UoPeople MATH 1302 U8

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Problem 1: Greatest Common Divisor using Division Algorithm

Problem: Choose a pair of two-digit numbers and find their greatest common divisor. Show the steps.

Solution: Let's find $\gcd(84, 36)$ using the Euclidean algorithm.

Using the division algorithm repeatedly:

$$84 = 36 \cdot 2 + 12 \tag{1}$$

$$36 = 12 \cdot 3 + 0 \tag{2}$$

Since the remainder is 0, we have $\gcd(84, 36) = 12$.

Verification: $84 = 12 \cdot 7$ and $36 = 12 \cdot 3$, where $\gcd(7, 3) = 1$.

Problem 2: Modular Arithmetic Operation

Problem: Choose any one modular arithmetic operation from the set of positive integers $\mathbb{Z}$ and find the identities and inverses of all elements.

Solution: I choose multiplication modulo 7, i.e., the operation $\times_7$ on the set $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$.

Operation Table for Multiplication modulo 7:

$\times_7$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Identity Element: The identity element is 1, since $a \times_7 1 = a$ for all $a \in \mathbb{Z}_7$.

Inverses:

- 0 has no inverse (since $0 \times_7 a = 0 \neq 1$ for any a)
- $1^{-1} = 1$ (since $1 \times_7 1 = 1$)
- $2^{-1} = 4$ (since $2 \times_7 4 = 8 \equiv 1 \pmod 7$)
- $3^{-1} = 5$ (since $3 \times_7 5 = 15 \equiv 1 \pmod 7$)
- $4^{-1} = 2$ (since $4 \times_7 2 = 8 \equiv 1 \pmod 7$)
- $5^{-1} = 3$ (since $5 \times_7 3 = 15 \equiv 1 \pmod 7$)
- $6^{-1} = 6$ (since $6 \times_7 6 = 36 \equiv 1 \pmod 7$)

All non-zero elements have multiplicative inverses because 7 is prime, making $(\mathbb{Z}_7 \setminus \{0\}, \times_7)$ a group.

Problem 3: Solving Quadratic Equation in Modular Arithmetic

Problem: Find and explain solution for the equation $x^2 + 5x + 1 = 0$ in $\mathbb{Z}_3$.

Solution: We need to solve $x^2 + 5x + 1 \equiv 0 \pmod 3$.

First, let's simplify the coefficients modulo 3:

$$5 \equiv 2 \pmod 3 \tag{3}$$
$$1 \equiv 1 \pmod 3 \tag{4}$$

So our equation becomes: $x^2 + 2x + 1 \equiv 0 \pmod 3$

Let's check each element in $\mathbb{Z}_3 = \{0, 1, 2\}$:

For $x = 0$: $0^2 + 2(0) + 1 = 0 + 0 + 1 = 1 \equiv 1 \pmod 3 \neq 0$

For $x = 1$: $1^2 + 2(1) + 1 = 1 + 2 + 1 = 4 \equiv 1 \pmod 3 \neq 0$

For $x = 2$: $2^2 + 2(2) + 1 = 4 + 4 + 1 = 9 \equiv 0 \pmod 3$

Answer: The equation $x^2+5x+1 \equiv 0 \pmod{3}$ has exactly one solution: $x \equiv 2 \pmod{3}$.

Verification: We can factor the simplified equation as: $x^2 + 2x + 1 = (x + 1)^2 \equiv 0 \pmod{3}$

This means $x + 1 \equiv 0 \pmod{3}$, so $x \equiv -1 \equiv 2 \pmod{3}$, confirming our solution.