lets say I want to find the value of the function at $x = 0.1$

$$\left| f(x) - P_6(x) \right| = \left| \frac{f^{(7)}(\xi)}{7!} (x-x_0)^7 \right|$$

$f(x) = \sin(x)$

$f^{(1)}(x) = \cos(x)$

$\vdots$

$f^{(7)}(x) = -\sin(x)$

$$\left| f(0.1) - P_6(0.1) \right| = \left| \frac{-\sin(\xi)}{7!} (0.1-0)^7 \right|$$

→ what will be the max^m value of $\sin(\xi)$ between $\overset{a}{0}$ & $\overset{b}{0.1}$?

↳ 1.

$$\left| f(0.1) - P_6(0.1) \right| \le \frac{1}{5040} (0.1)^7$$

$$\le 1.984 \times 10^{-11}$$ } estimation correct upto 11 decimal points.

→ the error is coming bcz of the truncation of the taylor series

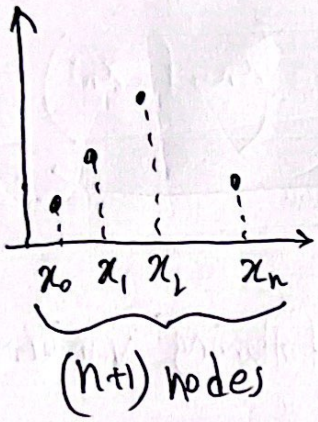
→ While doing numerical analysis, computer suffers from :

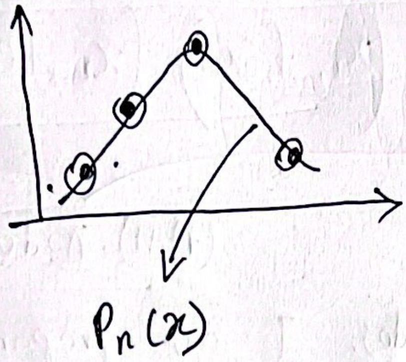
① Truncation error

② Rounding error of fl.

Polynomial Interpolation (Using Vandermonde Matrix)

If I am given $(n+1)$ number of nodes, the polynomial, that I can interpolate will be of degree, n





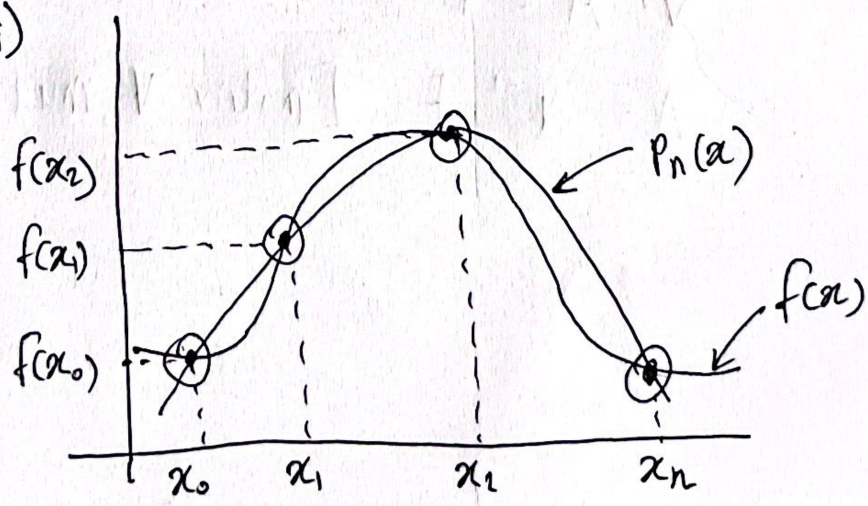
$$P_n(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots + a_n x^n$$

$$= \sum_{k=0}^{n} \underbrace{a_k}_{\text{constants/coefficients}} \underbrace{x^k}_{\text{natural basis}}$$

our task is to find these constants/coefficients.

Important point:

$$P_n(x_i) = f(x_i)$$



If I have $(n+1)$ nodes, I will have to satisfy $(n+1)$ conditions, I need to find $(n+1)$ nodes.

