University of the People

Discrete Mathematics Assignment
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1 Partial Ordering Relation

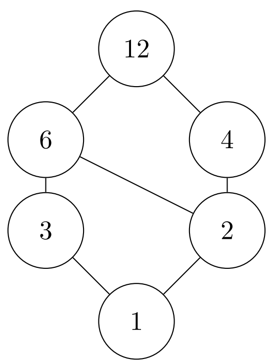
A partial ordering relation is a binary relation that is reflexive, antisymmetric, and transitive. I'll explain this using the "divides" relation ($|$) on the set $\{1, 2, 3, 6, 12\}$.

In the divides relation, we say that $a|b$ if a divides b evenly (without remainder). Formally, $a|b$ if there exists an integer k such that $b = a \cdot k$.

Let's verify that the divides relation is a partial ordering:

1. **Reflexive**: For any element a, $a|a$ (since $a = a \cdot 1$). ✓
2. **Antisymmetric**: If $a|b$ and $b|a$, then $a = b$. ✓
   (If a divides b and b divides a, they must be equal)
3. **Transitive**: If $a|b$ and $b|c$, then $a|c$. ✓
   (If a divides b and b divides c, then a divides c as well)

Here's the Hasse diagram for this relation:



Explanation of the Hasse Diagram:

- In a Hasse diagram, elements are represented as nodes, and the relation between elements is shown by lines connecting them.
- Elements are arranged vertically according to their "rank" - if $a|b$, then b appears above a.
- We omit drawing edges implied by transitivity and reflexivity to keep the diagram clean.
- In this diagram, 1 is at the bottom since it divides all other numbers.
- 12 is at the top since it's divisible by all other numbers in the set.
- If we can follow a path upward from element a to element b, then $a|b$.
- Note that not all pairs of elements are comparable. For example, 2 and 3 are not related by the divides relation (neither divides the other), which is why there's no path between them.

2 Transitive Closure of Relations

2.1 Definition and Uses

The transitive closure of a relation R on a set A is the smallest transitive relation that contains R. It's denoted as R^+ or R^* and includes all pairs that are connected directly or indirectly through the relation.

Formally, (a, c) is in the transitive closure of R if either:

- (a, c) is in R, or
- There exists some b such that (a, b) and (b, c) are both in R, or
- There's a longer chain connecting a to c through the relation.

Uses of Transitive Closure:

1. **Graph Theory**: Finding all reachable nodes from a given node in a graph
2. **Database Systems**: Computing all dependencies between attributes
3. **Network Analysis**: Determining all nodes that can be reached from a starting point
4. **Programming Language Semantics**: Defining the inheritance relation between classes

Example: Consider a relation $R = \{(a, b), (b, c), (d, e)\}$ on the set $\{a, b, c, d, e\}$. R itself is not transitive because (a, b) and (b, c) are in R, but (a, c) is not.

The transitive closure $R^+ = \{(a, b), (b, c), (d, e), (a, c)\}$ R^+ adds (a, c) to make the relation transitive.

2.2 Equivalence Relation

Consider the relation R defined on the set of integers as $R = \{(x, y) : x, y \in \mathbb{Z}, (x - y) \text{ is a multiple of } 11\}$.

To show R is an equivalence relation, I need to prove it's reflexive, symmetric, and transitive.

1. **Reflexive**: For any $x \in \mathbb{Z}$, $(x - x) = 0$, which is a multiple of 11 ($0 = 0 \times 11$). So $(x, x) \in R$ for all x. ✓
2. **Symmetric**: If $(x, y) \in R$, then $(x - y)$ is a multiple of 11.
   This means $(x - y) = 11k$ for some integer k.
   Then $(y - x) = -(x - y) = -11k = 11(-k)$, which is also a multiple of 11.
   So $(y, x) \in R$. ✓

3. **Transitive**: If $(x, y) \in R$ and $(y, z) \in R$, then:
   $(x - y) = 11k_1$ for some integer k_1
   $(y - z) = 11k_2$ for some integer k_2
   Adding: $(x - y) + (y - z) = (x - z) = 11k_1 + 11k_2 = 11(k_1 + k_2)$
   So $(x - z)$ is a multiple of 11, which means $(x, z) \in R$. ✓

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

2.3 Equivalence Class of 0

The equivalence class of 0, denoted [0], is the set of all integers y such that $(0, y) \in R$. This means $(0 - y) = -y$ must be a multiple of 11.

So $y = 11k$ for some integer k. Therefore, $[0] = \{\ldots, -33, -22, -11, 0, 11, 22, 33, \ldots\} = \{11k : k \in \mathbb{Z}\}$

2.4 Number of Equivalence Classes

For the relation R, two integers x and y are in the same equivalence class if and only if $(x - y)$ is a multiple of 11.

