University of the People

Math 1302-01 Discrete Mathematics

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Functions, Domains, and Sequences Assignment

Question 1: Function Analysis

For the function $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = \begin{cases} n+3 & \text{if } n \text{ is odd} \\ n-5 & \text{if } n \text{ is even} \end{cases}$

Injectivity: A function is injective if each output maps to at most one input.

- For odd inputs: $f(\text{odd}) = \text{odd} + 3$
- For even inputs: $f(\text{even}) = \text{even} - 5$

Let's check if different inputs can produce the same output:

- If $f(n_1) = f(n_2)$, we need to check if $n_1 = n_2$ must always be true
- If both n_1 and n_2 are odd: $n_1 + 3 = n_2 + 3 \Rightarrow n_1 = n_2$ ✓
- If both n_1 and n_2 are even: $n_1 - 5 = n_2 - 5 \Rightarrow n_1 = n_2$ ✓
- If n_1 is odd and n_2 is even: $n_1 + 3 = n_2 - 5 \Rightarrow n_1 = n_2 - 8$

Since n_1 is odd and n_2 is even, $n_1 = n_2 - 8$ would make n_1 even (since $n_2 - 8$ is even), which contradicts our assumption. So this case is impossible.

Therefore, the function is injective.

Surjectivity: A function is surjective if every element in the range has at least one preimage in the domain.

Let's check if every integer in $\mathbb{Z}$ can be obtained as an output:

- If m is even:
  - If we set $n = m + 5$, then $f(n) = m + 5 - 5 = m$ (when n is even)
- If m is odd:
  - If we set $n = m - 3$, then $f(n) = m - 3 + 3 = m$ (when n is odd)

Therefore, every integer has a preimage, making the function surjective.

Inverse Function: Since f is bijective (both injective and surjective), it has an inverse $f^{-1} : \mathbb{Z} \to \mathbb{Z}$

$$f^{-1}(m) = \begin{cases} m+5 & \text{if } m \text{ is even} \\ m-3 & \text{if } m \text{ is odd} \end{cases} \tag{1}$$

Question 2: Function Composition with 3-Element Sets

Let's define sets A, B, and C, each with three elements:

- $A = \{a_1, a_2, a_3\}$
- $B = \{b_1, b_2, b_3\}$
- $C = \{c_1, c_2, c_3\}$

Let's define functions $f : A \to B$ and $g : B \to C$ as:

- $f(a_1) = b_2,\ f(a_2) = b_1,\ f(a_3) = b_3$
- $g(b_1) = c_3,\ g(b_2) = c_1,\ g(b_3) = c_2$

Composition $g \circ f$ (gof): $(g \circ f)(a) = g(f(a))$

- $(g \circ f)(a_1) = g(f(a_1)) = g(b_2) = c_1$
- $(g \circ f)(a_2) = g(f(a_2)) = g(b_1) = c_3$
- $(g \circ f)(a_3) = g(f(a_3)) = g(b_3) = c_2$

So $g \circ f : A \to C$ is defined as $\{(a_1, c_1), (a_2, c_3), (a_3, c_2)\}$

Composition $f \circ g$ (fog): This would require $g : C \to B$ and $f : B \to A$, but we defined $f : A \to B$ and $g : B \to C$, so $f \circ g$ isn't defined for our functions.

Both $f \circ g$ and $g \circ f$ cannot be defined with the sets as given because the domains and codomains don't align properly for $f \circ g$. The composition $g \circ f$ is defined and maps from A to C.

Are they equal? Since $f \circ g$ cannot be defined with our setup, they cannot be equal.

Question 3: Student Grades Composition

We have 5 students with their scores and grades:

- Ani: 75 → B
- Leon: 60 → C
- Linh: 85 → B+
- Liam: 95 → A
- Abdul: 60 → C

Let's define:

- Set $S = \{\text{Ani}, \text{Leon}, \text{Linh}, \text{Liam}, \text{Abdul}\}$ (students)
- Set $M = \{60, 75, 85, 95\}$ (marks)
- Set $G = \{A, B, B+, C\}$ (grades)

Function $f : S \to M$ (student to marks)

- $f(\text{Ani}) = 75$
- $f(\text{Leon}) = 60$
- $f(\text{Linh}) = 85$
- $f(\text{Liam}) = 95$
- $f(\text{Abdul}) = 60$

Function $g : M \to G$ (marks to grades)

- $g(60) = C$
- $g(75) = B$
- $g(85) = B+$
- $g(95) = A$

Domains and Ranges:

- Domain of f: $S = \{\text{Ani}, \text{Leon}, \text{Linh}, \text{Liam}, \text{Abdul}\}$
- Range of f: $\{60, 75, 85, 95\} \subseteq M$
- Domain of g: $M = \{60, 75, 85, 95\}$
- Range of g: $\{A, B, B+, C\} = G$

Composite function $h = g \circ f$: This maps students directly to grades

- $h(\text{Ani}) = g(f(\text{Ani})) = g(75) = B$
- $h(\text{Leon}) = g(f(\text{Leon})) = g(60) = C$
- $h(\text{Linh}) = g(f(\text{Linh})) = g(85) = B+$
- $h(\text{Liam}) = g(f(\text{Liam})) = g(95) = A$
- $h(\text{Abdul}) = g(f(\text{Abdul})) = g(60) = C$

Is the composition commutative? For $f \circ g$ to exist, we would need $g : X \to S$ and $f : S \to Y$, but our functions don't fit this pattern. Therefore, $f \circ g$ is not defined, and composition is not commutative in this case.

Question 4: Sequences Concepts

Sequence: An ordered list of numbers following a specific pattern.

Example: The sequence $3, 9, 27, 81, \ldots$ where each term is found by multiplying the previous term by 3.

Recursive Function: Defines each term of a sequence based on previous terms.

Example: The Fibonacci-like sequence defined as: $a_1 = 4$, $a_2 = 7$, $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$ This gives: $4, 7, 15, 29, 59, \ldots$

Closed Formula: An explicit formula to find the nth term without calculating previous terms.

Example: Consider the sequence $5, 8, 11, 14, \ldots$ The closed formula is $a_n = 5 + 3(n-1) = 3n + 2$

Arithmetic Sequence: A sequence where the difference between consecutive terms is constant.

Example: $7, 13, 19, 25, \ldots$ has a common difference of 6 The formula is $a_n = a_1 + (n-1)d = 7 + (n-1)6 = 7 + 6n - 6 = 6n + 1$

Geometric Sequence: A sequence where the ratio between consecutive terms is constant.

Example: $5, 10, 20, 40, \ldots$ has a common ratio of 2 The formula is $a_n = a_1 r^{n-1} = 5 \times 2^{n-1}$

Question 5: Sequence Analysis

Part i: Finding the formula for $5, 6, 7, 8, 8, 9, 10, 10, 11, 12, 12 \ldots$

Looking at the pattern, we have: $5, 6, 7, 8, 8, 9, 10, 10, 11, 12, 12 \ldots$

The pattern seems to be that each number appears once until 8, then each number appears twice. Let's define a_n as the nth term:

For $n \leq 3$: $a_n = n + 4$ For $n > 3$: $a_n = \lfloor \frac{n+5}{2} \rfloor + 4$

This gives us: $a_1 = 5$, $a_2 = 6$, $a_3 = 7$, $a_4 = 8$, $a_5 = 8$, $a_6 = 9$, $a_7 = 10$, $a_8 = 10$, $a_9 = 11$, $a_{10} = 12$, $a_{11} = 12$

The next four terms would be: $13, 13, 14, 14$

Part ii: Series $6 + 36 + 216 + \ldots$

This is a geometric series with first term $a = 6$ and common ratio $r = 6$. The formula for the nth term is $a_n = 6 \times 6^{n-1} = 6^n$

Next three terms: $6^4 = 1,296$, $6^5 = 7,776$, $6^6 = 46,656$

Closed formula: $a_n = 6^n$

Sum of first n terms: $S_n = \frac{a(1-r^n)}{1-r} = \frac{6(1-6^n)}{1-6} = \frac{6(1-6^n)}{-5} = \frac{(6^n-1)\times 6}{5}$

Part iii: Series $21 + 24 + 27 + \ldots$

This is an arithmetic sequence with first term $a = 21$ and common difference $d = 3$. The formula for the nth term is $a_n = a + (n-1)d = 21 + (n-1)3 = 21 + 3n - 3 = 3n + 18$

Next three terms: $30, 33, 36$
Closed formula: $a_n = 3n + 18$
Sum of first n terms: $S_n = \frac{n(a_1+a_n)}{2} = \frac{n(21+(3n+18))}{2} = \frac{n(3n+39)}{2} = \frac{3n^2+39n}{2}$

References

[1] Doerr, A., & Levasseur, K. (2022). *Applied discrete structures* (3rd ed.). Licensed under CC BY-NC-SA.

[2] Levin, O. (2021). *Discrete mathematics: An open introduction* (3rd ed.). Licensed under CC 4.0.