Lets say, I am given the following $(n+1)$ nodes:

$$\underbrace{(x_0, f(x_0))\ (x_1, f(x_1))\ (x_2, f(x_2))\ \cdots\ (x_n, f(x_n))}_{(n+1)\text{ nodes}}$$

From the above $(n+1)$ nodes, I can prepare the following vandermonde matrix:

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}}_{\text{Vandermonde Matrix, } V} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}}_{F}$$

$$V \cdot A = F$$

$A = V^{-1} \cdot F$ [matrix V must be invertible

Example:

$x_0 = 0$ $\quad$ $x_1 = \frac{\pi}{2}$ $\quad$ $x_2 = \pi$

$f(x_0) = 1$ $\quad$ $f(x_1) = 0$ $\quad$ $f(x_2) = -1$

3 nodes $\longrightarrow$ $P_2(x)$

$\downarrow$

(3X3) Vandermonde Matrix

$$\begin{bmatrix} 1 & 0 & 0^2 \\ 1 & \pi/2 & (\pi/2)^2 \\ 1 & \pi & \pi^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{\pi}{2} & \frac{\pi^2}{4} \\ 1 & \pi & \pi^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

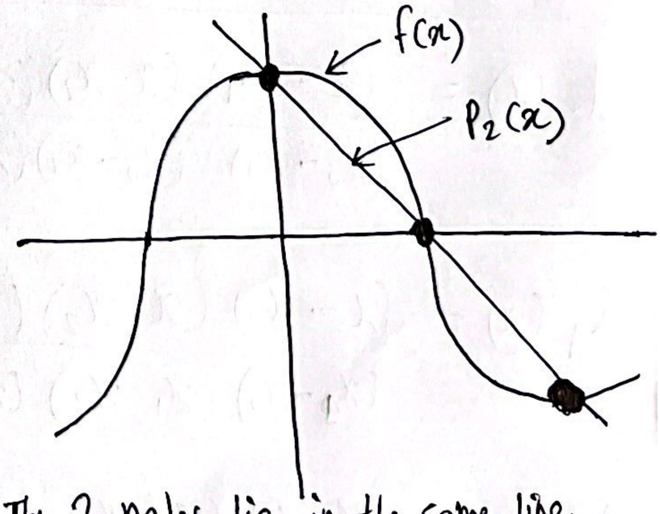
$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{\pi} \\ 0 \end{bmatrix}$$

$$P_2(x) = a_0 \cdot 1 + a_1 x^1 + a_2 x^2$$

$$= 1 + \left(-\frac{2}{\pi}\right)x + 0\,(x)^2$$

$$= 1 - \frac{2}{\pi}x$$

The above 3 nodes came from the function $\cos(x)$.



The 3 nodes lie in the same line, hence we got a straight line ($P_1(x)$) instead of a degree 2 polynomial, $P_2(x)$

Lagrange Basis:

Natural basis $\rightarrow \{1, x, x^2 \ldots\}$

~~lagrange basis~~ $\rightarrow \{\cancel{l_0(x)}\, l_1, l_2 \ldots\}$

lagrange basis $\rightarrow \{l_0(x), l_1(x), l_2(x) \ldots\}$

Previously, using natural basis:

$$P_n(x) = \sum_{k=0}^{n} a_k x^k$$

$x^k \rightarrow$ basis

$a_k \rightarrow$ coefficients [We need to calculate the coefficients]

$$= a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots a_n x^n$$

Now, using lagrange basis:

$l_k(x) \rightarrow$ basis [We need to calculate the basis]

$$P_n(x) = \sum_{k=0}^{n} f(x_k)\, l_k(x)$$

$f(x_k) \rightarrow$ coefficients

$$= f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + f(x_2)\, l_2(x) + \ldots f(x_n)\, l_n(x)$$

How to calculate Lagrange basis:

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}$$

$$\underbrace{}_{\text{will be provided in the question}}$$

$$P_n(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + \cdots + f(x_n)\, l_n(x)$$

need to calculate

Example

$x_0 = -\frac{\pi}{4}$ $\quad x_1 = 0$ $\quad x_2 = \frac{\pi}{4}$

$f(x_0) = \frac{1}{\sqrt{2}}$ $\quad f(x_1) = 1$ $\quad f(x_2) = \frac{1}{\sqrt{2}}$

3 nodes $\rightarrow P_2(x)$

$\downarrow$

3 coefficients

$\downarrow$

3 $l_k(x)$ s.

