University of the People

Math 1302-01 Discrete Mathematics

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Question 1

Imagine you received a scholarship that would cover only 3 courses out of the 11 courses related to your field of study at your university. How many ways will you have to choose the three courses and how many ways can you choose for the remaining two courses if one course- English (out of the 11 courses) is mandatory to take? Explain how you arrived at the answer.

Solution

For this problem, we need to consider that English is mandatory among the 3 courses covered by the scholarship.

Given:

- Total number of courses: 11
- Scholarship covers: 3 courses
- English is mandatory in those 3 courses

Since English is mandatory, we've already selected 1 course out of the 3 scholarship courses. We need to select 2 more courses from the remaining 10 courses.

Number of ways to choose 2 courses from 10 courses $= \binom{10}{2} = \frac{10!}{2!(10-2)!} = \frac{10\times 9}{2\times 1} = 45$

For the remaining two courses (not covered by scholarship), we need to select 2 courses from the 8 courses that were not selected for the scholarship:

Number of ways to choose 2 courses from 8 courses $= \binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{8\times 7}{2\times 1} = 28$

Therefore, there are 45 ways to choose the three courses covered by scholarship (including English) and 28 ways to choose the remaining two courses.

Question 2

Consider two sets A and B having cardinality of your choice. Explain how many injective and bijective functions are possible from set A to set B. Please avoid the examples given in textbooks or online resources and come up with your own unique example.

Solution

Let's choose set A with cardinality 3 and set B with cardinality 4.

So, $|A| = 3$ and $|B| = 4$

For my example, let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$

Injective Functions (One-to-One):

A function $f : A \to B$ is injective if distinct elements in A map to distinct elements in B.

Number of injective functions from A to B $= |B| \times (|B| - 1) \times (|B| - 2) \times \ldots \times (|B| - |A| + 1)$

In our case: $4 \times 3 \times 2 = 24$

We can verify this by considering the process:

- For element a, we have 4 choices in B
- For element b, we have 3 remaining choices in B
- For element c, we have 2 remaining choices in B

This gives us $4 \times 3 \times 2 = 24$ possible injective functions.

Bijective Functions (One-to-One and Onto):

A function is bijective if it is both injective and surjective (every element in B is mapped to).

For a bijective function to exist, sets A and B must have the same cardinality. In our example, $|A| = 3$ and $|B| = 4$, so no bijective functions are possible.

If we were to consider sets of equal cardinality, for example $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, then:
Number of bijective functions $= |B|! = 3! = 6$
These 6 bijective functions would be:

- $f_1 = \{(a, 1), (b, 2), (c, 3)\}$
- $f_2 = \{(a, 1), (b, 3), (c, 2)\}$
- $f_3 = \{(a, 2), (b, 1), (c, 3)\}$
- $f_4 = \{(a, 2), (b, 3), (c, 1)\}$
- $f_5 = \{(a, 3), (b, 1), (c, 2)\}$
- $f_6 = \{(a, 3), (b, 2), (c, 1)\}$

Question 3

Find the coefficient of x^7 in the expansion of $x^3(x + 2)^{10} + (x + 5)^7$

Solution

We need to find the coefficient of x^7 in the expression $x^3(x + 2)^{10} + (x + 5)^7$.

Let's consider each term separately:
For $x^3(x + 2)^{10}$:
We can rewrite this as $x^3 \sum_{k=0}^{10} \binom{10}{k} x^k 2^{10-k}$
For the coefficient of x^7, we need $x^3 \cdot x^k = x^7$, which means $k = 4$.
So we need the coefficient of x^4 in $(x+2)^{10}$, which is $\binom{10}{4} 2^{10-4} = \binom{10}{4} 2^6 = \frac{10!}{4!(10-4)!} \cdot 64 = 210 \cdot 64 = 13,440$
For $(x + 5)^7$:
Using the binomial theorem: $(x + 5)^7 = \sum_{k=0}^{7} \binom{7}{k} x^k 5^{7-k}$
For the coefficient of x^7, we need $k = 7$.
So the coefficient is $\binom{7}{7} 5^{7-7} = \binom{7}{7} 5^0 = 1$
Total coefficient of x^7:
Adding the coefficients from both terms: $13,440 + 1 = 13,441$
Therefore, the coefficient of x^7 in the expansion of $x^3(x + 2)^{10} + (x + 5)^7$ is 13,441.

Question 4

A newly constructed apartment has 30 club members. The club has planned to create a sports committee consisting of 7 club members. How many different sports committees are possible? How many committees are possible if it is mandatory to have the selected treasurer of the club members in the sports committee? Explain in detail.

Solution

Part 1: How many different sports committees are possible?

We need to select 7 members out of 30 club members. This is a combination problem, as the order of selection doesn't matter.

Number of ways to form the committee $= \binom{30}{7} = \frac{30!}{7!(30-7)!} = \frac{30!}{7!23!}$
Using the formula: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
$\binom{30}{7} = \frac{30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$
$= \frac{30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24}{5040}$
$= \frac{4,076,350,421,560}{5040}$
$= 2,035,800$

Therefore, 2,035,800 different sports committees are possible.

Part 2: How many committees are possible if it is mandatory to have the selected treasurer in the sports committee?

Given that the treasurer must be included in the committee, we have already selected 1 person out of the 7 needed. We need to select 6 more people from the remaining 29 club members.

