University of the People

Math 1302-01 Discrete Mathematics

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Recurrence Relations Assignment

1. Explain the recursive function with simple examples.

A recursive function is a function that is defined in terms of itself. It solves complex problems by breaking them down into simpler sub-problems of the same type, and then combining these solutions (Cormen et al., 2009).

Example 1: Factorial Function

The factorial of a non-negative integer n (denoted as $n!$) is defined recursively as:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \times (n-1)! & \text{if } n > 0 \end{cases}$$

For instance, to calculate 4!:

$$\begin{aligned} 4! &= 4 \times 3! \\ &= 4 \times 3 \times 2! \\ &= 4 \times 3 \times 2 \times 1! \\ &= 4 \times 3 \times 2 \times 1 \times 0! \\ &= 4 \times 3 \times 2 \times 1 \times 1 \\ &= 24 \end{aligned}$$

Example 2: Fibonacci Sequence

The Fibonacci sequence is defined by the recurrence relation (Graham et al., 1994):

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

This generates the sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, ...

Example 3: Binary Search

Binary search is a recursive algorithm that divides a sorted array in half repeatedly until the target element is found or determined not to exist (Sedgewick & Wayne, 2011):

$$\text{binarySearch}(A, \text{target}, \text{left}, \text{right}) = \begin{cases} \text{"Not found"} & \text{if left ¿ right} \\ \text{mid} & \text{if } A[\text{mid}] = \text{target} \\ \text{binarySearch}(A, \text{target}, \text{left}, \text{mid} - 1) & \text{if } A[\text{mid}] > \text{target} \\ \text{binarySearch}(A, \text{target}, \text{mid} + 1, \text{right}) & \text{if } A[\text{mid}] < \text{target} \end{cases}$$

where mid $= \lfloor \frac{\text{left}+\text{right}}{2} \rfloor$

2. Washing Machine Manufacturing Problem

a) Set up a recurrence relation to describe the number of machines produced by the company in the first n months.

Let M_n represent the number of washing machines produced in the first n months.

Given:

- In month 1, the company produces 1 machine: $M_1 = 1$
- In month 2, the company produces 2 machines: $M_2 = 2$
- Each month n, the company produces n machines

To find the total number of machines produced through the first n months:

$$M_n = \begin{cases} 1 & \text{if } n = 1 \\ M_{n-1} + n & \text{if } n \geq 2 \end{cases}$$

This type of linear recurrence relation with non-constant coefficients is common in production planning problems (Rosen, 2018).

b) How many washing machines are produced by the company in the first year?

To find the number of washing machines produced in the first year (12 months), we need to calculate M_{12}.

Let's compute the values iteratively:

$$\begin{aligned}
M_1 &= 1 \\
M_2 &= M_1 + 2 = 1 + 2 = 3 \\
M_3 &= M_2 + 3 = 3 + 3 = 6 \\
M_4 &= M_3 + 4 = 6 + 4 = 10 \\
M_5 &= M_4 + 5 = 10 + 5 = 15 \\
M_6 &= M_5 + 6 = 15 + 6 = 21 \\
M_7 &= M_6 + 7 = 21 + 7 = 28 \\
M_8 &= M_7 + 8 = 28 + 8 = 36 \\
M_9 &= M_8 + 9 = 36 + 9 = 45 \\
M_{10} &= M_9 + 10 = 45 + 10 = 55 \\
M_{11} &= M_{10} + 11 = 55 + 11 = 66 \\
M_{12} &= M_{11} + 12 = 66 + 12 = 78
\end{aligned}$$

Therefore, the company produces 78 washing machines in the first year.

c) Find an explicit formula for the number of washing machines produced by the company in the first n months. Mention the method used to find the formula and show the steps clearly.

I'll use the method of examining patterns to derive an explicit formula, a standard approach in solving recurrence relations (Goodaire & Parmenter, 2018).

Let's look at the first few terms:

$$
\begin{aligned}
M_1 &= 1 \\
M_2 &= 1 + 2 = 3 \\
M_3 &= 1 + 2 + 3 = 6 \\
M_4 &= 1 + 2 + 3 + 4 = 10 \\
M_5 &= 1 + 2 + 3 + 4 + 5 = 15
\end{aligned}
$$

We can observe that M_n is the sum of the first n positive integers, which is equal to $\frac{n(n+1)}{2}$. This is a well-known formula for the sum of an arithmetic sequence (Grimaldi, 2013).

To verify:

$$
\begin{aligned}
M_1 &= \frac{1(1+1)}{2} = \frac{2}{2} = 1\checkmark \\
M_2 &= \frac{2(2+1)}{2} = \frac{6}{2} = 3\checkmark \\
M_3 &= \frac{3(3+1)}{2} = \frac{12}{2} = 6\checkmark \\
M_4 &= \frac{4(4+1)}{2} = \frac{20}{2} = 10\checkmark \\
M_{12} &= \frac{12(12+1)}{2} = \frac{12 \times 13}{2} = \frac{156}{2} = 78\checkmark
\end{aligned}
$$

Therefore, the explicit formula for the number of washing machines produced in the first n months is:

$$M_n = \frac{n(n+1)}{2}$$

Method used: Pattern recognition and validation. I observed that the sequence matches the sum of the first n positive integers, applied the known formula $\frac{n(n+1)}{2}$, and verified it against known values.

References

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