We can classify any integer by its remainder when divided by 11. There are exactly 11 possible remainders: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.

So there are exactly 11 distinct equivalence classes:

$$
\begin{aligned}
[0] &= \{x \in \mathbb{Z} : x \equiv 0 \pmod{11}\} \\
[1] &= \{x \in \mathbb{Z} : x \equiv 1 \pmod{11}\} \\
[2] &= \{x \in \mathbb{Z} : x \equiv 2 \pmod{11}\} \\
&\vdots \\
[10] &= \{x \in \mathbb{Z} : x \equiv 10 \pmod{11}\}
\end{aligned}
$$

3 Mathematical Induction Proof

Prove that for all positive integers $n \geq 1$, $5n > 4n$ using mathematical induction.

Base Case ($n = 1$):

$$
\begin{aligned}
5(1) &= 5 \\
4(1) &= 4 \\
5 &> 4 \quad \checkmark
\end{aligned}
$$

Inductive Hypothesis: Assume that for some $k \geq 1$, $5k > 4k$.
Inductive Step: Need to prove that $5(k + 1) > 4(k + 1)$

$$
\begin{aligned}
5(k + 1) &= 5k + 5 \\
4(k + 1) &= 4k + 4
\end{aligned}
$$

From the inductive hypothesis, we know $5k > 4k$.
Adding 5 to both sides: $5k + 5 > 4k + 5$
Since $5 > 4$, we know $4k + 5 > 4k + 4$
Therefore, by transitivity: $5k + 5 > 4k + 4$

So $5(k + 1) > 4(k + 1)$ ✓

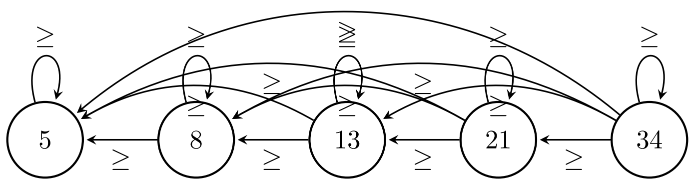
By the principle of mathematical induction, $5n > 4n$ holds for all positive integers $n \geq 1$.

4 Relation $R =$ ‘ $\geq'$ on a Set of Five Integers

Let's choose the set $A = \{5, 8, 13, 21, 34\}$ (five integers from the Fibonacci sequence).

The relation $R =$ ‘ $\geq'$ (greater than or equal to) is defined on A.

4.1 Diagrammatical Representation of R on A



4.2 Adjacency Matrix for R on A

For the relation $R =$ ‘ $\geq'$ on the set $A = \{5, 8, 13, 21, 34\}$, the adjacency matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

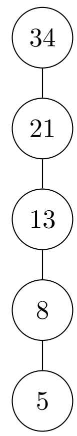
In this matrix:

- Row i and column j contain a 1 if element $i \geq$ element j
- Row i and column j contain a 0 if element $i <$ element j

For example:

- The entry at row 8, column 5 is 1 because $8 \geq 5$ (true)
- The entry at row 5, column 8 is 0 because $5 \geq 8$ (false)
- All diagonal entries are 1 because every number is equal to itself ($x \geq x$)

4.3 Hasse Diagram for $R = `\geq'$ on A



Explanation of the Hasse Diagram:

- In a Hasse diagram for the $\geq$ relation, larger elements appear higher in the diagram.
- The diagram shows the transitive reduction of the relation, omitting edges that can be inferred through transitivity.
- For this relation $R = `\geq'$ on $A = \{5, 8, 13, 21, 34\}$, the elements form a total ordering.
- The diagram shows a single vertical chain with 34 at the top (greatest) and 5 at the bottom (smallest).
- Each element is connected only to its immediate predecessor/successor, as the other relationships can be inferred through transitivity.
- $34 \geq 21 \geq 13 \geq 8 \geq 5$, and from this we can infer all other relations like $34 \geq 5$.

This Hasse diagram illustrates that ‘$\geq$’ is a total order on this set, meaning every pair of elements is comparable (either $a \geq b$ or $b \geq a$ for any a, b in the set).

References

[1] Rosen, K. H. (2019). *Discrete Mathematics and Its Applications* (8th ed.). McGraw-Hill Education.

[2] Grimaldi, R. P. (2017). *Discrete and Combinatorial Mathematics: An Applied Introduction* (5th ed.). Pearson.

[3] Epp, S. S. (2019). *Discrete Mathematics with Applications* (4th ed.). Cengage Learning.