Number of ways $= \binom{29}{6} = \frac{29!}{6!(29-6)!} = \frac{29!}{6!23!}$

$\binom{29}{6} = \frac{29\times28\times27\times26\times25\times24}{6\times5\times4\times3\times2\times1}$

$= \frac{29\times28\times27\times26\times25\times24}{720}$

$= \frac{4{,}076{,}350{,}421{,}560 \div 30}{720}$

$= \frac{135{,}878{,}347{,}385.33}{720}$

$= 475,020$

Therefore, 475,020 committees are possible if it is mandatory to have the treasurer in the sports committee.

Question 5

Answer the following questions:

Part i

Explain bit string in your own words.

A bit string is a sequence of binary digits (0s and 1s) arranged in a specific order. Each position in the string can only have one of two possible values: 0 or 1. Bit strings are fundamental in computer science and digital systems as they represent how information is stored and processed at the most basic level.

The length of a bit string refers to the number of binary digits it contains. For example, "1010" is a bit string of length 4, while "01101" is a bit string of length 5. Bit strings are used to represent data, instructions, memory addresses, and virtually all information in computing systems.

Part ii

Give an example of a bit string with any length and weight and explain how combinations help find the number of bit strings possible for the example.

Let's consider bit strings of length 6.

The weight of a bit string is defined as the number of 1s in the string. For my example, I'll choose bit strings of length 6 with weight 3, meaning they contain exactly three 1s and three 0s.

An example of such a bit string would be: 101010

To find the number of possible bit strings of length 6 with weight 3, we need to determine how many ways we can place three 1s in six positions. This is a combination problem, as the order of selection matters.

Number of possible bit strings $= \binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = \frac{6\times5\times4}{3\times2\times1} = 20$

Therefore, there are 20 possible bit strings of length 6 with weight 3.

Some examples include: 111000, 110100, 110010, 110001, 101100, 101010, 101001, 100110, 100101, 100011, 011100, 011010, 011001, 010110, 010101, 010011, 001110, 001101, 001011, 000111

The combination approach works because we're essentially asking: "In how many ways can we choose 3 positions out of 6 to place the 1s?" Once we've placed the 1s, the remaining positions must contain 0s.

Part iii

Choose a 3-digit number example and explain the number of derangements that can be formed from it.

Let's choose the 3-digit number 427 as our example.

A derangement is a permutation where no element appears in its original position. For our 3-digit number 427, we need to find permutations where: - 4 is not in the first position - 2 is not in the second position - 7 is not in the third position

For a 3-digit number, the total number of permutations is $3! = 6$. The possible permutations of 427 are: 427, 472, 247, 274, 724, 742

Now, let's check which of these are derangements: - 427: This is the original number, not a derangement. - 472: 4 is still in its original position, so not a derangement. - 247: 2 is not in its original position, but 7 is, so not a derangement. - 274: All digits are in new positions, so this is a derangement. - 724: 2 is in a new position, but 4 is in its original position, so not a derangement. - 742: All digits are in new positions, so this is a derangement.

Therefore, there are 2 derangements possible from the number 427: 274 and 742.

This matches the formula for calculating the number of derangements: $D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$

For $n = 3$: $D_3 = 3! \left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) = 6 \times \frac{1}{3} = 2$

Part iv

To create a 4-digit password for your Android phone,

Part a

How many ways are there to crack the password if no digit repeats?

If no digit repeats, we need to select 4 digits from the 10 possible digits (0-9) and arrange them in a specific order.

This is a permutation problem: $P(10, 4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10 \times 9 \times 8 \times 7 = 5,040$

Therefore, there are 5,040 possible 4-digit passwords with no repetition.

Part b

If the digits can be repeated, how many ways are there to crack the password?

If digits can be repeated, each of the 4 positions can be filled with any of the 10 digits (0-9).

Total number of possibilities $= 10^4 = 10,000$

Therefore, there are 10,000 possible 4-digit passwords when repetition is allowed.

Concept Justification

In each of these questions, what concept would you use, permutations or combinations? Justify?

1. **Question 1 (Scholarship courses):** This uses combinations because the order of selecting the courses doesn't matter. We're simply interested in which 3 courses are covered by the scholarship, not the order in which they are chosen. Similarly, for the remaining 2 courses, the order doesn't matter.

2. **Question 2 (Functions):** This uses permutations because when defining a function, we need to specify which element in B corresponds to each element in A. The order of assignment matters.

3. **Question 3 (Binomial expansion):** This uses combinations through the binomial coefficient $\binom{n}{k}$, which represents the number of ways to choose k objects from a set of n objects, where order doesn't matter.

4. **Question 4 (Sports committee):** This uses combinations because we are selecting a group of 7 people from 30, and the order of selection doesn't matter. What matters is who is in the committee, not the order they were selected.

5. **Question 5:**

   - **Part ii (Bit strings):** This uses combinations because we're choosing which positions in the string will contain 1s, and the order of selection doesn't matter.
   - **Part iii (Derangements):** This involves permutations with additional constraints. Derangements are a special type of permutation where no element appears in its original position.

- **Part iv (Password):**
  - **Part a (No repeats):** This uses permutations because we're arranging 4 digits from 10 possible digits, and the order matters for a password.
  - **Part b (With repeats):** This uses the multiplication principle with replacement, which is a type of permutation where repetition is allowed.

References

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2. Anderson, I. (2001). *A First Course in Discrete Mathematics*. Springer-Verlag London.
3. Grimaldi, R. P. (2013). *Discrete and Combinatorial Mathematics: An Applied Introduction* (5th ed.). Pearson.