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-\frac{\pi}{4})}{(-\frac{\pi}{4})(-\frac{\pi}{2})} = \frac{8}{\pi^2}\, x\left(x-\frac{\pi}{4}\right)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+\frac{\pi}{4})(x-\frac{\pi}{4})}{(-\frac{\pi}{4})(\frac{\pi}{4})} = -\frac{16}{\pi^2}\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)$$

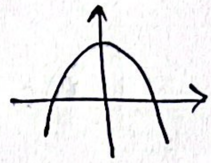
$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+\frac{\pi}{4})(x-0)}{(\frac{\pi}{2})(\frac{\pi}{4})} = \frac{8}{\pi^2}\, x\left(x+\frac{\pi}{4}\right)$$

$$P_2(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + f(x_2)\, l_2(x)$$

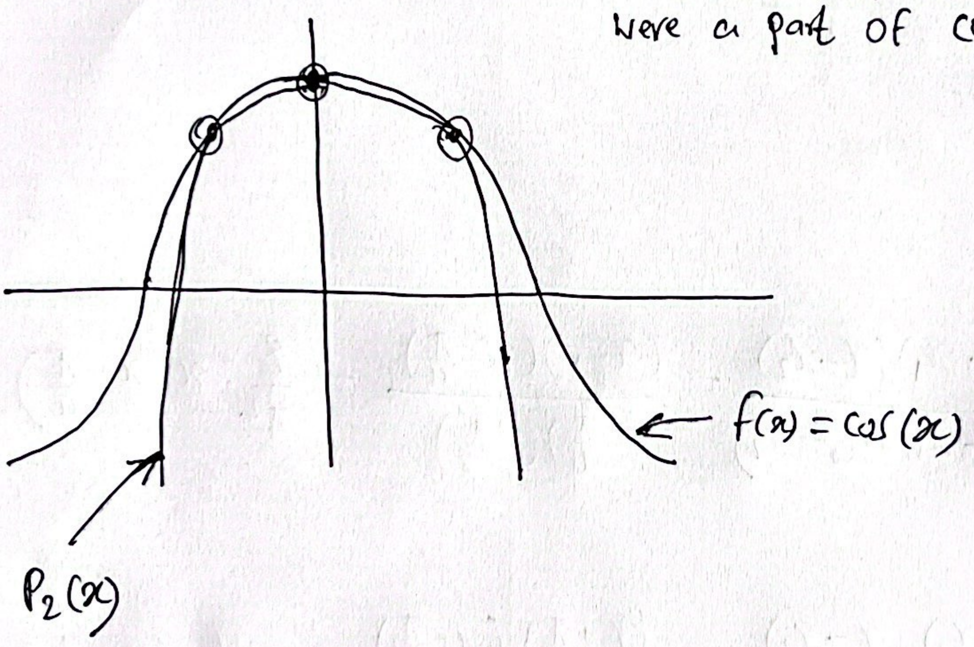
$$= \frac{1}{\sqrt{2}}\left(\frac{8}{\pi^2}\right)(x)\left(x-\frac{\pi}{4}\right) + 1\cdot\left(\frac{-16}{\pi^2}\right)\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)$$

$$+ \frac{1}{\sqrt{2}}\left(\frac{8}{\pi^2}\right)(x)\left(x+\frac{\pi}{4}\right)$$

$$= \underbrace{\frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}}-1\right)}_{-\text{ve number}} x^2 + 1 \quad \rightsquigarrow$$



The 3 nodes which was initially given were a part of cos (x) graph



Problem of Lagrange:

→ New nodes cannot be added

→ If added, need to do calculation of $l_k(x)$ newly again

Advantage of Lagrange:

→ No need to inverse a matrix (Inversing a matrix is computationally very expensive)